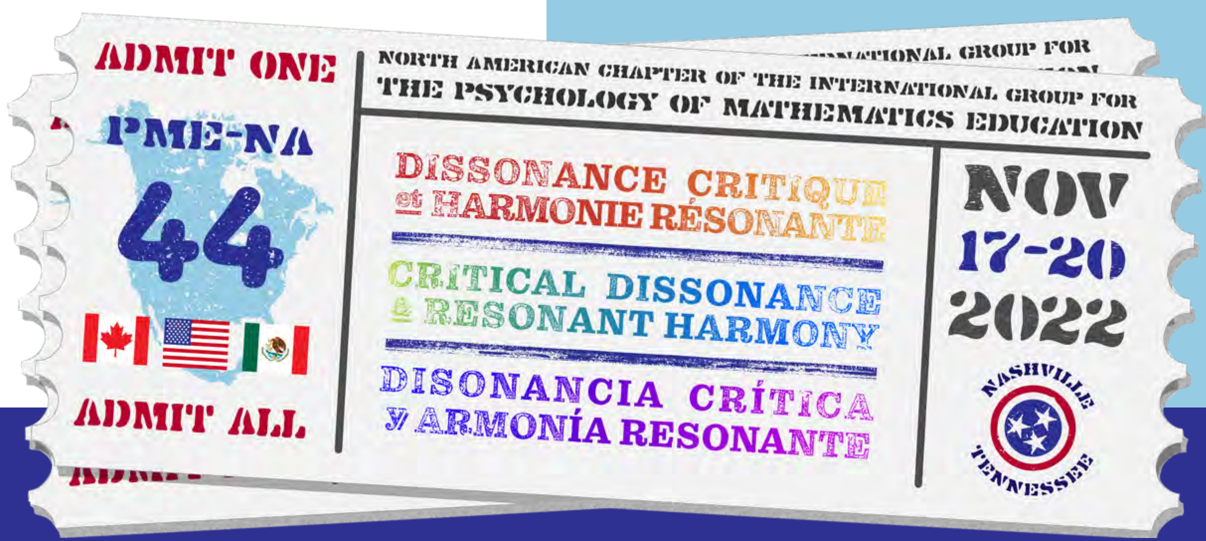
2022

PROCEEDINGS OF THE 44TH ANNUAL MEETING OF THE NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

CRITICAL DISSONANCE AND RESONANT HARMONY



EDITED BY

Alyson E. Lischka
Elizabeth B. Dyer
Ryan Seth Jones
Jennifer N. Lovett
Jeremy Strayer
Samantha Drown

MIDDLE TENNESSEE STATE UNIVERSITY

Proceedings of the Forty-Fourth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

Critical Dissonance and Resonant Harmony

Nashville, TN USA

November 17 – 20, 2022

Editors:

Alyson E. Lischka
Middle Tennessee State University
Alyson.Lischka@mtsu.edu

Elizabeth B. Dyer
University of Tennessee – Knoxville
edyer@utk.edu

Ryan Seth Jones
Middle Tennessee State University
Ryan.Jones@mtsu.edu

Jennifer N. Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Jeremy Strayer
Middle Tennessee State University
Jeremy.Strayer@mtsu.edu

Samantha Drown
Middle Tennessee State University
DrownS@rcschools.net

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J., Strayer, J., & Drown, S. (2022). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Middle Tennessee State University.

Citation:

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J., Strayer, J., & Drown, S. (2022). *Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Middle Tennessee State University.

ISBN: 978-1-7348057-1-0

DOI: 10.51272/pmena.44.2022

Copyright:

Land Acknowledgement:

The PME-NA 44 Conference is held on unceded Indigenous land including the traditional homelands of the Cherokee, Shawnee, and Yuchi. The connections of Indigenous Peoples to this land continues to the present day. As we begin our conference it is important to acknowledge our place, both geographically and historically, paying tribute to the land and our ancestors—and honoring both. We note that just speaking the word Tennessee is a tribute to a first nations' word for "where the river bends." The genocide, forced displacement, and cultural erasure of indigenous peoples resulting from the colonization of this land is particularly felt here, where the Trail of Tears cut through Middle Tennessee. In the midst of this history, Native American Indians tell their story today—including the joy of return. Founded in 1980, the Native American Indian Association of Tennessee is working to improve the quality of life for Indigenous People in this state. This includes raising funds to one day build the Circle of Life Indian Cultural Center, which will showcase a research library, exhibit halls, emergency relief support, job training, and education. These efforts help to close the circle of hatred and prejudice so that all Tennesseans can come together in freedom and pride.

An important goal of land acknowledgments is to increase support of local Indigenous communities. You can support the work of the Native American Indian Association of Tennessee by donating at naiatn.org. You can also learn more about the history of Tennessee's Indigenous communities by visiting the First Peoples exhibit at the Tennessee State Museum, which is about 3 miles from the conference site. More information is at tnmuseum.org.

This statement was created in conversation with local Indigenous leaders and informed by the Native Governance Center's Guide to Indigenous Land Acknowledgment.

PME-NA History and Goals

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of PME. The first PME-NA conference was held in Evanston, Illinois in 1979. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically oriented foundations.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME-NA Membership

Membership is open to people who are involved in active research consistent with PME-NA's aims or who are professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and visit the "Membership" page.

PME-NA Steering Committee

Elected Members

Karl W. Kosko (Chair) – Kent State University (2019 – 2022)
Doris Jeannotte (Chair Elect) – L'Université du Québec à Montréal (2021 – 2024)
Julie Amador – University of Idaho (2019 – 2022)
José Martínez Hinestroza – Texas State University (2019 – 2022)
Alyson E. Lischka – Middle Tennessee State University (2019 – 2022)
Oyemolade Osibodu – York University (2020 – 2023)
Angeles Dominquez – Tecnológico de Monterrey (2021 – 2024)
María S. García González – Universidad Autónoma de Guerrero (2020 – 2023)
Xiangquan (James) Yao – Pennsylvania State University (2021 – 2024)
Brent Jackson (graduate student rep) – Michigan State University / WestEd (2021 – 2022)
Megan M. Parise (graduate student rep) – University of Minnesota (2021 – 2022)

Appointed Members

Aaron Brakoniecki (Webmaster) – Boston University
Yi-Yin (Winnie) Ko (Treasurer) – Indiana State University

Conference Chairs

Past Conference Co-Chairs (2019 – 2022)

Sandy Spitzer
Towson University
sspitzer@towson.edu

Dana Olanoff
Widener University
dolanoff@widener.edu

Kim Johnson
West Chester University of Pennsylvania
Kjohnson2@wcupa.edu

Current Conference Co-Chairs (2020–2023)

Alyson Lischka
Middle Tennessee State University
Alyson.Lischka@mtsu.edu

Elizabeth B. Dyer
University of Tennessee Knoxville
edyer8@utk.edu

Ryan Seth Jones
Middle Tennessee State University
Ryan.Jones@mtsu.edu

Jennifer N. Lovett
Middle Tennessee State University
Jennifer.Lovett@mtsu.edu

Jeremy Strayer
Middle Tennessee State University
Jeremy.Strayer@mtsu.edu

Future Conference Chair (2021–2024)

Teruni Lamberg
University of Nevada-Reno
terunil@unr.edu

PME-NA 2022 Contributors

Local Organizing Committee

Alyson E. Lischka, *Middle Tennessee State University*
Elizabeth B. Dyer, *University of Tennessee-Knoxville*
Ryan "Seth" Jones, *Middle Tennessee State University*
Jennifer N. Lovett, *Middle Tennessee State University*
Jeremy Strayer, *Middle Tennessee State University*

pmenaLOC@gmail.com

Strand Leaders

The Local Organizing Committee is extremely appreciative of the following people for serving as Strand Leaders. They managed the reviewing process for their strand and made recommendations to the Local Organizing Committee. The conference would not have been possible without their efforts.

Nina Bailey
Pavneet Kaur Bharaj
Stephanie Casey
Charity Cayton
María de los Angeles Domínguez Cuenca
Lara Dick
Nick Fortune
Maria S. Gargonza
Natasha Gerstenschlager
Maureen Grady
Will Hall
Kristin Harbour
Jose Martinez Hinestroza
Doris Jeannotte
Signe Kastberg
Yvonne Lai
Valerie Long
LouAnn Lovin
Sararose Lynch
Marta Magiera
Ricardo Martinez
Allison McCulloch
Susanna Moitoris Miller
Cody Patterson
Priya Prasad
Rani Satyam
Patrick Sullivan
Alexis Stevens
Barb Swartz
Janet Walkoe
Lucy Watson
Doug Whitaker
Ashley Whitehead
Derek Williams
Chris Willingham

Sponsors

Middle Tennessee State University College of Basic and Applied Sciences
Middle Tennessee State University Department of Mathematical Sciences
Middle Tennessee State University College of Education
Middle Tennessee State University Center for Educational Media
Middle Tennessee State University Mathematics and Science Education Doctoral Program
Tennessee Stem Education Center

Reviewers

Abreu, Sofía
Acevedo, Carlos Ivan
Acuña, Claudia Margarita
Adams Corral, Melissa
Adams, Elizabeth
Adil, Aamina
Adjei, Emmanuel K
Aguilar Perdomo, Noé Fernando
Aguilar, Alexandra Rene
Aguilar, Jair
Akgul, Enisa
Akuom, Denish Ogweno
Al Hadi, Moosa Ali Abdullah
Alarcon, Jeannette
Aljarrah, Ayman
Alonso, Virginia A
Alqahtani, Muteb M.
Altshuler, Mari
Alyami, Hanan
Amick, Lisa
Amidon, Joel
An, Tuyin
Anantharajan, Madhuvanti
Anderson, Rick
Anderson, Robin Keturah
Anthony, Monica
Antonides, Joseph
Appova, Aina
Armstrong, Alayne
Armstrong, Amber Gelena
Arnold, Elizabeth G
Aryal, Harman Prasad
Atabas, Sebnem
Ataide Pinheiro, Weverton
Azimi Asmaroud, Seyedehkhadijeh
Bamat, David
Baniahmadi, Mona
Barabe, Genevieve
Barker, David
Barno, Erin
Barron, Victoria Jasmine
Basaldú Gutiérrez, Yesenia Esmeralda
Beesley, Andrea
Bentley, Brianna
Berger, Ashley
Bertolone-Smith, Claudia Marie
Bianco, Kathryn
Blakeman, Laurie
Bleiler-Baxter, Sarah K
Boileau, Nicolas
Bondurant, Liza
Borowski, Rebecca
Bosica, John
Bostic, Jonathan David
Botello, Megan
Bowen, Brian
Bowers, David Matthew
Boyce, Steven
Bozzano, Patricia Eva
Brafford, Tasia Lorissa
Bragelman, John
Brass, Amy
Brewer, Jacob
Brown, Rachael Eriksen
Bui, Mai
Burch, Lori
Burnett, Miriam
Byrne, Martha
Byun, Sunghwan
Cabañas-Sánchez, Guadalupe
Callahan, Pamela Catherine
Candela, Amber Grace
Cannon, Susan
Caro-Rora, Domonique
Carson, Cynthia
Castanheira, Brittney
Castellón, Libni
Castro Superfine, Alison
Champion, Joe
Chan, Jenny Yun-Chen
Chandler, Kayla
Chao, Theodore
Chen, Lizhen
Cheung, Matthew
Choi, Sun Jung
Cirillo, Michelle
Clark, Dan
Clark, Quintana M.
Closser, Avery Harrison
Colen, Jung Youn
Collier, Crystal
Conner, AnnaMarie
Conner, Kimberly
Cordero-Siy, Eric
Corum, Kimberly Chang
Corven, Julien
Cox, Wesley A
Crawford, Angela R
Crawford-Ferre, Heather
Cudd, Michele Denise
Czocher, Jennifer A
Davis, Alan
DeBay, Dennis J
DeJarnette, Anna Fricano
Delaney, Victoria
Desai, Siddhi
Dietiker, Leslie
Dillman, Brittany
DiNapoli, Joseph
Dinç, Emre
Dogan, Hamide
Doherty, Kristin
Dominguez, Higinio
Drimalla, James
Duarte Mejia, Iris Mariela
Duncan, Matthew David
Dyer, Elizabeth B
Earnest, Darrell
Eckman, Derek
Eisenreich, Heidi
El Chidiac, Fady
El Turkey, Houssein
Elliott, Lucas
Elliott, Rebekah
Ellis, Amy
Ellis, Ruby
Elmore, Traci
Ely, Robert
Engledowl, Christopher
Erwin, Alicia
Escobar Durán, Uriel
Ezaki, John
Fajardo, Maria del Carmen
Feikes, David
Fernandes, Anthony
Fink, Heather
Fisher, Molly
Flanagan, Karle
Flanagan, Kyle
Fletcher, Nicole
Fletcher, Samantha
Flores-Gasca, Carlos Enrique
Ford, Ben
Foster, Mike
Fox, Ryan D
Francis, Dionne Cross
Franks, Asia
Freeburn, Ben
Fuentes Acevedo, Patricia Cristina
Funkhouser, Kelsey
Galanti, Terrie
Gallagher, Keith
Gantt, Allison L.
García González, María
García López, Xochitl Josefina
García-Caro, Diana Carolina
García-Rodríguez, Martha Leticia
Gartland, Sara
Gehrtz, Jessica

Gerardo, Juan M
Gibbons, Alanna
Gilbertson, Nicholas J
Gillespie, Ryan
Gomez Marchant, Carlos Nicolas
Gonzalez, Gloriana
Gorham Blanco, Tracey
Graysay, Duane
Greenstein, Steven
Gregg, Jonathan
Grillo, Monica
Grissett, Noah E
Grundmeier, Dusty
Guajardo, Lino
Gualdron Pinto, Elgar
Guba, Taylor Paige
Gupta, Dittika
Habib, Kathryn
Haiduc, Ana-Maria
Haines, Cara
Hakansson, Susie W.
Hall, Jennifer
Hamilton, Michael
Han, Ahreum
Han, Jaepil
Han, Simon Byeonguk
Hardison, Hamilton
Harper, Frances K
Harrington, Cody
Heath, Amanda Lake
Heaton, Ruth
Heid, M. Kathleen
Heisler, Jennifer Lynn
Herbel-Eisenmann, Beth
Herbst, Pat G
Hernandez, Gabriela
Hernández-Cruz, Lucero
Hernandez-Zavaleta, Jesus Enrique
Hertel, Joshua
Hillman, Susan L.
Hinojosa, Denisse Maribel
Hjalmarson, Margret
Hollebrands, Karen
Holm, Jennifer
Hong, Dae S.
Houston, Scotty
Hoyos, Veronica
Hsiao, Mindy
Huang, Rongjin
Huang, Siqi
Huerta-Vazquez, Miguel Angel
Huffman, Amanda
Hunt, Jessica H.
Husband, Marc
Infante, Nicole
Ion, Michael
Islas-Ortiz, Erasmo
Jackson, Kara Jones
Jackson, Shani
Jacobs, Victoria R.
Jansen, Amanda
Jao, Limin
Jeannotte, Doris
Ji, Jane
Johns, Carolyn
Johnson, Heather Lynn
Johnson, Kim
Jones, Ryan Seth
Jong, Cindy
Joseph, Manjula
Joswick, Candace
Juárez-Ruiz, Estela
Jung, Hyunyi
Kalinec-Craig, Crystal
Kamlue, Nitchada
Kandasamy, Sindura Subanemy
Kar, Tuğrul
Karakok, Gulden
Kaschner, Scott
Kaufman, Bailey Anne
Kebreab, Lybrya
Kent, Laura Brinker
Kerrigan, Sarah
Khan, Ishtesa
King, Michelle Morgan
King, Shande
Kirby, Jordan
Kirkland, Patrick
Kirmizi, Mehmet
Kirwan, J Vince
Knapp, Melinda
Knotts, Angela
Ko, Yi-Yin
Koehne, Christina
Koestler, Courtney
Kokushkin, Vladislav
Kopparla, Mahati
Kosko, Karl Wesley
Krebs, Angela
Krejci, Brooke
Krupa, Erin E.
Küchle, Valentin Alexander Balthasar
Kwon, Minsung
Kwon, Oh Hoon
Lamberg, Teruni
Lanius, Melinda
Larios-Osorio, *Victor
LaValley, Bethany
Lawton, Carrie
Leatham, Keith R.
Lee, Boram
Lee, Hwa Young
Lee, Kihoon
Lee, Mi Yeon
Lee-Hassan, Alexa W.C.
Lei, Qingli
Levin-Guracar, Elise
Lewis, Catherine C.
Leyva, Luis Antonio
Liang, Su
Lim, Dexter
Lima, Cynthia
Lin, Ho-Chieh
Lindfors-Navarro, Heather
Lischka, Alyson E.
Litster, Kristy
Liu, Jinqing
Lloyd, Mary Elizabeth Riley
Lo, Jane Jane
Lolkus, Michael
Lopez-Betancourt, Alicia
Lu, Pierre
Luczak, Rileigh
Lunney Borden, Lisa
Lunt, Jana
Lyublinskaya, Irina
Ma, Yue
MacArthur, Kelly
MacDonald, Beth L.
Mainzer, Emily
Makumba, Paul Chepkongane
Maldonado Reynoso, Norma Patricia
Males, Lorraine M
Mannix, Joshua P
Marchionda, Hope
Marcia Rodriguez, Sindi Lorely
Marsh, Dalton Dayne
Martin, Kristi
Martinez Hinestroza, Jose
Martínez Pérez, Sandra Areli
Martínez-Miraval, Mihály André
Martinez-Soto, Eduan
Mask, Walker
Mason, Erica N.
Matranga, Anthony
Matute, Karla Valesca
Mbewe, Rose
McGraw, Rebecca
McHenry, Darleana
McMillan, Brandon
McNeill, R. Taylor
Meador, Audrey
Meagher, Michael S
Melhuish, Kathleen

Membreño Estrada, Sharon Samantha
Mendoza, Monica
Menke, Jenna
Metts, Elizabeth
Mirin, Alison
Mkhatshwa, Thembinkosi Peter
Moldavan, Alesia Mickle
Monroe, Ann
Montaño Ramos, Romario
Montero-Moguel, Luis E
Montiel Espinosa, Gisela
Mooney, Edward
Moore, Alexander S.
Moore, Kevin
Moore-Russo, Deborah
Moss, Diana L.
Mule, Taylor
Myers, Kayla
Myers, Marrielle
Naddor, Josh Evan
Nagle, Courtney
Naresh, Nirmala
Nathan, Mitchell J.
Navarrete, Jairo
Neihaus, Aubrey
Neumayer DePiper, Jill
Newton, Jill
Ngo, Vy Vo Hoang
Nikula, Johannah
Nirode, Wayne
Nitta, Kathleen
Norton, Anderson
Nuguid, Maria Eloisa
Nuñez-Gutierrez, Karina
Nurnberger-Haag, Julie
Obielodan, Florence F
Olanoff, Dana
Olarte, Royce
Olson, Gary
Oriowo, Olanrewaju
Orozco-Santiago, José
Orrill, Chandra
Orta Amaro, José Antonio
Ortiz Rocha, Yudi Andrea
Otten, Samuel
Özgün-Koca, Asli
Ozturk, Ayse
Pak, Byungeun
Palmer, Katrina
Panorkou, Nicole
Paoletti, Teo
Parise, Megan
Park, Hyejin
Park, Matthew
Parks, Amy Noelle
Parr, Erika David
Pate, Kaleigh
Pérez Martínez, Helen Mariel
Phelps-Gregory, Christine M
Piatek-Jimenez, Katrina
Pinilla, Robyn K
Pitvorec, Kathleen
Poetzel, Adam Roman
Polly, Drew
Pope, Derek
Powell, Arthur B.
Prough, Sam
Przybyla-Kuchek, Julia
Pynes, Kristen D'Anna
Quaisley, Kelsey
Quintero Ochoa, Rodil Eladio
Raja, Waleed Ashraf
Randazzo, Brooke
Reed, Samuel D
Reis, Jenna
Rice, Michaela Ann
Rich, Erin
Richman, Andrew S
Ricks, Thomas
Riling, Meghan
Ristroph, Ingrid
Roan, Elizabeth
Roberts, Sarah A.
Rodríguez, Flor Monserrat
Roehrig, Amy E
Roh, Kyeong Hah
Roscioli, Kate
Ross, Andrew Michael
Rouhani, Andre
Ruiz, Steven L
Rupnow, Rachel
Rush, Yolanda Gail
Russell, Bridgette
Saclarides, Evthokia Stephanie
Salaam, Bolanle
Saldaña, Mike
Sapkota, Bima Kumari
Scharfenberger, Adam
Schenck, Kelsey E.
Seago, Nanette
Seat, Jennifer
Sellers, Morgan Early
Sencindiver, Benjamin D.
Sepulveda, Francisco
Sevgi, Sevim
Sevier, John
Shaughnessy, Meghan
Shekell, Calli
Siciliano, Lisa M
Siebert, Daniel
Silla, Elena Marie
Skultety, Lisa
Smith, Amy
Smith, Ethan P.
Smith, Felicia
Smith, Hannah
Smith, Shawnda
Smucker, Karoline
Spitzer, Sandy M
Stack, Kamala
Stafford, Taylor E
Stephan, Michelle
Stephens, Ana
Stepp, Zachary
Stewart, Karon
Stockero, Shari L
Stoddard, Elyssa
Stokes, David
Strayer, Jeremy F
Street, Ciera
Strickland, Sharon
Stringer, Brinley Poulsen
Stroud, Rena
Suazo-Flores, Elizabeth
Suh, Jennifer M.
Sundrani, Anita
Sung, Hanall
Sung, Yewon
Sutter, Molly
Swartz, Micah
Tague, Jenna
Tanck, Hilary
Taqiyuddin, Muhammad
Tasova, Halil Ibrahim
Tate, Holly N.
Tchoshanov, Mourat
Tellos, Alison
Teuscher, Dawn
Thacker, Ian
Theobold, Allison S
Thomas, Jonathan Norris
Thompson, Jennifer
Throop-Robinson, Evan
Towers, Jo
Tremaine, Rachel
Turner, Blake O'Neal
Tyburski, Brady A
Tyminski, Andrew M.
Uhing, Karina
Valenzuela, Carlos
Van Zoest, Laura R.
Vargas Zambrano, Luis Carlos
Vargas-Alejo, Veronica
Villafañe-Cepeda, Wanda
Vo, Michael
Walker, William
Walsh, Patricia A

Ward, Jennifer
Warshauer, Hiroko Kawaguchi
Watanabe, Tad
Watkins, Jonathan D
Webb, Jessica
Webster, Amber
Wei, Xinyu
Weiland, Travis
Weingarden, Merav
Wessman-Enzinger, Nicole Marie
Wheeler, Ann
Whitacre, Ian
Whitaker, Nicole
Wieman, Rob
Wiest, Lynda R.
Wilhelm, Annie
Wilkerson, Trena
Wilson, Aaron Thomas
Witt, Nicholas
Wonsavage, F. Paul
Woodward, Jerry
Wrightsman, Elizabeth
Yalman Ozen, Demet
Yao, Xiangquan
Yaro, Kwesi
Yeo, Sheunghyun
Yilmaz, Zuhal
Yu, Xi
Zhang, Pingping
Zolfaghari, Maryam
Zolt, Holly
Zwanch, Karen

Preface

On behalf of the 2022 PME-NA Steering Committee, the 2022 PME-NA Local Organizing Committee, and Middle Tennessee State University, we welcome scholars to Nashville, Tennessee, USA, for the Forty-Fourth Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter, held at the Loews Vanderbilt Hotel.

The goal of PME-NA is to promote the international exchange of research on the psychology of mathematics education and to promote an ever-deepening understanding of the psychological aspects of teaching and learning mathematics. Psychology, the study of mind and behavior, encompasses biological influences, social pressures, and environmental factors that impact how learners think and act. As members of PME-NA, it is necessary that we attend both to the contexts in which the teaching and learning of mathematics takes place and to the experiences of individual participants, while considering the multiple voices, histories, systems, and social structures present in our learning spaces. In recent years, our contexts and experiences have been forever impacted by the world-wide COVID pandemic and a renewed struggle for civil rights in many of our communities.

This year's conference theme, ***Critical Dissonance and Resonant Harmony,*** reflects not only the time and place that we gather, but also the time and place in which we conduct our academic work. Dissonance can be jarring to experience, whereas harmony can be pleasing. We gather in Nashville - Music City - which is no stranger to both dissonance and harmony. Walk the streets and you will experience both, as vibrations felt in your body or heard in your ear. Look closer and you will see both, knitted in the very fabric of our identity. We are not only the site of lunch counter sit-ins, Freedom Rides, the final approval for women's suffrage (the 19th Amendment to the U.S. Constitution), the Teens for Equality Black Lives Matter rally, and the Annual March for Black Women in STEM, but also The Trail of Tears, countless acts of oppression, and reactionary legislation limiting citizen's rights—most recently those of LGBTQIA+, BIPOC, and Immigrant communities. We know that dissonance is necessary for change and liberation, and so has a critical component. We also know that harmony occurs when multiple voices and forces join simultaneously to amplify and enrich, achieving a resonance that pushes through to the other side and finds a better way. Still, in reality, dissonance and harmony cycle, neither seemingly fully formed, and often sharing contiguous or overlapping spaces. The PME-NA community will gather in this place in 2022, traveling from locations across North America that are experiencing similar dissonances and harmonies.

It is in this context that we conduct our work in the field of mathematics education, as researchers who value the contributions and experiences of each and every person. In preparation for PME-NA 44, we invited all presenters to reflect on how their work impacts the contexts and experiences of members in mathematics learning communities, particularly those who are on the margins of these communities, and to address what these reflections mean for their work. Reflection may have included consideration of the following questions: How does your work challenge a settled mathematics learning status quo? How does your work help to create more socially just contexts for learning and teaching mathematics? How does your work have an impact on society more broadly, beyond individual mathematics classrooms and school districts? How does your work improve learning conditions for each and every mathematics learner? Whose voice does your work center in the mathematics learning process? What can be learned from reflecting on this question? This proceedings represents the work of mathematics education scholars as they took up this call for reflection.

We hope the PME-NA 44 conference provides a communal space for critical reflection and conversation on how our individual and collective work contributes to dissonant and harmonious movements in our field as it relates to the psychological aspects of teaching and learning mathematics. We recognize that what is perceived as dissonant or harmonious varies among cultures and even individuals and is by no means universal. And so, we look forward to the opportunities for conversation and learning across our international community as we share instances of dissonance and harmony that can serve to propel research toward action and change.

This year's conference will be attended either in-person or remotely by close to 600 researchers, faculty members, and graduate students from around the world including Canada, Mexico, Colombia, and the USA. Approximately 800 authors are represented by the papers in this volume. Each paper was reviewed by multiple referees in a double-blind process. After initial reviews were submitted, Strand Leaders reviewed feedback and made selections for papers to include in the conference. Finally, the local conference committee made decisions based on reviewer and strand leader feedback and constraints of the conference space. The result was an overall acceptance rate of 66% with the final papers comprised of 101 research reports, 140 brief research reports, 84 Posters, and 16 Working Groups. In addition, we conducted a doctoral consortium in conjunction with the conference during which 23 late-stage doctoral researchers engaged with mentors and each other to support their research. These scholars presented their in-progress work as posters during our poster sessions.

In 2021, the PME-NA Steering Committee voted to require all working groups wishing to continue their work at a subsequent conference to produce a report on their work following the conference. Therefore, this proceedings contains a report of work conducted in 2021 along with a proposal for the work to be conducted at the 2022 conference for each returning working group. These reports are found in Chapter 17.

This conference and publication would not have been possible without the support of many in our field: strand leaders who put forth efforts to ensure the quality of the papers selected for the conference program, colleagues who supported the local organizing committee as we navigated the unchartered territories of planning a conference during a pandemic, plenary speakers who took up our vision for the theme, a Steering Committee which supported us while allowing space for us to follow our own vision, and the many scholars who share their brilliant work with our field through this outlet. We thank Middle Tennessee State University for supporting our efforts to lead this conference. We also thank Samantha Drown and Sarah Johnson for their work in compiling this proceedings.

The local organizing committee hopes that you engage with this research by looking for both dissonances and harmonies within these papers that enrich the ways in which your own work can continue to support the learning and teaching of mathematics.

Table of Contents

Chapter 1:
Plenaries

RE-MEMBERING PLACE: MATHEMATICAL ACTIONS FOR INNOVATIVE, RESILIENT, AND CULTURALLY RICH COMMUNITIES

Florence Glanfield
University of Alberta, Canada
glanfiel@ualberta.ca

Cynthia Nicol
University of British Columbia, Canada
cynthia.nicol@ubc.ca

Jennifer S. Thom
University of Victoria, Canada
jethom@uvic.ca

How might mathematics educators recognize discourses as resonating harmonies in their practices as researchers? In this paper we share individual experiential narratives guided by Ojibway author Richard Wagamese's Medicine Wheel teachings in the four directions of East (humility), South (trust), West (introspection), and North (wisdom). As we journey through (re)membering place we offer opportunities for recognizing resonating harmony(ies) and algo-rhythms in our practices as mathematics education researchers and for engaging with critically dissonance discourses and actions. This (re)membering supports relating with each other, mathematics, communities, and place in ways that are more sustainable, inter-connected, and kincentric.

Keywords: place and land education, Indigenous perspectives, ecological perspectives, mathematics education

Introduction

We, Florence, Cynthia, and Jennifer, all mathematics educators and researchers, use Ojibway author Richard Wagamese's (2011) four direction Medicine Wheel teachings to share our individual research journeys. We use stories (Clandinin & Connelly, 2002) to re-member and re-think our individual actions and the ways those individual actions have woven together to become collective work. Wagamese's teachings follow the direction that Earth travels each day. Just as humans have said that the start of the day is in the East, Wagamese's Medicine Wheel teachings also begin in the East. This paper is written in the four directions; East, South, West, and North. Wagamese's Medicine Wheel teachings start the section; followed by stories of each of the authors.

East

> **THE OLD ONES** say that humility is the foundation of everything. Nothing can exist without it. Humility is the ability to see yourself as an essential part of something larger. It is the act of living without grandiosity. Humility, in the Ojibway world, means "like the earth." The planet is the epitome of a humble being, with everything allowed the same opportunity to grow, to become. Without the spirit of humility there can be no unity, only discord. Humility lets us work together to achieve equality. Humility teaches that there are no greater or lesser beings or things. There is only the whole. There is only the great, grand clamour of our voices, our spirits, raised together in song. (Wagamese, 2011, p. 9)

Humility: The Power of Place (Cynthia)

It's common for cliffs on Haida Gwaii along Canada's Pacific northwest coast to receive 10m high waves, for spruce trees to dance in a southeast storm, and for the ocean to leave platters of cockles, clams, crab, and octopi on its beaches when the tide is out. Haida Gwaii is a land between sea and sky where the worldview Gin 'waadluuwan uu gu dil adiidaa "everything depends on everything else" speaks to the synergetic relationship between humans and nonhumans, natural and Supernatural Beings. It is a place where "culture" says Haida artist and activist Guujaaw "actually is our relationship to land" (Guujaaw as quoted in Jones, 2006 p. 29). The inlets and shores of Haida Gwaii are nutrient-rich and tempered with warm offshore currents feeding a diversity of life. In spring grey whales migrate between their winter breeding areas in California and their summer feeding waters in the Bering Sea. From our little log cabin on Haida Gwaii's North Beach, living without electricity or running water, we could see the whales along their route, releasing their breath with spectacular spray as they prepared for the next dive.

One spring afternoon out in our 12 ft Zodiac boat, we found ourselves surrounded by a pod of grey whales. A kilometre off-shore with no other boats in sight, we shut down the motor as the greys circled. Maybe they were curious about who had ventured into their territory. Maybe the bottom of our Zodiac looked familiar from a whale's perspective, but different enough to make them curious, take notice, and check us out. Three maybe four barnacled backs rose from the water, one flipping a tale wider than the length of our dwarfed boat. Their breath lingered while we held our own.

Without warning one broke from the circle and swam directly toward us. Grey whales can reach 15 m (about 50 ft) in length. I imagined this massive being flicking us in the air like a dog playing catch with a ball. We sat still, as still and humble as what Atleo calls an "insignificant leaf floating in a spring well" (Atleo 2011, p. 98). The power in this moment lay with this whale. As the whale slipped smoothly and quietly – planned and precise – under our boat and rose on the other side we acknowledged our respect with thanks. And, as if this magnificent animal heard us, the whale circled back. Resting its head on the water's surface, we met eye-to-eye, feeling once again the power of this relationship and our place within it. This whale's gaze and 'playful' interaction reminded us of our own fragility as visitors in their realm.

Haida oral stories speak of interactions between humans and nature and Supernatural Beings as sharing skins. Under our skins, both human and nonhuman, we are reminded that we are all connected, sharing this power of place.

Humility and Earth (Jennifer)

"Humility, in the Ojibway world, means 'like **the earth**. …[H]umility ... is **the foundation of everything. Nothing can exist without it.**" (Wagamese, 2011, rearranged, p. 9).

These words which describe the Ojibway world re-mind me of my Chinese grandmother's family name. "Chan" written as Chinese characters 陳 (see also Figure 1) animate familiar meanings of place, direction, and temporality to those of Ojibway meanings for East. 陳 means earthly place, landscape, ancient, and abundant. 東 as the sun rising from behind the tree is East which infers morning but also spring as the first season of the new year. Thus, not only does the direction of East locate Earth as place but it also posits Earth as originary and foundational. Just as light appears, day breaks night, and from winter comes spring, it is from the East where new beginnings and possibilities emerge.

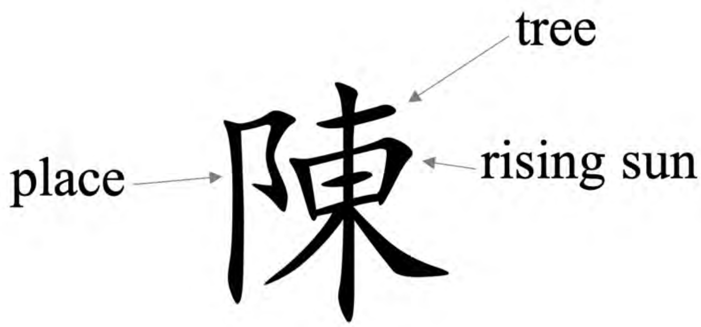


Figure 1. Chinese characters for "Chan"

"Humility, in the Ojibway world, **means 'like the earth.** …**[H]umility ... is the foundation of everything. Nothing can exist without it.**" (Wagamese, 2011, rearranged, p. 9).

Deeper within the layers of Wagamese's description, I hear the wor(l)ds of Japanese-Canadian curriculum scholar Ted Aoki. Aoki (2004a) remembers and reminds that the word *humility* is etymologically akin to earth. From *humilis* meaning "on the ground", *dhghem* or *humus* as "earth", and *don* as "place" (Humility, 2022), the very word has originated and evolved from *earth*. Humility as described by Aoki (2004a) is "concerned with lived space[s] where people dwell communally ... with others [and] earth under the sky." (p. 300) And just as *hum*ility comes from *humus*, Aoki (2004a) remembers and reminds that we *hum*ans do too.

While East marks first light, new beginnings, and starting points, humility calls for contemplation of what has come before. Humility makes returning to and reconsidering onto-epistemologies of place possible. In doing so, it allows for movement and change. *Returning* then is not simply 'turning around' or 'going back' but 'turning again'. Further, from the East turning again conceptually conjures notions of a turning point or starting anew.

For me, East signifies the first part of my journey as graduate student. Asked by my supervisor why I decided to pursue further studies in mathematics education, I explained to her that: "When I am in the mathematics classroom, there are certain things and events I can see and theoretically explain through constructivism. However, there are all sorts of other things and events I know or sense are happening but all I can do is point to them. I have no means by which to describe them yet I desperately want to understand them." Digging deeper, I was also conflicted with the idea that if "WE ARE CONNECTED TO THIS EARTH" (Thom, 2012, p. 2) then why is it that school mathematics feels so disconnected. These "things" and "events" were just a few of the items on what was becoming an ever-growing list. Many were contradictions I experienced between my home life and life lived as mathematics student and teacher. From ontologies to epistemologies to cultural discourses to the metaphorical nature of language to meanings of place. My hope was that these questions would be the start of a new journey for me. A journey that would require me to re(-)turn and explore the place of classroom mathematics and if necessary, bring forth possibilities for 're-rooting the learning space' and develop an understanding for what it means to 'mind where children's mathematics grow' (Thom, 2012).

Understanding Humility in Relation to the Boreal Forest (Florence)

I grew up on a ranger station in what is now called northeastern Alberta. My father was a forest officer and grew up in the prairies. My Métis mother was born and raised in the North primarily living in a variety of small communities around a huge fresh water lake. The ranger station was located on the edge of boreal forest and farmland. A boreal forest is full of deciduous and coniferous trees; land that includes muskeg (or bogs), freshwater lakes, rivers, and streams.

The land sustains a vast number of animal and plant species - species that also sustain human life. I grew up eating “wild meat” such as moose, deer, partridge, ducks, goose, and fish. I grew up picking and eating wild berries and learning about some medicines that were available in the forest.

I also grew up being in the boreal forest, learning from my Métis grandparents and from my parents about what it means to pay attention to the land. I grew up learning how to ‘find my way’ through the forest and learning to pay attention, for example, to the ways in which moss grows on trees as you might need the moss to build a fire for warmth. A large part of these early years was understanding forest fires. Forest fires were feared and at the same time celebrated. Forest fires were feared because of the way that they could impact human life. My father would be called away when a fire was spotted in order to begin to work with others on the initial attack; hoping that the fire could be contained in order to protect humans and human properties. At the same time there was a contradiction because a boreal forest relies on a forest fire for rejuvenation and growth. Nutrients are released, seeds are released for some species of trees, and fires open the canopy to allow new growth (Natural Resources Canada, 2022).

It was always fascinating to me to watch the way that a forest ‘rejuvenated’ following a fire. The years following a forest fire would mean that we needed to pay attention to reading the land in a different way because there would be new growth. You would notice the new jack pine seedlings that could now grow because of the fire. We would pay attention to the places that fireweed would grow across the burned area as fireweed was often the first ‘new growth’ we would see. We would spend time ‘reading’ the changes in the land after a fire and noticing the changes in the ways that the animals would live around the land that had been burned. Sometimes it would mean hunting in a new place for the moose or deer because the growth of the plants had changed because of the fire.

Often throughout these early years of my life we were taught to live with the forest because it was a way to learn about how to ‘survive’ with the forest. Even though we learned a lot about reading the land we also learned about respecting the land as the land can be changed quickly and dramatically by rain or no rain, fire or no fire, and extraordinary winds. Learning about ‘surviving’ taught me about humility; I was taught to respect the land, the weather, and the species that sustained our family. As I reflect on these early years now, I was learning about uncertainty and interdependence of humans and nonhumans living with earth or land.

South

> **TRUST IS THE** spiritual by-product of innocence. My people say that innocence is more than lack of knowledge and experience, it's learning to look at the world with wonder. When we do that, we live in a learning way. Trust, the ability to open yourself to teachings, is the gateway for each of us to becoming who we were created to be. All things bear teachings. Teachings are hidden in every leaf and rock. But only when we look at the world with wonder do the teachings reveal themselves, and trust is also the ability to put those teachings to work in our lives. Trust is, in fact, our first act of faith and our first step towards the principle of courage that will guide us (Wagamese, 2011, p. 57).

Trust: Relations Take the Time it Takes (Cynthia)

As shapeshifters, Supernatural Beings appear in Haida stories told for generations. They’re carved into poles, masks, and jewelry, and painted on bentwood boxes, woven into cedar hats, and tattooed on bodies. I’ve learned from Haida Elder Gwaaganad Diane Brown that it is here, Haida Gwaii, where multispecies kinship between human and nonhuman, natural and Supernatural is honored – where, for example, "we treat the ocean as our relative."

Supernatural Being Kugann Jaad (Figure 2) appeared for Haida artist Billy Yovanovich when working on his ideas for mathematics education on Haida Gwaii. It is Kugann Jaad, says Billy, who is known for her ability to restore balance and equity "with her strength, wisdom, and vision she guides us, speaking with both her hands and her eyes with knowledgment of what is to happen, what will happen, or even provide resources to meet challenging situations (Nicol et al., 2020, p. 17-18).

Figure 2. Kugann Jaad by Billy Yovanovich

Kugann Jaad for me helps tell the story of working with Haida communities, first as a beginning teacher, and then for the past 17 years as an academic to connect community, culture, and mathematics within the culture and place of Haida Gwaii. The task, when first started, seemed challenging but doable: work with six teachers (Indigenous and non-Indigenous; elementary and high school; from the north and southern parts of the Island; beginning and experienced) and the Principal of Indigenous Education to create practices and resources that ground and strengthen relationships between people, mathematics, and place. Teachers with more than 20 years of experience found themselves questioning their own assumptions, challenging what they noticed as deficit perspectives, and recognizing the complex work needed. We were excited about these modest results and humbled by the time it took. With community we took two years to develop a common workable set of questions that we called the PAIRS approach to highlight place, relationships and Indigenous storywork with mathematics, and another three years to develop a book as a collection of community Elders' stories alongside mathematical adventures.

Our efforts seemed to move forward and backward at the same time. Off-Island teachers typically left the Islands within 5 years, new teachers joined the project, and the work would begin again. Kugann Jaad loves a good puzzle, and as shapeshifter can slip into the skins of others to lend a hand when needed. Perhaps it was Kugann Jaad who helped us accept that the strength of this work was felt not within a couple years, or 5 years, or even 10. It came with intergenerational teachings – when for instance I found the daughter of one of my former Haida students in my university teacher education mathematics class. I had taught her mom, her aunties, and some of her uncles, all of whom, I have since learned, shared with her their good stories of learning mathematics in my class many years ago. Now here she was in my university class eager and ready to embrace mathematics as a new teacher. Kagann Jaad reminds us to trust the time some learnings take.

Trusting Wonder (Jennifer)

"[O]nly when we look at the world with wonder do the teachings reveal themselves" (Wagamese, 2011, p. 57).

Wonder as Wagamese (2011) explains allows for an openness to the world, opportunity to receive its teachings, and inspires hope as well as promise to "live in a learning way" (p. 57). For me, his teachings evoke insights and sensibilities which resonate with 詩 (see also Figure 3). Pronounced "shi" in Japanese and Cantonese, 寺 can be interpreted as "tera (sic) [or] [E]arth/measure [as] 'temple,' a sacred place where one may be allowed to hear the true measure of earth beings, mortals in the nearness of divinity." (Aoki, 2004d, p. 374) 言 is "to speak/to sound [w]ithin the 'mouth' that sounds forth or sings, over layered with three echoes and a lingering note." (Aoki, 2004d, p. 374). 詩 expresses reverence and awe for Earth as place and teacher. Moreover, given the opportunity to learn by "listening to the land" (Glanfield et al., 2021, p. 258), "to hear .... the inspirited beat of earth's rhythm" (Aoki, 2004d, p. 374-375) as "*land-guaging* algo-rhythms" (Glanfield et al., 2021, p. 258) opens possibilities for humans to speak and sound them forth into the world so they may endure. In trusting wonder, I am learning that "to [hear] and read the (con)texts of the natural world intimately and openly implies that 'language … is not a specifically human possession, but is a property of the animate earth, in which we humans participate' (Abram, 2010, p. 11)" (Thom, 2019a, p. 252).

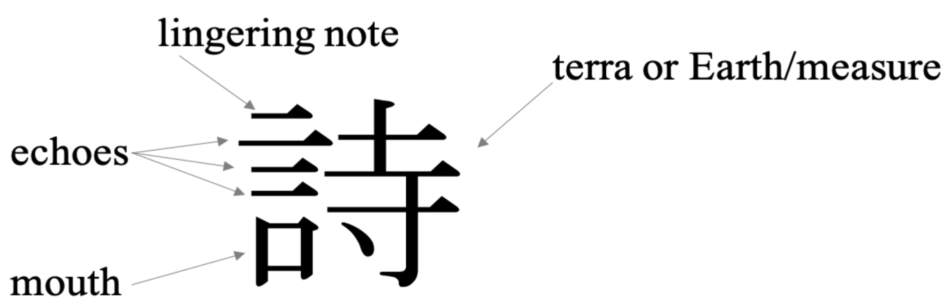


Figure 3. Chinese characters for "shi"

Doing so prompts me to revisit Alan Bishop's "assertion that all human cultures 'do' mathematics—that is, count, measure, design, play, locate, and explain (Bishop, 1988)" (It's Spring, 2022, p. 6). Here it is as if in its resurfacing, Bishop's assertion asks to be re(-)turned. Wondering as Wagamese does and as 詩 implies, I wonder:

[W]hat [inspirited algo-rhythms might be heard if] such a focus [for mathematics education] recognizes not only human activities as mathematical but emphasizes ways of knowing, doing, and being which are inherently grounded in and come from the land and Earth [such as when]:

~ Sun and moon rise (and set)
~ Seasons change
~ Spiders web
~ Raindrops dance upon the water's surface
~ Maple seedlings spin and spiral from tree to ground
~ Bat and fish and tree and pea tendril ... just "know" where they are and how to get to where they need to be (It's Spring, 2022, p. 6)
~ Humpback whales work together to net and feed.

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

What do I Trust (Florence)

The south direction in my life are the years that I was in formal postsecondary education. I first of all began to not trust as I was provided with implicit and explicit messages about silencing the teachings I had with my family in relation to the boreal forest. I wanted to be a mathematics teacher so when I finished high school I entered university to study mathematics first and then education second. Through those first two degrees I was taught a lot of different theories and practices. Theories about ways that children learned, theories about why humans act the way they do, theories about relationships, theories about the 'best' way to teach mathematics, theories about the best ways to assess student learning, and theories about the actions of the 'best' teachers. It felt like the years of formal education were a different world as I did not know or could not 'see' the ways in which the interdependencies that I learned about in my early years belonged in this world of theories that suggested an "if then" philosophy. That is, if x action is taken then y event will happen. This was contradictory to the philosophy that I grew up with as I was learning to 'read' the land, a philosophy that paid attention to interdependence and relationships.

As I began my teaching career I felt like I needed to implement all that I learned through my formal education. I often think about how, in my first 1.5 years of teaching, I spent hours and hours reviewing the "cumulative files" of each youth that were registered in my mathematics classes prior to the beginning of each school term. I did this because I believed that if I 'knew' about the youth in my classes then I could plan the 'best' program for each. I think about how I would spend hours and hours to make sure I could solve every mathematics problem in the assignments I planned to give because I thought that the "best" mathematics teacher was the mathematics teacher that had all of the correct answers. I think about how I would plan well in advance and how I wanted to keep up with my teaching colleagues in order to make sure I covered the curriculum in the same time frame. I wanted to quit teaching after those first 1.5 years because it was not anything like I had imagined.

As I began to learn about the youth that were with me in the classroom, I started to notice that the "cumulative files" didn't really tell me anything about who the youth were as individuals. The "cumulative files" often shared information about what the youth could not do and rarely did those notes reveal what the youth could do. I began to wonder if I was thinking about the youth in a deficit way. I began to realize that all of my planning ahead did not pay attention to what was happening with each collection of humans in any one class. So while planning ahead was important I also noticed I had to be prepared for the different observations from the youth in each class. I realized that my planning ahead did not recognize the interdependence of a classroom. I began to realize that questioning my practices was moving me away from the predominant "if then" philosophy of my university years and moving closer to the "interdependence and relationship" philosophy of my early years.

West

> **ON THE MEDICINE** Wheel, introspection is the "looks within place." Humility and trust offer many teachings, and introspection is a means of seeing how those apply to our lives. It's a place of vision. It's a resting place where the story, the song each of us has created up to this moment can be inspected and those things deemed unnecessary be let go. It's a place of courage, because the hardest place to look is within. Many people stop here, deterred by the trials of the journey and the sudden hurts that sometimes make life hard. But introspection is meant to bring us to balance. It is the place where all things are ordered, where all things ring

> true at the same time. Balance allows us to move forward, and when we do, the journey becomes wondrous again by virtue of our ability to see the whole trail. (Wagamese, 2011, p. 107).

Introspection: What are you? (Cynthia)

At a community potlatch feast in a northern BC Indigenous community a young girl sitting next to me turned and proudly declared her heritage: "I'm Nisga'a!" Then, with what seemed an obvious question to her, and looking straight at me asked, "What are you?"

What am I? I could answer *who* am I? or *Where* are you from? But *what* am I? It's taken a few years for me to learn how to respond to that question. Growing up as a non-Indigenous/settler in the Kootenay territory of southern British Columbia, in a predominantly white Euro-centric community, my heritage was rarely questioned and certainly invisible. We were taught nothing about the system of residential schools, lasting for over 160 years, involving more than 150,000 children in an education that offered more abuse than teaching. How is it possible that such atrocities occurred but then were erased from Canadian education? Even while I taught on Haida Gwaii these experiences were rarely spoken or only whispered.

My work with rural and Indigenous communities continues to deepen my own self-gaze to "disrupt molded images" (Dion, 2007), and settler colonial logics (Donald, 2012). This involves working with communities toward building relationships with land through personal experience, drawing upon stories and ancestral knowing through listening to Elders, and (re)membering respectful and responsive mathematical learning environments by connecting math, community and culture.

Through the teachings of Haida Elder Gwaaganad and Haida artist/activist Guujaaw I'm learning more about "what" I am. Gwaaganad spoke at our project meetings of the need to treat the ocean as our relative, as a member of the family. The ocean is neither an object, nor unending resource, nor even something that needs protection. The ocean is a relative, the ocean is kin. Similarly, Guujaaw spoke of the importance of ancient cedar trees to the Haida people. If "the trees are gone [due to industrial logging]" said Guujaaw, "then we're just like everyone else." The trees and the land including the ocean and waterways define what it means to be Haida.

Learning about relationships with the ocean and forests, as relatives, opened pathways for listening to the possibilities the land and waterways offer as teachers. Gwaaganad and Guujaaw led us in (re)membering long views of time and experiences of food harvesting with the changing ocean life, changing harvesting practices, and the revitalizing of ancient pedagogies of place. Opening myself up to embrace these kinds of obligations to the land/ocean, to be respectful and responsible for caring for the non-human world brings me closer to understanding what I am.

What am I? I'm still not sure I have a full response to this question, but I do know that it speaks to a more kincentric (Salmón, 2001) way of being, one that supports living in the world in more relational and emergent ways.

Simultaneous Complementarities (Jennifer)

"Introspection... 'looks within place.' ... [A] place of vision... where ... the song each of us has created up to this moment can be inspected .... [W]here all things ring true at the same time." (Wagamese, 2011, p. 107).

As I sit, "keeping watch" (Thom, 2012, p. 363), reflectively circling back and reflexively circling forth, I can appreciate and more deeply understand the onto-epistemological meanings of place that connect and (in)form my journey as mathematics education researcher. Now from the West, it is my Chinese grandfather's family name 譚 "Tàahm" and my surname "Thom" that appears, inviting my further contemplation. Similar to 詩 or “shi”, 言 depicts a mouth which sounds or sings forth, echoes, and holds a lingering note. 覃 translates as “early west.” (see also Figure 4). (How curious it is that 譚 foretells my grandfather’s journey from Hoiping China to Vancouver Island, singing forth as it echoes my family’s history in the “early west” during the turn of the 20th century onward.)

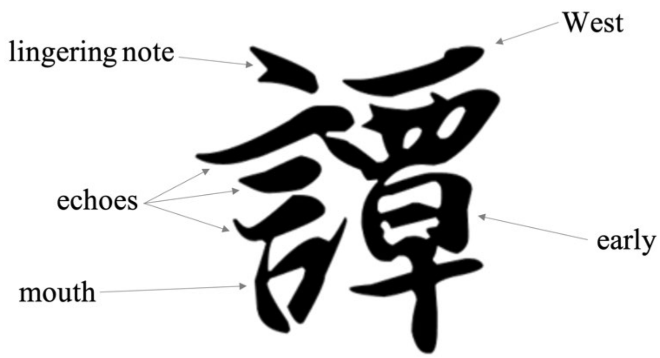


Figure 4. Chinese characters for Tàahm/Thom

Looking again, the Chinese characters for West bare other meanings which suggest reflection. For example, “dusk” as the time of day when the sun disappears below the horizon and the season of autumn. Both resonate with Wagamese’s description of West as “a resting place” and “a place of vision” that affords “look[ing] within place” and “introspection” (Wagamese, 2011, p. 107). Perhaps more intriguing is how the idea of “early west” presents as “all things ring[ing] true at the same time.” (Wagamese, 2011, p. 107). That is, *sunrise* and *sundown*; *early* or *soon* and *late*, *morning* and *evening*; even *beginning* and *end*. And as well, while autumn signifies a time of ripening, harvest, and plenty, it is also a time of decay, death, and decline.

These ‘opposites’ which could be viewed as contradictory can also been understood as wonderous spaces in which to “see the whole trail” (Wagamese, 2011, p. 107). Not as distinct or separate parts that compete with one another but in the harmonious *and* discordant ways they come together to dialectically (in)form “a space of conjoining and disrupting, indeed, a generative space of possibilities, a space wherein in tensioned ambiguity newness emerges.” (Aoki, 2004b, p. 318). So understood, dwelling within such edgy yet fecund spaces inspires my inquiry and occasions my learning into: how STEM *and* the cultural commons (Bowers, 2016) enable eco-centric intelligences within communities (Thom, 2019b); conceptualizing modern *and* ecological discourses in ways other than an impasse (Thom, 2021); alternative meanings and purposes for STEM within Indigenous *and* ecological perspectives (Glanfield et al, 2021); dynamics of mathematical ideas as individual *and* collective phenomena (Thom, 2012); mathematical drawings as act *and* artifact (Thom & McGarvey, 2015); mathematical perception *and* representation (Thom et al., 2021); and bodily experiences *and* mathematical conceptions (Thom, 2017, 2018; Thom et al., 2015; Thom & Hallenbeck, 2021; Thom & Pirie, 2002; Thom & Roth, 2011).

Indigenous Knowledge Systems and University Teachings (Florence)

I turn to university and to teachings of Indigenous knowledge systems in the west. After years of teaching I return to study a graduate degree in educational administration as I want to learn more about schools and how schools and school systems come to see youth as deficit. I imagine that I will become a school administrator that will make a change in the school system. I never become a school administrator. When I finish the graduate degree I begin to work with the provincial ministry of education in developing and implementing provincial mathematics programs and developing student assessment materials. I also began to notice that who I am as an Indigenous person made a difference when I was working with communities across Northern Canada around mathematics programs. I remember being invited to sit with Indigenous community members in the northwestern part of the Northwest Territories, outside of the normal mathematics program activities, when the community learned I was Métis. At another community in the Northwest Territories an Elder talked with me about how the number system in the Dene dialect was not a base 10 system when she learned I was Métis. The Elder told me about how numbers were important in communities traditionally but that it was about ‘enough’ and not always needing ‘more.’ These were ideas that I had not previously learned. Indigenous community members would tell me how they were working to have Indigenous languages present within schools. I was asked about the languages that my family spoke and I began to inquire within my family about the Indigenous languages that were spoken, as I could not remember Indigenous languages being spoken.

What I was learning through my living is that the predominant “if then” philosophy so evident in much of my formal education and the policy work was being replaced in my living with the “interdependent relational” philosophy of my early years. As I entered into a PhD program I searched for theoretical frameworks and methodologies that more closely aligned with an “interdependent relational” philosophy. The searching was not easy; but I had the opportunity to learn about narrative inquiry (Clandinin & Connelly, 2004) and an enactivist view of cognition (Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991). These views ‘aligned’ with what I was learning from Indigenous knowledge holders as I was making sense of doctoral studies and aligned with an “interdependent relational” ontological stance.

North

> **TO BE TRULY** wise is to understand that knowing and not knowing are one. Each has the power to transform. Wisdom is the culmination of teachings gleaned from the journey around the circle of life, the Medicine Wheel. Circles have no end. We are all spirit, we are all energy, and there is always more to gain. This is what my people say. When the story of our time here is completed and we return to spirit, we carry away with us all of the notes our song contains. The trick is to share all of that with those around us while we're here. We are all on the same journey, and we become more by giving away. That's the essential teaching each of us is here to learn. (Wagamese, 2011, p. 151).

Our journeys traveling through Wagamese’s Medicine Wheel teachings, East (humility) to South (trust) to West (introspection) bring us to North (wisdom) and our questions of what this journey means for mathematics education. We are aware of the colonizing role mathematics and mathematics education has played and continues to play as “one of the most powerful weapons in the imposition of Western culture … [and] a secret weapon of cultural imperialism” (Bishop, 1990, p. 51). And so, we ask: How does (re)membering place give rise to discourses and actions that are both resonating harmonies and critically dissonant in mathematics education? How can (re)membering place help challenge more dominant ways of being in relation – from exploitation, violence and oppression over land, animals, humans, language and cultures (Calderon, 2014; Seawright, 2014) to more intimate experiences of dwelling together humans and more-than-humans for the wellbeing of all (Abram, 2011).

We acknowledge calls in the literature to re-imagine mathematics education to address current global challenges (e.g., Adams, 2018; Barwell, 2018; Glanfield et al., 2019; Nicol et al., in press; Wolfmeyer et al., 2017). For instance, critiques of educating for STEM, are gaining attention in teacher education (Khan, 2020; Nicol et al., 2020), curriculum (Wolfmeyer et al., 2018a; Thom, 2021) and communities (Thom, 2019b; Wiseman et al., 2020); design (Glanfield et al., 2020), interdisciplinarity (Yaro, Amoah & Wagner, 2020), social justice (Davis & Renert, 2013; Wolfmeyer et al., 2018b); and mathematical formatting (Barwell, 2018; Skovsmose, 2021).

In addition, there are calls to address issues such as equity and the need to rehumanize mathematics education (Guitérrez, 2017, 2018); to teach and assess in ways that build upon the strengths of students from marginalized groups and communities (Aguirre & del Rosario Zavala, 2013; Celedón-Pattichis, Musanti & Marshall, 2010); and to recognize intersectionality (Bullock, 2017; Gholson 2016).

Yet, with Wagamese (2011), we have traveled the four directions to offer narratives for mathematical ways of being that support a more holistic engagement with human and natural environments. Where place, and land, is teacher.

In the Chinese language:

> wisdom is inscribed in a family of words: *human, humility, humus, and humor*, all etymologically related as they are, too, in our language. The Chinese characters of a wise leader read *sei-jin* 聖人–a person who, indwelling with others 人, stand between heaven and earth 土, listening 耳 to the silence, and who, upon hearing the wor[l]d, allows it to speak 口 to others so others may follow. (Aoki, 2004c, p. 214).

> What kind of a place is this? A place where there is room for words like *humor, human, humus, humility* to live together. In such a place, to be humiliated is to be reminded that we are communally ecologic, that the rhythmic measures of living [with] Earth come forth polyphonically in *humor* and *human* and *humus* and *humility*. (Aoki, 2004a, p. 300)

We end as we began with questions …

1. What are the conditions that make possible ethical and rigorous engagement across communities in resonant harmonies and critical dissonances that can help us move together towards improved relationships and wiser futures, as we face unprecedented global and local challenges?
2. What are some guidelines, approaches, and practices for ethical and respectful engagement with communities that can help us to work together in holding space for each other human and more-than-human in the place of mathematics education?
3. How do we learn together, to co-construct and learn to be with each other, the land and our more than human kin, in ways that are compassionate, sensory, interconnected, and with humility, courage, wonder and trust in caring for life over the long haul? And what is the role of mathematics education in this?

References

Abram, D. (2010). *Becoming animal: An earthly cosmology*. Pantheon.

Abram, D. (2011). Storytelling and wonder: On the rejuvenation of oral culture as an ecological imperative. In O. Urbain & D. Temple (Eds.), *Ethical transformations for a sustainable future* (pp. 9-17). Routledge. doi:10.4324/9781351311403.

Adams, K. (2018). STEM education and the miracle of life. In *A pedagogy of responsibility: Wendell Berry for ecojustice education* (pp. 117–135). Routledge.

Aguirre, J. M., & del Rosario Zavala, M. (2013). Making culturally responsive mathematics teaching explicit: A lesson analysis tool. *Pedagogies: An international journal, 8*(2), 163-190.
Aoki, T. T. (2004a). "Humiliating the Cartesian Ego." In W. F. Pinar & R. L. Irwin (Eds.), *Curriculum in a new key: The collected works of Ted T. Aoki* (pp. 291-301). Routledge.
Aoki, T. T. (2004b). "Imaginaries of "East and West": Slippery Curricular Signifiers in Education." In W. F. Pinar & R. L. Irwin (Eds.), *Curriculum in a new key: The collected works of Ted T. Aoki* (pp. 313-319). Routledge.
Aoki, T. T. (2004c). "Legitimating Lived Curriculum: Towards a Curricular Landscape of Multiplicity." In W. F. Pinar & R. L. Irwin (Eds.), *Curriculum in a new key: The collected works of Ted T. Aoki* (pp. 199-215). Routledge.
Aoki, T. T. (2004d). "Sonare and Videre: A Story, Three Echoes and a Lingering Note." In W. F. Pinar & R. L. Irwin (Eds.), *Curriculum in a new key: The collected works of Ted T. Aoki* (pp. 367-376). Routledge.
Atleo, E. R. (2011) *Principles of Tsawalk: An indigenous approach to global crisis*. UBC Press.
Barwell, R. (2018). Some thoughts on a mathematics education for environmental sustainability. In P. Ernest (Ed.) *The philosophy of mathematics education today* (pp. 145-160). Springer.
Bowers, C. (2016). *A critical examination of STEM: Issues and challenges*. Routledge.
Bullock, E. C. (2017). Beyond "ism" groups and figure hiding: Intersectional analysis and critical mathematics education. *Mathematics Education and Life at Times of Crisis, 29*, p. 29-44.
Celedón-Pattichis, S., Musanti, S. I., & Marshall, M. E. (2010). Bilingual elementary teachers' reflections on using students' native language and culture to teach mathematics. In *Mathematics Teaching and Learning in K-12* (pp. 7-24). Springer.
Clandinin, D. J. & Connelly, M. (2004). *Narrative inquiry: Experience and story in qualitative research*. John Wiley and Sons Inc.
Davis, B., & Renert, M. (2013). *The math teachers know: Profound understanding of emergent mathematics*. Routledge.
Dion, S. D. (2007). Disrupting molded images: Identities, responsibilities and relationships-teachers and indigenous subject material. *Teaching Education*, *18(*4), 329-342. https://doi.org/10.1080/10476210701687625.
Donald, D. (2012). Forts, colonial frontier logics, and Aboriginal-Canadian relations: Imagining decolonizing educational philosophies in Canadian contexts. In A. A. Abdi (Ed.) *Decolonizing philosophies of education* (pp. 91-111). Sense Publishers.
Gholson, M. L. (2016). Clean corners and algebra: A critical examination of the constructed invisibility of black girls and women in mathematics. *The Journal of Negro Education, 85*(3), 290-301.
Glanfield, F., Thom, J. S. & Ghostkeeper, E. (2019). Living landscapes and topographies [Conference presentation]. American Educational Research Association 2019, Toronto, ON, Canada.
Glanfield, F., Thom, J. S. & Ghostkeeper, E. (2021). Living landscapes, archi-text-ures, and land-guaging algo-rhythms. *Canadian Journal of Science, Mathematics, and Technology Education, 20*(2). https://doi.org/10.1007/s42330-020-00085-8.
Government of Canada. (2022). Why forests need fires, insects, and diseases. (Retrieved October 15, 2022, https://www.nrcan.gc.ca/our-natural-resources/forests/wildland-fires-insects-disturbances/why-forests-need-fires-insects-and-diseases/13081)
Gutiérrez, R. (2017). Why mathematics (education) was late to the backlash party: The need for a revolution. *Journal of Urban Mathematics Education, 10*(2), p. 8-24.
Gutiérrez, R. (2018). Introduction: The Need to Rehumanize Mathematics. In I. Goffney, R. Gutiérrez, & M. Boston (Eds.), *Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (Annual Perspectives in Mathematics Education; Vol. 2018). National Council of Teachers of Mathematics.
Humility. (2022). In *Online Etymology Dictionary*. Retrieved October 15, 2022, from https://www.etymonline.com/word/humility#etymonline_v_16050
Jones, M. (2006). Haida art and Haida Gwaii. In D. Augaitis, M. Jones, & P. Macnair (Eds.) *Raven travelling: Two centuries of Haida art* (pp. 28-39). Vancouver Art Gallery/Douglas & McIntyre.
Khan, S. (2020). After the M in STEM: Towards mathematics for multispecies flourishing. *Canadian Journal of Science, Mathematics, and Technology Education, 20*(2). https://doi.org/10.1007/s42330-020-00089-4.
Maturana, H. & Varela, F. J. (1992). *Tree of knowledge: The biological roots of human understanding*. (Rev Ed). Shambhala.
Nicol, C., Gerofsky, S., Nolan, K., Francis, K., & Fritzlan, A. (2020). Teacher educator professional learning with/in place: Storying the work of decolonizing mathematics education within a colonial structure. *Canadian Journal of Science, Mathematics, and Technology Education, 20*(2). https://doi.org/10.1007/s42330-020-00080-z.
Nicol, C., Thom J. S & Glanfield, F. (in press). Imagining STEM with and as Place. In R. Tierney, F. Rizvi, K.

Ercikan, & G. Smith (Eds.), *International Encyclopedia of Education* (4th edition) (12 pages). Elsevier.
Salmón, E. (2000). Kincentric ecology: Indigenous perceptions of the human-nature relationship. *Ecological Applications, 10*(5), 1327-1332.
Skovsmose, O. (2021). Mathematics and crises. *Educational Studies in Mathematics*, *108*(1-2), 369-383.
Styres, S. D. (2017). *Pathways for remembering and recognizing indigenous thought in education: Philosophies of iethi'nihsténha ohwentsia'kékha (Land).* University of Toronto Press.
Thom, J. S. (2012). *Re-rooting the learning space: Minding where children's mathematics grow*. Brill.
Thom, J. S. (2017). All about… embodied learning. *Nursery world*, *2017*(24), 17-21.
Thom, J. S. (2018). (Re)(con) figuring space: Three children's geometric reasonings. In *Contemporary research and perspectives on early childhood mathematics education* (pp. 131-158). Springer, Cham.
Thom, J. S. (2019a). Bending an ear in the quest of "what is it?". In *Living culturally responsive mathematics education with/in Indigenous communities* (pp. 231-258). Brill.
Thom, J. S. (2019b). Ecological Revelations: Recovering the Unseen. *Educational Studies, 55*(5), 510-530.
Thom, J. S. (2021). Co/inspiriting Ecological Conversations with Chet A. Bowers (1935–2017) and Ted T. Aoki (1919–2012). In A. Dentith, D. Flinders, J. Lupinacci, & J. S. Thom (Eds.) *Curriculum, environment, and the work of CA Bowers* (pp. 22-43). Routledge.
Thom, J. S. (2022, May). It's Spring. CMESG Newsletter/GCEDM *Infolettre*, *38*(2), 5-6. https://view.genial.ly/6271ce79a1ea070018534517
Thom, J. S., & Hallenbeck, T. (2021). Beyond words/signs: DHH learners' spatial reasoning in mathematics as embodied cognition. *American Annals of the Deaf, 166*(3), 384–414.
Thom, J. S., & McGarvey, L. M. (2015). The act and artifact of drawing (s): observing geometric thinking with, in, and through children's drawings. *ZDM*, *47*(3), 465-481.
Thom, J. S., & Pirie, S. E. (2002). Problems, perseverance, and mathematical residue. *Educational Studies in Mathematics*, *50*(1), 1-28.
Thom, J. S., & Roth, W. M. (2011). Radical embodiment and semiotics: Toward a theory of mathematics in the flesh. *Educational Studies in Mathematics*, *77*(2), 267-284.
Thom, J. S., D'Amour, L., Preciado, P., & Davis, B. (2015). Spatial knowing, doing, and being. In *Spatial reasoning in the early years* (pp. 73-92). Routledge.
Thom, J. S., McGarvey, L. M., & Lineham, N. D. (2021). Perspective taking: Spatial reasoning and projective geometry in the early years. *Mathematics Education Research Group of Australasia*.
Varela, F. J., Thompson, E. & Rosch, E. (1991). *The embodied mind: cognitive science and human experience*. MIT Press.
Wagamese, R. (2011). *One story, one song*. Douglas & McIntyre.
Wiseman, D., Lunney Borden, L., Beatty, R., Jao, L., & Carter, E. (2020). Whole-some artifacts: (STEM) teaching and learning emerging from and contributing to community. *Canadian Journal of Science, Mathematics, and Technology Education, 20*(2). https://doi.org/10.1007/s42330-020-00079-6.
Wolfmeyer, M., Lupinacci, J., & Chesky, N. (2017). Three ontologies of STEM education: An apolitical curricular trend, Eurocentric economic policy, and discursive episteme. *Critical Education, 8*(15), 68-79.
Wolfmeyer, M., Lupinacci, J., & Chesky, N. (2018a). EcoJustice mathematics education: An ecocritical (re) consideration for 21st century curricular challenges. *Journal of Curriculum Theorizing, 32*(2), 53-17.
Wolfmeyer, M., Lupinacci, J., & Chesky, N. (2018b). Inserting critical mathematics into STEM education. In R. Jorgensen & K. Larkin (Eds.), *STEM education in the junior secondary: The state of play* (pp. 273-294). Springer, Singapore.
Yaro, K., Amoah, E., & Wagner, D. (2020). Situated perspectives on creating mathematics tasks for peace and sustainability. *Canadian Journal of Science, Mathematics, and Technology Education, 20*(2). https://doi.org/10.1007/s42330-020-00083-w.

DECENTERING TO BUILD ASSET-BASED LEARNING TRAJECTORIES

Amy B. Ellis
University of Georgia
amyellis@uga.edu

The development and use of learning trajectories is a body of research that has made enormous contributions to the field of mathematics education, offering insight into the teaching and learning of topics at all levels. Simultaneously, the work of building learning trajectories can benefit from explicitly adopting an anti-deficit stance, incorporating ways to center student voices from an asset-based perspective. In this paper I propose two related constructs to support this work: decentering and second-order models. In decentering, researchers work to set aside their own knowledge to understand students' reasoning as viable. This can support models of student mathematics that position student thinking as rational, powerful, and productive. I provide one example of the work of decentering and discuss ways to build learning trajectories that emphasize students' strengths and competencies.

Learning trajectories research has played a prominent role in the field of mathematics education, and it continues to exert influence on the teaching and learning of mathematics. In a recent plenary address to PME-NA, Steffe (2017) remarked that the construction of learning trajectories is "one of the most daunting but urgent problems facing mathematics education today" (p. 39). The influence of this sphere of research is evident in funding priorities at the NSF and the IES, in special journal issues (Duncan & Hmelo-Silver, 2009), in topics conferences (e.g., the learning trajectories panel held at the VARGA 100 Conference in 2019), and in special reports (Daro et al., 2011; Taguma & Barrera, 2019). For instance, the National Research Committee (NRC) issued a special report in 2009 identifying a set of goals for young children based on learning trajectories, which ultimately led to the use of learning trajectories as a foundation for the Common Core standards in mathematics (Clements et al., 2019). We also see the prominence of learning trajectories research for PME-NA as reflected in plenary paper topics (e.g., Battista, 2010; Confrey, 2012; Sarama, 2018; Steffe, 2017).

Researchers have defined and theorized learning trajectories in a variety of ways. Simon (1995) initially coined the term "hypothetical learning trajectory" to describe "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Sarama (2012) described a learning trajectory as a depiction of students' thinking and learning in a specific mathematics domain and a "related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children though a developmental progression of levels of thinking" (p. 83), and Confrey and Maloney (2010) described a learning trajectory as a progression of cognition that represents ordered, expected tendencies developed through empirical research aimed at identifying the likely steps students follow. There is variation in the degree to which researchers characterize learning trajectories as being (a) connected to particular task sequences, (b) influenced by specific teaching actions or other contextual factors, and (c) depictions of strategies, skills, or performances versus concepts and operations; for a more expanded discussion of these differences, see Battista, 2010, or Ellis et al., 2014. For my work, I have found Steffe's (2012) characterization to be particularly useful. He described a learning trajectory as a model of students' initial concepts and operations, an account of the observable changes in those concepts and operations as a result of students' interactive mathematical activity in the situations of learning, and an account of the mathematical interactions that were involved in the changes. I consider task sequences to be a part of mathematical interactions, but the emphasis is on the interactions themselves, including particular teaching moves, students' activity and conversation with one another, and students' interactions with tools, artifacts, and representations.

As a body of research, learning trajectories have made enormous contributions to the field. They have offered insight into major milestones of students' conceptual development for a variety of topics, including measurement (Battista, 2010; Clements & Sarama, 2009; Sarama et al. 2011), composition of geometric figures (Clements et al., 2012), fractions (Maloney & Confrey, 2010; Steffe, 2012b; Steffe & Olive, 2010; Wright, 2014), early algebra (Blanton et al., 2015; Hackenberg & Lee, 2015), geometry (Fitri & Prahmana, 2020), function (Ellis et al., 2016; Fonger et al., 2020), and probability (Rahmi et al., 2020; Wijaya & Doorman, 2021), among others. Learning trajectories research informs not only standards development, but also curriculum, pedagogical decision making, teacher noticing, and professional development (Clements, 2007; Confrey et al., 2014; Hackenberg & Sevinc, 2022; Liss, 2019; Meyers et al., 2015; Suh et al., 2021; Steffe, 2004). However, this body of research has also weathered critiques. These critiques include concerns about an overfocus on tasks, cautions about the need to better attend to variation in students' progression, scrutiny of the basis for the construction of learning trajectories, and calls to more explicitly address equity and inclusion.

An overfocus on tasks can occur when learning trajectories offer only task sequences paired with learning goals, without attending to the teaching actions and other contextual factors that are important for supporting students' development. Relatedly, learning trajectories can be construed as generalizable or transportable from one situation or context to the next, as if students, teachers, classrooms, and cultures were interchangeable. It is important to recognize that trajectories developed in one context may not always appropriately depict students' learning in a markedly different context. Additionally, not all students will progress in the same way throughout any given trajectory. Learning is more individualized, context-dependent, and idiosyncratic than what could ever be depicted in a neat, ladder-like sequence. Certainly, many researchers who construct learning trajectories are aware of these constraints. For instance, Clements and Sarama (2012) wisely reminded the reader that their task sequences are not necessarily the only or even the best path for learning and teaching, but are instead merely hypothesized to be "one fecund route" (p. 84). Nevertheless, learning trajectories have, at times, been interpreted in overly broad or simplified ways.

A more central issue that I would like to tackle in this paper is the models that constitute the basis of learning trajectories. In particular, it is worth considering the affordances and constraints of these models for developing and using learning trajectories to highlight students' competencies. In order to do so, I now turn to a consideration of first-order and second-order models, advocating for the use of second-order models to advance an asset-based perspective.

The Potential Pitfalls of Building Learning Trajectories from First-Order Models

Learning trajectories that are built on the foundation of the researcher's understanding of the discipline are based on what we call first-order models (Steffe & Olive, 2010). First-order models, or first-order knowledge, are the models that one constructs "to order, comprehend, and control his or her own experience" (ibid, p. xvi). There is robust evidence of reliance on researchers' first-order knowledge of mathematics in learning trajectories research. For instance, Clements and Sarama (2012) described a hypothetical learning trajectory as one involving conjectures about a possible learning route that aims at significant mathematical ideas, and a specific means to support and organize learning along this route. Those mathematical ideas are the researcher's ideas: "The trajectory is conceived of through a thought experiment in which the historical development of mathematics is used as a heuristic" (p. 82). To offer a few other examples, Confrey and colleagues (2014) used the term learning trajectory to refer to "clusters and sequences of standards and their related descriptors" (p. 720), Baroody et al. (2022) depicted the goals of a learning trajectory to be based on "the structure of mathematics, societal needs, and research on children's thinking about and learning of mathematics" (p. 195), and Andrews-Larson et al. (2017) described their hypothetical learning trajectory as content-specific documentation of common milestones and learning environments supporting students' progression across those milestones. Certainly, not all studies reporting on learning trajectory development conceive of learning trajectories in this manner. For instance, Confrey (2006) underscored the importance of the learner in guiding this work, emphasizing the centrality of students' voices and disciplinary perspectives, and others have published learning trajectories that reflect this aim (e.g., Fonger et al., 2020; Steffe, 2012; Steffe & Olive, 2010). Nevertheless, there remains a strong emphasis on learning trajectories that are based on researchers' own mathematics as starting points.

Building learning trajectories from first-order knowledge can offer important affordances. Such trajectories reflect the researcher's nuanced, in-depth understanding of the relevant content and key learning goals, as well as research-based knowledge of how to support student learning. At the same time, trajectories developed from first-order knowledge may also position students in terms of how they measure up against researchers' knowledge of the discipline. Moreover, this framing runs the risk of depicting students in terms of falling short. Adiredja (2019) characterized this stance as "epistemological violence", particularly towards minoritized students and students from marginalized communities, when the research we conduct positions students' knowledge as inferior or problematic. Furthermore, such a stance centers the perspective of the expert rather than that of the student. Certainly, many thoughtful scholars are careful to consider these issues in both their construction and use of learning trajectories, emphasizing the potential of learning trajectories to be asset-based models (e.g., Clements & Sarama, 2012; Hunt et al., 2020; Meyers et al., 2015; Suh et al., 2021). There is nothing inherent in a learning trajectory that requires it to be constructed as a deficit-based tool. Nevertheless, learning trajectories built from first-order models may fail to identify, sufficiently explore, or acknowledge the competence and brilliance of student thinking. In fact, as a field we run the risk of learning trajectories being used to bolster deficit stances towards minoritized and marginalized students, particularly when the trajectories over-privilege formal language, consistency in understanding, or straightforward and direct change in understanding (Adiredja, 2019). Adiredja pointed out that it is not that these mathematical goals are bad, but rather, an inflexible privileging of such goals can interact with deficit master-narratives to devalue the mathematical sensemaking of students, particularly students of color.

Learning trajectories built from first-order knowledge can also run the risk of encouraging teachers and other stakeholders to use them in a manner that places students on a continuum, with some positioned as more advanced and others positioned as deficient. Such an emphasis is reminiscent of the studies focused on achievement gaps, which allow researchers to "unconsciously normalize, the 'low achievement' of Black, Latina/Latino, First Nations, English language learners, and working-class students without acknowledging racism in society or the racialization of students in schools" (Gutiérrez, 2008, p. 359). Moreover, this treatment of learning trajectories may miss important nuances, not only about student thinking and reasoning, but also about the ways in which students may shift from one understanding to another based on complex, interrelated factors.

What, then, is the alternative? Researchers can instead psychologize students' mathematics by constructing second-order models, which are the hypothetical models observers construct of their students' knowledge in order to explain their observations of students' states and activities (Steffe & Olive, 2010). They are referenced to the researcher's first-order mathematics, as well as the researcher's conceptions and interpretations of the language and actions of students. These second-order models are sometimes referred to as the mathematics of students; students' first-order models (their own models of mathematics) are referred to as students' mathematics (Steffe, 2017).

Building Learning Trajectories from Second-Order Models

In building learning trajectories that are elaborations of second-order knowledge, we concern ourselves with identifying the mathematics of students and elaborating students' mathematical concepts and operations. I consider these learning trajectories to be *coproduced* by students and researchers (Steffe, 2012). Although initial hypothetical learning trajectories may be informed by a researcher's first-order mathematical knowledge, in combination with their knowledge of student thinking, these trajectories are nascent, ill-formed, and flexible. The learning trajectories that are consequently built out of teaching actions with students are accounts of students' initial concepts and operations, an account of the observable changes in those concepts and operations as a result of teaching and learning actions, and an account of the teaching and learning actions that led to the changes.

Building learning trajectories as second-order models encourages, or perhaps even requires, a different epistemology of mathematics, one that deviates from Western naïve realism traditions. Drawing on Piaget's epistemological beliefs, von Glasersfeld (1982) wrote that "The cognitive organism is first and foremost an organizer who interprets experience and, by interpretation, shapes it into a structured world" (p. 612). This one sentence conveys a radical departure, as von Glasersfeld put it, from traditional ideas of not only knowledge, but of reality itself. Knowledge is not a more or less accurate representation of reality. We construct our conceptions of reality through perception, not directly, and we cannot maintain a belief about knowledge being a reflection of reality by simply acknowledging that our reflection may not always be very accurate. This is not to say that von Glasersfeld denied reality; rather, he considered it to emerge only through bumping up against constraints. From this perspective, it does not make sense to judge knowledge based on its accuracy; in fact, this would be impossible, because it would require comparing one's knowledge to an independently existing reality and judging the closeness of the match. How can any human do this without direct access to that reality? Instead, knowledge is successful if it is viable, i.e., when it is not impeded by constraints.

Within this framing, there is no such thing as a mathematics that resides outside of human experience. The very concept of the second-order model is based on an epistemology that considers mathematics to be a product of the functioning of human intelligence. Students' mathematics *is* the mathematics. Certainly, we can compare our second-order model of the mathematics of a student to our first-order model of our own mathematics. In doing so, it is productive to understand that there are two mathematics, and both are legitimate. This requires a rejection of the Platonist knowledge traditions that frame mathematics as universal and objective. It also requires one to position students' ways of knowing and thinking as rational, rather than inferior when compared to standard strategies, procedures, and conventions (Louie et al., 2021). As L. Steffe explained, "students are 'never wrong' even though their thinking may not appear as viable with respect to certain situations or ways of thinking. Mistakes are always an observer's concept" (personal communication, October 5, 2022). Louie and colleagues argued that a failure to position students' reasoning as legitimate can discourage teachers from attending closely to unconventional ways of thinking and seeking to understand them, much less valuing or inviting them. I argue that this can also be true of researchers' treatment of students' ideas. In contrast, if we understand that students' mathematics *is* the mathematics, then we will be compelled to take students' reasoning and competencies as the starting point for building any learning trajectory.

One way to build second-order models is through the process of conceptual analysis, which is a process guided by the question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (Steffe, 2017, p. 78). Thompson and Saldanha (2000) described conceptual analysis as articulating the conceptual operations that, "were people to have them, might result in them thinking the way they evidently do" (p. 315). Engaging in conceptual analysis draws on a researcher's ability to decenter, and can support the development of the epistemic student. Below I discuss each of these constructs in turn.

Decentering

Piaget (1955) introduced the idea of decentering to characterize the actions of an observer attempting to understand how an individual's perspective differs from their own (Teuscher et al., 2016). Piaget developed this idea to describe an aspect of a child's development: when a child learns to decenter, they begin to abandon egocentrism and develop the capacity to consider another's perspectives, thoughts, and feelings (Piaget & Inhelder, 1967). Steffe and Thompson (2000) then extended Piaget's construct to characterize a teacher's stance towards a student, particularly in terms of a teacher's ability to adjust their actions in order to understand a student's thinking.

Arcavi and Isoda (2007) described decentering as:

> the capacity to adopt the other's perspective, to 'wear her conceptual spectacles' (by keeping away as much as possible our own perspectives), to test in iterative cycles our understanding of what we hear, and possibly to pursue it and apply it for a while. Such a decentering involves a deep intellectual effort to be learned and exercised (p. 114).

Decentering is a stance that attends to both mathematical thinking and social interactions. It entails interacting with students reflectively, in a conscious attempt to set aside one's own knowledge to understand a student's reasoning as viable (Thompson, 2000). This reflective stance towards interactions with students is crucial for creating viable second-order models, and such efforts are hampered if a teacher – or a researcher – does not make concerted efforts to differentiate the mathematics of students from one's own mathematics. Steffe and Ulrich (2020), in distinguishing between responsive / intuitive interaction and analytic interaction, described the latter as a process of stepping out of analyzing students' thinking in ongoing interaction. All of the researcher's attention is absorbed in trying to think like the students, and produce and experience mathematical realities that are intersubjective with their own first-order models.

If researchers do not decenter, students' thinking and reasoning may not be considered worthwhile models of the environment in their own right, and instead may be positioned only in relation to standard models (i.e., the researcher's models) of mathematical knowledge. The construction of learning trajectories that are not a consequence of decentering may then position students as falling short, with insufficient attempt to understand or model students' thinking as viable, powerful, and potentially productive, even in times when it deviates from canonical mathematics. In contrast, a decentering researcher "always assumes that a student has some viable system of meanings that contribute to her or his actions" (Teuscher et al., 2016, p. 439).

Teuscher and colleagues went on to point out that the ideas of correct versus incorrect become largely irrelevant beyond informing one's future actions. This is not to suggest that correctness is unimportant. Rather, when engaged in the hard work of decentering, correctness is not a notion that contributes utility to building second-order models. A student does not position their own knowledge as incorrect, and decentering means seeing the world with the student's mathematical eyes.

The Epistemic Student

Hackenberg (2014) defined an epistemic student as an organization of schemes of action and operation that undergo change over time. The epistemic student is a model, one that is composed of the ways of operating common to all students at the same level of development "whose cognitive structures derive from the most general mechanisms of co-ordination of actions" (Beth & Piaget, 1966, p. 308). I see the epistemic student as a useful model of characteristic mathematical activity that is developmental, generalized, and dynamic (Ellis, 2014). It is an abstraction (Piaget, 1970), meant to explain some ways of operating that we suspect may be common across students.

The epistemic student is a helpful construct because students who share initial concepts and operations often respond in somewhat common ways to thoughtful instructional interactions. This does not mean that every student will respond identically, but typically there are a manageable number of ways of reasoning that bubble up repeatedly across participants and contexts. The epistemic student can be a useful model for trying to walk the tightrope between overgeneralization and over specificity. I acknowledge that it is not appropriate or even accurate to claim that my second-order models and resulting learning trajectories, which are developed from small numbers of students in specific contexts, would extend to all students in all contexts. To do so would ignore the variation in students' experiences, backgrounds, and positionalities, as well as the variation in classrooms, schools, and cultures. Simultaneously, the work of building learning trajectories necessarily entails a belief in the value of creating scientific (rather than experiential) models with the potential of being useful across different students and contexts.

Learning Trajectories Built from Second-Order Models Emphasize Anti-Deficit Stances

The body of learning trajectories research has been critiqued for not adequately considering equity or addressing student diversity (e.g., Zahner & Wynn, 2021). Some may even be used in ways that can reinforce deficit perspectives. A deficit perspective is "a propensity to locate the source of academic problems in deficiencies within students, their families, their communities, or their membership in social categories (such as race and gender)" (Peck, 2021, p. 941). In contrast, an anti-deficit perspective begins with the assumption that students are capable of reasoning mathematically and that they bring productive resources for learning mathematics. It acknowledges that learning is time-consuming, and that "imperfect articulations of mathematical ideas and some inconsistencies in the student's current conception are a natural part of the process" (Adiredja, 2019, pp. 416-417). Furthermore, adopting an anti-deficit perspective means locating the source of students' academic challenges within the racist, sexist, and ableist institutional structures that restrict, or even actively oppose, access to high-quality educational opportunities. When considering student thinking, a researcher considers and identifies the assets and competencies that students possess, rather than what students lack.

A goal of learning trajectory construction must be to position student thinking as rational, powerful, and viable, and from that position, seek to understand why students reason the way they do. It is our job, as researchers, to construct second-order models that reflect a value that student thinking is sensible and intelligent. In constructing learning trajectories, we must begin with that stance, identify student concepts and operations so that we can meet students where they are, and then consider productive teaching interactions that can support students' shifts from one way of thinking to the next. In doing so, we must also understand and acknowledge that these shifts may be idiosyncratic, time-consuming, and messy, as is learning itself. By starting from a model of the mathematics of students, we can then construct models for how teachers might interact with students to bring forth productive changes in their concepts and operations.

The learning trajectories that my colleagues and I produce (e.g., Ellis et al., 2016; Fonger et al., 2020) emphasize students' strengths and competencies, even when student thinking differs from canonical mathematics. We see an important outcome of our learning trajectory work to be that of highlighting those strengths and competencies with stakeholders. The goal of our work is to understand why students reason the way they do, and to show how students can and do think in ways that are thoughtful, reasonable, and nuanced, even if, at first glance, one might only see an incorrect answer or a puzzling strategy. Like many others (e.g., Clements & Sarama, 2012), our learning trajectories provide multiple viable paths and do not claim to represent the only (or even the best) route to learning. Centering the mathematics of students is explicit in our theoretical framing and constitutes the starting point for creating and refining trajectories.

An Example of Building a Learning Trajectory from Second-Order Models

Our learning trajectories are depictions of concepts and mental operations, in concert with teaching interactions and in relation to task sequences, set in specific teaching and learning concepts. The concepts and mental operations *are* the mathematics in our trajectories. As an example, my colleagues and I constructed a learning trajectory of students' understanding of exponential growth from a covariation perspective (Ellis et al., 2016). That trajectory in its entirety is beyond the scope of this paper, but I will highlight here four of the operations we identified: (1) Explicit coordination of change in y-values for 1-unit change in x-values, (2) Coordination of change in y-values for multiple-unit changes in x-values: repeated multiplication imagery, (3) Coordination of change in y-values for multiple-unit changes in x-values: exponentiation imagery, and (4) Coordination of change in y-values for any unit change in x-values, for any Δx. Mathematically, from our perspective, these are all the same operation. For an exponential function $y = ab^x$, it is possible to coordinate the change of any two y-values with any two corresponding x-values according to the relation $\frac{y_2}{y_1} = b^{x_2 - x_1}$, and the value of Δx is immaterial. Conceptually, however, these are not the same operation. In our work with students, my colleagues and I found that coordinating changes in x-values and corresponding y-values for unit changes in x is different from coordinating for large changes in x. Additionally, one can engage in coordination for large changes in x either by appealing to repeated multiplication imagery, or by appealing to a different set of stretching or scaling images. Furthermore, managing this type of coordination for cases when the change in x is less than 1 draws on a different set of concepts, and students can engage in operations (1) – (3) long before they can do operation (4).

What students can do and how they can reason is always rational from their perspective. Not yet being able to engage in operation (4) is something that a researcher would describe as a constraint for the student. But, from the student's perspective, there does not exist a more "advanced" way to coordinate exponential growth. Students are always reasoning with their available conceptual operations. For teachers, then, it is advantageous to understand how students may be operating, so that they do not impose ways of thinking on the students that run counter to the students' reasoning. Curricular treatments of exponential growth, however, to the extent that they might address a coordination approach at all, do not distinguish these forms of reasoning, because they can all be handled with the same formula. In contrast, my teaching interactions with students revealed that it is sensible for students to draw on different imagery when constructing these operations, and that transitioning from one form of reasoning to another can be effortful and may require specific instructional support. In short, such a transition is a significant intellectual achievement. Without this knowledge, teachers and textbooks will not distinguish them, and students may consequently experience challenges in making sense of expressions such as $2^{(1/7)}$; after all, when the meaning of exponents presented to students in school is only that of repeated multiplication what does it mean to engage in such multiplication 1/7 times? Now that we are aware that these operations are mathematically different for students, we can help improve the teaching and learning of exponential growth ideas.

As an example of the decentering work that supported our understanding of the mathematics of students, consider a task in which students are provided with a table of height values at certain times for a special plant called a Jactus, which grows exponentially (Figure 1).

This is a table for a Jactus that doubles every week. The entries are approximate. How much taller will the plant grow in a quarter of a week?

Week	Height
0	1"
0.25	???
0.5	1.414214"
0.75	1.681793"
1	2"
1.25	2.378414"
1.5	2.828427"

Figure 1: Table of Week and Height Values for a Doubling Jactus

If I were to solve this problem, I would take the ratio of any two consecutive height values in the table that were a quarter of a week apart. That ratio is approximately 1.189, and so I can divide the height at week 0.5 by 1.189 to find the missing height value. The question is written in an unusual way, because it asks students about how the plant grows in a quarter of a week, but the table has an empty spot for the height value at a specific time, 0.25 weeks, which is actually a slightly different question. This was a "serendipitous mistake" (Tasova et al., 2021), because it enabled us to identify a form of reasoning about which we had been unaware prior to students encountering the task. Our initial intention was to support the idea that the ratio of height values for any quarter-week gap will always be the same.

When working with 8th-grade participants who had never before had school instruction on exponential growth, we initially expected that they would use a strategy like the one I described, because they had already used that strategy with prior tables. For instance, when encountering tables with uniform gaps of 1 week, 2 weeks, or 5 weeks, our students had divided height values to determine the plant's growth for the corresponding amount of time. But in this case, they did not leverage this strategy. For instance, consider the work of one of our participants, Uditi (a self-chosen pseudonym). In describing herself, Uditi discussed her experiences as an immigrant to the midwestern United States from India. She shared that she enjoyed mathematics and science, which was why she had volunteered for our research study, and she preferred expressing her ideas in small groups rather than with the whole class. When encountering the task in Figure 1, Uditi wrote the expression "$1 \times \_\_\_\_^{0.5}$". She then proceeded to use a cumbersome guess and check strategy to determine the missing value that would go in the blank to yield the plant's height at 0.5 weeks, which she knew had to be approximately 1.414. By doing this, Uditi determined that the growth factor was 2, and then laughed ruefully as she saw that the task description had already told her that the Jactus doubled each week. She then wrote the equation, "$\text{Height} = 1 \bullet 2^{\text{week}}$", and then substituted 0.25 for the exponent to determine the height at 0.25 weeks.

Uditi's strategy was correct, and it was also creative. It revealed an understanding of many important ideas, including the idea that she could write a correspondence relation of the form $y = 1b^x$ because the initial height at Week 0 was 1 inch. Her strategy, however, also surprised me and my colleagues, because it was different from what she had done before, and it was also more cumbersome and difficult than just dividing. Moreover, Uditi was not the only student who approached the problem in this surprising way. Other students across two different teaching experiments did as well, which suggested to us that there was an important conceptual issue with that task that we had not anticipated. In combination with other students' responses to similar tasks, we began to realize that the value of Δx was critical. If one week is the period of time for the plant to double its growth, then it became clear that asking students to determine what happened *within* a week was a conceptually different task than asking students what happened across a span of multiple weeks – even though, from our perspective as researchers, the two tasks were mathematically identical.

Part of the job of creating second-order models is to engage with students reflectively, attempting to decenter in order to understand why their behavior and reasoning is sensible. Uditi and other students could already write expressions such as $y = x^{0.25}$, therefore presumably using decimal and fractional exponents to determine a fixed height value. What puzzled us was that they could also divide two height values to determine an amount of growth for a given time span. Why, then, did Uditi not do so with this task? Our goal was to now try to understand Uditi's reasoning that drove the unanticipated strategy. In doing so, we hypothesized that it was because Uditi could attribute two meanings to an equation such as height = 2^3, but only one meaning to an equation such as height = $2^{0.25}$. The expression 2^3 in the first equation meant two things to Uditi: It could be a static height value, such as the plant's height at 3 weeks, or it could be a measure of growth, i.e., how many times larger the plant grows in height for a time span of 3 weeks. But the expression $2^{0.25}$, we hypothesized, could be the plant's static height value at 0.25 weeks, but not how many times larger the plant would grow in a time span of 0.25 weeks. We suspected two reasons for this, both related to students' meanings for multiplication. The first is that determining growth across multiple weeks entails generalizing the operation of multiplication to an exponential context. This is fairly easy to do, as many students hold an expectation that multiplication makes bigger (Greer, 1987). They can extend their meaning of doubling by mentally repeating the operation multiple times across multiple weeks. In contrast, 0.25 weeks is less than a week, and there is no easy way to extend the operation of doubling to a fraction of a week. Furthermore, Uditi and the other students we worked with had all received school instruction that described exponential growth as repeated multiplication. The image of repeated multiplication does not easily lend itself to decimal exponents, as it is difficult to imagine such an operation.

Our efforts to construct a second-order model of Uditi's mathematics via her activity with this and related tasks led us to realize that constructing exponents less than 1 as a representation of growth is its own separate mathematical concept. This is not something I understood prior to working with Uditi and other students. We learned that in order to make sense of non-natural exponents as representations of growth, it is useful to invite students to shift to images that do not entail repeated multiplication. For our participants, this meant creating an image of change in the plant's height between weeks that represented an action of stretching, or scaling. For instance, Pei's drawing (Figure 2) shows a Jactus stretching as it doubled from Week 1 to Week 2 to Week 3, on the right, and then he was able to reverse his doubling operation to imagine halving the Jactus's height to see how tall it would be at Week ½ on the left. Once Uditi developed a similar scaling image, she was then able to answer the following question: "Say a plant grows 3 times as tall every week. How many times taller will it grow in 1 day?". Uditi wrote "$3^{.14}$" and explained, "There are seven day(s) in a week. So, I divided one week into seven parts, which represent one day." She could now conceive of an expression such as $3^{.14}$ to represent a measure of growth, not just a static height value.

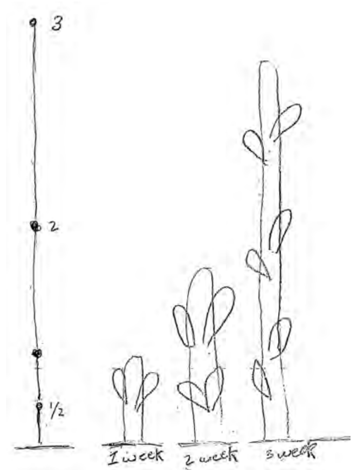


Figure 2: Pei's Drawing of a Doubling Jactus

Before my teaching interactions with Uditi, I was unaware that an expression such as $y = ab^x$ could represent two different ideas, a static value or an expression of growth. Certainly, this is not a big idea, and its truth is obvious to me now. Nevertheless, the concept was not originally part of my own first-order knowledge, nor was I aware that the mental imagery needed to undergird the second idea would need to be different from the first in order to accommodate non-natural exponents. Thus, my participants' mathematics served as a source of novel mathematics for me as a researcher, as it could also do for teachers who make use of the learning trajectory.

This approach toward the creation of learning trajectories shifts mathematical authority to the students. Uditi's mathematics served as a source of new mathematics for me. As a researcher, it was my job to understand her mathematics, why it made sense conceptually, and then determine ways to support her to create the meanings and images that would be productive for fostering an understanding of nonnatural exponents. This work centers student thinking as the core of our activity. Moreover, a productive stance to aid in my own decentering is to ask the question why. Why did Uditi's activity make sense? Why did she use the strategy that she did? Researchers must ask these questions with the assumption that there is always a sensible reason driving the student's activity. We simply need to be careful enough in our own research to find it. Moreover, in doing that work, we as researchers can grow in our own first-order knowledge: knowledge of the mathematics itself, of student thinking about mathematics, of the concepts and operations needed to make sense of particular ideas, and of the kinds of tasks and teaching moves that can support the development of those ideas.

I would like to close this extended example by pointing out that this model of a learning trajectory, with the four operations I shared, differs from learning trajectories that are (a) a proposal of the kinds of reasoning we should expect from students based on content analysis, (b) a specification of target performances, (c) a set of strategies, (d) a network of constructs that one might encounter through curriculum and/or instruction, or (e) a set of tasks that could be provided as stand-alone problems. An elaboration of a learning trajectory built from second-order models will identify student concepts and operations in relation to both tasks and teaching actions. I do not mean to denigrate or minimize the incredibly valuable contributions that prior learning trajectories have made, but rather, to articulate and clarify my vision of what a learning trajectory could be when it is built from second-order models.

Learning Cannot be Separated from Activity and Context

Helping our students develop stretching and scaling images for exponential growth turned out to be a productive route for their learning. Furthermore, because we saw the type of reasoning Uditi demonstrated in other students, our construction of the epistemic student from Uditi and her peers supported a model in which one may need explicit support to shift from a repeated multiplication image to an alternate image. Sharing second-order models in this manner can also help mathematically experienced adults, such as curriculum authors and teachers, understand and appreciate a different mathematics from the one they already know. In this manner, learning trajectories research can play a role not only in helping the field better understand how to support student learning of particular mathematics topics in the curriculum, it can – and should – determine what mathematics should be in the curriculum to begin with.

Nevertheless, emphasizing stretching and scaling images may not necessarily be a universally productive route. The degree to which it proves to be fruitful for other students in other contexts is an open question. As I alluded to above, researchers have raised concerns about the need to attend more explicitly, and more theoretically, to the role that teaching interactions play in influencing student learning (Empson, 2011; Simon et al., 2010). This body of work challenges the assumption that features of learning are transportable, or that it is possible to study effective teaching and learning in a particular context and tease out some key aspects that can then be generalizable to other contexts. Mathematics learning occurs in interaction, not only via teaching actions, but also via tasks, tool use, student discourse, classroom norms, school and community settings, and in relation to students' identities, histories, and positionalities (Nasir et al., 2008). Because mathematics learning does not occur in isolation from these sociocultural contexts, it is wise to avoid overly strong claims about the transportability of any particular finding. Learning trajectories research is a worthwhile endeavor not just for its potential to broadly improve the teaching and learning of particular topics (Baroody et al., 2022), but also to develop a set of contextualized, specific case studies of the types of reasoning that can exist and can be supported in particular ways.

Build Learning Trajectories that are Engines of Equity

I have advocated for the usefulness of the epistemic student as a construct that can help researchers navigate a balance between over generalizability and over specificity. In what might seem like an odd turn, I am now going to argue against my own argument, or, at least, consider an alternate stance. That stance is this: If the epistemic student is an idealized abstraction, a model composed of common ways of operating, this leads me to question what type of student we imagine when evoking the epistemic student. Analyzing student reasoning without attending to sociocultural diversity runs the risk of reinforcing deficit narratives about minoritized students and students from marginalized communities (Zahner & Wynn, 2021). And yet, the construct of the epistemic student may encourage this form of analysis.

Studies of cognition and equity are frequently positioned as separate areas of research in mathematics education (Adiredja, 2019). For instance, Adiredja has pointed out that racial and gender inequities are seldom considered in cognition studies, and, furthermore, engaging in analysis that does not include these positionalities of the students we study does not make our research apolitical: "Rather, it has the impact of maintaining the status quo that is the dominant master-narrative about White male exclusive membership in mathematics and centering education around their needs and concerns" (ibid, p. 426). One way to begin to address this limitation can be to extend the notion of the epistemic student to understand the identities and positionalities of our research participants who contribute to the epistemic student model. We can invite studies highlighting the powerful reasoning of marginalized and minoritized students, including being deliberate about who we include as participants in research opportunities, being thoughtful about the ways in which we engage our participants in research, and being explicit about our participants' positionalities.

We must build learning trajectories that are explicitly and theoretically organized from an asset-based perspective. Such trajectories can begin with efforts to understand our students' cultural competencies, and by drawing on our students' backgrounds and out-of-school knowledge and practices, rather than ignoring or even excluding them. Deliberately creating learning environments that leverage students' cultural and linguistic strengths supports their mathematical reasoning (Abdulrahim & Orosco, 2020). This means, then, broadening our starting point for the construction of hypothetical learning trajectories. Rather than beginning only with our first-order knowledge of mathematics, combined with research and pilot studies about student learning, we can also incorporate (a) hypothesized second-order models, (b) research on students' funds of knowledge relative to the topic at hand, and (c) information about our participants' values, interests, and knowledge. In this manner, learning trajectories research can meaningfully center student voices, and can serve as a bridge connecting research on cognition and equity.

Acknowledgments

The research reported in this paper was supported by the National Science Foundation (award no. 1920538). I would also like to thank Maria Blanton, Marrielle Myers, Leslie Steffe, and Jennifer Suh for their helpful conversation and feedback on the ideas in this paper, as well as Joel Amidon, Muhammed Fatih Dogan, Nicole Fonger, Torrey Kulow, and Zekiye Özgür for their contributions and brilliance in collaboratively constructing, refining, and writing about learning trajectories with me.

References

Abdulrahim, N. A., & Orosco, M. J. (2020). Culturally responsive mathematics teaching: A research synthesis. *The urban review*, *52*(1), 1-25.

Adiredja, A. P. (2019). Anti-deficit narratives: Engaging the politics of research on mathematical sense making. *Journal for Research in Mathematics Education*, *50*(4), 401-435.

Andrews-Larson, C., Wawro, M., & Zandieh, M. (2017). A hypothetical learning trajectory for conceptualizing matrices as linear transformations. *International Journal of Mathematical Education in Science and Technology*, *48*(6), 809-829.

Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice. *Educational studies in mathematics*, *66*(2), 111-129.
Baroody, A. J., Clements, D. H., & Sarama, J. (2022). Lessons Learned from 10 Experiments That Tested the Efficacy and Assumptions of Hypothetical Learning Trajectories. *Education Sciences*, *12*(3), 195.
Battista, M. (2010). Representations of learning for teaching: Learning progressions, learning trajectories, and levels of sophistication. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the Thirty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. pp. 60-71. The Ohio State University.
Beth, E. W., & Piaget, J. (1966). Mathematical epistemology and Psychology, trans. *W. Mays (Dordrecht: D. Reidel)*.
Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, *46*(5), 511-558.
Clements, D. H. (2007). Curriculum research: Toward a framework for "research-based curricula." *Journal for Research in Mathematics Education,* -.(1), 35–70. doi:10.2307/30034927
Clements, D. H., Fuson, K. C., & Sarama, J. (2019). Research commentary: Critiques of the Common Core in early math: A research-based response. *Journal for Research in Mathematics Education*, *50*(1), 11-22.
Clements, D. &Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Routledge.
Clements, D. H., & Sarama, J. (2012). Learning trajectories in mathematics education. In *Hypothetical Learning Trajectories* (pp. 81-90). Routledge.
Clements, D. H., Wilson, D. C., & Sarama, J. (2012). Young children's composition of geometric figures: A learning trajectory. In *Hypothetical Learning Trajectories* (pp. 163-184). Routledge.
Confrey, J. (2006). The evolution of design studies as methodology. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 135–151). Cambridge University Press.
Confrey, J. (2012). Articulating a Learning Sciences Foundation for Learning Trajectories in the CCSS-M. *North American Chapter of the International Group for the Psychology of Mathematics Education*.
Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. *International Conference of the Learning Sciences, 1*, 968-975.
Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: A framework for connecting standards with curriculum. *ZDM*, *46*(5), 719-733.
Daro, P., Mosher, F. A., & Corcoran, T. (2011, January). *Learning trajectories in mathematics* (Research Report #RR-68). Consortium for Policy Research in Education.
Duncan, R.G., & Hmelo-Silver, C.E. (2009). Learning progressions: Aligning curriculum, instruction, and assessment. *Journal of Research in Science Teaching, 46*, 606-609.
Ellis, A.B. (2014). What if we built learning trajectories for epistemic students? In L. Hatfield, K. Moore, & L. Steffe (Eds.), *Epistemic Algebraic Students: Emerging Models of Students' Algebraic Knowing*, WISDOM[e] Monographs (Vol. 4, pp. 199-207). University of Wyoming.
Ellis, A.B., Özgür, Z., Kulow, T., Dogan, M.F., & Amidon, J. (2016). An exponential growth learning trajectory: Students' emerging understanding of exponential growth through covariation. *Mathematical Thinking and Learning*, *18*(3), 151 – 181.
Ellis, A.B., Weber, E., & Lockwood, E. (2014). The case for learning trajectories research. In Oesterle, S., Liljedahl, P., Nicol, C., & Allan, D. (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36, Vol. 3,* pp. 1 – 8. PME.
Empson, S. B. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast*, *8*(3), 571-596.
Fitri, N. L., & Prahmana, R. C. I. (2020). Designing learning trajectory of circle using the context of Ferris wheel. *JRAMathEdu (Journal of Research and Advances in Mathematics Education)*, *5*(3), 247-261.
Fonger, N., Ellis, A.B., & Dogan, M.F. (2020). A quadratic growth learning trajectory. *Journal of Mathematical Behavior, 59*, 1-22.
Greer, B. (1987). Brief Report: Nonconservation of Multiplication and Division Involving Decimals. *Journal for Research in Mathematics Education*, *18*(1), 37-45.
Gutiérrez, R. (2008). Research commentary: A gap-gazing fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education*, *39*(4), 357-364.

Hackenberg, A. J. (2014). Musings on three epistemic algebraic students. In K. C. Moore, L. P. Steffe & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (Vol. 4) (pp. 81-119). WISDOMe Monographs, University of Wyoming.

Hackenberg, A. J., & Lee, M. Y. (2015) Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196–243.

Hackenberg, A. J., Sevinc, S. (2022). Middle school students' construction of reciprocal reasoning with quantitative unknowns. *Journal of Mathematical Behavior, 65*.

Hunt, J. H., Martin, K., Khounmeuang, A., Silva, J., Patterson, B., & Welch-Ptak, J. (2020). Design, development, and initial testing of asset-based intervention grounded in trajectories of student fraction learning. *Learning Disability Quarterly*, 0731948720963589.

Liss, D. R. (2019). The development of distributive partitioning operations. *Journal of Mathematical Behavior, 56*.

Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: the FAIR framework for anti-deficit noticing. *ZDM–Mathematics Education*, *53*(1), 95-107.

Maloney, A., & Confrey, J. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory.

Meyers, M., Sztajn, P., Wilson, P.H., & Edgington, C. (2015). From implicit to explicit: Articulating equitable learning trajectories based instruction. *Journal of Urban Mathematics Education, 8*(2), 11-22.

Nasir, N. I. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in the mathematics classroom and beyond. *Review of research in education*, *32*(1), 187-240.

National Research Council. (2009). *Mathematics in early childhood: learning paths toward excellence and equity*. National Academy Press.

Peck, F. (2021). Towards anti-deficit education in undergraduate mathematics education: How deficit perspectives work to structure inequality and what can be done about it. *PRIMUS*, *31*(9), 940-961.

Piaget, J. (1955). *The language and thought of the child*. Meridian Books.

Piaget, J. (1970). *Genetic epistemology*. Columbia University Press.

Piaget, J., & Inhelder, B. (1967). *The child's conception of space*. Norton.

Rahmi, F., Sampoerno, P. D., & Ambarwati, L. (2020, February). Probability learning trajectory: Students' emerging relational understanding of probability through ratio. In *Journal of Physics: Conference Series* (Vol. 1470, No. 1, p. 012067). IOP Publishing.

Sarama, J. (2018). Perspectives on the Nature of Mathematics and Research. *North American Chapter of the International Group for the Psychology of Mathematics Education*.

Sarama, J., Clements, D. H., Barrett, J., Van Dine, D. W., & McDonel, J. S. (2011). Evaluation of a learning trajectory for length in the early years. *ZDM*, *43*(5), 667-680.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145.

Simon, M. A., Saldanha, L., McClintock, E., Karagoz Akar, G., Watanabe, T., & Ozgur Zembat, I. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction*, *28*, 70-112.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, 6(2), 129–162

Steffe, L. P. (2012). On the construction of learning trajectories of children: The case of commensurate fractions. In *Hypothetical Learning Trajectories* (pp. 129-162). Routledge.

Steffe, L. P. (2012). On the construction of learning trajectories of children: The case of commensurate fractions. In *Hypothetical Learning Trajectories* (pp. 129-162). Routledge.

Steffe, L. P. (2017). Psychology in Mathematics Education: Past, Present, and Future. North American Chapter of the International Group for the Psychology of Mathematics Education.

Steffe, L. & Olive, J. (2010). *Children's fractional knowledge*. Springer.

Steffe, L. P., & Thompson, P. W. (2000). Interaction or intersubjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*(2), 191–209.

Steffe, L. P., & Ulrich, C. (2020). Constructivist teaching experiment. In *Encyclopedia of mathematics education* (pp. 134-141). Cham: Springer International Publishing.

Suh, J.M., Birkhead, S., Frank, T., Baker, C., Galanti, T., & Seshaiyer, P. (2021). Developing an asset-based view of students' mathematical competencies through learning trajectory-based lesson study. *Mathematics Teacher Educator, 9*(3), 229-245.

Taguma, M., & Barrera, M. (2019). OECD future of education and skills 2030: Curriculum analysis. *Disponibile su: https://www. oecd. org/education/2030-project/teaching-and-learning/learning/skills/Skills_for_2030. pdf*.

Tasova, H., Ellis, A.B., Hamilton, M., Moore, K., Wawsa, A., Çelik, A., & Ying, Y. (2021). A serendipitous mistake: How one teacher's beliefs and knowledge mediated her in-the-moment instruction. In Olanoff, D., Johnson, K., & Spitzer, S. (Eds.), *Proceedings of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1574-1579), Philadelphia, PA.

Teuscher, D., Moore, K. C., & Carlson, M. P. (2016). Decentering: A construct to analyze and explain teacher actions as they relate to student thinking. *Journal of Mathematics Teacher Education*, *19*(5), 433-456.

Thompson, P. W. (2000). Radical constructivism: Reflections and directions. In L. P. Steffe & P. W. Thompson (Eds.), *Radical constructivism in action: Building on the pioneering work of Ernst von Glaserfeld* (pp. 412–448). London: Falmer Press.

Thompson, P. W., & Saldanha, L. A. (2000). Epistemological analysis as a research methodology. In M. Fernandez (Ed.), *Proceedings of the annual meeting of the International Group for the Psychology of Mathematics Education - North America*, (pp. 313-319). Tucson, AZ: University of Arizona.

Von Glasersfeld, E. (1982). An interpretation of Piaget's constructivism. *Revue Internationale de Philosophie, 36*(4), 612 – 635.

Wijaya, A., & Doorman, M. (2021). A Learning Trajectory for Probability: A Case of Game-Based Learning. *Journal on Mathematics Education*, *12*(1), 1-16.

Wright, V. (2014). Towards a hypothetical learning trajectory for rational number. *Mathematics education research journal*, *26*(3), 635-657.

Zahner, W., & Wynn, L. (2021). Rethinking Learning Trajectories in Light of Student Linguistic Diversity. *Mathematical Thinking and Learning*, 1-15.

THE ROLE OF LEARNING PROGRESSIONS IN "DEMOCRATIZING" STUDENTS' ACCESS TO ALGEBRA

Maria Blanton
TERC
Maria_Blanton@terc.edu

Learning progressions have become an important construct in educational research, in part because of their ability to inform the design of coherent standards, curricula, assessments, and instruction. In this paper, I discuss how a learning progressions approach has guided our development of an early algebra innovation for the elementary grades and provide examples of how this approach can help challenge a settled mathematics learning status quo about the kind of algebra students can learn, when they can learn it, and how all students can be successful. Empirically derived learning progressions are an important part of designing early algebra innovations that can open new curricular pathways for teaching and learning algebra, creating accessible and effective avenues of learning for all students.

Keywords: Algebra and algebraic thinking, learning trajectories and progressions, elementary school education

Using a Learning Progressions Approach to Develop an Early Algebra Innovation

Over a decade ago, my research interests turned towards a question that I view as critically important in teaching and learning algebra: Does early algebra matter? Since I assume early algebra *does* matter, perhaps a better way to frame this question is to what extent does early algebra matter, in what ways does it matter, and how might we capture or measure this? There are deep implications for the answers to these questions. A truly effective integration of early algebra (or, algebraic thinking in the elementary grades) would entail significant costs because it requires "deep curriculum restructuring, changes in classroom practice and assessment, and changes in teacher education—each a major task" (Kaput, 2008, p. 6). Such costs highlight the need for carefully constructed models of early algebra instruction—models that have been missing from elementary grades mathematics. However, these models, as curricular roadmaps for developing children's algebraic thinking across elementary grades in a deep, systemic way, are essential to understanding early algebra's impact.

To build such a model, our research team turned to a *learning progressions approach*. Learning progressions have become an important construct in educational research (Clements & Sarama, 2004, 2009; Confrey et al., 2014; Simon, 1995; Stevens et al., 2009), in part because of their ability to inform the design of coherent standards, curricula, assessments, and instruction (Daro et al., 2011). We focused our work on the development of several core components aligned with this approach (e.g., Clements & Sarama, 2004; see also Fonger et al., 2018): (1) empirically-derived learning goals around algebraic thinking in elementary grades; (2) grade-level instructional sequences designed to address these learning goals; (3) validated assessments to measure students' understanding of core algebraic concepts and practices as they advance through the instructional sequences; and (4) progressions that specify increasingly sophisticated levels of thinking students exhibit about algebraic concepts and practices in response to an instructional sequence. A learning progressions approach served two purposes in our work. It provided an over-arching "large-grain-size-level" framework to guide our design of a Grades K–5 early algebra intervention from which we might measure the impact of early algebra on children's algebra readiness for middle grades. It also provided a theoretical mechanism for identifying "small-grain-size-level" cognitive foundations in children's algebraic thinking that, along with other existing research in the field, could inform the development of our intervention.

At a large-grain-size level, we used a learning progressions approach (e.g., Shin et al., 2009, Stevens et al., 2009) in the development of learning goals, grade-level instructional sequences, and grade-level assessments for K–5. Using Kaput's (2008) content analysis of algebra as a set of *key aspects* (ways of thinking algebraically) and *core strands* (content domains where algebraic thinking occurs), we identified core algebraic thinking practices of *generalizing*, *representing*, *justifying*, and *reasoning with* mathematical structure and relationships as an underlying conceptual framework for the design of our goals, sequences, and assessments (Blanton, Brizuela et al., 2018). We viewed a conceptual framework organized around algebraic thinking *practices* as critical to avoid designing instructional sequences based simply on the use of ubiquitous "algebra" tasks (e.g., solving equations) and not cohesively grounded in what it means to think algebraically.

To develop our learning goals, we analyzed the treatment of the core algebraic thinking practices in several dimensions: (1) empirical research on Grades K–8 students' algebraic thinking; (2) national curricular frameworks and standards such as the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000) and *Common Core State Standards* (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010); (3) Grades K–8 curricular materials (e.g., *Everyday Mathematics*, *Singapore Math, Investigations)*; and (4) formal algebra content at both secondary and postsecondary levels. We then organized early algebra content to align with strands in Kaput's (2008) algebra content analysis, in particular, "the study of structures and systems abstracted from computations and relations" (Strand 1) and the "study of functions, relations, and joint variation" (Strand 2) (p. 11). These strands also aligned with core content around which the early algebra research base had coalesced. Based on this, we structured our findings on early algebra content within several "Big Ideas" (e.g., Stevens et al., 2009), or content domains, where core algebraic thinking practices can occur: Generalized Arithmetic; Equivalence, Expressions, Equations, and Inequalities; and Functional Thinking. We then unpacked the role of algebraic thinking practices within each Big Idea by delineating core algebraic *concepts* related to the practices, *claims* that specify the nature of skills or understandings expected of students regarding a specific concept, *evidence* in students' work that would indicate they had developed the skills or understandings specified in our claims, and research-based *difficulties and misconceptions* that students have with a concept (Shin et al., 2009). The concepts, claims, evidence, and difficulties or misconceptions associated with the algebraic thinking practices within each Big Idea informed the development of grade-specific learning goals.

Using our learning goals, we then constructed grade-level instructional sequences for each of Grades K–5 by designing specific task structures that connected the algebraic practices with related claims about the skills or understandings that might reasonably be expected of students. For example, for Generalized Arithmetic we designed sequences of tasks that created opportunities to *generalize* arithmetic relationships, to *represent* these relationships in different ways, to develop appropriate general arguments for *justifying* the arithmetic relationships students observed, and to use arithmetic generalizations students developed as objects (Sfard, 1991) for *reasoning with* novel problems or properties of arithmetic that make computational work more efficient. Grade-level learning goals were used to guide content for these task structures. For example, the types of arithmetic relationships addressed at a particular grade, the types of representations used to express them, or the nature of arguments students might develop to show relationships were valid, were guided by our learning goals. We then used design research (Cobb et al., 2003) to field-test and refine our proposed grade-level instructional sequences. Finally, we used these task structures to develop validated, grade-level assessments by which we could measure within-grade and across-grade (longitudinal) growth in children's algebraic thinking in order to understand early algebra's impact.

Our effort to design a broader, multi-year (Grades K–5) approach to developing students' early algebraic thinking—essentially, components (1)–(3) above—aligns with what is sometimes characterized as "learning progressions" (Stevens et al., 2009). At the same time, a smaller grain size approach was needed to fill in "gaps" in empirical research on our understanding of children's algebraic thinking (essentially, component (4) above), particularly in the early elementary grades where the research base was less developed than that for Grades 3–5. Because of its narrow scale in terms of a focus on specific algebraic concepts or practices within short instructional timelines (e.g., weeks), this second aspect of our work might be seen as more akin to learning trajectories (e.g., Stevens et al., 2009). This "small-grain-size-level" research was critical for identifying increasingly sophisticated levels in students' thinking about a particular practice or concept within a Big Idea. For example, we identified trajectories in students' thinking about concepts such as a relational understanding of the equal sign (Blanton, Otalora Sevilla et al., 2018) and variable and variable notation (Blanton et al., 2017), as well as for practices such as generalizing functional relationships (Blanton, Brizuela et al., 2015; Stephens et al., 2017), generalizing arithmetic relationships (Ventura et al., 2021), and justifying claims about arithmetic relationships (Blanton et al., 2021).

Regardless of the nomenclature used or whether focusing on "big" ideas over a broad span of time (e.g., multiple years) or "small" concepts in a narrow span of time (e.g., weeks), a learning progressions approach has provided a flexible theoretical paradigm with key features aligned with our core research goals: identifying increasingly sophisticated ways students come to think about an algebraic concept or practice in response to an instructional sequence (Duschl et al., 2007; Simon, 1995; Smith et al., 2006); attending to specific content domains rather than general cognitive structures in how we design our instructional sequences to study early algebra's impact (Baroody et al., 2004); organizing content within these domains to facilitate the development in students' understanding of algebraic concepts and practices over time (Smith, et al., 2006); and relying on classroom-based empirical research, rather than just a logical analysis of the discipline, to understand how students' early algebraic thinking develops (Stevens et al., 2009).

The value of this approach extends beyond its role as a research paradigm, however. One of the organizing questions of the PME-NA 2022 conference—*How does your work challenge a settled mathematics learning status quo?*—is at the heart of early algebra's goal of "democratizing" students' access to algebra. By democratizing access to algebra, we mean opening up pathways to students for whom the traditional "arithmetic-then-algebra" approach—teaching arithmetic in elementary grades, followed by formal algebra in secondary grades—has been unsuccessful (e.g., Hiebert et al., 2005) and has limited students' access to STEM career and workforce opportunities (e.g., LaCampagne, 1995; NCTM, 2000). These challenges have been particularly felt among students from historically underserved communities (e.g., Moses & Cobb, 2001; Museus et al., 2011). For example, elementary grades students from lower socioeconomic (SES) backgrounds are two times more likely to be deficient in mathematics than students from higher SES backgrounds (U.S. Dept. of Education, NCES, 2007). This makes such students especially vulnerable in later formal algebra courses, which impacts their chance for success in college (U.S Dept. of Education, 2008) and access to STEM-related disciplines and careers. The promise of early algebra, then, is to address existing inequities in school mathematics and broaden students' access to STEM disciplines. In what follows, I consider a few examples of how our work, built around learning progressions, has helped challenge the status quo of settled mathematics learning around the kind of algebra students can learn, when they can learn it, and how all students can be successful.

Representing Generalizations Using Variable Notation

The traditional "arithmetic-then-algebra" approach to teaching and learning algebra has entailed certain views about what kind of algebra content should be taught and when. One aspect of algebra that has historically been largely outside the purview of elementary grades is variable and variable notation. Our work takes the view that variable notation is a useful tool that children can begin to understand and use from early elementary grades. From this perspective, we have sought to understand trajectories in students' thinking about variable and the use of variable notation to represent arithmetic and functional relationships. While the act of symbolizing a generalization is central to algebraic thinking, the way in which a generalization is represented can vary. In elementary grades, non-conventional forms such as natural language and drawings—symbol systems whose meanings are already available to young children—have historically been prioritized as a more productive way to represent generalizations (Resnick, 1982).

Part of the hesitation for the use of variable notation with young children has likely been due to strict interpretations around Piaget's formal stages of development, along with the concern that premature formalisms (Piaget, 1964) might lead to meaningless actions on symbols (Blanton et al., 2017). It is reasonable to assume that the well-documented challenges adolescents have with the concept of variable and the use of variable notation (e.g., Knuth et al., 2011; Küchemann, 1981) would be even more prominent among younger, elementary grades children. Yet, unlike younger children, adolescents are expected to build a mathematical understanding of literal symbols to notate variable quantities *after* they have deeply developed ways of thinking about letters in linguistic contexts (e.g., Braddon et al., 1993). This suggests that difficulties with variable notation may be more related to conflicts generated by students' use of literal symbols in mathematical contexts that rely on the understandings they already have about literal symbols in non-mathematical contexts (McNeil et al., 2010).

Research, however, increasingly supports that variable notation can be a valuable tool for young learners (e.g., Blanton, Stephens et al., 2015; Brizuela et al., 2015; Carpenter et al., 2003; Cooper & Warren, 2011; Dougherty, 2008; Fuji & Stephens, 2008). Learning progressions have helped us better understand why this might be the case. In a recent study (Blanton et al., 2017), we explored a trajectory in first graders' understanding of variable and use of variable notation to represent functional relationships. The conceptual space of interest here is the level at which children do not yet understand the concept of variable quantity nor how to use variable notation to represent a variable quantity, a level we characterize as *pre-variable/pre-symbolic*. The close analysis involved in mapping out a learning trajectory helped surface a more nuanced view of students' understandings about variable and variable notation. In particular, we observed that students' whose thinking was "pre-variable" did not yet *perceive* a variable quantity in a mathematical situation. As such, they naturally searched for other tools and ways of understanding within their conceptual field to make sense of a situation involving an unknown.

How might this manifest in students' mathematical actions? When young learners at a pre-variable/pre-symbolic level of thinking encounter a situation with a variable quantity, they typically assign a numerical value to the quantity, either randomly or based on some numerical feature of the situation. They might also propose (hypothetically) that the quantity be measured or counted to determine a specific value, even though it cannot be. This is not an unreasonable approach, given that students' mathematical experiences are, typically, fully centered on arithmetic at this point, where quantities are known or can be counted or measured and represented by a numerical value. Moreover, a child whose thinking is pre-variable (that is, the child cannot imagine or does not "see" an unknown within a situation) would not reasonably be expected to *look for* literal symbols (or even non-conventional representations such as natural language) to symbolize a quantity. Instead, we would expect them to use the tools and ways of understanding already available to them, which are arithmetic in nature.

As students progress in their thinking, some pick up the use of algebraic notation before they can perceive variable quantities. Such cases are indicated by the use of literal symbols as labels or to represent objects rather than quantities. That is, students recognize that a literal symbol can be used to recognize *something*, but not a variable quantity, since this is outside of their conceptual field. However, once variable and variable notation co-emerge in children's thinking, children begin to use variable notation in meaningful ways to symbolize variable quantities (Blanton et al., 2017). Through our construction of a learning trajectory around variable and variable notation, we came to see that the challenge was not that young learners cannot understand (and, thus, should not be exposed to) variable notation. It is, rather, that they have learned to interpret mathematical situations through an arithmetic lens which leads to certain ways of (arithmetic) problem solving that are inadequate for situations involving variable quantities. Instead, if they first learn to perceive a variable quantity, this motivates the need for a symbol system—whether conventional or not—to represent the quantity. Once they perceive a variable quantity, the symbolic system—including the use of literal symbols—can be meaningfully used to represent unknowns.

Our findings elsewhere support that children can learn to use variable notation in meaningful ways. For example, in a large-scale, randomized (CRT) study in 46 elementary schools on the effectiveness of our Grades 3–5 instructional sequences (i.e., early algebra intervention), we found significant differences in how treatment students, who were taught the intervention as part of their regular curriculum, were able to represent functional relationships with variable notation in comparison to control students, who were taught only their regular curriculum. Figure 1 compares treatment and control students' use of variable notation and natural language (words) to represent a functional relationship they observed using data they had culled from a problem situation. Not only were treatment students significantly better able to represent a function with variable notation (as well as with words) than control students, treatment students were also significantly better able to use variable notation than words. (Even students in control schools were better able to use variable notation than words, although the differences were not significant).

The point here is that a learning progressions approach has enabled us to better understand potential challenges to young learners' understanding of variable and variable notation and to design instruction that can significantly improve their understanding, both of which call into question the long-held view that young learners should focus on representational tools that are already available to them (e.g., natural language, diagrams). While we strongly support young learner's use of representational systems such as natural language, we equally support the introduction of variable and variable notation in appropriate ways to young learners and view learning progressions research as a means to help shift perceptions that variable and variable notation is beyond the grasp of young children.

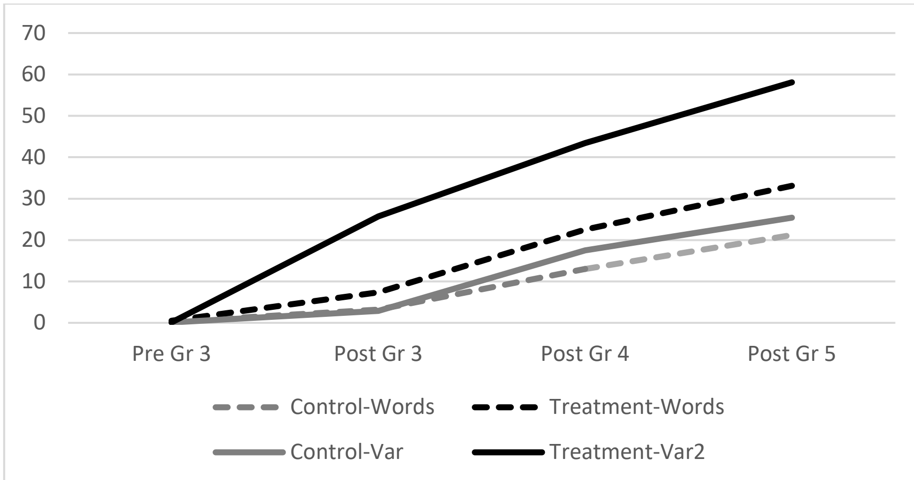


Figure 1. Comparison of treatment and control students use of variable notation to represent a functional relationship.

Developing Mathematical Arguments

Justifying claims about mathematical relationships is central to early algebraic thinking, yet the role of justifying or proving in school mathematics has historically been limited, particularly in elementary grades (Ball et al., 2002; Stylianides, 2016). However, studies suggest that deductive reasoning emerges in the elementary grades (Falmagne, 1980) and that with appropriate instruction, students can learn to use deductive—rather than empirical—reasoning to develop mathematical arguments (Stylianides & Stylianides, 2008). This challenges the long-held view that children's lack of ability to reason deductively is due to a developmental constraint and, instead, points to limited classroom opportunities as the more likely cause for their challenges with building good, grade-appropriate mathematical arguments (Stylianides, 2016; see also, e.g., Ball & Bass, 2003; Carpenter et al., 2003; Lampert, 1992; Maher & Martino, 1996). Studies suggest that the lack of argumentation in elementary grades has far-reaching implications in that it detaches students from sense-making (Staples et al., 2012) and can promote difficulties with proof and proving in high school (e.g., Coe & Ruthven, 1994; Knuth et al., 2002; Stylianides & Stylianides, 2008).

Stylianides and Stylianides (2008) and Stylianides (2007) suggest that research that details trajectories in children's thinking in response to instructional sequences focused on argumentation can help us understand how young children come to reason deductively in response to specific instructional conditions. Because justifying mathematical relationships is a core practice in our conceptual framework, we are interested in how students' come to build strong mathematical arguments in response to instruction. We recently conducted a study in which our goal was to identify progressions in Grades K–1 children's understanding of parity arguments and underlying concepts (e.g., even and odd numbers) in response to our Grade K and Grade 1 instructional sequences (Blanton et al., 2021). Our particular focus was on how children's understanding of representation-based proofs (Schifter, 2009)—versus empirical arguments—developed from the start of formal schooling, before their introduction to any parity concepts.

While our sample was small due to the design nature of our work, our findings were similar to those of other researchers who have conducted extensive work in this area (e.g., Schifter et al., 2009; Stylianides, 2007; Van Ness & Maher, 2019). For example, even in kindergarten, we found that students were able to construct informal structural parity arguments that did not rely on the use of specific or even generic numbers. What we found even more surprising was how rare empirical arguments were in these early grades, even though such arguments are a predominant proof strategy in secondary grades (Coe & Ruthven, 1994; Staples et al., 2012). For example, kindergarten students were unfamiliar with the concepts of pair and even and odd numbers prior to our instructional sequence in Grade K, but by Grade 1 pre-test (i.e., after the Grade K sequence), many students routinely used a pairs strategy (where numbers that can be represented as pairs of cubes are even and those that cannot be represented in this way are odd) to reason about the parity of numbers represented in concrete, visual, and abstract forms. More importantly, out of a group of 10 students interviewed, no student used an empirical argument to justify why the sum of an even and an odd would be odd. Six out of ten were able to correctly use a structural argument involving a pairs strategy (three students could not build either type of argument; one student was not asked this question).

In developing trajectories in children's thinking about particular algebraic concepts or practices, we have found that introducing algebraic thinking earlier—even as early as kindergarten—can help mitigate the development of misconceptions in students' thinking that can occur within an arithmetic-focused approach to instruction. We have observed this in students' understanding of variable quantity and variable notation, where engaging students in mathematical tasks that first help them perceive variable quantities can support their use of variable notation in meaningful ways. We have seen that the early introduction of representation-based arguments can help offset entrenched forms of empirical reasoning by providing students with an accessible, grade appropriate process for justifying their claims. We have seen early attention to functional relationships between quantities mitigate an ingrained focus on recursive patterns in function data that makes it difficult in later elementary grades to re-focus students' thinking on co-varying quantities. We have seen concrete and visual tools help students begin to think relationally about the equal sign even before symbols and equations are introduced, thereby disrupting the operational thinking that is often fostered through arithmetic work focused on standard forms of representations (Blanton, Otalora Sevilla et al., 2018). In this way, learning progressions in early algebra research has helped challenge historically settled notions (even some of our own) regarding what algebraic concepts should be addressed and when. But early algebra's mission to democratize access to algebra is broader than "what" or "when." It is ultimately about "who." In what follows, I briefly touch on some of our evidence around this.

Challenging Perceptions of *Who* Can Do Algebra

The underlying premise of early algebra is that developing children's informal notions about mathematical structure and relationships, beginning in kindergarten, will better prepare them for success in formal algebra in later grades. As described earlier, our work has focused on the use of a learning progressions approach to build the tools with which we could explore this premise, and we are starting to see the contours of an answer to the question of whether early algebra "matters." Our recent 3-year randomized (CRT) study on the effectiveness of the Grades 3–5 early algebra "intervention" (i.e., grade-level instructional sequences) offers some evidence of this. The study was conducted in 46 schools across urban, rural, and suburban settings, where the intervention was taught by classroom teachers as part of their regular mathematics instruction. We found overall that, at each of Grades 3–5, students who received the intervention as part of regular instruction significantly outperformed their peers who received only regular instruction in both their knowledge of algebraic concepts and practices and their use of algebraic strategies to solve tasks (Blanton et al., 2019). Moreover, treatment students maintained a significant advantage over control students in middle grades, one year after the intervention ended (Stephens et al., 2021).

Algebraic thinking in the elementary grades is now codified as essential to algebra education in frameworks such as the *Common Core State Standards for Mathematics* (NGA Center & CCSSO, 2010). Through the adoption of these standards, many states have elevated the role of algebra, leaving students potentially vulnerable to a persistent marginalization in school. With students from underserved communities already underrepresented in STEM professions (Oscos-Sanchez et al., 2008), the long-term implications for the inequity of access to educational on-ramps for students around early algebra are significant. As such, early algebra innovations should be designed to ensure that they are inclusive of learners across diverse classrooms. So, while our overall results are promising, it was (and continues to be) important for us to further unpack our findings for students in different demographics and learning conditions. One example of this was our comparison in the performance of a subset of participating treatment and control schools for which the majority of students were from underserved communities (i.e., 100% low SES, 94% underserved racial minorities). We found that, like our overall population, treatment students again significantly outperformed control students at each of Grades 3–5 on both their knowledge of algebraic concepts and practices and their use of algebraic strategies to solve tasks (Blanton et al., 2019). To visualize the significance of this finding, Figure 2 compares the performance of students in treatment and control schools for this subset of schools along with the performance of the overall control group. In addition, it also shows that, while treatment students from schools with underserved communities did underperform the overall control group prior to the intervention, by the end of Grade 3 they outperformed even the *overall* control group (although not significantly) and maintained this advantage throughout the end of the intervention in Grade 5.

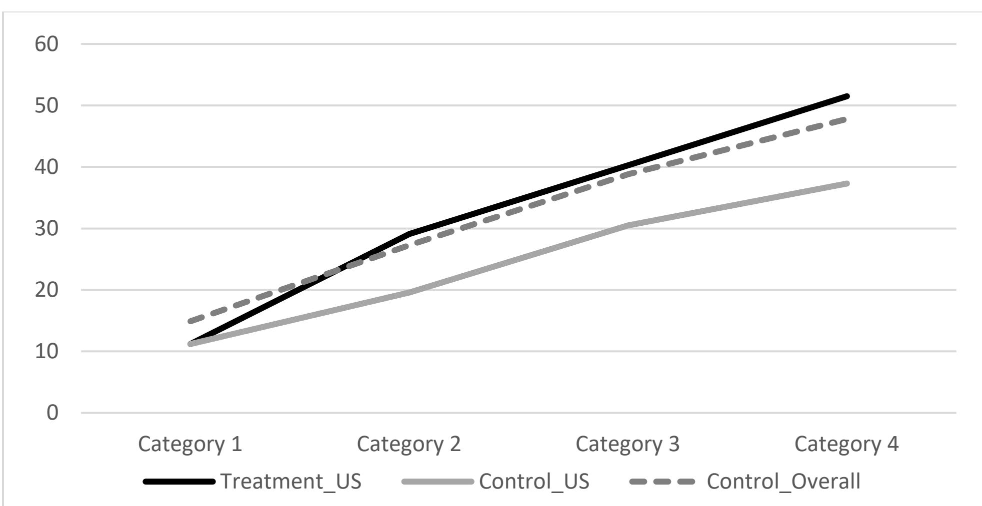


Figure 2. Comparison of performance for underserved students (US) in treatment and control schools with control students overall.

These promising results support findings from other studies that show providing students from historically underserved communities with more challenging learning environments in elementary grades can increase mathematics proficiency (Lee, 2011; Zilanawala et al., 2018). While it's difficult to tease out whether the role of a learning progressions approach was sufficient for these results, we can say that *when* a learning progressions approach was used, we have found a significant improvement in *all* learners' ability to think algebraically.

Building on this, we are continuing to explore how to develop a more inclusive early algebra intervention in Grades K–2. We expanded our recent design work around instructional sequences and assessments in Grades K–2 to include a more intentional focus on students with learning differences in order to understand how these students make sense of the algebraic concepts and practices in our intervention and how its features (e.g., concrete and visual tools) support learning. We have found that tools such as balance scales helped students analyze equations and reconsider unfamiliar equation forms (Stephens, Sung, Strachota et al., 2022), even prior to developing strong computational skills. Further, we have documented how these tools mediated the abilities of diverse learners to generalize and represent what they notice about structure and relationships involving parity concepts (Strachota et al., 2021)

While this work is ongoing, our Grades K–2 instructional sequences already show promise overall in developing children's algebraic thinking. We recently conducted a small, one-year cross-sectional study ($n = 80$) in each of Grades K–2 that examined the potential of each grade-level sequence when taught by classroom teachers. After only a one-year intervention in each grade, we found a marginally significant interaction between treatment condition and performance that showed gains favoring treatment students in their understanding of early algebra concepts such as the structure of evens and odds, mathematical equivalence and equations, properties of arithmetic, the representation of varying unknown quantities, and functional thinking [$F\ (1, 58) = 3.794$, p=.056] (Stephens, Sung, Blanton et al., 2022). Further, we found no significant three-way interaction among treatment condition, performance on the assessment, and grade level, suggesting that the impact of the intervention was similar across grade levels. With these findings for Grades K–2 and our more fully developed findings about the effectiveness of our Grades 3–5 intervention, we are optimistic about the innovation's potential to positively impact all students' experiences in algebra.

Conclusion

A learning progressions approach has been central in our efforts to develop an empirically grounded model for teaching and learning algebra that can increase all students' opportunities for success in algebra. It has provided both a framework for developing an innovation to measure early algebra's impact and a mechanism to examine fine-grained details in how children's algebraic thinking develops. It has also helped us think about how to reframe learning around an asset-based perspective in which instruction can build on the rich ways students think about algebraic concepts and practices, rather than from a deficit model built around students' misconceptions. Designing innovations from a perspective of what students *can* do (or, as Jim Kaput used to say, the "happy stories") can minimize the need to design for the purpose of "undoing" misconceptions in students' thinking that arise when elementary grades instruction does not attend to algebraic concepts and practices. Early algebra innovations based on learning progressions can open new curricular pathways for teachers and create effective avenues of learning that can democratize all students' access to algebra. We are hopeful that such models can continue to challenge the national discourse on teaching and learning algebra around the kind of algebra students can learn, when they can learn it, and how all students can be successful.

Acknowledgment

The research reported here was supported by the US Department of Education under IES Award # R305A170378 and the National Science Foundation under Award # 1720129. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the US Department of Education or the National Science Foundation.

References

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). National Council of Teachers of Mathematics.

Ball, D. L., Hoyles, C., Jahnke, H. N., & Movshovitz-Hadar, N. (2002). The teaching of proof. In L. I. Tatsien (Ed.), *Proceedings of the International Congress of Mathematicians* (Vol. III, pp. 907–920). Higher Education Press.

Baroody, A. J., Cibulskis, M., Lai, M.-L., Li, X. (2004). Comments on the use of learning trajectories in curriculum development and research. *Mathematical Thinking and Learning, 6*(2), 227–260. https://doi:10.1207/s15327833mtl0602_8

Blanton, M., Brizuela, B., Gardiner, A., & Sawrey, K. (2017). A progression in first-grade children's thinking about variable and variable notation in functional relationships. *Educational Studies in Mathematics 95*(2), 181–202, https://doi 10.1007/s10649-016-9745-0.

Blanton, M., Brizuela, B., Gardiner, A., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year-olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education, 46*(5), 511–558.

Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A., Stroud, R., Fonger, N., & Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice* (pp. 27–49). Springer.

Blanton, M., Gardiner, A., Ristroph, I., Stephens, A., Stroud, R., & Knuth, E. (2021). Progressions in young learners' understandings of parity arguments. *Mathematical Thinking and Learning*. https://doi.org/10.1080/10986065.2022.2053775

Blanton, M., Otalora Sevilla, Y., Brizuela, B., Gardiner, A., Sawrey, K., & Gibbons, A. (2018). Exploring kindergarten students' early understandings of the equal sign. *Mathematical Thinking and Learning, 20*(3), 167–201, https://doi: 10.1080/10986065.2018.1474534

Blanton, M., Stephens, A., Knuth, E., Gardiner, A., Isler, I., & Kim, J. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39-87. http://dx.doi.org/10.5951/jresematheduc.46.1.0039

Blanton, M., Stroud, R., Stephens, A., Gardiner, A., Stylianou, D., Knuth, E., Isler, I., & Strachota, S. (2019). Does early algebra matter? The effectiveness of an early algebra intervention in grades 3–5. *American Educational Research Journal*, *56*(5), 1930–1972.

Braddon, K. L., Hall, N. J., & Taylor, D. (1993). *Math through children's literature: Making the NCTM standards come alive.* Teacher Ideas Press.

Brizuela, B. M., Blanton, M., Sawrey, K., Newman-Owens, A., & Gardiner, A. (2015). Children's use of variables and variable notation to represent their algebraic ideas. *Mathematical Thinking and Learning, 17*, 1–30. http://dx.doi.org/10.1080/10986065.2015.981939

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Heinemann.

Clements, D. H. & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89. http://doi:10.1207/s15327833mtl0602_1

Clements, D., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Routledge.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Coe, R., & Ruthven, K. (1994). Proof practices and constructs of advanced mathematics students. *Educational Research Journal, 20*, 41–53.

Confrey, J., Maloney, A., & Nguyen, K. (2014). Introduction: Learning trajectories in mathematics. In A. Maloney, J. Confrey, & K. Nguyen (Eds.), *Learning over time: Learning trajectories in mathematics education* (pp. *xi–xxii*). Charlotte, NC: Information Age.

Cooper, T. J., & Warren, E. (2011). Years 2 to 6 students' ability to generalise: Models, representations and theory for teaching and learning. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 187–214). Springer.

Daro, P., Mosher, F. A., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction* (Research Report No. 68). Consortium for Policy Research in Education.

Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). Lawrence Erlbaum Associates.

Duschl, R. A., Schweingruber, H. A., & Shouse, A. W. (Eds.). (2007). *Taking science to school: Learning and teaching science in grades K–8.* National Academies Press.

Falmagne, R. J. (1980). The development of logical competence: A psycholinguistic perspective. In R. Kluwe & H. Spada (Eds.), *Developmental models of thinking* (pp. 171–197). Academic Press.

Fonger, N. L., Stephens, A., Blanton, M., Isler, I., Knuth, E., & Gardiner, A. (2018). Developing a learning progression for curriculum, instruction, and student learning: An example from mathematics education, *Cognition and Instruction,* 36:1, 30–55, doi: 10.1080/07370008.2017.1392965

Fuji, T., & Stephens, M. (2008). Using number sentences to introduce the idea of variable. In C. E. Greenes & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics: Seventieth yearbook* (pp. 127-140). National Council of Teachers of Mathematics.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., Hollingsworth, H., Manaster, A., Wearne, D., & Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 Video Study. *Educational Evaluation and Policy Analysis, 27(2),* 111–132.

Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Lawrence Erlbaum/Taylor & Francis Group & National Council of Teachers of Mathematics.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2011). Middle school students' understanding of core algebraic concepts: Equivalence and variable. In J. Cai & E. Knuth (Eds.), Early algebraization: A global dialogue from multiple perspectives. *Advances in Mathematics Education Monograph Series* (pp. 259-276). Springer. http://dx.doi.org/10.1007/978-3-642-17735-4_15

Knuth, E. J., Choppin, J., Slaughter, M. & Sutherland, J. (2002). Mapping the conceptual terrain of middle school students' competencies in justifying and proving. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Weigel, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 1693 – 1670). Clearinghouse for Science, Mathematics, and Environmental Education.

Küchemann, D. E. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (pp. 102-119). Murray.

LaCampagne, C. B. (1995). (Ed.). *The Algebra Initiative Colloquium, Vol. 2, Working Group Papers.* U.S. Department of Education, Office of Educational Research and Improvement.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. K. Oser, A. Dick, & J. Patry (Eds.), *Effective and responsible teaching: The new synthesis* (pp. 295–314). Jossey-Bass Publishers.

Lee, O. (2011, May). Effective STEM strategies for diverse and underserved learners. Paper presented at the National Research Council's Workshop on Successful STEM Education in K–12 Schools, Washington, DC.

Maher, C. A., & Martino, A. M. (1996). The development of the idea of a mathematical proof: A 5-year case study. *Journal for Research in Mathematics Education, 27,* 194–214.

McNeil, N., Weinberg, A., Hattikudar, S., Stephens, A.C., Asquith, P., Knuth, E., & Alibali, M. (2010). A is for apple: Mnemonic symbols hinder the interpretation of algebraic expressions, *Journal of Educational Psychology 102*(3), 625-634. http://dx.doi.org/10.1037/a0019105

Moses, R. P., & Cobb, C. E. (2001). *Radical equations: Math literacy and civil rights*. Beacon Press.

Museus, S., Palmer, R. T., Davis, R. J., & Maramba, D. C. (2011). *Racial and ethnic minority students' success in STEM education*. Jossey-Bass.

National Council of Teachers of Mathematics (2000). *Principle and standards for school mathematics*. Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010).

Common core state standards for mathematics. Council of Chief State School Officers. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf
Oscos-Sanchez, M.A., Oscos-Flores, L.D., Burge, S.K. (2008). The Teen Medical Academy: Using academic enhancement and instructional enrichment to address ethnic disparities in the American healthcare workforce. *Journal of Adolescent Health 42*(3), 284–293.
Piaget, J. (1964). *Developing and learning: Conference on cognitive studies and curriculum development.* Cornell University Press.
Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 136-155). Lawrence Erlbaum Associates.
Schifter, D. (2009). Representation-based proof in the elementary grades. In D. Stylianou, M. Blanton, & E. Knuth (Eds.), *Teaching and learning proof across grades: A K-16 Perspective* (pp. 71-86). Routledge.
Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. http://dx.doi.org/10.1007/BF00302715
Shin, N., Stevens, S. Y., Short, H., & Krajcik, J. S. (2009). *Learning progressions to support coherence curricula in instructional material, instruction, and assessment design.* Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA.
Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2)*,* 114–145. http://dx.doi.org/10.2307/749205
Smith, C. L., Wiser, M., Anderson, C. W., & Krajcik, J. (2006). Implications of research on children's learning for standards and assessment: A proposed learning progression for matter and the atomic-molecular theory. *Measurement, 14*(1&2), 1–98.
Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifaceted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior, 31*, 447 – 462.
Stephens, A. C., Fonger, N. L., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning, 19*(3), 143–166. https://doi.org/10.1080/10986065.2017.1328636
Stephens, A.C., Stroud, R., Strachota, S., Stylianou, D., Blanton, M., Knuth, E., & Gardiner, A. (2021). What early algebra knowledge persists one year after an elementary grades intervention? *Journal for Research in Mathematics Education*, *2*(3), 332–348.
Stephens, A., Sung, Y., Blanton, M., Gardiner, A., Stroud, R., & Knuth, E. (2022). *Kindergarten–grade 2 students' algebraic reasoning before and after a one-year early algebra intervention.* Conference proposal under review.
Stephens, A., Sung, Y., Strachota, S., Veltri Torres, R., Morton, K., Gardiner, A., Blanton, M., Knuth, E., & Stroud, R. (2022). The role of balance scales in supporting productive thinking about equations among diverse learners. *Mathematical Thinking and Learning, 24*(1), 1–18.
Stevens, S. Y., Shin, N., & Krajcik, J. S. (2009). Towards a model for the development of an empirically tested learning progression. Paper presented at the Learning Progressions in Science (LeaPS) Conference, Iowa City, IA. Retrieved from http://www.education.msu.edu/projects/leaps/proceedings/Stevens.pdf
Strachota, S., Morton, K., Veltri-Torres, R., Stephens, A., Sung, Y., Murphy Gardiner, A., Blanton, M., Stroud, R., & Knuth, E. (2021). Generalizing about even and odd numbers*.* Paper presented at the *14th Annual Conference of the International Congress on Mathematical Education*, Shanghai, China.
Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38,* 289–321.
Stylianides, A. J. (2016). *Proving in the elementary mathematics classroom*. Oxford University Press.
Stylianides, G. J., & Stylianides, A. J. (2008). Proof in school mathematics: Insights from psychological research into students' ability for deductive reasoning. *Mathematical Thinking and Learning, 10*(2), 103–133.
U.S. Department of Education. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf
U.S. Department of Education, National Center for Education Statistics. (2007). The condition of education 2007 (NCES 2007-064). U.S. Government Printing Office.
Van Ness, C. K., & Maher, C. A. (2019). Analysis of the argumentation of nine-year-olds engaged in discourse about comparing fraction models. *Journal of Mathematical Behavior, 53,* 13–41.
Ventura, A. C., Brizuela, B. M., Blanton, M., Sawrey, K., Gardiner, A., & Newman-Owens, A. (2021). A learning trajectory in kindergarten and first grade students' thinking of variable and use of variable notation to represent indeterminate quantities. *Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2021.100866.

Zilanawala, A., Martin, M., Noguera, P., & Mincy, R. (2018). Math achievement trajectories among black male students in the elementary and middle-school years. *Educational Studies 54,* 143–164. doi.org/10.1080/00131946.2017.1369414

INTENTIONALITY IN USING LEARNING TRAJECTORIES TO "REFRAME" TEACHER NOTICINGS TOWARDS ANTI-DEFICIT AND ASSET-BASED INSTRUCTION

Jennifer M. Suh
George Mason University
jsuh4@gmu.edu

Learning Trajectories have the potential to be used as a tool to advance equity by explicitly connecting to anti-deficit framing and asset-based instruction. This plenary paper highlights research on three use cases for learning trajectories (LT) with an intentionality around promoting equity: 1) the use of LT based Lesson Study with vertical teams of teachers to position students as capable and teachers as knowledgeable, 2) the use of LT coupled with anti-deficit framing in curriculum design research to provide students with access to rigorous educational resources and asset based instruction, 3) the use of LT with formative assessment to develop preservice teachers' equitable teaching practices to advance students understanding. The presenter invites the PMENA community to consider how learning trajectories can be coupled with powerful equity-focused research and frameworks to disrupt the status quo, broaden the notion of learning mathematics, eliminate labeling, and dismantle inequitable structures and hierarchy in the mathematics classroom.

Keywords: Learning Trajectories and Progressions, Anti-deficit Orientation, Asset-based Instruction, Teacher Noticing, Professional Development, Preservice Teacher Education

Introduction

In this paper, I discuss how learning trajectory (LT) research should attend to equity by providing access to rigorous educational resources, positioning students as capable and teachers as knowledgeable, and questioning the curriculum and high stakes assessment practices. I do this work by engaging and privileging the voices of teachers and coaches as co-designers and researchers in Lesson Study and Curriculum Design Research with the focus on rehumanizing mathematics for students (Gutiérrez, 2018).

I am a first generation Korean American mathematics education scholar, who attended my formative years of elementary schools in the 70s between two countries, experiencing Korean as a second language and also what was at the time called, English as a second language. I experienced first-hand differential learning experiences (NCTM, 2020; Jong et al., 2020) where some students were centered and others marginalized. Experiencing schooling in two countries, I also noticed the differential treatment of the teaching profession, one where it is a revered and noble profession and the other where the teacher's professional judgment is constantly questioned and viewed where anyone can teach. This has motivated me to focus my work on elevating the voices of teachers and the teaching profession in the US and building on students' mathematics strengths, particularly those who are marginalized in the mathematics classroom. My research is informed by a commitment to equity and culturally sustaining pedagogy in mathematics education. I work mostly with schools that are racially, culturally, and linguistically diverse and receive title 1 funding. My research has focused on LT use in Lesson Study and Community-based Math Modeling to connect mathematics to students' lived experiences, attending to both cognitive and socio-cultural perspectives. I lean on the work of Aguirre et al.'s (2013) centering their definition of equity where,

> All students in light of their humanity – personal experiences, backgrounds, histories, languages, physical and emotional well-being–must have the opportunity and support to learn rich mathematics that fosters meaning-making, empowers decision-making, and critiques, challenges and transforms inequities/injustices. Equity demands responsive instruction that promotes equitable access, attainment, and advancement for all students" (p. 9).

Given my research orientation, my plenary paper focuses on two questions-1) How do we use LT with teachers and coaches as a tool to deepen teacher knowledge and promote asset-based instruction, especially for students who have been historically marginalized? 2) How might we use LT to "reFrame" teacher noticings towards an anti-deficit orientation?

Shifting from Deficit to Anti-deficit Orientation by "reFraming" Teacher Noticings

This PMENA Plenary event in 2022 marks a significant time in our society, where we experienced the struggles and pain due to the Pandemic as well as systemic racism and escalation of racial tension leading to the Black Lives Matter movement. The pandemic unleashed hate, xenophobia and scapegoating leading to AAPI Hate with racist rhetoric. Labeling the "Asian" community as a monolith with an erasure of individual identity and the myth of the model minority or that "Asians are good at math" perpetuates a stereotype that is racist and dehumanizing (Shah, 2019), ignoring the huge diversity of linguistic, socio-economic, political and cultural backgrounds. It also masks the issues that different communities may need different supports in the school setting to succeed and excel. The Pandemic magnified the inequities that have long-existed in our society, education and communities. Deficit framed discourse streamed the media with outcries of "learning loss", while educational organizations worked hard to fight against this harmful language and discourse (i.e., Where is Manuel? A rejection of 'Learning Loss' TODOS, 2020). In addition, the danger in the discourse that marks the achievement of marginalized students being "more behind" in their learning, again perpetuates a pernicious mindset of achievement gap that our community has worked tirelessly to move away from (Gutierrez, 2008). Our professional organizations showed solidarity in fighting against systemic racism and this deficit framing and advocated for the "Mo(ve)ment to Prioritize Antiracist Mathematics: Planning for This and Every School Year" (TODOS, 2020), and AMTE's (2022) statement on "Equitable and Inclusive Mathematics Teaching and Learning" and the press release on systemic racism advocating for practices that draw on students' mathematical, cultural, and linguistic resources/strengths, and challenge policies and practices grounded in deficit- based thinking. The voices from our leading mathematics educational organizations (NCSM, NCTM & ASSM, 2020, 2021) "In Continuing the Journey: Mathematics Learning 2021 and Beyond", and "In Centering Our Humanity: Addressing Social and Emotional Needs in Schools and Mathematics Classrooms" (TODOS, 2020) advocated for math educators and school leaders to keep our focus on teachers, families and students well being, during this contentious socio political climate and Pandemic.

Rather than returning to the pre-pandemic status quo, Ladson-Billings (2021) argued for a "hard re-set" for a new "post-pandemic pedagogy" stating,

> In a re-set school environment, we will begin a school year with an accurate assessment of what students already know. The school year will have varied and regular formative assessments to determine how well students are understanding what they are taught, and an end of the year assessment would be keyed to what was actually taught in their classrooms. Assessment would no longer be a punitive tool to "catch" students but rather a diagnostic and

developmental tool that will tell teachers and schools how to adjust their curriculum and pedagogy (p. 74).

And yet, we know the opposite is happening where teachers are again being pressured to “catch students up” so that they can once again administer high-stakes tests. This is problematic as we know as scholars like Louie et al. (2021) describe the danger when a teacher with a framing around “closing the racial achievement gap” implicitly frames Black, Hispanic, and Indigenous students as mathematically lacking and White students’ achievement as the standard by which they should be measured (Gutiérrez 2008; Martin 2009). This framing makes one more likely to attend closely to Black, Hispanic, and Indigenous students’ errors without attending to their knowledge or strengths, to interpret these errors as evidence of misconceptions and failures, leading to deficit noticing.

Instead, we should be focused on varied and regular formative assessments to holistically determine how well students are understanding what they are taught and focus on asset based pedagogy like Complex Instruction (Horn, 2012; Cohen et al., 1999; Eli & Wood, 2016) where we develop teachers skills in assigning competence in student work. Research from Cohen et al.’s (1999) work on Complex Instruction showed that when teachers praised low-status students publicly for a task-related accomplishment, those students' participation increased; their status differences were mitigated or eliminated; and ultimately, their achievement increased.

When teachers better understand the learning trajectory continuum and anticipate a broader range of strategies, teachers can spot the strength of students along the LT continuum who may typically not get highlighted (Suh et al., 2018). In fact, Empson (2011) reflected on the 2010 PMENA and noted how LT impacts teacher professional noticings, “As teachers interact with students and decide how to proceed, there are many types of decisions to be made – how to gather information about children’s thinking, how to respond to it appropriately in the moment, how to design tasks that extend it, and even what to pay attention to” (p. 587).

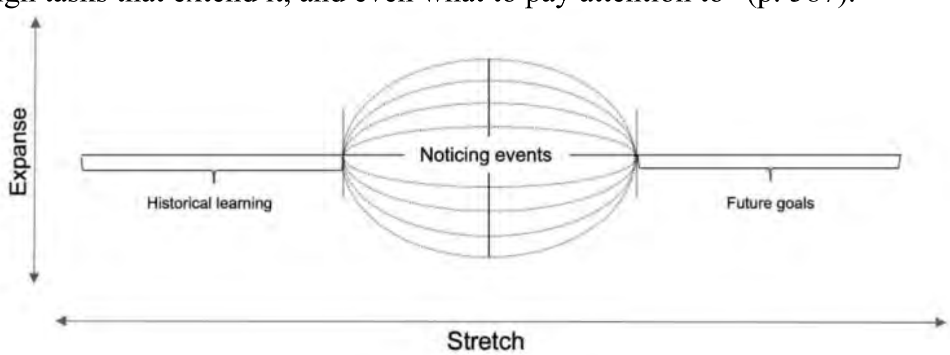


Figure 1: Reflecting Multi-dimensional Noticing for Equity (van Es et al., 2022)

Developing teachers' ability to assign competencies requires specialized skills of equitable noticing (Jacobs et al., 2010; Jacobs & Spangler, 2017; Kalinec-Craig et al., 2021; Jilk, 2016; Jong, 2017; Wager 2014). More recently, van Es et al. (2022) described a framework for Multidimensional Noticing for Equity, a system of noticing to disrupt inequities. Their framing towards a more multidimensional noticing for equity include the perspective of “taking account for how the histories and cultures of teachers, learners, and mathematics —and the broader historical, cultural, and political contexts in which they exist—are at play in moment-to-moment classroom interactions” (van Es et al, 2022, p.115 citing Louie, 2018; Mendoza et al., 2021; Shah & Coles, 2020). In their more expansive framework, the multidimensional noticing was used to interpret teachers' enactment of culturally sustaining instructional practices organized around a) Stretch, which captures the relation of teachers' noticing to both their own and students' past and futures, and (b) Expanse, which reflects the breadth and range of what teachers identify as noteworthy in moments of classroom interaction teaching and teacher noticing (see Figure 1).

Louie et al. (2021) discuss the importance of framing as a way to challenge deficit discourses about marginalized students that devalue the knowledge and abilities of students of color in classrooms in the US. They note,

> Deficit discourses may give rise to deficit noticing, wherein teachers attend almost obsessively to the errors and shortcomings of students of color; interpret errors and shortcomings as evidence of deficiencies in students, their families, or their cultures; erase students' assets; and disregard schooling practices and social structures that limit students' opportunities to learn and thrive. (p. 96)

Louie et al.'s (2021) most recent work framed anti-deficit noticing explicitly emphasizing how **F**raming is critically important in the ways teachers **A**ttend, **I**nterpret and decide to **R**espond (see Figure 2)

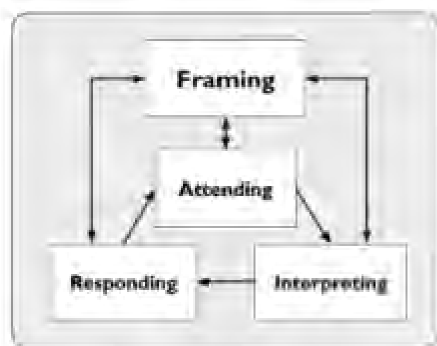


Figure 2: FAIR framework for Anti-deficit Noticing (Louie et al., 2021)

These two frameworks helped our research team think more broadly as we worked with our teachers in how we needed to frame anti-deficit orientation and asset-based instruction when working with learning trajectories. An important implication that we gleaned from the Multi-dimensional Noticing for Equity (van Es et al., 2022) is that although noticing captures the moment-to-moment events, what teachers notice and attune to is multidimensional in that it takes into account historical knowledge of the student, class and context and attends to the complexity of the instruction at play (i.e. Culturally Sustaining Pedagogy). The FAIR framework for Anti-deficit Noticing (Louie et al., 2021) noted the importance of being intentional and explicit when framing toward anti-deficit orientation to shift teachers noticing from deficit to anti-deficit noticing (See Figure 1). In particular, this FAIR framework's emphasis on framing allowed our team to pay close attention to *"how we notice"* students as full human beings with many mathematical strengths and resources, framing math learning as a creative exploration of ideas and framing interaction and interpersonal relationship as integral to learning (Louie et al., 2021).

Expanding the Notion of Math Competence using LT and Students' Assets

We emphasize in our LT PD work that teachers bring these multiple aspects and framing into view as they engage in multidimensional noticing. We focus on encouraging narrative that emphasizes the strengths students bring to the classroom including their mathematical thinking as well as their disposition and mathematics practices. LT has a long history in mathematics education research (Battista, 2011; Blanton et al., 2015; Clements & Sarama, 2004, 2009; Confrey et al., 2009, 2011; Ellis, 2014; Hackenberg, 2013; Ebby et al., 2020; Petit et al., 2020; Simon, 1995; Steffe & Olive, 2010; and others cited in the synthesis by Lobato & Walters, 2017) and well featured in past PMENA proceedings and has many metaphors and descriptors including a climbing wall (Confrey et al., 2021), in a conceptual corridor (Confrey, 2006);and levels of sophistication plateaus (Battista, 2004), just to name a few. Confrey and Maloney (2010) describe learning trajectories as

> a researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms, of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (p. 968)

Clements and Samara (2004) describe learning trajectories for early childhood mathematics for narrow sequences of topics as “a conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking” (p. 83).

The affordance of using LT deepens teachers' understanding of the progression of student learning-drawing upon their knowledge of the learning trajectories to make instructional decisions. More specifically, LTs have been used with teachers and researchers to better understand how students come to understand concepts (Battista 2004; Hackenberg & Tillema 2009) and to use “the learning goal, the learning activities, and the thinking and learning in which the students might engage” (Simon 1995, p. 133) to provide direction for teachers as they plan learning activities and predict the potential reasoning, misconceptions, and learning of students. Hypothetical learning trajectories (Simon 1995) have also been used in professional development settings to enhance instructional practices. Wilson et al. (2015) reported on a study using professional development where LTs bridged “guidelines for student-centered instruction with domain-specific understandings of students’ thinking for teachers” (p. 227).

According to Sztajn et al. (2012) existing research on teachers’ use of learning trajectories “shows that as teachers make sense of trajectories, these trajectories can support growth in mathematical knowledge, selection of instructional tasks, interactions with students in classroom contexts, and use of students’ responses to further learning” (p. 149). Research on Learning Trajectory based Instruction (LTBI, Sztajn et al., 2017) and the specific design decisions (Sztajn, 2010) the team attended to revealed the importance of setting discursive norms to focus on student thinking and teaching from a strength-based perspective particularly “at a time when deficit perspectives and language of differentiated instruction (such as having “high”, “medium”, and “low” children) to express ideas about student learning have been normalized.” (Sztajn et al., 2017, p. 30). With the many descriptors that include terms like “trajectory”, “progression”, “increasingly complex”, “levels of sophistication” (Battista, 2010), some caution that translated LT research can be misused as hierarchical levels that teachers use to sort students and dangerously place labels on students as being high or low. In fact, Myers et al. (2014) concluded that learning about LTs without additional support was insufficient to challenge deeply rooted ideas about student abilities. Even with attention to the design of PD, Myers (2014) found that teachers with severe deficit orientation used LTs to talk about what students could not do as opposed to thinking about moving students forward. Through discourse intervention, teachers started to use ability as a temporary descriptor to present students' current mathematical performance and used language from the LT to support these claims. The explicit attention to having teachers refer to the LT language instead of labels for students demonstrated the potential of LTs to support equitable instruction.

Celedòn-Pattichis et al. (2018) asset-based approaches to mathematics education are a conscious way to move away from deficit perspectives by teaching in ways that view students' language and culture as well as families', and communities' ways of knowing (Civil 2007; Bartell et al., 2017) as intellectual resources to engage with mathematics in the classroom. Asset based approaches offer a more humanizing view of student thinking that extends beyond school mathematics and recognizes that mathematics thinking and learning happens at home and in communities but is often unrecognized in school settings. Opening up learning trajectories to be able to recognize other forms of math thinking and experiences is key. Celedòn-Pattichis et al. (2018) also cautioned the community to recognize that not all communities and families focus on counting and operations in the specific way that Cognitively Guided Instruction (CGI) has described. Studies that combined CGI with culturally responsive instruction improved the mathematics performance of Native American students with learning disabilities (Hankes, et al., 2013) and other studies with culturally and linguistically diverse students engaged in complex CGI problem solving where teachers drew from language and culture as intellectual resources showed positive outcomes (Celedón-Pattichis et al., 2010; Turner et al., 2008).

Below I share use cases with LT PD and curriculum design research where coupling LT with asset-based approaches yielded anti-deficit teacher noticings. I detail how the use of vertical lesson study teams and other PD structures focused on learning trajectories and multidimensional noticing supported the development of anti-deficit professional noticings. I will refer to my research team as "we" in the case studies to represent the collaborative efforts of multiple researchers and doctoral students from the VDOE projects called TRANSITIONS and Bridging for Math Strength and an NSF project called IMMERSION.

Case #1: Synthesizing Previous Research on using Learning Trajectory-based Lesson Study- Appreciating Students' Robust Understanding

For over a decade, my colleagues and I used Learning Trajectory based Lesson Study (LTLS, Suh et al., 2021; Suh et al., 2019a; Suh et al., 2019b; Suh et al., 2018; Suh et al., 2017; Suh & Seshaiyer, 2014) in a series of multi-year state funded project called TRANSITIONS, where we worked with vertical teams of K-8 teachers, studying, planning, implementing and reflecting on teaching through rich tasks. In the Study phase of the Lesson Study (Lewis, 2012) instead of focusing on grade level standards, we used Confrey's (2012), five elements to unpack the LTs and to plan and anticipate strategies for a rich task starting with discussing: 1) the conceptual principles and the development of the ideas underlying a concept; 2) strategies, representations, and "conceptions"; 3) meaningful distinctions, definitions and multiple models; 4) recognizing coherent structure or pattern in the development of progressively complex mathematical ideas ; and 5) bridging standards or identifying the underlying concepts.

In Suh et al.'s article (2019a) we detailed a LTLS with a group of teachers ranging from Kindergarten to six grade, at an elementary school near a military base with the highest mobility rate of 33% in the district. With this transient population, the LTLS team wanted to use LT to bridge coherence in their curriculum. The team chose the often used submarine sandwich sharing task related to equipartitioning (Confrey et al., 2021; Confrey et al. 2009) typically used in Grades 3 or 4. They decided to launch the first lesson iteration in the Kindergarten classroom to study how very young students might approach this task. We framed the LTLS using elements from the Teaching for Robust Understanding (TRU) Framework (Schoenfeld & The Teaching for Robust Understanding Project, 2016), particularly focusing on the dimensions of *equitable access to content* using the LT and *agency, ownership, and identity* described as " the extent to which students are provided opportunities to contribute to conversations about disciplinary ideas, to build on others' ideas and have others build on theirs—in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners" (p. 9).

Though teachers studied the Equipartitioning LT during the Study phase of the LTLS, seeing how students approached equipartitioning through the Lesson Study surprised the educators and brought the LT to life. They noticed the strength in Kindergarten students being able to halve equal sized parts and share among friends, and understand the context of fair share. One group split the 6 sandwiches into 12 halves and made sure everyone of the 8 friends had equal sized parts (halves) and decided to remove 4 halves not eaten. Another group contributed the idea that there were some extras and wanted to use up the whole set of sandwiches and gestured cutting the halves into another half (fourths). Knowing how students approached this task allowed teachers to think about how to advance students to exhaust the whole without leaving any part of the sandwich unused. We used the LT "look fors" to help us better understand criteria for equipartitioning:

1. Having the correct number of parts
2. Exhausting the whole, leaving no parts unused
3. Having equal-size parts

(LT TurnonCC website, Confrey et al., n.d.).

Through lesson study, this vertical team saw this same lesson enacted in second, fourth and sixth grade classrooms. In our debrief, instead of focusing on grade level standards, the focus of the conversation came from observing how the students responded to the task related to the equipartitioning LT 'look fors' as well as using language from like non-anticipatory sharing, additive coordination to multiplicative coordination (Empson & Levi, 2011) as shown in Figure 3.

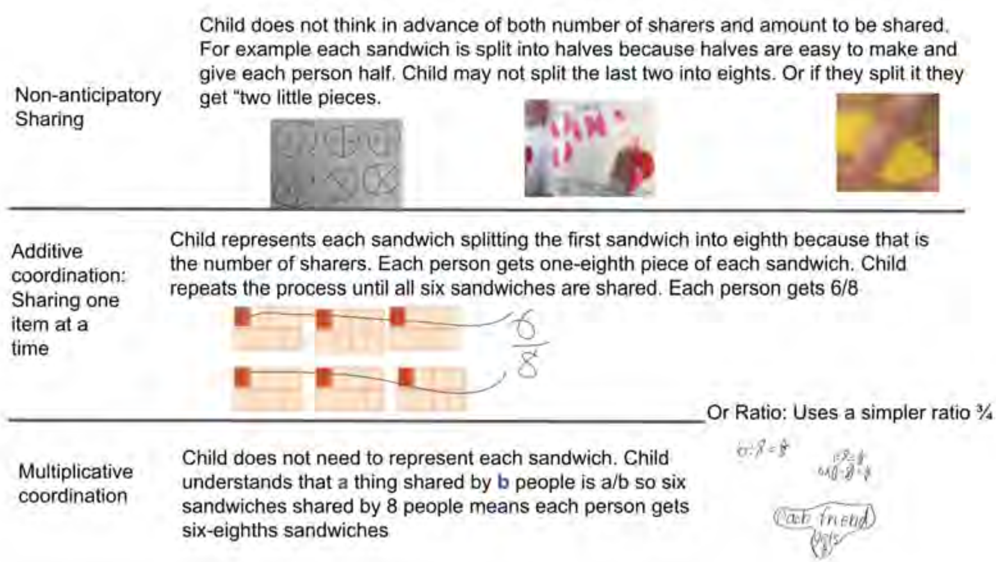


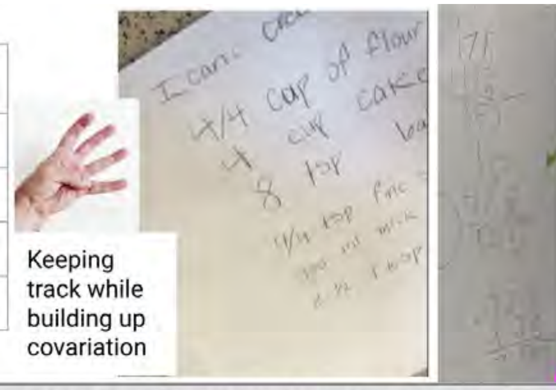
Figure 3: Analysis of Student Thinking using Equipartitioning LT on Sharing Tasks

The kindergarten teacher was proud to share the brilliance of her students and the LTLS team acknowledged and appreciated the informal understandings that emerged in earlier grades as valuable prerequisites for building more complex ideas.

In another vertical lesson study through an NSF project called IMMERSION (Suh et al., 2022), a team of 3rd through 6th grade teachers wanted to celebrate the Lunar New Year by making mooncakes with the many Asian students who celebrated the holiday. This school with its culturally and linguistically diverse student population identified 71% Latinx, 14% Asian, 11% White, 2 % Black and 3% others with 73% qualifying for free and reduced fee, embraced culturally sustaining pedagogy viewing students' home and community cultural practices as resources "to honor, explore, and extend" (Paris, 2012, p. 94). Scaling up a recipe is a rich task, typically classified as a middle grade proportional reasoning task but because there was a real need to scale up a recipe that was set for 6 servings, the teachers chose this task and decided that the first iteration would be launched in a 3rd grade classroom. As teachers discussed the learning trajectory continuum, they identified the skills that students have already developed like skip counting, as well as emerging skills like repeated addition and multiplication, connecting to future learning goals like scaling up using a ratio table. They discussed the connection to students' assets in terms of cultural funds of knowledge (Moll et al., 1992) and family practices in cooking and emphasized bringing in realia for measurements as well as the ingredients to connect to students' multiple knowledge bases (Turner et al., 2013). They anticipated some students using manipulatives/realia to make sense of figuring out how many times their recipe might have to be scaled up based on the serving size, in addition to repeated addition, using multiplication facts as well as using as manipulatives and tables. Just like the previous task with the sharing sandwich, students in third grade used their multiple knowledge bases (Turner et al., 2013) to figure out how much of the ingredients they would need to make enough mooncakes. Teachers noted how students used manipulatives to figure out how many batches they would need (the scale factor) and noted how they used their fingers or notes on paper as they scaled up the ingredients. This coordination through iterative skip counting is the precursor to recognizing the covariation nature of early proportional reasoning (Steinthorsdottir & Sriraman, 2014).

With this in mind, teachers recognized that these emergent ways of keeping track with their fingers and scaling up was a brilliant way of thinking and reflected formative strategies to show covariation. Teachers were seeing the LTs in action as they observed student work and thinking (see Figure 4).

batch	servings
1	6
2	12
3	18
4	24



Build-Up (repeated addition)	The given ratio of first-measure-space quantity to second-measure-space quantity is repeated and combined using addition or multiplication as a form of repeated addition.	
Multiplicative	A multiplicative relationship is explicitly used.	
	Between-Measure Space	The given between-measure-space ratio is maintained in the target ratio.
	Within-Measure Space	The scale factor is determined and applied within each measure space, a reduced rate is scaled, or a unit rate is scaled.

Figure 4: Analysis of Student Thinking using Early Proportional LT on Scaling up a Recipe with LT Chart from Riehl & Steinthorsdottir, 2014

Through both of these LTLS, we found that,

1. The observers as well as the host teacher acknowledged and appreciated the brilliance and quality of students' fraction and proportional reasoning and positioned students' conceptions and multiple strategies as strengths.
2. The coach facilitated a productive debrief with the participating teachers to verify, validate and sometimes dispute the hypothetical learning trajectories based on their observations noting that they saw some attributes of more advanced thinking with earlier formative strategies.
3. Bridging the learning trajectories through a rich task across multiple grade levels allowed teachers to better focus on LT and talk less of grade level standards. Seeing the students work vertically across grades k-6 allowed teachers to appreciate informal understandings as valuable prerequisites for building more complex ideas.
4. Teachers can play an active role in validating LTs with researchers and at times disrupt notions of traditional sequencing of mathematics prescribed by standards.
5. Rich tasks can go beyond the realm of standards and provide a low floor and high ceiling where teachers can use their knowledge of LT to highlight students' strengths and position students as capable along the LT.

Lessons learned from LTLS informed our most recent work below with LT based professional development and curricular design activities that highlight the ideas of using LT as a tool to build on students' assets and promote anti-deficit framing in our work.

Case #2: Using LT with Formative Assessment to Build on Math Strength- Multi-dimensional Professional Noticing focused around Anti-deficit Framing

This case study began in 2020 at the start of the pandemic when mathematics leaders in the state department of education approached our team to create curriculum resources for teachers to support deep conceptual learning around essential concepts. One of the problems of practice presented was that teachers were skillful at finding what gaps students had in their understanding but did not always know where to go next to advance student thinking. In reflecting on this problem of practice, I thought back to Shaun Harper's (2010) paper called *An Anti-Deficit Achievement Framework for Research on Students of Color in STEM*, where he states that the kinds of questions we ask can focus on failure and not successes. For example, we can reframe the question- "Why do so few pursue STEM majors?" (Deficit-Oriented Questions) to "What stimulates and sustains students' interest in attaining degrees in STEM fields?" (Anti-deficit reframing). He notes that it is both important to unearth systemic inequities and barriers as well as identify structures and strategies that support students of color to thrive. In this same vein, we wanted to reorient this problem of practice from, "How do we work with students once we know the gaps in their understanding?" to "How do we spot students' strengths and use that to advance their learning in mathematics?"

In our design research, we invited teachers and coaches as our co-designers, tapping into the geniuses in our schools (Wiseman et al., 2013) to build an LT based curricular resource site for educators. The key design components for our design institute included asset-based instruction, knowledge and integration of learning trajectories as our teacher designers created formative assessment with bridging activities. Bridging for Math Strength design work engaged teachers to unpack LT, use formative assessment to articulate the 'look-fors' for building on math strength and purposeful questions to advance student learning through a designed set of learning activities along that continuum. The participants in our Math Strength Design team included twenty-seven K-8 teachers and coaches working as teacher designers in teams of three. The summer design institute took place in June of 2020 with Implementation Cycles in Fall 2021 and Spring 2022. We used a rapid prototyping method with iterative design and refinement through implementation cycles.

Using a variety of research-based strength building strategies in our Design Institute, we equipped our teachers with the knowledge and research on Learning Trajectories and concrete strategies to support asset-based thinking. This included:

1) Locating relevant usable LT research translated for practitioners – (https://www.learningtrajectories.org/, http://www.ogapmath.com/, https://www.sudds.co/ Blanton, 2008; Empson & Levi, 2011; Ebby et al., 2020; Hackenberg, 2013; Petit, st al., 2020; Steffe & Olive, 2010)
2) Broadening the notion of math competence and smartness with Complex Instruction (Kobett, & Karp, 2020; NCTM, 2020; Jilk, 2016; Kalinec-Craig, et al., 2021; Horn, 2012; Lotan, 2003; Cohen et al., 1999; Featherstone et al, 2011)
3) Anti-deficit & Multi-dimensional Noticing frameworks (Louie et al., 2021; van Es et al., 2022)
4) Lesson Study to collectively learn about students' brilliance in thinking and strategies competence (Lewis, 2022; Suh & Seshaiyer, 2014)
5) Evaluating and sequencing learning activities to advance students' thinking

We used the data sources including the designed module, implementation narrated through Flipgrid, a recording platform, debrief webinars about the implementation cycles and interviews with teacher designers.

Our research questions focused on 1) How can we use LT with teachers and coaches as a tool to deepen teacher knowledge and promote asset-based instruction for students? 2) How might we use LT to "reFrame" teacher noticings towards anti-deficit orientation?

Each of the fall follow- up sessions with teacher designers included debriefs around implementing the modules and analyzing student work. Using Flipgrid, the teacher-designers narrated their implementation so other teachers could follow their process. They started by sharing the formative assessment they chose, and then outlined the sequence of activities picked based on the student work on the formative assessments. Knowledge of LT helped them sequence learning activities for students. These voices centered teachers and coaches in the LT and focused the lesson implementation to elevate the strengths of students.

Below Kelly shares her implementation of the LT based curricular resource with some of her students. The excerpt brings to life how a coach might use LT as a tool to build on students' strengths. In her noticings (see Figure 5), she honored this student's funds of knowledge when he shared with her, "Well I like 7s because I watch a lot of football so I'm good at counting by 7s". She interprets his strengths then decides that she would respond by asking questions to see how he might recognize patterns and apply reasoning strategies. She planned for questions such as, "What do you notice about the relationship of 7x2, 7x4, and 7x8? You mentioned you felt comfortable with 7s. What other numbers feel friendly to you? How could you use them to solve more challenging facts?" In deciding on rich educational learning experiences to strengthen his reasoning, she proposed a center activity called Strive to Derive which is a game that shows arrays that students can break apart to rehearse the strategy of using known derived facts or distributive property. In addition, she proposed visual number strings to build on patterns and relationships for multiplication.

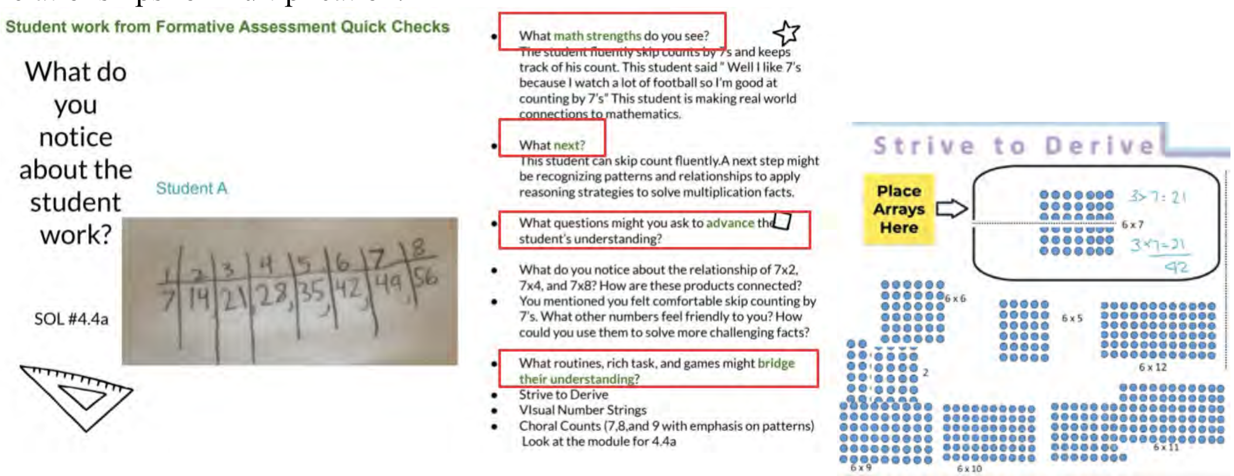


Figure 5: Providing question prompts to support asset based/anti-deficit noticings

In our analysis we found that using formative assessment and LT based bridging activities as a curricular resource during the pandemic revealed the power of removing high stakes testing and letting teachers use formative data in meaningful and humanizing ways. One of the teachers shared, "It was *liberating* because I didn't think so much about grade level standards and the state assessment. Instead, I focused on the LT and where students showed strength and built on their strength through routines, rich tasks and games." The LT-based curricular structure supported teachers and coaches in reframing how they view student learning beyond grade levels- moving away from language like "below grade level" and "at risk". Sienna, an instructional coach, found that the LT structure helped teachers see their students' learning as a progression and consider next steps rather than visualizing a gap between students' current understanding and the "final goal of the standard". Kara had a similar epiphany in her second grade classroom. She found that the LT structure allowed her to focus on conceptual understanding in her second grade students' work rather than simply quantifying the number of incorrect answers, stating "When you go to grade something or check it over, I'm not necessarily looking at 'Oh, they got 15 out of 20. They're missing a bunch.' ...I'm really zoning and honing in on what patterns I can find. I feel like that's what this cycle has taught me is that there are patterns in student work." Kara's attention to patterns in her students' work allowed her to identify their position on the LT and plan targeted instruction to support their learning as well as identify strengths and growth areas.

Case #3:PST Teachers Use of LT to Assess Student Thinking and Design Sequence of Activities

This next case illustrates ways LTs can be used with preservice teachers to prepare them as competent mathematics teachers. The term "learning trajector(ies)" appears at least sixty-three times in the document for the Standards for Preparing Teachers of Mathematics (SPTM) published by the Association of Mathematics Teacher (AMTE, 2017). For example, Standard EC3 and EC7 emphasizes the importance of LT in curricular knowledge as well as for assessment.

> EC.3. Mathematics Learning Trajectories: Paths for Excellence and Equity: Well-prepared beginning teachers of mathematics at the early childhood level understand learning trajectories for key mathematical topics, including how these learning trajectories connect to foundational knowledge, curriculum, and assessment frameworks. [Elaboration of C.1.4]
>
> EC.7. Seeing Mathematics Through Children's Eyes: Well-prepared beginning teachers of mathematics at the early childhood level are conversant in the developmental progressions that are the core components of learning trajectories and strive to see mathematical situations through children's eyes. [Elaboration of C.3.1]

Supporting PSTs on how to use LT and formative assessment to strengthen student thinking is a priority in a mathematics methods course. Focusing on these two standards, I designed an assignment called *Learning Trajectory-based Formative Assessment & Sequenced Digital Math Activities.* In this assignment, PSTs planned and enacted asset-based solutions that included digital tools as learning activities. The goal of the tasks was to re-engage students in mathematics as the students worked to develop their sense of agency, identity, and ownership in their mathematical learning. This case focuses on how PSTs used learning trajectory research to analyze formative assessment data and then sequenced bridging activities using digital technologies to enrich mathematics instruction for individual and collective learning. PSTs also learned to use digital technology teacher dashboards, which allowed them to be responsive by providing ease of analysis of student proficiency, facilitating immediate feedback, and providing information to form targeted small groups to support student learning. The *Learning Trajectory-based Formative Assessment and Digital Math Activities* (Suh et al., 2022) assignment was designed using two important frameworks, Learning Trajectory Based Instruction (LTBI, Sztajn et al., 2012) and Technological Pedagogical Content Knowledge (TPACK, Mishra, P., & Koehler, M. J., 2006))

Sara mapped out her learning trajectory concept map with the core idea of 'unitizing' (Lamon 1994, Steffe 1992, 1994) as "operating with singleton units to coordinating composite units" (Singh 2000, p. 273) highlighting the array as part of the development of spatial and numeric composite (Battista, 2012) necessary for multiplicative reasoning.

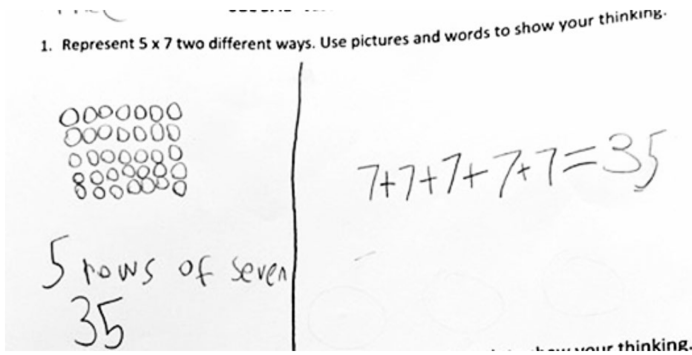


Figure 6: Sample Formative Assessment that Puzzled the Preservice Teacher

Sara, had been working with a student named Selena who could draw rows and columns with dots to model multiplication, and use repeated addition (see Figure 6), but oftentimes she would be off by 1 or 2 with her final answer. Sara was curious how to support Selena. She watched her during a formative assessment and noticed that for every problem, she would count every dot. Knowing that she wanted to move the student from counting all to seeing "many as one" as a composite unit, she found a technology tool called Bunny Times (see Figure 7) that worked with an array model with an added feature. Her analysis of the tool highlighted the affordance of the "fog feature" actually helped Selena advance to other strategies that were more efficient like skip counting or adding on from a known fact. She liked this applet because the visual helps students make connections between rows and columns. The game can be scaled in the size of math facts. Additionally, 'fog' can settle over the field obscuring some of the answers disallowing counting. Facts can be differentiated when starting the game.

Figure 7: Learning activity that supports unitizing and skip counting

Sara stated in her assessment report,

> For my target student, I plan to use Bunny Times math after working with her on skip counting. Bunny Times allows for multiplication facts to be scaled to learner readiness. Additionally, it can be played with all rows and columns visible or with some hidden under a layer of fog. For my student, with practice on unitizing, skip counting, and counting on, I hope that she will be able to complete problems using the "fog" feature.

Through the assignment, Sara learned that she could lean on her clinical faculty, a math coach in the school as well as her course instructor to assess where her student was in the multiplicative learning trajectory. She reflected on this practice-based assignment stating,

> I think this assessment will prove to be one the most important things which happened to this

> student in 3rd grade. Because of her fabulous math disposition and other areas of proficiency, it is likely that her struggles with unitizing and counting would have been masked and not observed. This project allowed for data collection, meeting with a math specialist, testing, and ultimately transferring that into specific intervention. -Sara, the PST

This PST noticed many strengths in this child's mathematical understanding related to multiplication and the intervention built on those strengths to stretch the student to be a stronger mathematician.

Implications for Mathematics Teacher Educators and Researchers

In reflecting on PMENA 2022's Theme on Dissonance and Harmony, I share some concluding thoughts and implications for mathematics teacher educators and researchers. First, we need to take the notion of a "hard reset" (Ladson-Billings, 2021) seriously to dismantle inequitable structures and practices that exist in mathematics teaching and learning and challenge the status quo. We found in our LTLS research how teachers can play an active role in validating researcher-conjectured LTs and at times challenge the traditional sequencing of mathematics prescribed by standards. We have viewed LT as a tool to help reframe teachers thinking about what students are capable of doing and finding a strength based asset orientation to instruction (Bartell et al., 2017). Building on the work of LTBI and our previous work around LTLS (Suh et al., 2018), the project that began during the pandemic called "Bridging for Math Strength" with the professional development and design research study continues to go through iterations of refinement with our model and products to support teachers teaching and student learning. With this work, we focused on changing the narrative and mindset of teachers, moving away from looking at gaps and solely focusing on error patterns (deficit-approach) to finding strengths in student thinking and using the LT to advance student thinking based on strengths and growth areas. Working on explicitly noticing and assigning competence (Gresfali et al., 2009) to shift classroom status and using LT has helped look for the strengths and where to move students forward in their learning. Too often data-talk focuses on looking at gaps using white performance as a standard to show how marginalized students are performing. Bridging the learning trajectories through a rich task across multiple grade levels allowed teachers to better focus on LT and talk less of grade level standards as the final arbiter of learning.

In order to create more socially just contexts for learning and teaching mathematics, we propose a paradigm shift in learning more deeply about the LT so that we can assess student strength and make a path of learning activities through rich tasks, place more emphasis on formative assessment and move away from gap gazing that continues to persist with state assessment (Gutierrez, 2008). Catalyzing Change Early Childhood (NCTM, 2020) - shows that students often marginalized are not given rich tasks instead given more rote learning. We advocate moving away from a "my students can't" narrative and the opportunity gaps that exist from students engaging in rigorous tasks. Continuing with data meetings with state assessments to "close the achievement gap" and catch students up will perpetuate deficit discourses, deficit noticing, obsession over errors and shortcomings of students of color blaming deficiencies in students, their families, or their cultures (Louie et al., 2017). Instead, Celedon-Pattichis et al., (2018) urges researchers and mathematics teachers to embrace asset-based approaches to mathematics education and to consciously move away from deficit perspectives that view students, parents, and communities as lacking in different aspects that enable them to be ready for schooling (Coleman et al., 2016). They encourage the mathematics education community to appreciate the math knowledge/experiences that students bring from home and communities and by doing so students bring ways of thinking that broaden mathematics beyond what is written in standards or embodied in curriculum.

To impact society more broadly, beyond individual mathematics classrooms and school districts our work must improve learning conditions for each and every mathematics learner. With a hard reset and a focus on asset-based instruction and anti-deficit noting with the intentional use of learning trajectory (LT) with equity focused PD, our work revealed that teachers felt liberated and empowered to open up varied and expansive ways to discuss students' mathematics competencies, name students' strength and position students as capable. LTLS allowed teachers to be researchers and share their expertise and validated research-conjectured LT with real classroom data. This positioned them as knowledgeable and elevated their status as learning scientists. With the Bridging for Math Strength project, the use of LT coupled with anti-deficit framing in curriculum design research provided teachers with a tool and the language to analyze student thinking and plan rigorous educational resources and asset-based instruction for their students.

Lastly, the use of LT with formative assessment in the practice-based assignment for preservice teachers provided a scaffolded learning experience in the field with multiple educators supporting them with LT research and equitable teaching strategies to advance students' understanding. As I close this paper, I invite the PMENA community to consider how learning trajectories can be coupled with powerful equity focused research (Gutierrez, 2007; Celedon-Pattichis et al., 2018; Hand, 2012; Wager, 2014) and frameworks (Aguirre & Zavala, 2013; Bartell et al., 2020; Yeh et al., 2020) to disrupt the status quo, broaden the purposes of learning mathematics (NCTM, 2020), eliminate labeling, and dismantle inequitable structures and hierarchy in the mathematics classroom. In continuing this work, I invite mathematics researchers and math teacher educators to consider border crossing (Silver & Lunsford, 2017) as boundary spanners (AACTE, 2018) to not only translate but engage teachers and researchers in the viewing the relationship between research and practice in education as bidirectional rather than unidirectional so that "research could/should influence/inform practice, but also that practice could/ should influence/inform research" (Silver & Lunsford, 2017, p. 36) while centering the voices of students who are at the margins and attending to the socio-political lens with their LT research.

Acknowledgments

I would like to acknowledge Dr. Toya Frank and Dr. Julia Aguirre for reviewing this plenary paper and for both being an awesome mentor and friend, Dr. Theresa Wills, Dr. Courtney Baker, Dr. Padhu Seshaiyer and our many doctoral students and colleagues at GMU who have been partners on these projects and my EQSTEMM research team, Erin Turner, Mary Alice Carlson, Elizabeth Fulton and Julia Aguirre who enrich my thinking every day. In addition, I acknowledge the funding agency, NSF STEM-C 1441024 and VDOE for the project mentioned above.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Aguirre, J., & Zavala, M. (2013). Making culturally responsive mathematics teaching explicit: A lesson analysis tool. *Pedagogies*, *8*(2), 163–190.

American Association of Colleges for Teacher Education. (2018). A pivot toward clinical practice, its lexicon, and the renewal of educator preparation: A report of the AACTE Clinical Practice Commission. Washington, DC: Authors.

Association of Mathematics Teacher Educators. (2020). AMTE statement on systemic racism [Position statement]. https://amte.net/files/AMTE%20Racism%20Press%20Release.pdf
Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. Journal for Research in Mathematics Education, 48(1), 7–21.
Bartell, T., Turner, E. E., Aguirre, J. M., Drake, C., Foote, M. Q., & Roth McDuffie, A. (2017). Connecting children's mathematical thinking with family and community knowledge in mathematics instruction. Teaching Children Mathematics, 23(6), 326–328.
Battista, M. (2010). Representations of learning for teaching: Learning progressions, learning trajectories, and levels of sophistication. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the Thirty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. pp. 60-71. The Ohio State University.
Blanton, M. (2008). Algebra and the Elementary Classroom Transforming Thinking, Transforming Practice. Heinemann.
Celedòn-Pattichis, S., Peters, S.A., Borden, L.L., Males, J.R., Pape, S.J., Chapman, O., Clements, D.H., Leonard, J.(2018). Asset-based approaches to equitable mathematics education research and practice. Journal for Research in Mathematics Education, 49(4), 373–389.
Celedón-Pattichis, S., Musanti, S. I., & Marshall, M. E. (2010). Bilingual elementary teachers' reflections on using students' native language and culture to teach mathematics. In M. Q. Foote (Ed.), Mathematics teaching & learning in K–12: Equity and professional development (pp.7–24). New York, NY: Palgrave Macmillan. doi:10.1057/9780230109889_2
Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. S. Nasir & P. Cobb (Eds.), Improving access to mathematics: Diversity and equity in the classroom (pp. 105–117). New York, NY: Teachers College Press.
Clements DH, Sarama J. (2018). Myths of Early Math. *Education Sciences*. 8(2):71. https://doi.org/10.3390/educsci8020071
Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. Mathematical Thinking and Learning, 6(2), 81–89. https://doi.org/10.1207/s15327833mtl0602_1
Clements, D. H., & Sarama, J. (2017/2019). Learning and teaching with learning trajectories [LT]2. Retrieved from Marsico Institute, Morgridge College of Education, University of Denver Website: LearningTrajectories.org
Coleman, S. T., Bruce, A. W., White, L. J., Boykin, A. W., & Tyler, K. (2016). Communal and individual learning contexts as they relate to mathematics achievement under simulated classroom conditions. Journal of Black Psychology, 43(6), 543–564. doi:10.1177/0095798416665966
Confrey, J., Shah, M., & Maloney, A. (2022). Learning Trajectories for Vertical Coherence, *Mathematics Teacher: Learning and Teaching PK-12*, *115*(2), 90-103. Retrieved Oct 12, 2022, from https://pubs.nctm.org/view/journals/mtlt/115/2/article-p90.xml
Confrey, J., Maloney, A. P., Nguyen, K. H., Mojica, G., & Myers, M. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. In Paper presented at the 33rd conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.
Confrey, J. (2012). Articulating a learning science foundation for learning trajectories in the CCSS-M. In Van Zoerst, L. R., Lo, J. J., & Kratky, J. L. (Eds.), Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology Mathematics Education (pp. 2–20). Western Michigan University.
Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In K. Gomez, L. Lyons, & J. Radinsky (Eds.), Learning in the Disciplines: Proceedings of the 9th International Conference of the Learning Sciences (ICLS 2010) – Volume 1, Full Papers. Chicago, IL: International Society of the Learning Sciences. Retrieved from http://dl.acm.org/citation.cfm?id=1854484.
Cohen, E., Lotan, R., Scarloss, B., & Arellano, A. (1999). Complex instruction: Equity in cooperative learning classrooms. Theory into Practice, 38(2), 80–86.
Ebby, C. B., Hulbert, E.T. & Broadhead, R.M. (2020). A focus on addition and subtraction. Routledge.
Eli, J. A., & Wood, M. B. (2016). Learning to facilitate groupwork through complex instruction. Tucson, AZ: University of Arizona.
Ellis, A. B., (2014). What if we built learning trajectories for epistemic students? In L. Hatfield, K. Moore, & L. Steffe (Eds.), Epistemic algebraic students: Emerging models of students' algebraic knowing, WISDOMe Monographs (Vol. 4, pp. 199–207). Laramie, WY: University of Wyoming.
Empson, S. B.. & Levi, L (2011). Extending Children's Mathematics. Hennemann.

Empson, S. (2011). On the idea of learning trajectories: Promises and pitfalls. The Mathematics Enthusiast, 8(3), 571–596.
Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. Educational Studies in Mathematics, 70(1), 49–70. https://doi.org/10.1007/s10649-008-9141-5
Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept.The Journal of Mathematical Behavior, 32(3), 538–563. doi:10.1016/j.jmathb.2013.06.007
Hankes, J., Skoning, S., Fast, G., & Mason-Williams, L. (2013). Closing the math gap of Native American students identified as learning disabled. Investigations in Mathematics Learning, 5(3), 44–59. doi:10.1080/24727466.2013.11790326
Harper, S.R. (2010), An anti-deficit achievement framework for research on students of color in STEM. New Directions for Institutional Research, 2010: 63-74. https://doi.org/10.1002/ir.362
Jilk, L. M.(2016). Supporting Teacher Noticing of Students' Mathematical Strengths. Mathematics Teacher Educator, 4(2), 188-199.
Gutiérrez, R. (2007). Context matters: Equity, success, and the future of mathematics education. In Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1–18).
Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. Educational Studies in Mathematics, 80, 233–247.
Horn, I. S. (2012). Strength in numbers. National Council of Teachers of Mathematics.
Jacobs, V., Lamb, L., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169–202. https://doi.org/10.5951 /jresematheduc.41.2.0169
Jacobs, V. R. & Spangler, D.A.(2017). Research on Core Practices in K–12 Mathematics Teaching. In Jinfa Cai, (Ed.) Compendium for research in mathematics education. Reston, VA : National Council of Teachers of Mathematics, 766-792.
Jong, C., Priddie, C., Roberts, T., Museus, S.D. (2020). Race-related factors in STEM: A review of research on educational experiences and outcomes for racial and ethnic minorities. In C.C. Johnson, M. Mohr-Schroeder, T. Moore, L. Bryan, & L. English (Eds), Handbook of Research in STEM Education, Routledge.
Jong, C. (2017). Extending equitable teaching practices in teacher noticing: Commentary. In Schack, E. O., Fisher, M. H., & Wilhelm, J.A. (Eds.). Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks (pp. 207-214). New York, NY: Springer.
Kalinec-Craig, C. A., Bannister, N., Bowen, D., Jacques, L. A., & Crespo, S. (2021). "It was smart when:" Supporting prospective teachers' noticing of students' mathematical strengths. *Journal of Mathematics Teacher Education*, *24*(4), 375–398. https://doi-org.mutex.gmu.edu/10.1007/s10857-020-09464-2
Kobett, B. M., & Karp, K. S. (2020). Strengths-based teaching and learning in mathematics: Five teaching turnarounds for grades K–6. Corwin.
Ladson-Billings, G. (2021) I'm here for the hard re-set: Post pandemic pedagogy to preserve our culture. *Equity & Excellence in Education, 54*(1), 68-78, DOI: 10.1080/10665684.2020.1863883
Learning and Teaching with Learning Trajectories. (n.d.). https://www.learningtrajectories.org/
Lewis, C. (2002). Lesson study: A handbook of teacher-led instructional change. Research for Better Schools.
Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 74–101). Reston: National Council of Teachers of Mathematics.
Lotan, R. (2003). Group-worthy tasks. Educational Leadership, 60(6): 72–75.
Louie, N., Adiredja, A., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. ZDMMathematics Education. Advance online publication. https://doi.org/10.1007/s11858-021-01229-2
Louie, N. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. Journal for Research in Mathematics Education, 28(5), 488–519.
Louie, N. (2018). Culture and ideology in mathematics teacher noticing. Educational Studies in Mathematics, 97(1), 55–69.
Math Mapper. (n.d.). Math Mapper Formative Diagnostic Assessment Real-Time Data and Feedback for Students and Teachers https://www.sudds.co/
Mendoza, E., Hand, V., van Es, E. A., Hoos, S., & Frierson, M. (2021). "The ability to lay yourself bare": Centering rupture, inherited conversations, and vulnerability in professional development. Professional Development in Education, 47(2–3), 243–256. https://doi.org/10.1080/19415257.2021.1891955

Moll, L.C., Amanti, C., Neff, D., Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. Theory into Practice, 31,. 132–141.

Myers, M., Sztajn, P., Wilson, P. H., & Edgington, C. (2015). From implicit to explicit: Articulating equitable learning trajectories based instruction. Journal of Urban Mathematics Education, 8(2), 11–22.

Myers, M.(2014). The use of learning trajectory based instruction in supporting equitable teaching practices in elementary classrooms: North Carolina State University, Raleigh, NC.

NCSM & NCTM. 2020. Moving Forward: Mathematics Learning in the Era of COVID-19. https://www.mathedleadership.org/docs/resources/NCTM_NCSM_Moving_Forward.pdf.

NCSM, NCTM, & ASSM. (2021). Continuing the Journey: Mathematics Learning 2021 and Beyond https://www.nctm.org/mathematics2021/

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2020). Catalyzing change in early childhood and elementary. Reston, VA: Author

NCTM Research Committee. (2018). Asset-based approaches to equitable mathematics education research and practice. Journal for Research in Mathematics Education, 49(4), 373–389.

Ongoing Assessment Project OGAP Math. (n.d). https://ogap.com /

Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. Educational Researcher, 41(3), 93–97.

Petit, M. (2011). Learning trajectories and adaptive instruction meet the realities of practice. In P. Daro, F. A. Mosher, & T. Corcoran (Eds.), Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction (Research Report No. RR-68; pp. 35–40). Consortium for Policy Research in Education.

Petit, M.M., Laird, R., Wyneken, M.F., Huntoon, F.R. Abele-Austin, M.D. & Sequeira, J.D. (2020). A focus on ratios and proportions. Routledge.

Riehl, S.M. & Steinthorsdottir, O.B. (2014).Revisiting Mr. Tall and Mr. Short. Mathematics Teaching in the Middle School , 20(4), 220-228.

Schoenfeld, A. H., & The Teaching for Robust Understanding Project. (2016). An introduction to the teaching for robust understanding (TRU) framework. Retrieved on August 1, 2017 from http://truframework .org or http://map.mathshell.org/trumath.php

Shah, N. (2019). "Asians are good at math" is not a compliment: STEM success as a threat to personhood. Harvard Educational Review, 89(4), 661-686.

Shah, N., & Coles, J. A. (2020). Preparing teachers to notice race in classrooms: Contextualizing the competencies of preservice teachers with antiracist inclinations. Journal of Teacher Education, 71(5), 584–599. https://doi.org/10.1177/0022487119900204

Silver, E. & Lunsford, C. (2017). Linking Research and Practice in Mathematics Education: Perspectives and Pathways. In Jinfa Cai, (Ed.) Compendium for research in mathematics education. Reston, VA : National Council of Teachers of Mathematics, 28-47.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. Journal for Research in Mathematics Education, 26(2), 114–145.

Steffe, L. P., & Olive, J. (2010). Children's fractional knowledge. Springer: New York.

Steinthorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th-grade girls' developmental trajectories in proportional reasoning. Mathematics Education Research Journal, 21(1), 6-30.

Suh, J. M. & Gallagher, M. & Capen, L. & Birkhead, S. (2021). Enhancing teachers' noticing around mathematics teaching practices through video-based lesson study with peer coaching. International Journal for Lesson & Learning Studies. ahead-of-print. 10.1108/IJLLS-09-2020-0073.

Suh, J.M., Birkhead, S., Farmer, R. R., Galanti, T., Nietert, A., Bauer, T., & Seshaiyer, P. (2019a). Split it! Unpacking the equipartitioning learning trajectory. Teaching Children Mathematics, 25(6), 362–369.

Suh, J.M., Birkhead, S., Galanti, T., Farmer, R., & Seshaiyer, P. (2019b). The Use of Lesson Study to Unpack Learning Trajectories and Deepen Teachers' Horizon Knowledge (756-781). In R. Huang & Takahashi, A. (Eds.), *Theory and Practices of Lesson Study in Mathematics: An International Perspective*. Springer.

Suh, J. M., Birkhead, S., Baker, C., Frank, T., & Seshaiyer, P. (2017). Leveraging Coach-Facilitated Professional Development to Create Teacher Social Networks for Enhancing Professional Practice. In M. Boston & L. West (Eds.), *Annual Perspectives in Mathematics Education: Reflective and Collaborative Processes to Improve Mathematics Teaching*. (pp. 89-100). Reston, VA: National Council of Teachers of Mathematics.

Suh, J. M. & Seshaiyer, P. (2014). Examining teachers' understanding of the mathematical learning progression through vertical articulation during Lesson Study. *Journal of Mathematics Teacher Education*, *18*(3), 217-229.
Suh, J., Roscioli, K., Morrow-Leong, K., Tate, H. (2022). Transformative technology for equity-centered instruction. Proceedings of the Society for Information Technology & Teacher Education International Conference, San Diego, CA, USA.
Suh, J.M. (2022, April 23). Engaging in Learning Trajectory-Based Lesson Study and Formative Assessment to Promote Asset-Based Instruction. Paper presented at the 2022 annual meeting of the American Educational Research Association. Retrieved October 15, 2022, from the AERA Online Paper Repository.
Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. Educational Researcher, 41(5), 147–156.
Sztajn, P., Edgington, C., Wilson, P. H., Webb, J. & Myers, M. (2012). Learning trajectory based instruction project In Sztajn, P (Eds.), *Learning Trajectories for Teachers: Designing Effective professional Development for Math Instruction*. (pp. 15-47). Reston, VA: National Council of Teachers of Mathematics.
TODOS. (2020). The Mo(ve)ment to Prioritize Antiracist Mathematics: Planning for This and Every School Year [Position statement]. https://www.todos-math.org/statement
TODOS. (2020). Centering Our Humanity: Addressing Social and Emotional Needs in Schools and Mathematics Classrooms [Position statement]. https://www.todos-math.org/statements
TODOS. (2020). Where is Manuel? A rejection of 'Learning Loss' https://www.todos-math.org/
Turner, E., Foote, M., Stoehr, K., Roth McDuffie, A., Aguirre, J., Bartell, T., & Drake, C. (2016). Learning to leverage children's multiple mathematical knowledge bases in mathematics instruction. Journal of Urban Mathematics Education, 9(1), 48–78.
Turner, E., Dominguez, H., Maldonado, L., Empson, S. (2013). English Language Learners identity-enhancing participation in mathematical discussion. Journal for Research in Mathematics Education, 44, 1, 199-234. Special Equity Issue, R. Gutiérrez (Ed.).
Turner, E. E., Celedón-Pattichis, S., & Marshall, M. (2008). Cultural and linguistic resources to promote problem solving and mathematical discourse among Hispanic kindergarten students. In R. Kitchen & E. Silver (Eds.), The inaugural TODOS: Promoting high participation and success in mathematics by Hispanic students: Examining opportunities and probing promising practices (Vol. 1, pp. 19–42). Washington, DC: National Education Association and TODOS: Mathematics for ALL.
van Es, E. A., Hand, V., Agarwal, P., & Sandoval, C. (2022). Multidimensional Noticing for Equity: Theorizing Mathematics Teachers' Systems of Noticing to Disrupt Inequities. *Journal for Research in Mathematics Education*, *53*(2), 114–132. https://doi-org.mutex.gmu.edu/10.5951/jresematheduc-2019-0018
Wager, A. A. (2014). Noticing children's participation: Insights into teacher positionality toward equitable mathematics pedagogy. Journal for Research in Mathematics Education, 45(3), 312–350.
Wilson, P. H., Sztajn, P., Edgington, C., & Myers, M. (2015). Teachers' uses of a learning trajectory in student-centered teaching practices. Journal of Teacher Education, 66(3), 227–244.
Wiseman, L., Allen, L.N., & Foster, E.(2013). The Multiplier Effect: Tapping the Genius Inside Our Schools. Corwin.

LEARNING TRAJECTORIES RESEARCH NEEDS A HARD RE-SET: USING PCTM TO CENTER COGNITION, CONTEXT, AND CULTURE

Marrielle Myers
Kennesaw State University
mmyers22@kennesaw.edu

Research on LTs remains a central topic in mathematics education. In this plenary paper, I argue that LT-based research needs a hard re-set if it is to play a role in creating more equitable and anti-oppressive experiences for historically marginalized students. I begin with an overview of LT-based research presented during PME-NA plenary sessions, which I examine through a lens of cognition, context, and culture. I assert that a continued focus on cognition reproduces the status quo and causes dissonance for many learners. I then discuss equity in LT research and how it has evolved throughout the years. Next, I offer Political Conocimiento in Teaching Mathematics (PCTM) as a framework that can support us in asking the complex sociopolitical questions needed to create liberatory spaces in mathematics teaching and learning. I end by inviting the field to commit to centering equity in their LT-based research as a political act.

Keywords: Equity, Inclusion, and Diversity; Social Justice; Teacher Educators; Learning Trajectories and Progressions

Introduction

For many of us, the past two years have been challenging mentally, emotionally, physically, and spiritually. There are multiple pandemics affecting our students. And while news of a looming recession and midterm elections may be dominating media outlets, the pandemics of racism, cisheteropatriarchy, redlining, xenophobia, ableism, wealth inequality, food insecurity, and climate change are still alive and well. Right-leaning states, politicians, and media are openly attacking supports aimed at remedying these pandemics. Capitalism continues to thrive at the expense of the very Black and Brown people whose ancestors built this country. Wars and threats of wars are happening around us. And while the United States stands ready to send money overseas to maintain its interests, it fails to protect its own citizens from police brutality or ensure clean drinking water as a human right.

Teachers and schools are also facing some of the greatest attacks we have seen in decades. Texas, Georgia, and Florida continue to compete in a "race for the bottom" as they seek to define and ban "divisive topics," create anti-woke laws, ban books, whitewash this nation's history, and further marginalize students who identify as LGBTQIA+ by developing policies intended to destroy their safety. Teachers are left exhausted by the pandemics, ongoing attacks, and fears of being sued or otherwise humiliated in any attempt to support students specifically harmed by these pandemics. Education remains a political pawn. Schools are underfunded. And politicians would rather spend tax dollars enforcing racist laws instead of paying teachers what they deserve. These conditions further exacerbate teacher burnout leading many school districts to start the year with an unprecedented number of vacancies.

Instead of critical mathematics education scholars being consistently asked to defend the relevance of their work as if the context of concurrent pandemics doesn't impact the teaching and learning of mathematics, we need to shift the conversation to examining how each of these cells of mathematics education research serves to maintain or liberate us from the multiple pandemics plaguing our nation. As we meet on these stolen lands, I implore our community to take up Aguirre et al.'s (2017) call for engaging in equity-oriented mathematics education research as a political act as well as what Ladson-Billings (2021) termed a "hard re-set." Ladson-Billings (2021) used the term hard re-set as a mantra and called for us to center students and culture in an effort to build a more humane future. Specifically, she stated:

> We must re-think the purposes of education in a society that is straining from the problems of anti-Black racism, police brutality, mass incarceration, and economic inequality. The point of the hard re-set is to reconsider what kind of human beings/citizens we are seeking to produce. (Ladson-Billings, 2021, p. 72)

LT-based research needs a hard re-set that can only be achieved if our community adopts a critical stance that centers equity. Doing so requires holding cognition, culture, and context together while using a critical lens. To date, LT research has privileged cognition at the expense of culture and context, and that disregard has led to inequitable uses. To make this point, I first discuss the history of LTs in plenary sessions at PME-NA in relation to cognition, culture, and context. Next, I establish my positionality and history with LT-based research. I then discuss the current state of equity-based approaches in LT research, and offer Political Conocimiento in Teaching Mathematics (PCTM) as a framework for our field to consider in order to enact the hard re-set needed (Gutiérrez, 2017). I end with questions and implications for the field.

Learning Trajectories at PME-NA

The North American Chapter of the Psychology of Mathematics Education (PME-NA) has a detailed history of centering LTs in its conference. In his 2010 plenary session and paper, Mike Battista spoke about the similarities and differences between Learning Progressions (LPs) and LTs, and what it meant for students to move through LPs/LTs (e.g., milestones, levels of sophistication). After highlighting differences in theoretical framings, the nature of levels, and the inclusion of instruction as a way to differentiate LTs and LPs, Battista turned to his work with Cognitive Based Assessments (CBAs). He noted that a CBA LP outlines students' conceptions, obstacles, plateaus, and mental processes needed to advance for a given topic (2010, p. 66). While Battista notes that movement through progressions is not unilateral because "students' learning backgrounds and mental processing differ[s]," there was no specific mention or acknowledgment of the sociocultural and political environment this instruction, assessment, and research validation occurred in. We are also left to wonder how Battista defines learning backgrounds and to what extent, if any, that definition captures the rich knowledge students bring from their homes and communities. Battista ended his paper by calling on researchers to exercise caution when using quantitative techniques to develop and validate LPs so as to not misapply such techniques or ignore how this thread of work interacts with research on learning (Battista, 2010, p. 69). It is noteworthy that Battista's caution to the field centered cognition and upholding principles of research methodology while tangentially addressing context and ignoring culture.

Susan Empson (2010) questioned the novelty and usefulness of LTs in Battista's plenary in an invited critique and reaction. After offering a summary of LTs and drawing on Simon's (1995) discussion of a hypothetical learning trajectory, Empson pondered LTs' place in teaching and research. In addition to posing questions for consideration, Empson acknowledged the importance of context when she cautioned the field not to underestimate the role of tasks, teachers, and teaching in LT research. She also reminded us to acknowledge disciplinary practices in the same way we focus on content, which is critical since LTs are tools used for teaching and ultimately derive meaning from classroom contexts. Finally, this paper noted that LTs ultimately need to be useful for teachers and that the creation and use of LTs must be an interactive process that involves careful study of how teachers use them. Empson's paper reminded us of the importance of context in cognition-based research. This paper did not explicitly address culture.

In 2012, Confrey offered a plenary paper summarizing how LTs were used in the development of the Common Core State Standards for Mathematics (CCSM). A major premise of this work was to support state leaders in distilling elements of the new standards into smaller pieces of information supported by research on student learning over time. Another key element of her paper was presenting components of LTs in ways that were useful for teachers, resulting in the creation of a hexagon map of K-8 mathematics standards. As Confrey described the hexagon map, she provided a rationale for its design, noting how different big ideas were connected (e.g., counting, addition, and subtraction), how some content supported the learning of other ideas (e.g., equipartitioning supporting the development of division and multiplication), and why some topics were visually clustered between others (e.g., length area and volume are nestled between equipartitioning on one side and shapes/angles on the other). This plenary offered a detailed analysis of a multiplication and division LT coupled with figures, strategies, and multiple representations. Confrey and team unpacked each trajectory by articulating conceptual principles, strategies, representations, misconceptions, meaningful distinctions in language, coherent structure, and bridging standards, which centered cognition.

Confrey explicitly stated that the hexagon map did not address the standards for mathematical practice (which could have inserted relevant connections to culture and context) but noted that students would surely use various practices as they progressed through the trajectory (2012, p. 8). She then posited that when LTs are properly unpacked and coordinated with standards, teachers can be better supported to connect underlying mathematical principles. She ended this paper by inviting the field to consider the usefulness of the hexagon maps as one example of what coherence across standards could look like, with the ultimate goal of supporting teachers as they transitioned to using the CCSS (Confrey, 2012). While the hexagon maps and associated unpacking offered valuable information related to cognition, a sociopolitical lens was not evident. As such culture was not discussed and context was not addressed. Given the social and political nature of teaching and how the CCSS were created, such connections would have been valuable as they could have contextualized this national movement and addressed teachers' concerns around new mandated curricula and tests that accompanied these standards.

The next plenary talk on LTs featured Julie Sarama in 2018. In a response to the conference theme, *Looking back, looking ahead: Celebrating 40 years*, Sarama discussed how mathematical knowledge developed in young children and shared brief highlights from her work. Her definition of LTs, which she noted is rooted in constructivism, acknowledged the importance of instruction and mathematical tasks. As Sarama unpacked the tenets of *Hierarchic Interactionalism*, she noted several points that were central to young childrens' innate skills and environment. She ended this chapter by offering an example of a student named Justin progressing through the LT-based *Building Blocks* curriculum. Sarama highlighted this student's growth in counting, addition, and subtraction as one example of how student thinking across multiple LTs is interactive and can grow concurrently (Sarama, 2018). Again, this plenary paper, and the original paper from which the example was drawn, centered cognition (and cognitive science) and did not provide insight on the context of Justin's "growth," his culture, who the teacher was, and the broader sociopolitical context this study occurred in.

In reflecting on these plenaries, Battista's took a cognitive approach. Empson pushed back and acknowledged context but left much unsaid about culture. Using a constructivist paradigm, Confrey and Sarama attend to context in limited ways (e.g., standards, curriculum) but did not include culture or other sociopolitical factors. My goal in providing this brief historical overview through a lens of cognition, context, and culture was not to freeze any of these scholars in time as working toward equity is a journey and not a destination. Rather, I sought to highlight how each of the plenary papers privileged cognition (when written) at the expense of context and culture, thus casting equity to the sidelines. While these scholars moved the field forward in monumental ways by disrupting understandings of how mathematical content is organized and reframing students as capable of rich mathematical thinking, they did not go far enough to disrupt other problematic strongholds in mathematics teaching and learning (e.g., tracking, low expectations). This review of previous LT-focused plenaries at PME-NA highlights the timeliness and necessity of the current conference theme, as scholars were specifically invited to consider their work through a sociopolitical lens. I hope this paper and resulting discussion contribute to a collective examination of why a hard re-set is needed in LT-based research if it is to play *any* role in leading toward a more "antioppressive and equitable human experience" in mathematics teaching and learning (Aguirre et al., 2017, p. 127).

Positionality

Before moving forward, it is important that I provide context on who I am and how I came to this work. I am not an outsider to LT-based research. In fact, I have an intimate history with LTs, and most of my time as a graduate research assistant for my master's and doctoral programs was spent on large-scale, NSF-funded, LT-focused grants. Earlier in my program, I worked on a research team to develop LTs for equipartitioning and rational numbers. As a graduate assistant, I conducted numerous clinical interviews and worked with my teammates to construct and validate LTs. Our team regularly met and engaged with other LT experts in the field, many of whom I cite in this paper, and worked to support the development of our state's mathematics standards. During that time, several concerns began to arise in the field around the construction and validation of LTs. Many of these questions were aimed at the diversity, or lack thereof, of the student population upon whom LTs were constructed and validated. Our team considered that feedback and began to intentionally recruit research participants in various settings to diversify our student sample. I recall being curious about this critique and wondering if students from different racial, cultural, and linguistic backgrounds would demonstrate different pathways through our sets of tasks. At that time, my equity lens was not sophisticated enough to recognize that engaging diverse students in a fairly "rigid" set of tasks was unlikely to produce different outcomes. In fact, as I reflect on this approach to diversifying the student sample in our research, I now see that we worked to accumulate more in the sample, rather than pause and fundamentally reorganize the research design. I was also only beginning to understand the impact of context and interlocking systems of oppression. And as such, I was not yet able to a) question the rationale for centering cognition, b) form arguments about how the clinical interview structure excluded context, c) understand how students intersectional identities influenced their work on our tasks, or d) understand how the social and political context of standards, funding, and other external forces impacted our research work.

After transitioning to another project, my focus shifted from developing and validating LTs in rational numbers to designing LT-based professional development. While on this project, our team worked on translating and coordinating early grades LTs focused on number, counting, addition, and subtraction, into useful tools for teachers across grades K-5. This research project, titled *Learning Trajectories Based Instruction* (LTBI), served as the basis for my dissertation. As we sought to understand and develop a model for *how* teachers learned to use LTs, I, along with other team members, became interested in how teachers' implementation of LTBI looked different across various subsets of their student population. I had grown as a person and a scholar and was much more aware of equity, justice, and how the systemic nature of marginalization in schools maintained opportunity gaps and systematically excluded minoritized learners.

After ending my high school teaching career to complete my doctorate full-time, I was much more attentive to culture and context. I had also grown increasingly frustrated with the ways "new trends" and "innovative curricular materials" in mathematics teaching and learning yielded the same results year after year (e.g., opportunity gaps, tracking). Because of the national attention LTs were receiving at the time, and how they were being used to develop curriculum and assessment, I wanted to be proactive in considering how they could be used equitably. Therefore, I worked with my team to articulate a theory of Equitable Learning Trajectory Based Instruction (E-LTBI), which I then investigated in a case study of four teachers for my dissertation (Myers, 2014). This E-LTBI framework resulted from simultaneously considering Gutiérrez's (2007) four dimensions of equity and our existing LTBI framework (Myers et al., 2015). I share more about this work in my review of LTs and equity after providing a brief overview of LTs/LPs and critiques.

Learning Trajectories

Several definitions of LTs have been offered in the field. Clements and Sarama (2004) define LTs as

> descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (p. 83)

Confrey and Maloney's (2010) definition of LTs features similar language but notes that trajectories a) are empirically supported, b) include activities, tools, and assessments in addition to tasks, and c) highlight the iterative nature movement, reflection, and refinement as students move from informal understandings to formal ideas (p. 2). Research around LTs exists in three primary areas: development and validation (constructing LTs in different domains and content strands) (Battista, 2004; Blanton et al., 2015; Confrey et al., 2009; Maloney & Confrey, 2010; Gravemeijer et al., 2003; Fonger et al., 2020), informing instructional tools (e.g., standards, curriculum, and assessment) (Clements, 2002; Clements & Sarama, 1998; Confrey, 2012; Daro, Mosher, & Corcoran, 2011; Mosher, 2011), and, more recently, professional learning for teachers (Bargagliotti & Anderson, 2017; Clements & Sarama, 2009; Edgington, 2012; Sarama et al., 2016; Suh & Seshaiyer, 2015; Sztajn et al., 2012; Wickstrom, 2014; Wilson et al., 2013; Wilson et al., 2015; Wilson et al., 2017). Lobato and Walters (2017) conducted a detailed review of research on LTs and LPs in mathematics and science education. They produced a taxonomy of approaches to learning trajectories and progressions, which they refer to as LT/Ps. At each level, they described the approach, offered an example, highlighted the features, outlined the methods used, and discussed the purpose, benefits, and tradeoffs. I invite readers to study the full paper to learn more about the breadth and depth of research around developing and validating various LT/Ps.

It is important to note that I do not seek to offer a distinction between LTs and LPs nor advocate for one over the other. I encourage readers interested in the LT vs. LP discussion to read Battista (2010) and Ellis, Weber, & Lockwood (2014), as both papers offer a detailed account. The relevant similarity from my perspective is that while LTs and LPs center cognition and narrowly reference context (in noting the importance of carefully selected tasks and pedagogical moves), neither body of research explicitly addresses the social, cultural, or political context in which the research, validation, creation, and intended uses occurred. Moreover, I argue that both LTs and LPs offer a narrow definition of what mathematics is, whose mathematics is privileged, and why we engage with it, thus missing an opportunity to expand the view of what counts as mathematics (Aguirre et al., 2017).

Critiques of LTs (and LPs)

Critiques of LTs and LPs in mathematics and science education are not new. In the National Research Council (NRC) report *Taking Science to School*, the authors provide an overview of teaching and learning science in grades K-8. In their chapter on learning progressions, much of which aligns with LTs mathematics education, the committee highlighted how LPs can be used to map students' understandings and unify science topics that have previously been disconnected. The committee ended this chapter by discussing the design challenges of LPs and stated,

> No single learning progression will be ideal for all children, since they have different instructional histories, bring different personal and cultural resources to the process of learning science, and learn in different social and material environments. The best learning progressions are those that make effective use of the resources available to different children and in different environments. This is the challenge that we are farthest from responding to effectively with the current research base. (NRC, 2007, p. 222)

The committee later noted that although they recognized inequities in science education and the dire need to address them, they were unsure about what recommendations to make related to modifying instruction for diverse learners. Their suggested agenda for future action included focusing on the effectiveness of different instructional strategies, the unpacking of systemic inequities across schools, and the need for specific research that examined the complexity of culture, language, and socioeconomic status (NRC, 2007). This report summarized the cognitive aspects of LPs in science and pointedly expressed the absence of context, culture, and other sociopolitical constructs.

In a 2011 paper, an expansion of her 2010 plenary response, Empson questioned what LTs afforded, foregrounded, and obscured, which parallels the current conference theme of dissonance and harmony. I appreciated that this more detailed analysis considered both promises and pitfalls of LTs by making the case that learning is as much contextual and social as it is cognitive. Empson went on and acknowledged that teaching was a relational act that "depends fundamentally on interpersonal relationships of trust and respect." (Empson, 2011, p. 587). What was underdeveloped in this paper was the explicit unpacking of the word *contextual* and the historical and political nature of those contexts. Additionally, when children's differences were alluded to in the text, the words race, gender, culture, language, sexual orientation, or ability were not explicitly mentioned. When we do not intentionally name the different elements of students' intersectional identities we can inadvertently reify some scholars' beliefs that LTs only need to focus on "cognitive differences." I argue that while this critique pushed for the inclusion of context (and culture to some degree) in LT research, a sociopolitical could have strengthened this critique. Just as Empson argued that learning cannot be separated from teaching, I argue that teaching cannot be separated from the teacher. Since teachers hold a range of beliefs and biases about who can do mathematics and who should be afforded opportunities for “rigorous” mathematics, neglecting to use a sociopolitical lens secures an oppressive, anti-black, anti-immigrant, anti-poor, anti-LGBTQ, ableist system. We must explicitly engage teachers’ beliefs as we center context and culture in conversations about LTs.

In their examination of equity in LT/LP research, Delgado and Morton (2012) analyzed existing research using a cognitive constructivist framework. They noted that this framework was aligned with their definition of equity. These authors pushed against “equity for all” (or dominant framing of equity) and embraced a postmodern definition of equity that “acknowledges existing inequities in society [and] proposes responsive, individualized attention to students in order to compensate for past lack of opportunities and to promote social justice” (Delgado & Morton, 2012, p. 205). As they examined LT/LP research that explicitly considered “issues of equity,” the authors found that student populations lacked diversity (or didn’t report any demographic information), only focused on common themes amongst students’ mathematical ideas instead of capturing all students’ ideas, and failed to consider how students’ family and community knowledge shaped their engagement in tasks and resulting movement through a trajectory. The authors concluded their paper and stated,

> Research groups developing LPs and LTs should ideally include advocates for certain groups of students, for example, an expert on special education and team members that are deeply knowledgeable about the culture of minority students. Developing learning progressions and learning trajectories that do not address inequity in educational opportunities in math and science for students will only exacerbate the current problem. As the learning sciences, science education, and mathematics education fields continue to negotiate and define the nature of LPs and LTs, an expansion to include equity concerns at the forefront can greatly benefit groups that have been traditionally underserved. (Delgado & Morton, 2012, p. 209)

LTs and Equity

In this section, I present three ways “equity” has been addressed in LT-based research. First, I highlight Sarama & Clements’ body of work as an example of a dominant framing of equity as it primarily focuses on access and achievement (Gutiérrez, 2007). I also discuss how their attention to equity has shifted over time to include some critical framings of equity. Next, I discuss equitable uses of LTs by highlighting two cases: LTBI and Suh et al. (2022). After showing how Clements and Sarama’s work influenced my dissertation study and led to the development of the E-LTBI framework, I transition to the work of Suh and colleagues who built from our LTBI findings and intentionally embedded equity in their LT-based PD model. I conclude by highlighting the work of Zahner and Wynn (2021), who centered equity in an attempt to address gaps in representation in LT development.

Dominant Framing. One area of LT & equity research focuses on how LTs can offer access and support achievement for minoritized students. The body of work of Clements and Sarama represents decades of research and tens of millions of dollars of grant funding from large-scale funders (e.g., The National Science Foundation and Institution of Education Sciences), which led the development of curriculum (Clements & Sarama, 1998; Sarama & Clements, 2019), conferences, and the creation of research centers. Because these scholars' definition of LTs and the resulting body of work has been so influential in LT-based research, I draw on it as one example of a dominant framing of equity. Consider the large-scale randomized trial that was conducted and published in several venues (Clements et al., 2013; Sarama et al., 2012). The authors noted that they chose their research site because “children from low-resource communities and who are members of ethnic and linguistic minority groups demonstrate significantly lower levels of mathematics achievement than children from higher-resource, nonminority communities" (Clements et al., 2012, p. 2). The authors went on to note that their LT-based PD model, "include[d] guidelines for promoting equity through the use of curriculum and instructional strategies that have demonstrated success with underrepresented populations" (Clements et al, 2012, p. 4). Findings from this and similar studies indicated that African-American students in their experimental groups scored significantly higher than their counterparts in control groups. One conclusion of this study was that "centering instruction around LTs may focus teachers' attention on students' thinking and learning in mathematics rather than their memberships in ethnic groups and thus avoids perceptions that negatively affect teaching and learning" (Clements, Sarama, Wolfe & Spitler, 2012, p. 26). In their 2014 book chapter, Clements and Sarama stated, "several "gold standard" randomized control trial studies have shown that curricula and professional development based on learning trajectories increase children's achievement more than those that do not." (p. 7). They went on to say, "learning math at an early age is critically important for young children, especially those from disadvantaged communities" (Clements & Sarama, 2014, p. 8).

Clements and Sarama's attention to and expression of equity continued to grow and expand throughout the years. They sharpened their perspective by explicitly addressing six myths about LTs, three of which are germane to this analysis (Clements & Sarama, 2017/2019). First, they argued that LTs are asset-based because they help teachers recognize and build upon students' thinking. Next, they defended critics' notions of LTs being narrowly focused by noting that LTs are "deeply constructivist" and address broad ranges of ideas. Finally, they stated, "Learning trajectories are expressly built to be adaptable to different cultures, groups, and individuals. One important adaptation is for different cultures. Learning trajectories take funds of knowledge from all communities seriously and encourage using such funds" (Clements & Sarama, 2017/2019, p. 2).

More recently, Clements et al. (2020) suggested that teachers who know how to use the three components of an LT are better suited to understand the complexity of early mathematics content and offer instruction that is more closely aligned with students' current conceptions, thus providing more robust mathematics experiences for all children. They stated that such environments are necessary for "vulnerable children who live in poverty, are members of linguistic and ethnic minority groups, or…children with disabilities" (Clements et al., 2020, p. 1). They suggested that early-childhood teachers could benefit from sustained PD focused on learning trajectories that also included direct support for engaging children with learning disabilities. The authors ended this paper by announcing their STEM Innovation for Inclusion in Early Education Center, which they noted is a critical step in ensuring equity and excellence in early STEM experiences. Ongoing work from this team continues to suggest that LT-based PD positively impacts students from "low-resource communities" (Sarama, Clements & Guss, 2021).

As I followed this and other bodies of LT-based research over the years, I observed how the discussions of context and culture have both evolved by expanding the attention given to equity and, in some cases, remained stagnant by only considering equity in relation to student achievement. I have paid particular attention to how students were described and positioned in these and other studies. I would encourage Clements and Sarama to consider how the language they used to dispel myths in their 2017/2019 resource document may be at odds with how often students and their communities are referred to as low resource, vulnerable, and minority. Although those phrases were often used as demographic descriptions, doing so without a sociopolitical lens may serve to reify the deficit orientations Clements & Sarama seek to disrupt. It is also important to note how context was included (e.g., as a description or as a mediating variable in a statistical model, etc.) and whether or not the context of the study was situated in the historical context of schools and schooling in the United States (e.g., critical explanations of sociopolitical factors that intentionally created disadvantaged or low-resource communities). Finally, I invite the reader to carefully consider how this dominant view of equity and its evolution was inextricably tied to cognition, which overwhelmingly excluded critical discussions of context and culture.

Equitable Usage of LTs. A second area of equity and LT research focuses on the ways LTs are used in instruction. The Learning Trajectories Based Instruction (LTBI) research project (for which I was a graduate research assistant) is one such example (Sztajn et al., 2012). This study used Clements and Sarama's early number, counting, addition, and subtraction LTs in a multi-year research project with K-2 teachers. In our 2015 paper *From implicit to explicit: Articulating equitable learning trajectories based instruction*, my colleagues and I argued that although we initially considered our LT-based research with teachers to attend to issues of equity, what we learned in our work caused us to reconsider some of those assumptions (Myers et al., 2015). Similar to other cognition-focused teacher learning models, our project centered students' thinking, disrupted notions about the "traditional sequencing of mathematics," and created space for students' individual thinking to emerge and be positioned as valuable along various trajectories. Teachers in our study deepened their knowledge of K-2 mathematics content, appreciated the language the trajectory afforded them, and began to recognize and value a range of students' mathematical contributions (Edgington, 2012; Myers, 2014; Wilson et al., 2015). In this sense, one assumption of our work built on Clements et al. (2012) suggestion that by using LTBI, teachers would see that all children were capable of mathematics, potentially reducing their focus on other demographic factors.

Unfortunately, the suggestion that focusing on cognition could reduce attention to other factors did not hold true in our work, highlighting how good intentions and race-evasive approaches are not enough to effect radical change across diverse groups (Rodriguez, 2003). As a result, we saw LT language taken up and used oppressively such as when teachers replaced the language of "low students" and "high students" with LT-based vocabulary (e.g., the low students being renamed the direct counters). Results from my dissertation highlighted that most LT-based work aligned with a dominant approach to equity (Myers, 2014) and that the elements that connected to Gutiérrez's (2007) critical axis were shallow or maintained dominant framing. Moreover, in my paper titled, *The unintended consequences of a learning trajectories approach*, I reported on a teacher, Elizabeth, who possessed several deficit orientations about her students. This teacher also shared that she wasn't confident in her own mathematics knowledge and therefore taught science instead of mathematics to her kindergarten students. Although this teacher made "gains" in her content knowledge and began offering LT-based instruction in her classroom, several issues emerged. A primary concern was that this teacher's deficit orientations about students overshadowed what she learned about LT-based instruction. This teacher ultimately used knowledge gained in the PD to justify retaining kindergarten students from minoritized groups, and the LT-based language she acquired in our sessions provided "credibility" to her decisions as a teacher (Myers, 2015).

It is important to note that my dissertation study sought to examine what equity may look like as a *by-product* of participating in LT-based PD, as equity was not explicitly centered in the PD design. Building upon findings from LTBI-based research, Suh and colleagues considered that

LT-based PD was not enough to disrupt teachers' beliefs about minoritized students. Therefore, she and her team intentionally embedded equity in their LT-based teacher learning study design, simultaneously centering cognition, context, and culture. They engaged teachers in professional development focused on LTs, asset-based instruction, and cognitive demand (Suh et al., 2022). They hypothesized that pairing LTs with equity-oriented and anti-deficit frameworks for noticing (Louie et al., 2021; van Es et al., 2022) would help teachers recognize students' multiple knowledge bases. They suggested that this framework would support teachers to assign mathematical competence to their students. The authors noted that teachers in their study moved beyond discussing "gaps" in students' understanding to using what they referred to as "strength-based language" (Suh et al., 2022). It is unclear from these findings if the shift from "gap-based language" to "strength-based language" reflected a change in teachers' beliefs or if teachers were merely using the "new language" that had become normalized in the professional learning space, potentially as a proxy for previously held viewpoints (Myers et al., 2013; Wilson et al., 2017). The field can benefit from continued research that examines how similar findings (e.g., strength-based language) relate to broader conceptions of socially just and anti-oppressive learning environments.

Representation in LT Development. A final area of equity and LT research focuses on who is used in constructing LTs. In noting the absence of diversity in the student population of much LT-based work, Zahner & Wynn (2021) conducted clinical interviews with 23 multilingual students using LT-based tasks focused on proportional reasoning and linear functions. This study of twenty-five ninth-grade students (primarily Latinx, Asian, and African American), ten of which were multilingual, provided valuable insights into how the linguistic complexity in mathematics tasks impacted how students approached tasks and explained their reasoning, thereby influencing their potential "ranking" or placement on a content-focused LT. Their findings call into question the role of language in previous large-scale work conducted to create, norm, and validate LTs. Given that many initial LT-based studies did not consider language or the linguistic complexity of tasks in the study design, we are left to wonder how students interpreted tasks. Even when scholars acknowledged demographics (e.g., language status) in large-scale studies they typically neglected to present analysis around how language mediated performance on standardized tests. Zahner and Wynn's research is an example that culture and context can be centered while simultaneously investigating cognition.

Summary

For organizational purposes, I presented these two sections (critiques of LTs and equity in LT-based research) separately. Readers should note that some of this work occurred concurrently and that some of the shifts in how equity was presented in LT-based research were in direct response to the ongoing critiques of equity in LT research (e.g., Empson, 2010, NRC, 2007, Sztajn and Wilson, 2019), LT-based conferences and working groups, and other conversations that have continued in the field. Despite a small shift in how scholars have attended to equity in their LT work, cognition is still the focus, and I argue that we have not yet met the call of centering equity in LT research as both a political act and a collective responsibility. I also note that in order for an LT-based re-set to happen in ways that honor Aguirre et al.'s (2017) call, we must pause to unpack a) how discussions of "equity" have (or have not) evolved in LT-based work, b) whether we have been intentional about acquiring the knowledge necessary to make the shifts in genuine ways, c) if our work focuses on the full humanity of students' experiences and moves beyond acknowledging their test scores, and d) how we have chosen to foster and cultivate critical collaborations that value the expertise of a range of scholars, colleagues, and families. In what follows, I describe PCTM and suggest that this framework can help us ask the complex, sociopolitical questions needed (Gutiérrez, 2013) for us to examine LT-based as we consider how cognition, context, and culture are necessarily entangled.

Using PCTM to Critically Consider Cognition, Culture, and Context

In honoring the conference theme and Aguirre et al.'s (2017) call to be intentional about discussing interlocking systems of oppression with colleagues, I suggest that Political Conocimiento in Teaching Mathematics (PCTM) can provide the field with a lens to examine LT-based research and ask complex questions. Because PCTM explicitly engages content, pedagogy, social context, and politics, it is suitable for examining LT-based research across its three primary domains (e.g., development and validation, informing instructional tools, and professional learning with teachers). For too long, large-scale research projects (e.g., LTs and Cognitively Guided Instruction) have been conducted in mathematics education and taken up by the mainstream without engaging a sociopolitical lens during conception. Equity-oriented scholars have then dedicated their careers to proving how the initial work was not grounded in equity while also considering how to "re-mix" the research and curricula to meet their justice-oriented agendas (Maldonado et al., 2022). My goal in offering this framework is to call for an end to this two-phased approach that centers cognition first and then leaves equity-oriented scholars to consider culture and context. After describing PCTM, I unpack questions posed in the conference theme to illuminate the power of this theoretical framework.

Political Conocimiento in Teaching Mathematics

PCTM is a theoretical framework that highlights the unique ways that mathematical knowledge for teaching (MKT), pedagogical content knowledge (PCK), knowledge of students, and political knowledge are entangled (see Figure 1) to produce a unique way of *knowing* that teachers (and researchers) need to consider as they teach and conduct research, especially with historically marginalized students. Here, I list two elements of this framework that make it useful to critically analyze LT-based teaching and learning. First, PCTM explicitly links content, pedagogy, knowledge with students and communities, and political knowledge (e.g., power dynamics) and situates the entanglement of each of these in a social and historical context (Gutiérrez, 2012). This linkage does two things. One, it dispels the "false dichotomy" between mathematics and equity that some scholars assert (Aguirre et al., 2017), making it inconceivable to conduct research on cognition and teaching without attending to context and culture. Two, this linkage asserts that political knowledge is not merely added to the other dimensions. On the contrary, engaging a political lens causes us to re-examine the other components (Gutiérrez, 2012). As such, one would not consider developments related to content and pedagogy (e.g., develop a LT-based curriculum) and then question how to use it with diverse learners. Instead, this framework embraces the tensions that exist when these individual pieces are entangled, thereby allowing us to ask richer questions and reflect on decisions we make to foreground or background different dimensions (Myers, Gutiérrez & Kokka, in press). Using such a framework at the onset of LT-based research would have eliminated many of the critiques that came later since culture, race, language, communication patterns, task design, teacher and researcher identity, etc., would have all been considered in the development and validation phases. Embracing this tension allows us to see that dissonance and harmony can coexist while we also question who experiences dissonance and who experiences harmony (Gutiérrez, 2009a).

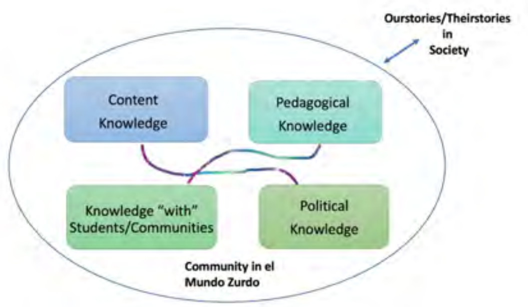


Figure 1. Political Conocimiento in Teaching Mathematics (Gutiérrez, 2012)

Second, the word conocimiento is drawn from the Spanish verb conocer. The Spanish word *saber* represents a "fixed" kind of knowing (e.g., to know facts, how to follow steps, to know a piece of information). The verb *conocer,* however, represents a fluid type of knowing that is contextual (e.g., to know a person, to know of a place). This distinction is critical as *conocer* asserts that context and conditions are essential to framing *how* we know (Gutiérrez, 2017). This kind of knowledge prevents us from objectively knowing "low-income students," "minority students," or "English language learners" in a fixed or homogenous way and, as such, applying a "best practice" to that population with the hopes of minimizing an achievement gap. A teacher or researcher using PCTM as a lens would first question how a "best practice" was developed. That person would ask, "best for whom? Best under which conditions? Best to what end? Best for what outcome?" Using this type of framework also problematizes the treatment of groups of students as static objects of our research and suggests that knowledge must be co-constructed with learners, families, and community partners. Reframing knowing as relational also reminds us that how we *know* is always changing because our environments and sociopolitical contexts are ever-changing. Moreover, as scholars, our *knowing* about LTs is also fluid as we continue to re-negotiate our work in light of new understandings about our intersectional identities, new scholarship, etc. PCTM is undoubtedly a powerful theoretical framework that the field can use to consider complex questions similar to those posed in this year's conference theme. In the next few paragraphs, I consider two of the thematic questions through the lens of PCTM.

How does LT-based teaching challenge a settled mathematics learning status quo?

Using PCTM allows us to answer this question in two ways. First, it is important to acknowledge the historical and political contexts that afforded the "mathematics learning status quo" to be created and maintained for so many years, leading to dissonance. Our field can benefit from engaging in discussions around how mathematics has been socially constructed in a way to maintain systems of oppression (Gutiérrez, Myers & Kokka, 2022, in review). Using PCTM forces us to unpack the political nature of the status quo, understand who is negatively affected by it, and develop a comprehensive approach to address the problem instead of accepting it as the norm or rushing to a "quick fix."

Second, PCTM allows us to consider how questioning mathematical content (thereby questioning mathematical content knowledge) can lead us to work toward harmony. Given that the four elements of the PCTM framework are entangled and that the resulting knowledge is relational, pulling one thread in an attempt to disrupt a status quo necessarily challenges our ways of knowing, allowing us to consider the other dimensions and reimagine mathematics learning more broadly. For example, several studies demonstrated that LT-based professional development supported teachers in understanding the complexity of mathematics content, how underlying ideas were connected, that mastery was not a prerequisite for more sophisticated ideas, and that informal understandings were valuable prerequisites for building more complex ideas. This work is important. What was missing from those conversations is specific research on how changes in teachers' MCK interacts with the other elements of PCTM, to potentially disrupt other status quos in mathematics teaching and learning. For example, does the “asset-based” lens teachers acquired during LT-based PD support them in reframing their understanding of the mathematical practices (e.g., centering a range of communication and argumentation styles, debunking traditional notions of precision) through a sociopolitical lens? Moreover, even when teachers develop an “asset-based” lens as a result of LTs and constructed counternarratives about students’ thinking, what did this mean for how teachers recognized and valued students’ humanity? Did “disrupting inequity” in mathematics thinking lead teachers to advocate for greater change? And did changes in beliefs, if any, persist over time? If LT-based teaching continues to treat mathematics as disconnected from students, families, communities, and contexts, how can we expect to truly eradicate inequity in mathematics teaching and learning?

How does LT-based teaching have an impact on society more broadly, beyond individual mathematics classrooms and school districts?

This question naturally engages each element of the PCTM framework as we consider how LT-based teaching and learning (content and pedagogical knowledge) *might* impact society (students, communities, politics) more broadly (our/theirstories). I argue that, to date, much of what we have seen in LT-based teaching supports long-standing notions that mathematics is a neutral and culture-free domain. As we examine mainstream curricula and approaches to teaching mathematics, we see that there is still a focus on drill and memorization, even though worksheets have been traded for digital tools. PCTM helps us see that school-based mathematics is still privileged at the expense of home, community, and place-based mathematical knowledge. We have not considered how to use our collective power to disrupt standardized testing and its oppressive effects. Instead, much LT-based work has been advertised in support of helping our students perform better on tests. PCTM can support scholars in thinking about using LT-based research to “play the game” and “change the game” (Gutiérrez, 2009b). This question reminds us that because teaching happens in classrooms, which are housed in schools, which are located in communities, which are a part of society, any classroom level teaching and research ultimately has an impact on these other spaces. Whether that impact upholds the status quo or redistributes power is a question scholars delve into while remembering our moral obligations (Stephan et al., 2015) and embracing a “productively self-critical” disposition (Kilpatrick, 2013, p. 73 as cited in Larnell et al., 2016).

Conclusion

Before I close, I return to the case of Elizabeth, who used what she learned in LT-based PD to justify retaining historically marginalized students in kindergarten. As I mentioned, this teacher showed growth in her content knowledge. She also used what she learned to attend to students’ thinking and plan next steps aligned with the trajectory. But when her “content-focused reform efforts” didn’t produce the results she expected to see on students’ quarterly benchmark tests, deficit narratives entered the conversation (e.g., if they weren’t eating free breakfast in the morning we could do extra practice, their parents don’t spend enough time with them at home). What was missing? How did Elizabeth need to be supported to question the usefulness of an LT-based approach across all students? What tools did Elizabeth need to support her so that she could ask *questions about the nature of standardized testing* instead of asking *questions about her students and their families*? Was Elizabeth ever providing “equitable and high-quality instruction” if these comments were indicative of her beliefs about students? And what does it mean that Elizabeth chose to retain students instead of advocating for them? Frameworks like PCTM help us ask these questions about Elizabeth's case. Part of our moral obligation as scholars is to ensure that we consider how cognition, context, and culture are entangled and the type of professional learning experiences teachers need to understand the political nature of their work, not just the cognitive aspects. LT-based PD alone is insufficient for ensuring equity and justice.

Mathematics teaching, learning, and research are political acts (Aguirre et al., 2017; Larnell et al., 2016). And as such, we must use care in conducting our work and consider what's at stake when we don't approach our research critically. In this paper, I built the case for using PCTM as a theoretical framework to examine LT-based research to support the hard re-set needed if LTs are ever going to be relevant in creating a more humane and just mathematics experience for historically marginalized students. I also submit that it is necessary to pause, reflect, and engage in the self-work and education needed to prepare for this re-set. This paper contributes to that pause by adding to critical discussions about LT-based research and suggesting a theoretical framework that can support our collective efforts. Despite increasing explicit attention to equity in LT research, a sociopolitical lens is still needed as we grapple with considering LTs at the intersection of cognition, context, and culture. We need to continue to unpack the various definitions of equity that guide our work and engage in conversations across research paradigms to build critical LT-based research models. And while we cannot change the LT research that has come before us, we can strengthen our commitment to equity and justice by asking more complex questions that critically hold cognition, context, and culture together.

References

Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements, D. H. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education*, *48*(2), 124–147. https://doi.org/10.5951/jresematheduc.48.2.0124

Bargagliotti, A. & Anderson, C. (2017) Using Learning Trajectories for Teacher Learning to Structure Professional Development. *Mathematical Thinking and Learning*, *19*(4), 237-259, DOI: 10.1080/10986065.2017.1365222

Battista, M. (2010). Representations of learning for teaching: Learning progressions, learning trajectories, and levels of sophistication. In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), *Proceedings of the 32nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 60-71. Columbus, OH: The Ohio State University.

Battista, M. T. (2004). Applying cognition-based assessment to elementary school students' development of understanding area and volume measurement. *Mathematical Thinking and Learning, 6*(2), 185-204.

Blanton, M., Brizuela, B. M., Gardiner, A. M., Sawrey, K., & Newman-Owens, A. (2015). A learning trajectory in 6-year olds' thinking about generalizing functional relationships. *Journal for Research in Mathematics Education*, 46(5), 511–558.

Clements, D. H. (2002). Linking research and curriculum development. *In L. D. English (Ed.), Handbook of international research in mathematics education* (pp. 599–636). Mahwah, NJ: Erlbaum.

Clements, D. H. & Sarama, J. (2017/2019). Learning and teaching with learning trajectories. Retrieved from Marisco Institute, Morgridge College of Education, University of Denver. Website: https://www.learningtrajectories.org/documents/1602020021691.pdf

Clements, D. H. & Sarama, J. (2014). The importance of the early years. *In R. E. Slavin (Ed.), Science, technology & mathematics (STEM)* (pp. 5-9). Thousand Oaks, CA: Corwin.

Clements, D. H. & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York, NY: Routledge.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89. https://doi.org/10.1207/s15327833mtl0602_1

Clements, D. H., & Sarama, J. (1998). Building Blocks—Foundations for mathematical thinking, pre-kindergarten to grade 2: Research-based materials development. Buffalo, NY: State University of New York at Buffalo.

Clements, D. H., Sarama, J., Wolfe, C. B., & Spitler, M. E. (2012). Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies: Persistence of effects in the third year. *American Educational Research Journal,* Retrieved from: https://www.researchgate.net/publication/254325922_Longitudinal_Evaluation_of_a_Scale-Up_Model_for_Teaching_Mathematics_With_Trajectories_and_Technologies_Persistence_of_Effects_in_the_Third_Year

Clements, D. H., Vinh, M., Lim, C.I., & Sarama, J. (2020). Stem for inclusive excellence and equity. *Early Education and Development*. https://doi.org/10.1080/10409289.2020.1755776

Confrey, J. (2012). Articulating a learning science foundation for learning trajectories in the CCSS-M. In Van Zoerst, L. R. Lo, J. J. & Kratky, J.L. (Eds.). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology Mathematics Education* (pp. 2-20). Kalamazoo, MI. Western Michigan University.

Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. *In Gomez, K., Lyons, L., & Radinsky, J. (Eds.) Learning in the disciplines: Proceedings of the 9th international conference of the learning sciences (ICLS 2010),* Vol. 1. International Society of the Learning Sciences: Chicago IL.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009). Equipartitioning/Splitting as a Foundation of Rational Number Reasoning Using Learning Trajectories. *Paper presented at the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Thessaloniki, Greece.

Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction*. (Research Report No. 68). Madison, WI: Consortium for Policy Research in Education.

Delgado, C. & Morton, K. (2012). Learning Progressions, Learning Trajectories, and Equity. In van Aalst, J., Thompson, K., Jacobson, M. J., & Reimann, P. (Eds.), The Future of Learning: *Proceedings of the 10th International Conference of the Learning Sciences (ICLS 2012)*, Vol. 1, (pp. 204-211). Sydney, Australia.

Edgington, C. (2012). Teachers' uses of a learning trajectory to support attention to student thinking in the mathematics classroom. (Unpublished doctoral dissertation). North Carolina State University, Raleigh NC.

Ellis, A. B., Weber, E., & Lockwood, E. (2014). The case for learning trajectories research. *In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 1–8). Vancouver, Canada: PME.

Empson, S. (2011). On the idea of learning trajectories: Promises and pitfalls. *The Mathematics Enthusiast, 8*(3), 571–596.

Empson, S. (2010). A Critique and Reaction to Representations of learning for teaching: Learning progressions, learning trajectories, and levels of sophistication (Optimizing research on learning trajectories). *In P. Brosnan, D. B. Erchick, & L. Flevares (Eds.), Proceedings of the Thirty-Second Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 60-71. Columbus, OH: The Ohio State University.

Fonger, N., Ellis, A., Dogan, M. (2020). A quadratic growth learning trajectory. *Journal of Mathematical Behavior, 59*, 1-22. DOI: https://doi.org/10.1016/j.jmathb.2020.100795

Gravemeijer, K., Bowers, J., & Stephan, M. (2003). A hypothetical learning trajectory on measurement and flexible arithmetic. *Journal for Research in Mathematics Education*. Monograph, 12, 51–66.

Gutiérrez, R. (2017). Political conocimiento for teaching mathematics: Why teachers need it and how to develop it. In Kastberg, S., Tyminski, A. M., Lischka, A., & Sanchez, W. (eds.), *Building support for scholarly practices in mathematics methods (pp. 11-38)*. Information Age Publishing.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, *44*, 37–68.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In Choppin, J., Herbel-Eisenmann, B., & Wagner, D., (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies*, pp. 17-33. New York: Springer.

Gutiérrez, R. (2009a). Embracing the inherent tensions in teaching mathematics from an equity stance. *Democracy & Education*, 18(3), 9–16.

Gutiérrez, R. (2009b). Framing Equity: Helping Students "Play the Game" and "Change the Game." *Teaching for Excellence and Equity in Mathematics*, 1(1), 4-8.

Gutiérrez, R. (2007). Context matters: Equity, success, and the future of mathematics education. *Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1–18). Lake Tahoe, NV.

Gutiérrez, R., Myers, M. & Kokka, K. (2022, in review). The Stories We Tell: Why Unpacking Narratives of Mathematics is Important for Teacher Conocimiento.

Ladson-Billings, G. (2021) I'm here for the hard re-set: Post pandemic pedagogy to preserve our culture. *Equity & Excellence in Education, 54*(1), 68-78, DOI: 10.1080/10665684.2020.1863883

Larnell, G. V., Bullock, E. C., & Jett, C. C. (2016). Rethinking Teaching and Learning Mathematics for Social Justice from a Critical Race Perspective. *Journal of Education*, *196*(1), 19–29. https://doi.org/10.1177/002205741619600104

Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 74–101). NCTM.

Louie, N., Adiredja, A., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. *ZDM Mathematics Education*. https://doi.org/10.1007/s11858-021-01229-2

Maldonado Rodríguez, L.A., Jessup, N., Myers, M., Louie, N., & Chao, T. (2022). A critical lens on Cognitively Guided Instruction: Perspectives from mathematics teacher educators of color. *Mathematics Teacher Educator*, 10(3), 191-203. https://doi.org/10.5951/MTE.2020.0015

Maloney, A. & Confrey, J. (2010). The Construction, Refinement, and Early Validation of the Equipartitioning Learning Trajectory. In Gomez, K., Lyons, L., & Radinsky, J. (Eds.), *Learning in the Disciplines: Proceedings of the 9th International Conference of the Learning Sciences* (ICLS 2010). Vol. 1, (pp. 968-975). Chicago IL: International Society of the Learning Sciences.

Mosher, F. A. (2011, September). The role of learning progressions in standards-based education reform. *Consortium for Policy Research in Education: Policy Briefs*. Retrieved from http://repository.upenn.edu/cpre_policybriefs/40/

Myers, M. (2014). The use of learning trajectory based instruction (LTBI) in supporting equitable teaching practices in elementary classrooms: A multi-case study (Unpublished doctoral dissertation). North Carolina State University, Raleigh, NC.

Myers, M., Edgington, C., Wilson, P.H., & Sztajn, P. (2013). Teachers' positioning of students in relation to ability/achievement in a professional development setting. *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Chicago, IL.

Myers, M., Kokka, K. & R. Gutiérrez. (in press). Maintaining tensions: Braiding as an analogy for mathematics teacher educators' political work. *Association of Mathematics Teacher Educators Professional Book Series* (volume five).

Myers, M., Sztajn, P., Wilson, P. H., & Edgington, C. (2015). From implicit to explicit: articulating equitable learning trajectories based instruction. *Journal of Urban Mathematics Education, 8*(2), 11–22.

National Research Council (2007). Taking science to school. Washington, DC: National Academy Press.

Rodriguez, A. J. (2003). "Science for All" and Invisible Ethnicities: How the Discourse of Power and Good Intentions Undermine the National Science Education Standards. *Multicultural Science Education: Theory, Practice, and Promise,* 120, 21-35.

Sarama, J. (2018). Perspectives on the nature of mathematics and research. *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Greenville, SC: University of South Carolina & Clemson University.

Sarama, J., & Clements, D. (2019). The Building Blocks and TRIAD projects. In P. Sztajn & P. H. Wilson (Eds.), *Learning trajectories for teachers: Designing effective professional development for math instruction* (pp. 104-131). New York, NY: Teachers College Press.

Sarama, J., & Clements, D. H. (2009). Early childhood mathematics education research: Learning trajectories for young children. Taylor & Francis.

Sarama, J., Clements, D. H., & Guss, S. (2021). Longitudinal evaluation of a scale-up model for professional development in early mathematics. In Early Childhood Teachers' Professional Competence in Mathematics. Taylor & Francis.

Sarama, J., Clements, D.., Wolfe, C. B., & Spitler, M. E. (2016). Professional development in early mathematics: Effects of an intervention based on learning trajectories on teachers' practices. *Nordic Studies in Mathematics Education*, 21(4), 29–55.

Sarama, J., Clements, D. H., Wolfe, C. B., & Spitler, M. E. (2012). Longitudinal evaluation of a scale-up model for teaching mathematics with trajectories and technologies. *Journal of Research in Educational Effectiveness, 5*(2), 105–135.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26(2), 114-145.

Stephan, M. L., Chval, K. B., Wanko, J. J., Civil, M., Fish, M. C., Herbel-Eisenmann, B., Konold, C. & Wilkerson, T. L. (2015). Grand challenges and opportunities in mathematics education re-search. *Journal for Research in Mathematics Education, 46*(2), 134–146. DOI:10.5951/jresematheduc.46.2.0134

Suh, J., & Seshaiyer, P. (2015). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. Journal of Mathematics Teacher Education, 18(3), 207–229. DOI:10.1007/s10857-014-9282-7

Suh, J., Wills, T., Kirschner, S., Vora, M., Roscioli, K. & Wearly, A. (2022). Developing asset-based instruction through learning trajectory-based curricular design.

Sztajn, P. & Wilson, P. H. (2019) Learning trajectories and professional development. In P. Sztajn & P. H. Wilson (Ed), *Learning Trajectories for Teachers: Designing Professional Development for Mathematics Instruction*. New York: Teachers' College Press.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156.

Wickstrom, M. (2014). An examination of teachers' perceptions and implementation of learning trajectory based professional development (Unpublished doctoral dissertation). Illinois State University, Normal, IL.

Wilson, P. H., Mojica, G. F., & Confrey, J. (2013). Learning trajectories in teacher education: Supporting teachers' understandings of students' mathematical thinking. *The Journal of Mathematical Behavior*, *32*(2), 103–121.

Wilson, P. H., Sztajn, P., Edgington, C., & Myers, M. (2015). Teachers' uses of a learning trajectory in student-centered teaching practices. *Journal of Teacher Education, 66*(3), 227–244.

Wilson, P. H., Sztajn, P., Edgington, C., Webb, J., & Myers, M. (2017). Change in teachers' discourse about students in a professional development on learning trajectories. American Educational Research Journal, 54(3), 568–604.

Zahner, W. & Wynn, L. (2021). Rethinking learning trajectories in light of student linguistic diversity, *Mathematical Thinking and Learning*, DOI: 10.1080/10986065.2021.1931650

van Es, E. A., Hand, V., Agarwal, P. & Sandoval, C. (2022). Multidimensional noticing for equity: Theorizing mathematics teachers' systems of noticing to disrupt inequities, *53*(2), 114-132.

UNDERGRADUATE LATIN* QUEER STUDENTS' INTERSECTIONALITY OF MATHEMATICS EXPERIENCES: A BORDERLANDS PERSPECTIVE

Luis A. Leyva
Vanderbilt University
luis.a.leyva@vanderbilt.edu

The cisheteropatriarchal climate of STEM education shapes oppressive experiences for queer and trans (QT) students majoring in the sciences. Intersectionality of STEM experiences for QT students of color is missing in the literature. Thus, it has been unexplored how undergraduate STEM as a racialized space shapes variation in experiences among QT students. Such intersectional analyses are especially necessary in mathematics -- a discipline socially constructed as 'neutral' despite being a gatekeeper to STEM degrees for historically marginalized groups. To address this area of needed research, this paper presents findings from an analysis of undergraduate Latin* QT students' intersectionality of mathematics experiences as STEM majors with a focus on peer relationships. I conclude with implications for research and practice to disrupt mathematics education as white, cisheteropatriarchal space.*

Keywords: gender, intersectionality, Latin*, sexuality

Introduction

Ximena (they/them), a Mexican agender[1] pansexual person in their senior year of college studying mechanical engineering, considered dropping the major due to invisibility of their identity that made them question their ability. They saw mathematics education as valuable for increasing historically marginalized populations' access to engineering and other STEM careers.

> A lot of it [dropping engineering] had to do with imposter syndrome. And I think a big influence within that was lack of visibility. I really felt like I didn't fit in because everyone in my class is a male or white… When I was going to drop out, I was going to switch to math teaching… Even when I become an engineer, I want to help students understand math because I feel math can be a way to gatekeep these really good jobs… Math should be accessible, so that's why I was like, 'I'll just pursue that'… I want to inspire people of color and people in the Latinx community and queer folks to go into these STEM jobs… I want to stop it from being all just straight white men.

This perspective reflecting Ximena's use of their role as a mathematics educator to diversify scientific professions was shared during their individual interview as a participant in my research study about the experiences of queer and trans*[2] (QT) students of color in STEM majors. The narratives of undergraduate QT students of color in STEM like Ximena are central to the study's analysis to understand their experiences of intersectionality (Crenshaw, 1991), which refers to

[1] *Agender* is a gender identity of being genderless.

[2] *Queer* refers to all marginalized sexual and nonnormative gender identities. *Trans** describes individuals who depart from their assigned gender at birth and move across socially-constructed boundaries in normative views of gender, including those who do and do not pursue medical gender-affirming treatment as well as those who identify with binary and nonbinary genders (Nicolazzo, 2017). The term's asterisk, much like in computer searches, ensures broad inclusion of gender identities and expressions (e.g., transgender, transfeminine, trans man, transsexual). To avoid reinforcing trans* erasure, I use the phrase *queer and trans** and the corresponding acronym *QT*.

multiply-marginalized people's unique forms of oppression and agency at the juncture of racism and other overlapping systems of power. Ximena's quote conveys agency to use mathematics education in addressing the systemic issue of limited diversity in STEM, which transpired into a distinctly intersectional experience of invisibility as a Mexican queer engineering student.

Prior research has importantly shown how cisheteropatriarchy (i.e., a system of oppression that marginalizes QT people and cisgender women by reinforcing heterosexual and cisgender identities as normative and upholding misogyny) shapes undergraduate STEM as oppressive for QT students (Cech & Waidzunas, 2011; Miller et al., 2020). However, QT students of color were underrepresented in study samples, and the lack of intersectional analyses left implicit how racism figured into QT students' experiences of cisheteropatriarchy in STEM. The present analysis addresses these limitations in extant research by exploring variation in experiences of oppression and agency among undergraduate Latin*[3] queer STEM majors like Ximena. This analysis comes from a larger study that exclusively sampled Black, Latin*, and Asian QT students in STEM to examine variation in intersectionality specific to race, gender, and sexuality.

My analysis pays particular attention to the influence of mathematical contexts on Latin* QT students' experiences while pursuing STEM majors. As raised in Ximena's opening quote, mathematics has played a long-standing role in higher education as a gatekeeper to STEM degrees and careers for historically marginalized groups. For example, research has shown that Latin* students' racialized experiences in undergraduate mathematics have a major influence on perceptions of their academic ability and decisions to continue coursework required for STEM majors (e.g., Leyva, 2016; Oppland-Cordell, 2014). Epistemologies of neutrality and objectivity in mathematics, which are rooted in ideologies of antiBlack racism (e.g., colorblindness) and cisheteropatriarchy (e.g., gender neutrality), insidiously reinforce the discipline's oppressive nature through educational opportunities that frame social identities and experiences as irrelevant (Gutiérrez, 2017; Leyva, McNeill & Duran, 2022; Martin, 2009). Constructions of mathematics as 'neutral,' therefore, also marginalize QT learners who may internalize that their queerness is distracting or unwelcome due to pervasive silence about their identities (Kersey & Voigt, 2020; Yeh & Rubel, 2020), which may negatively impact their STEM persistence (Leyva, McNeill, Balmer, et al., 2022). Mathematics education must be interrogated to inform instructional and support practices that affirm historically marginalized identities in STEM. An intersectional lens of analysis allows for complex insights that are essential to developing such practices, which are less readily attainable when attending to only a single axis of social oppression. The present study advances such inquiry in undergraduate mathematics research where QT students' racialized experiences went unaddressed (e.g., Voigt, 2022) and intersectionality among Latin* students specific to QT learners was an unexplored area of study (e.g., Oppland-Cordell, 2014).

The PME-NA 44 conference theme, "Critical Dissonance and Resonant Harmony," addresses how experiences of dissonance are "necessary for change and liberation" in struggles for justice across educational and societal contexts. Harmonious justice is characterized as resonance in multiple stakeholders amplifying each other's voices in disrupting oppressive forces tied to racism and overlapping systems of power. Applying the PME-NA 44 theme as a lens to interpret Ximena's opening quote, the invisibility of Latin* and queer students in engineering and STEM more broadly shaped dissonance between their intersectional identity and mathematical contexts

[3] The asterisk in *Latin** considers fluidity in gender identities across the Latin American diaspora (Salinas, 2020). The term *Latin** responds to (mis)use of Latinx, a term reserved for gender nonconforming peoples of Latin American origin and descent (Salinas & Lozano, 2019).When describing student participants in the present analysis who did not identify as trans*, I use the phrase *Latin* queer*. Otherwise, I used the broader descriptor *Latin* QT*.

that produced feelings of imposter syndrome. Ximena's agency through their motivation to teach mathematics depicts an effort to seek justice by dismantling gatekeeping to STEM careers. Such effort fosters resonance for future generations of Latin* QT students between their identities and STEM pursuits. The following research questions for the present study are framed to examine dissonance and agency across Latin* QT students' intersectionality of mathematics experiences:

1. What oppressive contexts of mathematics education contribute to Latin* QT students' experiences of dissonance as STEM majors?
2. What agency do Latin* QT students exhibit to manage intersectional oppression from dissonant experiences in mathematics and protect their identities while pursuing STEM?

In what follows, I present the study's guiding theoretical perspective and methods to address the research questions. This paper presents findings specific to peer relationships in mathematics that shaped Latin* QT students' intersectionality of experiences. I conclude by discussing the study's scholarly significance and implications for educational research and practice.

Theoretical Perspective

The theory of *borderlands* (Anzaldúa, 1987/2007) from Chicana feminist thought served as the guiding theoretical perspective for the study. As a theory of the flesh (Moraga & Anzaldúa, 1981/2021), Gloria Anzaldúa drew on her lived experience to develop borderlands as a lens for theorizing intersectional oppression and agency. Borderlands theory captures how an individual at the juncture of multiple, contradictory systems of power (e.g., racism, misogyny, homophobia) can experience an oppressive sense of liminality and ambivalence that can also be a site for transformative resistance. Anzaldúa, a sixth-generation Chicana lesbian born in Texas near the U.S.-Mexico border, described this push-pull dynamic between racialized constructions of nation-state and cisheteropatriarchy at the juncture of these two power systems:

> As a mestiza, I have no country, my homeland cast me out; yet all countries are mine because I am every woman's sister or potential lover. (As a lesbian, I have no race, my own people disclaim me; but I am all races because there is the queer of me in all races). (p. 102)

This experience of liminality for Anzaldúa and other multiply-marginalized individuals is theorized to be existing at the borderlands, wherein la frontera (the border) is a "metaphor for all types of crossings – between geopolitical boundaries, sexual transgressions, and the crossings necessary to exist in multiple linguistic and cultural contexts" (Cantú & Hurtado, 2012, p. 6). Borderlands theory "seek[s] enlightenment of the ambiguity and contradiction of all social experience" (Cantú & Hurtado, 2012, p. 5) to generate complex insights into multiply-marginalized individuals' intersectionality of experiences.

There are three key constructs from borderlands theory that provided a foundation for the present study. First, *Nepantla* (liminality) refers to the space of being neither here nor there and the multiplicity of realities, as depicted in Anzaldúa's quote above, where new knowledge is produced. This space is characterized by la mezcla (the hybridity) at the borderlands of competing sources of power, which generates new understandings of the world that inform multiply-marginalized individuals' ability to engage in border-crossings. Second, *mestiza consciousness* refers to individuals' outsider-within knowledge at the borderlands. This critical awareness of not being fully accepted on either side of the border is an empowering source of knowledge for challenging oppressive dualities and coming into one's full, intersectional identity. Anzaldúa (1987/2007) argues that mestiza consciousness provides individuals with a sense of conocimiento (familiarity with) that allows for an interconnected understanding of others' experiences at the borderlands. This knowledge production is a foundation for building coalitions and advancing collective action against intersectional injustices. Finally, *la facultad* (ability) refers to individual agency in leveraging knowledge produced through a sense of liminality at the borderlands to engage in border-crossings and resist oppressive systems.

Borderlands is a foundational theoretical perspective in scholarship advancing justice for QT people of color (Brockenbrough, 2015; Ferguson, 2004; Kumashiro, 2001), including work specific to Latin* QT communities (Aguilar-Hernández & Cruz, 2020; Hames-García & Martínez, 2011; Hernández et al., 2021; Rodríguez, 2003). To illustrate, the concept of mestiza consciousness provided an orientation for the development of gay Latino male studies from a space of solidarity with Chicana and Latina lesbian scholarship (Hames-García & Martínez, 2011). Latin* trans and nonbinary scholars have extended borderlands theory by disrupting its binary conceptualization of gender and sexual oppression to make visible the intersectional realities of border-crossing among Latin* gender nonconforming people (Cuevas, 2018; Hernández et al., 2021). Thus, borderlands theory offers a promising foundation for the present study of Latin* QT students' intersectionality of mathematics experiences as STEM majors.

Methods

The analysis presented in this paper comes from a larger study exploring intersectionality of experiences among undergraduate queer and trans* (QT) students of color pursuing STEM majors. A total of 39 students who identify as Black, Latin*, and Asian, including mixed-race, are included in the sample. The larger study examines students' narratives of oppression and agency to elucidate features of STEM classroom instruction and co-curricular support spaces experienced as affirming or marginalizing of their intersectional identities. For the present paper, I focus on an analysis specific to Latin* QT participants' mathematics experiences.

Study Context and Participants

Participants in the larger study were recruited across four U.S. universities in 2019-2021. The study began in 2019 at Lorde University– a large, research-intensive, elite, and private historically white institution (HWI; see Leyva, McNeill, Balmer, et al., 2022). The research team expanded the study in 2020 to include three large, research-intensive, and public HWI contexts – Ferguson University, Moraga University, and Rivera University. The three public HWIs were purposefully selected for their strong records of success with enrolling and granting bachelor's degrees to Black and Latin* students according to recent higher education policy reports (Excelencia in Education, 2018; Harper & Simmons, 2019). Rivera University received the federal designation as a Hispanic-Serving Institution (HSI). In addition, Moraga University and Rivera University were selected as institutions recognized for their efforts in promoting positive campus life experiences for QT students (Campus Pride, 2020). Ferguson University was also selected for its institutional legacy of preparing undergraduate students from racially minoritized backgrounds to pursue STEM graduate degrees and professional careers.

The purposeful selection of university sites made space for variation in how different institutional contexts and student support offerings shaped intersectionality of STEM experiences among QT students of color. For example, the multi-institutional study design allowed our team to look across student experiences in private and public HWIs across different U.S. regions. Inclusion of the three public universities with strong records of racial equity, including Moraga University as a HSI and Ferguson University with its history of addressing racial inequities for STEM access, served to explore the extent to which culturally-affirming institutional missions and efforts provided intersectional support in STEM for queer of color identities. Our selection of Moraga University and Rivera University allowed us to consider how nationally-recognized support for minoritized gender and sexual identities impacted QT students of color in STEM.

For participant recruitment, the research team asked staff and student leaders for university offices and organizations to share information about the study with undergraduate networks. These offices and organizations had missions relevant to the scope of the study, including those of social affinity (e.g., queer student alliances), STEM (e.g., American Society of Civil Engineers chapters), and intersections of social affinity and STEM (e.g., Society for the Advancement of Chicanos/Hispanics and Native Americans chapters). See Leyva, McNeill, Balmer, et al. (2022) for details about recruitment across the four universities. The analytical sample for the present paper includes all 16 Latin* QT students from the study. This sample reflected variation in ethnoracial identity (e.g., Colombian, Mexican, Puerto Rican), queer sexual identity (e.g., bisexual, gay, pansexual) and STEM major (e.g., computer science, mathematics, mechanical engineering). Six Latin* participants held gender-expansive identities, such as agender, female/questioning, genderfluid, nonbinary, and transmasculine. Four students attended Lorde University, three attended Ferguson, four attended Moraga, and five attended Rivera.

Data Collection

The research team used five data sources for rich, multidimensional portraitures of Latin* QT students' intersectionality of STEM experiences. First, participants completed a demographic survey to collect information about their identities (race, gender, and sexuality), year of study, STEM major, course enrollment, and campus involvement. For participants recruited in 2020-2021 during the COVID-19 pandemic and the transition to remote learning in higher education, the survey asked participants to indicate any preferred accommodations of the study design to ensure comfort with their participation (e.g., addressing privacy concerns about discussing their gender and sexuality via Zoom from home). Second, each participant submitted a STEM autobiography. The autobiographies were written reflections about being QT students of color in STEM, including responses to prompts about memorably positive and negative experiences, socially-affirming academic and co-curricular spaces, and influential people. One autobiography prompt asked participants to reflect on the role that mathematics played in their STEM trajectories as QT students of color and vivid memories from their experiences. The third data source was event journaling (Leyva, Quea, et al., 2021; Leyva, McNeill, Balmer, et al., 2022). Throughout the study, participants kept an ongoing record of events across STEM spaces (e.g., classrooms, study groups) experienced as supportive/encouraging or unsupportive/discouraging of their identities (see Leyva, McNeill, Balmer, et al., 2022 for more details about journaling).

The two final data sources were an individual interview (60-90 minutes) and group interview (90-120 minutes). Participants completed the survey and autobiography prior to the individual interview (Interview 1). Interviews with Lorde University participants were completed in person and before the start of the COVID-19 pandemic. Interviews with participants enrolled in other universities were scheduled after the start of the pandemic in 2021 and completed on Zoom. All interviews were semi-structured, audiotaped, and transcribed. Two research team members conducted each interview. To the extent possible, we matched participants and interviewers with similar racial, gender, and sexual identities. Such matching was an effort to increase comfort in discussions about racism, misogyny, cissexism, and heteronormativity in STEM.

Interview 1 explored participants' perspectives on being QT students of color in and out of STEM, major influences on their STEM pursuits, coping strategies for STEM persistence, and recommendations for queer of color inclusion in STEM. The protocol revisited excerpts from participants' STEM autobiographies and event journaling related to these themes. For participants interviewed during the pandemic, we also explored participants' views on the influence of COVID-19 on their STEM experiences. In particular, participants were asked about standout moments in online/hybrid classrooms that positively or negatively impacted their identities, nature of their campus involvement, and relationships with peers and family members.

After Interview 1, participants completed a group interview (Interview 2). I adopted my group interview methodology of presenting 3-4 prompts for stimulated-responses to explore variation in participants' intersectionality of STEM experiences (see Leyva, 2021 and Leyva, McNeill, Balmer, et al., 2022 for more details about the interview methodology). The prompts in Interview 2 featured excerpts from research about QT students' STEM experiences (e.g., Cech & Waidzunas, 2011; Kersey & Voigt, 2020) and intersectionality of campus experiences among QT students of color (e.g., Nicolazzo, 2016; Vega, 2016). The excerpts provided concrete starting points to stimulate group dialogue about emergent themes from preliminary data analysis, including cisheteronormativity in STEM instruction and perceptions of STEM ability linked to race and gender. To explore variation in opportunities for intersectional support across different universities in our study, one prompt included excerpts that featured reflections from QT students of color in prior research about their experiences of campus climate and identity-related support offerings at their institutions. We followed the presentation of these excerpts with statements about the advancement of diversity and equity at participants' home institutions. These statements were used to probe the extent to which participants perceived opportunities for identity support in their broader campus contexts influencing their STEM pursuits. After participants were presented with each interview prompt, we asked them about their general interpretations, degree of perceived relevance to their experiences, and recommendations for increasing queer of color inclusion in relevant practices of STEM education.

To the best of our ability, each participant who completed Interview 2 was paired with at least one other participant of a similar race-gender identity. Such identity-matching across the group interviews was an attempt to avoid participants feeling their perspectives were tokenized and create space for variation in responses to interview prompts among students with a similar intersectional identity. When we conducted Interview 2 via Zoom with participants after the onset of the COVID-19 pandemic, we also tried to have representation of participants enrolled in different universities for each interview to allow for discussion of similarities and differences in their STEM experiences across institutional contexts. We completed three group interviews with Latin* QT participants who attended different universities but shared a similar race-gender identity (namely, Latina cisgender women, mixed-race Hispanic and Latina cisgender women, and Latine gender-expansive people). For example, the group interview for mixed-race Hispanic and Latina participants included Erica (mixed, cisgender, bisexual Latina of Brazilian descent[4] in Rivera University), Laura (Mexican and Chinese pansexual female at Ferguson University) and Tamara (mixed-race Hispanic, Peruvian/white masculine woman lesbian at Moraga University).

Data Analysis

The research team used the framework of *STEM Education as a White, Cisheteropatriarchal Space* (WCHPS, Figure 1; Leyva, 2021; Leyva, McNeill, Balmer, et al., 2022) to guide data analysis. The WCHPS framework provides a lens for examining how interplay between racism and cisheteropatriarchy in STEM educational contexts shapes intersectional experiences of oppression and agency for multiply-marginalized individuals, including QT students of color.

[4] I use wording from participants in describing their identities.

Each framework dimension (ideological, institutional, and relational) attends to a level of influence in STEM education at which white cisheteropatriarchy (the juncture of white supremacy and cisheteropatriarchy) impacts multiply-marginalized individuals' experiences. The ideological dimension addresses beliefs, norms, and values that organize STEM educational practices. The institutional dimension explores structural inequities that constrain opportunities for achievement and participation in STEM, in addition to forms of individual agency in navigating such oppressive structures. The relational dimension addresses interactional forms of oppression and agency in STEM education. These dimensions are interconnected, thus allowing the WCHPS framework to examine how white cisheteropatriarchy shapes intersectionality of STEM experiences in complex ways. Elsewhere (Leyva, McNeill, Balmer, et al., 2022), I provide more details about developing the WCHPS framework.

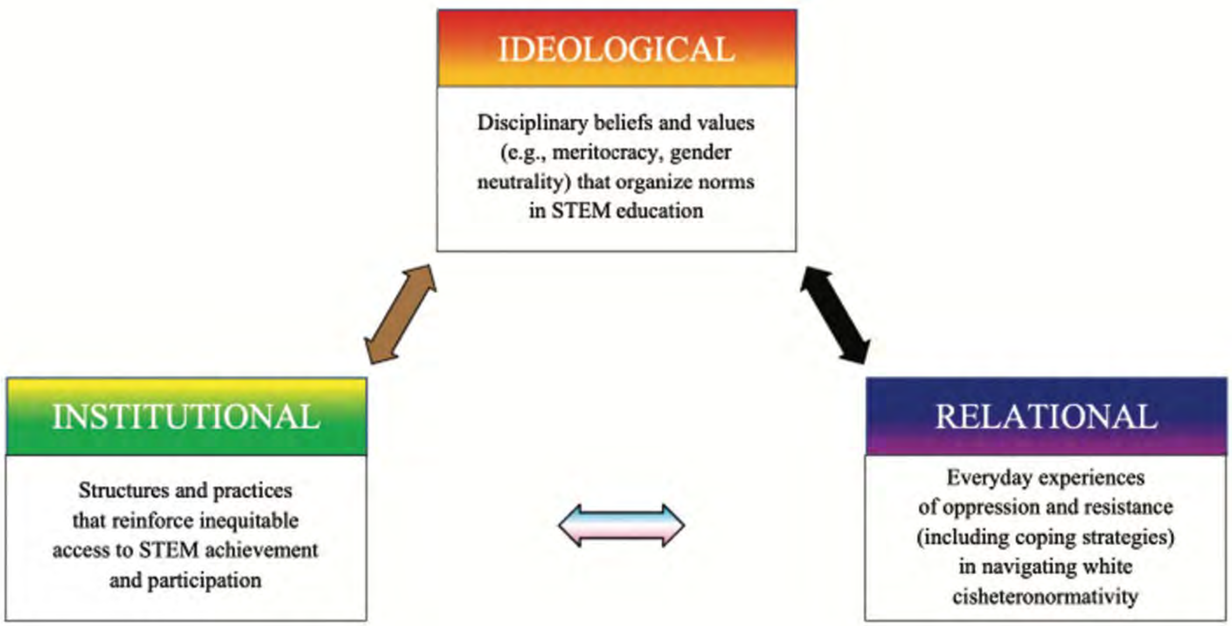


Figure 1: Framework of STEM Education as a White, Cisheteropatriarchal Space

Data analysis was completed in four stages. First, the research team inductively coded participants' data to flag aspects of their mathematics experiences corresponding to each WCHPS dimension. We coded for instances of oppression, supportive forms of structural disruption, and agency across ideological, institutional, and relational levels of Latin* QT students' experiences. (See Leyva, Balmer, et al., 2021 and Leyva, McNeill, Balmer, et al., 2022 for more details about coding.) The second stage of analysis was constructing an analytical narrative for each participant of navigating white cisheteropatriarchy in mathematical spaces as Latin* QT students majoring in STEM. These narratives were developed by following the critical race methodology of counter-storytelling (Solórzano & Yosso, 2022), which centers racially minoritized people's experiences of oppression for theorizing resistance to racism and other interlocking systems of power. Each participant's counter-story was structured around oppression, structural disruptions, and agency across WCHPS dimensions identified in coding.

The third stage of analysis was identifying themes across the 16 Latin* QT participants' counter-stories that address our two research questions. To address the first research question, we identified themes about oppression through experiences of dissonance between participants' identities and mathematical contexts. We addressed the second research question by identifying themes of agency specific to participants' behaviors and motivations for navigating dissonance to protect their Latin* QT identities and success as STEM majors. The fourth and final stage of data analysis was interpreting the themes through the lens of borderlands theory (Anzaldúa, 1987/2007) and its uptake in Latin* queer studies with an expansive theorizing of gender (e.g., Cuevas, 2018). In particular, the borderlands concept of Nepantla served to elucidate how dissonance in Latin* QT participants' mathematics experiences reflected liminality at the juncture of power systems that limited full affirmation of their intersectional identities. Mestiza consciousness supplied a lens to account for participants' critical awareness of how white cisheteropatriarchy operated in mathematical contexts, including how it was reinforced through dissonant educational practices. The construct of la facultad elucidated Latin* QT students' agency to manage dissonance in mathematics as a white, cisheteropatriarchal space. Overall, our analysis captured how white cisheteropatriarchy in STEM educational contexts shaped the borderlands of Latin* QT students' intersectionality of mathematics experiences.

To sharpen our data analysis, the research team conducted member-check interviews with available participants after data collection was complete. Eleven of the 16 Latin* QT students in the study sample completed a member check. We structured member-check interviews in three parts corresponding to each of the framework dimensions. During each interview part, the interviewers read a section from the participant's counter-story to convey our understanding of oppression, disruptions, and agency at each level of the WCHPS framework. Participants, who were positioned as experts of their lived experiences during the interviews, were prompted to suggest edits and add content to ensure an accurate analysis in their counter-stories. The interviewers also asked participants a series of questions to clarify and further elaborate on important ideas raised in each counter-story section read to them.

Positionality

Our research team (one faculty member, three doctoral students, eight master's students, and two undergraduate students) are members of the Power, Resistance and Identity in STEM Education (PRISM) Lab at Vanderbilt University. The team has robust social diversity across intersections of race (African American, Black, Latin*, biracial, white), gender (cisgender, nonbinary, transmasculine), and sexuality (demipansexual, gay, lesbian, pansexual, queer, heterosexual, unsure). Most of the team collected and analyzed data from Latin* QT participants. Milner's (2017) framework on positionality in educational research guided the team's self-reflections to avoid dangers of approaching our study without consciousness for the influence of our identities and experiences. These reflections avoided the seen danger of not interrogating our respective areas of privilege and oppression. The team adopted an asset-based research approach by making space for understanding Latin* QT participants' agency and resistance in mathematical contexts, thus avoiding the unforeseen danger of readers perceiving QT students of color as powerless victims of oppressive educational systems. To mitigate the unseen danger of misinterpreting participants' sensemaking about intersectionality of their experiences, we completed interviews and coding in pairs to have multiple researcher perspectives present in data collection and analysis procedures. We also completed member checks described earlier to strengthen the trustworthiness of our findings. Team members bracketed their lived experiences from those of participants to avoid the unseen danger of distorting participants' realities, all while remaining critical of oppressive STEM structures and systems through use of the counter-storytelling methodology. I avoided the unseen danger of flattening variation in oppression and agency that Latin* QT students experienced by infusing voices from multiple participants in the findings, including a cross-case analysis of counter-stories for two participants with different ethnoracial, gender, and sexual identities. Although this is a solo-authored piece, I solicited feedback from team members who analyzed Latin* QT participants' data to ensure the final draft of our analysis accurately reflects our collective work.

Findings

Themes across the 16 Latin* QT participants' counter-stories reflected how two mathematical contexts shaped experiences of dissonance and agency: (i) curricula and instruction and (ii) peer relationships. To respect space limitations, I present themes about peer relationships in this paper. I develop themes for curricula and instruction elsewhere (Leyva, 2022, 2023).

Half of the participants (Daniela, Erica, Koyotl, Laura, Ros, Steven, Teresa, Ximena) reflected on how peer relationships produced dissonance with their Latin QT identities across mathematical contexts. One theme was the lack of social diversity in STEM majors and mathematics classrooms, which brought forth feelings of isolation and imposter syndrome as well as struggles to build identity-affirming networks of support. Koyotl (he/him; Indigenous American, gay, transmasculine person of Mexican descent; third-year computer science major in Rivera University), for example, shared how the main impact of mathematics on his experience as a computer science major was making him feel "lonely as a LGBTQ+ student in STEM" (Autobiography). Being in mathematics classrooms, where he was the only queer student to his knowledge and often felt "alone in a room full of white students" (Autobiography), brought him to feel academically inferior to students from majority groups. Similarly, Erica (she/her; mixed, cisgender bisexual Latina of Brazilian descent; third-year computer science major in Rivera University) managed a sense of imposter syndrome due to a lack of peer diversity, which was not the case at the community college from where she transferred and "actively [saw] older students or more students of color" (Interview 1). She reflected on building a strong connection at the community college with an older student who identifies as a lesbian in her pre-calculus course, "We were able to connect over the fact that she was also around my age and came from an arts [background]… That was a really great way of [feeling like] 'Okay, well, clearly I'm not the only older student or I'm not the only queer one'" (Interview 1). Opportunities to build diverse networks of peer support in STEM were limited, which shaped dissonance in Erica's experience.

The second theme specific to peer relationships was marginalization through interactions during collaborations with peers in mathematical contexts, including microaggressions of ability as well as instances of cisgender women being fetishized or hypersexualized. To develop this theme, we look across counter-stories from two Latin* queer STEM participants (Ros and Daniela). The counter-stories are structured in three parts. I open each counter-story with a short biographical sketch of Ros and Daniela. To address the study's first research question about experiences of dissonance, the second part of each counter-story depicts peer relationships that limited affirmation and support for Latin* queer participants' identities as mathematics learners. The final part of each counter-story answers the second research question about agency. In this portion of the counter-stories, I account for participants' behaviors and motivations for managing and resisting dissonance through peer relationships to protect their identities and STEM success. I conclude the findings section by applying borderlands theory and the WCHPS framework in a cross-case analysis of Ros's and Daniela's counter-stories.

Ros's Counter-Story

Ros (they/she/he) is a Mexican, bisexual, and genderfluid person majoring in mechanical engineering as a senior in Rivera University. They saw themself as being female-presenting and recognized their privilege of passing as white. Family played a central role in Ros's motivation to succeed as a future engineer. They perceived their academic accomplishments as giving back to their parents for their sacrifices, including how their mother was unable to use her computer engineering degree from Mexico after immigrating to the U.S. Success in mathematics allowed Ros to serve as a role model for younger cousins aspiring to become engineers, "The only introduction my cousins and I had to it [calculus] was through media, and now it's nice to be one of the family members they can come to for help" (Autobiography). Ros felt a strong sense of acceptance as a queer person from his family, which he described as different from "traditional Latino ones" (Interview 1) where homophobia and transphobia rooted in religious beliefs resulted in some of her Latin* friends being disowned or having their queerness disregarded.

In addition to family, peer relationships that affirmed Ros's full identity as a Mexican queer engineering student were important for Ros's persistence in navigating the cisheteropatriarchal culture of STEM. His most positive STEM educational experience was having a peer network of support in high school engineering classes, which he described as "all Latinx and a few were LGBTQ+ as well so [he] always felt like [they] were all on the same plane" (Autobiography). Establishing such peer connections was more difficult as an engineering major with students largely coming from white, wealthy backgrounds and where the space felt "very cishet male-dominated… [and] very masculine" (Interview 1). As further developed in the remainder of Ros's counter-story, the exclusionary space of undergraduate STEM contributed to Ros's sense of vulnerability when meeting and working with unfamiliar peers in different contexts (e.g., study groups, engineering organization meetings). The constant threat of facing homophobic and transphobic microaggressions about their academic ability and queer identity in peer interactions shaped dissonance for Ros in mathematical spaces. To navigate such dissonance as a Latin* queer student, Ros carefully studied unfamiliar STEM peers' interactions to assess if they would be accepting of their queerness before sharing their pronouns and discussing their identities.

Dissonant influences. Dissonance in Ros's mathematics experience as a mechanical engineering major arose from navigating cisheteronormativity in peer interactions. Ros described how their queerness was often overlooked or stigmatized and how they felt underestimated in terms of their academic ability. Ros described facing peers' microaggressions about her mathematical ability couched in humor, "I have heard some jokes along the lines of people finding it interesting that I am skilled in math when 'Queer people are bad at math' is apparently a stereotype" (Autobiography). He expressed uncertainty if such peer humor was coming from a space of homophobia and transphobia or whether it served as a coping mechanism for fellow queer students in STEM, "If you're saying that weird gay joke, does that mean that you're going to be against my identity? Or is it one of those things where… you're also queer and you're just joking about it to cope?" (Interview 1). The ambiguity of humor that invoked microaggressions of mathematical ability produced dissonance in Ros's experience as a queer engineer.

Queer oppression in peer humor was one aspect of a broader culture of cisheteronormativity that Ros experienced in mathematical and scientific spaces. This oppressive culture made Ros feel vulnerable to facing homophobia and transphobia when joining new STEM study groups with unfamiliar peers. To illustrate, Ros recounted their most negative experience as a Latin* queer student in STEM as being when they were misgendered and assumed to be straight during a small study group session for an introductory calculus or physics course.

> I went out of my comfort zone to sit at a group with people I didn't know… all of which seemed like cis men minus one woman. As we were getting along, the woman turned to me and said something along the lines of 'It's nice to see another regular girl at these things, it's

> always all guys or the girls who come are lesbians.' It was really jarring to me because I neither identify as a woman nor am I straight so as she said this to me I had to sit with the knowledge that my identity was something she was bothered by. And because I wasn't familiar with this group, I just chuckled awkwardly and switched the topic. (Autobiography)

The groupmate's assumption that Ros was a cisgender, heterosexual woman suggests a feeling of relief from patriarchal, male-dominated STEM spaces. At the same time, this assumption points to the cisheteronormative culture of STEM that erases queer people as well as produces undue labor for queer students like Ros about disclosing or concealing their gender and sexuality. Building peer networks in STEM was a taxing endeavor for Ros in terms of constantly preparing themself for students "making assumptions based on what you look like… [and] mak[ing] comments that are probably going to be hurtful and homophobic and transphobic" (Interview 1).

The dissonance that Ros experienced as a queer person navigating cisheteronormativity in STEM peer relationships also took form in campus meetings for identity-based engineering organizations and programs. Although Ros felt a reprieve from the hypermasculinity of their major in events for women in engineering, the focus on cisgender women produced tensions about sharing their genderfluid identity and made them feel like an imposter, "When I do go to events that are specifically for women, I'm kind of like, 'Do I say that I don't identify as a woman? Does that make me not belong in this space even though I appear female?'" (Interview 1). Ros also felt conflict with her queer identity in the university's chapter for the Society of Hispanic Professional Engineers (SHPE). He described how SHPE's strong focus on racial affinity, in addition to traditional, culturally-mediated views on gender and sexuality among student members, made expression of his queer identity inaccessible in this Latin* affinity space.

> With SHPE, I didn't so much expand on my gender and sexual identity because… everybody in those spaces bonds over the fact that they identify as Hispanic or Latino, so it's not as prioritized. They are a little bit better about having pronouns available and all of that at some of their meetings, but most of them, it's just traditional. And there's also this stigma... with people who are more traditionally Hispanic… especially first-generation immigrants aren't always onboard and as progressive when it comes to gender and sexuality. (Interview 1)

SHPE, as a single-identity affinity space that left cisheteronormative beliefs of gender and sexuality in Latin* culture uninterrogated, limited opportunities for Ros to find community with Latin* peers that embraced her full identity. With Ros having received the "biggest backlash… from people with Latino backgrounds against [his] sexuality" (Interview 1), including family members in Mexico, he chose not to engage his queerness with SHPE peers. Dissonance between Ros's Latin* queer identity and STEM peer groups, including study groups and spaces for identity affinity in engineering, limited peer support that affirmed their intersectional identity.

Agency. One strategy that Ros adopted to navigate cisheteronormativity in peer relationships when working in study groups was not disclosing their queerness right away to protect themself from homophobia and transphobia, "If I do go into these spaces, I'm not really like, 'Hey, these are my pronouns. Please respect them. Please respect me.' I'll let you assume whatever you want to assume" (Interview 1). Ros described entering new spaces of STEM peer collaboration as "learning how to read the room and learning… what's going to be acceptable and what's not going to be acceptable" (Interview 1) in terms of identity expression. Although Ros also experienced transphobia during groupwork in high school, the presence of Latin* QT peers in her engineering classes who "reaffirmed what [she] was feeling and helped [her] work through that whole gender identity thing" (Interview 1) mitigated oppression in collaborations. With the challenge of building diverse networks of peers who would provide similar forms of support in the engineering major, Ros readily concealed his queerness, much like he did when misgendered in the study group, to cope with the dissonance experienced in STEM peer relationships.

Another behavioral strategy that reflected Ros's agency in managing dissonance as a Latin* queer engineering student was entering new spaces with unfamiliar individuals alongside peers with whom they feel comfort and trust. Ros reflected on the significance of such peer connections as a STEM major, "Making connections has been the biggest thing in finding support with people that you feel comfortable with. It is one of the most essential things I've found to try to get through college with my identities" (Interview 1). To illustrate, Ros developed a close relationship with a queer classmate and SHPE leader whose openness about her identity increased her comfort with attending SHPE meetings. This strategy of "mak[ing] a connection first with the person before going into a space" (Interview 1) depicts Ros's agency in protecting himself from cisheteronormativity at intersections of STEM and Latin* cultures to overcome dissonance that limited access to peer support networks across mathematical spaces.

Daniela's Counter-Story

Daniela (she/her) is a bisexual, mixed-race Latina (Colombian/Cuban and white) in her third-year as a computer science major at Lorde University. Being a mixed-race queer woman meant that "the thing that impacts a lot of [her] experiences is a lack of identity" (Interview 1), especially as someone who sees herself as "be[ing] a part of two cultures and then a part of neither culture at the same time" (Interview 1). This sense of liminality in Daniela's experience made it initially difficult for her to find community on campus as a STEM student. At the same time, Daniela faced peers' racialized and gendered assumptions that her accomplishments were not based on merit but rather affirmative action. This "recipe for imposter syndrome" (Journal) as a mixed-race Latina in STEM made her feel constantly devalued and defeated.

In managing identity-specific challenges and academic struggle in computer science as a difficult major, Daniela built a supportive community of diverse, like-minded peers in and beyond the engineering school. She reflected on how such community supported her STEM persistence as a mixed-race queer Latina, "If there are people that are like you, you can ask them for help and form a community… If I didn't have an engineering community, I don't know what I would do. I'd probably fail" (Interview 1). She identified three sources of peer community: (i) the university's SHPE chapter; (ii) a professional engineering fraternity; and (iii) a group of women of color who met at a conference for gender equity in technology. Representation of queer people and opportunities to exchange stories of identity-based struggles created space for Daniela to comfortably engage her bisexuality in these STEM collectives, which contrasted her STEM classroom experiences where "you leave yourself at the door" (Interview 1). SHPE was also a space where she could be open about the "feeling of not fully being able to own up to [being] Latinx" (Interview 1) and could learn more about Latin* culture, which she felt was taken from her as a child raised to believe that being white was more socially acceptable. The collective of women of color in technology fields, which included another queer student, was a space for openly sharing and processing personal histories of misogyny in STEM. Daniela described how this community of healing and empowerment in STEM conveyed how "from the struggle you can form really good bonds with people" (Interview 1), which they sustained through regular check-ins and sharing professional development resources.

I now highlight how peer relationships in mathematical contexts (namely, fetishizing and sexualizing behaviors from male peers) contributed to Daniela's experiences of dissonance as a mixed-race bisexual Latina in STEM, which exemplifies the gendered struggles she processed in the women of color collective. Next, I address Daniela's agency in coping with such dissonance.

Dissonant influences. Daniela reflected on how the overrepresentation of white male students in her mathematics classes made her hyperconscious of being fetishized and hypersexualized when interacting with them. Such hypersexualization felt racialized to Daniela as a mixed-race Latina, "In terms of my sexuality… I've had to be very conscious of people fetishizing. So, it's very common for guys to be, 'Oh, I think Latinas are really hot, and I want their mom to make me tacos,' which I find super offensive and creepy" (Interview 1). Daniela described the risk of being hypersexualized when seeking male peers' support with mathematics, "I typically have to [ask] a man, who always makes a sexual advance on me… These advances make it really hard to focus and feel comfortable in math classes. This is especially annoying because math classes are incredibly difficult" (Autobiography). The discomfort Daniela felt from being hypersexualized added to the challenge of understanding content in mathematics courses.

As a computer science major, Daniela was often one of the only women in collaborative groups, which made her feel vulnerable to being hypersexualized, "If I'm working on an assignment with a group of people, that group of people is going to be predominantly male, and they're all going to be making advances, typically. And that's something that's pretty obnoxious" (Interview 1). To depict how "mathematics has always been an awkward experience" (Autobiography) navigating male peers' fetishization and hypersexualization, she recalled being the only woman in a study group where one male placed his foot on hers to keep her from moving away, another invited her to study in his room, and another asked for her phone number. Male peers' behaviors capture how Daniela was perceived as a sexual object before she was seen as a mathematics classmate. Despite this discomfort, she reflected on the "pressure to not turn guys down" (Interview 1) in her study groups and classes because rejecting their sexual advances jeopardized access to peer support in mathematics. In the following section addressing Daniela's agency, I highlight how she strategically used her bisexuality to keep male peers interested in helping her with mathematics despite their oppressive advances. The white, heterosexual male gaze in mathematics, thus, shaped Daniela's experience of dissonance as a bisexual Latina navigating racialized fetishization and hypersexualization in peer relationships.

Agency. Agency in Daniela's experience to protect her academic success in the masculinized spaces of mathematics courses is reflected in strategic expressions of her sexual identity. She shared how although claiming to be a lesbian could stop male peers from hypersexualizing her, presenting as bisexual ensured that they would remain interested in her and continue to help her.

> Maybe it would be easier if I just told them that I was a hundred percent lesbian, not interested in them, and they wouldn't make advances at all. But then I also consider, maybe then they wouldn't help me study, you know? And then they wouldn't be interested in assisting me anymore, or interacting with me anymore. (Interview 1)

Daniela used her bisexuality strategically by playing into male classmates' fetishization and hypersexualization as a way of maintaining access to peer support in mathematics. Such agency, while protective of Daniela's academic success, reinforced dissonance in her STEM experience.

Another way that Daniela exhibited agency in managing unwanted sexual attention was toning down feminine self-expression through dress, particularly in STEM classrooms where she was underrepresented as a femme Latina. Daniela described preferring to wear crop tops and artsy, quirky clothing that departed from what she described as the "engineering uniform" (Interview 1), a hoodie and T-shirt. However, she worried that her preferred dress would make her hypersexualized and subjected to negative judgment from STEM peers and faculty as "some dumb girl that is dressed inappropriately for class" (Interview 1) who put more attention into her appearance than academics. As a result, Daniela saw sacrificing her femininity through dress as protecting her not only from being hypersexualized, but also having her intellect undermined.

> One thing I leave of myself behind is… I don't wear the clothes I wanted to wear to school a lot because… I don't want other students to think I look dumb… If you look like you took care of yourself [as an engineering student], people say you're not smart… I also don't want to be wearing anything that could possibly be sexualized by people. (Interview 1)

Making intentional choices of dress was a strategy that Daniela adopted to protect herself from the dissonance between her intersectional identity and masculinized environments of STEM classrooms. By "going for as asexual as possible in classes" (Interview 1), Daniela's dress captured a self-preserving form of agency in response to how femininity was disassociated from STEM ability as well as male peers' racialized hypersexualization in mathematical contexts. Although Daniela's selectivity in dress alleviated dissonance rooted in the white, heterosexual male gaze in classrooms, she had to sacrifice aspects of her identity for such self-preservation.

Cross-Case Analysis

Looking across Ros's and Daniela's counter-stories through the lens of borderlands, their experiences of dissonance in peer relationships demonstrate Nepantla through a sense of liminality about bringing their full Latin* queer identities into mathematical contexts, especially collaborative settings like study groups. Ros's and Daniela's experiences at the juncture of racism and cisheteropatriarchy (e.g., navigating culturally-mediated tensions of disclosing queerness in SHPE, managing racialized fetishization from male peers) made them feel a sense of being "neither here nor there." The counter-stories also depict mestiza consciousness through critical awareness of the white, cisheteropatriarchal gaze in peer interactions that shaped dissonance across the intersectionality of their experiences. This awareness also informed agency or la facultad in using their motivations and strategic forms of self-expression to protect their Latin* queer identities and ensure academic success. In what follows, I develop two conclusions from a cross-case analysis of Ros's and Daniela's counter-stories through the lens of borderlands and the WCHPS framework. These conclusions address how Latin* queer students' experiences of dissonance and agency exemplify navigating the borderlands of mathematics education as a white, cisheteropatriarchal space across ideological, institutional, and relational levels.

First, the counter-stories show how dominant constructions of mathematics and STEM as neutral can leave racialized, cisheteropatriarchal climates of educational contexts unchallenged. Daniela's perception of STEM classrooms as disconnected from identities and social experiences, for instance, left her grappling with the liminality of being a mixed-race queer Latina struggling to find socially-affirming community as a computer science major. She ultimately found relational support in STEM co-curricular contexts, including SHPE and a conference for gender equity in technology fields, that made space to process her experiences of Nepantla and coming into her intersectional identity. These spaces, thus, disrupted ideological notions of STEM as neutral to offer Daniela opportunities to heal from the dissonance of being a mixed-race queer Latina in computer science. Unfortunately, such ideological disruptions were absent in formal institutional contexts, including peer collaboration in mathematics courses, that subjected Daniela to being fetishized and hypersexualized as a queer woman of color working with predominantly white, heterosexual male peers. Daniela's awareness of the white, cisheteropatriarchal gaze across classrooms and groupwork in mathematics illustrates mestiza consciousness that guided her agency (la facultad) through strategic self-expression in dress and disclosure of her queer sexuality. Ros's counter-story similarly depicts such awareness of facing intersectional marginalization as a Mexican genderfluid, bisexual person in STEM peer interactions where homophobic and transphobic perspectives went unchecked (e.g., jokes about queer people lacking mathematics ability, having their queer identity overlooked in SHPE meetings). Both counter-stories capture how ideological assumptions of neutrality in mathematical spaces can perpetuate Latin* queer oppression in exclusionary climates of peer relationships linked to a lack of social diversity (institutional influence) and toxic interactions involving microaggressions of ability and sexual harassment (relational influences). Therefore, being at the borderlands for Daniela and Ros as Latin* queer STEM majors meant managing oppressive peer relationships reinforced through assumptions of mathematics as a neutral space.

Second, both Latin* queer students' counter-stories portray the undue cognitive labor placed upon them to adopt strategies for protecting their identities and success across oppressive contexts of peer relationships in mathematics. To illustrate, Ros's keen awareness of how homophobic and transphobic ideologies influenced peer interactions, such as their most negative STEM experience of being misgendered and assumed to be straight in a study group, taxed them with the burden of concealing their queerness until peers said or did something that signaled being accepting of their genderfluid and bisexual identities. Ros exhibited similar awareness of the cisheteropatriarchal culture in the SHPE organization that informed her decision to attend meetings when accompanied by a peer with whom she can confide her queer identity. Critical awareness of study groups and SHPE meetings as potentially oppressive spaces depicts Ros's mestiza consciousness that led to investing cognitive energy in determining how to present himself with unfamiliar peers and ensuring the presence of queer-affirming peers for his safety. While these behavioral strategies depict agency or la facultad from Ros in managing queer marginalization through peer relationships, such self-protection came at a cost with Ros only being able to embrace certain aspects of their identity as a Latin* queer student in STEM. Daniela faced a similar reality in terms of sacrificing her gender expression through feminine dress to protect herself from hypersexualization in mathematics classrooms. Furthermore, Ros's counter-story conveys how institutional spaces of identity support for STEM majors often failed to account for experiences of queerness and intersectionality. The cisnormative framing of gender support in programs for women in engineering left Ros with a sense of liminality as a genderfluid person who presents as female and the labor of concealing their queerness to avoid being deemed as not belonging. She described how tensions of being a queer SHPE participant were due to traditional views of gender and sexuality in Latin* culture that also shaped dismissal of her sexuality from family, specifically cousins living in Mexico who were also sources of motivation for excelling in mathematics. Ros's liminality in SHPE and programs for women engineers captures his experiences of Nepantla in terms of being "neither here nor there" as a beneficiary of institutional support. The affirmation that Ros felt for their full identity from immediate family and in high school engineering with Latin* queer peers was missing. Ros, thus, was taxed with the same cognitive labor of protecting their queerness in peer collaborations for mathematics coursework even in institutional spaces like SHPE designed to resist oppression in formal STEM contexts. The labor imposed on Ros and Daniela to cope with intersectional oppression across peer relationships in groupwork and co-curricular programs reflects realities at the borderlands with needed disruptions of white cisheteropatriarchy in mathematical contexts.

Discussion

The present study's scholarly significance is twofold. First, the focus on Latin* queer students contributes insights about intersectionality that went unexamined in research about Latin* students and QT students in undergraduate STEM (e.g., Convertino et al., 2022; Hughes, 2017 McGee, 2016). Findings uncovered intersectional forms of dissonance that Ros and Daniela managed between their Latin* queer identities and peer relationships in mathematical spaces. For example, Daniela's counter-story depicts her strategic ways of sharing her queer sexuality and expressing her gender through dress to navigate the white, cisheteropatriarchal gaze in mathematics courses, which subjected her to unwanted sexual attention from male peers who were also sources of academic support. Second, representation of Latin* gender nonconforming students in the study extends understandings of gender equity in mathematics education. Over the years, equity-oriented research in mathematics education has framed gender as a binary and disconnected from other dimensions of social experience, including race and sexuality (Leyva, 2017; Leyva & Mahtab, in press). By centering intersectionality of mathematics experiences for Latin* students with queer gender identities, this study disrupts the long-standing erasure of trans* and nonbinary students of color in mathematics education for more complex insights on how racialized forms of trans* oppression impact gender equity. To illustrate, Ros's counter-story conveys struggle in finding peer support across study groups and pre-professional engineering societies that affirmed their genderfluid identity. Along with feeling like an imposter in programs for women in engineering that mainly catered to cisgender women, Ros felt tensions about participating in SHPE where cisheteronormative ideologies of gender and sexuality in Latin* culture influenced their perceptions of peers' acceptance of their queer identity.

Findings from the present study raise implications for research to generate more nuanced insights into Latin* QT students' intersectionality of mathematics experiences. One implication is exploring how differences in ethnoracial backgrounds, immigration history, and ability to pass as white, cisgender, and/or heterosexual shape variation in intersectional oppression and agency. Although Ros and Daniela both identified as members of the Latin* community, their racialized experiences differed. Social norms of gender and sexuality in Mexico shaped Ros's concerns about facing homophobia and transphobia from cousins living there, which can be likened to the vulnerability he felt about engaging his queerness with Latin* engineering peers. Daniela's upbringing in a mixed-race household with Colombian-Cuban and white parents, who she felt distanced her from Latin* culture, produced conflict in identifying as Latina. Peer relationships in SHPE helped her come into her intersectional identity and overcome struggles in finding community as a mixed-race, queer woman in STEM. Future research that continues unpacking within-group differences in participants' racial backgrounds, including mixed-race experiences, disrupts monolithic representations of Latin* QT students in mathematics and contributes robust understandings of intersectionality for this population. A second implication for research is further exploring variation in Latin* QT students' intersectionality of mathematics and STEM experiences across different types of higher education institutions, including HSIs. Although Ros attended a HSI with a mission for providing culturally-affirming opportunities to Latin* students, the cisheteropatriarchal culture of STEM environments, even in spaces of racial affinity like SHPE, limited their access to networks of peer support like they had in high school engineering, This finding illustrates the importance of future research that continues to explore how structures and practices at HSIs and other institutions with missions for serving minoritized populations (e.g., women's colleges) disrupt intersectional marginalization in mathematics classrooms and other STEM contexts for Latin* QT students. Such research in HSIs responds to calls for insights about queer-affirming institutional support that serve Latin* QT students (Vega et al., 2022).

The study's findings also raise implications for educational practice. Both counter-stories addressed peer collaboration for mathematics coursework as a context where participants experienced dissonance with their Latin* queer identities. This finding points to the importance of instructors co-constructing norms of collaboration with their class that resist influences of white cisheteropatriarchy and expand Latin* QT students' comfort with groupwork. Examples of norms include students reflecting on how much space they have occupied, checking in on students who have not shared ideas, using peers' chosen names and pronouns, and avoiding assumptions of peers' gender, sexual, and other identities. These norms foster accountability to ensure that group members, including Latin* QT students, feel safe and comfortable to participate. This accountability alleviates the vulnerability that Ros and Daniela reported about navigating racialized-gendered judgments of ability, possibilities of being hypersexualized, and uncertainties of homophobia and transphobia that negatively impacted their experiences in peer collaboration. Instructors can ensure students engage in alignment with co-constructed norms by re-visiting them as a class when they observe groups departing from them, modeling norms through behavior and discourse during instruction, and encouraging adoption of these norms to structure collaboration in peer study groups and other contexts outside of the classroom.

Another practice implication is adopting an expansive approach to identity-based support in affinity spaces for STEM majors from historically marginalized groups. Findings captured limitations to peer support that participants experienced in spaces like SHPE and programs for women in engineering, where they were unable to bring their full intersectional identities as Latin* queer individuals. Leaders for STEM affinity spaces must interrogate how they frame gender inclusion to avoid reinforcing cisnormative exclusion like Ros had experienced. In addition, the design of STEM affinity spaces for Latin* students (e.g., SHPE) and QT students (e.g., Out in STEM) must give explicit attention to experiences of intersectionality to ensure a robust sense of inclusion and support for Latin* QT students. Daniela's ability to process complexities of being a mixed-race queer Latina in SHPE exemplifies how such intersectional support was available to her in a space for racial affinity.

Conclusion

This study portrays undergraduate Latin* QT students' experiences at the borderlands as aspiring STEM majors navigating white cisheteropatriarchy in mathematical contexts. Ros's and Daniela's counter-stories shed light on dissonance between their intersectional identities and exclusionary peer relationships, in addition to their individual agency for coping with such dissonance to protect their identities and success in STEM. With participants' accounts of agency involving the sacrifice of their queer identities, disruptions of mathematics education as a white, cisheteropatriarchal space are needed to advance intersectional justice for Latin* QT students.

Acknowledgments

Thank you to the organizers of the PME-NA 44 conference for the invitation to deliver a plenary based on the study presented in this paper. The reported research was funded by the National Academy of Education/Spencer Foundation Postdoctoral Fellowship and the Vanderbilt-Peabody College Small Research Grant initiative. I am grateful for feedback from Elsie Kindall, Taylor McNeill, and fellow PRISM Lab members on an earlier draft of the paper.

References

Aguilar-Hernández, J. M., & Cruz, C. (2020). Grounding emerging scholarship on queer/trans* Chicana/o/x and Latina/o/x pedagogies. *Association of Mexican American Educators Journal, 14*(2), 5-22.

Anzaldúa, G. (1987/2007). *Borderlands/La frontera: The new mestiza*. Aunt Lute Books.

Brockenbrough, E. (2015). Queer of color agency in educational contexts: Analytic frameworks from a queer of color critique. *Educational Studies, 51*(1), 28-44.

Campus Pride (2020). *Campus Pride's 2020 Best of the Best LGBTQ-friendly colleges & universities. https://www.campuspride.org/2020BESTOFTHEBEST/.*

Cantú, N., & Hurtado, A. (2012). Introduction to the fourth edition. In *Borderlands/La frontera: The new mestiza*. Aunt Lute Books.

Cech, E. A., & Waidzunas, T. J. (2011). Navigating the heteronormativity of engineering: The experiences of lesbian, gay, and bisexual students. *Engineering Studies, 3*(1), 1-24.

Convertino, C., Alvidrez, M., Pickett, J., & Camberos, D. (2022). Exploring worksheets and attendance-taking as gatekeeping features in a gateway course for Latinx STEM majors. *Journal of Latinos and Education*, 1-17.

Crenshaw, K. (1991). Mapping the margins: Identity politics, intersectionality, and violence against women. *Stanford Law Review, 43*(6), 1241-1299.

Cuevas, T. J. (2018). *Post-Borderlandia: Chicana literature and gender variant critique*. Rutgers University Press.

Excelencia in Education (2018). *Latino college completion: United States. Excelencia in Education*. Retrieved from https://www.edexcelencia.org/research/latino-college-completion.

Ferguson, R. A. (2004). *Aberrations in Black: Toward a queer of color critique*. University of Minnesota Press.

Gutiérrez, R. (2017). Why mathematics (education) was late to the backlash party: The need for a revolution. *Journal of Urban Mathematics Education, 10*(2), 8-24.

Hames-García, M., & Martínez, E. J. (2011). *Gay Latino studies: A critical reader*. Duke University Press.

Harper, S. R., & Simmons, I. (2019). *Black students at public colleges and universities: A 50-state report card*. Los Angeles: University of Southern California, Race and Equity Center.

Hernández, E. D., Alvarez, E. F., & García, M. (2021). *Transmovimientos: Latinx queer migrations, bodies, and and spaces*. University of Nebraska Press.

Hughes, B. E. (2017). " Managing by not managing": How gay engineering students manage sexual orientation identity. *Journal of College Student Development, 58*(3), 385-401.

Kersey, E., & Voigt, M. (2020). Finding community and overcoming barriers: Experiences of queer and transgender postsecondary students in mathematics and other STEM fields. *Mathematics Education Research Journal*.

Kumashiro, K. K. (2001). *Troubling intersections of race and sexuality: Queer students of color and anti-oppressive education*. Rowman & Littlefield.

Leyva, L. A. (2016). An intersectional analysis of Latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education, 9*(2), 81-121.

Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 397-433.

Leyva, L. A. (2021). Black women's counter-stories of resilience and within-group tensions in the white, patriarchal space of mathematics education. *Journal for Research in Mathematics Education, 52*(2), 117–151.

Leyva, L. A. (2022). "We can't just turn that off and then do some physics": A counter-storytelling analysis of introductory physics as a white, cisheteropatriarchal space in undergraduate STEM education. In B. W. Frank, D. Jones & Q. Ryan (Eds.), *2022 Physics Education Research Conference (PERC) Proceedings*. American Association of Physics Teachers. Grand Rapids, MI. https://doi.org/10.1119/perc.2022.plenary.pr.Leyva.

Leyva, L. A. (2023). *Undergraduate mathematics education as a white, cisheteropatriarchal space and opportunities for structural disruption to advance queer of color justice*. Invited presentation delivered as the Spectra Lavender Lecture for the 2023 Joint Mathematics Meetings. Boston, MA.

Leyva, L. A., Balmer, B. R., Marshall, B. L., McNeill, R. T., Jefferson, A., & Roditti, N. (2022). Characterizing agency among gender-expansive Black and Latin* students with queer sexual identities in undergraduate STEM. Paper presented annual meeting of the American Educational Research Association. San Diego, CA.

Leyva, L. A., McNeill, R. T., Balmer, B. R., Marshall, B. L., King, V. E., & Alley, Z. D. (2022). Black queer students' counter-stories of invisibility in undergraduate STEM as a white, cisheteropatriarchal space. *American Educational Research Journal, 59*(5), 863–904.

Leyva, L. A., McNeill, R. T., & Duran, A. (2022). A queer of color challenge to neutrality in undergraduate STEM pedagogy as a White, cisheteropatriarchal space. *Journal of Women and Minorities in Science and Engineering, 28*(2), 79-94.

Leyva, L. A., & Nazemi, M. (in press). Toward intersectional understandings of gender in mathematics education. In M. A. Clements, B. Kaur, T. Lowrie, V. Mesa, & J. Prytz (Eds.), *Fourth international handbook of mathematics education*. Springer.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition & Instruction, 39*(1), 1-34.

Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record, 111*(2), 295-338.

McGee, E. O. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture?. *American Educational Research Journal, 53*(6), 1626-1662.

Miller, R. A., Vaccaro, A., Kimball, E. W., & Forester, R. (2020). "It's dude culture": Students with minoritized identities of sexuality and/or gender navigating STEM majors. *Journal of Diversity in Higher Education, 14*(3), 340.

Milner, R. H. (2007). Race, culture, and researcher positionality: Working through dangers seen, unseen, and unforeseen. *Educational Researcher, 36*(7), 388–400.

Moraga, C., & Anzaldúa, G. (1981/2021). *This bridge called my back: Writings by radical women of color*. SUNY Press.

Nicolazzo, Z. (2016). 'It's a hard line to walk': Black non-binary trans* collegians' perspectives on passing, realness, and trans*-normativity. *International Journal of Qualitative Studies in Education, 29*(9), 1173-1188.

Nicolazzo, Z. (2017). *Trans* in college: Transgender students' strategies for navigating campus life and the institutional politics of inclusion*. Stylus.

Oppland-Cordell, S. (2014). Urban Latina/o undergraduate students' negotiations of identities and participation in an Emerging Scholars Calculus I workshop. *Journal of Urban Mathematics Education, 7*(1), 19-54.

Rodríguez, J. M. (2003). *Queer Latinidad: Identity practices, discursive spaces*. NYU Press.

Salinas, C. Jr. (2020). The complexity of the "x" in Latinx: How Latinx/a/o students relate to, identify with, and under- stand the term Latinx. *Journal of Hispanic Higher Education, 19*(2), 149–168.

Salinas, C., Jr., & Lozano, A. (2019). Mapping and recontextualizing the evolution of the term Latinx: An environmental scanning in higher education. *Journal of Latinos and Education, 18*(4), 302–315.

Solórzano, D., & Yosso, T. (2002). Critical race methodology: Counter-storytelling as an analytical frame- work for education research. *Qualitative Inquiry, 8*(1), 23–44.

Vega, G. P. (2016). *Latina lesbian students: Understanding their experiences and perceived sexual identity development at a Hispanic-serving institution*. Unpublished doctoral dissertation.

Vega, G. P., Duran, A., McGill, C. M., & Rocco, T. S. (2022). Attending to sexuality in servingness: A phenomenological exploration of the experiences of Latina lesbians at a Hispanic-serving institution. *Journal of Diversity in Higher Education*.

Voigt, M. (2022). A quantitative exploration of Queer-spectrum students' experiences in introductory undergraduate mathematics courses. *PLoS ONE, 17*(10), e0275325.

Yeh, C., & Rubel, L. (2020). Queering mathematics: Disrupting binary oppositions in mathematics pre-service teacher education. In N. Radakovic & L. Jao (Eds.), *Borders in mathematics pre-service teacher education* (p. 227-243). Springer.

Chapter 2:

Curriculum, Assessment, and Related Topics

“MIRROR LOGIC”: A PRESERVICE MATHEMATICS TEACHER’S THINKING ABOUT RADIAN IN THE CONTEXT OF LIGHT REFLECTION

Hanan Alyami
Purdue University
alyamih@purdue.edu

Lynn Bryan
Purdue University
labryan@purdue.edu

Integrated science, technology, engineering, and mathematics (iSTEM) education allow learners to utilize multiple disciplinary perspectives. However, the discipline of mathematics remains underrepresented in iSTEM curriculum. To explore the nature of mathematical thinking with an iSTEM curricular approach that emphasizes mathematics, we investigated the thinking of a preservice mathematics teacher, Alex (pseudonym), who engaged in a task-based digital activity involving radian angle measure in the context of light reflection. Findings suggest that Alex’s ways of thinking comprise mathematical terminology, concepts, and processes, including mathematical ways of thinking about light reflection. The findings in this report suggest that emphasizing mathematics in this iSTEM context provided an opportunity for new ways of thinking about radian angle measure, and about how angle measure relates to light reflection.

Keywords: curriculum, geometry and spatial thinking, integrated STEM

Despite more than a decade of science, technology, engineering, and mathematics (STEM) reform initiatives toward integrated STEM (iSTEM) approaches (National Academy of Engineering [NAE] and National Research Council [NRC], 2014; National Council of Supervisors of Mathematics [NCSM] and National Council of Teachers of Mathematics [NCTM], 2018; NRC, 2013), the underrepresentation of mathematics in iSTEM education curriculum remains (English, 2016; Fitzallen, 2015). Given the need to develop iSTEM curriculum where mathematics holds equal importance as other disciplines (Baker & Galanti, 2017), it is incumbent upon educators to find ways of foregrounding mathematics within iSTEM experiences to better develop learners’ understanding not only of core mathematics content and practices but also about how core mathematics content and practices meaningfully relate to other disciplines. English (2016) and Fitzallen (2015) called for approaches that address how mathematical concepts and practices contribute to the learning and understanding of other STEM disciplines in iSTEM instructional contexts. Additionally, Li et al. (2019) called for research that attends to how thinking in content-based approaches relates to thinking in other disciplines. We take up these calls (English, 2016; Fitzallen, 2015; Li et al., 2019) with purposeful attention to situating mathematics as pivotal in the iSTEM experience. We argue that iSTEM experiences that foreground mathematics can contribute to mathematical thinking, and we consider how mathematical thinking relates to other STEM disciplines. We specifically explore a preservice mathematics teacher’s [PMT’s] thinking about the mathematical concept of radian angle measure in the context of light reflection. The research question guiding this report is “What ways of thinking does a PMT demonstrate upon interacting with a digital task that involves radian angle measure and light reflection?”

Theoretical Framing

To answer the research question, we took a constructivist perspective (Schunk, 2012) on the construct of thinking, and spatial thinking in particular, which we describe in the following sections. Additionally, we clarify our perspective and definition of iSTEM curricular approaches.

iSTEM Curriculum

Curricular approaches that involve iSTEM have been defined in various ways in the literature (Navy et al., 2021), with teachers, administrators, and policy makers having different views on iSTEM education (Breiner et al., 2012; Holmlund et al., 2018). In this report, our perspective of iSTEM curriculum involves instructional activities with learning goals of content and/or practices from one or more of the STEM disciplines, as anchors, along with engineering and/or engineering design practices, as integrators. Additionally, iSTEM curriculum activities involve opportunities to emphasize twenty-first century skills in a real-world, authentic context, to be solved through collaboration, communication, and teamwork (Bryan & Guzey, 2020).

There are many challenges when it comes to implementing iSTEM curricular approaches (English, 2016; Fitzallen, 2015). One of these involves distinctions in the knowledge base between disciplines (Williams et al., 2016). For example, discipline-specific words have explicit definitions and are used in unique ways in that discipline (Morgan & Sfard, 2016) despite use and overlap of such words in other disciplines. For example, the light reflection principle (Figure 1) is understood as the equality of the angle of incidence and the angle of reflection relative to a perpendicular to the mirror known as the normal ($\alpha = \beta$). The light reflection principle can also be understood as the equality of the angles of incidence and reflection relative to the mirror ($\gamma = \delta$). In this context, the term *normal line* refers to perpendicularity in relation to the scientific phenomenon of light reflection. A person with a mathematical perspective might refer to the normal line in the context of light reflection, using its mathematical property of perpendicularity, rather than using the term itself.

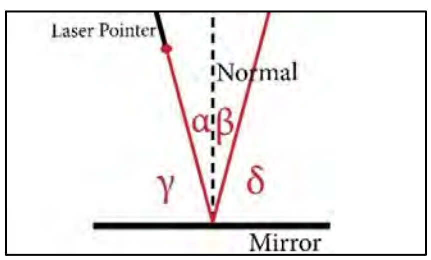


Figure 1. Demonstration of the principle of light reflection

Despite the challenges, research and reviews have reported the effectiveness of iSTEM education approaches on learners' engagement, motivation, interest in STEM, and increased mathematical achievement (Honey et al., 2014; Stohlmann, 2018; Zhong & Xia, 2020). However, little is known about learners' mathematical thinking as they engage in iSTEM instruction (Li et al., 2019). Hence, this study focuses on the ways of thinking that are involved in an iSTEM task that involves radian angle measure and light reflection.

Ways of Thinking

Our definition of ways of thinking builds on Harel's (2008) description of thinking as a learner's established cognitive characteristics, and Thompson et al.'s (2014) extension of Harel's (2008) definition, where thinking is the consistency in a learner's reasoning about mathematical situations. Building on these descriptions, we interpret ways of thinking as the thought patterns a learner demonstrates when reasoning about a particular concept given a specific situation that evokes such reasoning. For example, researchers demonstrated that PMTs think of radian angle measure as angles expressed in terms of π (Akkoc, 2008; Fi, 2003). Additionally, Moore et al. (2016) reported that PMTs' thinking about radian angle measure incorporates a unit circle diagram (Figure 2) to perform calculations. These studies suggest that through their prior coursework and experiences, PMTs may have developed a thought pattern to reason about radian angle measure. Such reasoning makes the use of special angles expressed in terms of π, and/or using calculational strategies an established way of thinking about radian angle measure.

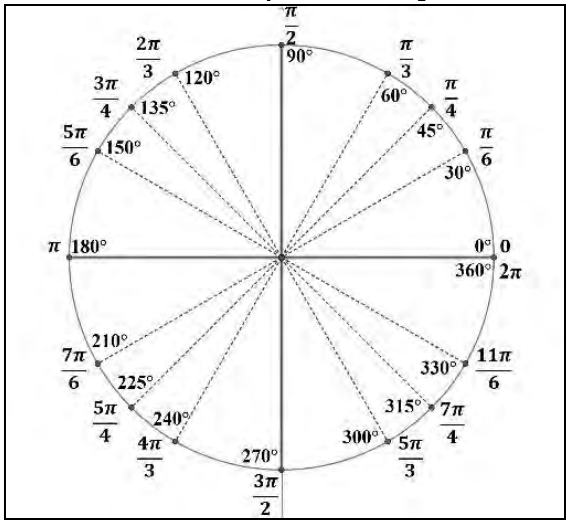


Figure 2. A typical diagram of the unit circle

Spatial Thinking

We characterize ways of thinking with particular attention to spatial ways of thinking. Commonly known as spatial reasoning, we use the term spatial ways of thinking to refer to thought patterns that include “the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects” (Bruce et al., 2017, p. 146), through “a collection of cognitive skills comprised of knowing concepts of space, using tools of representation, and reasoning processes” (NRC, 2006, 12). Spatial thinking is associated with various disciplines and is correlated with achievements in both mathematics (e.g., Clements & Sarama, 2009; Mix, 2019; Mulligan et al., 2018), and other STEM disciplines (e.g., Newcombe, 2010, 2013; Newcombe & Shipley, 2015; Pruden et al., 2011). However, there are few opportunities for students to engage in spatial thinking in school (Clements & Sarama, 2011; Sinclair & Bruce, 2015; Whiteley et al., 2015). Because spatial thinking is associated with achievements in mathematics in addition to achievements in other disciplines (Bruce et al., 2017), it is appropriate to investigate the ways of thinking involved in an iSTEM curricular approach with attention to spatial ways of thinking.

While spatial thinking is usually associated with visualization, Whiteley et al. (2015) suggested addressing and legitimizing broader spatial ways of thinking, including symmetrizing, comparing, decomposing-recomposing, situating, orienting, and scaling. We describe these spatial ways of thinking in the methods section (Table 1), however, we note that the spatial ways of thinking we mentioned do not represent all spatial ways of thinking, nor do they exist in isolation of each other and/or other ways of mathematical thinking (Davis et al., 2015). For example, Munier and Merle (2009) built on NCTM’s (2000) recommendation to interrelate geometry and spatial thinking to provide 3-5 graders the opportunity to explore angle measure through physics-based teaching sequences, one of which included light reflection. Through an iterative process of spatial experimentation and geometric knowledge development, the 3-5 graders were able to discover light reflection principle, by attending to the symmetry between the angle of incidence and the angle of reflection relative to the mirror (γ = δ in Figure 1). This illustrates students’ use of symmetrizing as a form of spatial thinking in conjunction with the notion of angle as a form of mathematical thinking to discover light reflection principle.

Methods

Research Design

We employed a qualitative case study design (Flyvbjerg, 2011) to examine a PMTs mathematical thinking during a lesson that was part of an iSTEM unit embedding the design of a periscope (Alyami, in-press). The lesson entailed a digital task that involved radian angle measure in the context of a light reflection scenario (described further in the following section).

Participant and Task

The PMT participating in this study was Alex (pseudonym), who was enrolled in a mathematics teacher preparation program at a large Midwestern university. Alex volunteered and was compensated for his time after the first author briefly presented the opportunity in his secondary mathematics methods course. While Alex likely encountered the concepts of radian angle measure and light reflection during his K-16 schooling, he was not offered a formal learning session about radian or light reflection prior to participating in this study.

A Desmos activity (i.e., *Radian Lasers*) comprised the task in this report, where Alex typed values of angle measure (in radian) to adjust a laser and one or two mirrors so the laser beam would successfully pass through three stationary targets at once (Figure 1). The angles of the laser and the mirror are relative to the horizontal and in standard position, where positive angle values are counterclockwise and negative angle values are clockwise. The task consisted of two warm-up activities to familiarize Alex with the functionality of the digital interface, followed by six challenges. A benefit of the *Radian Laser* task is that the angles needed to situate the mirror were not limited to the common special angle (e.g., π/6, π/3, π/2). For example, one way of solving Challenge 1 is by positioning the laser upwards at an angle that is 5π/6 radian, with the mirror angled at a 5π/12 radian, which is not a common special angle (Figure 4).

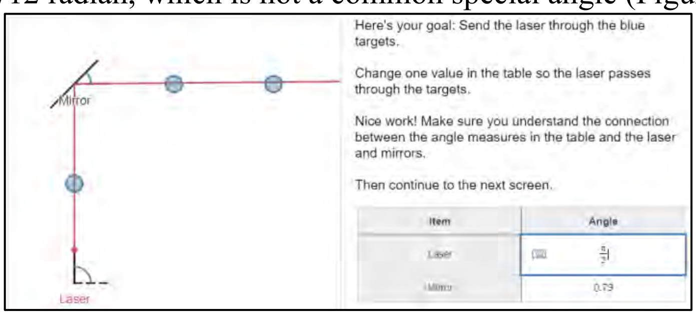


Figure 3. A challenge from the *Radian Lasers* activity

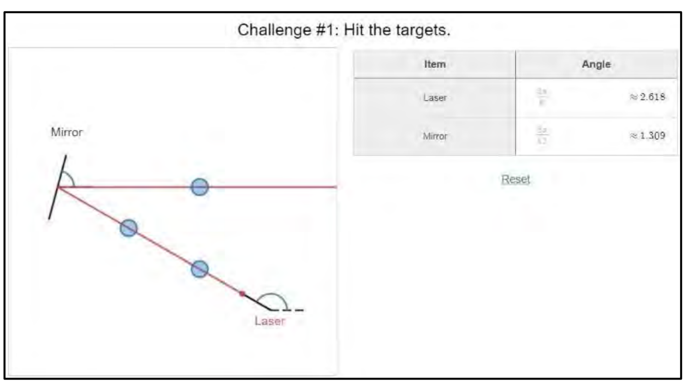


Figure 4. Challenge 1, where the angle of the mirror is not a common special angle

Following principles of structured, task-based interviews (Goldin, 2000), Alex engaged with the *Radian Lasers* task in a semi-structured, think-aloud interview setting, where he could use his own language to make sense of the task (van Someren et al., 1994). The semi-structured setting provided an opportunity to ask for elaborations (e.g., How do you know that? How would you represent your thinking?), which were informed by the responses Alex provided throughout the interview to encourage him to clarify his thinking.

Data and Analysis

The think-aloud, semi-structured interviews, led by the first author, took place virtually, were video recorded, and lasted approximately one hour. The interview video and time-stamped transcript comprise the data for this study. To analyze the data, the first author used a whole-to-part inductive approach for coding (Erickson, 2006), beginning with playing and watching the whole video without coding, stopping, or pausing. Then, NVivo software was used to code the media file, as described by Wainwright and Russell (2010). At this stage, the unit of analysis consisted of one or more sentences that formed coherent statements in which Alex described his thinking about how to reposition the angle of the laser and/or the mirror. To further analyze these statements, we used thematic analysis, which is a coding strategy that involves identifying themes in the data that are informed by the research questions, theoretical framework, and literature review (Saldaña, 2013). Specifically, we coded Alex's statements with attention to spatial ways of thinking described in Whiteley et al. (2015). As part of thematic analysis, we were open to the development of new categories that emerged from the data. Table 1 contains the codes that were evident in the data.

Table 1: Coding Scheme of All Spatial Ways of Thinking Utilized by Alex

Code	Description
Locating	Thinking about *where* objects are situated and/or positioned.
Orienting	Thinking of *how* objects are situated and/or positioned in relation to each other.
Comparing	Thinking about angle size in relation to itself or another angle (bigger, smaller, etc.)
Decomposing-Recomposing	Thinking of a whole as spatially broken into a specific number of parts, and/or spatially adding up parts to form a specific whole.
Symmetrizing	Thinking and applying properties such as congruence and symmetry with similar parts spatially facing each other around an axis.
Visualizing	Thinking visually of geometric objects and managing their characteristics. Includes: Managing both visible and imagined visual information.
Diagramming	Thinking of and managing geometric objects and patterns through drawing Includes: Gestures depicting semantic content (e.g., tracing angles as if to represent the angles on a typical unit-circle diagram) (Sinclair et al., 2018).

After coding all the transcript, the first author reviewed statements that were coded with multiple codes to provide a meaningful interpretation of the coded data "so that more can be gleaned from the data than would be available from merely reading, viewing, or listening carefully to the data multiple times" (Simon, 2019, p. 112). To answer the research question and provide evidence for our argument, the interpretation of the data focused on Alex's ways of thinking in relation to radian angle measure, and in relation to light reflection.

Findings

We describe in this section the mathematical and spatial ways of thinking Alex demonstrated upon engagement with the digital task, *Radian Lasers*. To argue that this iSTEM experience, which foregrounds mathematics, contributes to mathematical thinking and brings mathematics in relation to other STEM disciplines, we organize the two sections of the findings to start with Alex's ways of thinking in relation to radian angle measure. We then describe his ways of thinking in relation to making sense of light reflection.

Ways of Thinking about Radian Angle Measure Beyond the Special Angles

Since the angles needed in *Radian Lasers* are not limited to common special angle (e.g., π/6, π/3, π/2), Alex needed to think beyond the special angles commonly represented on a typical diagram of the unit circle (Figure 2). When a challenge required a noncommon angle, Alex estimated the measure of angle based on spatial comparisons and estimating the angles in between. For example, to reason about Challenge 1 (Figure 4), Alex initially situated the mirror at π/3 radian (left side of Figure 5), and noted that "the laser is hitting the mirror at a perpendicular angle … and that tells me that this angle needs to be slightly bigger." The previous statement suggests that Alex is observing the result of situating the mirror at π/3 radian angle, and then comparing the size of the resulting angle in relation to the angle that would lead the laser to hit the third target. Alex then entered π/2 for the mirror (right side of Figure 5), as he stated that "maybe π/2 would increase the angle," which caused the laser to reflect beyond where the third target is located. Upon missing the third target, Alex said "Okay, so I know it's between π/3 and π/2."

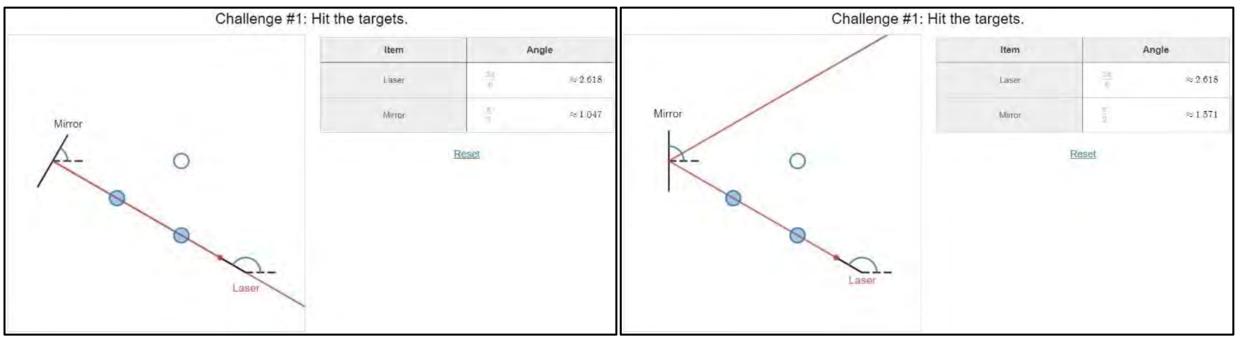


Figure 5. Alex's trials of π/3 & π/2, sending the laser respectively below & above the target

However, Alex was not familiar with a special angle that is between π/2 and π/3, and asked the first author if he could try an input such as 2.5π/6 for the angle. When he entered 2.5π/6, he observed the laser hit the third target. The interviewer asked Alex to explain why he questioned his ability to use the fraction, 2.5π/6. Alex explained:

> The 2.5π/6 was not like an option in my head because that's not, like one of the things that are usually mentioned or like associated with like radians. The reason why I got there is because I knew it was in between π/2 and π/3 but with the list of all, like the radians that I know, π/2 and π/3 are, like right next to each other, and there's, like no whole number π over anything in between those two numbers.

Alex's explanation is in reference to a typical diagram of a unit circle (Figure 2), where π/3 and π/2 are represented without depicting other angles between them. Alex recognized the need for an angle between π/3 and π/2, which led him to try the value between 2π/6 and 3π/6. He concluded that the angle would be 2.5π/6.

Mathematical Ways of Thinking about Light Reflection

In this section, we provide an analysis of Alex's ways of thinking as he makes sense of the scientific phenomenon of light reflection. We describe Challenge 3 of the *Radian Lasers* which could be solved by positioning the laser downward at an angle that is -π/6 radian, with the mirror being angled at a 5π/12 radian, to reflect the laser beam to the third target at the bottom (Figure 6).

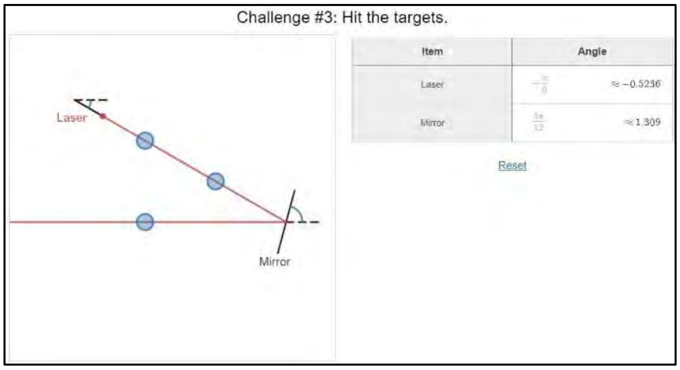


Figure 6. A possible solution for Challenge 3 from the *Radian Lasers* task

Alex used diagramming to represent his visualization of the challenge as alternate interior angles, then extended his diagramming with a focus on the mirror (Figure 7). Alex explained that "this entire thing [points to the mirror in Figure 7] is π, and so then I tried to find out what these two [the angles of incidence and reflection relative to the mirror] would be remaining, and I got … 5π/12 because there's two of them so I have to split the angle in half, because these two angles are equal." Alex's explanation of the mirror as a straight angle where its measure represents the "entire thing is π," suggests his thinking about the straight angle as a whole. Alex then decomposed the straight angle into π/6, which he concluded from the alternate interior angle theorem, and two angles that "would be remaining." Alex's elaboration demonstrates his attention to symmetry as he has "to split the angle in half because these two angles are equal."

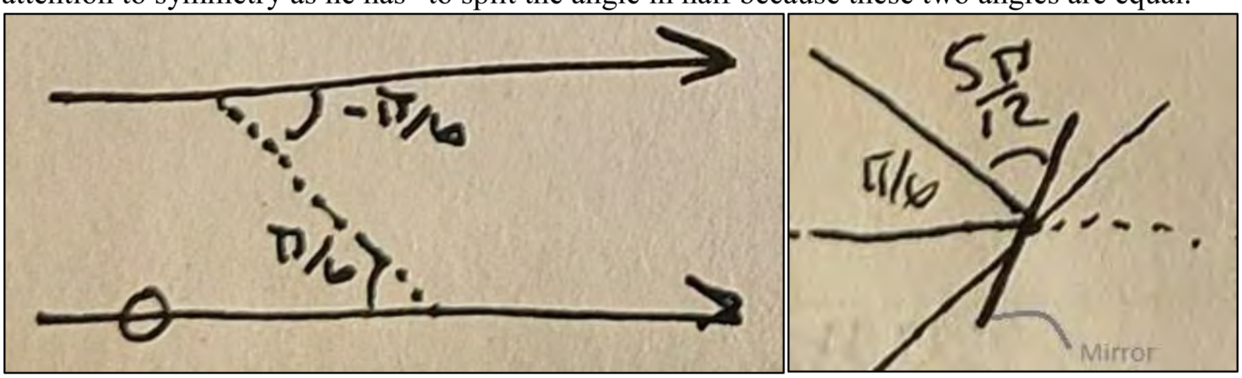


Figure 7. Alex's diagramming of Challenge 3 of the *Radian Lasers*

This example illustrates Alex's mathematical ways of thinking to make sense of the angle at which to situate the mirror. Alex's ways of thinking involved mathematical concepts (i.e., alternate interior angles), as well as various spatial ways of thinking (i.e., diagramming, decomposing, and symmetrizing). Alex went further to describe the light reflection principle from a mathematics perspective. Specifically, when Alex described his diagramming for Challenge 3 (Figure 7), he pointed at the angle the laser makes with the mirror upon reflection and stated, "this whole entire angle is π/6. I know the bisector, it, each angle would be like π/12. Then I know that this angle bisector is perpendicular with, you know, the mirror." Alex's description of the bisector of π/6 as "perpendicular" to the mirror illustrates his mathematical thinking about the science of light reflection, which he referred to as "adjust[ing] for mirror logic." The significance of the angle's bisector is because the angle the laser makes upon reflecting from the mirror (i.e., π/6) is the sum of the angles of incidence and reflection relative to the normal ($\gamma = \delta$ in Figure 1). However, Alex described the mathematical property of the

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

normal line as perpendicular to the mirror, instead of using the science terminology. Alex's explicit description of the science of light reflection through his mathematical perspectives was not elicited by the interviewer. Alex utilized his mathematical ways of thinking to make sense of the principle of light reflection. While Alex did not explicitly use the terminology, "angle of incidence" or "angle of reflection," he was meaningfully incorporating mathematical thinking to make sense of the science of light reflection.

Discussion

To date, there are few iSTEM curriculum materials that emphasize mathematical concepts as the anchor discipline (English, 2016; Fitzallen, 2015), despite evidence of the benefits of iSTEM curricular approaches on mathematical achievement and development of mathematical understanding (Stohlmann, 2018). Additionally, there are few opportunities to engage in spatial thinking in schools (Clements & Sarama, 2011; Sinclair & Bruce, 2015; Whiteley et al., 2015), despite the role of spatial thinking in understanding both mathematics (e.g., Mix, 2019) and other disciplines (e.g., Newcombe & Shipley, 2015; Pruden et al., 2011). Our report illustrates a purposeful integration approach with a focus on mathematics in the context of science within an iSTEM unit. We argue that iSTEM experiences that foreground mathematics can meaningfully contribute to mathematical thinking, in addition to enhancing how mathematics relates to other STEM disciplines. The *Radian Lasers* as an iSTEM approach that emphasized mathematics provided Alex, a PMT, the opportunity to utilize mathematical and spatial ways of thinking about radian angle measure (e.g., alternate interior angle, perpendicular lines, angle bisector, visualization, diagramming, comparing), and to relate angle measure to light reflection principle.

Previous studies suggest that PMTs' thinking about radian angle measure is limited to special angles expressed in terms of π (Akkoc, 2006; Fi, 2003) and calculational strategies using the unit circle (Moore et al., 2016). Similarly, Alex initially referred to some of the special angles on the unit circle. However, the *Radian Lasers* task constrained these established ways of thinking as Alex was not able to only depend on few special angles. Alex used spatial comparison to think beyond the special angles that are associated with the unit circle. This suggests that the *Radian Lasers* as an iSTEM activity that focused on radian angle measure in a science context provided an opportunity for Alex to reason about radian angle measure beyond the special angles. Alex's use of multiple spatial ways of thinking reflects Davis et al.'s (2015) discussion that the spatial ways of thinking do not exist in isolation of each other and/or in isolation of other mathematical ways of thinking. Specifically, to reason about the functionality of the mirror, Alex used diagramming as a spatial way of thinking in relation to a mathematical concept (i.e., alternate interior angle theorem), and in relation to other spatial ways of thinking (i.e., Symmetry and Decomposing-Recomposing).

Alex's mathematical ways of thinking assisted him in not only applying mathematical content and processes (e.g., alternate interior angles, spatial thinking), but also in making sense of light reflection. The iSTEM activity in which Alex engaged brought mathematics to bear in a situation that represents a scientific phenomenon, which allowed for the construction of a relationship between a scientific phenomenon and a mathematical concept. Our findings align with English (2016) and Fitzallen's (2015) call for educators and curriculum developers to capitalize on iSTEM approaches that emphasize mathematics, as well as Li et al.'s (2019) call for research that explores thinking in iSTEM contexts. The report illustrates how iSTEM approaches that foreground mathematics have the potential to support learners' thinking of not only mathematical content, but also meaningful mathematical applications and connections to other STEM disciplines.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science and Technology, 39*(7), 857–878. https://doi.org/10.1080/00207390802054458

Alyami, H. (in-press). A radian angle measure and light reflection activity. *Mathematics Teacher: Learning and Teaching Pre-K–12.*

Baker, C. K., & Galanti, T. M. (2017). Integrating STEM in elementary classrooms using model-eliciting activities: Responsive professional development for mathematics coaches and teachers. *International Journal of STEM Education*, *4*(10), 1–15. https://doi.org/10.1186/s40594-017-0066-3

Breiner, J. M., Harkness, S. S., Johnson, C. C., & Koehler, C. M. (2012). What is STEM? A discussion about conceptions of STEM in education and partnerships. *School Science and Mathematics*, *112*(1), 3–11. https://doi.org/10.1111/j.1949-8594.2011.00109.x

Bruce, C. D., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, M., Francis, K., Hawes, Z., Moss, J., Mulligan, J., Okamoto, Y., Whiteley, W., & Woolcott, G. (2017). Understanding gaps in research networks: Using "spatial reasoning" as a window into the importance of networked educational research. *Educational Studies in Mathematics, 95*(2), 143–161. https://doi.org/10.1007/s10649-016-9743-2

Bryan, L., & Guzey, S. S. (2020). K-12 STEM Education: An overview of perspectives and considerations. *Hellenic Journal of STEM Education, 1*(1), 5–15. https://doi.org/10.51724/hjstemed.v1i1.5

Clements, D. H. & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. Taylor & Francis.

Clements, D. H. & Sarama, J. (2011). Early childhood teacher education: The case of geometry. *Journal of Mathematics Teacher Education, 14*(2), 133–148.

Davis, B. Okamoto, Y. & Whiteley, W. (2015). Spatializing school mathematics. In B. Davis & the Spatial Reasoning Study Group (Eds.), *Spatial reasoning in the early years: Principles, assertions, and speculations*. (pp. 139-150). Routledge.

English, L. D. (2016). STEM education K-12: Perspectives on integration. *International journal of STEM education, 3*(1), 1–8. https://doi.org/10.1186/s40594-016-0036-1

Erickson, F. (2006). Definition and analysis of data from videotape: Some research procedures and their rationales. In J. Green, G. Camilli, & P. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 177–192). Taylor & Francis.

Fi, C. (2003). *Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge and envisioned pedagogy* [Unpublished doctoral dissertation]. University of Iowa.

Fitzallen, N. (2015). STEM education: What does mathematics have to offer? In M. Marshman (Ed.). *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia, Sunshine Coast, June 28-July 2* (pp. 237–244). MERGA.

Flyvbjerg, B. (2011). Case study. In Denzin, N., & Lincoln, Y. (Eds.). *The Sage handbook of qualitative research* (4th ed.). (pp. 301–316). Sage Publications.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates.

Harel, G. (2008). DNR perspective on mathematics curriculum and instruction, part I: Focus on proving. *ZDM, 40*(3), 487–500. https://doi.org/10.1007/s11858-008-0104-1

Holmlund, T. H., Lesseig, K., & Slavit, D. (2018). Making sense of "STEM education" in K-12 contexts. *International Journal of STEM Education*, *5*(32), 1–18. https://doi.org/10.1186/s40594-018-0127-2

Honey, M., Pearson, G., & Schweingruber, H. A. (2014). *STEM integration in K-12 education: Status, prospects, and an agenda for research*. The National Academies Press.

Li, Y., Schoenfeld, A. H., diSessa, A. A., Graesser, A. C., Benson, L. C., English, L. D., & Duschl, R. A. (2019). On thinking and STEM education. *Journal for STEM Education Research*, *2*, 1–13. https://doi.org/10.1007/s41979-019-00014-x

Mix, K. S. (2019). Why are spatial skill and mathematics related? *Child Development Perspectives, 13*, 121–126. http://dx.doi.org/10.1111/cdep .12323

Morgan, C., & Sfard, A. (2016). Investigating changes in high-stakes mathematics examinations: A discursive approach. *Research in Mathematics Education*, *18*(2), 92–119. doi:10.1080/14794802.2016.1176596

Moore, K., LaForest, K., & Kim, H. (2016). Putting the unit in pre-service secondary teachers' unit circle. *Educational Studies in Mathematics, 92*(2), 221–243. https://doi.org/10.1007/s10649-015-9671-6

Mulligan, J., Woolcott, G., Mitchelmore, M., & Davis, B. (2018). Connecting mathematics learning through spatial reasoning. *Mathematics Education Research Journal, 30*, 77–87.
Munier, V., & Merle, H. (2009). Interdisciplinary mathematics–physics approaches to teaching the concept of angle in elementary school. *International Journal of Science Education, 31*, 1857–1895.
National Academy of Engineering and National Research Council (2014). *STEM integration in K-12 education: Status, prospects, and an agenda for research*. The National Academies Press.
National Council of Supervisors of Mathematics and National Council of Teachers of Mathematics. (2018). *Building STEM education on a sound mathematical foundation: A joint position statement on STEM.* Author.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics* (Vol. 1). Authors.
National Research Council. (2006). *Learning to think spatially: GIS as a support system in the K–12 curriculum*. National Academic Press.
National Research Council. (2013). *Monitoring progress toward successful K-12 STEM education: A nation advancing?* National Academies Press.
Navy, S., Kaya, F., Boone, B., Brewster, C., Calvelage, K., Ferdous, T., Hood, E., Sass, L., & Zimmerman, M. (2021). "Beyond an acronym, STEM is . . . ": Perceptions of STEM. *School Science and Mathematics, 121*(1), 36–45.
Newcombe, N. S. (2010). Picture this: Increasing math and science learning by improving spatial thinking. *American Educator*, *34*(2), 29–35.
Newcombe, N. S. (2013). Seeing relationships: Using spatial thinking to teach science, mathematics, and social studies. *American Educator*, *57*(1), 26–31.
Newcombe, N. S., & Shipley, T. F. (2015). Thinking about spatial thinking: New typology, new assessments. In J. S. Gero (Ed.), *Studying visual and spatial reasoning for design creativity* (pp. 179–192). Springer.
Pruden, S. M., Levine, S. C., & Huttenlocher, J. (2011). Children's spatial thinking: does talk about the spatial world matter? *Developmental Science, 14*(6), 1417– 1430. https://doi.org/10.1111/j.1467-7687.2011.01088.x
Saldaña, J. (2013). *The coding manual for qualitative researchers*. Sage.
Schunk, D. H. (2012). *Learning theories: An educational perspective* (6th ed.). Pearson.
Simon, M. A. (2019). Analyzing qualitative data in mathematics education. In K. R. Leatham (Ed.), *Designing, conducting, and publishing quality research in mathematics education* (pp. 111–123). Springer.
Sinclair, N., & Bruce, C. (2015). New opportunities in geometry education at the primary school. *ZDM Mathematics Education*, *47*(3). doi:10.1007/s11858-015-0693-4
Sinclair, N., Moss, J., Hawes, Z., & Stephenson, C. (2018). Learning through and from drawing in early years geometry. In K. S. Mix & M. T. Battista (Eds.), *Visualizing mathematics: The role of spatial reasoning in mathematical thought*. (pp. 229–252). Springer.
Stohlmann, M. (2018). A vision for future work to focus on the "M" in integrated STEM. *School Science and Mathematics*, *118*, 310–319. https://doi.org/10.1111/ssm.12301
Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In K. C. Moore, L. P. Steffe, & L. L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (Vol. 4, pp. 1–24). University of Wyoming.
Van Someren, M. W., Barnard, Y. F., & Sandberg, J. A. C. (1994). *The think aloud method: A practical approach to modelling cognitive processes.* Academic Press.
Wainwright, M., & Russell, A. (2010). Using nvivo audio-coding: Practical, sensorial and epistemological considerations. *Social Research Update*, (60), 1–4.
Whiteley, W., Sinclair, N., & Davis B. (2015). What is spatial reasoning? In B. Davis & the Spatial Reasoning Study Group (Eds.), *Spatial reasoning in the early years: Principles, assertions, and speculations*. (pp. 3–14). Routledge.
Williams, J., Roth, W.-M., Swanson, D., Doig, B., Groves, S., Omuvwie, M., Ferri, R. B., & Mousoulides, N. (2016). *Interdisciplinary mathematics education: A state of the art.* Springer Open. https://doi.org/10.1007/978-3-319-42267-1_1
Zhong, B. & Xia, L. (2020). A systematic review on exploring the potential of educational robotics in mathematics education. *International Journal of Science and Mathematics Education*, *18*(1), 79–101. https://doi.org/10.1007/s10763-018-09939-y

COLLECTIVE CURRICULAR NOTICING WITHIN A MATHEMATICS PROFESSIONAL LEARNING COMMUNITY

Erin Barno
Boston University
ebarno@bu.edu

Leslie Dietiker
Boston University
dietiker@bu.edu

This paper explores how a professional learning community (PLC) redesigns high school mathematics lessons towards a shared commitment. We describe the nature of a PLC's collective curricular vision to illuminate how teachers can come to new understandings as a group in order to shift the ways students experience mathematics. Using the curricular noticing framework (attending, interpreting, and responding), we analyzed the meetings of a PLC with six teachers as they individually presented lessons to be redesigned with a focus on the group's shared commitment. Findings indicate three ways ideas were introduced that led to expansive responses, which suggests this analytic approach could identify ways in which a PLC can work towards new curricular decisions.

Keywords: Curriculum, Teacher Noticing, Teacher Beliefs

A central part of mathematics teaching is the design of lessons within a system of constraints (Brown, 2009). After teaching a lesson, teachers may reconsider specific design decisions based on their individual interpretations of the enacted lesson (what we refer to as *redesign*). Recognizing new curricular opportunities, however, is then limited to what prior experiences this individual teacher has had and what personal frameworks they use to make sense of their curriculum (what we refer to as *curricular vision*) (Darling-Hammond et al., 2005; Drake & Sherin, 2009). Teachers work under the demands of larger systemic constraints (e.g., policy, social messages of learning loss) as well as local constraints (e.g., administrative agendas with standardized testing, co-planning opportunities, curriculum access). Teachers, therefore, are limited within their redesign decisions — not by choice, but by pressure.

We are concerned that these constraints on curricular decision-making may prevent teachers from making ambitious changes to their teaching, thus maintaining the status quo for their students' experience and learning. Redesigning a lesson with other teachers that reimagines a given individual's curricular vision by challenging the given constraints can enable different types of experiences to inform teacher curricular decisions. When teachers are members of a professional learning community ("PLC") committed to a shared curricular vision, individuals then have new things to consider, negotiate, and think about in their own redesign choices. Therefore, we are interested in understanding the nature of collective redesign to illuminate how PLCs can come to new understandings as a group with a shared commitment in order to shift the ways students experience mathematics.

In this paper, we begin to answer the questions: (1) *What is the nature of curricular decision making for teachers with a shared design commitment in a PLC?* and (2) *What enables an expansive curricular decision?* We will illustrate how a group of teachers collectively arrived at and engaged with decision decisions that expanded potential opportunities for students. With better understanding of the potential ways groups of teachers can collectively redesign lessons, we hope to inform how PLCs with a shared commitment can potentially catalyze broader opportunities for students in mathematics classrooms.

Theoretical Framework

We use the *curricular noticing framework* (attending, interpreting, and responding) as a way to trace and describe the nature of lesson redesign in the PLC individually and across a group.

Curricular Noticing

Any decision that a teacher makes as part of the profession is shaped by the lenses (*interpretation)* with which they engage. Additionally, teachers' designs are also influenced by the phenomena (e.g., student work, textbook tasks, assessment data) to which they pay attention (*attending)*. Together, these contribute to a teacher's professional decisions to act (*responding*). Teacher's shifting lenses can contribute to teachers *attending* to, *interpreting*, and making pedagogical *responses* consistent with a particular goal when reflecting on a lesson enactment (Baldinger, 2017; Louie, 2018; Louie et al., 2021; van Es et al., 2014). *Curricular noticing* builds on the teacher noticing framework to describe how the phases of noticing (attending, interpreting, and responding) take place when teachers are designing or adopting any form of curricular materials (e.g., textbooks, enacted videos of lessons) (author, year). Although teachers may pass through a sequence of attending-interpreting-responding ("A-I-R"), the phases are not strictly sequential so that, for example, an interpretation may cause a teacher to redirect their attention. In addition, not all may lead to curricular decisions. For example, a teacher's interpretation that follows their attention may lead them to decide to not move forward with a particular action (i.e., response).

Increasingly, the noticing framework has begun to provide insight into how teachers identify patterns towards challenging systemic inequities in mathematics classrooms. Research in teacher noticing has begun to explore how individual teaching stances, such as deficit perspectives (Louie, 2018; Louie et al., 2021) or particular goals (Hand, 2012) can influence the extent to which teachers' interpretations lead to disrupting or perpetuating inequitable practices via responses. However, curricular noticing has not yet explored how a teacher's frameworks (i.e., curricular vision) can impact possible curricular responses. We argue that curricular noticing can support our understanding of how teachers challenge and disrupt systematic patterns of thinking due to the structural systems of schooling and testing, particularly within mathematics education.

Curricular Vision as a Commitment within PLCs

An individual's curricular vision, when interacting with others' own curricular vision, can create disruptive *responses* within group lesson design. Because curricular noticing happens from the standpoint of an individual's curricular vision (Dietiker et al., 2018), a group of individuals working together in a PLC means there are multiple curricular visions (potentially overlapping or shared in some cases). These curricular visions are positioned to interact and influence what responses are possible from individuals within the PLC and the PLC as a collective. When a group of teachers choose to align their own curricular visions towards new possibilities (which we will call *collective curricular vision*), their collaboration can support curricular responses related to students' potential mathematical learning experiences in new ways.

We conceptualize this shared stance for a collective curricular vision as a form of *commitment*. Evidence of a member's commitment to the collective curricular vision, therefore, is the explicit intention of aligning to a shared stance based not only on personal frameworks but also shared frameworks. The connection between a groups' shared commitment and the alignment to that commitment can either make space for new types of curricular responses (what we will call *expansive* responses) or prevent curricular responses from taking form during a group conversation (what we will call *restrictive* responses).

Methods

This study is a qualitative analysis of a lesson redesign meeting with a group of teachers with a shared commitment for how their interactions led to potential expansive opportunities for students. Pairs of teachers from three high schools (one urban comprehensive public school, one suburban comprehensive public school, and one urban private charter) in the Northeast were selected to participate. The 6 teachers, along with 2 researchers, participated in the redesign meeting. At this meeting, teachers took turns sharing video clips and data from a lesson they had taught the previous year and which they wanted to redesign. The data used here is based on audio recordings of workshops between the teacher who shared the lesson (the "lead teacher") and other teachers in the professional learning community. This design group was part of a larger project aimed at creating aesthetic opportunities for their students. Using video of enacted lessons as a form of curricular materials allows for curricular noticing to take in embodied, emotional, and verbal expressions of mathematics engagement.

Case Selection Process

Of the six redesign meetings, two were selected to be analyzed. To select, we looked for three qualities: (1) the participation of multiple members of the group; (2) conversation that specifically attended to the shape of content (i.e., curricular); and (3) conversation that included reference to the shared commitment: improving students' aesthetic opportunities.

In one session, Ms. Elm presents a lesson she previously designed and taught about the Rational Root Theorem (RRT). In another, Ms. Willow, presented an introduction to inverses lesson that she designed, taught, and wanted to redesign. Both of these audio recordings met the above three criteria, and had 5-minute segments that included all three pieces of criteria. These 5-minute segments were chosen for in-detailed analysis because of clear evidence of individual's members noticing and the noticing across members of the PLC. Therefore, the described segments show examples of exchanges towards expansive or restrictive responses.

Analytical Methods

The audio recordings were transcribed and analyzed for their curricular noticing (Dietiker et al., 2018). To identify instances of *curricular attending*, we analyzed the discourse for evidence of teachers "taking in" (ibid, p. 525) what was under discussion (e.g., presented by the lead teacher or introduced by another member of the PLC). To identify instances of *curricular interpreting*, we analyzed the discourse for evidence that a teacher made sense of what was attended to using their knowledge base in relation to their goals (e.g., evaluating a suggestion for its benefits). Finally, we identified instances of *curricular responding* by looking at decisions of action, both proposed and accepted (i.e., selecting a polynomial).

For each coded utterance, we analyzed how it related to the prior coded utterances. For example, when analyzing a coded utterance, we asked "was there a prior moment of attending, interpreting, or responding to another utterance that supported this utterance?" After compiling these threads of connected utterances, we identified the types of threads that emerged most frequently and connected to the groups' expansive responses.

Findings

With a collective curricular vision in a PLC, we found that the curricular noticing of the group of teachers were collectively shaped as follows: 1) prior responses were pulled back into the discourse as something to attend to, 2) prior interpretations were pulled back into the discourse as something to attend to with new interpretation, and 3) prior responses were impacted by the shifts in collective attending and interpreting. These characteristics, in turn, enabled this group to generate expansive responses. To show this, we present two exchanges from different parts of the design meeting to illustrate each of these characteristics, and describe how those either led to restrictive or expansive responses. In the first, based on a lesson on the RRT, all three characteristics were present and described. In the second, focused on redesigning a lesson that is an introduction to inverse functions, two of the three characteristics were present and described. In the first, based on a lesson on the RRT, all three characteristics were present and described. In the second, focused on redesigning a lesson that is an introduction to inverses, two of the three characteristics were present and described.

The Collective Redesign of Ms. Elm's Lesson on the Rational Root Theorem

Summary of Design Meeting. [1] At the start, the group watched an abbreviated video of key moments of the lesson enacted the prior year, selected by Ms. Elm. In this lesson, Ms. Elm had students use guess and check to test potential roots of given polynomials (cubic functions and quartic functions) in order to conjecture the RRT. Within this context, Ms. Elm presented the guess and check process as identifying "suspects" towards solving the mystery of the corresponding polynomial roots. She grappled with a tension; how could the lesson have students eliminate roots in such a way that they have time to understand the concept behind the RRT while keeping the aesthetic potential of mystery intact.

[2] After viewing the video, Ms. Spruce says:

> I loved the four guesses part…that was where you could definitely see like the kids [snapping], getting excited. What I wondered was... could you implement that earlier? To like generate more excitement…as you go up in degree and complexity, start taking away the number of guesses, because I think that would also generate the sense of…excitement, but in between there do some stuff to facilitate…better guesses

[3] Immediately after this comment, Mr. Palm points out that there was no evidence of students' verbally articulating what the learning objective is, and raises a question about the curricular goal of the lesson (ex: "do you want them, by the end of this…?"), offering multiple potential learning objective goals. [4] Ms. Elm replies:

> I mean ideally, I would love them to do all the things you said…that's really a tall order for one period...the key piece…sometimes there are so many potential factors that [the RRT] doesn't seem very efficient, but as the polynomial becomes more depressed, like that process is super-efficient when you're looking at your new *p* and *q*… If there's any way... um... but like I don't know if we ever got to that idea, so, um...I just don't know.

[5] As the conversation progresses, Ms. Spruce continues thinking aloud about the polynomial's leading coefficient (*p*) and its constant (*q*), suggesting that students, although not yet understanding the relationship between the roots and the polynomials, might get a sense of what could be a root. Ms. Willow and Ms. Elm continue to discuss how students' intuitive sense of numbers can lead them to making connections. [6] Ms. Willow suggests changing the polynomials to encourage students to focus on the coefficients. [7] Ms. Elm builds on this by saying that the new choice of coefficients should not visually mislead students (such as having p=1 and q=5 when the roots are not 1 and 5. [8] Ms. Spruce jumps in and describes this option as "kinda interesting," which [9] is echoed by Mr. Ash as a potential moment of "beauty."

Characteristic 1: Attending to a Prior Response. In [2], Ms. Spruce drew *attention* to the students' embodied reaction to the prior design choice of guesses. This focus on students' emotional and aesthetic reaction is connected to the larger groups' commitment toward captivating lessons. Then in [5], Ms. Spruce *attended* to her prior *response*, interpreting it as a way to connect to students' intuitive sense around numbers (i.e., connecting roots with the coefficients). Others continue *attending* to students' aesthetic reactions via numerical attunement [6, 7] and embodied excitement [9]. Collectively, these responses change the direction of the redesign; the focus on the original lesson in terms of students' articulating learning objectives shifted toward how the problems can make space for an increased aesthetic experience in a way that also contributes to students' learning.

Alternatively, this characteristic can limit curricular responding, such as in the case of Ms. Elm [4] attending to Mr. Palm's response [3]. The expansiveness of future responses began to close — her use of "I don't know" signifies that Mr. Palm's utterance overwhelmed her from responding at all. Unlike the previous example, here an A-I-R chain begun in [2] was interrupted.

Characteristic 2: New Attending and Interpretation of a Prior Response. Ms. Willow's prior *response* in [6] suggests that coefficients should be selected in such a way that students are learning through their intuitive sense. However, Ms. Elm in [7] *attends* to and *interprets* numerical attunement as misleading and unsupportive of sensemaking. This leads to a new *response* of a lesson design that does not have the potential to mislead students. Although this example can be read as one that restricts responses, the fact that the utterance [7] named a design challenge actually positioned the group to consider responding in two ways one, to either resolve that problem through a design-related response, or two, to shift what was attended to how it was interpreted that allows for a new expansive response. Therefore, the act of attending and interpreting a prior response enabled the group to name a problem, creating an opportunity to think more deeply about lesson redesign towards the shared commitment of student aesthetic experience [8-9].

Characteristic 3: Attending to Prior Interpretation. When Ms. Elm's *interpretation* in [7] was then *attended* to by Ms. Spruce [8], a new opportunity for students' aesthetic responses was created (i.e., the tension could potentially lead to a moment of excitement when students are able to find a pattern). This made space to consider aesthetic opportunities within the redesign. So, although the interpretation of the problem set in relation to students' numerical intuition was named as a potential issue [7], that interpretation was expansive because it was considering aesthetic characteristics within a lesson redesign. So, not only did that initial interpretation encourage continued noticing around aesthetic, but also made space for the reframing of the interpretation as an opportunity [8], which, in turn, acts as an example of enabling a subsequent expansive response.

The Collective Redesign of Ms. Willow's Lesson on Introduction to Inverses

Summary of Design Meeting. [1] In the beginning of this episode, the PLC is discussing how to adapt the opening task from the original lesson. [2] Ms. Dogwood proposes prompting students to compose $x^2 + 3$ and $\sqrt{x-3}$. [3] Ms. Spruce follows by saying, "Do you want them to have that in their minds when they go to [Problem] three?", which asks students to match functions that are inverses from a list of linear functions. Ms. Willow share this concern, saying:

[4] Ms. Willow: Oh, I see, because then I might be giving them stuff that has squaring…
[5] Ms. Spruce: Which isn't necessarily the end of the world, if you allow yourself to have awareness…
[6] Mr. Ash: You could just ask for linear(s). Like on [Problem] 3…you could restrict [the given types of functions].

Following this moment, the group discusses the difficulty students might have with simplifying the composed functions. This includes [7] Ms. Dogwood wondering if students would know to "square out" a radical, [8] Ms. Willow naming that doing so would be "really hard," and [9] Ms. Dogwood describes this process as "ugly."

[10] Ms. Willow then suggests creating a problem within the set that requires three operations to simplify the composed functions. She continues [11] by saying the three steps could more clearly illustrate the "doing and undoing" that is key to inverse functions, compared to prior suggestions that potentially resulted in the composition not "coming out nice." [12] Ms. Dogwood, in turn, articulates how inputs could influence whether the composition is "nice" and responds by suggesting that the group thinks of problems that would have a "nice" composition with the original problem's "yucky" elements (such as fractions). Ms. Willow continues:

> [13] Ms. Willow: I'm worried though, because if they try composing these…it would be three halves times two-thirds *x* plus one minus one… they're gonna be like, 'Oh, okay, so that's like *x* plus one minus one, it cancels out…
>
> [14] Mr. Ash: But that would be a nice opportunity for you to…because that's a common mistake that you saw happening later on…[say] maybe 'here's an opportunity, don't forget to distribute.'
>
> [15] Ms. Dogwood: But I think Ms. Willow's wondering…how can [the students] trouble that's not right? Then your question begs, like, 'how come? We have to distribute."
>
> [16] Ms. Willow: I don't know how that will make them feel when we do all the canceling…like, if they just tried to cancel and it didn't work, then suddenly all this canceling *does* work, does that feel better or worse?
>
> [17] Ms. Cherry: Well maybe they understand *why* it's so special, like *woah*, you know? Because everything cancels out.

Characteristic 1: Attending to a Prior Response. When considering Mr. Ash's [6] use of the prior *response* [3], we see how he encouraged the PLC to design in such a way that considers students' intuitive number sense to understand why the functions are inverses. When bringing Ms. Spruce's [3] response back into as something to *attend* to, Mr. Ash [6] opens the community to consider how clarifying what students have in their minds could be influenced by a restriction to the types of problems being presented. So, Mr. Ash *responds* by suggesting that restricting the problem set to linear equations (as opposed to incorporating quadratics and square root functions as inverse pairs) could serve as a better way to focus students' attention.

In another moment, we see a sequence of utterances where what is attended to stems from a prior utterance's response. An example of this starts when Ms. Willow [13] *attends* to Ms. Dogwood's *response* in [12] to create a pair of functions that, when composed, is "nice," although the coefficients may be "yucky." Ms. Willow *responds* [13] by naming that students' intuitive sense of what cancels could also be an issue towards their understanding of inverses and what actually cancels. Then Ms. Willow's *response* [13] was then attended to by Mr. Ash in [14], where this could be leveraged to support their conceptual understanding of the need to distribute [16]. This illustrates how prior responses that are pulled back into the discourse can result in expansive responses when the process is compounded by happening multiple times in a row.

In another example, the PLC *attends* to prior *responses* of multiple members of the group at the same time. Ms. Dogwood [15] draws *attention* to both Ms. Willow's [13] and Mr. Ash's [14] prior *responses* about students' intuitive sense by interpreting students' need to challenge their intuitive sense of canceling (as opposed to just refocusing the attention). This happens again when Ms. Cherry [17] *attends* to the same two *responses* that Ms. Dogwood just *attended* to. Here, Ms. Cherry interprets those responses by naming the importance of students understanding why the expressions cancel in order to have a positive aesthetic experience while canceling. This interpretation results in a new response where Ms. Cherry suggests that some of the tensions named in the prior responses could potentially be a moment where students "understand why [composing inverses] is so special."

Characteristic 2: New Attending and Interpretation of a Prior Response. Ms. Spruce, in [3], *responded* when she suggested clarifying what the lead teacher (Ms. Willow) wanted students to have in their minds while solving the subsequent problems. Ms. Willow [4] *attended* to Ms. Dogwood's original *attending* in [2], but *interprets* it using Ms. Spruce's [3] *interpretation* of how prior problems in the set could influence their approach to new problems. Ms. Willow's *response* [4], in turn, becomes the same *response* that Ms. Spruce previously had — that is, that the lead teacher (here, being Ms. Willow herself) needs to clarify what she wants students to have in their mind. So, although it could be said that Ms. Willow did not contribute anything novel to the conversation, she was able to elevate the prior response in a way where members of the PLC could recognize the design challenge as Ms. Willow was *interpreting* it in relation to the commitment.

Another instance where prior *responses* are reimagined with new *attending* and *interpreting* is when Ms. Willow [16] refines the *response* of students' intuitive sense of what cancels as an issue. Here, she highlights the need to design problems that do not confuse students by *attending* to students' embodied feelings of when canceling "works" or "doesn't" work. So, Ms. Willow brings in the *interpretation* that students can feel better or worse when there are "inconsistent" responses with canceling. This new *attending* and *interpreting* of prior *responses* now brings in students' affective reactions when considering their intuitive sense of numbers and the relationship to the problem set. So, instead of only focusing on how students can mislead themselves or on the "ah-ha!" moments that students might have due to their intuition, Ms. Willow draws *attention* to the tensions that might arise during the lesson. This reimagining oriented teachers to the broader affective experience that students may have when engaging with this topic through new expansive *responses*. Students' opportunities to have positive aesthetic experiences, on the other hand, may have been restricted if the PLC had not considered the broader affective experience as presented within the redesign.

Discussion

By highlighting the role of collective curricular vision, this paper argues how teachers who collectively redesign lessons with a shared commitment can shift their attention and interpretation of what other teachers offer towards more expansive views of mathematics learning. When a PLC's collective curricular noticing is guided by a commitment that intentionally disrupts constraints within the design process, such as attending to students' emotional reaction to a mathematical concept over attending to standardized testing scores, the way in which noticing functions within the collective can lead to *expansive* responses. In addition, when a PLC's curricular noticing, despite being guided by a shared commitment, tends to stray away from the collective beliefs in any capacity (such as reinforcing design habits that are traditionally inequitable, redirecting the group from an expansive response), *restrictive* responses can emerge. In terms of the curricular noticing framework, we can see an alignment to the collective curricular vision's commitment through (a) what an individual decides to attend, interpret, or respond to with during PLC conversations, and (b) the expansiveness or restrictiveness of resulting responses. This, in turn, informs curricular responses toward reimagined ways for students to think about, engage with, and experience mathematics.

Analyzing curricular noticing as a collective practice within the curricular noticing framework sheds light on the way in which teachers are listening to each other, considering different ideas, and hearing other teachers' own attending, interpreting, and responding to the *same* noticing within the same discussion. Being able to trace the way in which members of a PLC build upon each other's contributions shows that a response helps move the group towards deeply understanding what they care about in terms of the shared commitment; we are now able to see how this is done by tracing the process of fine-tuning the question they are *really* vexed with to ensure the PLC's commitment is salient in how decisions are made within the redesign. Although individuals in a PLC bring in their own interpretations through their experiences, they are being exposed to and building off of the noticing that is happening within the collective.

This analysis illustrates that, through the product of years of teaching, collective sensemaking around aesthetic, and co-design work as part of a PLC, expansive curricular responses are possible through emergent discourse. Identifying this contributes to the mathematics education field by shedding light on the nature of collective curricular noticing and how expansive curricular responding develops in response to a shared commitment and a space to design lessons aligned to that commitment. Knowing this, there is a call for PLCs to develop a curricular vision around a shared commitment, such as increasing aesthetic opportunities for students. With this in place, we can begin to think about the emergence of expansive curricular noticing across redesign sessions between educators. This focus opens up potential opportunities for students — when a teachers' curricular noticing is including and expanding beyond content goals, a students' learning experience can begin to include a wider range of experiences is equally if not more important towards shaping an enjoyable, expansive, and engaging learning experience for young people in a space that typically marginalizes them. A shared attunement to an expansive ideology (be it aesthetic, or another touchstone towards a more equitable learning space), and the resulting curricular responses, has the potential to become expansive when rooted in a professional learning community.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1652513. We also appreciate our teacher participants and other research team members for their support in this study.

References

Baldinger, E. M. (2017). "Maybe It's a Status Problem." Development of Mathematics Teacher Noticing for Equity. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks (pp. 231–249). Springer International Publishing. https://doi.org/10.1007/978-3-319-46753-5_14

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp. 17–36). Routledge.

Darling-Hammond, L., Banks, J., Zumwalt, K., Gomez, L., Sherin, M. G., Griesdorn, J., & Finn, L. (2005). Educational goals and purposes: Developing a curricular vision for teaching. In L. Darling-Hammond & J. Bransford (Eds.), Preparing teachers for a changing world: What teachers should learn and be able to do (pp. 169–200). Jossey-Bass.

Dietiker, L., Males, L. M., Amador, J. M., & Earnest, D. (2018). Research Commentary: Curricular Noticing: A Framework to Describe Teachers' Interactions With Curriculum Materials. Journal for Research in Mathematics Education, 49(5), 521–532. https://doi.org/10.5951/jresematheduc.49.5.0521

Drake, C., & Sherin, M. G. (2009). Developing curriculum vision and trust. In J. T. Remillard, B. Herbel-Eisenmann, & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp. 321–337). Routledge.

Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. Educational Studies in Mathematics, 80(1–2), 233–247. https://doi.org/10.1007/s10649-012-9387-9
Louie, N. (2018). Culture and ideology in mathematics teacher noticing. Educational Studies in Mathematics, 97(1), 55–69. https://doi.org/10.1007/s10649-017-9775-2
Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: The FAIR framework for anti-deficit noticing. ZDM – Mathematics Education, 53(1), 95–107. https://doi.org/10.1007/s11858-021-01229-2
van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A Framework for the Facilitation of Teachers' Analysis of Video. Journal of Teacher Education, 65(4), 340–356. https://doi.org/10.1177/0022487114534266

GATHERING VALIDITY EVIDENCE TO SUPPORT MATHEMATICS EDUCATION SCHOLARSHIP

Jonathan D. Bostic
Bowling Green State Univ.
bosticj@bgsu.edu

Erin Krupa
North Carolina State Univ.
eekrupa@ncsu.edu

Timothy Folger
Bowling Green State Univ.
tdfolge@bgsu.edu

Brianna Bentley
North Carolina State Univ.
blbentle@ncsu.edu

David Stokes
North Carolina State Univ.
djstokes@ncsu.edu

Validity and validation is central to conducting high quality quantitative mathematics education scholarship. This presentation aims to support scholars engaged in quantitative research by providing information about the degrees to which validity evidence related to their instrument use or interpretation, were found in mathematics education scholarship. Findings have potential to steer future quantitatively focused scholarship and support equity aims.

Keywords: assessment, research methods

The inferences and interpretations drawn from quantitative assessments are largely grounded in the validity evidence and their associated claims (AERA et al., 2014; Carney et al., in press; Kane, 2016). Mathematics education scholarship using quantitative assessments has not consistently adhered to strong validity and validation practices as seen by the limited presence of validity evidence related to many instruments used with K-12 students and teachers (Bostic et al., 2019, Bostic et al., 2021; Krupa et al., 2019). A purpose for the present study is to serve as an educative piece for mathematics educators who intend to develop or use quantitative assessments in their research. We offer results about the degree to which validity has been taken up in mathematics education scholarship that uses quantitative instruments.

Related Literature

The *Standards for Educational and Psychological Testing* ([*Standards*] AERA et al., 2014) describe five sources for validity: test content, response process, relations to other variables, internal structure, and consequences from testing. Reliability is a related but not sufficient condition of validity (AERA et al., 2014; Kane, 2012). While the *Standards* describe some approaches for each source, those descriptions are not intended to be exhaustive. A special issue of *Psicotherma* (2014, 26(1)) includes articles related to each of those sources. Here again, the authors of those articles indicate that their description of each validity source is to introduce readers to that source and are not intended to be comprehensive. Thus, there is a need for a more comprehensive list of data collection approaches related to each validity source.

Mathematics education scholarship has started to address validity within the context of quantitative research across K-20 students, as well as preservice and inservice teacher settings (e.g., Bostic & Sondergeld, 2015; Carney et al., 2017; Gleason et al., 2019; Hill & Shih, 2009; Melhuish & Hicks, 2019; Walkowiak et al., 2014; Wilhelm & Berebitsky, 2019). These authors provide discussions about how they explored validity evidence and serve as potential roadmaps for others doing validation work. There are some common approaches to gathering validity evidence. The present study intends to summarize literature about approaches to gathering validity evidence so that quantitative assessment developers, users, and reviewers have a more comprehensive understanding of what has been done previously, and what approaches are viable. Our research question was: In what ways is validity evidence and arguments present in mathematics education scholarship that uses quantitative instruments between 2000-2020?

Methods

Context for study

In 2019, 39 mathematics education scholars convened at a 2-day conference led by the authors. Scholars included mathematics educators, psychometricians, special educators, and policy experts who had previously conducted quantitative mathematics or statistics education assessment work. Conference leaders asked attendees to form small groups and brainstorm viable data collection approaches that might generate validity evidence for that source. After sharing ideas, groups rotated to each approach and left feedback, followed by whole-group discussions. The product was an extensive list with at least five unique approaches to gathering validity evidence for each validity source. There were still questions about whether there were other approaches and how to define some of these approaches for a broad audience (e.g., factor analysis). To that end, the authors of this submission reviewed literature and sought definitions to create an Evidence Types Guidebook. The definitions were sent to conference attendees for feedback and revised as needed. Independent of that work, conference attendees have been conducting syntheses of literature across a variety of contexts including teacher education, elementary and secondary students, and statistics education.

An Evidence Types Guidebook served to facilitate the identification of approaches typically utilized within a validity evidence type. For each of the evidence types (i.e., test content, response processes, internal structure, relation to other variables, and consequences of testing) within the Guidebook, a general definition of the type was given, followed by a list of methods commonly used to support a validity claim within the respective category. For example, the internal structure section included approaches such as factor analysis, item response theory, latent class analysis, and other approaches commonly used to assess and support claims of validity related to the internal structure of a quantitative instrument. Each of these methods also contained definitions and citations for further exploration and information. The Guidebook content was aligned with the validity evidence repository framework and served to support participants, in general, throughout the framework application process.

Data Sources and Analysis

Our data collection and analysis process is summarized here; more details are provided in Bostic et al. (2022). The PRISMA statement guided the literature search (Rethlefsen et al., 2021). The top 24 mathematics education journals (Williams & Latham, 2017) were searched for studies using quantitative instruments. Articles that included quantitative instruments were culled to create a list of instruments. Next, validity evidence was sought for each instrument through a literature search using google scholar. Instrument names and keywords were used to generate an appropriate sample space. As an example from teacher education, over 3,000 articles were examined, which led to over 300 instruments Evidence was coded as being connected to a validity source or reliability. Validity claims in support of arguments were also coded. We share results from those syntheses as a means to illuminate the frequencies of various approaches as well as opportunities for use of new approaches.

Findings

Overall, synthesis groups searched for validity evidence of 190 instruments and found 278 articles with descriptions of validity evidence (see Table 1). The majority of articles (83%) did not contain an interpretation statement or a use statement. In addition, 73% of the articles that contained validity evidence did not specify any claims. An example of a use statement comes from the Statistics Education synthesis group: "The availability of an instrument such as the Attitudes Toward Research (ATR) scale which has been designed for students, may provide information concerning motivational aspects associated with learning research, and might also have potential for identifying distinctive attitude profiles of students who find research problematic. Overall however, this study's results validate the utility of the ATR scale in measuring student attitudes toward research" (Papanastasiou, 2005, p. 23).

Table 1. Instrument and Article Overview

	Elementary (K - 6) Tests & Instruments	Secondary (7 - 12) Tests & Instruments	Statistics (K - 20) Tests & Instruments	Teacher Education Instruments	Combined
Number of instruments	59	27	16	88	190
Number of articles	92	36	52	98	278
Articles with an interpretation statement	17 18.48%	4 11.11%	19 36.54%	7 7.14%	47 16.91%
Articles with a use statement	18 19.57%	2 5.56%	20 38.46%	2 2.04%	42 15.11%
Articles with a claim	37 40.22%	6 16.67%	15 28.85%	15 15.31%	73 26.26%

Considering the distribution of the five types of validity as well as reliability: Internal structure, reliability, and test content were the most frequently located. Table 2 shows the frequency of each evidence type across different areas. The most frequently used method for each evidence type is displayed in Table 3. Some frequencies are quite high (e.g., alignment with frameworks) whereas the mode for other validity sources was quite low (e.g., quantitative DIF analysis).

Table 2. Evidence Type Frequency

	Elementary (K - 6) Tests & Instruments	Secondary (7 - 12) Tests & Instruments	Statistics (K - 20) Tests & Instruments	Teacher Education Instruments	Combined
Consequences of Testing	4 1.73%	0 0.00%	5 2.81%	1 0.69%	10 1.71%
Internal Structure	48 20.78%	7 21.88%	34 19.10%	29 20.14%	118 20.17%
Relations to Other Variables	33 14.29%	1 3.13%	27 15.17%	19 13.19%	80 13.68%
Reliability	68	9	29	53	159

	29.44%	28.13%	16.29%	36.81%	27.18%
Response Process	27 11.69%	3 9.38%	13 7.30%	6 4.17%	49 8.38%
Test Content	51 22.08%	12 37.50%	70 39.33%	36 25.00%	169 28.89%
Total Number of Evidence Types	231	32	178	144	585

Table 3. Mode of approach for each validity source and reliability

Validity Source or Reliability	Most common type of approach	(count, %)
Test content	Alignment with frameworks	n = 45, 26%
Response Process	Student written work	n = 19, 38%
Relations to Other Variables	Correlation analysis	n = 41, 51%
Internal Structure	Confirmatory factor analysis	n = 41, 34%
Consequences of Testing	Item functioning such as DIF	n = 3, 30%
Reliability	Internal consistency, alpha	n = 19, 38%

Discussion

It is clear from analyzing the validity evidence from this sample of instruments that modern notions of validity and validation arguments (AERA et al., 2014; Author, in press; Kane, 2016), have not necessarily been taken up by the field. We do not blame authors for this omission. It may be that validity evidence is removed during the editing process. Authors may not be prepared to conduct validation work. It has also been shown that 75% of mathematics education graduates take two or less quantitative research courses, where validity and validation might be discussed (Shih et al., 2019). Few instruments presented in existing scholarship are accompanied with an explicit statement describing the intended interpretation and use of test scores. Further, we found little validity evidence based on consequences of testing and response processes. Given current equity issues in mathematics education, it is a concern that there is not more evidence of consequences from testing and bias, especially to ensure fair use of the score interpretations from the tests.

Validity is naturally an equity issue (AERA et al., 2014; Cronbach, 1988). Otherwise, tests may have bias and test scores may be used unfairly. Cronbach (1988) proclaimed, “Tests that impinge on the rights and life chances of individuals are inherently disputable” (p. 6). Furthermore, the inferences drawn from tests that lack a validity argument may not be accurate (Carney et al., in press). To yield accurate inferences about student learning or teacher practice, it is critical for scholars to have tests and instruments with strong validity evidence and robust validity arguments.

Acknowledgments

This research is funded in part by grants from the National Science Foundation (1920619, 1920621). All findings are those of the authors and not necessarily of the funding agency.

References

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education. (2014). *Standards for educational and psychological testing*. Washington, DC: American Educational Research Association.

Bostic, J.D., Carney, M., Casey, S, Engledowl, C., Folger, T., Gallagher, M., Howell, H., Smith, W., Tjoe, H., & Wilhelm, A. (2022, February). *Choose your instruments wisely: Supporting mathematics teacher educators' research and practice.* Symposium presented at annual Association of Mathematics Teacher Educators Conference. Henderson, NV.

Bostic, J., Lesseig, K., Sherman, M., & Boston, M. (2021). Classroom observation and mathematics education research. *Journal of Mathematics Teacher Education, 24, 5-31, https://*doi.org/10.1007/s10857-019-09445-0.

Bostic, J., Krupa, E., Carney, M., & Shih, J. (2019). Reflecting on the past and thinking ahead in the measurement of students' outcomes. In J. Bostic, E. Krupa, & J. Shih (Eds.), *Quantitative measures of mathematical knowledge: Researching instruments and perspectives* (pp. 205-229). New York, NY: Routledge.

Bostic, J., & Sondergeld, T. (2015). Measuring sixth-grade students' problem solving: Validating an instrument addressing the mathematics Common Core. *School Science and Mathematics Journal, 115, 281-291*.

Carney, M., Bostic, J., Krupa, E., & Shih, J. (in press). Instruments and use statements for instruments in mathematics education. *Journal for Research in Mathematics Education*. Accepted for publication.

Carney, M., Cavey, L., & Hughes, G. (2017). Assessing teacher attentiveness: Validity claims and evidence. *Elementary School Journal, 118*(2), 281-309.

Cronbach, L. J. (1988). Five perspectives on validity argument. In H. Wainer & H. Braun (Eds.), *Test validity* (pp. 3-17). Erlbaum.

Gleason, J., Livers, S., & Zelkowski, J. (2017) Mathematics Classroom Observation Protocol for Practices (MCOP2): A validation study. *Investigations in Mathematics Learning, 9*(3), 111-129.

Hill, H., & Shih, J. (2009). Examining the quality of statistical mathematics education research. *Journal of Research in Mathematics Education, 40*(3), 241-250.

Kane, M.T. (2012). All validity is construct validity. Or is it? *Measurement: Interdisciplinary Research and Perspectives, 10*(1-2), 66-70.

Kane, M. T. (2016). Validation strategies: Delineating and validating proposed interpretations and uses of test scores. In S. Lane, M. R. Raymond, & T. M. Haladyna (Eds.), *Handbook of test development. (2nd ed., pp. 64-80).* Routledge.

Krupa, E., Carney, M., & Bostic, J. (2019). Approaches to instrument validation. *Applied Measurement in Education, 32*(1), 1-9.

Melhuish, K., & Hicks, M. (2019). Title redacted for blinding. In In J. Bostic, E. Krupa, & J. Shih (Eds.), *Quantitative measures of mathematical knowledge: Researching instruments and perspectives* (pp. 121-151). New York, NY: Routledge.

Papanastasiou, E. C. (2005). Factor Structure of the "Attitudes toward Research" Scale. *Statistics Education Research Journal, 4*(1), 16–26.

Rethlefsen, M. L., Kirtley, S., Waffenschmidt, S., Ayala, A. P., Moher, D., Page, M. J., & Koffel, J. B. (2021). PRISMA-S: an extension to the PRISMA statement for reporting literature searches in systematic reviews. *Systematic reviews*, *10*(1), 1-19.

Shih, J., Reys, R., Reys, B., & Engledowl, C. (2019). A profile of mathematics education doctoral graduates' background and preparation in the United States. *Investigations in Mathematics Learning, 11(*1), 16-28.

Walkowiak, T. A., Berry, R. Q., Meyer, J. P., Rimm-Kaufman, S. E., & Ottmar, E. R. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*(1), 109-128.

Wilhelm, A. G., & Berebitsky, D. (2019) Validation of the mathematics teachers' sense of efficacy scale. *Investigations in Mathematics Learning, 11*(1), 29-43.

Williams, S. R., & Leatham, K. R. (2017). Journal quality in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 369-396.

INVESTIGATING ELEMENTARY MATHEMATICS TEACHERS' ADAPTATION OF ACTIVITIES FOUND ON VIRTUAL RESOURCE POOLS

Lara K. Dick
Bucknell University
lara.dick@bucknell.edu

Amanda G. Sawyer
James Madison University
sawyerag@jmu.edu

Over the past decade, teachers have increasingly turned online to supplement their mathematics curriculum and some research regarding where and how teachers search has been completed. However, there are many open questions regarding this rising phenomenon, one of which is learning more about what teachers do after finding a mathematics activity online. We report on a subset of a survey of elementary school teachers across the United States about their use of elementary mathematics activities found via virtual resource pools; for this manuscript, we share results on two questions related to teachers' adaptation of elementary mathematics activities they obtained.

Keywords: Curriculum; Elementary School Education; Instructional Activities and Practices

Prior to the rise of the internet, most schools adopted elementary school mathematics textbooks and provided teachers with associated teaching materials with the understanding that this was "all" teachers needed to have at their disposal to teach mathematics successfully (Browne & Haylock, 2004; Remillard, 2005). However, over the last decade, many teachers have turned online to various virtual resource pools to supplement their elementary mathematics curriculum. Teachers may search for activities on online teacher marketplaces such as Amazon Inspire, Pinterest or TeachersPayTeachers, or they may search on more vetted sites such as professional organizations like the National Council of Teachers of Mathematics. Recent research shows ever-increasing popularity of teachers' curriculum supplementation via virtual resource pools (Hu & Torphy, 2020) but overall, this supplementation is not well-understood by the mathematics education research community. According to Silver who completed a literature review on online resource supplementation, most research "tends to center phenomena closely related to supplementation (e.g. virtual resource use, teacher social media use) rather than supplementation itself" (2021, p. 2). This paper seeks to study the act of supplementation, specifically that of adaptation of supplemental mathematics activities found online by elementary teachers.

Literature Review

To situate this work, we share literature related to lesson supplementation with resources found on virtual resource pools and lesson adaption more generally. We then share the focus of this study.

Lesson Supplementation from Virtual Resource Pools

Teacher supplementation can be described as any change teachers make to curriculum resources provided by officials at their schools or districts (Silver, 2021). Teachers have always supplemented their classroom resources but Hodge et al. (2019) discusses the increase of teacher supplementation as a recent phenomenon due to increasing internet use and the implementation of the Common Core Standards. With the rise of the internet came a variety of virtual resource pools which Silver (2021) defines as "websites that host curriculum resources for download by teachers to use either alongside or instead of their school-adopted textbooks" (p. 1). Thus, VRPs include websites hosted by textbook companies, professional organizations, open educational resources, online teacher marketplaces and more general sharing sites like Pinterest that teachers use to search for supplementary curriculum resources.

One finding of a recent review of research is that "teachers curriculum supplementation is massive in scope" (Silver, 2021, p. 6). Surveys of teachers looking at their VRP supplemental usage has shown this increase with Davis et al. (2013) reporting 60% of teachers reporting supplementing their curriculum to more recent studies reporting over 80% (e.g. Sawyer et al., 2019a; Carpenter & Shelton, 2021; Tosh et al., 2020). Research shows that teachers often also turn to social media sites such as Instagram and Twitter as spaces to search for materials on various VPRs (e.g. Carpenter & Kruka, 2014: Carpenter et al., 2020) which is further complicated by the number of teacherpreneurs who promote their self-created resources on these and other social media sites (e.g. Hu & Torphy, 2020; Shelton & Archambault, 2019).

One question researchers have studied is teachers' reasons for supplementing their classroom curriculum. From his synthesis of research Silver (2021) describes teachers' reported motivations are largely to fill holes they perceive with their provided curriculum, to assist their students or make teaching better for themselves. Sawyer et al. (2019a) found that teachers reported alignment to standards and student engagement to be the most important factors for teacher supplementation via VPRs. In contrast, Schroeder et al. (2019) found "idea gathering" as the main purpose for teacher supplementation with other reasons related to improving the classroom environment and increasing student engagement (p. 171). Specific to the TeachersPayTeachers VRP, teachers cited the following reasons for supplementation: to fill gaps in existing curriculum related to concepts or skills, to make learning fun, to provide additional practice for students and for inspiration when they were not sure how to approach a particular lesson (Carpenter and Shelton, 2021). Taking together, these results point to teachers' carefully searching for and choosing supplementary resources that meet their curricular needs and the needs of their students. For this study, we are interested in what happens after teachers choose a supplemental curricular resource. Specifically, do they use the supplement as they found it or do they adapt it in some way.

Lesson Adaptation

For the purpose of our study, we utilize Davis et al.'s (2011) description of adaptation, "Adaptations can include insertions, deletions, or substitutions, for example, and may be based on aspects of the teachers' contexts, their students' needs and strengths, and their learning goals, knowledge, beliefs, identities, and orientations" (p. 797). Studies researching adaptation have focused on teachers' reasoning (e.g. Choppin, 2011; Drake & Sherin, 2006; Carpenter & Shelton, 2021) with results showing reasons for adaptation falling into two main categories: improving instruction for students and making teaching better for teachers.

Methods of adaptation vary across elementary school subjects; researchers have investigated methods of adaptation in reading (e.g. Parsons, 2012), science (e.g. Forbes, 2011), as well as the subject studied for this paper, mathematics (e.g. Choppin, 2011; Remillard, 1999; Sherin & Drake, 2009). Sherin & Drake (2009) found that mathematics teachers typically adapt their activities in a three-step cycle: omit, replace, and create. They explain that teachers first remove items from a mathematics activity and then replace it with something that they find useful and that they believe will better support their students' learning. If this is not deemed sufficient, mathematics teachers create new curricular elements to support student learning in their classroom (Sherin & Drake, 2009). In a more recent study, elementary preservice and inservice teachers reported adapting resources they found on Pinterest often with the top reported reason for adapting given as "meeting student needs and attending to grade-level standards." (Schroeder et al., 2019, p. 173). Because Schroeder et al.'s (2019) study was not subject specific and was specific to teachers searching for supplemental resources on one particular VRP, Pinterest, the current study both broadens and narrows the question of adaptation by looking at elementary teachers' supplemental mathematics activities found on any VPR. Specifically, this study seeks to answer the following two research questions:

1. How often do US elementary teachers report adapting mathematics activities they find on VPRs?
2. How do US elementary teachers describe how they adapt mathematics activities they find on VPRs to fit the needs of their students?

Conceptual Framework

For this study, we adopt the Teacher Curriculum Supplementation Framework (Silver, 2021) that includes four dimensions surrounding teacher supplementation as a tool to guide research: the teacher's reasons for supplementation, the supplement's source, the teacher's supplemental use pattern and features of the supplement itself. In the literature review, we discussed what is currently known about teacher's reasons for supplementation, the fact that teachers' use of supplemental resources is massive and have indicated that for this study, we are not considering specific VRP sources from which supplements are sought. We are, however, interested in the supplement itself, specifically how often and how teachers choose to adapt a supplemental elementary mathematics activity.

When considering the supplement itself, researchers may study features like the visual appeal, educational quality and how these affect student learning. Silver (2021) notes that most research in this area is researcher determined meaning researchers use their expertise to study the curricular supplements. Research in these areas regarding elementary mathematics supplemental activities have shown supplements tend to be low in quality (e.g. Hertel & Wessman-Enzinger, 2017; Sawyer et al., 2019b; Shapiro et al., 2021), but that does not mean teachers use the supplement as they find it. Because of this, Silver (2021) suggests that more work is needed looking at "a teacher's supplementation process from start to finish" (p. 24). While this study does not follow teachers' full process, it does begin to consider what happens to a supplement after it is chosen which meets the call by Hu and Torphy (2020) to focus more on the compilation, distribution and diffusion of supplements into the classroom.

Methodology

This study reports on a subset of a survey conducted in June 2018 that asked teachers 23 questions about their elementary mathematics VPRs search practices including how often, where they searched, what they were looking for when they searched, and how often and how they adapted resources they found. Results on how often, where and what teachers reported looking for when searching for supplemental activities was reported in Sawyer et al. (2019a).

Data Collection

To distribute the survey, we emailed the presidents of all 50 state affiliates for both the National Council of Teachers of Mathematics and the Association of Math Teacher Educators, thus obtaining our participants through a snowball sampling technique (Weiss, 1994). We also posted the survey link on multiple social media sites including Twitter, Facebook, and Instagram using #elemmathchat, #edchat, #mathchat, #elemchat, #mtbos, #iteach, #iteachmath, and #numbersenseroutines. After the emails and social media posts were sent, the survey was open for seven weeks. 601 teachers responded; they hailed from 48 states, excluding Delaware and

South Dakota, as well as three US territories. They were currently teaching kindergarten through sixth grade and had between zero and 36+ years of teaching experience.

There were two survey questions specifically focused on teacher adaptation. The survey questions that we report on in this paper include:

1. How often do you adapt mathematics activities you find online?
2. How do you adapt activities you find online to fit the needs of your students?

For responding to frequency of adaptation, teachers choose one of the following five scaled responses, “Always; Most of the time; About half the time; Sometimes; Never”. For responding to how they adapt, we purposefully did not define what we meant by what constitutes “how” to not limit our study to a specific definition of the term. However, we did focus the question on how teachers adapted to fit student needs which is in line with Schroeder et al. (2019)’s finding that the majority of adaptions focused on student needs. We were interested in how adaptations were made to elementary mathematics supplements for students, not for teachers.

Of the 601 elementary teachers who responded to the survey, 496 answered the question about how often they adapted online elementary mathematics activities but only 290 teachers provided a descriptive narrative about their adaptation process. We therefore answer research question 1 using the 496 teacher responses, but answer research question 2 with the 290 teacher responses.

Data Analysis

This mixed methods study used both quantitative and qualitative data analysis. We implemented descriptive statistics when describing how often teachers reported adapting elementary mathematics activities. We applied the constant comparative method through grounded theory to analyze qualitative data provided by the teachers in their descriptive narratives explaining how they approach adapting to meet the needs of their students (Glaser & Strauss, 1968; Miles, Huberman, & Saldaña, 2013). For the qualitative analysis, we used Qualtrics’s coding tool. First, we ran a keyword analysis of the open-ended responses. We then used the most common terms along with the constant comparative method through grounded theory to allow for additional terms to emerge (Glaser & Strauss, 1968; Miles, Huberman, & Saldaña, 2013). One researcher did this initial work. Once data saturation was reached, an additional researcher worked with the coder to collapse keywords and identify overarching themes. At this point, keywords were collapsed into larger categories. Teachers whose responses did not include common keywords were individually coded into the four overarching categories by the two researchers. Questions about responses or instances where the two researchers could not agree were brought to a third researcher for consultation.

Findings

Research Question One: How Often

Less than 1% of the teachers (n=4) said that they never adapt a supplemental math activity. The majority of the teachers reported that they adapt sometimes (21.6%, n=107), about half of the time (21.6%, n=107) or most of the time (40.9%, n=203). 15% (n=75) of the teachers reported adapting their supplemental mathematics activities all of the time. Taken together, these results show adaptation to be a common practice with 78% of the teachers indicating that they adapt elementary mathematics activities found on VRPs at least half of the time.

Research Question Two: Teacher’s Reported Reasons for Adapting

While the teachers were asked how they adapt supplemental activities to fit the needs of their students, the four overarching categories regarding teacher responses did not all relate specifically to student needs. The four categories were 1) Adapted for different learning needs;

2) Adapted for classroom implementation; 3) Adapted for mathematics content; 4) Adapted for visual appeal. Keywords and definitions for each of these categories are provided in Table 1.

Table 1: Teacher's Collapsed Adaptation Categories with Keywords and Definition

Category (count, %)	Keywords	Definition
Adapted for Different Learners	English Language Learners Needs** Special Education Level of Difficulty** Easy/Hard/Difficult/Levels/ Rigor/Advance/Struggle Differentiation** Scaffold	The teachers adapted the activities to meet different learner needs.
Adapted for Classroom Implementation	Manipulatives Directions* Time Structure** Independent/Group/Partner /One-on-One/Centers Technology Video/PowerPoint/iPad/ Smart Board/Computer/Tech	The teachers adapted the activities by adding elements or taking elements away by altering the implementation of the activity in their classroom.
Adapted for Mathematics Content	Standards Common Core/ District/State Numbers* Content Alignment Curriculum	The teachers adapted the activities by adding elements or taking elements away to by altering the mathematical content.
Adapted for Visual Appeal	Format Font Design Size Picture/Image Visual	The teachers adapted the visual aesthetics of the activity.

*The keyword was coded in 10% or more of responses. **15% or more

Adapted for different learning needs. We found 64% (n=191) of the teachers reported adapting elementary mathematics to meet the needs of different types of learners. This category included teacher descriptions about adapting the online activity through differentiation or through making accommodations for students with special needs, often involving discussions of levels of difficulty. For example, a teacher stated that they adapted online mathematics activities by "Change[ing] level as necessary, add[ing] scaffolding and modeling" which we coded as adapting for different learning needs. Multiple teachers reported focusing adaptations on “increasing the rigor,” and others reported adapting “to differentiate.” Other teachers discussed adaptation for specific student groups (e.g. English-language learners (ELLs)) or based on cultural needs as seen in the following response, “I rewrite to provide access for English learners. I remove unnecessary language to assure that I am measuring math, not language. I also rewrite lame attempts at ‘multiculturalness,’”. Responses such as these highlight the main finding that these teachers are sensitive to their different learners’ needs both mathematical and outside of the mathematics itself and place these needs at the forefront of their minds when they consider how to adapt elementary mathematics activities they find on VPRs.

Adapted for classroom implementation. While not necessarily specific to student needs, we found that 45% (n=134) of the teachers reported adapting supplemental activities to adjust to the teachers’ facilitation or more specifically, how they would implement the activity in their classroom. Responses in this category included reported adaptations such as adding manipulatives, changing directions, modifying suggested timing, or changing the activity from individual to a group game. These examples show implementation changes that directly affect student engagement with the activity. For example, a teacher stated they adapted by, “adding follow up activities, or changing the directions.” Others teachers described changing methods of implementation to meet their teaching style or classroom structure such as the following teacher who described, “If it is a PDF, I may turn it into a Smart Notebook that I can utilize for whole group instruction,” or another teacher who reported adapting to “tie it [the activity] to a lesson or activity from our current math standard and another subject we might be studying, such as science, social studies, health, etc.” While there were additional responses similar to the previous two, the majority in the classroom implementation category related to changing implementation structure regarding directions such as “cross out questions” or “extending time” … and “print them on task cards for a game.” These examples show minor adaptations allowing the teacher to change a task to make it structurally work in their classrooms but also point to structural changes that they deem necessary to assist student learning and engagement.

Adapted for mathematics content. We found that 26.5% (n=79) of teachers reported adapting their supplemental materials specifically for the mathematics content. These teachers indicated they changed the specific mathematical ideas represented in the online activities. For example, one teacher stated that they, “Use larger or smaller numbers in place of what’s provided [or] use fractions or decimals instead of whole numbers.” These adaptations allow for the structure of the activity to be used across mathematical topics. Another teacher reported that they “change numbers to fit a more appropriate range for students” thus altering the mathematics specifically. Yet others reported changing the mathematics to make activities easier or more difficult depending on their students’ needs.

Teachers also identified changing content to address specific standards. A teacher stated that they adapted the activities for different standards, “to ensure alignment.” Teachers also indicated adapting activities by changing the style of questions that were presented. For example, one teacher stated, “I reformat questions so they align with the style of questions students will see on State tests.” Another teacher reported,

> The advertised grade level items I find online tend to be too easy for my grade level. If I go up a grade level those tend to not totally align with my state standards. Usually I can only use about 70% of what is provided [from VRPs].

These show teachers adapting the level of the mathematics content and the way questions are posed to students because of state or district requirements.

Adapted for visual appeal. The least common thematic category was adaptation for visual appearance accounting for only 22 teacher responses (7.4%). Responses in this category included teachers who described their adaptations regarding superficial aspects of the activities not related to content or implementation. For example, a teacher described changing "color to black and white" and another referred to making activities "more durable by laminating," and yet another, "Sometimes I retype it to make it in a format I like." One teacher provided reasoning for visual changes related to student needs; they explained that they adapted by adding "larger spacing, fonts and pictures. I made it easier for my students to see it and fill it in." Most of the responses in this category were accompanied by descriptions that fell into the other three categories as well, such as this one that explicates visual appeal along with adapting for different learners. Importantly, only two teachers, including the first one presented in this paragraph, only discussed adaptations based on visual appeal.

Overall. Many teachers provided short, simplistic responses when describing how they adapt their supplemental elementary math activities for their students' needs, but despite their brevity, the responses indicated an overlap between the four identified thematic categories of adaptation for many teachers. For example, one teacher described their adaptation process as, "Increase the rigor and retype it". Other teachers provided additional details, such as the following teacher who explained,

> I may make the activity whole group so everyone can benefit. I also will sometimes change or clear-up the directions. I find that I need to often increase the rigor. I use TpT [TeachersPayTeachers] a lot and there tends to be a 3-5 band that is often too basic.

This response was unique, not because of the overlap of various ways they adapt, but because the response provides rationale for the need to adapt the rigor.

Discussion

These results indicated that 99% of teachers identified some form of adaptation of their classroom activities. When asked how they adapt these activities to meet student needs, 64% of the teachers who responded indicated adapting elementary mathematics activities found on VPRs to meet the needs of specific learners, such as English-language learners or special education students. This is a promising result because research has shown online supplements can cause harm to certain students, with Polikoff and Dean (2019) specifically discussing this issue for ELLs and Harris et al. (2020) discussing supplements that embody racist ideas. These studies were for history and English supplements, but we find it encouraging that some elementary teachers report adapting for these specific learners for mathematics.

Many teachers discussed changing the implementation structure of supplemental activities which mirrors an often-cited reasons teachers provide for searching for supplemental activities (e.g. Sawyer et al., 2019a; Carpenter and Shelton, 2021) regarding searching to increase student engagement. Teachers also reported adapting the mathematics content. Given that the quality of elementary mathematics supplements themselves tends to be low (Sawyer et al., 2019b; Shapiro, 2021), it is good to learn that teachers are adapting the mathematics content. These results also mirror the finding that teachers (Schroeder et al., 2019) and preservice teachers (Schroeder & Curcio, 2022) often adapt supplemental resources to align with state or district standards. The findings seem to indicate that teachers are not finding what they want when searching for elementary mathematics activities on VRPs. We find it interesting and encouraging that teachers appear to be taking a critical stance when evaluating and selecting supplemental elementary mathematics activities and that they expect to adapt what they find to better meet the needs of their students.

Through this study, we have gained further insight into what teachers do with elementary mathematics curricular supplements they find on VRPs and the answer is that they tend to adapt these supplements prior to classroom implementation. We agree with Silver (2021) that "it is essential that we understand teachers' supplementation decisions in order to develop teachers who supplement skillfully and responsibly" (p. 17). We believe the teachers would benefit from training regarding mathematics supplementation that goes beyond how to search and includes information on the adaptation process, but also moves further and follows teachers to the implementation process of these supplemental activities.

Limitations

This research does come with limitations. For example, it must be noted that a teacher's perception of what an activity is suggesting to do in a classroom could be incongruent with the creators' intent for the activity. Because we did not have access to either the supplemental activities that the teachers originally found or to teachers' adapted supplemental activities, we are unable to compare. The research field could learn from a comparative study to learn whether or not teachers are, for example, successful at increasing the rigor of a supplemental activity or if an adaptation for a specific learner actual increases the student's learning. Finally, this investigation was conducted prior to the pandemic; it is possible that teachers' adaptation of supplemental activities has changed how they adapt elementary mathematics activities in their in-person classrooms. More research is needed in this area.

Conclusion

Teachers have become their own curators of supplemental curricular resources and part of this curation process involves adaptation. Our research suggests that, in general, teachers adapt the supplemental elementary mathematics activities they find on VPRs. They adapt for various reasons related to student support but the most reported reasons relate to different learners and classroom implementation. Given the wide variance of student needs and structural parameters within each classroom, adaptation seems perhaps inevitable for effective teaching, but more remains to be learned about how adaptations are enacted and the effects on student learning. We believe it would benefit teachers to be explicit about their adaptation processes and discuss it with fellow teachers as a way to provide more information to the overall VPR supplementation process.

References

Browne, A., & Haylock, D. (Eds.). (2004). Professional issues for primary teachers. United Kingdom: SAGE Publications.

Carpenter, J., & Krutka, D. (2014). How and why educators use twitter: A survey of the field. Journal of Research of Technology in Education, 46(4), 414–434. https://doi.org/10.1080/15391523.2014.925701

Carpenter, J., Morrison, S., Craft, M., & Lee, M. (2020). How and why are educators using Instagram? Teaching and Teacher Education, 96, 103149. https://doi.org/10.1016/j.tate.2020.103149

Carpenter, J. P., & Shelton, C. C. (2021, March). Educators' perspectives on and motivations for using an online education marketplace: The case of TeachersPayTeachers.com. In Society for Information Technology & Teacher Education International Conference (pp. 722-727). Association for the Advancement of Computing in Education (AACE).

Choppin, J. (2011). Learned adaptations: Teachers' understanding and use of curriculum resources. Journal of Math Teacher Education 14, 331–353. https://doi.org/10.1007/s10857-011-9170-3

Davis, E., Beyer, C., Forbes, C., Stevens, S. (2011). Understanding pedagogical design capacity through teachers'

narratives. Teaching and Teacher Education 27(4), 797-810. https://doi.org/10.1016/j.tate.2011.01.005
Davis, J., Choppin, J., Roth McDuffie, A., & Drake, C. (2013). Common Core State Standards for mathematics: Middle school mathematics teachers' perceptions (Technical report). University of Rochester, Warner School of Education.
Drake, C., & Sherin, M. (2006). Practicing change: Curriculum adaptation and teacher narrative in the context of mathematics education reform. Curriculum Inquiry, 36(2), 153–187. https://doi.org/10.1111/j.1467-873X.2006.00351.
Forbes, C. (2011). Preservice elementary teachers' adaptation of science curriculum materials for inquiry-based elementary science. Science Education 95(5). 927-955. https://doi.org/10.1002/sce.20444.
Glaser, B., Strauss, A., Strutzel, E. (1968). The Discovery of grounded theory; Strategies for qualitative research. Nursing Research 17(4) 364.
Harris, L., Shelton, C., & Archambault, L. (2020). Examining issues of quality on teachers pay teachers: An exploration of best-selling U.S. history resources [Paper presentation]. National Council for the Social Studies, College and University Faculty Assembly. https://academicexperts.org/conf/site/2021/papers/58537/
Hertel, J. T., & Wessman-Enzinger, N. M. (2017). Examining Pinterest as a curriculum resource for negative integers: An initial investigation. Education Sciences, 7(2), 45.
Hodge, E., Salloum, S., & Benko, S. (2019). The changing ecology of the curriculum marketplace in the era of the Common Core State Standards. Journal of Educational Change, 20(4), 425–446. https://doi.org/10.1007/s10833-019-09347-1
Hu, S., & Torphy, K. (2020). Social media and education: Curation within and outside the schoolhouse (Working paper). Michigan State University, Teachers in Social Media.
Miles M., Huberman M., & Saldana, J. (2013). Qualitative data analysis: A methods sourcebook (3rd ed.). Thousand Oaks, CA: SAGE Publications.
Parsons, S., (2012). Adaptive teaching in literacy instruction: Case studies of two teachers. Journal of Literacy Research 44(2). 149-170. https://doi.org/10.1177/1086296X12440261
Polikoff, M. S. & Dean, J. (2019) The supplemental-curriculum bazaar: Is what's online any good? Thomas B. Fordham Institute. https://fordhaminstitute.org/ national/research/supplemental-curriculum-bazaar
Remillard, J. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. Curriculum Inquiry 29(3). 315-342. https://doi.org/10.1111/0362-6784.00130.
Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. Review of Educational Research, 75(2), 211–246. https://doi.org/10.3102/00346543075002211
Sawyer, A., Shapiro, E., Dick, L, & Wismer, T., &. (2019). Just what online resources are elementary mathematics teachers using? Contemporary Issues in Technology and Mathematics Teacher Education,19(4), 670-686.
Sawyer, A., Dick, L., Wismer, T.* & Shapiro, E.* (2019). The top 500 mathematics pins: Analysis of elementary mathematics activities on Pinterest. Journal of Technology and Teacher Education, 27(2), 235-263.
Schroeder, S., & Curcio, R. (2022). Critiquing, curating, and adapting: Cultivating 21st-century critical curriculum literacy with teacher candidates. Journal of Teacher Education, 00224871221075279.
Schroeder, S., Curcio, R., & Lundgren, L. (2019). Expanding the learning network: How teachers use Pinterest. Journal of Research on Technology in Education, 51(2), 166–186. https://doi.org/10.1080/15391523.2019.1573354
Shapiro, E. J.*, Dick, L. K., Sawyer, A. (2021) Critical consumption of online resources. Texas Mathematics Teacher, 47(1), 6-11
Sherin, M., & Drake, C. (2009). Curriculum strategy framework: Investigating patterns in teachers' use of a reform-based elementary mathematics curriculum. Journal of Curriculum Studies, 41(4), 467–500. https://doi.org/10.1080/00220270802696115
Shelton, C., & Archambault, L. (2019). Who are online Teacherpreneurs and what do they do? A survey of content creators on teacherspayteachers.com. Journal of Research on Technology in Education, 51(4), 398–414. https://doi.org/10.1080/15391523.2019.1666757
Silver, D. (2021). A Theoretical Framework for Studying Teachers' Curriculum Supplementation. Review of Educational Research 20(10), 1-35. https://journals.sagepub.com/doi/10.3102/00346543211063930.
Tosh, K., Doan, S., Woo, A., & Henry, D. (2020). Digital instructional materials: What are teachers using and what barriers exist? (Technical report). RAND Corporation. https://doi.org/10.7249/RR2575.17
Weiss, R. S. (1994). Learning from strangers: The art and method of qualitative interview studies. New York: Free Press.

INTERACTIVE DYNAMICS IN TASK-BASED CURRICULAR MATERIALS

Allison L. Gantt
University of Delaware
agantt@udel.edu

Given the importance of students' interactions to their opportunities to learn mathematics, I investigate the interactive dynamics that are encouraged in two U.S. high school task-based written curricular materials and the purposes for interaction that they communicate. I conduct a thematic analysis to organize the functions of dynamics that encourage students to interact with peers' ideas (e.g., through dialogue) and with their own ideas (e.g., through reflection). Peer-interactive dynamics in this sample focused on sense-making, negotiation, and comparison. Self-interactive dynamics focused on developing awareness of thinking, formalization of new algebraic ideas, exploration, and precise language. I discuss how patterns among these themes, across who interaction is designed for and across curricular materials, could generate future lines of inquiry connecting the design of curriculum to student learning.

Keywords: Curriculum, Instructional Activities and Practices, Metacognition

Written curricular materials can organize the landmarks of subject matter together with a plan for learners' thinking and development (e.g., Remillard & Heck, 2014); in other words, they can communicate intended opportunities to learn for students (Brown et al., 2009). The actualization of opportunities to learn mathematics, however, is strongly linked with the quality of students' *interactions* with mathematical ideas in the classroom (Cohen et al., 2003; Gresalfi, 2009; Haertel et al., 2008; Hiebert & Grouws, 2007). Because of the strong influence of written curricular materials on teachers' enactment and students' experiences (Choppin et al., 2018; Remillard & Heck, 2014), we could better understand how curriculum connects to learning by examining the interactions curricular materials encourage.

In the following paper, I present findings from an analysis of the types of student interactions recommended in two U.S. high school mathematics curricula (i.e., *interactive dynamics*). I chose to examine task-based curricula as their approach depends upon students' activity as a primary instructional method. I was guided by the following research question: *What functions can be ascribed to the interactive dynamics in teachers' guides of task-based curricula?* I begin with a review of what is known about the relationship between interactions and mathematics teaching learning and follow with a description of my method and results. I use these findings to exemplify some of the current purposes of interaction in task-based curricula, discussing some of the patterns in results and connecting with literature to recommend areas for future exploration.

Interactions in Mathematics Teaching and Learning

Hiebert and Grouws (2007) argue that opportunities to learn mathematics are generated through interacting and grappling with key conceptual ideas. In classrooms that center student discourse and activity, students may grapple as they engage their peers' contributions. Webb et al. (2014) found that students' engagement with one another's thinking (beyond giving explanations) was significantly related to student achievement. Moreover, they found that teachers' actions (namely, their follow-up questioning strategies) were influential in establishing a classroom where such engagement could develop. Research has shown, however, that even in classrooms with consistent discourse among students, not all students experience the same potential for interactions with peers' ideas (Gresalfi, 2009; Reinholz & Shah, 2018). These findings suggest open questions about how curricular materials could support and structure teachers to facilitate all students' high-level engagement with their peers' contributions.

Interaction that promotes learning in the classroom may not be limited to students engaging others' ideas; research has also shown the importance of students interacting with their own thinking around mathematics content (e.g., Cohen et al., 2003). Piaget's theory of reflective abstraction (2001), for example, suggests that a learner must engage with their own thinking after accommodating a new idea to encourage stable meanings. Furthermore, Jansen (2020) suggests that teachers play an active role in designing instructional structures that encourage consideration of how one's own ideas have changed over time and through experiences with revision and reflection. Although these interactions are considered to be central to teaching practice, studies have yet to explore how curricular materials might support or communicate particular strategies.

Method

To respond to my research question, I conducted a qualitative content analysis (Mayring, 2015) and accompanying thematic analysis (Terry et al., 2017) of the interactive dynamics constructed in teachers' guides of one unit each in two task-based curricular materials: *Illustrative Mathematics* (IM) and the *Interactive Mathematics Program* (IMP). I selected these texts as they each satisfied my conditions for a task-based curriculum: 1) the student-facing materials took the form of bounded activity with an established instructional goal (i.e., a task), and 2) the teacher's guide encouraged facilitation of student activity and dialogue as the primary mechanism of instruction (i.e., learning was framed as task-based). Moreover, the two curricula selected each contained an instructional unit focused on the same content. Specifically, I examined an instructional unit from each set of curriculum materials that encouraged students' work with standard form linear equations, systems, and inequalities ("Unit 2: Linear Equations, Inequalities, and Systems" in IM Algebra 1 and "Cookies" in IMP Year 2). This choice allowed me to account for variation in teacher's guide recommendations due to mathematical topic.

I analyzed the teacher's guide text surrounding a task. In IM, I considered each numbered activity within a lesson (except the cool-down) a task. In IMP, I considered each titled activity listed in the table of contents to be a task. I focused on the text for each task that was marked as an expected component of teacher's facilitation (e.g., "Activity Synthesis" or "Doing the Activity" sections). I omitted sections that were described as additional supports.

To begin my analysis, I first coded sentence-by-sentence within a task for *interactive dynamics*. I assigned this code if the teacher's text described enactment in a way that would explicitly encourage a student (or students) to respond to a contribution related to the algebraic content of the mathematical task. For example, a request for a student to share a content-related response to a peer's mathematical contribution would constitute an interactive dynamic; a request for a student to simply share an idea with the class would not (e.g., Webb et al., 2014).

After that, I assigned additional codes to sentences describing interactive dynamics. First, if the text expected a teacher to encourage their students to respond to their peers' ideas (e.g., a dialogue), I coded it as a *peer-interactive dynamic*. A response to students' own ideas (e.g., a reflection) would constitute a *self-interactive dynamic*. I also coded if an interactive dynamic encouraged teachers to engage *all* of their students or *some* of their students. If such engagement was not specified (e.g., the teacher asks a question to "students"), it was classified as intended for some students, as no structure was described that would ensure reaching all.

Once interactive dynamics had been catalogued in both curricula, I constructed themes from the data to identify patterns and relationships across and within codes (Terry et al., 2017). Through my themes, I sought to explore structures that were common and unique to each task-based curriculum and considered what they expressed about learning mathematics. In the results that follow, I describe and exemplify these themes. I note that these findings are preliminary.

Results

In my results, I characterize themes in the functions of peer-interactive and self-interactive dynamics in the sampled texts. Peer-interactive dynamics in this sample focused on sense-making, negotiation, and comparison. Self-interactive dynamics focused on developing awareness of thinking, formalization of new algebraic ideas, exploration, and precise language.

Peer-Interactive Dynamics: Sense-Making, Negotiation, and Comparison

In the teachers' guides of these task-based curricula, I found that peer-interactive dynamics directed toward *all* students emphasized understanding each other's ideas. Teachers' guide prompts in both IM and IMP encouraged teachers to facilitate *making sense of one another's solutions* (e.g., "Have students discuss their solutions in groups and try to understand one another's work," IMP, 2010, p. 25). In IM specifically, this sense-making process was always accompanied by a request for *reaching consensus around solutions*. For example, the "Launch" of one lesson gave teachers the following instruction:

> Tell students that when one student explains, the partner's job is to listen and make sure that they agree and that the explanation makes sense. If they don't agree, the partners discuss until they come to an agreement. (IM, 2019, p. 140)

As this quote shows, this peer-interactive dynamic centered a negotiation-based process (i.e., "discuss until they come to an agreement") and engaged all students with partners in the process. Peer-interactive dynamics reaching *some* students, however, occurred frequently in full-class discussions facilitated by the teacher and less commonly encouraged consensus-building. Rather, I identified one central theme in such peer-interactive dynamics: *looking for similarities and differences*. This theme was common across both curricula.

Examples of peer-interactive dynamics encouraging some students to look for similarities and differences ranged from short, general requests to more specific, task-focused requests. For instance, a prompt in IM encouraged the teacher, "After each student presents, ask if others solved it in the same way" (IM, 2019, p. 244). I note how this prompt, aimed to engage student volunteers, could be applicable across a variety of tasks. Other prompts, however, encouraged student volunteers in full-class discussions to center on "the variations among the graphs for each inequality" (IMP, 2010, p. 26) or "whether these equations all represent the same relationship" (IM, 2019, p. 53). In contrast, none of the peer-interactive dynamics described for *all* students (as shown above) were task-specific requests.

Self-Interactive Dynamics: Awareness, Formalization, Exploration, and Definition

Self-interactive dynamics reaching *all* students differed between IM and IMP texts. In the IM sample, self-interactive dynamics promoted *metacognitive awareness* for all individuals during full-class Math Talk routines. Math Talk routines encouraged students to engage a short series of mental math exercises to launch a central idea of the lesson. All Math Talks included the following instruction: "Give students quiet think time for each problem and ask them to give a signal when they have an answer and a strategy" (IM, 2019, p. 49). In this way, despite the full-class setting of the activity, IM encouraged all students to reflect on their own ideas and included a structure to support this behavior.

IMP, on the other hand, employed self-interactive dynamics of *developing new algebraic structures* for all students, which was strongly connected to the structure of the curriculum around a unit problem. In IMP, students begin the "Cookies" unit working an open problem and recording their progress; they then re-engage that problem at checkpoint moments during the unit as they build related mathematical ideas through other tasks. For example, students' first engagement with the "Cookies" unit problem asks teachers to "have groups use the inequalities to check that the combinations discussed earlier really do satisfy the constraints" (IMP, 2010, p. 4). This self-interactive dynamic for all students (through groups) could support students to transition from open strategies to the application of an algebraic structure (i.e., inequality statements), supporting their progress in the unit problem task.

I identified self-interactive dynamics reaching *some* students primarily in the form of questions the teacher could pose to individuals or the full-class during group work or discussions. Such questions were most often highly task-specific. Moreover, many of the themes for self-interactive dynamics reaching some students reflected the same themes reaching all students, falling again along curricular lines. IM continued to include prompts that encouraged *metacognitive awareness* (e.g., "How do you know that the expression gives us the liters of water in Tank A after *m* minutes?" IM, 2019, p. 165). IMP continued to include prompts directed toward *developing new algebraic structures* (e.g., "Encourage students to keep plotting points…until they realize that the boundary between satisfying the inequality is the line corresponding to the equation P + I =140," IMP, 2010, p. 19).

However, I also interpreted new themes in the samples of self-interactive dynamics reaching *some* students: namely, *exploring current algebraic structures* and *defining terminology and concepts*. Exploring current algebraic structures could be found in both curricula, encouraging students to investigate, apply, or interpret the situations and structures of the problem (e.g., "Ask whether there is any way to adjust the statement 4(-2) > 3(-2) to make it true," IMP, 2010, p. 9). Defining terminology and concepts, on the other hand, was most salient in IM: "If students refer to edges and vertices as 'lines' and 'points,' ask if they remember the 'math names' for these things" (IM, 2019, p. 51). As the example illustrates, these self-interactive dynamics in IM pressed students who used these terms to interact with ideas to develop precise language skills.

Discussion

My findings highlight how two task-based curricular materials communicate the functions of students' interactions in the classroom, both with students' peers' ideas and their own. Collectively, these findings illustrate what interactive dynamics can be designed to achieve with students; they also generate new questions about how interactions can be structured to support all students' opportunities to learn. I described that, although peer-interactive dynamics for all students encouraged general understanding and negotiation, dynamics for some students encouraged comparison in task-specific scenarios. Given that opportunities to learn may be experienced differentially across classrooms (e.g., Gresalfi, 2009), this finding suggests a factor of influence and leverage. Future research could investigate how peer-interactive dynamics in curricula could support goals of both sense-making and comparison through structures that support all students to engage with specific mathematical ideas.

Self-interactive dynamics, on the other hand, differed by curriculum, suggesting that interactive dynamics may also support curricular priorities. IM emphasized building students' metacognitive awareness and precision of mathematical language. IMP strived to leverage students' own ideas to build new algebraic structures. Future study could examine the relationship between these interactive goals and teachers' enactment (e.g., Brown et al., 2009) and students' ultimate experiences of opportunities to learn in the classroom. By examining the intentions for interactive dynamics in written curriculum, we can better understand how to design support for the interactions that occur during student learning.

References

Brown, S. A., Pitvorec, K., Ditto, C., & Kelso, C. R. (2009). Reconceiving fidelity of implementation: An investigation of elementary whole-number lessons. *Journal for Research in Mathematics Education*, *40*(4), 33. https://doi.org/10.5951/jresematheduc.40.4.0363

Choppin, J., Roth McDuffie, A., Drake, C., & Davis, J. (2018). Curriculum ergonomics: Conceptualizing the interactions between curriculum design and use. *International Journal of Educational Research*, *92*, 75–85. https://doi.org/10.1016/j.ijer.2018.09.015

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119–142. https://doi.org/10.3102/01623737025002119

Gresalfi, M. S. (2009). Taking up opportunities to learn: Constructing dispositions in mathematics classrooms. *Journal of the Learning Sciences, 18*(3), 327–369. https://doi.org/10.1080/10508400903013470

Haertel, E. H., Moss, P. A., Pullin, D. C., & Gee, J. P. (2008). Introduction. In P. A. Moss, D. C. Pullin, J. P. Gee, E. H. Haertel, & L. Jones Young (Eds.), *Assessment, equity, and opportunity to learn* (pp. 1–16). Cambridge University Press. https://doi.org/10.1017/CBO9780511802157.003

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 378–392). Information Age Publishing.

Illustrative Mathematics. (2019). *Algebra 1 unit 2 teacher guide*. Illustrative Mathematics. illustrativemathematics.org

Interactive Mathematics Program. (2010). *Cookies teacher guide.* Key Curriculum Press.

Jansen, A. (2020). *Rough draft math: Revising to learn*. Stenhouse Publishers.

Mayring, P. (2015). Qualitative content analysis: Theoretical background and procedures. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education*. Springer. https://doi.org/10.1007/978-94-017-9181-6_13

Piaget, J. (2001). *Studies in reflecting abstraction*. Psychology Press Ltd.

Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education*, *49*(2), 140–177. https://doi.org/10.5951/jresematheduc.49.2.0140

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM*, *46*(5), 705–718. https://doi.org/10.1007/s11858-014-0600-4

Terry, G., Hayfield, N., Clarke, V., & Braun, V. (2017). Thematic analysis. In C. Willig & W. Stainton-Rogers (Eds.), *The SAGE handbook of qualitative research in psychology*. SAGE Publications Ltd. https://doi.org/10.4135/9781526405555

Webb, N. M., Franke, M. L., Ing, M., Wong, J., Fernandez, C. H., Shin, N., & Turrou, A. C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning. *International Journal of Educational Research*, *63*, 79–93. https://doi.org/10.1016/j.ijer.2013.02.001

HOW THE TEACHER AND STUDENTS IMPACT THE UNFOLDING OF MATHEMATICAL IDEAS ACROSS A LESSON

Amanda Huffman	Leslie Dietiker	Andrew Richman
Purdue University	Boston University	Boston University
huffma33@purdue.edu	dietiker@bu.edu	asrich@bu.edu

By highlighting the curriculum modifications that lead to maintaining, or enhancing, the mathematical quality of an algebra lesson introducing the substitution method for solving systems of equations from an algebra textbook, we present an analysis of how a teacher and her students impact how the mathematical ideas unfold across the lesson and how they are experienced. Using a narrative-based analytical approach to write the stories of the written and enacted lessons, we found key similarities and differences in the lessons. In comparing the mathematical plots, we found evidence of how the teacher and students alter the unfolding story with the incorporation of more jamming than seen in the text and more questions developed based on the students' needs and their responses.

Keywords: Curriculum; Teaching Practice; Algebra and Algebraic Thinking

Research has demonstrated that teachers combine pedagogical practice, contextual factors, and the curriculum when teaching a lesson (e.g., Remillard & Heck, 2014). Teachers make pedagogical and contextual decisions (e.g., restricting calculators) based on the needs of their students, the context of learning, and their own prior experiences. While the text offers resources for the lesson and can certainly impact the teacher's instruction and students' learning (Remillard et al., 2014), what happens in the classroom often appears quite different than what the textbook authors intended since the actions of both teachers and students are influenced by a variety of prior experiences, goals, and perspectives. Ben-Peretz (1990) refers to the set of intended and unintended curricular uses of written curriculum materials as *curriculum potential* and describes the subset of these that are viewed favorably by its authors in the *curriculum envelope*. This theoretical framing implies there are a range of appropriate and beneficial interpretations and uses of all written curriculum materials.

However, there is little understanding of the ways in which lessons can vary within the curriculum envelope. Therefore, in this study, we illustrate how complex interactions between teachers and students in an enacted lesson shifts how the lesson unfolds while still drawing from the elements of the textbook lesson and maintaining its overall intentions. We present the case of one enacted algebra lesson implemented to a group of students from a set of written materials that were explicitly designed to increase the mathematical quality of courses by providing students access to rich, conceptually connected mathematical ideas. In this case, the enacted lesson maintained (or even enhanced) the mathematical quality of the lesson as written and thus is within the curriculum envelope. By highlighting the curriculum modifications that lead to maintaining, or enhancing, the mathematical quality of the written materials (e.g., its aesthetic opportunities, rigor), we begin to address the research question: *In an enacted lesson that is within the curriculum envelope, in what ways can the teacher and students impact how the mathematical ideas unfold across the lesson and how they are experienced?* We present a new methodology to comparing enacted and written curriculum that focuses not only on content, but also on how the content unfolds across the lesson. This comparison allows us to see how enacted lessons can maintain or even enhance the mathematical quality of the lesson for students and could reveal potential strategies for taking advantage of the design of the curriculum.

Theoretical Framework

To compare mathematics lessons how mathematical ideas unfold, we interpret the lessons as a form of narrative. Similar to how literary stories can captivate a reader by the withholding and revelation of information (Nodelman and Reimer, 2003), mathematics lessons can similarly structure how information emerges, enabling curiosity and encouraging a reader to build interest in learning more (Dietiker, 2015). A *mathematical story* is a metaphorical interpretation of how the mathematical ideas unfold across a lesson, connecting the beginning with the end. Note, however, that mathematical stories are not limited to contextualized word problems, but rather describe the twists and turns of the mathematical revelations throughout a lesson. As with literary stories, mathematical stories have characters (i.e., mathematical objects, such as algebraic expressions) that are manipulated and changed through *action* (i.e., mathematical transformations, such as substituting an expression to make an equivalent equation). In addition, these mathematical characters and actions play out in one or more mathematical *settings* (i.e., the representational space, such as symbols on paper or manipulatives) (Dietiker, 2015).

In a literary story, the text communicates via a narrator, and its interpretation focuses on how the text communicates to a reader in ways that are both logical (i.e., what makes sense) and aesthetic (i.e., what it makes a reader feel) (Bal, 2009). Mathematical stories in written curriculum materials operate similarly; the text narrates the story, and it can be read for how it communicates to potential readers—who in this case are teachers and students. Reading a written mathematical story for how it communicates requires identifying how the mathematical ideas, which are assumed to be unknown from the start, emerge and change as the story proceeds. For example, if a new definition or a theorem is presented that has not yet entered the story, it is interpreted as a new revelation even if it is already familiar to the researcher.

Enacted mathematical stories (i.e., those that unfold within classroom) also have narrators and can be interpreted by how the mathematical ideas emerge and change across a lesson. However, in this case, the narration occurs both by the written curriculum (e.g., when a task is read from a worksheet) or by a human (e.g., when a teacher introduces a topic or when a student describes how they solved a problem). Thus, enacted mathematical stories are more like guerilla theater, where the teachers and students are both actors and audience members contributing to and experiencing the story (Dietiker et al., in progress). The teachers in enacted mathematical stories can also have a role in *storytelling*, as they may have specific intentions for how the mathematical story will progress, which impacts their decisions in the moment. For example, when a student raises a new mathematical idea that alters what is known at that point in a story, the teacher can choose what to do with that information (e.g., suppress, elaborate) based on how they want the mathematical story to progress.

The way a mathematical story unfolds has a potential aesthetic impact on its readers, what we refer to as its *mathematical plot*. A mathematical plot "describes the aesthetic response of a reader as he or she experiences a mathematical story, perceives its structure, and anticipates what is ahead" (Dietiker, 2015, p. 298). The way in which the mathematical ideas play out over time can spur curiosity in a reader, leading them to pose new questions (e.g., asking *how can that be?*) and seek information (e.g., recognizing that two equations are equivalent) that answer them. As new facts are revealed, a reader has an opportunity to make progress on what is known about the questions they have adopted at that point in the story. According to literary theory, stories that offer numerous sustained questions simultaneously are more compelling (Barthes, 1974). The transition of what is known about a question, from when it opens to how it is answered is referred to as a *story arc*. A story arc can be short-lived when a question is answered quickly after being asked, or can be longer when a question remains unanswered and is still in consideration throughout portions of the lesson. Since some questions may never be resolved, some story arcs may remain open at the end of the story. Others may be abandoned when it is clear to a reader that the story has moved on. With multiple story arcs open, a student may sense a growing mystery, while a decrease may cause a reader to sense relief (Dietiker, 2015).

Methods

This study is part of a larger exploratory study that initially examined the different ways experienced (i.e., more than five years of experience) Algebra I teachers enacted the same textbook lessons. Using the mathematical story framework, this study compares one teacher's enacted lesson juxtaposed with the written lesson through a narrative-based analytical approach. In this section, we describe the context of the written and enacted lessons and describe how we interpreted and compared their mathematical plots. Note that we share our interpretations and they may vary from other groups completing this same process. Therefore, a key to this analysis is that both the written and enacted stories were analyzed using a consistent process. Our interpretation of these mathematical stories represents a consensus of a diverse mixture of mathematics education researchers, practicing teachers, mathematicians, and graduate students, and thus were informed by a mixture of perspectives regarding curriculum, mathematics, and teaching. Our interpretations represent a *potential* interpretation of a novice learner, as we took into consideration the story's previous revelations while setting aside our prior knowledge.

Sources of Data

The written lesson in this study, Lesson 4.2.1 from the *CPM Core Connections Algebra* textbook (Dietiker et al., 2006), introduces the substitution method for solving a system of linear equations. All elements of the lesson as provided in the teacher's edition were analyzed, which includes the student-facing materials (i.e., tasks, explanations) and suggestions to the teacher.

The enacted lesson, which was based on lesson 4.2.1 from the *CPM Core Connections Algebra* textbook (Dietiker et al., 2006), was observed in Ms. Turner's (all names are pseudonyms) Southern United States high school Algebra I course for special education students during the Fall of 2015. Ms. Turner's classroom consisted of six students, all of whom were Black, seated in small groups. The lesson was videotaped using three cameras: one facing the front board and two facing the students. In addition, audio recorders were placed on student desks about the room. The main video recording facing the front board, along with the audio recorders, was used to build the lesson transcript.

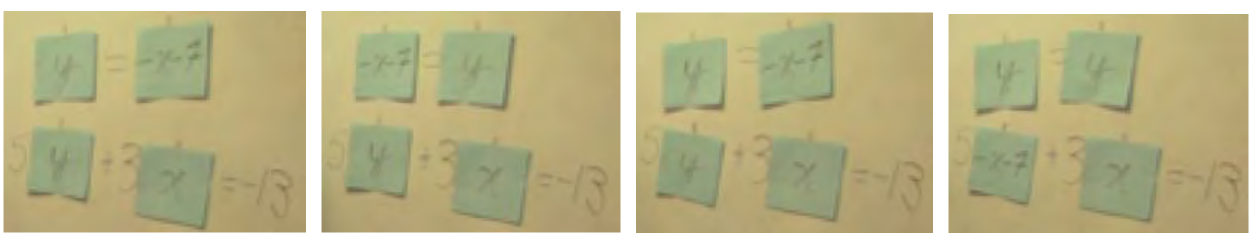
Ms. Turner, who is White, was in her 19th year of teaching and was selected to be observed for this study based on her expertise. For six years prior to this study, Ms. Turner led professional development for mathematics teachers in addition to coaching fellow teachers in her district for the previous seven years. Prior to the observation of this lesson, Ms. Turner was interviewed for information on her school demographics, her goals for the lesson, and her anticipated challenges during the lesson. According to Ms. Turner, from the pre-interview, the learning goal for this lesson was for students to rewrite the two equations with two variables as a single equation in a single variable by using a suitable substitution expression. After the observation, she was interviewed again about her reflections and was asked to discuss her curricular decisions that were made throughout the lesson.

The Written Mathematical Story

The lesson begins with the students solving the system $y = -x - 7$ and $5y + 3x = -13$ using a method that was introduced in a prior lesson, namely, the *equal values method* (i.e., a process that involves solving each of two equations for the same variable and then setting the two resulting expressions equal to each other). After providing the students a few minutes to solve the system of equations, the teacher is encouraged to stop the students and launch a discussion about whether it would be useful to have an easier method. A task then introduces the substitution method for the same system of equations, and the teacher is encouraged to use strips of paper with parts of the expressions to build the equations to aid in a discussion of the logic of the new method. Four more systems of equations are then given for students to solve using this new method. One of these results in a new situation (i.e., no solution). The lesson concludes with the task prompting students to figure out a way to know if the solution of a fictional student is correct or not.

The Enacted Mathematical Story

Ms. Turner started the lesson by reviewing the equal values method. She then prompted students to solve a system of equations (i.e., $y = -x - 7$ and $5y + 3x = -13$) with this method. After the students struggled to solve the problem for a few minutes, Ms. Turner stopped the students and offered a new method to the students with a reasoning of why this new method was beneficial (i.e., you can avoid ugly numbers). Using the paper switching method described in the teacher's guide, Ms. Turner introduced the students to the substitution method. Figure 1 shows the paper switching process with (a) the original system of equations formed by cards and symbols, (b) the system after switching the cards in the top equation, (c) the same systems with the *y* cards switched in the top and bottom equations, and (d) the same system after switching the bottom *y* card with the $-x - 7$ card in the top. Before Ms. Turner even started through the process one student indicated insight, but she quickly quieted the student. She continued to guide the students through the substitution method with the paper switching visual. After "showing her (Ms. Turner's) magic" (i.e., the substitution method), the students practiced solving two more systems of equations using their new method.



(a) (b) (c) (d)

Figure 1: Paper switching process.

Interpreting the Mathematical Story and Plot

In order to analyze and compare the mathematical plots of the written and enacted lessons, the researchers analyzed each portion of both stories for opportunities for new understanding. To interpret the stories, we analyzed the text of the story; the text of the written lesson included the suggested components of the lesson in the teacher's guide including all tasks and statements, while the text of the enacted lesson was the transcript from the classroom.

Analyzing the mathematical plots required three passes through each text. Each pass was first performed individually and then the group of researchers met to resolve differences. First, the researchers divided the text into agreed upon *acts*, where each act represented a portion of the story during which the mathematical story advanced and different acts were marked by changes in the mathematical characters, actions, or settings. Second, the researchers identified all the questions raised, either explicitly or implicitly, by the curriculum, teacher, or students in the text.

Third, we used Barthes' (1974) codes to describe the transition from question to answer and any suspense or surprise in between: *question formulation, promise of an answer, snare* (misleading information), *jamming* (unanswerable question), *suspension* (delayed answers), *partial answer* (progress), and *disclosure of the answer* (endorsing the answer).

With both the written and enacted mathematics lessons coded for their mathematical plots, the researchers analyzed how the mathematical plots of the written and enacted stories were similar and different.

Findings

The mathematical plots, which reveal how the stories unfolded in the lessons, are provided in Figure 2 (as written) and Figure 3 (as enacted). The acts (in columns) are presented in order from left to right, while the mathematical questions are listed in the order they emerged in the left column. The shaded cells, representing the story arcs, contain letters for Barthes' (1974) plot codes, where a: Question by Teacher or Environment, b: Question by Student, c: Promise, d: Progress by Teacher or Environment, e: Progress by Student; f: Snare, g: Jamming, h: Suspension, and i: Disclosure.

	ACT	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	How do you solve systems using the substitution method?	a	c		c	de	de	ed	d	e		c	e	e	
2	Why do we need another method for solving systems of equations?	a	d	e	di										
3	What makes a system of equations too messy for the equal values method?		a	e	i										
4	What is the solution to the system of equations (y=-x-7 and 5y+3x=-17)?			aeg		ed	de		d	i					
5	How does the substitution method eliminate the fractions that arise when using the equal values method?				a	e	de		i						
6	Can you switch the y with the -x-7 in the first equation? Why or why not?					aei									
7	Can we switch the y in the second equation with the -x-7? Why or why not?						aei								
8	Could we switch the -x-7 with the x in the second equation? Why or why not?							aei							
9	How do I decide what to substitute for?							ade					e	e	
10	What is the solution to 4-33a (y=3x and 2y-5x=4)?										aei				
11	What is the solution to 4-33b (x-4=y and -5y+8x=29)?											aei			
12	What is the solution to 4-33c (2x+2y=18 and x=3-y)?												aeg		
13	What does it mean to get a false equation?												be		
14	What is the solution to 4-33d (c=-b-11 and 3c+6=6b)?													aei	
15	How do you know that Mei's solution is correct?														ae

Figure 2: Mathematical plot of the *Core Connections Algebra* textbook Lesson 4.2.1.

	ACT	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	What is the substitution method?	ad				ceg	de	de	ed	ded	ed	de	de					
2	What makes a system too messy to solve with equal values?	a			e	d						d						
3	How do you solve a system of equations with the equal values method?	a	d	de		g												
4	How do you solve the system y=-x-7 and 5y+3x=-13 using the equal values method?			ade	ed	gd												
5	What is the solution to the system of equations y=-x-7 and 5y+3x=-13?			ade	ed	e			ed	ed	ed	dei						
6	Can I set these two equations equal to each other and start solving?			aei														
7	What do we need to do to one of these equations to solve with equal values method?			aei														
8	How do we isolate y in the equation 5y+3x=-13?				ae	gd												
9	What should I do with -13 and -3x, and why?				aei													
10	How do we use algebra tiles to help us solve equations?				af	gh				d	d							
11	What makes this new method easier?					aeg			d	d		i						
12	What does that student mean by "you put it in the place of y?"					bg	d		di									
13	What does the "=" symbol represent?					aed	de		e									
14	Is y=-x-7 equivalent to -x-7=y? Why?						aei											
15	Is the system equivalent if we switch the [y] terms? Why?							aei										
16	Can the y in the bottom equation be switched with the -x-7 in the top equation? Why?								aei									
17	Is the equation y=y true?								ae									
18	How do we solve this new one-variable equation?									aed	ei							
19	Why do we have to use the distributive property?									ae								
20	How do I multiply 5 by -x?									bei								
21	How do I solve -2x=22?										aei							
22	Once I have a value for one variable, how can I find the other variable?											aed			def			
23	What mistake is she predicting?											adei						
24	What is -x when x is negative?											bei						
25	What is the solution for the system x=2y+4 and 3x+2y=-28 when using substitution?												aed	ed	dei			
26	What is the solution to the system y=3x and 2y-5x=4 when using substitution (4-33a)?															adei		
27	How can you tell if the solution x=4 and y=2 is correct for the system without solving?																aegc	
28	What is the solution of 3(3x+5)=42?																	aed

Figure 3: The Mathematical Plot of the Enacted Lesson.

In comparing the mathematical plots, we found evidence of how the teacher and students alter an unfolding story within the curriculum envelope for the benefit of student experiences. Specifically, we discuss three shifts in the enacted mathematical story: (a) additional jamming by Ms. Turner's added questions and interactions with her students, (b) Ms. Turner's students' increased engagement through her verbal declaration of knowing what mistake her student was about to make and evidence of her understanding of her students and the rapport she has built with them, and (c) added questions asked by the teacher in response to her students' needs while addressing the algebraic processes of solving the systems of equations.

Added Jamming in the Enacted Story

In the enacted lesson, specifically in Act 5, Ms. Turner's actions of posing the problem to solve a system of equations (i.e., $y = -x - 7$ and $5y + 3x = -13$) using the equal values method created multiple instances of *jamming*, or an experience where you think you are going to get an answer and then the story threatens and drops hints that you may not get the answer. Both the written lesson Question 5 and the enacted lesson Question 4 (*What is the solution to the system of equations* $y = -x - 7$ and $5y + 3x = -13$*?*) created a similar instance of jamming as the posed problem became too messy to solve. However, Ms. Turner added additional questions around this problem which added more jamming to the enacted story:

3	How do you solve a system of equations with the equal values method?
8	How do we isolate *y* in the equation $5y + 3x = -13$?
10	How do we use algebra tiles to help us solve equations?
12	What does that student mean by "you put it in the place of the *y*"?

That is, these additional questions (i.e., Question 3, Question 8, Question 10) offered support for her students as they tried to solve the given system of equations using the equal values method. Then, when Ms. Turner stopped the class as the problem became too messy, abandoning the equal values method to solve the given system of equations, this simultaneously disrupted Question 3, Question 4, and Question 8. As Ms. Turner moved on to introduce the substitution method, Question 12 was introduced by a student, but this question was quickly jammed in order for Ms. Turner to walk through the entire process of the substitution method step-by-step with the paper switching method.

During the paper switching demonstration, beginning in Act 5, another element of jamming occurred as one student quickly picked up on what to do. Ms. Turner stopped the student from sharing his complete thought, to continue explaining the process and to ensure the other students followed the substitution method. Ms. Turner enabled this aesthetic moment in her classroom by deciding in the moment to quiet her students' *aha* moment, as the other students in the class would benefit from a guided introduction to substitution. After Ms. Turner explained a little more, she returned to the student who was eager to share and quickly saw how the substitution method worked when doing the paper switching visual provided by the curriculum. These added elements of jamming and Ms. Turner's deviations enhanced the lesson for her students while providing her students access to rich, conceptually connected mathematical ideas as the author(s) intended in the written lesson.

Predicting a Student's Error in the Enacted Story

In Act 11, Ms. Turner's response, "I guarantee you've made a mistake and I'll eat my words if you didn't," while the students are solving for y to finish solving the system of equations after as a class they found the value for x created question 23. Question 23, *What mistake is she predicting?* is not in the written lesson, but enhanced the lesson as Ms. Turner knew what her student was going to do. This student had mixed up the double negative in the problem, and this student action led to the formulation of Question 26, *What is -x when x is negative?* These two questions were added to the enacted story based on the teacher's predictions and the student's mistake. Again, these questions were created in the moment and based on the context of the lesson and Ms. Turner's students, something not seen in the written story.

Additional Questions to Solve the System in the Enacted Story

In the enacted story, Ms. Turner also added questions not found in the written story. These added questions supported the students as they worked to understand the substitution method and solve systems of equations, as the students needed reminders and guidance in solving the equation for one variable once they made the substitution (e.g., reviewing the distributive property). These additional questions demonstrate how Ms. Turner's pedagogical practice and contextual factors impacted the enacted lesson. The first occurrence of added questions is at the very beginning as Ms. Turner guided the students in recalling the equal values method from the previous lesson. Ms. Turner added the following questions:

6	Can I set these two equations equal to each other and start solving?
7	What do we need to do to one of these equations to solve with equal values?
8	How do we isolate y in the equation $5y + 3x = -13$?
9	What should I do with -13 and -3x, and why?
10	How do we use algebra tiles to help us solve equations?

These additional questions helped students to start solving the system of equations. These questions were not raised when the written substitution lesson was analyzed, since both the equal values method and how to solve an equation for one variable were presumed prior knowledge. Seeing Ms. Turner's introduction of these questions indicates the need for the students' prior knowledge to be recalled as a class. This also demonstrates how Ms. Turner knew her students and the context of the lesson and made adjustments to meet them where they were in supporting their development of the substitution method.

This is not the only time Ms. Turner added additional questions to the enacted lesson. The next occurrence was seen when guiding the students through the substitution method. Ms. Turner added additional questions (i.e., Questions 18 - 21) to assist the students in solving the system of equations using substitution. Ms. Turner even referred back to a visual representation, algebra tiles, and also discussed the distributive property while solving the equation for x after substituting in the expression for y. These added questions made the lesson fit the students and allowed the students to engage in the rich mathematical task.

18	How do we solve this new one variable equation?
19	Why do we have to use the distributive property?
20	How do I multiply 5 by $-x$?
21	How do I solve $-2x=22$?

Discussion

Our findings reveal how, in the enacted lesson, the teacher's questioning and the questions and responses of the students altered the story of the written lesson while still maintaining its overall intentions. The students' needs were known by the teacher and the students' responses and actions (i.e., "I already know what y is", forgetting a negative) enhanced the lesson. Ms. Turner's use of the introductory problem, the paper switching visual, and the practice problems showed her use of the curriculum when teaching the lesson, while the added jamming, prediction of a students' error, and added questioning demonstrated Ms. Turner's use of pedagogical practice and contextual factors when teaching the lesson. These experiences enhanced the lesson for these students and allowed these students to complete the mathematical task and learning goal envisioned by the authors. Ms. Turner's crafting and improvisation of the lesson in the moment supported the written lesson.

Through our narrative-analytical approach, we illustrated how complex interactions between a teacher and students in an enacted lesson shifted how the lesson unfolded while the lesson still drew from the elements of the textbook lesson. We demonstrated how one enacted lesson maintained, and even enhanced, the lesson for the students. Additional research is needed to highlight other enacted lessons also maintaining, and possibly even enhancing, the written lesson. Additionally, further research is needed to explore how to support teachers in recognizing the potential strategies of implementation that take advantage of the design of the curriculum materials so that more enacted lessons maintain or enhance the mathematical quality for students.

Acknowledgments

An author of this paper receives research funding from CPM Educational Program, which sells products involved in this research. This author also receives book royalties from this publisher for the textbooks used in the study. The terms of this arrangement have been reviewed and approved by Boston University in accordance with its conflict of interest policies.

References

Bal, M. (2009). Narratology: Introduction to the theory of narrative (C. Van Boheemen, Trans.; 3rd ed.). University of Toronto Press.

Barthes, R. (1974). *S/Z* (R. Miller, Trans.) Macmillan.

Ben-Peretz, M. (1990). *The teacher-curriculum encounter: Freeing teachers from the tyranny of texts.* SUNY Press.

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education, 119-156.* https://www.jstor.org/stable/40539329

Dietiker, L., Miller, E. R., Brakoniecki, A., & Richman, A. S. (in progress). "I'm Feelin' It!" Curricular contributions for aesthetic experiences in mathematics.

Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics, 90*(3), 285-302. https://doi.org/10.1007/s10649-015-9627-x

Dietiker, L., Kysh, J., Thomas Sallee, G., & Hoey, B. (2006). *Core connections: Algebra.* CPM Educational Program.

Nodelman, P., & Reimer, M. (2003). *The pleasures of children's literature* (3rd ed.). Allyn and Bacon.

Remillard, J. T., Harris, B., & Agodini, R. (2014). The influence of curriculum material design on opportunities for student learning. *ZDM Mathematics Education, 46,* 735-749. https://doi.org/10.1007/s11858-014-0585-z

Remillard, J. T., & Heck. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education, 46,* 705-718. https://doi.org/10.1007/s11858-014-0600-4

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* Teachers College Press.

MATHEMATICS EDUCATION DESIGN PRINCIPLES TO BUILD AN EVIDENCE BASE FOR FRACTION INTERVENTION

Jessica H. Hunt, Ph.D.
North Carolina State University
Jhunt5@ncsu.edu

Kristi Martin, Ph.D.
Sam Houston State University
Kxm138@shsu.edu

We report the results of the second of two pilot studies from an intervention designed and developed based on a validated trajectory of students' fraction concepts. Results suggest that designing and developing interventions that draw upon practices based in mathematics education and theory (e.g., utilizing students' own problem-solving actions, basing activities on learning trajectories, and supporting students to notice and discuss their problem-solving strategies) was an effective means of bolstering conceptual understanding and performance for the students we learned from in this work.

Keywords: students with disabilities; curriculum, equity, inclusion, and diversity.

Designing fraction instruction to meet the needs of diverse learners is a challenge facing educators today. One need often outlined in the literature is designing and testing curricular materials that build conceptual understanding, especially for students with learning difficulty (MD) and learning disability (LD). Observed performance differences that students with MD experience are concerning because conceptual knowledge mediates differences in overall performance in fraction skills between students with and without MD (Vukovic, 2012). Yet, we contend that the inadequate progress is likely the result of an opportunity gap caused not by a lack of knowledge or sub-optimal cognitive ability but instruction that does not provide these students access to their conceptual capacities or advance their understanding.

One innovative approach to bolstering access and advancement of conceptual knowledge of fractions is the use of learning trajectories as a grounding for intervention design and innovation. Learning trajectories include a goal, developmental stages of thinking, and activities designed to explicitly promote the stages of thinking to bolster concepts of fractions (Clements et al., 2020). Unfortunately, although basing instruction on learning trajectories is often recommended, "there is little direct evidence to support this approach" (p. 3).

We report the results of the second of two pilot studies (see Hunt et al., 2020) from an intervention designed and developed based on a validated trajectory of students' unit fraction, partitive fraction (i.e., non-unit fractions less than one), and iterative fraction (i.e., non-unit fractions greater than one) concepts. The research questions for this study are: (a) To what extent does a supplemental fraction intervention implemented in a school intervention setting and based on trajectories of students' fraction learning demonstrate evidence of increased student outcomes, defined as conceptual advance evidenced by units coordination, for students with LD and MD? And (b) Is there a statistically significant increase in score on a standardized measure of fractional concepts and operations after participating in a supplemental fraction intervention implemented in a school intervention setting for students with LD and MD?

Conceptual Framework

The learning trajectory utilized in the current study addresses the concepts of fractions as a coordination of units. Units coordination is defined as the number of units children can bring into fraction problems to think and reason with (Hackenberg et al., 2016) and involves the processes of partitioning, iterating, and the eventual combination of partitioning and iterating into a single process (Hackenberg, 2007; Norton & Wilkins, 2010; Wilkins et al., 2013). Partitioning is the mental or enacted action of dividing a unit into equal-sized parts. As students become more fluid with partitioning, they can mentally pull out a created fractional unit from the whole and understand it in relation to the whole without destroying the whole. They begin to iterate, or repeat, the fractional unit to make larger units (e.g., use $\frac{1}{3}$ to make $\frac{2}{3}$).

At first, students make sense of larger fractional units within the bounds of the whole (two-level units coordination). Over time, students will learn to combine partitioning and iterating into a single process called splitting. Splitting involves anticipating the results of partitioning a whole concurrently with iterating a fractional unit (Hackenberg, 2007; Norton et al., 2018). When students can split, they can understand larger fractional units outside of the bounds of a whole (three-level units coordination). Students will often combine the two processes sequentially before combining them into a single operation. We hypothesized students with LD could utilize similar processes to build a concept of fractions toward the learning goal, "Fractions are numbers that have magnitude determined by the coordination of the numerator with the denominator."

Methods

Participants and Data Sources

The pilot study took place in one suburban elementary school located in the Southeastern United States. The school enrolled approximately 1,050 students: of these students, 5.1% received special education services and 75.8% were from underrepresented groups. Students were selected based upon five criteria: (a) currently in fourth or fifth grade, (b) previously identified as requiring at least Tier II intervention in mathematics in response to sustained low performance and poor progress on grade-level curriculum, (c) have a weakness is fraction concepts and applications as identified by the classroom teacher, (d) performance of less than 30% on a screener of fraction concepts from the state's end-of-grade test, and (e) provide information parental consent and student assent. When students who met the criteria were identified, the school selected participants.

The process yielded a total of 10 students – two of whom received Special Education services (LD) and four who received 504 or Tier 3 services for math difficulty (MD) - participated in the study. These 10 students comprised two intervention groups, with five students in Group 1 and five students in Group 2. Of the 10 students, three were females, seven were males, nine were People of Color, and one was White. Students were between nine and 11 years of age. Intervention attendance for the nine out of the 10 students was high (100%).

Our intervention was designed to support the fraction learning of students with LD. Yet, schools often include both students with LD and MD in supplemental mathematics instruction. We included students with LD and MD to match school norms and procedures.

Data sources for conceptual advance included video data, accompanying student work, and researcher field notes for each session. For performance change, 16 items pulled from the 2018-2019 North Carolina Department of Public Instruction (DPI) released third, fourth, and fifth grade End-of-Grade (EOG) Exams were used as a distal measure of performance because they measured the Number and Operations Fraction Standards students were assessed upon across elementary school. The measure was also group-administered and included six items measuring third grade standards, two items measuring fourth grade standards, and eight items measuring fifth grade standards. Internal consistency ranged from 0.90 to 0.94.

Study Design & Data Analysis

A one phase, mixed methods quasi-experimental approach drove this research (Creswell & Plano Clark, 2018). Qualitative data were considered primary in the design, consistent with the study emphasis on changing students' conception and appropriate with small sample sizes.

Data analysis to detect conceptual change was done on several levels. First, constant comparison methods (Leech & Onwuegbuzie, 2007) and classical content analysis (Grbich, 2007) were used to document and quantify students' partitioning and iterating processes within each problem for each student. Researchers examined each student's (a) partitioning and iterating actions used to solve each problem (b) the student's utterances as they solved problems along with how they named, or quantified, their solution, (c) whether the student appeared to have a plan/strategy before having to act, and (d) any observable evidence related to the student's units coordinating. We used peer debriefing and collaboration to search for evidence that confirmed or refuted claims made across the data to build reliability and validity in the coding process. We had 86% interrater reliability for individual analyses of codes across the entire set. Researchers then quantified the number of occurrences for each code (e.g., how many times partitioning was observed) by dividing the totals by the total number of all codes.

Second, researchers used emergent coding to document how students' observable processes across the tasks and after the second, fourth, sixth, and tenth instructional sessions were or were not consistent with the stages of units coordinating learning trajectory confirmed in previous years of the project. Generally, researchers looked for evidence of the stages of units students could either (a) describe before activity in the problems or (b) used with activity in the problems. The analysis was iterative and included multiple perspectives to make confirmatory claims about patterns in the data consistent with conceptual advance. We established 100% interrater reliability after peer debriefing and collaborative work. Finally, to detect initial differences in performance on fraction state standards as a result of the instructional trajectory (i.e., before and after the intervention within the same group of students), a one-tailed paired sample t-test was used to evaluate statistically significant differences on the distal measure. The IV was time; the DV was score. The level of significance was set at 0.05.

Results

Three nuances in partitioning consistent with our learning trajectory were coded across the intervention sessions: (a) In action halving (i.e., students engaging in halving strategies in the midst of problem-solving), (b) In action linked (i.e., students engaging in partitioning linked the number of sharers in the midst of solving problems), and (c) Before action linked partitioning (i.e., students describing a partitioning plan that was linked to the number of sharers before solving the problem). After session two of the intervention, 56% of students used in action halving-based partitioning, and 23% and 21%, respectively, used in action planned or before action linked partitioning. After Session Four, 11% of students continued to use in action halving-based partitioning, 34% of students used in action linked partitioning; before action linked partitioning (55% of students) became the dominant method. After the eighth session, only 23% of students were in action partitioning. The final session showed a slight uptake in students' use of in action (33%) versus before action (67%) partitioning. The two students with LD both started the intervention using in-action halving partitioning. After session two, one continued to use in-action halving while the other progressed to in action linked partitioning. After session six, both progressed to and consistently used before action linked partitioning. Increased use of sophisticated partitioning alongside decreases in rudimentary partitioning showcases growth.

The forms of iteration uncovered and coded in the analysis were (a) in action trial and error (i.e., students created an equal share and did not correctly adjust it longer or shorter), (b) in action adjustment (i.e., students created an equal share and corrected adjusted its magnitude) and (c) planned (i.e., students accurately created an equal share and used iterating to confirm the part as $\frac{1}{n}$). After Session Two, 45% of students used in action trial and error iteration, 55% used in action adjustment, and 0% used planned iteration. After Session Four, the percentage of students using in action adjustment for iteration increased to 34%, and students using planned iteration increased to 22%, and the use of in action trial and error iteration was relatively unchanged (44%). After the eighth session, trial and error iteration was used by only one student (11%), while in action adjustment and (44%) planned (45%) iterating was dominant. Both students with LD began the sessions iterating in action trial and error. After session four, one student progressed to a stable use of planned iterating for the remaining sessions. The other student evidenced planned iteration in the tenth session.

Advances in units coordination emerged in the data that were consistent with students' advances in conceptual processes. After Session 4, most (89%) of students coordinated two levels of units in action (i.e., had to act within a problem to think about fractions as coordinated in magnitude with a whole), while one student (11%) came into the problems coordinating one level of units. After Session 8, students who bring in two levels of units into problem activity increased to 33%; students coordinating two levels of units in action fell to 56%. In the final session, 67% of students (inclusive of both students with LD) brought two levels of units into the problems. Process advances and units coordination mark students' conceptual advances.

A paired-samples t-test was conducted to compare performance score on a distal measure of students' fraction knowledge before and after participating in the instructional trajectory. A histogram confirmed the normality of the data. There was a statistically significant difference in group scores from pre-test ($m = 4.56$, $sd = 2.07$) to post-test ($m = 6.78$, $sd = 3.87$), $t = 2.23$, $p = 0.03$. Our results suggest that engaging in our fractions intervention positively impacted students' performance. When examining the performance of the students with LD on an individual basis, there was a statistically significant difference in scores from pre-test to post-test, $t = 2.24$, $p = 0.03$ and $t = 7.85$, $p < 0.0001$, respectively, for both students. Our results suggest that our intervention also positively impacted the performance of students with LD.

Discussion

Very few studies in mathematics education or special education evaluate the effectiveness of student-centered interventions based on mathematics education instructional practices and theory on improving student outcomes (see Hwang et al., 2019; Shin & Bryant, 2015). We provide evidence regarding an intervention designed and developed based on a validated trajectory of students' unit fraction, partitive fraction, and iterative fraction concepts (Hunt et al.., 2020). Designing and developing interventions that draw upon practices based in mathematics education and theory (e.g., utilizing students' own problem-solving actions, basing activities on learning trajectories, and supporting students to notice and discuss their problem-solving strategies) was an effective means of bolstering conceptual understanding and performance for the students that we worked with in this study. More work is needed to substantiate our findings.

References

Clements, D. H., Sarama, J., Baroody, A. J., & Joswick, C. (2020). Efficacy of a learning trajectory approach compared to a teach-to-target approach for addition and subtraction. ZDM - Mathematics Education. https://doi.org/10.1007/s11858-019-01122-z

Creswell, J. W., & Plano Clark, V. L. (2018). Designing and conducting mixed methods research (3rd ed.). Sage.
Grbich, C. (2007). Qualitative data analysis: An introduction. Sage Publications, Inc.
Hackenberg, A. J., Norton, A., & Wright, R. J. (2016). Developing fractions knowledge. Sage Publications, Inc.
Hunt, J. H., Martin, K., Khounmeuang, A., Silva, J., Patterson, B., & Welch-Ptak, J. (2020). Design, development, and initial testing of asset-based intervention grounded in trajectories of student fraction learning. *Learning Disability Quarterly*, 0731948720963589.
Leech, N. L., & Onwuegbuzie, A. J. (2007). An array of qualitative data analysis tools: A call for data analysis triangulation. School Psychology Quarterly, 22(4), 557–584.
Shin, M., & Bryant, D. P. (2015b). Fraction interventions for students struggling to learn mathematics: A research synthesis. Remedial and Special Education, 36(6), 374–387. https://doi.org/10.1177/0741932515572910.
Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: Participatory and anticipatory stages in learning a new mathematical conception. Educational Studies in Mathematics, 66, 273–291. https://doi.org/10.1007/s10649-007-9082-4
Vukovic, R. K. (2012). Mathematics difficulty with and without reading difficulty: Findings and implications from a four-year longitudinal study. Exceptional Children, 78(3), 280–300. https://doi.org/10.1177/001440291207800302.

'ME OR IDME': HOW INTERDISCIPLINARY MATHEMATICS EDUCATION (IDME) CONTRIBUTES TO STUDENTS' LEARNING EXPERIENCES WITH MATHEMATICS

Midhat Noor Kiyani
McGill University
midhat.kiyani@mail.mcgill.ca

Limin Jao
McGill University
limin.jao@mcgill.ca

Cinzia Di Placido
McGill University
cinzia.diplacido@mail.mcgill.ca

Sun Jung Choi
McGill University
sun.j.choi@mail.mcgill.ca

Dawn Wiseman
Bishop's University
dwiseman@ubishops.ca

Traditionally, mathematics is taught without a connection to the real world which makes it abstract and difficult for students to understand, thereby resulting in low mathematics achievement. This study investigated the learning experiences of Grade 8 students as they participated in project-based learning (PBL) as part of an interdisciplinary mathematics education (IdME) unit of study. Qualitative data were collected in the form of student self-reflections and interviews. Findings suggest that students were able to improve their mathematics knowledge and understanding by implementing mathematics concepts in real-world contexts. We argue that students not only understand mathematics better but also realize its importance as a discipline when it is taught through real-world projects.

Keywords: Instructional Activities and Practices, Middle School Education, High School Education, Integrated STEM / STEAM

Mathematics knowledge and processes are used in everyday life. Furthermore, skills developed through engaging with mathematics, such as logical thinking and problem solving, are critical in tackling complex global problems of the 21st century. Despite the importance of mathematics in our society, Canadian secondary students' mathematics achievement is declining (Allison & Geloso, 2021). A majority of the students find mathematics education vague, abstract, and detached from reality and real-life situations (Mosvold, 2008). Indeed, in secondary schools, mathematics has traditionally been taught without any connection to other subjects (Chi, 2021) despite curricula calling for interdisciplinary approaches to teaching and learning (e.g., Ontario Ministry of Education, 2016; Québec Education Program, 2001). Moreover, there is a dearth of literature that provides concrete examples and evidence of mathematics being taught in connection to other subjects, especially in the context of Canada. This paper presents a case of interdisciplinary mathematics education (IdME) being implemented in a Grade 8 mathematics course at a small, independent school in the province of Québec. Specifically, our study is guided by the following research question: How does IdME contribute to students' learning experiences with mathematics?

Perspectives

An interdisciplinary approach to teaching and learning involves curriculum organized around common learning across various disciplines where students learn concepts and skills that are common to two or more than two disciplines (Costley, 2015; Helmane & Briška, 2017). Specifically, in IdME, mathematics is taught in an interdisciplinary way (Chao-Fernández et al., 2019). IdME presents mathematics in a wider context (Chi, 2021), combined with one or more disciplines and everyday knowledge to encourage problem solving and inquiry (Williams et al., 2016). In IdME, mathematics can be combined with a wide range of disciplines including science, arts, or languages (Brante & Brunosson, 2014; Lovemore et al., 2021; Serrano Corkin et al., 2020). Studies indicate that IdME increases students' engagement (Madkins & Morton, 2021; Satterthwait, 2019; Serrano Corkin et al., 2020) and improves students' mathematics knowledge and understanding (Altin et al., 2021; Brante & Brunosson, 2014); both serving as crucial predictors of students' mathematical achievement.

A common approach for IdME is project-based learning (PBL)(Chi, 2021). PBL is an instructional method that allows students to engage with the content at a deeper level and bridge theory and practice (Condliffe, 2017). In PBL, students engage in complex tasks based on challenging questions or problems in authentic contexts (Bell, 2010; Laur, 2013). Students design, problem-solve, make decisions, and investigate topics of personal interest (Aydın-Günbatar, 2020; Lin et al., 2015). PBL helps students in "establishing conceptual associations between the learnt knowledge" (DemiRel, 2010, p. 48) and hence, improves their academic achievement in mathematics (Gürgil & Çetin, 2018). Indeed, a study by Holmes and Hwang (2016) found that students who initially had little belief in their mathematics skills "expressed mastery or learning goals" as result of engaging with PBL (p. 459). PBL has also shown to increase students' understanding of mathematics concepts by making students capable of applying mathematics knowledge to real-life situations (Boaler, 1997; Fain et al., 2015; Holmes & Hwang, 2016).

Research Context and Methods

The study took place within a context of a research-practice partnership with a small, independent girls' school in Québec. The Grade 8 mathematics teacher, Stephanie (all names are pseudonyms), was keen to integrate IdME into her teaching practice, and more specifically, used PBL as a means to achieving this goal. As such, she developed the "Lifestyles Project", comprised of three tasks: the Hobby, Careers and, Bedroom Design tasks. Tasks were purposefully interdisciplinary and designed in collaboration with teachers from other subject areas (science, English language arts, and visual arts, respectively). In the first task, students chose a hobby of personal interest and identified and explored science and mathematics concepts associated to the hobby. The second task required students to select a future career, find a job in this career, and calculate the salary that they would receive (given required tax deductions). In the final task, students designed a 3D, scaled model of the bedroom of their dream apartment considering mathematical constrains like size and surface area. Tasks were interspersed throughout the academic year, each spanning 2-3 weeks in length. Study participants were all of the students in Stephanie's Grade 8 mathematics course (*n*=16).

A qualitative case study method (Denzin & Lincoln, 2011; Stake, 2006), was used to explore the Grade 8 students' experiences with the Lifestyles Project. There were two sources of data for this study: 1) student self-reflections, and 2) student interviews. Students completed written self-reflection after most PBL classes to capture their experiences during the Lifestyles Project. Semi-structured interviews (Gubrium & Holstein, 2002) were conducted at the end of the year in which students were asked about their thoughts and feedback about the Lifestyles Project and IdME more generally. Interviews were audio-recorded and transcribed verbatim. Interview transcripts and student self-reflections were coded in a series of coding cycles using an inductive thematic analysis approach (Guest et al., 2011). Qualitative data analysis software NVivo was used to efficiently organize and code the data using a code book developed by the research team. We then selected codes specific to students' learning experiences in mathematics through an iterative process and categorized this data by seeking out emerging patterns and themes.

Findings

The aim of this study was to explore how IdME, as enacted through the Lifestyles Project, supported Grade 8 students' learning experiences with mathematics. In this section, we present two major themes that emerged from the data: 1) how the Lifestyles Project allowed students to connect mathematics to the real world, and 2) how real-world contexts supported the students' mathematics learning.

Connecting Mathematics to the Real World

Through the Lifestyles Project, students applied mathematics to realistic contexts. Students reflected on how the Lifestyles Project allowed them to see how mathematics could be used in the real world. For example, Raza shared,

> I learned that interior designing has math in it, because interior designers need to know how much the total surface area is of something. And they can't just pick up like, go buy a couch or something, and hope it fits. They really need to measure it, so it fits.

Or as another example, Cymbi described how the Careers task allowed her to learn about taxes. She said, "I loved learning new things like what QPIP (Québec Parental Insurance Plan) taxes were and why were they used. Because I never knew about these things, it was all so new!" Students recognized and discovered that many mathematics concepts are immediately applicable and helpful in problems and tasks that they may encounter in their daily lives.

Real-World Contexts to Support Students' Mathematics Learning

In addition to allowing students to see the usefulness of mathematics, real-world applications helped the students deepen their mathematics knowledge. Students' mathematical understanding was strengthened when they could connect mathematics with real-world applications relevant to their own lives. For example, in reflecting on the Bedroom Design task, Junan said,

> I think [the task] was also really good for understanding total surface areas a lot. On a paper, I don't count the backside (of furniture)…we can't see it. But, like actually seeing it (in 3D) because now if I were to look at my closet or something, I could say like, oh, I wouldn't count this because I can't see it is touching the wall.

Here, Junan described how calculating the surface area of a real 3D model instead of a 2D drawing on a paper allowed her to better understand the concept of surface area. Similarly, Cymbi described how designing her bedroom reinforced the importance of mathematics knowledge. She said, "Measurements really are important and proportions because like, I found myself making a table that was like, shorter than the bed and then I was like, oh my god, that doesn't make any sense." Here, Cymbi shared how measurement errors would result in major problems in this real-world context. Indeed, through the Lifestyles Project, students realized the value and importance of both mathematics and mathematics knowledge in broader, non-academic contexts.

Students also shared that the Lifestyles Project provided them an opportunity to develop a stronger understanding of mathematics concepts that they had previously learned in the course. For example, Junan said, "I was okay at [surface area] before, but especially now, like when I was looking at the actual shape, I was like, 'Oh, wait, I don't count this, I don't count that." A similar response was given by Aura who felt that the Bedroom Design task "helped with like, what we were learning in class was like, lateral area and total surface area."

Although the majority of students enjoyed the Lifestyles Project, some felt that the real-world contexts were a hinderance to their mathematics learning. As Rose said, "I wouldn't want [class] always to be these projects, because I would want to learn some math and stuff...I would prefer the majority of the year to be math". This response indicates that some students drew a clear boundary between 'mathematics' and these '(Lifestyle) projects'. These students did not see the Lifestyles Project as incorporating mathematics, whether it be the mathematics concepts or processes that they previously learned, or the students' conceptions of what mathematics is and how to do it. Kobu explained, "We're learning about stuff [in a way] that we wouldn't like, conventionally learn about in math class." Indeed, for some students, learning mathematics in a way that they were not accustomed to was challenging.

Discussion and Conclusion

The focus of this study was to explore how interdisciplinary mathematics education (IdME), implemented through project-based learning (PBL), contributed to the students' learning experiences with mathematics. In the "Lifestyles Project", Grade 8 students participated in a variety of tasks that involved integration of mathematics with other disciplines (i.e., science, English language arts, visual arts). Findings suggest that the students were able to make sense of mathematics concepts by applying them to real-world contexts, which provided them with an opportunity to enhance their mathematics learning and improve their mathematical understanding.

The relevant and authentic contexts of the Lifestyles Project allowed students to realize how mathematics can be implemented in the real world (Brante & Brunosson, 2014). Furthermore, these contexts underscored the importance of mathematics as a discipline that is inherently present and correlated to all disciplines. When engaging in IdME through the Lifestyles Project, the students recognized the need to apply integrated mathematics knowledge to engage in real-world design and problem solving tasks. Indeed, the students seemed to "understand how mathematics inter-relates with other school subjects and with the real life" (Chi, 2021, p. 667). We argue that IdME encouraged students to learn by "demonstrating the real-world relevance of their education" (Newell, 2010, p. 11). In connecting mathematics to real-world contexts, the students seemed to be better able to understand mathematics concepts that were initially perceived to be complex and abstract. This strengthened understanding of new and previously learned mathematics concepts, in turn, improves students' academic achievement, both in mathematics and more generally across disciplines (Applebee et al., 2007). Despite its positive impact on students' mathematics learning, some students found IdME to challenge their perception of mathematics education. Indeed, some students had preconceived notions that mathematics should be taught in isolation and not connected to real-world contexts. This is in contrast to research indicating that students are generally more positively inclined towards an interdisciplinary learning approach as compared to traditional approaches to teaching and learning (Ghisla et al., 2010; Zhan et al., 2017).

While these findings may not be generalizable to wider contexts, our study provides further evidence of the positive potential of IdME as a means to enhancing students' learning experiences with mathematics. We encourage researchers and educators alike to further consider the ways that IdME can be used to transform mathematics instruction and allow students to thrive.

References

Allison, D. J., & Geloso, V. (2021). *Math performance in Canada* (pp. 1–56). Fraser Institute. https://www.fraserinstitute.org/sites/default/files/math-performance-in-canada.pdf

Altin, R., Tokel, T., & Delialioglu, O. (2021). The effects of mathematics on programming skills and its retention: An experimental study. *Journal of Computers in Mathematics and Science Teaching*, *40*(3), 183–199.

Applebee, A. N., Adler, M., & Flihan, S. (2007). Interdisciplinary Curricula in Middle and High School Classrooms: Case Studies of Approaches to Curriculum and Instruction. *American Educational Research Journal*, *44*(4), 1002–1039. https://doi.org/10.3102/0002831207308219

Aydın-Günbatar, S. (2020). Making homemade indicator and strips: A STEM + activity for acid-base chemistry with entrepreneurship applications. *Science Activities*, *57*(3), 132–141. https://doi.org/10.1080/00368121.2020.1828794

Bell, S. (2010). Project-based learning for the 21st century: Skills for the future. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas*, *83*(2), 39–43. https://doi.org/10.1080/00098650903505415

Boaler, J. (1997). *Experiencing school mathematics: Teaching styles, sex and setting* (Vol. 24). Open University Press. https://www.proquest.com/docview/204850325/abstract/8770D5DFE584316PQ/1

Brante, G., & Brunosson, A. (2014). To double a recipe—Interdisciplinary teaching and learning of mathematical content knowledge in a home economics setting. *Education Inquiry*, *5*(2), 301–318. https://doi.org/10.3402/edui.v5.23925

Chao-Fernández, R., Mato-Vázquez, D., & Chao-Fernández, A. (2019). Fractions and pythagorean tuning—An Interdisciplinary Study in Secondary Education. *Mathematics*, *7*(12), 1–13. https://doi.org/10.3390/math7121227

Chi, N. P. (2021). Teaching mathematics through interdisciplinary projects: A case study of Vietnam. *International Journal of Education and Practice*, *9*(4), 656–669. https://doi.org/10.18488/journal.61.2021.94.656.669

Condliffe, B. (2017). *Project-Based Learning: A Literature Review. Working Paper.* (pp. 1–84) [Research report]. MDRC. https://www.semanticscholar.org/paper/Project-Based-Learning%3A-A-Literature-Review.-Paper.-Condliffe/bf8ffb27f742af21340c5d26e312ff185b44bb8e

Costley, K. C. (2015). *Research Supporting Integrated Curriculum: Evidence for using this Method of Instruction in Public School Classrooms* (pp. 1–11) [Evaluative Report]. https://www.semanticscholar.org/paper/Research-Supporting-Integrated-Curriculum%3A-Evidence-Costley/094aa3e2318893ac74cc77c4803921186e2ff761

DemiRel, M., & Coskun, Y. D. (2010). Case study on interdisciplinary teaching approach supported by project based learning. *International Journal of Research in Teacher Education*, *2*(3), 28–53.

Denzin, N. K., & Lincoln, Y. S. (2011). *The SAGE handbook of qualitative research*. SAGE.

Fain, N., Wagner, B., & Vukašinović, N. (2015, June 28). *A project-based approach to learning: Comparative study of two disciplines*. LearnxDesign, Chicago. https://www.researchgate.net/publication/280553841_A_project-based_approach_to_learning_comparative_study_of_two_disciplines

Ghisla, G., Bausch, L., & Bonoli, L. (2010). Interdisciplinarity in Swiss schools: A difficult step into the future. *Issues in Integrative Studies*, 295–331.

Gubrium, P. of S. and C. J. F., & Holstein, P. J. A. (2002). *Handbook of Interview Research: Context and Method*. SAGE.

Guest, G., MacQueen, K. M., & Namey, E. E. (2011). *Applied Thematic Analysis*. SAGE.

Gürgil, F., & Çetin, T. (2018). BÜTÜNLEŞTİRİLMİŞ PROGRAM UYGULAMALARININ ÖĞRENCİLERİN SOSYAL BİLGİLER DERSİ MOTİVASYONLARINA ETKİSİ. *Turkish Studies (Elektronik)*, *13*(27), 777–794. https://doi.org/10.7827/TurkishStudies.14312

Helmane, I., & Briška, I. (2017). What is Developing Integrated or Interdisciplinary or Multidisciplinary or Transdisciplinary Education in School? *Journal of Pedagogy and Psychology "Signum Temporis,"* *9*(1), 7–15. https://doi.org/10.1515/sigtem-2017-0010

Holmes, V.-L., & Hwang, Y. (2016). Exploring the effects of project-based learning in secondary mathematics education. *The Journal of Educational Research*, *109*, 1–15. https://doi.org/10.1080/00220671.2014.979911

Laur, D. (2013). *Authentic learning experiences: A real-world approach to project-based learning*. Routledge.

Lin, C.-S., Ma, J.-T., Kuo, K. Y.-C., & Chou, C.-T. C. (2015). Examining the efficacy of project-based learning on cultivating the 21st century skills among high school students in a global context. *Journal on School Educational Technology*, *11*(1), 1–9.

Lovemore, T. S., Robertson, S.-A., & Graven, M. (2021). Enriching the teaching of fractions through integrating mathematics and music. *South African Journal of Childhood Education*, *11*(1), 1–14. https://doi.org/10.4102/sajce.v11i1.899

Madkins, T. C., & Morton, K. (2021). Disrupting anti-blackness with young learners in STEM: Strategies for elementary science and mathematics teacher education. *Canadian Journal of Science, Mathematics and Technology Education*, *21*(2), 239–256. https://doi.org/10.1007/s42330-021-00159-1

Mosvold, R. (2008). Real-life connections in Japan and the Netherlands: National teaching patterns and cultural beliefs. *Journal for Mathematics Teaching and Learning*.

https://www.academia.edu/496349/Real_life_Connections_in_Japan_and_the_Netherlands_National_teaching_patterns_and_cultural_beliefs
Newell, W. H. (2010). Educating for a complex world. *Liberal Education*, *96*(4), 6–11.
Ontario Ministry of Education. (2016). *Ontario schools: Kindergarten to grade 12*. http://www.edu.gov.on.ca/eng/document/policy/os/onschools_2016e.pdf
Québec Education Program. (2001). *Chapter 2: Cross-curricular competencies*. http://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/PFEQ_competencestransversales- primaire_EN.pdf
Satterthwait, D. (2019). Making biology count: Integrating mathematics into the teaching of inheritance. *Journal of Biological Education*, *53*(1), 92–97. https://doi.org/10.1080/00219266.2018.1427613
Serrano Corkin, D. M., Ekmekci, A., & Fisher, A. (2020). Integrating culture, art, geometry, and coding to enhance computer science motivation among underrepresented minoritized high school students. *Urban Review: Issues and Ideas in Public Education*, *52*(5), 950–969. https://doi.org/10.1007/s11256-020-00586-8
Stake, R. E. (2006). Single Cases. In *Multiple Case Study Analysis* (pp. 1–16). Guilford Press.
Williams, J., Roth, W.-M., Swanson, D., Doig, B., Groves, S., Omuvwie, M., Borromeo Ferri, R., & Mousoulides, N. (2016). *Interdisciplinary mathematics education: A state of the art*. Springer International Publishing. https://doi.org/10.1007/978-3-319-42267-1_1
Zhan, Y., So, W. W. M., & Cheng, I. N. Y. (2017). Students' beliefs and experiences of interdisciplinary learning. *Asia Pacific Journal of Education*, *37*(3), 375–388. https://doi.org/10.1080/02188791.2017.1301880

CONTRASTING CASES IN GEOMETRY: OPPORTUNITIES TO EXPLORE DIFFERENT STUDENT SOLUTION STRATEGIES

Erin E. Krupa
NC State University
eekrupa@ncsu.edu

Brianna Bentley
NC State University
blbentle@ncsu.edu

Joshua P. Mannix
NC State University
jpmannix@ncsu.edu

Utilizing an innovative and theoretically-grounded approach, we extend the work of cognitive scientists and mathematics educators who have previously documented the impact of comparison on students' learning in algebra with the goal of transforming the learning that occurs in eighth-grade geometry classrooms. The purpose of this paper is to examine the types of comparisons participants made during think aloud interviews when engaging with curricular materials that have them examine multiple solution strategies. This research seeks to extend the work of using comparisons in algebra to determine if using comparisons in geometry will help improve students' mathematical understanding.

Keywords: Curriculum, Geometry and Spatial Reasoning, Middle School Education

Introduction

Dissonance is defined, in part, as "an instance of such inconsistency or disagreement" (Merriam-Webster, 2022). There is much to be learned through these inconsistencies, especially in regards to learning mathematics, for it is through this dissonance that students work to make sense of the inconsistencies and develop stronger mathematical arguments. By simply comparing different solution strategies to a mathematics problem, students can draw out inconsistencies to deepen their mathematical thinking.

Utilizing an innovative and theoretically grounded approach, we extend the work of cognitive scientists and mathematics educators who have previously documented the impact of comparison on students' learning in algebra (Star, Pollack, et al., 2015), with the goal of transforming the learning that occurs in middle grade geometry classrooms. Comparing and contrasting objects is a powerful learning tool with deep roots in cognitive science literature. Goldstone, Day, and Son (2010) stated, "research has demonstrated that the simple act of comparing two things can produce important changes in our knowledge" (p. 103). There is empirical support from cognitive scientist literature for the use of comparing contrasting examples for learning about business negotiations (Gentner et al., 2003), heat flow in science (Kurtz et al., 2001), children's learning (Loewenstein & Gentner, 2001; Namy & Gentner, 2002), and in studies of infants 4 to 6-months old (Oakes & Ribar, 2005). In mathematics education, research on comparing has proven effective in learning: estimation (Star & Rittle-Johnson, 2008, March, 2009), the concept of an altitude for a triangle (Guo & Pang, 2011), and equation solving (Rittle-Johnson & Star, 2007, 2009; Rittle-Johnson, Star, & Durkin, 2012).

Theoretical Framework

Comparisons are a powerful way to improve learning across disciplines; having students compare and contrast different strategies has produced gains in students' knowledge of algebra (Lynch & Star, 2014; Star, Newton, et al., 2015; Star, Pollack, et al., 2015). Furthermore, comparison of multiple strategies also plays a prominent role in policy documents. In particular, one of the five recommendations for improving mathematical problem solving for middle grade students noted in the released Practice Guide from the U.S. Department of Education

(Woodward et al., 2012) was to have students examine multiple problem solving strategies. Similarly, one of the Common Core State Standards for Mathematics (NGA, 2010) mathematical practice standards is for students to be able to construct viable arguments and critique the reasoning of others, specifically that, “Mathematically proficient students are also able to compare the effectiveness of two plausible arguments” (p. 10). When students share their ideas and evaluate the thinking of others in class discussions, they develop the ability to construct mathematical arguments; the practice of critiquing peers is thought to enhance mathematical understanding (Lampert, 1990; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005).

Researchers have developed contrasting case materials to address typical Algebra I content that students struggle with or that elicit student misconceptions (see Star, Pollack, et al., 2015; Star, Rittle-Johnson, & Durkin, 2016). Star and colleagues’ Algebra I curriculum is focused around worked example pairs (WEPs), where each WEP shows two fictitious students solving one or more algebra problems. The intent of each WEP is for students to directly compare and contrast, line-by-line, the methods that the two fictitious students use in solving each problem. Star and colleagues (2015) have demonstrated that teachers’ implementation of the contrasting cases materials, including whole class and small group discussions around the WEPs, can lead to increased procedural knowledge, procedural flexibility, and conceptual knowledge of Algebra I topics.

The research in this proposal seeks to extend this work to geometry content and to further enhance the effectiveness of this approach for geometry by creating digital materials with animations. Because a major goal of the project is to establish a scientific foundation for animated contrasting cases as a basis for the learning of geometry, this paper seeks to answer an initial question: *What types of comparisons between mathematical strategies do students make?*

Design of Materials

Our digital curricular materials center two fictitious students’ voices at the center of mathematics learning, and each lesson includes five unique features: a page for the first fictitious student’s solution strategy on a given geometry task, a page for the second fictitious student’s solution to a geometry task (which could be the same or different task shown on first student’s page), a page with both students’ strategies side-by-side, a discussion sheet with four questions for the students to answer, and a thought bubble page summarizing the key mathematical concepts in the problem. The discussion sheet and thought bubble page are designed to make the instructional goal of each WEP more explicit and for students to summarize their work from the WEPs (Star, Pollack, et al., 2015).

There are four units, each containing either five or six WEPs, that cover the geometry content in the 8th grade CCSSM (NGA, 2010). Each unit introduces two new characters; for this paper we will refer to the characters as Jaxon and Maxine. As we created the materials, we considered several design features: animations and colors to draw student’s attention to the geometric content in meaningful ways, characters’ methods purposefully selected to spark comparisons, geometric thinking of the WEP characters, and diversity of characters throughout the units.

Methods

After fully developing the 8th grade geometry materials, and due to shifts in classroom-based research due to COVID, we conducted 56 hour-long open-ended semi-structured clinical interviews (Piaget, 1976; Opper, 1977), in the form of think alouds, with individual participants ($n = 42$). Our goal was to elicit student thinking as participants engaged with the materials and discussion questions, not to get them to a “correct” response (Opper, 1977). Consistent with

Opper's description of the clinical method, we used the students' language, paced the interview to each student's speed, and encouraged students to elaborate on their thinking. In order to engage participants in each phase of the WEP during the interviews, we followed a detailed protocol: examine Jaxon's method, examine Maxine's method, horizontally compare Jaxon and Maxine's methods, solve the problems on the discussion page, and read the thought bubble at the end. If needed, we had questions for each phase of the protocol to probe student thinking. At the end, we asked if they had any questions for us or any final thoughts on the materials.

We transcribed each interview and began *a priori* (Saldaña, 2013) coding based on our key design features (Animations, Colors, Comparison Between Characters, Diversity of Characters, Geometric-Thinking of WEP Characters). We then added emergent (Saldaña, 2013) Level 1 codes for the students' geometric thinking and curricular form and Level 2, 3, and 4 codes as appropriate. We met to develop the codebook and then each coded several interviews individually. We then met to discuss any discrepancies and came to agreement on all codes. Once we had finalized our codebook, two coders independently coded each transcript and the third researcher resolved all disagreements. The average initial agreement between the pairs of coders was: 88.92% for Level 1 codes, 82.76 for Level 2, 81.04% for Level 3, and 88.09% for Level 4. This paper specifically analyzes the Level 1 Comparison Between Characters codes to determine what types of comparisons the participants made regarding Jaxon and Maxine's solutions strategies. Below, we report on the types of comparisons they made and give examples of each type.

Findings

Comparisons Between Characters

We observed 756 (23.27% of all Level 1 codes) instances where students were making comparisons between the WEP characters (Table 1). Most often they were discussing differences between the characters ($n = 484$), but they also noted similarities ($n = 267$) and used both WEP characters' strategies to verify a mathematical idea ($n = 5$). Looking at the Level 3 codes, regardless if students were pointing out a similarity or difference in Jaxon and Maxine's strategies, participants often referred directly to the method they were using to solve a problem.

Specifically, when pointing out differences, students most often described differences in the methods the characters used to solve a problem ($n = 380$). For example, when analyzing strategies related to translating a figure, one student stated, "Jackson is more plotting it out, while Maxine is subtracting the values to go left or down. They both had it in the same spot, which is good; I think that's the idea." This student realized Jackson is using a visual geometric method, while Maxine is using an algebraic approach, yet they arrive at the same answer. This student was attending to the visual/algebraic aspects of Jaxon and Maxine's approaches. Students noted differences in the students' methods regarding WEP specific content. For example, in a WEP designed to have students understand why the interior angle sum in a triangle is 180 degrees, one student said, "Alex like ripped his triangle apart and… what did Morgan do? And Morgan, just drew the line and just used like the parallel cut by transversal stuff to figure everything out. To figure out that it was 180 degrees." Here the student is attending to specific mathematics content in the WEP.

Students noted similarities based on the method students were utilizing in the WEP. In a WEP focused on verifying similarity using transformations, one student noted, "they both show that the triangle is similar, I guess. They both did show that the side lengths are proportional. As you can see." Another student, working in a WEP focused on reflections, stated, "They're both, they're both flipped to the other side, depending on if it's the x or y axis, they're both being flipped." Here the student noticed that Jaxon reflected his triangle over the x-axis, whereas Maxine reflected hers over the y-axis. This student was later able to make a generalization about what happens to the coordinates of a figure when it is reflected over the x-axis and y-axis. These comparisons helped students analyze benefits and drawbacks of the WEP characters' methods and make informed decisions about their preferred strategy for solving similar problems.

Table 1: Number Codes at Each Level

Level 1	*n*	Level 2	*n*	Level 3	*n*	Level 4	*n*
Comparison Between Characters	756	Similarities	267	Answer	55		
				Method	162		
				Problem	50		
		Differences	484	Answer	69		
				Method	380	Liked Better	37
						Same Result	20
						Way I Would Do It	33
						Easier	55
						Unsure	3
						Visual/Algebraic	26
						WEP Content Specific	206
				Problem	35		
		Led to Understanding	5				

Conclusions and Implications for Future Work

We have begun to document the types of comparisons students made during think aloud interviews regarding fictitious student methods to mathematics problems. This research shows a viable scientific basis for using comparisons to explore multiple solution strategies of students, as students were able to note similarities and differences in the strategies. Given critiquing reasoning is important to deepening mathematical understanding (Lampert, 1990; Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005), these findings are a step towards documenting the ways in which contrasting cases can be used in geometry. Future research will analyze if these comparisons of the fictitious students' solution strategies advance students' knowledge of geometry content, provide them with more flexible solution strategies, and equip them to better critique the reasoning of their peers.

Acknowledgments

This material is based upon work supported by the National Science Foundation agency under Grant No. DRL-1907745. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the NSF agency.

References

Gentner, D., Loewenstein, J., & Thompson, L. (2003). Learning and transfer: A general role for analogical encoding. *Journal of Educational Psychology, 95*(2), 393.

Goldstone, R. L., Day, S., & Son, J. Y. (2010). Comparison. In *Towards a theory of thinking* (pp. 103-121): Springer.

Guo, J.-p., & Pang, M. F. (2011). Learning a mathematical concept from comparing examples: The importance of variation and prior knowledge. *European Journal of Psychology of Education, 26*(4), 495-525.

Kurtz, K. J., Miao, C.-H., & Gentner, D. (2001). Learning by analogical bootstrapping. *The Journal of the Learning Sciences, 10*(4), 417-446.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63.

Loewenstein, J., & Gentner, D. (2001). Spatial mapping in preschoolers: Close comparisons facilitate far mappings. *Journal of Cognition and Development, 2*(2), 189-219.

Lynch, K., & Star, J. R. (2014). Views of struggling students on instruction incorporating multiple strategies in Algebra I: An exploratory study. *Journal for Research in Mathematics Education, 45*(1), 6-18.

Merriam-Webster. (n.d.). Dissonance. In *Merriam-Webster.com dictionary*. Retrieved February 15, 2022, from https://www.merriam-webster.com/dictionary/dissonance

Namy, L., & Gentner, D. (2002). Making a silk purse out of two sow's ears: Young children's use of comparison in category learning. *Journal of Experimental Psychology: General, 131*(1), 5.

National Governors Association Center for Best Practices, & Council of Chief State School Officers (NGA). (2010). *Common Core State Standards for mathematics.* Washington, DC: Authors.

Oakes, L. M., & Ribar, R. J. (2005). A comparison of infants' categorization in paired and successive presentation familiarization tasks. *Infancy, 7*(1), 85-98.

Opper, S. (1977). Piaget's clinical method. Journal of children's Mathematical Behavior, 5, 90-107.

Piaget, J. (1976). *The child's conception of the world* (J. Tomlinson and A. Tomlinson, Trans.). Littlefield, Adams & Co. (1926).

Reys, B. J., Dougherty, B. J., Olson, T., & Thomas, A. (2012). Common Core State Standards for Middle Grades Mathematics: Implications for Curriculum and Instruction. In C. R. Hirsch, G. Lappan, & B. J. Reys (Eds.), *Curriculum Issues in an Era of Common Core State Standards for Mathematics*. Reston, Virgina: NCTM.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561.

Rittle-Johnson, B., & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*(3), 529.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2012). Developing procedural flexibility: Are novices prepared to learn from comparing procedures? *British Journal of Educational Psychology, 82*(3), 436-455.

Saldaña, J. (2013). *The Coding Manual for Qualitative Researchers* (2nd ed.). Thousand Oaks: Sage.

Silver, E. A., Ghousseini, H., Gosen, D., Charalambous, C., & Strawhun, B. T. F. (2005). Moving from rhetoric to praxis: Issues faced by teachers in having students consider multiple solutions for problems in the mathematics classroom. *The Journal of Mathematical Behavior, 24*(3), 287-301.

Star, J. R., Newton, K., Pollack, C., Kokka, K., Rittle-Johnson, B., & Durkin, K. (2015). Student, teacher, and instructional characteristics related to students' gains in flexibility. *Contemporary Educational Psychology, 41*, 198-208.

Star, J. R., Pollack, C., Durkin, K., Rittle-Johnson, B., Lynch, K., Newton, K., & Gogolen, C. (2015). Learning from comparison in algebra. *Contemporary Educational Psychology, 40*, 41-54.

Star, J. R., & Rittle-Johnson, B. (2008, March). *The role of comparison in the development of flexible knowledge of computational estimation.* Paper presented at the Annual Meeting of the American Educational Research Association, New York.

Star, J. R., & Rittle-Johnson, B. (2009). It pays to compare: An experimental study on computational estimation. *Journal of experimental child psychology, 102*(4), 408-426.

Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., . . . Ogbuehi, P. (2012). Improving Mathematical Problem Solving in Grades 4 through 8. IES Practice Guide. NCEE 2012-4055. *What Works Clearinghouse*.

ANALYZING CURRICULUM FOR LEARNING TO PROFESSIONALLY NOTICE: A VARIATION THEORY OF LEARNING APPROACH

Alees Lee
Weber State University
aleeslee@weber.edu

John Bragelman
University of North Georgia
john.bragelman@ung.edu

Teaching teachers how to notice student mathematical thinking has received growing attention over the years. While descriptions of preservice teachers' experiences while learning to notice have been documented, there are still questions about how the curriculum can afford preservice teachers the best opportunities to develop their skill to professionally notice. In this paper, we examine a curriculum designed to teach preservice teachers how to professionally notice through the lens of Variation Theory of Learning. We posit Variation Theory of Learning provides insight into critical elements preservice teachers should be exposed when learning to professionally notice.

Keywords: Preservice Teacher Education; Content Courses; Professional Noticing; Variation Theory of Learning

Research on prospective teacher noticing has grown considerably over the past decade (Amador et al., 2021; Dindyal et al., 2021). Like practicing teachers, prospective teachers are required to understand children's mathematical thinking, interpret what their thinking implies for their current understanding, and decide where children need to go next in order to advance their learning (Jacobs et al., 2010; Leatham et al., 2015; Van Zoest & Stockero, 2012). The skill of attending to, interpreting, and responding to children's mathematical thinking is known as professional noticing (Jacobs et al., 2010). Intentionally developing the aforementioned skills in content courses for preservice teachers has shown mixed results (Superfine et al., 2015), and there are still questions regarding what experiences preservice teachers should have in content courses in order to grow their capacity to professionally notice students' mathematical thinking.

In this paper, we examine an experimental curriculum designed to start preservice elementary teachers on their journey to learning how to professionally notice. To investigate the potential learning opportunities afforded by the designed curriculum, we adopt Variation Theory of Learning (Marton, 2015). We argue Variation Theory of Learning (VTL) provides a useful lens for analyzing curriculum in order to determine what could be made possible to learn about professionally noticing.

The Context and Curriculum

Development of preservice teachers' capacity to professionally notice typically occurs within methods courses (e.g. McDuffie et al., 2014; Schack et al., 2013); however, the intricate relationship between what teachers notice and their content knowledge for teaching (Dick, 2017; Jong et al., 2021) suggests preservice teachers may benefit from intentional experiences to develop noticing skills simultaneously with content knowledge. As the integration into content courses is relatively new, designing instruction with professionally noticing in mind needs careful thought and consideration.

The noticing curriculum discussed in this paper is experimental and contains multiple content modules. Each module has four phases designed to weave the learning of content and the learning of noticing together. The focus of this paper is on the potential learning of professional noticing provided by the structure of the various modules within the four phases of module 1 as examined through the lens of VTL. We discuss the potential learning within the context of the first content module: counting strategies related to addition and subtraction as used with word problems.

Within Phase 1, the preservice teachers are prompted to solve a task (Figure 1) in as many ways as possible. The various solution strategies will then be shared and discussed as a class, and together they will hypothesize other ways students might solve the task. The goal of this phase is for preservice teachers to see and understand multiple counting strategies related to a single task their future students might use.

Professor Peabody took a sample of sixteen beans from a container. He recorded two lima beans and nine pinto beans. He forgot to record the number of navy beans before he dumped them back into the container. How many navy beans were in his sample?

Figure 1: Task from Counting Strategies Module

In Phase 2, the preservice teachers watch a video case of a teacher implementing the same task with her elementary students in Phase 2. The preservice teachers read a description of the elementary classroom context and are encouraged watch the video twice: first without a specific purpose and again to answer reflecting questions. The goal of this phase is to support the preservice teachers in noticing children's mathematical thinking, attending to the various solution strategies and making interpretations about them, and in providing evidence to support their claims. Phase 3 has preservice teachers analyze student work samples for a task with the same context but new numbers. The work samples came from the same students in the video case. The purpose of this phase is to have preservice teachers recognize there are multiple representations within which they can notice and attend to student mathematical thinking. Finally, in Phase 4, the preservice teachers review student work samples from the same students of a new task with a different context, but accessible using similar counting strategies. The intent of this phase is to extend preservice teachers' noticing of student work related to counting strategies with respect to other problem contexts.

The Variation Theory of Learning

A theory developed from empirical research in the classroom, the Variation Theory of Learning (VTL) (Marton, 2015) is uniquely suited to the investigation of learning opportunities in a postsecondary classroom for prospective teachers. VTL has two main tenants: humans discern elements of our world by attending to difference against a background of sameness; and one cannot learn unless there is something to be learned, which is called the *object of learning* (OL). VTL examines learning through the lens of variation that occurs with respect to the OL (Runesson, 2005). To discern the OL, one must see the OLs necessary *dimensions of variation* and their *features*. Dimensions of variation (DoVs) are important components to understand in order to make sense of the OL, while features are characteristics of the DoVs (Figure 2).

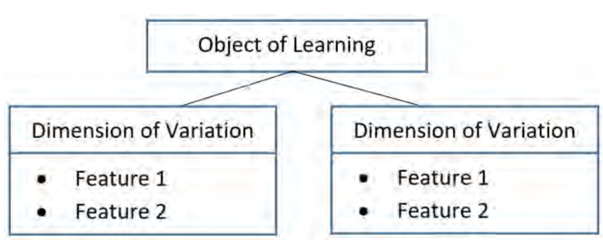


Figure 2: Structure of VTL Terminology

Marton (2015) describes four types of variation: 1) contrast, 2) generalization, 3) fusion, and 4) repetition. The type of variation depends on which dimension is in focus for the learner (Table 1). In order to discern the OL, one must first separate the necessary DoVs from each other. To separate a dimension from the others, the learner must experience a contrast followed by a generalization in relation to that DoV. The progression from contrast to generalization separates the focus DoV from the others, which is best done one dimension at a time (Pang & Marton, 2013). The learner must then bring the dimensions back together again with a fusion.

Table 1: Types of Variation

Type of Variation	Dimension in Focus	Other Dimensions
Contrast	Variant	Invariant
Generalization	Invariant	Variant
Fusion	Variant	Variant
Repetition	Invariant	Invariant

Beyond articulating the relationships between the object of learning, dimensions, features, and types of variation, Marton (2015) describes the various perspectives from which one can see these elements. An OL can be identified or determined from three different perspectives: the intended object of learning, the enacted object of learning, and the lived object of learning. The distinctions between these three elements are akin to recognizing what the teacher or curriculum designer intends to be learned (intended OL) is potentially different from what is made possible to learn during instruction (enacted OL) and from what the individual students actually learn (lived OL). In this paper, we analyze the curriculum from the perspective of the intended OL.

Seeing the Curriculum through the Lens of Variation Theory of Learning

In this section, we analyze the elements of the curriculum from the lens of VTL in order to articulate what the curriculum could afford the preservice teachers to learn about professional noticing. To that end, we establish the intended OL as: professionally noticing student thinking related to counting strategies and consider how the variation within the curriculum addresses that intended OL.

For the aforementioned intended OL, we envision two necessary dimensions (Figure 3). The first dimension relates to a teacher's understanding of counting strategies (DoV1), while the second dimension is the act of noticing student mathematical thinking (DoV2). Whether or not prospective teachers have opportunities to learn the OL depends on the careful structuring of variation which make the dimensions and OL visible to learners in the curriculum.

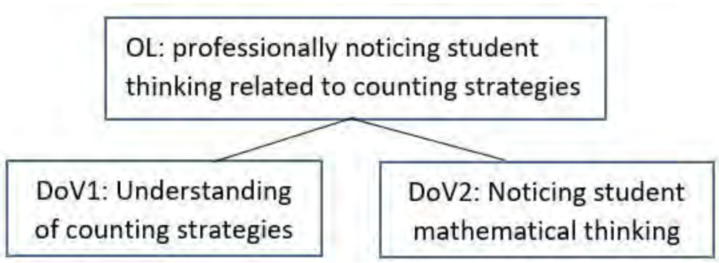


Figure 3: Learning to Notice VTL Structure

To uncover what the curriculum affords preservice teachers to learn about professional noticing, we examined the phases through the lens of VTL. Specifically, we considered the variation that could arise both within and across DoV1 and DoV2 as the preservice teachers progress through the four phases (Table 2). We hypothesize preservice teachers will have ample opportunity to discern elements surrounding DoV1. In particular, experiences within and from Phase 1 and Phase 2 surface a contrast and a generalization related to the idea that the strategies elementary students might employ to solve problems differ from how the preservice teachers themselves would solve the problem. Furthermore, the preservice teachers have opportunities to grapple with and understand the multiple counting strategies students might employ.

On the other hand, the curriculum does not explicitly provide the preservice teachers an opportunity to separate DoV2, noticing student thinking, from other dimensions through a contrast. To provide a contrast and open DoV2, one could encourage preservice teachers to see the authentic classroom instruction through various lenses such as classroom management, or how the students are grouped, or who is given authority to talk, in order to discern what it means to notice a specific element of instruction. The curriculum does afford the preservice teachers an opportunity to experience the dimension in Phase 3 through an induction, which is the same type of variation as a generalization without a contrast happening first. Finally, the curriculum provides an opportunity for preservice teachers to bring the dimensions together through a fusion in Phase 4.

Table 2: Variation across Phases

Variation Type	DoV1: Counting Strategies	DoV2: Noticing of Student Thinking
Contrast	Phase 1 – preservice teachers see different solution strategies for the same task. The solutions are from their classmates.	Does not occur
Generalization	Phase 2 - When watching the classroom video, the focus is still on the different solution strategies, however actual elementary students are now surfacing those strategies.	Phase 3 – Moving from Phase 2 to Phase 3, solution strategies and students remain invariant, but representations of student thinking (verbal/written) changes. ***Technically an induction.
Fusion	Phase 4 – A new task context is provided that allows for the solution strategies and what the preservice teachers notice to vary.	

Conclusion and Discussion

In this brief, we show how VTL as a lens to examine curriculum explicitly identifies and illuminates what the preservice teachers could have the opportunity to learn about professional noticing through engaging with the phases in the curriculum. Furthermore, the theory provides insight into elements that may be missing from the curriculum, such as providing an opportunity for students to learn what it means to notice a particular element emerging during instruction, like student thinking. In this, we offer insights into how VTL may support researchers and teacher educators in their development of preservice teachers' professional noticing of children's mathematical thinking.

References

Amador, J. M., Bragelman, J., & Superfine, A. C. (2021). Prospective teachers' noticing: A literature review of methodological approaches to support and analyze noticing. *Teaching and Teacher Education*, *99*, 103256.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. http://dx.doi.org/10.1177/0022487108324554

Dick, L. K. (2017). Investigating the relationship between profes- sional noticing and specialized content knowledge. In E. O. Schack, M. H. Fisher, & J. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 339–358). Cham: Springer.

Dindyal, J., Schack, E., Choy, B., & Sherin, M. (2021). Exploring the terrains of mathematics teacher noticing. *ZDM: Mathematics Education, 53*(1), 1-16. https://doi.org/10.1007/s11858-021-01249-y

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169-202. https://www.jstor.org/stable/20720130

Jong, C., Schack, E. O., Fisher, M. H., Thomas, J., & Dueber, D. (2021). What role does professional noticing play? Examining connections with affect and mathematical knowledge for teaching among preservice teachers. *ZDM – Mathematics Education, 53*(1), 151-164. https://doi.org/10.1007/s11858-020-01210-5

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88-124. https://doi.org/10.5951/jresematheduc.46.1.0088

Marton, F. (2015). *Necessary conditions of learning*. New York, NY: Routledge.

Pang, Ming Fai, and Ference Marton. "Interaction between the learners' initial grasp of the object of learning and the learning resource afforded." Instructional Science 41.6 (2013): 1065-1082.

Roth McDuffie, A., Foote, M. Q., Bolson, C., Turner, E. E., Aguirre, J., Bartell, T. G., . . . Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students' multiple knowledge bases. *Journal of Mathematics Teacher Education, 17*, 245-270. doi:10.1007/s10857-013-9257-0

Runesson, Ulla. "Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics." *Cambridge journal of education* 35.1 (2005): 69-87.Schack, E. O., Fisher, M. H., Thomas, J. N., Eisenhardt, S., Tassell, J., & Yoder, M. (2013). Prospective elementary school teachers' professional noticing of children's early numeracy. *Journal of Mathematics Teacher Education, 16*, 379-397.

Superfine, A. C., Li, W., Bragelman, J., & Fisher, A. (2015). Examining the use of video to support preservice elementary teachers' noticing of children's thinking. *Journal of Technology and Teacher Education*, *23*(2), 137-157.

Van Zoest, L. R. & Stockero, S. L. (2012). Capitalizing on productive norms to support teacher learning. *Mathematics Teacher Educator*, *1*(1), 41-52. http://dx.doi.org/10.5951/mathteaceduc.1.1.0041

A STORY OF TRIANGLE CONGRUENCY IN A CHINESE TEXTBOOK

Jane-Jane Lo
Western Michigan University
jane-jane.lo@wmich.edu

Jinqing Liu
University of New Hampshire
Jinqing.Liu@unh.edu

Lili Zhou
California State University, Los Angeles
zhoulili756@gmail.com

In this paper, we report findings from an analysis of four triangle congruency lessons from a grade 8 mathematics textbook from China, using the Mathematics Curriculum as A Story framework developed by Dietiker (2015). We found that the Chinese textbook organized the four lessons under a four-level-plot structure to answer only one overarching explorative question: "Can we guarantee triangle congruence using a sub-set of conditions?" which connects triangle congruence criteria well for easier sense making. We also found that the textbook provides students with varied reasoning and proof opportunities by embodying rich actions and real-life context in the tasks. The findings of this study demonstrate the illuminating power of the Curriculum as a Story framework to fill the gap of current textbook analysis research.

Keywords: Curriculum, Geometry and Spatial Reasoning, Middle School Education

Objectives

Mathematics textbooks have long been regarded as a bridge between the intended curriculum, such as national curriculum standards, and the implemented curriculum, i.e., the actual teaching in the classroom (Valverde et al., 2002). They determine, to a high degree, what teachers teach and what students learn (Stein et al., 2007). Fan et al. (2013) proposed a framework for classifying the literature on mathematics textbook research studies that includes four main categories: role of textbooks, textbook analysis and comparison, textbook use, and other areas. The primary goal of all these analyses is to provide a proxy measure for the potential learning opportunities afforded by each textbook.

Using approaches described above, research on mathematics textbooks have made significant contributions to identify and report features that can be used as the basis for comparisons. However, these findings collectively, still fail to provide a sense of the changes and flows of the mathematical content throughout a textbook nor are they able to account for the aesthetic of sequencing and presenting ideas in one way verse another way. As an old Chinese fable says, it would not be possible for a group of blind people to get an idea of what an elephant looks like by just sharing their individual experience obtained from touching parts of the real elephant. Recognizing this limitation of the current research on mathematics textbook analysis, Dietiker (2015) proposed a conceptualization of mathematics textbook as a story that provides a holistic view of curriculum to examine the connections as well as sequences between mathematical ideas. Thus, applying Dietiker's (2015) narrative framework, our current investigation is guided by a general question, "how does a textbook tell a story of triangle congruency?" Specifically, we focus on understanding how an eighth-grade mathematics textbook published by People's Education Press (PEP, 2013) in China tell the story of triangle congruency.

We chose the concept of congruency as the content focus for this study for two main reasons. First, it is an important and common topic in geometry curriculum worldwide (Jones & Fujita, 2013). The idea that a partial set of all angle and side length measures can guarantee congruency between two given polygons is a unique property of triangles, which provides a rich setting for developing students' geometric intuition and ability to perform deductive proof (Wang et al., 2018). Second, previous studies found triangle congruence a challenging topic for students worldwide (e.g., Wang et al., 2018). A recent review comparing the introduction of triangle of congruency in a Chinese and a U.S textbook found that while both textbooks introduced the same set of congruent triangle conditions to their students, the sequence of how these criteria were introduced were quite different (Lo, Zhou & Liu, 2021). The findings of the present study on Chinese lessons will lay the beginning of the groundwork for the future investigation on the effect of different approaches mathematics textbooks use to present triangle congruence theorems on students' learning.

Theoretical Perspectives

According to Dietiker (2015), a mathematics story contains four main elements: mathematical characters, mathematical actions, mathematical setting, and mathematical plot. Mathematical characters are "figures" that were brought into existence by descriptive naming that could be in a variety of forms such as words, graphs, tables, or symbolic forms. In a story, mathematical characters can take on multiple forms via the process of mathematical character development which could be the main focus of the story. In addition, a story can also be about the relationships among multiple mathematical characters. Mathematical actions are manipulations taken on a mathematical character that results in a mathematical change. Other than moving a mathematical story forward, mathematical actions can create a new mathematical character for a story. Dietiker (2015) gave an example of a story of counting to find the quantity of the number of remaining objects after removing one at a time. This sequence of action will result in a surprising result of naming a quantity when nothing is left to be counted.

Dietiker (2015) defines mathematical setting as the space in which the mathematical character emerged and developed via mathematical actions. One way to conceptualize a mathematical setting is the mathematical representation in which mathematical characters move about which can affect both the richness and the interpretations of the story. For example, a "function" character that lives in a mathematical setting of multiple representations afford richer learning opportunities for students to know this character.

A mathematical plot is the "potential temporal dynamics of the story that encourages (or discourages) a reader to continue to advance through the mathematical story" (Dietiker, 2015, p. 299). Dietiker and Richman (2017) identify two main features of a mathematical plot: the sequence of the events unfolded within the story, and both the known and unknown felt by the reader as they were propelled to read on by the moment-to-moment tension. Dietiker (2015) suggested that a careful analysis of a mathematics curriculum via mathematical characters, actions, settings, and plots would help teachers and curriculum developers identify the potential challenging spots for the students which could then guide the future curriculum improvement.

Methodology

The design of the study was informed by the theoretical perspectives and framing outline above and the analysis was conducted by the constant comparative analysis (Stake, 2000).

Setting:

The Chinese education system is highly centralized. There is only one set of national mathematics curriculum standards: Compulsory Education Mathematics Curriculum Standards (CMCS) (Ministry of Education, 2012) which have to be followed strictly by all textbook series. In CMCS, the content of triangle congruence is considered as a part of a larger topic, Triangles. Under the topic of Triangle, CMCS has six expectations: 1) understand special lines in triangles, 2) explore and prove triangle sum theorem and its deduction, 3) master triangle congruence, 4) explore and prove the properties of angle bisector line and perpendicular bisector, 5) explore isosceles triangle and right triangle, and 6) explore the Pythagorean theorem and its converse theorem. In this study, the lessons we focused on belong to third expectation: master triangle congruence.

Data Set

An eighth-grade math textbook published by People's Education Press (PEP, 2013) was selected for this analysis. Among Chinese textbook publishers, People's Education Press has the longest history in developing K-12 curriculum in all subjects, and their mathematics textbooks were most widely analyzed (e.g., Cai & Jiang, 2017; Fan & Zhu, 2007). The primary data set for this study is the section 12.2 of the eight-grade mathematics textbook. PEP Math devoted four lessons to the five congruent triangle criteria: side-side-side (SSS; Lesson 1), side-angle-side (SAS; Lesson 2), angle-side-angle and angle-angle-side (ASA & AAS; Lesson 3), and hypotenuse-Leg (HL; Lesson 4). There was also additional investigation of the criteria that did not work such as side-side-angle (SSA) in Lesson 2 and in angle-angle-angle (AAA; Lesson 3).

Data Analysis

The analyses of the congruent triangle units were carried out jointly by the three authors based on the narrative framework developed by Dietiker (2012, 2015). Each of the authors first read the textbook unit independently following the reading practices discussed in Dietiker (2012) as discussed below. We each wrote a story summary based on our notes taken during this first-read with an eye toward prompts, questions and anticipations, as well as character development. We shared and discussed our interpretations multiple times to meet a consensus. Constant comparisons (Stake, 2000) were used throughout the process to look for emerging themes and evidence that either support or inconsistent with our ongoing narrative. Then we met on-line to develop the first joint story summary by combining all the distinct details as well as resolving any differences in our own reading of the story. The same process was repeated to fine-tuning each story by focusing on the other constructs of the framework, like actions and settings.

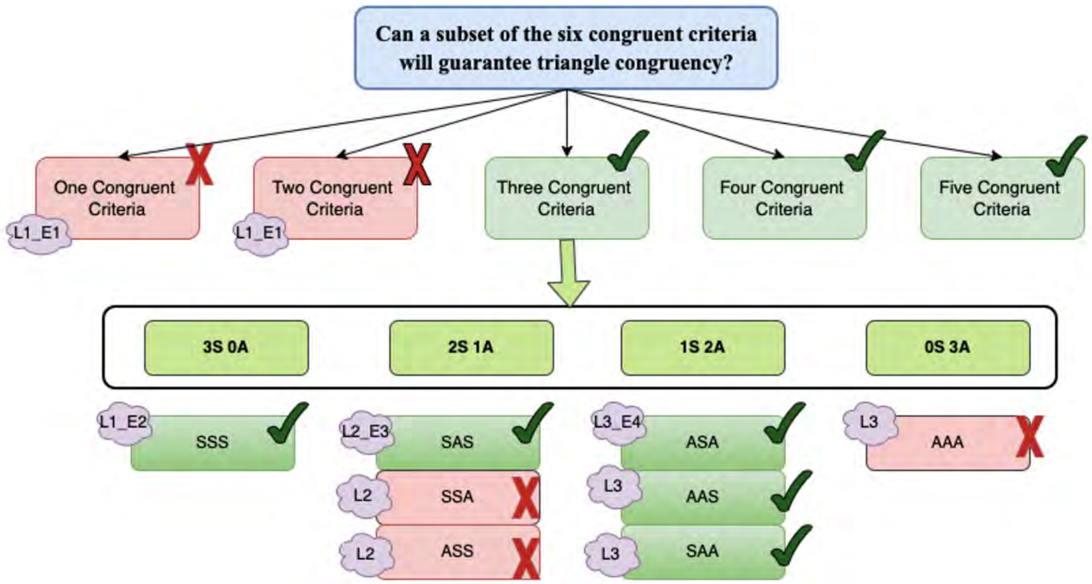
The mathematical story of triangle congruency we will be sharing with you is influenced by our own unique prior experiences with textbooks, mathematics, and education systems and cultures. All of us received K-16 education in Chinese education systems and came to the United States for our graduate degrees in mathematics education. Crossing cultural learning and teaching provided us a comparative perspective on mathematics curriculum which motivates us as mathematics teacher educators to conduct curriculum analysis and provide suggestions for developing effective curriculum. when we developed our knowledge and skills of mathematics curriculum analysis.

Findings

Because of the space limitation, we will limit our findings on the plots and actions of the stories. We will first describe the four-level plots of the congruent triangle story. Then, we will take a closer look at the plots around the introduction and development of the first explored condition: SSS. Finally, we identify several different types of actions employed by the students when investigating various mathematical characters: triangles with various types of properties.

Plots

PEP Math introduces the congruent triangles as a special case of congruent figures that have the same sizes and shapes. Therefore, when two triangles are congruent ($\triangle ABC \cong \triangle A'B'C'$), their three corresponding sides and angles are congruent. That is, the following six conditions exist: $AB=A'B'$, $BC=B'C'$, $CA=C'A'$; $\angle A=\angle A'$, $\angle B=\angle B'$, $\angle C=\angle C'$. Our read of the story uncovered four levels of plots as seen from Figure 1.



Note: L1_E1 means Lesson 1 Exploration 1; 2S1A means two pairs of congruent sides and one pair of congruent angles; A tick off means could guarantee triangle congruency; Cross means could not guarantee triangle congruency.

Figure 1: Four level mathematical plots of the congruent triangle story

First-level plot. The students were invited to explore the first-level, the overarching, plot via the following question: "Is it possible to identify various subsets of these six conditions that will also make two triangles congruent?"

Second level plots. In order to answer the questions posted in the first-level plot, students need to identify the potential subsets of these six conditions. Using a listing strategy, students may know the possible subsets including only one set, two sets, three sets, four sets, and five sets of congruent sides or angle conditions may guarantee triangle congruency. Therefore, to support the development of the first-level plot, there are five potential second level plots: will one, two, three, four, or five sets of congruent sides or angle conditions guarantee triangle congruency. The textbook first invited the students to ponder the following question "Will one or two sets of congruent sides or angle conditions guarantee triangle congruency?" by a series of drawings: "Start with an arbitrary triangle △ABC. Then draw a triangle △A'B'C' with just 1 criteria (one pair of congruent sides or one pair of congruent angles) or 2 criteria (two pairs of congruent angles, two pairs of congruent sides or one pair of congruent angles and one pair of congruent sides) of △ABC. Is △A'B'C' guaranteed to △ABC? (p. 35). The students were expected to provide counterexamples to reach the conclusion "no".

Third-level plots. The story then asked students to explore the question: "Will three sets of congruent conditions guarantee triangle congruency?" Answering this question led to an exploration of the third-level of plots. There are four kinds of combinations of three subsets of these six congruent conditions could be three pairs of congruent sides (3S0A), two pairs of congruent sides and one pair of congruent angles (2S1A), two pairs of congruent angles and one pair of congruent sides(1S2A), three pairs of congruent angles (0S3A). For each kind of combination, different order of congruent conditions leads to different arrangements, which leads to the fourth-level of plots.

Fourth-level plots. As Figure 1 shows, we identified eight different arrangements, including SSS; SSA, SAS, ASS; SAA, ASA, AAS; and AAA, all these can be considered different characters of the story. However, because a triangle is a closed figure with three sides and three angles, SSA can be understood as ASS, SAA can be understood as AAS. Therefore, we shaded ASS and AAS in grey, which are not explored by the textbook. After exploring all the six general arrangements, the textbook explored hypodense-leg (HL) which was a special case of SSA that was rejected earlier. Specifically, PEP confirmed five triangle congruency criteria in the following sequence: SSS, SAS, ASA, AAS, and HL It also rejected two criteria: SSA along with SAS, and AAA right after AAS.

The story ends with an exploration of a special case of right triangles (HL), a new character. The text invited students with the following question "With a pair of right triangles that already shared a pair of congruent right angles, what other conditions would be needed to establish their congruency?" The story then pointed to the students that based on the triangle congruent theorems already learned, two right triangles will be congruent if either 1) the pairs of two sides of the right angles were congruent or 2) there is a pair of congruent sides and a pair of congruent acute angles. Notice that the first condition will lead to SAS with the right angles be the congruent angles, and the second condition will lead either to ASA or AAS. However, the textbook did not provide these reasoning directly. Those two conditions were left for the students to explore. After those two conditions were confirmed, there was only one case left: What if the two right triangles shared a pair of congruent hypotheses and another pair of side. Note that this is the condition of SSA which was rejected as a valid congruency condition in the earlier lesson, so the students were likely to be intrigued.

A Closer Look at the Plots of The First Lesson on SSS

The story asks students to conduct the following exploration: Draw an arbitrary triangle. Then draw another triangle such that their three side measures are equal. Cut the triangle out and place it on top of the original triangle. Are they congruent? While the narrative initially leaves it open to the various approach students may use to answer this question, the story soon provides detailed step-by-step straight-edge & compass constructions to create a new triangle that has three sides congruent to those of the original triangle. Then students are asked if their steps are as same as the narrative and what conclusion they can reach from this exploration. The story then affirms the SSS condition. Further, the students are reminded of a previous experiment of creating a triangle with three wooden sticks. Once the triangle is created, both the shape and size of the triangle is determined.

The story then provides two worked-out examples to help students apply the newly learned SSS condition. The first example can be seen in Figure 2.

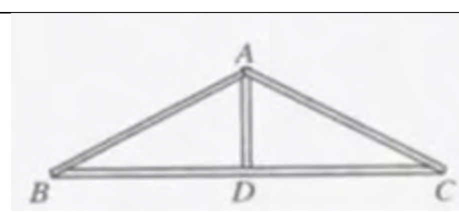
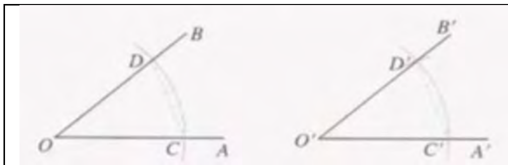
In $\Delta ABC, AB = AC,$ $Segment\ AD\ connects\ A\ and\ D,$ $where\ D\ is\ the\ midpoint\ of\ BC.$ $Prove\ that\ \Delta ABD\ \cong \Delta ACD$	

Figure 2. The first worked-out example after SSS is introduced (p. 36, PEP, 2013)

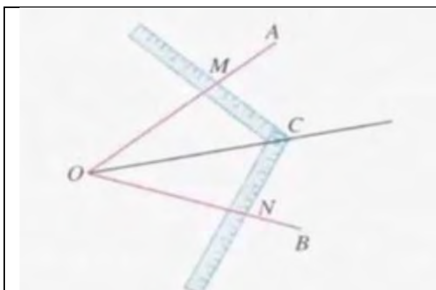
Before showing students the complete proof, the story first provides a pre-analysis, asking students to identify SSS conditions in the given problem. One pair of congruent sides, AB=AC, was already given. Segment AD, as a shared side, provides the second pair of congruent sides, AD=AD. The condition that D is the midpoint of side BC provides the third pair of congruent sides, BD=CD. The second worked-out example shows students how to construct a congruent angle with the straightedge & compass -using the SSS condition as follows (Figure 3).



1) As in the figure, use O as the center of the circle, arbitrary length as the radius, draw an arc that intersects OA and OB at points C and D.
2) Draw a ray $O'A'$. Use O' as the center, the length of OC as the radius, draw an arc intersects $O'A'$ at point C'.
3) Use C' as the center, the length CD as the radius, intersects the arc drawn in Step 2 at point D.
4) From D', draw a ray $O'B'$. Then$\angle A'O'B' = \angle AOB$.

Figure 3. The second worked-out example after SSS is introduced (p. 37, PEP, 2013)

The story of the SSS ends with two exercises that could be completed by students. The first exercise is another deductive proof similar to the worked-out example in Figure 2. The second exercise is a construction of an angle bisector of a given angle in the context of a construction trade as seen in Figure 4.



Construction workers often use a carpenter square to bisect an arbitrary angle. They used the following method as shown in the diagram: $\angle AOB$ is an arbitrary angle. On the segments OA and OB, pick OM=ON. Move the carpenter square so that the same distance markers on the two legs of the carpenter square will coincide with M and N. Then the ray formed by the vertex of the carpenter angle, C, and point O will be the angle bisector of $\angle AOB$. Why does this method work?

Figure 4. The second exercise at the end of SSS story (p. 37, PEP, 2013)

Actions

The story of congruent triangles in PEP Math engaged students in several different types of actions when exploring conjectures, establishing congruent criteria, constructing deductive proofs or to solving problems embedded in the real-life context that require the use of triangle congruent criteria they have just learned. Table 1 includes the frequency of each action type needed for the 20 tasks analyzed in this units. All tasks require multiple actions to complete.

Table 1. Frequency of each action within the story

Action Type	Drawing and direct comparison	Straight-edge & compass construction	Translating	Applying	Corres-ponding	Joining & separating
Frequency	9	6	28	18	11	2

The first action is “drawing and direct comparing”. When a conjecture was first posted, the students were advised to draw a triangle with the given condition to see if different ones could be created. If so, the original condition was rejected. If no different triangle could be drawn, the story then guided students to the second action, “the straight-edge & compass construction.” This action was used to prove SSS, SAS, ASA and HL. The third action is “translating”, for example, when students read the statement “…D which is the midpoint of AB” in Figure 2, they need to translate that statement into the existence of two congruent line segments, AD and BD. And when they see the diagram in Figure 2, they need to translate the image of a “shared side” of two triangles into the existence of a congruent side relationship in these two triangles.

The fourth action is “applying”. This includes both applying various congruent conditions to perform conductive deductive proofs as well as solve word problems embedded in real-life context. It also includes applying previous learned properties such as vertical angles are congruent and sum of the three interior angle measures of any triangle is 180° were used to identify pairs of congruent angles. The vertical angle property was first used in a worked-out example in Lesson One and then in an exercise in Lesson Three. And the property that the sum of the three interior angle measures of any triangle is 180° was used in a worked-out example in Lesson Three when applying ASA condition to prove AAS.

The fifth action is “corresponding”, students need to mentally manipulating the figures in the problem in order to establish the corresponding sides and angles correctly. The sixth action is “joining or separating”, for example, a pair of congruent sides of two triangles could be constructed from two congruent sub-segments and a shared segment via either the addition or subtraction property. Example in Figure 5 requires both the fifth and the sixth actions. The problem statement already contains a pair of congruent sides, AB=DC, and one pair of congruent angles, $\angle B = \angle C$. It is likely that students would think of using the SAS condition. They needed to mentally orient those two triangles given in the figure based on the locations of the corresponding angles and sides to identify the pair of two triangles, ΔABF *and* ΔDCE, that are likely to be congruent. To establish the congruency between these two triangles, they would need an additional pair of congruent sides: BF and CE. Here is where they need to use the constant addition property BE=CF on a shared line segment EF so that BE+EF=CF+FE. Thus, BF = CE.

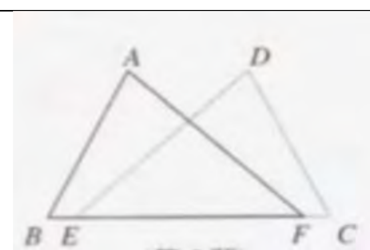
The points E and F are on the segment BC, BE=CF, AB=DC, ∠B=∠C. Prove ∠A=∠D	

Figure 5. An example of an exercise requires two different actions p. 39, PEP, 2013)

Conclusion and Implications

In this paper, we tell a story of how a Chinese textbook, PEP Math, helps their students to develop their concepts of triangle congruency based on the “Mathematics curriculum as a mathematical story” framework outlined by Dietiker (2012, 2015). The main plot started with SSS, a condition with three congruent sides, and proceeded systematically through the conditions with two sides and an angle, one sides and two angles, and three angles. Both the valid and invalid conditions were investigated under a specific case. The entire story ended with an investigation on right triangles which is a special case as one congruent condition was guaranteed. A variety of mathematical actions were used both in real life context and deductive proofs to provide students with rich opportunities to make conjectures, conduct systematic investigation, and apply what they have learned in new explorations.

These findings have two significances. First, it shows that using the curriculum as a story framework, especially the plot component, brought new affordance for examining curriculum coherence in a structural perspective that went beyond tasks analysis. Second, it shows the possibility of organizing the exploration of different triangle congruence criteria as an inquiry that could be systematically examined. All the lessons we analyzed are designed and organized to answer only one overarching explorative question: can we guarantee triangle congruence using a sub-set of conditions? Doing so, the criteria are well connected which is easier to make sense of.

This story is told from the perspectives of three mathematics education researchers who grew up with Chinese education systems. We acknowledge the influence of this background. The mathematics story framework provided us a holistic lens to investigate the topic of congruency. Even though we are familiar with the content, the story framework not only allow us to view mathematics content through a literary lens but also help us to capture nuance information that were hidden from us before this investigation. For example, PEP follows the similar lesson structure and provide multiple opportunities for students to use the same sets of mathematical actions while investigating conjectures and solving problems that are embedded in a variety of different settings. Using straightedge & compass construction as the only action to establish the congruent condition makes it necessary to start the story with SSS condition. These approaches reflect a distinct philosophy of the curriculum design including how the content is organized to follow the logical sequence and how the topic connects with other previous mathematical ideas and prepare for the future learning. They also open a variety of research questions that worth investigation. For example, how might our read of the congruent triangle story similar or different from the reads by teachers and students? Do stories on other important mathematics topics such as quadratic functions told by PEP math share the same design features such as arranging content based on logical sequence and high intensity of repeatedly utilizing the same mathematical actions? Similar investigation on the stories told by other mathematics textbooks may uncovered more curriculum design features that can be used to create appealing and have the high potential to help students. particularly those who are on the margins of these communities, to reach the learning goals as called by the 2022 PMENA Conference Theme.

Future studies could use these kinds of rich textbook analyses to develop assessment items that are more sensitive to the variations in the story told by different mathematics textbook that might contribute to the difference in student performances. For example, Common Core State Standards defines congruent figures via rigid transformations, which is different from the straightedge & compass construction typically used by the Chinese textbooks. Fan et al. (2017) found no significant difference in students' ability to solve general proof questions from their quasi-experimental study with two groups of eighth-grade Chinese students. The control group received regular instruction based on straight-edge & compass constructions, while the transformation instruction was integrated into their regular lessons on writing proof in the experimental group. However, the experimental group performed much better than the control group on challenging problems involving constructing auxiliary lines. More stories of the introduction and development of other important mathematics topics would make it possible to deepen the investigation on the link between textbook and student learning, suggested by Fan et al. (2013) as a critical research area that needs more work.

References

Cai, J., & Jiang, C. (2017). An analysis of problem-posing tasks in Chinese and US Elementary Mathematics textbooks. *International Journal of Science and Mathematics Education*, *15(8)*, 1521-1540.

Dietiker, L. C. (2012). The Mathematics Textbook as a Story: A Novel Approach to the Interrogation of Mathematics Curriculum (Ph.D. Dissertation). Michigan State University, East Lansing MI. Retrieved from http://search.proquest.com.ezproxy.bu.edu/pqdtft/docview/1282404212/abstract/1 4137FC43A333250B9/6?accountid=9676

Dietiker, L., (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics*, *90*(3), 285-302.

Dietiker, L., & Richman, A. (2017). When is an exploration exploratory? A comparative analysis of geometry lessons. In G. Schubring, L. Fan, & V. Geraldo (Eds.), Proceedings of the Second International Conference on Mathematics Textbook Research and Development (pp. 191–198). Universidade Federal do Rio de Janeiro: Instituto de Matemática.

Fan, L., Qi, C., Liu, X., Wang, Y., & Lin, M. (2017). Does a transformation approach improve students' ability in constructing auxiliary lines for solving geometric problems? An intervention-based study with two Chinese classrooms. *Educational Studies in Mathematics, 96*, 229–248.

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. *Educational studies in Mathematics, 66*(1), 61-75.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: Development status and directions. *ZDM – The International Journal on Mathematics Education, 45(5)*, 633–646.

Jones, K., & Fujita, T. (2013). Characterising triangle congruency in lower secondary school: The case of Japan. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (CERME8) (pp. 655–664). Ankara.

Lo, J., Zhou, L, & Liu, J. (2021). The introduction and development of triangle congruency in Chinese and US textbooks. In Olanoff, D., Johnson, K., Spizer, S. Proceedings of the 43th annual conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 73-77). Philadelphia, PA.

Ministry of Education Basic Education Curriculum Committee. (2012).义务教育数学课程标准解读 [*Compulsory Education Mathematics Curriculum Standard Interpretation.* 北京师范大学出版社 [Beijing Normal University Press].

Stake, R. E. (2000). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), Handbook of qualitative research (pp. 435–454). Thousand Oaks .

People's Education Press (PEP). (2013). 数学八年级上 [Mathematics 8th grade, the first volume]. Beijing, China: Author.

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319–369). Information Age Publishing.

Valverde, G. A., Bianchi, L. J., Wolfe, R. G., Schmidt, W. H., & Houang, R. T. (2002). According to the book: Using TIMSS to investigate the translation of policy into practice through the world of textbooks. Dordrecht.

Wang, Z., Wang, Z., & An, S. (2018). Error analysis of 8th graders' reasoning and proof of congruent triangles in China. *Journal of Mathematics Education*, *11*(2), 85–120.

USING ABSTRACTION AS A LENS TO ANALYZE INSTRUCTIONAL MATERIALS

Kevin C. Moore
University of Georgia
kvcmoore@uga.edu

Amy Ellis
University of Georgia
amyellis@uga.edu

Anne Wawsa
University of Georgia
anne.wawsa@uga.edu

Mike Hamilton
University of Georgia
mwhamilton@uga.edu

Halil I. Tasova
CSU-San Bernardino
halil.tasova@csusb.edu

Aytuğ Özaltun Çelik
Pamukkale University
aozaltun@pau.edu.tr

Erin Wood
University of Georgia
erin.wood@uga.edu

Over the past few decades, researchers have adopted forms of abstraction introduced by Piaget to build explanatory models of student and teacher knowledge. Although Piaget's forms of abstraction have proved productive for developing models of knowledge, their broader applicability to mathematics education remains an open question. In this brief report, we extend these forms of abstraction in order to analyze hypothetical outcomes of teachers' enactment of instructional materials.

Keywords: Cognition, Learning Theory, Curriculum

Piaget's (1970, 2001) genetic epistemology has played a critical role in mathematics education via researchers adopting his theory to develop models of students' mathematics, models of teachers' mathematics, and models of student-teacher interactions. Researchers carrying out this work have provided important insights into those meanings that prove productive for students' mathematical development, as well as those meanings that constrain students' mathematical development (Steffe & Olive, 2010; Thompson, 2013). Furthermore, these researchers have provided useful ways to characterize teaching in terms of teacher knowledge necessary to build upon students' ways of thinking (Liang, 2021; Tallman, 2015).

An important construct spanning these contributions is that of abstraction. Stated generally, abstraction is the process by which an individual develops stable, generalized knowledge structures. To Piaget, abstraction provided a vehicle to developing precise accounts of knowledge development while also articulating generalized differentiated characteristics of knowledge structures. Piaget proposed several forms of abstraction including empirical, pseudo-empirical, reflecting, and reflected abstraction (Montangero & Maurice-Naville, 1997; Piaget, 2001). Mathematics educators have adopted these forms to provide differentiated accounts of student and teacher knowledge in numerous contexts (Ellis et al., in preparation; Tallman & O'Bryan, in preparation; Thompson, 1994).

Given the usefulness of Piaget's forms of abstraction for developing accounts of student and teacher knowledge, it is plausible that the forms of abstraction are productive for analyzing other aspects contributing to the teaching and learning of mathematics. In this brief report, we extend Piaget's forms of abstraction in order to analyze instructional materials. Specifically, we analyze two secondary teachers' instructional materials for teaching quadratic growth in order to develop hypotheses of the knowledge students may abstract from engaging in those materials. Because this is a brief report, we close with potential implications of this work and future directions building on this preliminary analysis.

Background

Ellis et al. (in preparation) and Tallman and O'Bryan (in preparation) synthesized Piaget's forms of abstraction and described how mathematics education researchers have adapted these forms of abstraction to be viable in their areas of research. Empirical abstractions primarily concern observables and foreground sensory-motor experience, and reflected abstractions rest on a subject's consciousness of their ways of operating. These two forms of abstraction are critical aspects of knowledge development, but they are less relevant to the analysis of secondary mathematics instructional materials when compared to the two forms of reflective abstraction that are pseudo-empirical abstraction and reflecting abstraction.

Speaking on pseudo-empirical abstraction, Piaget (1977) explained, "When the object has been modified by the subject's actions and enriched by the properties drawn from their coordinations…the abstraction bearing upon these properties is called 'pseudo- empirical' because…the facts it reveals concern, in reality, the products of the coordination of the subject's actions…" (p. 303). To Piaget, a critical aspect of pseudo-empirical abstraction is that such an abstraction requires the presence of perceptual material or observables and foregrounds actions on that available material. Drawing on the work of Moore (2014), Ellis et al. (in preparation) argued for extending Piaget's construct of pseudo-empirical abstraction so that "perceptual material" or "observables" includes the products of activity, even if these products of activity are purely cognitive. As they illustrated, such an extension of pseudo-empirical abstraction is productive for developing viable models of students' mathematics at numerous levels.

Piaget's distinction between pseudo-empirical abstraction and reflecting abstraction rested on the extent perceptual material or observables are required. Ellis et al. (in preparation) noted that the broader interpretation of pseudo-empirical abstraction provided above requires a more restrictive framing of reflecting abstraction. A primary difference between these two forms of abstraction is that while the source material for pseudo-empirical abstractions is perceptual material or the result of actions, the source material for reflecting abstractions is the coordination of a subject's actions themselves. Reflecting abstractions thus involve differentiating an action from the effect of an action so that the actions themselves can be projected to a level of representation and taken as objects of thought (Ellis et al., in preparation; Tallman & O'Bryan, in preparation; Thompson, 1994). As we illustrate with our task analysis, these differences in the source material for a subject's abstractions have important implications for their learning.

Project Setting and Methods

The current work is situated in a multi-year project investigating students' generalizing including the ways in which teachers support generalizing in their teaching (Ellis et al., in press; Ellis et al., 2017). Our approach to generalization is cognitive, drawing on an actor-oriented perspective as detailed by Ellis et al. (in press). The project's guiding research questions are: What are the opportunities for generalizing in classroom settings? Specifically, what types of instructional moves, student engagement, and enacted tasks support classroom generalizing? The current paper addresses these questions by investigating the abstractions potentially supported during the implementation of instructional materials.

The project involves two high school teachers and two middle school teachers. We concentrate this paper on the two high school teachers' instructional materials in order to restrict our focus to one content area. We analyzed the instructional materials using conceptual analysis (Thompson, 2008) with a guiding framework of the forms of abstraction identified above. At its most general level, conceptual analysis involves answering the question, "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" (von Glasersfeld, 1995, p. 78). With respect to analyzing curricular materials, conceptual analysis involves developing hypothetical accounts of *realized curriculum* (Kilpatrick, 2011) or *conveyed meanings (Tallman & Frank, 2020)*. This is accomplished via generating and interpreting “typical” solutions to the instructional materials using the lens of abstraction in combination with ways of reasoning held by secondary mathematics students as suggested by research (Ellis, 2011; Ellis & Grinstead, 2008; Fonger et al., 2020; Moore et al., 2019).

Tasks and Task Analysis

The two secondary teachers’ instructional activity focused on quadratic growth. The instructional activity explored a sequence of discretely growing shapes (see one example in Figure 1) with the intention that students identify patterns in quantities’ values including their first- and second-differences. The primary goal and generalization of the activity was to identify that for successive equal increases in Q_A of a situation (e.g., sail size), Q_B (e.g., sail area) increases by constantly increasing amounts, and that such a covariational relationship is modeled by a quadratic relationship. We discuss hypothetical pseudo-empirical and reflecting abstractions against the backdrop of the aforementioned goal. Underscoring that the forms of abstraction are cognitive constructs, we discuss each form of abstraction using a typical solution that involves a student generating a table of values, first-differences, second-differences, and a formula.

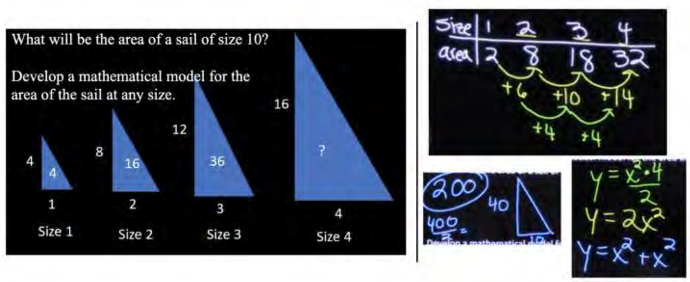


Figure 1: Example Activity (left) and a “Typical” Student Solution (right)

Pseudo-Empirical Abstraction

After working a series of activities like that presented in Figure 1, a student might observe that each time they obtain constant second-differences in a quantity, a quadratic formula models the situation. Recall that pseudo-empirical abstractions are those abstractions that foreground “perceptual material” or “observables” including the products of activity. In the case of the example activity (Figure 1), the products of activity include a table of values and a quadratic formula. Thus, the observation of the student would be a pseudo-empirical abstraction if their association is strictly based on noticing that constant second-differences were accompanied by a quadratic formula. The abstraction consists of an indexical association between constant-second differences and a quadratic formula with no logico-mathematical operations forming the basis for that association. The actions that produced the table of values and formula are inconsequential to the abstraction except in that they yielded an outcome or product to act as source material for the student’s abstraction. We contrast this with a reflecting abstraction in the next section.

Reflecting Abstraction

A limitation of pseudo-empirical abstractions stems from the abstraction foregrounding the products of actions rather than the actions themselves. For instance, and based on our experiences with students, the abstraction described in the previous section often results in the student associating a quadratic formula with constant second-differences regardless of how the other quantity’s values are ordered in a table (e.g., non-constant first-differences in Q_A that produce constant second-differences in Q_B). In the case of the example activity and solution in Figure 1, a reflecting abstraction that foregrounds the coordination of actions and their results would involve a student reflecting upon both the quantitative referents of their tabular activity, as well as how their relationship necessitates a quadratic formula.

With respect to the tabular activity, this would involve the student conceiving first-differences as the amount by which a quantity increases (or decreases) and second-differences as the amount by which a quantity's increase increases (or decreases) as shown in Figure 2. Furthermore, because a reflecting abstraction foregrounds the coordination of actions as opposed to the products of actions, the student's abstraction would include awareness that the constant "+1" increases in size are intrinsic to the constantly increasing increase in area. With respect to a quadratic formula model, a reflecting abstraction involves understanding how the aforementioned quantitative relationship necessitates a second-degree polynomial. Although the connection between the two is not trivial, researchers (Ellis, 2011; Ellis & Grinstead, 2008; Fonger et al., 2020) have illustrated its feasibility for students including those in middle grades.

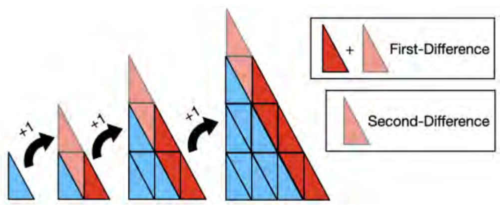


Figure 2: Conceiving first- and second-differences quantitatively

Discussion and Future Work

Although we are not aware of studies that have used the aforementioned forms of abstraction to develop hypothetical accounts of student activity in the context of teachers' instructional materials, mathematics education researchers and teachers have been sensitive to the role of abstraction in instructional design. For example, it is impossible to read the collective works of Steffe or Thompson and not sense the forms of abstraction directly informing their work even when not explicitly mentioned. As another example, Oehrtman (2008) provided a more general description of how Piaget's notion of abstraction can inform a layered sequence of activities so that students have the opportunity to reflect upon and identify common structures in their actions across a variety of contexts. In each of these cases, researchers leveraged abstraction in the context of their own research-based work and design. We find it important to include a complementary focus on teachers' extant instructional materials, as those materials play a significant role in students' educational experiences.

We illustrated that an abstraction framing provides a way to analyze instructional materials and produce differentiated accounts of knowledge development. Moving forward, we envision several productive avenues to continue investigating the viability of this framing. First, the current report is limited to one content area. Future work should look to extend the framing to other content areas including those not within secondary mathematics. Second, our analysis of the instructional materials consists of *hypotheses*. A more holistic account should include a focus on students' realized abstractions, as well as the role of teacher knowledge and instructional moves in students' construction of those abstractions. Third, we envision that an abstraction framing can provide a cognitive-focused approach to modifying instructional materials and their implementation. For instance, based on analysis like that provided here and then investigations into students' realized abstractions and aspects of instruction contributing to the construction of those abstraction, researchers and teachers can look to modify instructional materials to better support students' reflecting (and reflective) abstraction of productive meanings.

Acknowledgments

The research reported was supported by the National Science Foundation No. 1920538.

References

Ellis, A., Lockwood, E., Tillema, E., & Moore, K. C. (in press). Generalization across multiple mathematical domains: Relating, forming, and extending. Cognition and Instruction.

Ellis, A., Paoletti, T., & Lockwood, E. (in preparation). Empirical and reflective abstraction. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), Piaget's genetic epistemology in and for ongoing mathematics education research.

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. Journal for Research in Mathematics Education, 42(4), 308-345. http://www.eric.ed.gov/ERICWebPortal/detail?accno=EJ932248

Ellis, A. B., & Grinstead, P. (2008). Hidden lessons: How a focus on slope-like properties of quadratic functions encouraged unexpected generalizations. The Journal of Mathematical Behavior, 27(4), 277-296. https://doi.org/https://doi.org/10.1016/j.jmathb.2008.11.002

Ellis, A. B., Tillema, E., Lockwood, E., & Moore, K. C. (2017). Generalization across domains: The relating-forming-extending framework. In E. Galindo & J. Newton (Eds.), Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 677-684). Hoosier Association of Mathematics Teacher Educators.

Fonger, N. L., Ellis, A. B., & Dogan, M. F. (2020). A quadratic growth learning trajectory. The Journal of Mathematical Behavior, 59, 100795. https://doi.org/https://doi.org/10.1016/j.jmathb.2020.100795

Kilpatrick, J. (2011). Slouching toward a national curriculum. Journal of Mathematics Education at Teachers College, 2(1).

Liang, B. (2021). Learning about and learning from students: Two teachers' constructions of students' mathematical meanings through student-teacher interactions [Ph.D. Dissertation].

Montangero, J., & Maurice-Naville, D. (1997). Piaget, or, the advance of knowledge. In. Mahwah, N.J.: L. Erlbaum Associates.

Moore, K. C. (2014). Signals, symbols, and representational activity. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), Epistemic algebraic students: Emerging models of students' algebraic knowing (pp. 211-235). University of Wyoming.

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. The Journal of Mathematical Behavior, 56. https://doi.org/10.1016/j.jmathb.2019.01.008

Oehrtman, M. (2008). Layers of abstraction: Theory and design for the instruction of limit concepts. In M. P. Carlson & C. L. Rasmussen (Eds.), Making the connection: Research and teaching in undergraduate mathematics education (pp. 65-80). Mathematical Association of America.

Piaget, J. (1970). Genetic Epistemology. W. W. Norton & Company, Inc.

Piaget, J. (1977). Recherches sur l'abstraction réfléchissante (Vol. I & II). Presses univ. de France.

Piaget, J. (2001). Studies in reflecting abstraction. Psychology Press Ltd.

Steffe, L. P., & Olive, J. (2010). Children's Fractional Knowledge. Springer.

Tallman, M. A. (2015). An examination of the effect of a secondary teacher's image of instructional constraints on his enacted subject matter knowledge [Unpublished Ph.D. dissertation].

Tallman, M. A., & Frank, K. M. (2020). Angle measure, quantitative reasoning, and instructional coherence: an examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. Journal of Mathematics Teacher Education, 23, 69-95.

Tallman, M. A., & O'Bryan, A. E. (in preparation). Reflected abstraction. In P. C. Dawkins, A. J. Hackenberg, & A. Norton (Eds.), Piaget's genetic epistemology in and for ongoing mathematics education research.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), The development of multiplicative reasoning in the learning of mathematics. SUNY Press.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 31-49). PME.

Thompson, P. W. (2013). In the absence of meaning. In K. Leatham (Ed.), Vital directions for research in mathematics education (pp. 57-93). Springer. https://doi.org/10.1007/978-1-4614-6977-3_4
von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Falmer Press. https://doi.org/10.4324/9780203454220

DESIGNING INSTRUCTIONAL SEQUENCES FOR ETHICAL, CRITICAL AND MATHEMATICAL REASONING

Luke T. Reinke
UNC Charlotte
LReinke@uncc.edu

Jordan T. Register
UNC Charlotte
jtrombly@uncc.edu

Michelle Stephan
UNC Charlotte
Michelle.Stephan@uncc.edu

Designers of critical mathematics instruction have documented difficulties in simultaneously fostering the development of critical consciousness while supporting students in developing understandings of new mathematics. However, confining justice-oriented tasks to applications of previously learned mathematics limits the degree to which these tasks will be taken up by teachers. We describe our attempt to employ heuristics from the instructional design theory of realistic mathematics education [RME] to create a sequence aimed at developing students' critical and ethical reasoning while also developing new mathematical understandings of ratio, proportion and percents. We propose emergent adaptations to two of the realistic mathematics education design principles then propose an additional ethical principle to guide the development of future RME sequences.

Keywords: Curriculum; Learning Trajectories and Progressions; Rational Numbers & Proportional Reasoning; Social Justice

Ernest's work in the philosophy of mathematics education has set the stage for the argument of ethics as its first philosophy (Ernest, 2013). For Ernest (1998) "ethics arises from the ways in which persons live together and treat each other" (p. 9) including what is deemed right or wrong when making decisions and how we generate knowledge. He proposes that ethics should serve as the foundation for mathematizing and philosophizing and calls for "an ethics of mathematics" that acknowledges its social responsibility as well as its implications for freedom, justice, and cooperation (Ernest, 1998, 2013). "Ethics in mathematics education supports, and lays the foundation for, concerns about social justice" in that issues of social justice are concerned with the social activity of groups and the fair enjoyment of social benefits, while issues of ethics are concerned with interactions between people more generally (Atweh & Brady, 2009, p. 268). From this perspective, ethical considerations are based upon people's moral responsibility to one another, establishing "social justice concerns as a moral obligation, rather than charity, good will or convenient politics" (Atweh & Brady, 2009, p. 268). Accordingly, we view ethical reasoning as notions of what is right or wrong in social contexts, with considerations of social justice as one of its key components.

A primary goal of an ethical mathematics education is fostering students' epistemological empowerment through mathematics (Ernest, 2002). In other words, developing both mathematical and social empowerment in which the learner establishes a sense of self-efficacy in "the language, skills and practices of using and applying mathematics" in school settings, and their ability to use mathematics to "participate more fully in society through critical mathematical citizenship" (Ernest, 2002, p. 1). Such an education concerns the individuals' growth of confidence in using mathematics, creating knowledge, validating it, and transforming the world through mathematics (Ernest, 2002). Educators have attempted to develop students' awareness of the power that mathematics holds to both help and harm society through the teaching of critical mathematics, which aims to develop students' critical consciousness, or awareness of social, political, and economic oppression (Frankenstein, 1983; Freire, 1970; Gutstein, 2006). If ethical reasoning is primarily concerned with notions of what is right or wrong, we see critical consciousness as the component of ethical reasoning concerned with identifying oppressive systems. While critical mathematics focuses on responding to these systems, ethical mathematics also includes the responsible creation of mathematical products. Many efforts at implementing critical mathematics have done so with students who experience intersecting oppressions; this is understandable, given that developing critical consciousness is a pillar of culturally relevant teaching (Ladson-Billings, 1995). Some scholars, though, have argued for the importance of developing critical consciousness of students from communities of privilege, proposing that without such consciousness, students from these communities may, wittingly or unwittingly, abuse the disproportionate power they wield, which continues the dehumanizing effects of oppression on both the powerful and those with less power (Kokka, 2020; Stephan et al., 2021). These scholars propose that critical mathematics, like any critical pedagogy, works toward liberating the oppressed and oppressors (Freire, 1970).

Tensions in Designing for Ethical and Mathematical Reasoning

Reports on efforts to design and implement critical mathematics lessons aimed at developing students' mathematical power and critical consciousness highlight the difficulty of the work. One primary tension, raised by numerous scholars, involves balancing the goals of developing students' mathematical power so that they can succeed in dominant mathematics while also developing their critical consciousness (Gutiérrez, 2002). Gutstein (2006) proposes that productively navigating this tension is possible but admits that he relied primarily on a reform-based but non-critical textbook series, *Mathematics in Context*, to develop students' mathematical power. Brantlinger (2013) also grappled with the tension between the two goals. After finding himself unable to navigate the tension by simultaneously addressing social justice and mathematical concepts, he then tried separating them hoping for later syntheses. Brantlinger ultimately concluded that he was unable to resolve the tension between the two goals, at least in the secondary mathematical setting of geometry which the project addressed. Other scholars document teachers' (rather than their own) efforts to implement social justice themed mathematical lessons and their attempts to navigate the tension by separating social justice contexts and the development of mathematical ideas (Turner et al., 2009; Bartell, 2013) or simply focusing primarily on either the mathematical goals or social justice goals, leaving the other goal as more of a background possibility if the opportunity arises (Bartell, 2013).

In our reading of the literature on critical mathematics and teaching math for social justice, we noticed that studies of implementation were well grounded in the literature on critical mathematics and tenets of teaching math for social justice, but these studies tend not to cite instructional design theories for developing mathematical ideas. In other words, while the social justice aspect of these lessons was grounded in theory, the mathematical aspect relied on the intuition of the teachers or instructional designers rather than an established instructional design theory. We hypothesized that this theoretical imbalance may contribute to the difficulty in designing for new mathematical concept development simultaneous with critical consciousness. In our attempt to address the goals of developing new mathematical ideas through ethical and critical contexts, we relied heavily on an instructional design theory that uses realistic contexts to ground the development of mathematical ideas.

Instructional Design Principles

To reconcile the aforementioned tensions, we anchored our study to critical mathematics frameworks (Frankenstein, 1983; Gutstein, 2006; Skovsmose, 1994) as well as the Dutch instructional theory of realistic mathematics education [RME] (Gravemeijer, 1994), which provides a robust set of design principles for developing sequences that develop students' mathematical understanding. Van den Heuvel-Panhuizen and Drijvers (2014) present six core teaching principles which correspond to instructional design heuristics: activity, reality, level, intertwinement, interactivity, and guidance. Rather than elaborate all six, we focus on the reality and level principles to illustrate how we foregrounded these particular heuristics in our design activity when confronted with critical and ethical challenges.

The *reality* principle advises that instructional sequences begin in *experientially real scenarios*. The term experientially real does not mean that problem contexts need to be authentic, or real-world, but rather the problem context needs to be imaginable, so that students can act on the problem elements in sensible ways. An earlier interview study of similarly aged students (Stephan et al., 2021) inspired an initial conjecture of how this principle might be adapted for critical mathematics instruction: we hypothesized that problem contexts in which students felt like they themselves were targets of oppressions could be a more experientially real starting point for students than situations where others were targeted.

The *leveling* principle describes the notion that students' understanding passes through various stages. Their early conceptions often involve concrete, context-bound solutions to initial problems. As their understanding develops, their *models of* these situations become more abstract and less attached to these problem situations, and become *models for* use in other, mathematically isomorphic situations (Gravemeijer, 1999). Our earlier work provided a hypothesis of how this principle might be adapted for the development of critical mathematics instructional sequences as well; we conjectured that students' initial understandings *of* how they themselves were targeted might lead to an empathy *for* other vulnerable populations.

Given our experience in designing RME sequences, we were hopeful that we could develop an instructional sequence that simultaneously developed students' understanding of new mathematical concepts (namely proportional reasoning and percent change) while also developing their critical and ethical reasoning. The following questions guided our study:

Research Question

When designing a mathematics instructional sequence for developing ethical and critical reasoning,

1. What adaptations to the existing RME design heuristics emerged and why?
2. What additions to the existing RME design heuristics emerged and why?

Method

Setting

In identifying a suitable context for our study, we drew on Kokka's (2020) conceptualization of privilege as the "set of advantages one group has over others, granted because of membership or perceived membership in social categories" (Kokka, 2020, p.3). This conceptualization considers race, socioeconomic status, and educational privilege, as well as other intersectional identity traits including, but not limited to, sexual orientation, gender, and ability status, that are both interrelated and influence students' experiences of privilege and marginalization (Kokka, 2020, p.3). As such, the instructional sequence was designed with partners who work at a school we refer to as Lakeview Charter Middle School (LCMS). The student population at LCMS is approximately 68% White, 13% Black, 8% Asian or Pacific Islander, 6% multiracial, 6% Hispanic and less than 1% Native American. Families are responsible for providing transportation for their students to and from school. There is no possibility for students to have free or reduced lunch due to the requirement that students bring their lunch from home. Due to these requirements, the student population generally draws students from economically advantaged families. The Design Team consisted of the three co-authors (two white females and a white male in comfortable socioeconomic positions) as well as two LCMS teachers (one male and one female) and their principal (female), all three being people of color. The team met four times over the summer to brainstorm contexts and develop an outline of the unit, then the three co-authors worked and met intermittently over two months to develop the unit.

Data Analysis

Data for this paper consisted of field notes that captured design decisions as well as 13 different iterations of the instructional sequence, which aimed to develop students' understanding of ratio, proportion, and percent change within a context of nicotine vaping. To analyze our design decisions related to ethical and critical challenges, two co-authors independently coded the documents by marking each instance in which an ethical or critical challenge was raised. We then compiled the results and resolved any inconsistencies. For instance, both coders identified tensions related to a task asking students to analyze a graph showing the correlation between the amount of nicotine users ingest per day and the proportion of those users who meet the established criteria for addiction; students were asked "how many mg of Nicotine would you say leads to addiction in most people?" One author coded this as an *ethical* challenge, recalling the team's discussion of whether the idea of a threshold might encourage students who are interested in vaping to try to vape 'just under the amount' they think would get them addicted. The other author noted a *critical* tension with this same task recalling that the 5 mg threshold proposed by epidemiologists seemed somewhat arbitrary to the design team, who did not want the students to accept this threshold uncritically. Upon discussion, both codes were deemed appropriate.

Findings

Our analysis suggests some extensions to the RME reality and leveling design principles as well as the inclusion of an additional principle. We first discuss some new perspectives on the reality and leveling principles and then introduce the ethics principle.

Navigating the Reality Principle in Choosing Critical Contexts

When searching for an experientially real context to serve as the semantic grounding for the instructional sequence, we found ourselves rejecting many of the contexts that would have been rich for critical investigation because they were not *didactically rich,* in the mathematical sense. For example, the mathematics involved in water contamination, lead poisoning, and air pollution were too complex to support our ratio and percent learning goals. At other times, the context may have been didactically rich, and from the designers' point of view, *critically rich*, but when considering it from the students' points of view, they would not be engaging or may produce anxiety in students. Exploring critical issues around SAT [a college entrance exam] scores was one such context that the teachers rejected for two reasons. First, at this middle school, the SAT is introduced very early to students and is a topic that students are confronted with frequently. The teachers felt that students hear about it so often that they may not be interested in it. Furthermore, they felt their students are overly pressured to perform well on the SAT and this might produce anxiety.

After two brainstorming sessions, we decided on developing a unit related to the context of vaping, because we deemed the context to be both didactically rich and critically rich from the students' perspective. Because researchers had documented efforts to target teenagers with candy and fruit vaping flavors as well as sensational advertising (e.g., Center for Disease Control, 2016), the context also fit our initial hypotheses about how an experientially real situation, where students themselves were targeted, could provide initial access to critical perspectives. Tobacco companies' targeting of African American communities with advertisements and promotions for highly addictive menthol cigarettes (Henriksen et al., 2012) and their efforts to attract Native American customers with ethically questionable promotions (Lempert & Glantz, 2019) provided potential opportunities for students to practice empathy and critical intuition about situations where other populations are targeted.

Navigating the Leveling Principle in Choosing an Entry Point

Once we settled on the vaping contexts, we grappled with the joint goals of staying authentic to the context, while also following the RME modeling principle of beginning with more informal models and progressively introducing more formal models. In our attempt to find an accessible starting point for the unit, we initially looked for authentic ratios related to vaping that students could visualize using contextual imagery. We learned that nicotine channels require two molecules of nicotine to open the receptor, but quickly realized that that particular 2:1 ratio was likely too simple to prompt students to develop creative informal solutions; furthermore, the ratio does not change, so that particular relationship did not support an instructional sequence. Aware of the frequent use of varying concentrations of lemonade or other beverages at the beginning of ratio instructional sequences, we also attempted to begin the sequence by investigating the concentration of nicotine in e-cigarette cartridges, but we could not find a way to avoid the use of fractions or decimals because of the small concentration of nicotine relative to the other ingredients, so the authentic situation proved too complex.

Ultimately, we decided that the ratio of puffs per hour presented an accessible and rich possibility. Students could draw pictures of puffs of smoke and either hourglasses or small clock faces to represent hours, so that they could visualize and iterate the ratio. Once we had identified an accessible entry point, we attempted to identify realistic problems that involve this relationship.

- Given the data for an individual, can we determine if they are puffing more than they were? In other words, is the individual possibly becoming addicted?
- Who puffs more, one individual or another?
- How long would one cartridge last, given a set puff rate?

Navigating both Reality and Leveling Principles in Developing the Sequence

Given our desire to use the puff per hour ratio and a dearth of authentic data for individual vapers, we decided that for the outset of the instructional unit, we would not use actual data. Rather we would create fictional data within the vaping context, to provide maximum flexibility to meet our needs in developing the mathematical concept. Leaning on the RME principle of experientially real or imaginable scenarios, we created a fictional story of a teen named Sara, who tracked her puffing using a smartphone app. This fictional story allowed us the flexibility to start with appropriately accessible ratio (3 puffs:1 hour), ask a variety of questions (How many puffs would she take in 8, 24 hours?) and adjust that ratio as needed to increase complexity (5 puffs:2 hours) Although we incorporated real-world data (one cartridge contains enough vape liquid for approximately 200 puffs), much of the data was hypothetical, yet positioned students as friends, wondering if Sara should worry about increases in her puffing.

Once we had progressed from informal (pictorial) to preformal (long ratio tables) to formal (short ratio tables analogous to formal proportional notation using equations) representations of ratios, we then made a design decision to transition to an authentic and not hypothetical scenario: determining a reasonable puff rate threshold for different concentrations of vaping liquid that would correlate to a scientifically-backed definition of addiction.

An emergent design heuristic. The tensions between authenticity and the desire to attend to the leveling principle and our decisions in navigating the tension illustrate an emergent design heuristic that represents our efforts to adapt the reality and leveling principles to a critical mathematical instructional sequence. When the authentic data and mathematical relationships proved more complex to provide an accessible and didactically rich starting point, we employed the use of relevant, believable, fictional stories in early stages of the progression to provide access to informal strategies and more accessible number choices. Then, once students were able to use more formal representations that could be used for other situations, we transitioned to more authentic, non-fictional inquiry.

Our efforts to design with and for ethical reasoning resulted in a number of situations for which existing RME principles provided little guidance. In response, we propose an additional principle.

The ethics principle: A seventh principle

In designing mathematics instruction for ethical and critical reasoning, we found ourselves facing design challenges that were not addressed in the six principles. The analysis of our design field notes indicated that we were attending not only to the nature of the problems we chose to provoke students' ethical awareness, but also calling into question the ethics of our choices as designers. Thus, we humbly introduce the ethics principle that has emerged from our work. The *ethics principle* refers to the fact that mathematics is done by human beings and thus has the potential for bias, at best, and oppression, at worst. Conversely, doing mathematics can be liberatory and students must learn to recognize the impact that their mathematical solutions may have on the world. This principle can be viewed in two ways. First, designers must intentionally build ethical problem solving into instruction to provide opportunities for ethical decision making to arise. Second, designers themselves must problematize the ethical dimensions of their design activity and products. The ethics principle manifested itself in two ways during our design work: attending to ethics in the grounding context and attending to ethics when writing problems.

Attending to ethics in the grounding context. Using the context of vaping presented our first ethical challenge. We wondered if it was ethical to have discussions about vaping with teenagers, and worried in particular about introducing a potentially harmful practice to students who may not have been aware of the practice before our instruction. We considered whether we should use the names of actual companies, such as JUUL, in our materials, as that might promote the company's product to students. Another ethical consideration that emerged was that some students might be vaping and become anxious when they learn of the potential negative health consequences.

Potential design solution. We resolved these ethical dilemmas by consulting outside stakeholders such as the school's principal, teachers, and teenagers. As a member of the design team herself, the principal agreed that vaping might be a controversial topic for some parents, but revealed that the health teacher teaches about nicotine, and many students will already know about it. The teachers on the design team talked with other teachers, one of whom had a child in the class. The teacher-parents indicated that they did not think the vaping context would be harmful to their children. Finally, we talked with some teenagers who indicated that most teens know about vaping, adding that many middle schoolers they know are actually engaging in the practice. This feedback from stakeholders at the school led us to settle on the vaping context.

Attending to ethics in the problems. Once we committed to vaping as the context, there were ethical issues that arose as we wrote specific problems. We considered launching the instructional sequence with a video from the Truth Initiative (Shank, n.d.) that shows teens glorifying vaping, participating in vape challenges, and getting a buzz from the nicotine. Our intention was to acknowledge upfront that teens are vaping and to invite them into a serious conversation. From an ethics perspective however, we worried that the content of the video might inadvertently encourage teens to vape because it *looks cool and fun.*

A second time we were confronted with an ethical dilemma as designers occurred when we were considering introducing the idea of a "nicotine threshold" amount. A nicotine threshold refers to the minimum amount of nicotine an individual could ingest without becoming addicted. We worried that students who want to vape might think that, as long as they stay under this threshold, they can vape without getting addicted.

A third ethical issue arose as we were writing problems that might help students see themselves as the target of vaping companies. We created an instructional task that showed actual advertising photos by vaping companies. These advertisements contained pictures of candy flavored vape juice products, the Sesame Street Muppet Elmo encouraging vaping, young people partying with vape products, and famous entertainers holding vape pods. Our intent was to have students analyze the advertisements to see that vape companies were intentionally promoting their product to youth, but we worried the ads would attract students to the product. They might be inclined to think that *vaping looks cool;* or *if my favorite stars do it, it must be ok*.

Potential design solution. One way we attended to these ethical dilemmas involved creating what we refer to as a *contextual storyline*. We intentionally introduced a fictional character named Sara who became increasingly aware that she might be vaping enough to become addicted to nicotine. The mathematical problems we introduced not only supported our conjectured *mathematical* learning trajectory, but were couched in an ongoing narrative of Sara thinking about her vaping habit. We weaved the mathematical and contextual trajectories together so that students would develop proportional reasoning as they helped Sara and other fictional teen characters think about addiction. As the mathematical learning goals shifted to percent, the storyline changed to using a fictional app called the iVapeless meter so students could analyze how close Sara was to her self-imposed vaping limit. In this way, we used Sara's storyline to both develop students' mathematical understanding of ratio and percent while also learning about how vaping, even in small amounts, might lead to nicotine addiction. We also elicited students' knowledge from health class about the potential health consequences of nicotine throughout the storyline. In this way, we hoped the storyline we created would, at minimum, de-glorify vaping and maximally, deter students from using nicotine.

Discussion and Implications

In this study, we attempted to address an identified tension from previous studies of critical mathematical lessons: the difficulty in simultaneously teaching new mathematical concepts as well as ethical/critical reasoning. Unable to find critical mathematics studies that utilize established mathematics instructional design theories, we attempted to use tenets of realistic mathematics education to support our design toward the mathematical goals, while also attending to critical mathematics frameworks. One notable difference that emerged between the lessons we developed and lessons present in the existing literature is that we designed a *sequence* of lessons, built on researched *learning* trajectories for ratio and proportion (Civak, 2020; Stephan, 2021) and critical *contexts*. Our instructional sequence also began with contexts that positioned the students we worked with as targets of oppressive acts, and then proceeded to inquiries into how tobacco companies targeted populations with marginalized identities that many of our students did not possess. Prior reports and published resources tend to represent individual lessons, rather than carefully sequenced tasks built upon learning trajectories.

Through our work, we found that the RME design heuristics were helpful for supporting our design efforts, yet we needed to adapt two of the existing principles (the leveling and realistic principles) and add a new principle. First, our adaptation to the reality principle led us to a better understanding of the demands of selecting an appropriate anchoring, experientially-real context. Others have identified the step of determining justice-oriented contexts for a given student population and determining the mathematical topics that are relevant to that context. In addition to this, we also needed to determine whether contexts were didactically rich, meaning that problems inherent to the context lent themselves to develop a variety of models, from informal, context-bound models to more formal models. Furthermore, we needed to identify contexts that presented a variety of critical and mathematical questions; vaping for instance, had many different considerations including how to determine whether someone is vaping more, data related to teens' vaping practices, and the targeting of various groups with advertising.

Our second adaptation involved the leveling principle. Once we had determined a didactically rich, critical context, we found ourselves oscillating between posing problems with complete fidelity to the context and fictionalizing certain aspects of the situations. For example, we attempted to use real stories and actual data from social media and research articles for problem contexts, but sometimes the numbers and procedures in the reports were too complex to support students at that particular point in the mathematical learning trajectory. Thus, in order to develop a sequence that adhered to the leveling principle, we needed to fictionalize aspects of that context at times.

Finally, in addition to adapting the leveling and reality principles to design instruction with authentic critical contexts, we found the need for an additional RME heuristic. Ernest (2013) argues that the ethical considerations of problem contexts must be considered and should be the foundation for mathematization, yet none of the current RME principles provide much guidance to designers who create instructional sequences with attention to ethics and social justice. We illuminated the ways in which ethical considerations emerged for us during the design process and presented several instances when the design team encountered ethical dilemmas and the ways we resolved them. Consequently, we suggest the addition of an ethics design principle to RME and argue that, not only should instructional materials support students' development of ethical reasoning but that designers themselves must also be alert to and problematize the ethical dimensions of their designs for students.

In our description of these adaptations and additions to the RME heuristics, we must also clarify that these are proposals at an early stage of the design research project. We have developed and piloted an initial version of the instructional sequence, but we have yet to analyze the data on implementation. Thus, we are careful not to propose that the adaptations to the RME principles are final, rather they are emergent and likely to need further revision. As our work progresses, it will be important to apply and continue to revise these principles in other settings with other students, mathematical topics, and justice-oriented contexts.

Acknowledgments

This work was supported, in part, by funds provided by UNC Charlotte.

References

Civak, Rukiye A. (2020) The evolution of mathematical practices in a seventh-grade classroom: Analyzing students' development of proportional reasoning. Unpublished doctoral dissertation. Middle East Technical University.

Atweh, B., Brady, K. (2009). Socially response-able mathematics education: Implications of an ethical approach. *Eurasia Journal of Mathematics, Science & Technology Education, 5*(3), 267-276.

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, *44*(1), 129-163.

Brantlinger, A. (2013). Between politics and equations: Teaching critical mathematics in a remedial secondary classroom. *American Educational Research Journal*, *50*(5), 1050-1080.

Center for Disease Control (2016). *E-cigarette ads and yout*h. Retrieved February 7, 2022 from https://www.cdc.gov/vitalsigns/ecigarette-ads/index.html.

Civak, R. A. (2020) The evolution of mathematical practices in a seventh-grade classroom: Analyzing students' development of proportional reasoning. Unpublished doctoral dissertation. Middle East Technical University.

Ernest, P. (1998). *Social Constructivism as a Philosophy of mathematics*, NewYork: SUNY Press.

Ernest, P. (2002). Empowerment in mathematics education. *Philosophy of mathematics education journal*, *15*(1), 1-16.

Ernest, P. (2013). What is 'first philosophy' in mathematics education? *The Philosophy of Mathematics Education Journal, 27*.

Ernest, P. (2018). The Ethics of Mathematics: Is Mathematics Harmful? In *The Philosophy of Mathematics Education Today* (pp. 187–216). Springer International Publishing.

Frankenstein,M. (1983). Critical mathematics education: An application of Paulo Freire's epistemology. Journal of Education, 165(4), 315–339.

Freire, P. (1970). *Pedagogy of the oppressed*. London: Penguin Books.

Gravemeijer, K. (1994). *Developing realistic mathematics education*. Freudenthal Institute.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. Mathematical Thinking and Learning, 1(2), 155–177.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Towards a new equity research agenda. Mathematical Thinking and Learning, 4(2 & 3), 145–187.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. New York: Routledge.

Henriksen L, Schleicher NC, Dauphinee AL, Fortmann SP. (2012) Targeted advertising, promotion, and price for menthol cigarettes in California high school neighborhoods. *Nicotine Tobacco Research*. 14(1). 116-21

Kokka, K. (2020). Social justice pedagogy for whom? Developing privileged students' critical mathematics consciousness. *The Urban Review, 51*(1), 101–122

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal, 32*(3), 465-491.

Lempert, L. K., & Glantz, S. A. (2019). Tobacco industry promotional strategies targeting American Indians/Alaska Natives and exploiting tribal sovereignty. *Nicotine and Tobacco Research*, 21(7), 940–948.

Olver, J. & Foulds, J. (2021). Association between cigarette smoking frequency and tobacco use disorder in U.S. Adults. *American Journal of Preventative Medicine, 60*(5), 726-728.

Shank, S. (n.d.). *New study reveals teens 16 times more likely to use juul than older age groups.* Truth Initiative. https://truthinitiative.org/press/press-release/new-study-reveals-teens-16-times-more-likely-use-juul-older-age-groups

Skovsmose, O. (1994). Towards a critical mathematics education. Educ Stud Math, 27, 35–57.

Stephan, M. (2021). Classroom design-based research: Designing for proportional reasoning. In Z. Philippakos, E. Howell, & A. Pellegrino (Eds.), Design Based Research in Education: Theory and Applications (pp. 83–102). Guilford Press.

Stephan, M., Register, J., Reinke, L., Robinson, C., Pugalenthi, P., & Pugalee, D. (2021). People use math as a weapon: Critical mathematics consciousness in the time of COVID-19. Educational Studies in Mathematics, 108, 513-532.

Turner, E. E., Gutiérrez, M. V., Simic-Muller, K., & Díez-Palomar, J. (2009). "Everything is math in the whole world": Integrating critical and community knowledge in authentic mathematical investigations with elementary

Latina/o students. *Mathematical Thinking and Learning*, *11*(3), 136-157.
Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic Mathematics Education. In S. Lerman (Ed.), Encyclopedia of Mathematics Education (pp. 521-525). Dordrecht, Heidelberg, New York, London: Springer.

A STUDY OF THE CONSISTENCY IN NEW YORK STATE FIRST YEAR MATH EXAMS

Stephanie Schaefer
American College of Education
stephanie.schaefer@ace.edu

Deborah Moore-Russo
University of Oklahoma
dmr@ou.edu

As standards documents have been introduced over the past 20 years, many states have seen an evolution in both the standards and related high stakes exams. For many teachers across the U.S., the rollout of standards and exams has not been an experience that builds trust in state education leaders. In this study, we consider three major changes in the first-year high school math exams in New York State since 2002, looking at consistency in item types, topics addressed, and student performance. Shifts in all were noted, but the changes in topics, especially when not obvious by the names given to standards, are suggested as the mostly likely to misinform or misguide teachers. We consider how state educational leaders are working to build trust for the next iteration of standards. While this study is particular to one state, the methods and findings should be of interest to others who study curriculum and testing in high schools.

Keywords: Algebra and Algebraic Thinking; Assessment; High School Education; Standards

The National Council of Teachers of Mathematics (NCTM, 1989) introduced its first standards document just over 20 years ago. This launched an evolution of mathematics curriculum standards and related assessments. NCTM (2000) published a subsequent set of standards in just over 10 years, and the Common Core Standards were published 10 years later (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). While having a national organization, such as NCTM, put forth standards helped bring more coherence to mathematics instruction across the United States, this led to significant disruptions at the state level. State education leaders subsequently made changes in their standards and state assessments that were rolled out to teachers, while legislation (e.g., No Child Left Behind, 2002) put in place mandates that implemented state assessment accountability.

Researchers report that state assessments have been associated with intense stress for teachers, and teachers report feeling that the high-stakes state exams often undermine meaningful learning and instruction (Barksdale-Ladd & Thomas, 2000). This feeling of pressure is regardless of the state in which teachers work (von der Embse, Pendergast, Segool, Saeki & Ryan, 2016), and studies have reported on state assessment accountability and its impact on teacher stress and teachers' intentions to leave the teaching profession (e.g., Ryan et al., 2017).

A reconstitution of the mathematics curriculum based on standards has brought about research that compares states' curricula and adoption (Senk & Thompson, 2020) as well as research that considers states' suggested coverage and placement of key topics in the curricula (e.g., Nagle & Moore-Russo, 2014; Stanton & Moore-Russo, 2012). However, little has been published about the high-stakes state exams themselves. How much did changing standards impact state assessments, particularly those that were being administered for the first year of high school mathematics? In order to consider this question in light of a new set of state mathematics standards that will soon impact state assessments, the overarching research goal of this study is to look at changes in first-year high school mathematics state exams in New York State (NYS) over the past 20 years. The three specific research questions include:

1. How consistent have state exams been over the past 20 years, in terms of the topics covered, types of items included, and student performance on the exams?
2. How do previous exam topics compare to current first year mathematics state standards?
3. How are state education leaders preparing teachers for the upcoming changes in standards and state exams?

Theoretical Framework

Social capital theory can focus on notions ranging from power (Fine, 2001) to economic transactions (Hardin, 1999) to mechanisms that build common values (McNiell, 2007). In all of these, there is interconnectivity that involves some level of trust. *Trust* is the belief that an entity (be it an institution or an individual) will act in ways that are consistent with one's expectations of positive behavior (Algan, 2018). It is an expectation that arises that is based on shared norms (Fukuyama, 1996). Trust helps reduce uncertainty (Luhmann, 1979) and helps deliver optimal outcomes by strengthening relationships and preventing defection (Six, van Zimmeren, Popa & Frison, 2015). Hardin (2002) suggests that norms produce trust, which he describes in the first person as the perception that you have "an interest to take my interests in the relevant matter seriously…you value the continuation of our relationship" (p. 2).

In organizations, subordinates' trust in those over them has been found to be related to employee commitment and job satisfaction (Colquitt, Scott, & LePine, 2007; Dirks & Ferrin, 2002). Burke, Sims, Lazzara, and Salas (2007) suggest that this trust is based on aspects such as accountability, transparency, and consistency (Kim & Lee, 2018). Consider the mathematics education community in a state as an organized entity with teachers in roles that might be considered subordinate to those who create high-stakes exams. The state's educational leaders show trust in teachers with their administration and marking of the exams. However, if one has listened to teacher lounge grumblings or has read media reports (Domanico, 2021; Strauss, 2012; Taylor, 2016), it calls to question how much this trust has been reciprocated. To begin to understand teacher trust in state education leaders and mandated state testing, there are many aspects that could be studied. A logical initial investigation is to study how consistent state exams have been looking at different measures of consistency. For that reason, this study considers looks the three iterations of the NYS exams for first-year high school math in terms of consistency and using the current math standards as a lens. It also considers how NYS education leaders are preparing teachers for future revisions to the standards and related exams.

Current Study

Since this study focuses on a particular state, we first detail the sequential shifts that have occurred in first-year high school mathematics in NYS. Next, we situate ourselves and experiences as researchers for this study.

Context.

From 2002 to 2020, there were three changes in standards that affected the first-year math exams in NYS: the *old* standards (Math A) from 2002-2009, the *recent* standards (Integrated Algebra) from 2008-2015, and the *current* standards from 2014-2020. Each exam administration occurred in January, June, and August. Each exam consisted of multiple choice and constructed response items corresponding to the relevant standards. In some years, two versions of the exam were available during transitional periods between standards. There were a total of 2287 items that were graded and publicly available on the NYS Regents' website (NYS Office of State Assessment, 2021): 764 *old* items (2002-2009), 857 *recent* items (2008-2015), and 666 *current* items (2014-2020).

Research team.

The research team consists of two members who have firsthand experience with the changes in NYS standards and assessments in high school mathematics. The lead researcher is a former NYS secondary mathematics teacher who experienced: the *old* exams as a student, the *recent* exams as a pre-service teacher, and the *current* exams as an in-service teacher. As part of her dissertation work (Schaefer, 2020), Schaefer did an in-depth study of NYS state assessments focusing on readability. While focusing on readability, other changes across the years became obvious. The second researcher worked in a Department of Learning and Instruction in a university located in NYS; she taught the secondary mathematics education classes for pre-service teachers from 2004 to 2014, a period that overlapped the *old*, *recent* and *current* standards.

Data Collection and Standard Classification

All items from first-year *old* (n=764), *recent* (n=857), and *current* (n=666) NYS exams were used as data. A sample of 10% of *current* exam items were classified using the *current* standards to obtain interrater reliability with NYS standard classification (Schaefer, 2020). Cohen's κ was calculated to determine agreement ($\kappa = 0.903$, $p < 0.05$). Discrepancies were then analyzed, and the NYS classification was accepted in each area of discrepancy. All *old* and *recent* items were then coded by the first using the *current* standards. Both authors discussed any items that were difficult to code until consensus was reached.

Findings

In this section we present findings in the order of the research questions. The findings are then followed by relevant discussion.

Exam Consistency

We first consider how consistent the first-year mathematics exams have been in NYS. We look at the topics covered by the exams in light of the standards that applied at the time of the exams. We then consider the types of items on the exams. Finally, we investigate any changes in student performance.

Item Topics. The *old*, *recent*, and *current* exams had a wide variety of topics based on their individual standard systems. Table 1 notes the different topic areas for each of the exams in terms of the distributions by the relevant standards at the time. The *recent* standards show the emphasis on an integrated curriculum, with the inclusion of a Geometry standard, that was not present in either the *old* or *current* standards. The *old* exams had the greatest difference in topic names as compared to *recent* and *current* exams. Consider the *current* standard of Number and Quantity. On *old* exams, the Number and Numeration standard was the focus for 8% of exam items, and the Operations standard was the focus for 19% of exam items. On *recent* exams, Number Sense and Operations was the focus of 8% of exam items. On *current* exams, Number and Quantity was the focus of 5% of exam items. Measurement decreased from being the focus on 19% of items on the *old* exam to 6% on *recent* exams, and it was not considered a high school mathematics standard and was therefore not the focus of any items on the *current* exam. The emphasis on functions shifted from accounting for 18% of the items on *old* exams, under the Patterns/Functions standard, to being melded in with Algebra in *recent* exams. Functions is a standard by itself and represented the focus of 37% of the items on *current* exams.

Table 1: Topic Areas and Relative Distributions of Exams.

Old Exam		*Recent* Exam		*Current* Exam	
Standard	Dist.	Standard	Dist.	Standard	Dist.
Math Reasoning	7%	Algebra	54%	Algebra	50%
Operations	19%	Geometry	16%	Functions	37%
Number & Numeration	8%	Number Sense & Ops	8%	Number & Quant	5%
Model/Multiple Reps	20%	Stats & Probability	16%	Stats & Probability	9%
Measurement	19%	Measurement	6%		
Uncertainty	8%				
Patterns/Functions	18%				

Item Types. Exams since 2004 but prior to *current* exams seemed to be more heavily weighted toward multiple-choice items. When the *old* exams were initially administered, the relative weighting (in terms of point distribution) of multiple-choice items was 47%. Then, the *old* exam was revised so that the relative weighting of multiple-choice items was 71% of exam points. *Recent* exams had a similar 69% relative weighting of multiple-choice items. The *current* exams relative weighting decreased to 56% of the total points. Most notably, there was also a six-point constructed response item added to *current* exams, replacing the three-point items that were on previous exams. Table 2 outlines the differences in item type distributions with the relative weighting for each item type in relation to the overall total exam point value.

Table 2: Item Type Distributions and Relative Weighting to Total Point Values for Exams

Item Type & Value	*Old* Exam Jun02-Jun03		*Old* Exam Jan04-Jun09		*Recent* Exam Jun08-Jun15		*Current* Exam Jun14-Jan20	
	n	Rel Wt*	*n*	Rel Wt	*n*	Rel Wt	*n*	Rel Wt
Multiple-Choice (MC Items)								
2-point	20	47%	30	71%	30	69%	24	56%
Constructed Response (CR) Items								
2-point	5	12%	5	12%	3	7%	8	19%
3-point	5	18%	2	7%	3	10%	0	0%
4-point	5	24%	2	10%	3	14%	4	19%
6-point	0	0%	0	0%	0	0%	1	7%

Student Performance. In order to consider exam consistency, we now look at student performance on the exams. For the NYS Regents, passing an exam is equivalent to receiving at least 65% of the total points on an exam. However, some students with disabilities are given a 55% passing rate, depending on their Individualized Education Programs (IEPs). A score at or above 85% is considered passing with distinction. For this reason, Table 3 displays the passing rates at or above 65% and 85% for all students taking the first-year math exams in NYS as well as the passing rates at or above 55%, 65%, and 85% for students with documented disabilities who took these exams (NYS Education Department, 2022).

Note the passing rate of scores that are 65% or higher on exams has decreased for all students going from 73% on *old* exams to 72% on *recent* exams to 70% on *current* exams. The passing with distinction rates (i.e., scores of 85% or higher) have decreased more markedly for all students going from 25% on *old* exams to 16% on *recent* exams to 13% on *current* exams. Now consider only the population of NYS students with disabilities and their performance on the *old*, *recent*, and *current* exams. Students with disabilities' passing rates of 55% or higher have increased from 64% to 66% to 67%, respectively, while passing their passing rates of 65% have decreased from 45% to 42% to 39%, respectively. There has also been a decline in student with disabilities who pass with distinction (i.e., with scores of 85% or higher) from 6% to 2% to 1% respectively.

Table 3: Relative Passing Rates on Exams for All Students and Students with Disabilities

Item Type & Value	*Old* Exam n = 1,552,177			*Recent* Exam n = 1,933,213			*Current* Exam n = 1,324,731		
	$\geq$55%	$\geq$65%	$\geq$85%	$\geq$55%	$\geq$65%	$\geq$85%	$\geq$55%	$\geq$65%	$\geq$85%
All Students n = 4,980,809	-	73%	25%	-	72%	16%	-	70%	13%
Students with Disabilities n = 700,048	64%	45%	6%	66%	42%	2%	67%	39%	1%

Comparison to *Current* Standards

We now use the *current* standards as a lens to consider shifts in the topics covered on exams for the past 20 years. When considering the items on *old*, *recent*, and *current* exams (as displayed in Table 4), it is obvious that there have been shifts in the topics that are covered. This is most notable when looking that the number of items that were on the *old* exams (73%) and *recent* exams (57%) that do not apply to the *current* standards. For example, measurement and geometry are no longer part of the first-year mathematics curriculum in NYS. For the Algebra standard, there has been a respective shift from 22% to 29% to 50% respectively on *old*, *recent*, and *current* exams, with the items on the *current* exams primarily focusing on this standard. For the Functions standard, there has also been a notable increase from 1% to 4% to 37% respectively on *old*, *recent*, and *current* exams. This is due to introducing the concept of function as well as an emphasis on introducing and using functional notation (rather than only considering equations, such as $y = 2x + 3$). There have been less dramatic shifts in the percentages of items that address the Number and Quantity and the Statistics and Probability standards. There have also been some shifts within individual topics as to which subtopics are emphasized. However, most of these shifts are based on the large increases in all the subtopics under the Algebra and Function standards.

Table 4: Counts of Items on Exams by Current Standards' Topics/Subtopics

	Old		*Recent*		*Current*	
Topic and Subtopic Areas	*n*	%	*n*	%	*n*	%
Algebra	**166**	**22%**	**248**	**29%**	**331**	**50%**
Arith w/ Polynomials & Ratl Expressions	19	2%	21	2%	34	5%
Creating Equations	22	3%	38	4%	82	12%
Reasoning w/ Equations & Inequalities	109	14%	139	16%	144	22%
Seeing Structure in Expressions	16	2%	50	6%	71	11%
Functions	**7**	**1%**	**34**	**4%**	**244**	**37%**
Building Functions	-	-	1	< 1%	37	6%
Interpreting Functions	7	1%	25	3%	153	23%
Linear, Quadratic, & Exponential Models	-	-	8	1%	54	8%
Number and Quantity	**19**	**2%**	**27**	**3%**	**33**	**5%**
The Real Number System	2	< 1%	-	-	18	3%
Quantities	17	2%	27	3%	15	2%
Statistics and Probability	**13**	**2%**	**58**	**7%**	**58**	**9%**
Interpreting Categorical & Quantitative Data	13	2%	58	7%	58	9%
None Applicable	**559**	**73%**	**490**	**57%**	**-**	**-**

Preparations for Upcoming Changes in Standards

Another change in NYS is coming, with the Next Generation Math Standards (NYS Education Department, 2017a). When the *old* curriculum evolved to the *recent* curriculum, there was a significant shift in topic areas covered in the standards. In the shift from the *recent* to the *current* curriculum, similar topic areas were used; yet standards were condensed and there was a move away from an integrated curriculum. Now, from the *current* to the *future* Next Generation Math Standards, NYS educational leaders have created resources (i.e., a snapshot document and a crosswalk) to better explain how standards have been added, modified, or removed.

In the Snapshot document (NYS Education Department, 2019a), there were three categories that highlighted the major changes from *current* to *future* standards. These categories including mapping the *future* standards to the *future* curriculum, outlining which standards moved to another level of the mathematics curriculum, and additional clarification on *current* standards pertinent to the *future* curriculum. The Crosswalk document (NYS Education Department, 2019b) included detail clarifying the standards based on input from conference calls hosted by the NYS Education Department. These comments included notes that clarified more information about the standards. For instance, there was specific information regarding fluency expectations and how the first-year standards differed from those for the third-year math course, Algebra II.

The NYS Education Department created two committees that included both educators and parents for the *future* standards. There were also opportunities for public commentary on the draft document with over 750,000 comments from more than 10,500 people on the AIMHighNY 2016 survey (NYS Education Department, 2017b). This information was first used to create an initial version of the future standards in 2017 that has since been revised and disseminated in June 2019.

Probably most importantly, there has also been a longer roll-out period for the *future* standards than existed in past years when transitioning between standards. Initial information on the *future* standards was provided to teachers in 2017. The NYS Department of Education has also considered the impact of the pandemic and are taking its impact into account regarding the implementation of the *future* standards, which are not slated to be implemented and used for Algebra 1 exams until September 2023 (NYS Education Department, 2021).

Discussion

In conclusion, there have been some shifts in first-year high school math exams in NYS over the past 20 years. This is noted both in the item types and their weightings. It is also noted in student performance on the exams, especially in students passing with distinction. However, the areas where the revisions have been the most dramatic, especially in the impact on teachers, are in the topics that are covered on these exams.

Much of the discontent with the roll out of standards in the past is likely due to the lack of communication and a disruption in the continuity of topics that were included in the standards and on the exams. Change, by its very nature, disrupts consistency. However, there are mechanisms that can make change easier. There are also practices that can downplay change so that it is less evident initially, but this has consequences when it is realized that the change was more dramatic than initially believed. Having standards with similar topic names helped teachers transition from *recent* to *current* standards. However, what counted as Algebra for the *recent* exams was not the same as what counted as Algebra in the *current* exams. While 54% of the *recent* exam items were classified as addressing Algebra topics under the *recent* standards; only 29% of the *recent* exam items were classified as Algebra topics under the *current* standards. For example, ratios and proportions (including the three basic right triangle trigonometric ratios) were considered part of the subtopics covered in the *recent* Algebra standards, but they are not part of the *current* Algebra standards. Using the same topic name for a standard that represents different subtopics misinforms and can misguide teachers. When Algebra means one thing for a few years and then something else, it could have felt like a breach of trust for many teachers.

It is heartening that the NYS Department of Education has been more transparent and open to suggestions from teachers (and others) in the transition to future math standards that will be adopted soon. One way that this is noted in the inclusion of accessible documents that carefully map out the continuity across and evolution in topics. Optimistically, the research team is hopeful that the rollout of the future standards will be smoother than it was for past transitions of standards in NYS.

The input of teachers (as well as of other constituents in the learning process) is key to building trust. But bi-directional communication between teachers and state education leaders is not all that is needed to help teachers transition between curricular standards. There needs to be a logical, coherent evolution that builds on, rather than totally replaces, existing standards. Moreover, this evolution needs to be clearly communicated to all, but especially to teachers. Consistent transparency as to the coming changes and how they are line up with, build on, and improve the current standards would be of benefit and would help develop more trust between teachers and state education leaders.

Limitations, Future Study

While an obvious limitation is that this study only looks at NYS, the methods used and the underlying message are pertinent to those who study high school mathematics in other states, especially those interested in curricular standards and state exams. Other studies that investigate how different states have handled curricular shifts in standards and state assessment exams would be of great value. It would be interesting to see how future work could springboard from studies such as this to delve into more qualitative work on how trust is built or breached between teachers and state education leaders, especially in light of high stakes testing.

References

Algan Y. (2018). Trust and social capital. In J. E. Stiglitz, J.-P. Fitoussi, & M. Durand (Eds.), *For good measure advancing research on well-being metrics beyond GDP* (pp. 283-320). Paris: OECD (Organisation for Economic Co-operation and Development).

Barksdale-Ladd, M. A., & Thomas, K. F. (2000). What's at stake in high-stakes testing: Teachers and parents speak out. *Journal of Teacher Education, 51*(5), 384-397.

Burke, C. S., Sims, D. E., Lazzara, E. H., & Salas, E. (2007). Trust in leadership: A multilevel review and integration. *The Leadership Quarterly, 18*(6), 606–632.

Colquitt, J. A., Scott, B. A., & LePine, J. F. (2007). Trust, trustworthiness, and trust propensity: Meta-analytic test of their unique relationship with risk-taking and job performance. *Journal of Applied Psychology, 92*(4), 909–927.

Domanico, R. (2021, September 13). *Reset N.Y. state tests to recognize reality.* New York Daily News. http://nydailynews.com/

Dirks, K. T., & Ferrin, D. L. (2002). Trust in leadership: Meta-analysis findings and implications for research and practice. *Journal of Applied Psychology, 87(*4), 611–628.

Fine, B. (2001). *Social capital versus social theory: Political economy and social science at the turn of the millennium.* London: Routledge.

Kim, H., & Lee, T. H. (2018). Strategic CSR communication: A moderating role of transparency in trust building. *International Journal of Strategic Communication, 12(*2), 107-124.

Fukuyama, F. (1996). *Trust: The social virtues and the creation of prosperity.* New York: The Free Press.

Hardin, R. (1999). Social capital. In J. E. Alt & M. Levi (Eds.). *Competition and cooperation: Conversations with Nobelists about economics and political science.* New York: Russel Sage Foundation.

Hardin, R. (2002). *Trust and trustworthiness.* New York: Russel Sage Foundation.

Luhmann, N. (1979). *Trust and power*. Chichester: John Wiley & Sons.

McNiell, D. (2007). Social capital or sociality? Methodological contrast between economics and other social sciences. In S. Ioannides & K. Nielsen (Eds.). *Economics and the social sciences: Boundaries, interaction and integration* (pp. 163-184). North Hampton: Edward Elgar.

Nagle, C., & Moore-Russo, D. (2014). Slope across the curriculum: Principles and Standards for School Mathematics and Common Core State Standards. *The Mathematics Educator, 23*(2), 40-59.

National Council for Teachers of Mathematics. (1989). *Commission on Standards for School Mathematics. Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2000). *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.

New York State Education Department. (2017a). *New York State Next Generation Mathematics Learning Standards.* http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-next-generation-mathematics-p-12-standards.pdf
New York State Education Department. (2017b). *Board of Regents P-12 Committee Approves Next Generation Learning Standards.* http://www.nysed.gov/news/2017/board-regents-p-12-committee-approves-next-generation-learning-standards
New York State Education Department. (2019a). *New York State Next Generation Mathematics Learning Standards Algebra 1 Snapshot.* http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-math-standards-algebra-i-snapshot.pdf
New York State Education Department. (2019b). *New York State Next Generation Mathematics Learning Standards Algebra 1 Crosswalk.* http://www.nysed.gov/common/nysed/files/programs/curriculum-instruction/nys-math-standards-algebra-i-crosswalk.pdf
New York State Education Department. (2021). *New York State Next Generation Mathematics Learning Standards Instruction and Assessment Implementation Timeline*. http://www.nysed.gov/common/nysed/files/next-gen-mathematics-instruction-assessment-timeline.pdf
New York State Education Department. (2022). *NY State Data.* https://data.nysed.gov/
New York State Office of State Assessment. (2021). *Past examinations.* https://www.nysedregents.org/
No Child Left Behind Act of 2001, P.L. 107-110, 20 U.S.C. § 6319 (2002).
Ryan, S. V., von der Embse, N. P., Pendergast, L. L., Saeki, E., Segool, N., & Schwing, S. (2017). Leaving the teaching profession: The role of teacher stress and educational accountability policies on turnover intent. *Teaching and Teacher Education, 66,* 1-11.
Schaefer, S. (2020). *Readability, complexity, and representations: A holistic analysis of New York state algebra exams.* ProQuest Dissertations Publishing.
Senk, S. L., & Thompson, D. R. (2020). *Standards-based school mathematics curricula: What are they? What do students learn?* New York: Routledge.
Six, B., van Zimmeren, E., Popa, F., & Frison, C. (2015). Trust and social capital in the design and evolution of institutions for collective action. *International Journal of the Commons, 9*(1), 151–176.
Stanton, M., & Moore-Russo, D. (2012). Conceptualizations of slope: A look at state standards. *School Science and Mathematics, 112*(5), 270-277.
Strauss, V. (2012, May 16). *Meet the 'worst' 8th grade math teacher in NYC.* The Washington Post. http://www.washingtonpost.com/
Taylor, K. (2016, March 24). *Teachers are warned about criticizing New York State Tests.* New York Times. http://nytimes.com/
von der Embse, N. P., Pendergast, L. L., Segool, N., Saeki, E., & Ryan, S. (2016). The influence of test-based accountability policies on school climate and teacher stress across four states. *Teaching and Teacher Education, 59,* 492-502.

HOW DO MULTI-DIGIT MULTIPLICATION PROBLEMS PROMOTE PROCEDURAL FLEXIBILITY? AN ANALYSIS OF TWO FOURTH GRADE TEXTBOOKS

Elena M. Silla
University of Delaware
esilla@udel.edu

Procedural flexibility promotes efficient problem solving in mathematics. However, it is unclear whether and how elementary-grades textbooks promote this skill. The current study investigated lessons within two fourth grade math textbooks to determine how math tasks promote procedural flexibility within the domain of multi-digit multiplication. I developed and applied two frameworks to analyze the multiplication strategies and instructional strategies that might promote procedural flexibility that were presented in textbooks The textbooks differed in the number and types of multiplication strategies included. Neither textbook showed much variability in terms of instructional strategies, and there was little alignment between multiplication and instructional strategies. Future research could investigate additional textbook features such as teachers' guides study additional avenues for promoting procedural flexibility.

Keywords: Elementary School Education, Curriculum, Number Concepts and Operations, Standards

Historically, students in the United States have underperformed in mathematics (see the National Center for Educational Statistics, 2021). This lack of proficiency in math leaves many students unprepared for more advanced math courses such as Algebra. To remediate this, math researchers and educators have been targeting strands of mathematical thinking that may help students develop deeper understanding of mathematical concepts as well as more effective problem-solving skills (National Research Council, 2001).

One such strand of math learning is procedural flexibility. Procedural flexibility refers to students' abilities to think flexibly and adaptively about strategies for problem solving, thereby facilitating students' abilities to discern between multiple strategies and apply the most efficient strategy (Digital Promise Global, 2021; National Research Council, 2001; Star, 2005; Verschaffel et al., 2007). Procedural flexibility is a crucial skill for later math subjects such as Algebra (Rittle-Johnson et al., 2020; Star, 2005). Recent research has also emphasized the importance of procedural flexibility in its relation to both procedural and conceptual knowledge of problem solving (see Newton et al., 2020; Schneider et al., 2011). Having the procedural knowledge to accurately and fluently solve problems allows students to discern efficient problem-solving strategies and thus evoke procedural flexibility when selecting which strategy to use (Star, 2005). Similarly, conceptual knowledge of a topic can help students reason flexibly about problems (Durkin et al., 2017; Schneider et al., 2011). For example, a student who does not understand the standard algorithm for multi-digit multiplication but who has conceptual understanding of multiplication might be able to think flexibly and develop their own strategy for solving the problem. Improvements in procedural flexibility is also associated with increases in conceptual understanding (DeCaro, 2016; Durkin et al., 2017, 2021; Newton et al., 2020). Thus, procedural flexibility is an important aspect of mathematical thinking.

Procedural flexibility is also embedded in the Common Core State Standards for Mathematics (CCSS-M; 2016) and is emphasized across grade levels, with standards stating that students should have "skill in carrying out procedures flexibly, accurately, effectively, and appropriately" (CCSS-M, 2016). Its prominence both in the CCSS-M and in research suggests procedural flexibility may be a crucial skill to emphasize in mathematics instruction.

Conceptual Framework

While procedural flexibility is integral to math learning, it is unclear whether and how students learn procedural flexibility in the classroom. A central artifact to a math classroom is the textbook, which provide curricular materials that guide teachers' instruction (Lloyd et al., 2017). Therefore, this analysis will examine textbooks to explore in what ways textbooks promote the use of multiple strategies and procedural flexibility in problem solving.

Procedural Flexibility and Multiple Strategies

The ability to use and apply multiple strategies while problem solving is intricately tied to procedural flexibility. The CCSS-M (2016) emphasize the use of multiple strategies in problem solving. For example, the standard 4.NBT.B.5 mentions five different multiplication strategies students should be able to apply when solving problems. Other standards similarly include multiple strategies, implicitly emphasizing the importance of procedural flexibility. Equally important is the ability to apply strategies effectively. DeCaro (2016) found that students who were primed to use a specific strategy solved later problems less efficiently than those who were not primed, suggesting that this mental constraint affected students' abilities to efficiently apply strategies to solve problems. Promoting flexibility in thinking, then, may help students apply strategies more efficiently when problem solving.

Instruction of Procedural Flexibility

Researchers and educators have developed ways for procedural flexibility to be taught through instruction. One method is through comparing multiple strategies (Durkin et al., 2021; Rittle-Johnson et al., 2020), often through comparing two worked examples that show the same problem solved in two different ways. In one study, Durkin et al. (2021) tested curricular materials designed to foster comparison between multiple strategies for solving linear equations. They provided teachers supplemental worked example pairs (WEPs) that presented multiple strategies for solving the same problem as well as questions that prompted students to compare and analyze each strategy. Students whose teachers used multiple strategies in instruction and asked students to compare multiple strategies performed better on the posttest, particularly in the domains of procedural flexibility, than students whose teachers did not engage in these practices.

Other instructional strategies could be associated with increased procedural flexibility as well. A method often used in textbooks is to require students to solve a problem using a specific strategy, allowing students to engage with a multitude of strategies while problem solving (see Great Minds, 2016; MacMillian-McGraw Hill, 2009 for examples). Other techniques include encouraging students to reflect on strategies used during problem solving (Star et al., 2015; Woodward et al., 2012) and asking students to choose between strategies when problem solving.

Multi-Digit Multiplication

The current study of multiple strategy use and procedural flexibility is situated in the domain of multi-digit multiplication. Prior work on procedural flexibility has been done at the pre-Algebra or Algebra level (e.g., Durkin et al., 2021; Rittle-Johnson et al., 2020; Star, 2005), and less work has been done at the elementary level. Yet there is reason to believe that procedural flexibility is important for early grades, as well. As mentioned above, the CCSS-M standards (particularly for fourth and fifth grade) emphasize procedural flexibility through prompting students to perform operations fluently (CCSS-M, 2016). In particular, standards for multi-digit multiplication include multiple problem-solving strategies, suggesting this may be a fruitful domain to introduce strategy choice. Additionally, developing procedural flexibility when solving multiplication problems can help students solve problems more accurately and quickly (Digital Promise Global, 2021), which can help students solve more complex problems in later subjects. Finally, the role of procedural flexibility in developing conceptual knowledge (Rittle-Johnson et al., 2015; Schneider et al., 2011), as well as the prominence of flexible thinking in more advanced problem-solvers (Star & Newton, 2009) makes it a crucial skill for students to learn early on. Thus, it is important to explore in what ways procedural flexibility is taught in elementary mathematics classes and how curricular materials such as textbooks might promote this skill.

Research Questions

The current study examines the viability of two novel frameworks, consisting of multi-digit multiplication strategies and instructional strategies promoting procedural flexibility, through applying them to two fourth grade mathematics textbooks. My specific research questions are:

1. What strategies do fourth grade mathematics textbooks specify for students to use when multiplying multi-digit numbers?
2. In what ways do problems in fourth grade mathematics textbooks promote procedural flexibility of multiplication strategies?
3. How aligned are strategies used for solving multi-digit multiplication problems and practices promoting procedural flexibility?

Methods

Data Sources

Two textbooks were analyzed for this study: *Math Connects* (MacMillian-McGraw Hill, 2009) and *Eureka Math* (Great Minds, 2016). These textbooks were selected because of their accessibility to classrooms; *Math Connects* is written by a major U.S. textbook manufacturer, and *Eureka Math* is available for free online. Lessons within each textbook that focused specifically on multi-digit multiplication were selected for analysis. Within those lessons, I coded all independent practice problems for whether they contained multi-digit multiplication. These problems, hereby called math tasks, included both arithmetic problems that students had to solve and worked examples that asked questions about a worked-out problem.

Analytic Framework

To analyze what multiplication strategies and instructional techniques promoting procedural flexibility were promoted in fourth grade mathematics textbooks, I developed two frameworks with which to code math tasks. Math tasks were coded holistically, meaning that both the instructions and the task itself were included in the codes and analysis. Additionally, codes were additive, and a math task could have more than one code from each framework. Codes within the two frameworks were both research-based and data-based (DeCuir-Gunby et al., 2011). An initial list of codes was developed from prior literature. Additional codes that emerged through patterns in the data were then added to the frameworks, and previously coded problems were recoded to align with the updated framework. These frameworks are detailed below.

Strategies for Multi-Digit Multiplication. Math tasks were coded as prompting a specific multi-digit multiplication strategy if the strategy was explicitly included in the instruction (e.g., “Solve this problem using the standard algorithm”) or prompted via a visual scaffold (e.g., inclusion of an area model). Codes generated from prior literature included codes based on memorization (Lampert, 1986) and mental math (Rathgeb-Schnierer & Green, 2018), the use of physical objects when solving problems (Lampert, 1986), the use of area models (Kwon & Son, 2019), and the use of conceptual strategies such as the distributive property and other multiplication properties (e.g., the commutative property, decomposition; Ambrose et al., 2003; Lampert, 1998), and additional abstract strategies (Baek, 2006). Emergent codes included multiplication strategies such as the standard algorithm, using partial products, using a place value chart, or using drawings to solve the problem, all of which were common in the data sources (e.g., Great Minds, 2016; MacMillian-McGraw Hill, 2009) but were not represented in the literature. Finally, repeated addition was added based on the author's prior knowledge of multiplication strategies.

Instruction Promoting Procedural Flexibility. Math tasks coded as including instructions promoting procedural flexibility if there were specific instructions (e.g., "Choose a strategy to solve the problem below") instructing students to *how* to apply multiplication strategies to solve the problem. Initial codes emerging from the literature included asking students to compare strategies (Durkin et al., 2021) and reflect on strategy use (Star et al., 2015; Woodward et al., 2012). Later codes that emerged from the data sources included specifying a strategy for use while solving a problem and choosing between multiple strategies (Great Minds, 2016; MacMillian-McGraw Hill, 2009).

Connection Between Strategies and Procedural Flexibility. To address the third research question, the multiplication strategy framework and the instructional strategy framework were aligned to determine if there were patterns in co-occurrences of multiplication strategies and instructional strategies. It is plausible that some multiplication strategies may align better with certain instructional techniques to promote procedural flexibility. For example, more advanced strategies may be paired with promoting strategy choice because students have a greater repertoire of strategies from which to choose. Likewise, strategies that employ visual scaffolds often promote student reflection on the strategy used to help students align the scaffold with the numerical representation.

Results

Lessons in *Math Connects* ranged from having 21 to 47 multi-digit multiplication tasks, while lessons in *Eureka Math* ranged from having 4 to 19 multi-digit multiplication tasks. There were 302 math tasks in the specified *Math Connects* lessons, compared with 142 tasks in the specified *Eureka Math* lessons. Of those, multi-digit multiplication was included in 266 tasks in *Math Connects* (266 problems, no worked examples) and 140 tasks in *Eureka Math* (138 problems, two worked examples). These were the math tasks used in the analyses that follow.

Research Question 1: Fourth Grade Multiplication Strategies

First, I examined whether the strategies included in the framework appeared in the textbooks. The number of math tasks that explicitly included a multiplication strategy differed between the two textbooks. Overall, 14.3% (n = 38) of multi-digit multiplication tasks in *Math Connects* included multiplication strategies, while 74.3% (n = 104) tasks in *Eureka Math* contained at least one multiplication strategy for student use. Table 1 shows the number and proportion of multiplication strategies that appear in each textbook. Twelve out of the 14 strategies included in the framework were found in at least one of the textbooks. Two strategies, repeated addition and physical objects, were not found explicitly in the tasks in either textbook.

The amount and distribution of multiplication strategies differed between the two textbooks. Within the nine coded lessons in *Math Connects*, four different strategies were explicitly stated, with only two (Memorization and Mental Math) stated more than once. In contrast, among the 14 *Eureka Math* lessons coded, ten different strategies were evident, and all were present multiple times. The types of strategies varied between the two textbooks, as well. *Math Connects* contained a high number of math tasks that prompted students to use memorization (n = 27) or mental math skills (n = 9), strategies that were not stated in *Eureka Math*. The most prevalent strategies in *Eureka Math* were related to decomposition and visual aids, including partial products (n = 45), use of a place value chart (n = 28), and use of an area model (n = 24). These strategies were largely absent from *Math Connects*. *Eureka Math* also suggested that students use the standard algorithm (n = 23), often in conjunction with other strategies, such as using a place value chart. This strategy was similarly absent from *Math Connects*. Finally, the number of math tasks that included multiple strategies differed between the two textbooks, as well. Half of the 104 math tasks in in *Eureka Math* (n = 52) included multiple strategies. This included asking students to choose between strategies, including multiple strategies to solve the problem, or specifying multiple strategies to solve the problem. No math tasks in *Math Connects* included more than one strategy.

Table 1: Number and Percentage of Math Tasks that Suggest Specific Strategies

	Textbook	
Strategy Code	**Math Connects (n = 38 tasks)**	**Eureka Math (n = 104 tasks)**
Memorization	27 (71.1%)	-
Mental Math	9 (23.7%)	-
Standard Algorithm	-	23 (22.1%)
Physical Objects	-	-
Area Model	-	24 (23.1%)
Place Value Chart	-	28 (27.0%)
Other Drawings/Symbols	-	9 (8.7%)
Partial Products	1 (2.6%)	45 (43.3%)
Repeated Addition	-	-
Distributive Property	-	9 (8.7%)
Relational Comparison	-	10 (9.6%)
Decomposition	1 (2.6%)	8 (7.7%)
Additional Abstract Strategies	-	7 (6.7%)
Other Strategies	-	6 (5.8%)

Note. Percentage add up to more than 100 because some tasks prompted students to use more than one strategy.

Creating overall categories of strategies. As strategies were coded, patterns emerged between the categories. Some categories required memorization, some included visual scaffolds, some required the use of underlying conceptual knowledge, and some simply required use of a rote procedure. To facilitate comparison, I created four overarching categories within which strategies were sorted. Strategies were sorted by similarities in knowledge needed by students to apply the strategies. The four categories of Memorization, Procedures, Visual Scaffold, and Conceptual Understanding can be seen in Table 2.

Table 2: Multiplication Strategies Categorized by Knowledge Needed

Category	**Explanations**	**Strategies/Original Codes**

Memorization	Relies on memorized facts or prior knowledge	Memorization, Mental Math
Procedure	Using a rote procedure to solve problem	Standard Algorithm
Visual Scaffold	Requires use of a visual scaffold (either physical object or symbolic)	Area Model, Place Value, Physical Object, Other Drawings/Symbols
Conceptual Knowledge	Incorporates conceptual knowledge or understanding of multiplication	Repeated Addition, Partial Products, Distributive Property, Decomposition, Additional Abstract Strategies
Other Strategies	Strategies that do not fit into one of the four above strategies	Relational Comparison, Other

The overall strategy categories presented in Table 2 were then graphed to display the number of problems within each category presented in *Math Connects* and *Eureka Math*, depicted in Figure 1. The *Math Connects* column shows primarily Memorization strategies; indeed, these strategies make up 94.7% of coded strategies in the textbook (n = 36). In contrast, the Memorization category does not appear explicitly in *Eureka Math*. Instead, most problems coded involve conceptual understanding (n = 69, 66.9%) or present visual scaffolds (n = 54, 51.9%).

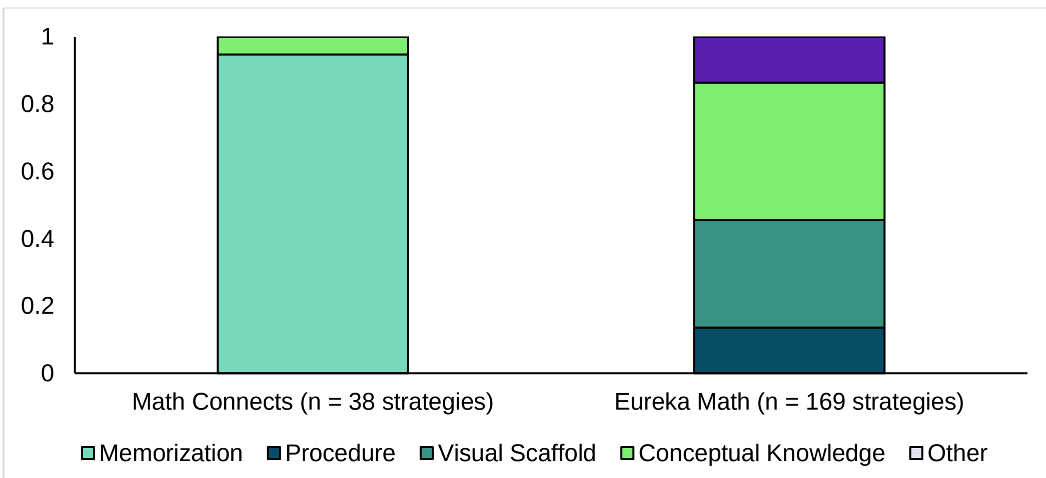


Figure 1: Percentage of Multiplication Strategies within Overall Categories

Note. The number of strategies in *Eureka Math* totaled to more than 104 because some math tasks prompted the use of multiple strategies.

Research Question 2: Instructional Strategies that Promote Procedural Flexibility

I next looked at whether the math tasks incorporated instructional strategies that promoted procedural flexibility. Figure 2 shows the percentage of tasks in each textbook that included problems that might promote procedural flexibility. All instructional techniques included in the framework were seen in the two textbooks; however, the percentages of instructional strategies differed. Within *Math Connects,* 28.3% (n = 47) of tasks employed instructional strategies that may promote procedural flexibility, compared with 69.0% (n = 98) of tasks in *Eureka Math.* In both textbooks, the most frequently used instructional component was to specify a strategy for students to practice (e.g., use the standard algorithm to solve this problem). The distribution of the other types of instructional techniques between the two textbooks was similar, with little variability between the techniques.

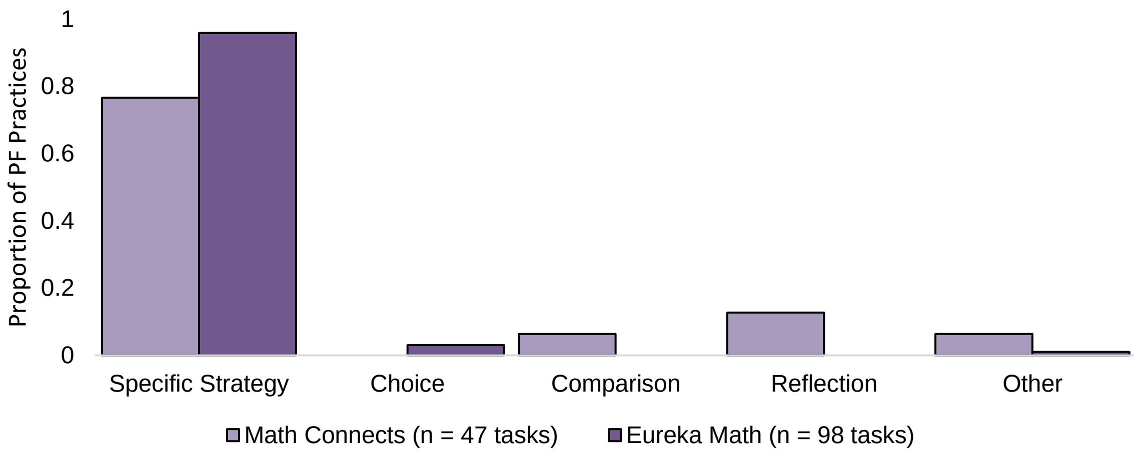


Figure 2: Number of Instructional Techniques that Promote Procedural Flexibility

Research Question 3: Alignment of Multiplication Strategies and Procedural Flexibility

Finally, I looked at whether there were relationships between instructional strategies multiplication strategies. Table 3 shows the alignment between these two axes in both textbooks. Overall, there was not a certain strategy that was paired a certain instructional strategy. However, the two strategies that were paired with reflecting on strategy (partial products and decomposition) in *Math Connects* were categorized as Conceptual Strategies. Interestingly, three tasks in *Eureka Math* promoted students to choose between four different strategies (standard algorithm, partial products, student-created area model, and distributive property).

Table 3: Alignment Between Instructional Strategies and Multiplication Strategies

Instructional Strategy	*Math Connects* Multiplication Strategies	*Eureka Math* Multiplication Strategies
Specify Strategy	Memorization (n = 36)	Conceptual Strategies (n = 61) Visual Scaffold (n = 56) Procedural (n = 20) Other (n = 12)
Choice	-	Conceptual Strategies (n = 6) Visual Scaffold (n = 3) Standard Algorithm (n = 3) Other (n = 1)
Comparison	No Strategy (n = 1)	-
Reflection	No Strategy (n = 6) Conceptual Strategies (n = 2)	-

Discussion

The current study explored what multiplication strategies and instructional strategies promoting procedural flexibility are presented in fourth grade mathematics textbooks and whether the presentation of multiplication strategies and instructional strategies were related. Using novel frameworks developed through prior literature and patterns in the data, I found that most multiplication and instructional strategies included in the frameworks were found in at least one of the two textbooks. Some multiplication strategies included in the framework, such as repeated addition and using physical objects, were not explicitly included in either textbook. Additionally, there was not much variability in the instructional techniques, nor was there overall alignment between multiplication strategies and included and instructional techniques to promote procedural flexibility. These results illustrate what can be learned from applying these frameworks to examine curriculum materials.

The frameworks illuminated great differences between the two textbooks coded. The majority of math tasks coded in *Eureka Math* prompted students to use at least one multiplication strategy when solving problems, yet over 85% of math tasks coded in *Math Connects* did not specify a strategy, instead often asking students simply to “Multiply.” Additionally, although both textbooks overwhelmingly used the “Specify Strategy” instructional technique, the other instructional strategies promoted in the two textbooks differed, with tasks in *Eureka Math* prompting student strategy choice and tasks in *Math* prompting reflection and comparison. Interestingly, neither textbook showed a strong alignment between multiplication strategies and instructional techniques for procedural flexibility. This was likely because there was not much variety in instructional techniques, as both textbooks primarily promoted the “Specify Strategy” technique and did not have many tasks that promoted other techniques. Although presenting multiple strategies for solving multiplication problems might inadvertently promote procedural flexibility, pairing this with instructional strategies that explicitly promote procedural flexibility might lead to greater gains in this skill. Future research might investigate whether this alignment is present in other parts of the textbook or within parts of the enacted lesson.

Limitations and Future Directions

A limitation of this study is that only two textbooks were used to demonstrate the viability of the constructed frameworks. Using a broader variety of textbooks as well as including more recent textbooks as data sources for the developed frameworks would give a more complete picture of what multiplication and instructional strategies are presented in fourth grade mathematics textbooks. Additionally, only independent practice problems were considered in the described analyses. Multi-digit multiplication problems also appeared in other parts of the lessons, such as parts developed for whole-class instruction. Future studies should consider analyzing these textbook features, as well as prompts included in teachers’ guides, to analyze how procedural flexibility is promoted through curriculum enactment.

Conclusion

Procedural flexibility is an important, yet often underemphasized, facet of mathematics learning. This skill can be emphasized explicitly in textbooks, and teachers may be able to further emphasize it through instruction. Small changes in the multiplication and instructional strategies seen in textbooks would likely be beneficial to students in helping them think flexibly and adaptively about problem solving. In learning flexible thinking at an early age, students may be better able to think flexibly about problem solving throughout their mathematics careers.

References

Ambrose, R., Baek, J., & Carpenter, T. P. (2003). Children's invention of multidigit multiplication and division algorithms. In A. J. Baroody & A. Dowker (Eds.), *The Development of Arithmetic Concepts and Skills: Constructing Adaptive Expertise.* Lawrence Erlbaum Associates, Inc., Publishers: Mahwah, NJ.

Baek, J. M. (2006). Children's mathematical understanding and invented strategies for multidigit multiplication. *Teaching Children Mathematics, 12*(5), 242-247.

Common Core State Standards (2016). Grade 4, Numbers & Operations in Base Ten. [link]

DeCaro, M. S. (2016). *Inducing mental set constrains procedural flexibility and conceptual understanding in mathematics*. 11.

DeCuir-Gunby, J. T., Marshall, P. L., & McCulloch, A. W. (2011). Developing and using a codebook for the analysis of interview data: An example from a professional development research project. *Field Methods*, *23*(2), 136–155.

Digital Promise Global (2021). Learner Variability Navigator: Mathematical Flexibility. [link]

Durkin, K., Rittle-Johnson, B., Star, J. R., & Loehr, A. (2021). Comparing and discussing multiple strategies: An approach to improving algebra instruction. *The Journal of Experimental Education,* 1-19.

Durkin, K., Star, J. R., & Rittle-Johnson, B. (2017). Using comparison of multiple strategies in the mathematics classroom: Lesson learned and next steps. *ZDM Mathematics Education, 49*, 585-597.

Great Minds (2016). Eureka Math, Grade 4, Teacher Edition, Module 3. [link]

Kwon, O. H., & Son, J. W. (2019). Revisiting Multiplication Area Models for Whole Numbers. *The Mathematics Enthusiast, 16(*1, 2 & 3), 359.

Lampert, M. (1998). Knowing teaching from the inside out: Implications of inquiry in practice for teacher education. *Yearbook-National Society for the Study of Education, 1*, 167-184.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction, 3*(4), 305-342.

Lloyd, G. M., Cai, J., & Tarr, J. E. (2017). Issues in curriculum studies: Evidence-based insights and future directions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 824-852).

MacMillian-McGraw Hill (2009). Math Connects: Grade 4, Teacher Edition.

National Center for Education Statistics (2021). The NAEP Mathematics Achievement Levels by Grade. [link]

National Research Council (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: The National Academies Press.

Newton, K. J., Lange, K., & Booth, J. L. (2020). Mathematical flexibility: Aspects of a continuum and the role of prior knowledge. *The Journal of Experimental Education*, *88*(4), 503–515.

Rathgeb-Schnierer, E., & Green, M. (2018). Developing flexibility in mathematical calculation. *Educação & Realidade, 44*(2).

Rittle-Johnson, B., Schneider, M., & Star, J. R. (2015). Not a one-way street: Bidirectional relations between procedural and conceptual knowledge of mathematics. *Educational Psychology Review, 27*(4), 587-597.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2020). How can cognitive-science research help improve education? The case of comparing multiple strategies to improve mathematics learning and teaching. *Current Directions in Psychological Science, 29*(6), 599-609.

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental psychology, 47*(6), 1525.

Star, J. R. (2005). Re-conceptualizing procedural knowledge. *Journal for Research in Mathematics Education, 36*(5), 404-411.

Star, J. R., & Newton, K. J. (2009). The nature and development of experts' strategy flexibility for solving equations. Zentralblatt Didaktik für Mathematik (ZDM), 41(5), 557-567.

Star, J. R., Newton, K., Pollack, C., Kokka, K., Rittle-Johnson, B., & Durkin, K. (2015). Student, teacher, and instructional characteristics related to students' gains in flexibility. *Contemporary Educational Psychology, 41*, 198-208.

Verschaffel, L., Torbeyns, J., De Smedt, B., Luwel, K., & Van Dooren, W. (2007). Strategy flexibility in children with low achievement in mathematics. *Educational & Child Psychology, 24*(2), 16-27.

Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2012). "Improving Mathematical Problem Solving in Grades 4 through 8. IES Practice Guide. NCEE 2012-4055.*" What Works Clearinghouse.*

HOW PROPORTIONAL REASONING IS PRESENTED IN U.S. AND KOREAN MATHEMATICS TEXTBOOKS

Ji-Won Son
University at Buffalo - SUNY
jiwonson@buffalo.edu

Seungjung Jo
SUNY Geneseo
Seungjung.jo@gmail.com

Samantha Wallace
University at Buffalo - SUNY
Sw232@buffalo.edu

Florence F Obielodan
University at Buffalo - SUNY
ffobielo@buffalo.edu

The purpose of this study is to examine how proportional reasoning is introduced and developed in two widely used U.S. and Korean mathematics textbooks for grades 6-7. Seven research-based frameworks that identify student learning opportunities for understanding of proportional reasoning were used to analyze the textbooks. The results showed that American textbooks include more problems that require explanations and make use of more effective contextual and number structure of problems than Korean textbooks. In contrast, Korean textbooks make a shift from providing highly contextualized problems to presenting abstract and purely computational problems, which aligns with the process of concreteness fading. In addition, Korean textbooks contain more unique types of topics and representations.

Keywords: Proportional reasoning, textbook analysis, cognitive demands of math tasks

Proportional reasoning has been recognized as a key concept in mathematics for middle grades students to develop (Common Core State Standards Initiative, 2011). However, research indicates that students have considerable difficulties understanding proportional reasoning because they tend to apply additive or subtractive thinking processes rather than multiplicative processes (Karplus, Pulos, & Stage, 1983). Students' difficulties with proportional reasoning may be largely attributed to the quality of their learning environments, such as textbooks that influence what is to be taught and what students learn (Alajmi, 2009; Stigler, Fuson, Ham, & Kim, 1986; Weiss, Pasley, Smith, Sanilower, & Heck, 2003). Given the important role of textbook in mathematics teaching and learning, this study, focusing on the case of proportional reasoning, examines learning opportunities presented in representative American and Korean textbooks. The study aims to compare various aspects of the structure and sequence of the lessons on proportional reasoning, and the characteristics of the problems presented in the lessons in American and Korean textbooks. Specifically, the study addressed the following questions: (1) When and how are ratios, rates, and proportional reasoning introduced and developed in American and Korean textbooks?; (2) What similarities and differences are observed in the content of ratios, rates, and proportional reasoning in American and Korean textbooks?

Theoretical perspectives

Student difficulties and recommended strategies for proportional reasoning

Proportional reasoning has been seen as a cornerstone of secondary mathematics curricula because it is important for understanding of percentages, gradient, trigonometry, and algebra

(NCTM, 2006). Accordingly, students' concepts of proportion have long been a focus of mathematics education research and have been explored about students' errors and difficulties in relation to proportional reasoning tasks (Lo & Watanabe, 1997). One of the roots of difficulties with proportional reasoning is that students have a weak understanding of the part/whole relationships described in fraction notation. Students often struggle with solving some proportional reasoning problems that involve the use of fraction notations (Norton, 2006).

Research has consistently emphasized students' difficulties with proportion and proportion-related tasks and applications and explored pedagogical ways to improve students' development of proportional reasoning (Behr, Harel, Post & Lesh, 1992; Lo & Watanabe, 1997). First, research has recommended to contextualize problems in real-life situations, as it can activate students' familiar experiences and informal knowledge for sense-making (Resnick & Omanson, 1987). Representing proportion concepts by using various models rather than numbers and symbols may increase students' conceptual understanding. Thus, concreteness fading, which refers to the process of beginning with concrete representations and then fading into more abstract ones, is found to be effective in developing students' conceptual understanding (Goldstone & Son, 2005). In addition, providing problems with high levels of cognitive demand gives students more opportunities to think and reason in given mathematical tasks. Research indicates that using high-level and cognitively complex tasks is important to develop the capacity to think, reason, and solve problems (Stein & Lane, 1996). Furthermore, it is also recommended to provide proportion tasks in a wide range of contextual (e.g., part-part-whole, scaling, well-chunked) and number (e.g., integer or non-integer answers) structures so that students can apply multiplicative thinking into various types of situations (Lamon, 1999). We use these research-based instructional strategies to identify student learning opportunities for understanding of proportional reasoning.

Textbook comparison

Analyses of mathematics textbooks have examined textbooks across countries and have brought many alternatives and insights to the field for improving instruction on challenging mathematical ideas. Prior international comparative studies have shown that curricula in Asian countries contain more tasks that are framed in concrete and real-life situations and provide more cognitively difficult problems, compared to the U.S. For example, Murata (2008) examined the presentation of addition and subtraction in the U.S. and Japanese textbooks and found that Japanese textbooks included more contextualized problems than the US textbooks, which mainly utilized computation problems. Similarly, Ding and Li (2010) compared Chinese textbook series with the two U.S. series on the topic of distributive property and found that the main problem context was computation problems in the U.S. textbooks, whereas it was word problems in the Chinese textbooks. Son and Senk (2010) also compared Korean and American textbooks with standards-based and traditional American textbooks and found that Korean textbooks include more problems that required students to explain than American textbooks. However, some studies have shown inconsistent results [see Fan and Zhu (2007), Li (2000), Hong and Choi (2014), Son and Senk (2010)]. While some studies revealed that American textbooks contained more problems with higher level cognitive demand, problems that required students to provide explanation, and multiple representation than either Chinese or Korean textbooks, other studies reported different findings. Examining whether these findings are consistent with the results of the previous international comparative textbook studies in the present study will enhance the current understanding of what students learn in the U.S. and Korea.

Methods

Representative and widely used American and Korean textbooks were chosen for this international comparative analysis. For American textbooks, *Eureka Math* or Engage NY modules (EM) (www.engageny.org) were chosen for its popularity (Opfer, Kaufman, & Thompson, 2016). There is only one set of textbook series developed on the national Korean curriculum standards by the Ministry of Education (KM).

The textbook analysis in this study focused on two aspects of textbooks: (1) the structure of the lessons and topics, and (2) the nature of the problems. For the analysis of the textbooks' structure of the lessons and topics, the introduction and development of the concepts of ratio, rate, and proportional reasoning as well as topics arrangement were examined. The analytical framework shown in **Table 1** was utilized to analyze the nature of the problems in depth. The analytical framework consists of the following seven categories: concrete fading (Ding & Li, 2010), cognitive demand (Stein, Smith, Henningsen & Silver, 2000), perspectives (Beckmann & Izsak, 2015; Shield & Dole, 2012; Thompson, 1994), task types (Cramer, Post, & Currier, 1993), contextual and number structure (Lamon, 1993), response types (Charalambous et al., 2010; Mayer et al., 1995), and problem solving difficulty (Hsu & Silver, 2014).

Table 1. Categories and subcategories of analytical framework

Analytical framework	Subcategories
1. Concreteness Fading	Word Problem Visual Representation Word Problem with Visual Representation Abstract
2. Cognitive Complexity	Memorization Procedures without Connections Procedures with Connections Doing Mathematics
3. Perspectives	Multiple Batches Variable Parts
4-a. Contextual Structure	Well-chunked Part-part-whole Associated Sets Stretcher/Shrinker Symbolic
4-b. Number Structure	I-I-I I-W-I I-B-I I-B-N N-B-N N-N-I N-N-N

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

4-c. Response Type	Numerical Answers Only Algebraic Expressions Required Explanation Required
5. Problem-solving Difficulty	Easy Medium Hard

Problems are coded as instances, according to the definitions of each analytical framework. After coding, we counted the frequency of all the problems in each subcategory of the analytical frameworks. A Microsoft Excel spreadsheet was used to record the frequency and percentage. The number and percentage of problems that were demonstrated in each subcategory were recorded in the spreadsheet. The counts and percentages of problems were summed, and are reported in the Findings section.

Summary of findings

The nature of problems with ratio, rate, and proportional reasoning

Table 2 presents the total number of problems in both textbooks counted. In total, there are 679 problems and 236 problems in EM and KM, respectively. Further, when the frequency and percentage distribution of total problems were categorized per concept, EM present percent problems most frequently and ratio problems least frequently. In contrast, KM include proportional reasoning problems most frequently and percent problems least frequently.

Table 2. Total frequency of problems in Korean and EM

EM (n=679)						KM (n=236)					
Grade (Module)	Ratio (%)	Rate (%)	Percent (%)	Proportional reasoning (%)	Total	Grade (Vol.)	Ratio (%)	Rate (%)	Percent (%)	Proportional reasoning (%)	Total
6 (1)	80 (34)	109 (47)	45 (19)	0 (0)	234 (100%)	6 (1)	48 (53)	25 (28)	17 (19)	0 (0)	90 (100%)
7 (1)	15 (9)	28 (16)	0 (0)	133 (75)	176 (100%)	6 (2)	5 (6)	0 (0)	0 (0)	80 (94)	85 (100%)
7 (4)	0 (0)	0 (0)	269 (100)	0 (0)	269 (100%)	6 (2)	0 (0)	0 (0)	0 (0)	61 (100)	61 (100%)
	95 (14)	137 (20)	314 (46)	133 (20)	679 (100%)		53 (22)	25 (11)	17 (7)	141 (60)	236 (100%)

Concreteness fading and visual representation types

Table 3 shows the frequency and percentage of concrete and abstract problems in EM and KM. The results showed that both textbooks provide most problems in concrete contexts (e.g., EM: 93% and KM: 81%). For abstract problems, only 7 % of the total problems in EM were situated in purely mathematical contexts, while 19 % in KM were framed in abstract contexts. It may seem that EM used more concrete problems than KM. However, it should be noted that the majority of problems in KM was word problems with visual representation, whereas the greater part of problems in EM was word problems. In KM, there were 45 % word problems with visual representation, 33% word problems, and 3% visual representation problems. In EM, 50 % word problems, 28% word problems with visual representation, and 15 % visual representation problems. This may indicate that KM situate the majority of their problems in concrete situations by using visual representation, while EM contextualize their problems through word problems.

In addition, the process of concreteness fading is not obvious in EM. Although concrete representations outnumbered abstract ones in EM, word problems were used most frequently (50%) followed by word problems with visual representation (28%) and visual representation problems (15%). This trend indicates that the frequency of concrete representation types used may not necessarily indicate the transfer from concreteness to abstractness. In contrast, the frequency of concrete representation types used in KM decreases in the following order: word problem with visual representation (45%) - word problem (33%) - visual representation problem (3%). This may show that there is a concreteness fading process within the concrete problems in KM. Moreover, given that KM contained 45% word problems with visual representation, 33 % word problems, and 19% abstract problems, there was a gradual fading process from concrete to abstract representations across problem types. Research shows that although providing learning opportunities in more concrete representations may activate students' familiar experiences for sense-making (Resnick & Omanson, 1987), making connections between concrete and abstract representations than just using concrete representations is found to be more effective in developing students' conceptual understanding (Goldstone & Son, 2005). This may imply that KM may be more advantageous in facilitating students' conceptual development on such abstract concept as proportional reasoning.

Table 3. The frequency and percentage of concrete and abstract problems

	Concrete				
	Word problem (%)	Visual representation (%)	Word problem with visual representation (%)	Abstract (%)	Total (%)
			EM		
Grade 6 M1	98 (42%)	37 (16%)	70 (30%)	29 (12%)	234 (100%)
Grade 7 M1	65 (37%)	28 (21%)	72 (41%)	2 (1%)	176 (100%)
Grade 7 M4	177 (65%)	28 (10%)	51 (19%)	16 (6%)	269 (100%)
Total	340 (50%)	103 (15%)	193 (28%)	47 (7%)	679 (100%)
			KM		
Grade 6 Vol.1	38 (42%)	5 (6%)	37 (41%)	10 (11%)	90 (100%)
Grade 6 Vol. 2	25 (29%)	1 (1%)	21 (25%)	38 (45%)	85 (100%)
Grade 6 Vol. 2	15 (25%)	1 (1%)	45 (74%)	0 (0%)	61 (100%)
Total	73 (33%)	7 (3%)	98 (45%)	47 (19%)	236 (100%)

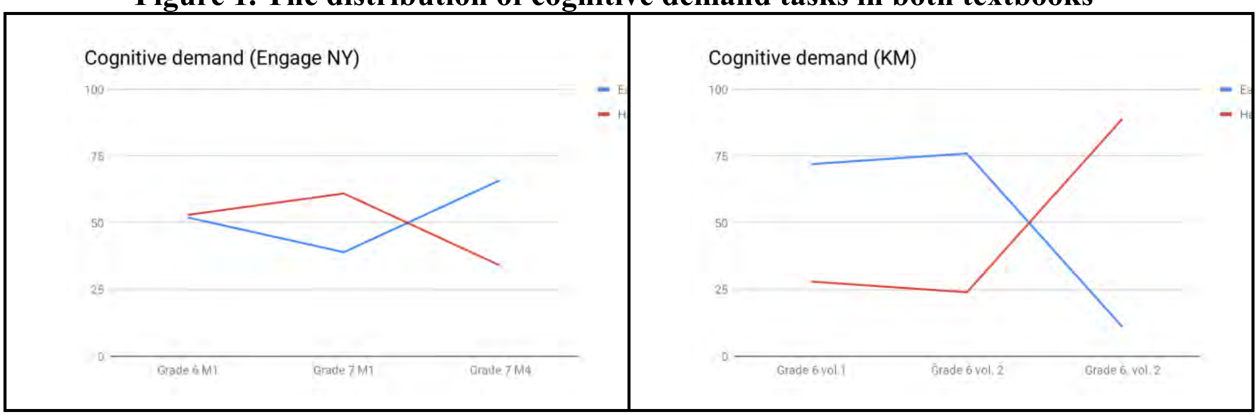
Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

Cognitive demand and problem-solving difficulty

The majority of the tasks is procedures without connections and procedures with connections in both textbooks. Problems with the highest level cognitive demand, doing mathematics, are little represented in both textbooks. EM have a higher percentage of low cognitive demand tasks (54%) than that in KM (42%). This result does not align with the finding from Hong and Choi (2013) reporting that the majority of problems in American and KM require lower level cognitive demand and more than 80% of problems only require simple algorithms or formulas.

Note that EM have more “procedures without connections” (43%) than “procedures with connections” (37%), while KM have more “procedures with connections” (39%) than “procedures without connections” (36%). This may show that KM contain more cognitively challenging problems that require students to use their understanding of concepts and underlying principles and procedures. However, it also should be noted that the problems in KM are generally either a problem with a series of easy subproblems. For example, the subproblems require students to follow at least four steps to complete the problem in EM, while the subproblems in KM ask for one step to complete the problem. Also, based on the analysis of problems in terms of problem solving difficulty, the majority of problems in KM (88%) are easy-level, which requires only one step to complete the problem, whereas EM contain four times more problems that are at least medium-level (49%), which consisted of two or four steps, than KM (12%). This indicates that EM are expected to complete tasks with more steps than Korean students.

Figure 1. The distribution of cognitive demand tasks in both textbooks



Perspectives: Multiple Batches vs Variable Parts

Table 4 illustrates the percentage distribution of problems based on the perspective of multiple batches and variable parts drawn from Beckman and Izsak (2015). A higher percentage of problems that utilized the multiple batches perspective than the variable parts perspective in both textbooks. EM included a higher percentage of problems with the multiple batches perspective (61%) than KM (24%). By contrast, KM included a higher percentage of problems that utilized both the multiple batches and variable parts perspective (36%) than EM (5%). This may show that EM focus on the development of the multiple batches perspective, while KM intend to develop both perspectives. Different from Beckman, the results of our study show that KM, as the curriculum of one of the mathematically high-performing countries, not only utilized the multiple batch perspective, but also the variable parts perspective in developing students’ proportional reasoning ability through providing the problems that utilized both perspectives.

Table 4. Frequency and percentage of multiple batches and variable parts perspectives

		EM			
Grade	Multiple Batches	Variable Parts	Both	Neither	Total
6 (1)	95 (41%)	13 (6%)	29 (12%)	97 (41%)	234 (100%)
7 (1)	93 (53%)	0 (0%)	0 (0%)	83 (47%)	176 (100%)
7 (4)	229 (85%)	7 (3%)	3 (1%)	30 (11%)	269 (100%)
Total	417 (61%)	20 (3%)	32 (5%)	210 (31%)	679 (100%)
		KM			
6 (1)	25 (28%)	0 (0%)	35 (39%)	30 (33%)	90 (100%)
6 (2)	3(4%)	6 (7%)	49 (57%)	27 (32%)	85 (100%)
6 (2)	52 (85%)	0 (0%)	2 (3%)	7 (12%)	61(100%)
Total	80 (24%)	6 (3%)	86 (36%)	64 (27%)	236 (100%)

Contextual and number structure

We further explored contextual and number structure in missing value problems to explore the learning opportunities for proportional reasoning concepts. We found that the main contextual structure of missing value problems in EM is the stretcher/shrinker problems (47%), whereas the symbolic problems (1%) were minimally represented. In KM, the majority of problems are well-chucked (32%) and part-part-whole (30%) problems, while the least frequently represented problems were associated sets (9%). In EM, all four semantic types were evenly utilized in grade 6. The well-chunked and stretcher/shrinker type were most frequently utilized in grade 7. This may suggest that EM began with a balance of all four semantic types and then moved to the stretcher/shrinker problem type in missing value problems. In contrast, KM initially used the stretcher/shrinker problem type, and then heavily relied on using the part-part-whole and the well-chunked problem type. This finding may show that that EM utilized more appropriate contextual structures of their missing value problems than KM, as their students develop conceptual understanding of proportional reasoning, based on the level of difficulty.

Table 5. Percentage of contextual structure of missing value problems in both textbooks

			EM (n=192)			
Grade (Module)	Well Chunked (%)	Part-Part-Whole (%)	Associated Sets (%)	Stretcher /Shrinker (%)	Symbolic (%)	Total (%)
6 (1)	28 (32)	18 (21)	18 (21)	20 (23)	3 (3)	87 (100)

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

7 (1)	29 (60)	5 (11)	1 (2)	13 (27)	0 (0)	48 (100)
7 (4)	0 (0)	0 (0)	0 (0)	57 (100)	0 (0)	57 (100)
Total	57 (30)	23 (12)	19 (10)	90 (47)	3 (1)	192 (100)
KM (n=90)						
Grade (Vol)	Well Chunked (%)	Part-Part-Whole (%)	Associated Sets (%)	Stretcher /Shrinker (%)	Symbolic (%)	Total (%)
6 (1)	1 (5)	7 (31)	1 (5)	13 (59)	0 (0)	22 (100)
6 (2)	6 (14)	20 (45)	6 (14)	3 (7)	9 (20)	44 (100)
6 (2)	22 (92)	0 (0)	1 (4)	1 (4)	0 (0)	24 (100)
Total	29 (32)	27 (30)	8 (9)	17 (19)	9 (10)	90 (100)

Discussion and Implications

This study showed notable similarities but also striking differences between EM and KM. The goal of cross-cultural comparison is to know in what measure the learning opportunities provided by textbooks get enacted in classroom practice and student learning. Our findings indicated that how this comparative study of textbooks may contribute insights to improve the learning environments of proportional reasoning. Korean textbooks' emphasis on the process of concreteness fading and skillful use of high level cognitive demanding problems are consistent with prior findings that Asian students are involved in more meaningful and desirable material to learn mathematics. Korean approaches in developing students' explicit understanding, such as the unique construction of lessons, may be helpful for textbook designers in America and other countries. Our study also showed that EM provide more opportunities for students to solve mathematics problems with complex number structures, but also to explain and reason about the problems than KM. EM encourage their students to be independent in solving mathematics problems by asking them to create visual representations to justify their reasoning. These findings seem to conflict with the findings that Asian textbooks present more problems requiring explanation and problems with multiple visual representations than American textbooks. Developers of KM can benefit from EM' approaches in using various strategies, such as a wider range of number structures and visual representations, and stressing more critical thinking.

Reference

Alajmi, A. H. (2009). Addressing computational estimation in the Kuwaiti curriculum: teachers' views. Journal of Mathematics Teacher Education, 12(4), 263-283.

Alajmi, A. H. (2012). How do elementary textbooks address fractions? A review of mathematics textbooks in the USA, Japan, and Kuwait. Educational Studies in Mathematics, 79(2), 239-261.

Beckmann, S., & Izsák, A. (2015). Two perspectives on proportional relationships: Extending complementary origins of multiplication in terms of quantities. Journal for Research in Mathematics Education, 46(1), 17-38.

Common Core State Standards Initiative. (2010). National Governors Association Center for Best Practices and Council of Chief State School Officers. Retrieved December, 11, 2012.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. Research ideas for the classroom: Middle grades mathematics, 159-178

Fan, L., & Zhu, Y. (2007). Representation of problem-solving procedures: A comparative look at China, Singapore, and US mathematics textbooks. Educational Studies in Mathematics, 66(1), 61-75.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. The Journal of the Learning Sciences, 14(1), 69-110.

Jeong, Y., Levine, S. C., & Huttenlocher, J. (2007). The development of proportional reasoning: Effect of continuous versus discrete quantities. Journal of Cognition and Development, 8(2), 237-256.

Lamon, S. (1999). Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers. Mahwah, NJ: Lawrence Erlbaum.
National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: National Council of Teachers of Mathematics.
Norton, S. (2006). Pedagogies for the engagement of girls in the learning of proportional reasoning through technology practice. Mathematics Education Research Journal, 18(3), 69-99.
Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. Mathematics teaching in the middle school, 3(4), 268-275.
Shield, M., & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. Educational Studies in Mathematics, 82(2), 183-199.

INFLUENCE OF CONTEXT ON TEACHERS' ASSESSMENT PRACTICES

Kamie K. Stack
University of Minnesota
stack180@umn.edu

Erin E. Baldinger
University of Minnesota
eebaldinger@umn.edu

The process of assessing students is a fundamental part of teaching and learning mathematics. The assessment practices a teacher chooses are shaped by their values while also being shaped by the context of the school, district, state, and country where the teaching takes place. This can result in gaps between teachers' values and practices. In this study, we use student work sample interviews with five secondary mathematics teachers to illustrate their values around assessment, the factors that influence their assessment practices, and how their agency influences their assessment decisions. We focus on the important role contextual factors can play in shaping teachers' agency and assessment choices. These findings have implications for teacher education and further research around how assessments are used.

Keywords: Assessment, teacher beliefs

The process of assessing students is a fundamental part of teaching and learning mathematics. The assessment practices a teacher chooses are shaped by their values while also being shaped by the context of the school, district, state, and country where the teaching takes place. Enacting high-quality assessment practices has been identified as a key factor in preparing future mathematics teachers (Association of Mathematics Teacher Educators, 2017). Prior research has highlighted teachers' assessment practices and their values related to assessment within and outside of mathematics education (e.g., Barnes et al., 2014; Beswick, 2011; Brown, 2004; Davis & Neitzel, 2011; Remesal, 2007). Just as there may be gaps between mathematics teachers' values and instructional practices more generally (Beswick, 2011), such gaps may extend to assessment practices in particular. Despite this, teachers' assessment practices are used as factors in measuring teaching effectiveness (e.g., Sato, 2014; Sato et al., 2008).

Given these conditions, researchers in the Network for Excellence in Teaching (NExT) sought to investigate multiple measures of teaching effectiveness (NExT Teacher Effectiveness Work Group, 2018). One such measure built on D'Souza's (2012) Teacher Assessment/Pupil Learning Protocol (TAPL) which was used to examine assessment practices and how they evolve with early career high school English teachers over a period of five years. Using D'Souza's TAPL, NExT created a student work sample interview protocol to measure teacher effectiveness across disciplines. As part of our participation in NExT, we implemented the student work sample interview with recent graduates of the secondary mathematics licensure program in a Midwest University.

In a student work sample interview, teachers select student work from a recent assessment they implemented and then discuss the pieces of student work with an interviewer. Through piloting this interview protocol, we gathered important data for the NExT consortium and the teacher preparation program. Beyond programmatic evaluation, we also noticed interesting patterns related to teachers' values and assessment practices. We therefore engaged in a process of open coding, drawing on the principles of grounded theory (Charmaz, 2006). We arrived at two research questions: (1) What factors influence how secondary mathematics teachers enact assessment practices? (2) How does their enactment align with their values about assessment?

Literature Review

In this section, we highlight relevant literature related to assessment and teacher agency. These two constructs are key to understanding the themes that emerged from our interviews.

Assessments Serve Multiple Purposes for Multiple Audiences

Assessments generate artifacts that serve multiple purposes for multiple audiences (Davis & Neitzel, 2011; Remesal, 2007). In addition to students and teachers, audiences for assessment artifacts include parents, caregivers and what Davis and Neitzel (2011) call the "higher-ups"—people such as school administrators, district leaders, and policy makers. With multiple and potentially conflicting audiences come multiple conceptions of the purposes of assessment. Teachers' conceptions exist along a continuum from a focus on the pedagogical purposes of assessment to a focus on the accountability purposes of assessment (Barnes et al., 2014).

At the pedagogy end of the continuum, students and teachers are the primary audiences for assessments. Pedagogical purposes of assessment include informing instruction, facilitating learning, and providing evidence of student understanding. For students, assessments align with the content in the curriculum (Davis & Neitzel, 2011), demonstrate their progress toward learning goals (Karp & Woods, 2008), and offer an opportunity to receive feedback. On the alternative extreme of the continuum, the "higher-ups" are the driving audience. Here, assessments are used as measures for teacher and school accountability (Brown, 2004). As an example, teachers may choose to use classroom assessments as intentional preparation for high stakes, standardized assessments to adhere to expectations from the "higher-ups" audience and their focus on accountability (Davis & Neitzel, 2011). In this example, the audience of the "higher-ups" and their beliefs about the purpose of assessment influence classroom assessment practices. The accounting role of assessment is punctuated in mathematics education because most students in the United States take a high stakes mathematics assessment at the end of every school year starting in third or fourth grade.

Enacting Assessment Values

Teacher values around assessment, including where such conceptions might fall along the continuum, influence classroom decisions (Biesta et al., 2015). However, a teacher may hold competing goals and values for their approach to assessment (Thomas & Yoon, 2014). For instance, a mathematics teacher may value teaching practices that support a deep conceptual understanding of essential learning goals. They may also value preparing their students for success on high stakes assessments. The first value may require teachers to allocate extended instructional time to a single learning goal. The second value may require teachers to follow a particular curriculum or set of standards within a specific time frame. These two values may come into conflict when students need more time to master a particular learning goal but also need to move on to the next learning goal to complete the curriculum before the high stakes assessment. How a teacher decides which value to prioritize depends on teacher agency. Teacher agency is the degree to which teachers can enact their values in classroom practices within the culture and context of the school (Biesta et al., 2015).

Teacher agency is situated by context and a complex relationship between educational theory, practice, and environment (Biesta et al., 2015). Thomas and Yoon (2014) found that teachers prioritized their pedagogical values when making instructional decisions until one of three conflicting values came into play: (1) requirements about time, curriculum, or assessment; (2) potential success for future learning; and (3) respect for students' cultures. For instance, a secondary mathematics teacher who was committed to student-centered learning might not enact that value when it came in conflict with preparing students for assessments by completing the national curriculum in the available time. In this case, the teacher switched to teacher-led instruction and limited the time allocated for a specific learning goal, despite their personal feelings that the concept was important, because it was not part of the required curriculum.

When there is tension between conflicting assessment values, particularly between pedagogical and accountability purposes, teachers may feel less agency around the accountability purposes of assessment. As a result, teachers may prioritize accountability purposes even to the detriment of their pedagogical values. Teachers' choices related to navigating conflicting values are influenced by the amount of agency they perceive in decision making. In this study we connect the research by Barnes et al. (2014), Thomas and Yoon (2014) and Biesta et al (2015) to discuss agency and how it relates to assessment values and practices with early career teachers in mathematics.

Methods

This study emerged as a result of piloting the student work sample interview protocol from the NExT digital handbook (NExT Teacher Effectiveness Work Group, 2018). The interview protocol was based on D'Souza's (2012) case study of early career teachers' assessment practices. D'Souza found that a student work sample interview protocol encouraged reflective practices with early career teachers that supported growth and development around assessment practices. Those findings motivated our choice to pilot this interview protocol, in particular because we were interested in how a protocol designed to work across disciplines might be used in the context of early career mathematics teachers. Unlike D'Souza, who implemented the TAPL with participating teachers routinely over five years, we piloted the NExT student work sample interview protocol once with each of our participating mathematics teachers.

Data Collection

Participants for this study were drawn from the approximately 40 secondary mathematics teachers who were in their first three years of teaching and had graduated from the secondary mathematics teacher licensure program at a Midwest university. All 40 teachers were invited to participate in the study and five teachers elected to participate. Four teachers taught at middle or high schools in the same Midwest state as the licensure program; the fifth taught in a school in the southeast United States.

Each teacher selected a recent assessment they had used in their mathematics courses and submitted a set of student work samples to the researchers prior to an individual interview. The only parameters for the teachers in this study was to select an assessment that had enough student work to drive a conversation about the learning that was present. The selected assessments ranged from two questions on a single skill to a summative unit test or project. The student work samples were graded and represented a range of student mastery.

The semi-structured interview included five main sections: (1) context questions focused on understanding the school setting, (2) description of the assessment, (3) discussion of student work samples, (4) implications for advancing student learning, and (5) a final reflection. The context questions asked teachers to share anything they felt was important for the researchers to know about their students and school. In the second section, teachers described the assessment and the learning activities that led to or followed the assessment. The majority of the interview time was dedicated to the third section, reviewing and analyzing the student work samples. During this portion of the interview, teachers were asked to organize the student work into three categories: *demonstrated mastery, approaching mastery,* and *still needs support*. The teachers then described what they noticed in each sample and explained why they categorized the student work as they did. Teachers were then asked about the next steps they wanted to take to further student learning. Finally, each interview concluded with an opportunity to reflect on the experience of participating in a student work sample interview, as well as any reflections about the licensure program. Each interview lasted approximately 60-90 minutes. All interviews were video recorded and transcribed for analysis.

Data Analysis

To analyze the interviews, we used two distinct but complementary approaches. The first approach followed the NExT digital handbook (NExT Teacher Effectiveness Work Group, 2018) which suggests using rubrics provided to measure the effectiveness of teachers' assessment practices. These rubrics were adapted from edTPA. They evaluated (1) assessment planning, (2) analysis of student learning, (3) feedback on student learning, and (4) analysis of student learning to inform teaching. Using these rubrics for the initial purpose of piloting the materials from the handbook, we began to notice additional patterns in the data.

These initial observations led us to engage in a more systematic, open analytic process. Our second approach applied grounded theory to code and analyze the interviews. Grounded theory starts with a general area of research interests and uses the coding of qualitative data to identify patterns (Charmaz, 2006). Those initial patterns inform subsequent research questions and coding to arrive at a theory from the data. We concluded by looking at the results from these two analytic approaches to identify any further patterns evident in the data.

Results

Our analysis resulted in three key findings. First, we found that the mathematics teachers demonstrated high-level indicators of effectiveness relative to the pre-existing rubrics. In particular, they demonstrated values related to assessment that align with high-quality instructional practices. Second, through our use of open coding, we identified a wide variety of factors that influence how teachers enact their assessment values. Finally, we describe how teachers revealed tension between their assessment values and their perceived agency to enact those values.

Teachers' Assessment Views

We found evidence that all five teachers in the study demonstrated assessment practices and values that aligned with high-level indicators described in the NExT handbook. Teachers demonstrated that they (1) believe all their students can do well on mathematics assessments, (2) can differentiate feedback and analysis based on knowledge of individual students, (3) view assessment as ongoing, and (4) can design and implement assessments that provide students with opportunities for deep mathematical learning.

Mastery and ongoing assessments. A high-level performance indicator states that teachers use an "assets-focused approach to describing student progress towards learning goals". We found that all five teachers demonstrated this indicator in how they responded to interview prompts to discuss the student work samples. The interview protocol asked teachers to categorize their work samples based on student mastery. Despite this prompt, all five teachers resisted grouping the samples and instead discussed each piece of student work individually. The teachers consistently tailored their discussion and ideas for further instruction based on their knowledge of each individual student. They described the student understanding they saw demonstrated in the work and discussed how that aligned with the individual's progress towards their overall mathematics goals. We found their collective decision to discuss students individually rather than as a group to be evidence that the teachers were focused on each students' mastery level on the assessment and their progress toward the learning goals. The teachers discussed student work in a way that consistently indicated that they believed all students would master the learning goal eventually.

Tied to the concept that all students would eventually master the learning goal, we also found that teachers approached assessment as ongoing. Teacher A, for example, said, “I like giving at least two chances on each test.” This teacher went on to describe how additional opportunities for students to take assessments combined with standards-based grading provides Teacher A with a more comprehensive understanding of student mastery. Like Teacher A, all participants discussed intentionally providing ongoing assessment opportunities that enable students to demonstrate mastery of the learning goals beyond the first assessment.

Assessments as opportunities for deep mathematical learning. Another indicator of productive assessment practices from the student impact rubrics was opportunities for deep learning. To demonstrate valuing deep mathematical learning through assessment practices, teachers might require students to communicate mathematical arguments through their work and/or develop mathematical ideas. Teachers in our study demonstrated this in multiple ways. Teachers C and D analyzed student learning based on the ability to communicate valid arguments to support their work. Teacher B designed assessments that encouraged students to construct new mathematical ideas alongside checks for understanding.

To Teacher D, having the correct steps in a geometric proof was only part of demonstrating learning mastery. They also felt students must be able to create and communicate a valid argument through their work in a way that others can understand. Teacher D explained that they were not only looking for an accurate answer. “I also want them to understand that this is communication and it should be written to be read. I'm looking to motivate the idea that we're doing math as both a deductive and social activity.” To assess that, Teacher D designed an open-ended project with student choice that allowed students to prove geometric theorems using paragraphs, annotated diagrams, or a two-column format. Teacher B’s assessment design also encouraged students to think critically and advance conceptual understanding. Their assessment included questions that asked how a data summary would change if a new value was added to the data table. Students were also asked to make predictions about outliers, which served as a pre-assessment to the next lesson. These examples demonstrate that these early career teachers were prepared to design assessments with opportunities for deep learning by developing mathematical ideas and communicating mathematical arguments rather than focusing assessments solely on procedural fluency.

Factors that Influence Enactment of Assessment Values

The teachers indicated that there are a variety of factors that influenced how they enact their assessment values in practice. These factors included influence from administration, influence from their teacher preparation program, and their perceived agency in classroom decisions.

Influence from the teacher preparation program. All five teachers discussed assessment practices and values that were influenced by their teacher preparation program. These influences include practical elements of teaching and learning of mathematics, such as implementing multiple forms of assessments, incorporating unit plans they designed during their university coursework, and considering specific examples of student misconceptions in assessment design. They also demonstrated theoretical understanding of assessment practices by citing specific readings they had studied and the socioemotional needs of students.

For example, Teacher C reflected on their assessment and discussed designing assessments that were accurate measures of student understanding, which requires questions that can tease out misconceptions. The assessment selected by Teacher C was part of a routine created by the mathematics team at their school to improve student skills on selected standards. In this assessment, students solved two questions using the distance formula. Both questions involved finding the distance of a horizontal line segment from the origin to another point. Teacher C critiqued their assessment, referencing a specific task from their teacher preparation program which revealed that students can develop misconceptions about triangles when the base is always horizontal to the x-axis. They applied that experience to this assessment and explained that future versions of this assessment should include questions that solve for the distance of both horizontal and diagonal lines. This example highlights the high-level indicators of designing and implementing assessments that are accurate measures of student understanding and using evidence in student work to measure nuanced growth towards the learning goal.

Teacher C explained, however, that adjusting assessment questions created a new set of challenges for department collaboration and measuring student growth. Their assessment was designed as an intervention strategy with multiple opportunities for reassessment. As such, one of the goals was consistency both in assessments/reassessments and across classrooms. Teacher C described the complexity of wanting to design assessments collaboratively and implement assessments that were consistent enough to support student growth through reassessments, while also varying the questions to be a true measure of what the students know. Teacher C's commitment to each of these values were influenced by their program, but the reality of navigating those values while collaborating with a department sometimes created tension.

Influence from school administrators. The interview data revealed that the teachers' assessment practices were strongly influenced by their school administrators' view of assessment in mathematics teaching and learning. The teachers expressed feeling tension between their assessment values and the assessment practices encouraged or enforced by their school administrators. The teachers described a complex and contextualized reality that influenced their ability to implement high level assessment practices.

Multiple teachers described tension between the values developed during their program and their administrator's views on assessment. For example, Teacher E discussed how their district's "data driven" assessment plan felt at odds with the values learned during their program. "We were taught about all the awesome ways that you could do things. Which is great...But I don't know if that effectively prepared me for walking into a school where that's the exact opposite of what they do". Teacher E felt that pressure from the district to maximize student performance on standardized tests influenced instructional and assessment decisions that went against the practices learned as a pre-service teacher.

Similarly, Teacher C shared that their school leadership viewed mathematics learning through a procedural lens.

> The program prepared me wonderfully for actually teaching mathematics, but navigating [my] district's and administration's attitude towards mathematics is more difficult...the decision makers in our administration conceptualize mathematics as all procedural. [...] How to navigate that while pushing towards the ultimate goal of making mathematics education about understanding is a thing that I feel like I don't know.

Through their work in the teacher preparation program, Teacher C learned to value understanding in mathematics teaching and learning, yet their administration viewed mathematics as "all procedural". Both Teachers C and E described feeling frustrated about how the "higher ups" views of assessment purposes influenced their ability to enact practices that align with their personal assessment values. They offset those tensions by creating ongoing assessment opportunities that provided students with additional chances. The teachers explained that in their respective departments, the mathematics teachers chose to integrate reassessments and test corrections to better align their values with the required assessment expectations.

Finite learning time. Limited learning time also appeared to be a significant factor for how teachers enacted their assessment values. We define *learning time* as available time for instruction and intervention based on scope and sequence constraints. All five teachers talked about learning time. The teachers described immense pressure to maintain scope-and-sequence pacing for the year. This is particularly notable since our interview questions focused strictly on assessment without any reference to instructional time. Teacher B reflected on this from their position as the sole mathematics teacher at their school, saying, "I feel kind of blessed almost to be the only teacher because I get to work on my own schedule...and really use data from assessments to help guide my teaching." This contrasts with teachers who felt bound by their pacing guides. These teachers described feeling pressure to "push through the content" faster than the students could handle and having to "hustle to convince" students to get extra help during lunch or after-school. These time barriers also affected how much time teachers felt they had to provide feedback and modify their instruction based on assessment data.

These influences from the teacher preparation program, school administrators, and learning time all contributed to situations where teachers found themselves with competing goals relating to assessment. In the next section, we turn to the construct of agency to help us understand how teachers navigated these competing values.

Complex and Contextualized Teacher Agency About Assessment

Despite our finding that these five teachers demonstrated high-level indicators with respect to their assessment values, how they were able to enact those values was contextualized through the complex realities of their schools. Our data highlights complexities around administrators' view of assessment purposes in mathematics and the relationship between learning time and instructional decisions. Teachers navigated these contexts based on their perception of agency over teaching practices. In some circumstances teachers felt agency to align their assessment values with practices, such as using assessment data to inform classroom decisions and designing assessments with opportunities for deep mathematical learning. At the same time, all of the teachers described situations where they felt pressure to enact assessment practices that went against their values. One area where teachers demonstrated agency was in designing year-long assessment plans with ongoing assessments. All of the teachers were required to meet assessment expectations set by the "higher ups". This included weekly department-wide skill assessments, a minimum of two summative assessments every four weeks, or administering assessments that were closely tied to high stakes standardized assessments. However, all five teachers also incorporated ongoing assessment practices or assessments designed to allow solutions in multiple representations within their school's larger assessment plan. Teachers were able to find space within those expectations to enact their assessment values based on the degree of agency.

Discussion

Our study had three findings. First, all five teachers demonstrated high-level performance indicators for assessment values. Second, factors influenced how teachers enacted their values - including tension created by the "higher ups" views and expectations about assessments in mathematics. Third, teachers navigated this tension by agency that was contextual. A key finding from this study showed that mathematics teachers recognized a misalignment between their assessment values and their agency to enact those values.

We argue that the degree of teacher agency to enact assessment values depends on where the conflicting values lie on the continuum of purposes of assessment (Barnes et al., 2014). When the purpose of assessment was at the pedagogical end of the continuum, teachers were doing the assessing and had agency to align their values with classroom practices. In our study, we saw this when teachers used multiple forms of assessment, created ongoing assessment practices, and individualized feedback. However, when the conflicting values were between accountability purposes from the "higher ups" and pedagogical purposes, teachers yielded their values to conform to expectations. At this end of the continuum, teachers were *being assessed* while they were assessing student understanding. The "higher ups" use assessment as a form of control and accountability to evaluate teacher effectiveness (Barnes et al., 2014). When there is tension between the teachers' assessment values and higher ups views of mathematics assessments, teachers are put in a position where they must choose between their evaluation of teacher effectiveness or pedagogical practices that align with their values.

One of the most striking themes from our data was the influence learning time had on teachers' assessment practices. All of the teachers in our study cited the pressures of learning time as a driving factor for instructional decisions that misalign with their values. Thomas and Yoon (2014) found that time was one of three factors that influenced a teacher to abandon their pedagogical values in order to conform to accountability purposes of assessment. In that study, time was grouped with curriculum and assessment. Echoing Thomas and Yoon, we posit that required curriculum and high stakes assessments, as well as learning time, are factors that influence teachers' agency to enact their assessment values. Furthermore, we argue that teachers felt pressured to disregard their values because time, curriculum, and high stakes assessments are at the accountability end of the continuum of assessment purposes.

"Higher ups" make decisions and set policies that they believe will help students learn mathematics, such as scope-and-sequencing and departmental teaching strategies. However, their lens of understanding these practices is often driven by standardized assessments, which means that it is still largely procedural. Mathematics teachers, through their licensure programs, have a complex and nuanced understanding of teaching mathematics for conceptual understanding that align more strongly with pedagogical purposes of assessment. Our findings support research that continues to think about ways to use available learning time for deep learning rather than focusing solely on procedural skills. Our data shows that less pressure around maintaining a scope-and-sequence may enable teachers to better enact their assessment values.

Our study piloted a teacher interview protocol that was adapted from D'Souza's (2012) TAPL with early career mathematics teachers. Similar to D'Souza, we found that this protocol was an effective tool in understanding teachers' practices and values around assessment. We found that when implemented with secondary mathematics teachers, the student work protocol revealed both indictors of high-level assessment practices and contextual factors that might prevent teachers from enacting those practices. We posit that teacher agency is a key component to enacting assessment values and is tied to the continuum of purposes of assessment (Barnes et al., 2014). Since accountability purposes influence teachers' agency, evaluations of teacher effectiveness should include opportunities for teachers to identify when they felt pressured to act against their values. The NExT student work sample protocol is one alternative to evaluate teachers' assessment values and practices.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Author. http://amte.net/standards

Barnes, N., Fives, H., & Dacey, C. M. (2014). Teachers' beliefs about assessment. In *International Handbook of Research on Teachers' Beliefs*. Routledge.

Beswick, K. (2011). Teachers' beliefs about school mathematics and mathematicians' mathematics and their relationship to practice. *Educational Studies in Mathematics*, *79*(1), 127–147. https://doi.org/10.1007/s10649-011-9333-2

Biesta, G. J. J., Priestley, M., & Robinson, S. (2015). The role of beliefs in teacher agency. *Teachers and Teaching*, *21*(6), 624–640. https://doi.org/10.1080/13540602.2015.1044325

Brown, G. T. L. (2004). Teachers' conceptions of assessment: Implications for policy and professional development. *Assessment in Education: Principles, Policy & Practice*, *11*(3), 301–318. https://doi.org/10.1080/0969594042000304609

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis*. SAGE Publications.

Davis, D. S., & Neitzel, C. (2011). A self-regulated learning perspective on middle grades classroom assessment. *The Journal of Educational Research*, *104*(3), 202–215. https://doi.org/10.1080/00220671003690148

D'Souza, L. A. (2012). Assessing student learning through guided inquiry: A case study of a beginning teacher. *Journal of Education*, *192*(2–3), 79–87. https://doi.org/10.1177/0022057412192002-310

Karp, G. G., & Woods, M. L. (2008). Preservice teachers' perceptions about assessment and its implementation. *Journal of Teaching in Physical Education*, *27*(3), 327–346. https://doi.org/10.1123/jtpe.27.3.327

NExT Teacher Effectiveness Work Group. (2018). *Preparing effective teachers: Multiple approaches to ensuring teaching quality and impact* (M. Sato & Lloyd, Eds.). Network for Excellence in Teaching. https://www.nexteachers.org/next-digital-handbook

Remesal, A. (2007). Educational reform and primary and secondary teachers' conceptions of assessment: The Spanish instance, building upon Black and Wiliam (2005). *The Curriculum Journal*, *18*(1), 27–38. https://doi.org/10.1080/09585170701292133

Sato, M. (2014). What Is the underlying conception of teaching of the edTPA? *Journal of Teacher Education*, *65*(5), 421–434. https://doi.org/10.1177/0022487114542518

Sato, M., Wei, R. C., & Darling-Hammond, L. (2008). Improving teachers' assessment practices through professional development: The case of National Board certification. *American Educational Research Journal*, *45*(3), 669–700. https://doi.org/10.3102/0002831208316955

Thomas, M., & Yoon, C. (2014). The impact of conflicting goals on mathematical teaching decisions. *Journal of Mathematics Teacher Education*, *17*(3), 227–243. https://doi.org/10.1007/s10857-013-9241-8

HOW DO TEACHERS AND DISTRICTS IMPLEMENT STATEWIDE CO-DESIGNED INSTRUCTIONAL FRAMEWORKS?

Michelle Stephan
UNC Charlotte
mstepha1@uncc.edu

Catherine Schwartz
East Carolina University
schwartzca@ecu.edu

Katherine Mawhinney
Appalachian State University
mawhinneykj@appstate.edu

We report the findings from our efforts to co-design statewide instructional frameworks to support elementary and middle school mathematics teachers' and leaders' implementation of state standards in ways that align with visions of high quality mathematics instruction. In this paper, we explore whether districts as well as individual teachers took up the instructional frameworks, their reasons for doing so, and the ways they used them to support instruction. Our findings indicate that the instructional frameworks were widely adopted across the state, supported teachers' pacing and sequencing efforts, and provided opportunities for professional learning. However, school/district leaders and classroom teachers had different views on why their district decided to take the frameworks up, indicating a communications divide that needs to be addressed in future co-design efforts.

Keywords: Systemic Change, Curriculum, Design Experiments

The notion of vision has long been researched as an important part of a teacher's ability to enact thoughtfully adaptive teaching (Duffy, 1998). Additionally, a shared vision of high-quality instruction among administrators, teachers, and other stakeholders, is essential for professional development and collaborations to be effective in schools (Birkeland & Feiman-Nemser, 2012; Fulton, et al., 2010) and for the implementation of new programs or policies (Gamoran, et al., 2003). The importance of common vision of mathematics instruction is reflected in Cobb and Jackson's (2011) theory of action for large scale instructional improvement in mathematics, which includes vision of high quality mathematics instruction (VHQMI) underlying a coherent instructional system as one of five key elements. What counts as "high quality" in mathematics education is debatable, but a large body of research suggests that HQMI involves listening to students and building on their representations and strategies to encourage sense making. Teachers who aspire to this vision are often constrained by district policies (e.g., scripted curricula, strict pacing and sequencing) and state mandates (e.g., assessments, standards). Although these conditions have constrained mathematics teachers across the country, we describe a research-practice partnership (RPP; Coburn, et al., 2013) that aims to co-design infrastructures and resources that support North Carolina mathematics teachers to develop and enact common visions of high-quality mathematics instruction (VHQMI). In this paper, we report the findings from our efforts to co-design statewide instructional frameworks to support elementary and middle school mathematics teachers' and leaders' implementation of state standards in ways that align with VHQMI. Specifically, we explore whether districts as well as individual teachers took up the frameworks and why.

Co-designing for Shared VHQMI Through Instructional Frameworks

Research suggests that pacing guides are often viewed as a constraint to enacting visions of practice (Duggan, et al., 2018). They undermine teachers' flexibility to meet individual student needs as district expectations of strict adherence to weekly or daily schedules contribute to the need to "cover everything," often by dropping conceptual-based, student-led activities in favor of teacher-directed activities (Bauml, 2015). In North Carolina [NC], school districts had each created their own pacing guides to sequence instruction based on statewide standards. Differing levels of capital and expertise among districts resulted in widely varied pacing guides, some co-created by district-wide teams of elementary math specialists and, in other districts, created by K-12 district curriculum leaders with no particular expertise in mathematics education. North Carolina teachers and leaders identified this as a problem of practice that could be leveraged to build towards a more coherent vision across the state. Thus, we convened teams including teachers, instructional coaches, administrators, and curriculum leaders from 19 diverse and representative school districts, higher education faculty from eight universities and state leaders from the NC Department of Public Instruction [NC DPI] to co-design an elementary and a middle school pacing guide that would be grounded in research on learning progressions (Common Core Standards Writing Team, 2013) and could be adapted *statewide*. The co-design teams decided to address their concerns about common VHQMI, and the productive vs. non-productive uses of pacing guides by envisioning a pacing document that goes beyond "what to teach when." Focused on the idea that curriculum materials can themselves can be educative (Drake, et al., 2014; Davis & Krajcik, 2005) the co-design teams set a goal of promoting a shared instructional vision by developing suggested state-wide, grade-level pacing guides, re-named **instructional frameworks** (IFs) to denote their role beyond traditional pacing documents that: 1) Emphasize curriculum guidance, not prescriptive pacing, 2) Focus on central ideas with links to exemplary curriculum materials, lessons, and instructional strategies, 3) Allow for flexibility and unpredictability based on differences in teachers, students, and contexts, 4) Address development of student reasoning and how to build upon it (i.e., learning progressions), and 5) Are adjusted frequently based on feedback from teachers. The co-design teams completed first drafts of the IFs in fall 2017, sought feedback through a statewide survey from stakeholders, made revisions and rolled the IFs out in spring 2018. Importantly, the NC DPI adjusted the content of their free, optional interim assessments [Check-Ins] to align with the IFs sequencing and pacing. Like the Check-Ins, districts had the option to use the IFs to guide implementation of the mathematics standards for the 2018-2019 school year. At this time, the elementary and middle school IFs have been available and adapted statewide for three years. In this paper, we share findings related to three research questions:

RQ1: For districts that adapted the IFs, what did participants think was the reason for uptake?
RQ2: What aspects of the frameworks did individual teachers report were their favorite?
RQ3: How did teachers report using the frameworks?

Research Methods

This work is part of an ongoing Design-based Implementation Research (DBIR) state-wide project that is in its seventh year, having partnered with hundreds of NC mathematics educators. In 2019, a survey was distributed through the state agency's listservs to approximately 20,000 mathematics teachers, school administrators, and district mathematics leaders. Of the 813 educators that responded to the survey, we analyzed only the responses of participants who worked with grades that would potentially use the IFs. These 538 responses represented 74% of the NC school districts. Of the respondents, 60 were School-based Coaches/Curriculum Facilitators, 38 District Curriculum Personnel, 4 Principals/Asst. Principals, and 436 Classroom Teachers. The respondents worked in elementary (320), middle (181), elem/middle (11), middle/high (16), and elem/middle/high school (10) settings. In this report, we analyzed the answers to a block of questions/prompts related to implementation of the IFs:

1. **Question 1:** Did your district use the instructional frameworks?
2. **Prompt 1:** If you know what influenced your district's decision to use the IFs, rank and order the reasons below: match/align to the state quarterly assessments, match the textbook or math program used in my district, align to State resources, match our current sequencing and pacing of instruction, address the development of student reasoning and how to build upon it based upon research, were co-designed by a large number of people, were created with feedback from teachers and leaders, and I do not know.
3. **Question 2**: What are your favorite aspects of the instructional frameworks? (Pick up to 3 choices): [same choices as Prompt 1].
4. **Question 3**: How did you use the IFs (Select all that apply): as a resource among a variety of resources, a resource to supplement a textbook or math program, to determine pacing and/or to sequence instruction, to gain a better understanding of what the standards mean, to gain a better understanding of the mathematics, to understand vertical alignment.

Findings

Of the 75 district leaders, school coaches, and principals/asst. principals who answered question 1, 85% (n=64) reported that their district was using the IFs, 13% (n=10) said they were not using the IFs, and 2% (n=1) said that they did not know. These 75 mathematics leaders represented 53 of 115 districts or roughly 46%. We used responses from district and school leaders only because we believe they have the most accurate knowledge of the district's decisions, whereas a teacher does not always have direct knowledge of their district's intentions. Interestingly, of the 227 classroom teachers who answered this question, 78% of respondents say that their district uses the IFs, 8% say that they do not, and 14% do not know. This suggests some coherency between teachers' and leaders' knowledge of whether the IFs are being used within their districts. The fact that 14% report that they do not know whether their district adapted the IFs or not suggests there may be a communication challenge between districts and teachers. Overall, we are encouraged by the high number of districts that report using the IFs.

Why Did Districts Take Up the Instructional Frameworks?

We separated responses to survey prompt 1, if you know why your district decided to use the IFs, by education role. According to district leaders, school coaches and principals/asst. principals, the top three ranked reasons that districts chose to use the IFs were that they matched the state quarterly assessments, matched the district's textbook/curriculum and incorporated feedback from teachers and leaders. In contrast, classroom teachers reported that their district chose to use the IFs because they matched the textbook/curriculum, matched their district's current pacing guide, and aligned to the State's resources. In this case, teachers and teacher leaders only agree that the district used the IFs because they match the textbook/curriculum of their district. Otherwise, there is little agreement. It is significant to notice that educators who are closer to their district's decision making process, report that assessments were an important part of their decision while classroom teachers ranked assessments in the bottom three reasons they thought their district used the IFs. This finding indicates that there is miscommunication between districts and teachers and suggests infrastructuring as a potential problem to address in working towards shared VHQMI.

What Were Individuals' Favorite Aspects of the IFs?

Of the 302 respondents to Question 2, approximately 50% chose *matched the state quarterly assessments* as one of their top three favorite aspects of the IFs. Not far behind was the fact that the IFs *address the development of student reasoning and how to build upon it based on research*

(40%) and that they *match their district's current pacing guide* (36%). Disaggregating the responses by role group (teachers and leaders), there is no difference which suggests regardless of role responsibilities, participants valued the same aspects of the IFs. In particular, the IFs seem to be valued for their intended purpose: suggested pacing of standards. Second, the IFs are valued for aligning standards with the assessments. Surprisingly, the second most valued characteristics of the IFs involved the fact that they are based upon the development of student reasoning and research. Taken together, these findings suggest that educators value the resources for addressing their practical challenges, *assessments and pacing,* and must be attended to in resource design. Further, grounding resources in research was a characteristic that was valued by those who took it up.

How did individuals use the IFs?

Question 3 asked participants to select among a variety of ways that they used the IFs in their classroom instruction or professional work (if leaders). From the 226 responses, the three most prevalent uses of the frameworks were to gain a better understanding of what the standards mean (51%), as a resource among a variety of resources (51%), and to determine pacing and/or to sequence instruction (47%). Less common responses included using the frameworks to gain a better understanding of the mathematics (36%), to understand vertical alignment (31%), and as a resource to supplement a textbook or math program (14%).

Discussion

The goal of this DBIR project was to work towards statewide shared VHQMI by co-designing instructional frameworks to address the shared problem of practice of non-productive uses of pacing guides and the many disparate, district-created pacing guides across the state. The IFs were taken up widely and served their purpose as a guide for sequencing and pacing and also as a tool to promote professional learning. Findings about why and how districts used the IFs and what was most valued about the tool have implications for future co-design. First, perceptions about why districts took up the IFs differed between leaders and teachers, indicating a lack of communication. Continued co-design to promote shared VHQMI needs to support communication across role groups. Second, respondents valued that the IFs addressed the development of student reasoning and how to build upon it based on research, and yet this component was not among the top reasons respondents perceived as a reason districts chose to use the frameworks. In other words, respondents valued that the tool was research-based and focused on student reasoning, but it was not a determiner of uptake in the way that alignment to state assessments was. Thus, the co-design process needs to include negotiating the existing structures within the communities for which tools are being created. Finally, intentionally seeking feedback from all role groups in the system is essential in co-design. In the case of the IFs, getting feedback at multiple levels allowed for attention to the differing needs that were expressed, resulting in high uptake and buy-in, opportunities for learning, and movement toward shared VHQMI. As one respondent noted:

> The instructional frameworks could be the beginning of a powerful movement in North Carolina towards Student Centered Mathematics Instruction. The frameworks are so detailed regarding rationale for clusters that it serves to support teacher professional learning. I think that we need so many more professional learning opportunities where teachers from all over the state can come and collaborate and learn from each other.

Acknowledgements

This work was supported by the National Science Foundation (NSF) under grant DRL-2100895 awarded to East Carolina University, DRL-2100903 awarded to UNC Greensboro, DRL-2100947 awarded to UNC Charlotte, and DRL-2100833 awarded to Appalachian State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the NSF.

References

Bauml, M. (2015). Beginning primary teachers' experiences with curriculum guides and pacing calendars for math and science instruction. *Journal of Research in Childhood Education*, *29*, 390–409,

Birkeland, S., & Feiman-Nemser, S. (2012). Helping school leaders help new teachers: A tool for transforming school-based induction. *The New Educator*, *8*(2), 109-138.

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development*, *13*(1), 6-33.

Coburn, C. E., Penuel, W. R., & Geil, K. E. (2013). *Research-practice partnerships. A strategy for leveraging research for educational improvement in school districts*. William T. Grant Foundation.

Common Core Standards Writing Team (2013). *Progressions Documents for the Common Core Math* Standards. Retrieved from http://ime.math.arizona.edu/progressions/#products, February 13, 2022.

Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational researcher*, *34*(3), 3-14.

Drake, C., Land, T. J., & Tyminski, A. M. (2014). Using educative curriculum materials to support the development of prospective teachers' knowledge. *Educational Researcher*, *43*(3), 154-162.

Duffy, G.G. (1998). Teaching and the balancing of round stones. *Phi Delta Kappan, 79*(10), 777-780.

Duggan, A., Schwartz, C., Stephan, M., Schulz, D., & Mawhinney, K. (2018). Examining critical conversations during co-design of instructional guidance documents. In Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 129)*. Greenville, SC: University of South Carolina & Clemson University.

Fulton, K., Doerr, H., & Britton, T. (2010). STEM teachers in professional learning communities: A knowledge synthesis. *National Commission on Teaching and America's Future*.

Gamoran, A., Anderson, C., Quiroz, P., Cicada, W., Williams, T., & Ashmann, S. (2003). *Transforming Math and Sceicne: How schools and districts can support change*. Teachers College Press.

DEVELOPING ASSET-BASED INSTRUCTION THROUGH LEARNING TRAJECTORY-BASED CURRICULAR DESIGN

Jennifer M. Suh
George Mason University
jsuh4@gmu.edu

Theresa Wills
George Mason University
twills@gmu.edu

Sara Kirschner
George Mason University
skirschn@gmu.edu

Alayna Wearly
George Mason University
awearly@gmu.edu

Maureen E. Vora
George Mason University
mvora@gmu.edu

Kate Roscioli
George Mason University
krosciol@gmu.edu

This research report describes a Learning Trajectory-based Curricular Design project that engaged teachers and coaches in the design and implementation process. As the project team, we focused on deepening teacher designers' understanding of the learning trajectory (LT) while situating student learning along a continuum to advance student thinking. Analysis of the design and implementation cycle demonstrated that teacher designers used their professional judgment and knowledge of LTs to assess the quality and appropriateness of curricular resources as they made instructional decisions to meet the needs of diverse learners. School-based coaches used these teaching resources as a type of professional development for identifying student strengths and "packaged" the resources for teachers who were overwhelmed from teaching during the pandemic. We discuss the importance of applying LT research for asset-based instruction.

Keywords: Learning Trajectories and Progressions, Teacher Knowledge, Instructional Activities and Practices, Standards (Strand: Curriculum, Assessment, and Related Topics; Mathematical Knowledge for Teaching)

COVID-19 interrupted teaching and learning in unprecedented ways and presented multifaceted challenges for students and teachers. As educators worked hard to support student learning in mathematics, the field looked for innovative ways to mitigate the challenges. In the spirit of PMENA 44's theme, Critical Dissonance and Resonant Harmony, we share how researchers and teacher designers worked collaboratively in design-based research to move beyond the "dissonance" created by COVID to build a curricular resource framed by learning trajectories (LT) and asset-based instruction to bring "harmony" to educators striving to meet the needs of every student.

The Need to Translate Learning Trajectory Research to Practitioners

We situate our work in uncertain times, as Ladson-Billings (2021) calls in a "re-set school environment", where she asks educators to use an accurate assessment of what students already know with varied and regular formative assessments to determine how well students are understanding what they are taught. In this way, assessment is not a "punitive tool to 'catch' students but rather a diagnostic and developmental tool that will tell teachers and schools how to adjust their curriculum and pedagogy"(Ladson-Billings, 2021, p.75). This re-set requires teachers to be deeply knowledgeable about the learning trajectory (LT). It is critically important to introduce LT research to practitioners due to the scale of disruption and overwhelm of teachers in the post COVID-19 learning environment. Teaching that incorporates an understanding of LTs has the potential to make instruction more efficient and supportive of learning mathematics for understanding (Suh et al., 2014; 2021). Clements & Sarama state that a LT describes:

> Children's thinking and learning in a specific mathematical domain, and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain. (Clements & Sarama, 2004, p. 83)

In many academic and practitioner resources, the terms LTs and learning progressions are used interchangeably with the emphasis on the developmental progression of levels of thinking within a conceptual domain. Confrey's notion of LT/progression is described as:

> A researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (Confrey & Maloney, 2010, p. 1)

According to Confrey (2012), there are five elements of LTs that teachers need to understand: 1) the conceptual principles and the development of the ideas underlying a concept; 2) strategies, representations, and "conceptions"; 3) meaningful distinctions, definitions and multiple models; 4) recognizing coherent structure or pattern in the development of progressively complex mathematical ideas; and 5) bridging standards or identifying the underlying concepts that "bridge the gap" between standards. Focusing on the five elements of LTs can improve instructional planning as teachers anticipate student strategies, representations, and conceptions that can be attributed to students' strengths and resources for building on their understanding. The potential of Learning Trajectory-Based Instruction (LTBI) in professional development (PD) settings has also been explored by the collective work of Sztajn et al. (2012), Wilson et al. (2015), and Myers et al. (2015), who examined how teachers' discursive patterns about students as mathematics learners changed as the teachers engaged with the LT in PD. In particular, they noted that teachers initially voiced expectations about students' mathematical ability related to student age or grade level. But as teachers' understandings of LT developed, their voiced expectations began to acknowledge that students' prior experiences influenced students' performance.

Asset-based instruction and the use of rigorous mathematics were central to the framing of our project and aligns with one of the key recommendations in *Catalyzing Change in Early Childhood and Elementary School* (NCTM, 2020) to develop "deep mathematical understanding" and build students "as confident and capable learners" (p. 11). Using Asset-based approaches to planning instruction is a conscious way to move away from deficit perspectives (Celedón-Pattichis et al., 2018). Teachers' explicit attention to focusing on strength in students' thinking and what children are capable of doing helps teachers in avoiding biases that impair teaching and learning. According to Gresalfi et al. (2009), what counts as "competent" gets constructed through an interaction between the opportunities that a student has to participate in a particular mathematics classroom and the student's uptake of those opportunities, meaning that structures that promote equitable participation and interaction are key. Positive and discourse-rich classrooms (NCTM, 2000; Stein & Smith, 2011) allow each student to have feelings of success and pride (NCTM, 2020). The instructional decision to formatively assess and highlight student thinking during discussions has important implications for assigning competence, as it suggests *what* students are accountable for and to *whom* they are responsible for sharing their thinking with (Gresalfi et al., 2009). We believe that by finding strength in students' multiple knowledge bases (Turner et al., 2016; Kobett & Karp, 2020), teachers are better able to assign competence in student thinking, while broadening the notion of what competence means and building student agency and a positive sense of identity (Civil, 2007; Gonzalez, Moll, & Amanti, 2005; Aguirre et al., 2013; Lotan 2003; Cohen et al., 1999; Gresalfi et al., 2009).

Figure 1: Mathematics Learning Trajectory-based Curricular Design Framework

Context for our Math Learning Trajectory-based Curricular Design for Practitioners

The professional development design institute used a LT-based Curricular Design framework (Figure 1) which incorporated LT research, asset-based instruction, and rigorous instructional resources with high levels of cognitive demand. The team of teacher designers curated curriculum modules aligned to state-selected bridging standards. Bridging standards connect content across units within grade levels and articulate prerequisite knowledge for standards in future grades. The modules designed for each bridging standard consisted of five components: a) a zoomed-in LT bridge that illustrated the connection between students' strengths, bridging concepts, and the targeted learning standard; b) "big ideas" about the principles and development of each LT; c) important assessment "look-fors" that included strategies, representations, and "conceptions" to use for formative assessment; d) purposeful questions to assess, clarify, and advance students' mathematical ideas; and e) cognitively-demanding bridging activities, specifically routines, rich tasks, and games.

The LT research embedded in the modules provided direction for teachers to predict their students' potential reasoning, misconceptions, and learning. The modules were designed to support teachers in examining student thinking according to levels of cognitive proficiency rather than age or grade level. The curricular focus on asset-based instruction was intended to challenge and expand what teachers value and consider to be mathematical competence. Based on formative assessment of students' strengths, the modules guide teachers to select targeted activities in response to students' understandings and to support further learning.

Methods

Context and Participants

Using design-based implementation research, this qualitative case study (Stake, 1995) followed six early elementary mathematics educators from a professional development program to understand how these teacher designers applied a LT framework and asset-based lens while designing and testing curricular materials through two implementation cycles. After each implementation cycle, teacher designers attended an implementation debrief meeting to share their experiences and change recommendations.

The teacher designers were purposefully recruited for the professional development based on their leadership and teaching experiences. The design process began in the midst of the Pandemic in the summer of 2021 with a 30 hour-one week Design Institute. Teacher designers then implemented the design modules in their classrooms during the fall of 2021 with iterative cycles of refinements. We met three sessions online to debrief each cycle of implementation. In the spring of 2022, we interviewed a core group of our teacher designers to learn about how they continued to use the LT based instructional modules to support diverse learners. This case study followed six total participants: three math coaches, Jana, who taught 11 years in the classroom and 7 years as an instructional coach; Mia, who taught 14 years as a classroom teacher and 14 as a coach; and Sienna, who taught 9 years as a classroom teacher and 18 years as a coach; a first grade teacher, Rebecca, who has taught 3 years, and two second grade teachers, Kara, who has taught 9 years, and Naomi, who has taught for 8 years.

Data Collection and Analysis

Three data sources were analyzed for this case study: video clips from implementation debrief meetings that served as focus group meetings, student work, and teacher reflection forms. The implementation debriefs were conducted in focus groups with smaller groups of the entire teacher designers which allowed us to invite individual comments while also situating those comments in context of the group that worked together during the design process (Morgan, 2011). Based on the data analysis from this focus group debriefs, we selected six teacher designers to conduct one on one interviews. The research team used open coding first individually, keeping analytical memos which provided preliminary analysis allowing “processes of discovery in the material” (Morgan, 2011, p.14). Next, the research team employed Knodel’s (1993) grid analysis to ensure researcher fidelity to the transcripts during subsequent analysis. Grid analysis allowed the research team to review transcript segments associated with each subtopic and calibrate the codes and categories to ultimately identify recurring themes.

Overarching Research Questions

Our design based implementation research questions included:

RQ1) How do teacher designers use the Mathematics LT-based Curricular Modules during their implementation cycles?

RQ2a) How does involvement in this design project influence teacher designers’ future work? b) In what ways did teacher designers’ deep dive into LTs translate into their use of LTBI in coaching or leading school districts? c) How does the focus on strengths-based instruction influence teacher designers’ approach to their instruction or work with teachers?

Results

In addressing the first research question, how do teachers use the Mathematics LT Framework/LT-based Curricular modules during their implementation cycles, our findings revealed two themes. First, teacher designers used their professional judgment and knowledge of LTs to assess the quality and appropriateness of curricular resources for supporting students in meeting specified learning goals. Secondly, curricular materials designed using the LT framework supported teacher designers’ understanding of students’ learning and informed their instructional decision making.

Teacher designers' demonstrated their professional judgment by assessing the quality and appropriateness of the curricular materials. Teacher designers implemented formative assessments and bridging activities to assess the quality and requirement criteria of each teaching resource. For example, through implementation and assessment of a teaching resource, Kara determined that it did not meet the requirement criteria of a rich task and recommended it be used instead as a formative assessment. Kara stated that "a lot of those tasks, they feel like worksheets. It was basically just a list of questions. …That feels like a quick check. No richer. It didn't feel like anything too different than a quiz-like question." Kara questioned the "richness" of the teaching resource and determined that it would not be appropriate as a "rich task" but instead could be useful in other ways. Kara's assessment was affirmed by another teacher designer.

Teacher designers also used their knowledge of their students to assess the value of teaching resources. Rebecca found it difficult to implement a computer-based game with her first grade class due to the technological expertise needed to play it. Although the mathematical concepts of the game were well-aligned to her students' needs, they struggled with the website. Rebecca adapted the game to a paper-based format which was more accessible to her students. Mia used her knowledge of her students in a slightly different way. The substantial changes she witnessed in her students' mathematical confidence helped her realize the "richness" of the teaching resource. She reflected on how the high-quality teaching resources changed the ways students engaged in their mathematics learning, stating that "it was a huge difference just within a week's time. They felt more confident. That was the biggest thing I took away, the kids were not afraid of being wrong anymore, and they were very comfortable with being able to manipulate and do the work."

Additionally, the teacher designers critiqued teaching resources according to their alignment along the LT. After playing the game "Race to 100" with her second grade class, Kara determined that the game was a strong teaching resource for moving her students along the place value LT. Other teacher designers noted that while some games did not align well to a bridging standard, other games could be used for several bridging standards.

Teacher designers' reflections on their students' learning in response to the LT-based curricular modules revealed a second theme and provided a lens for teacher designers to notice their students' understanding and make instructional decisions. Jana, an instructional coach, discussed how the resources helped teachers in her school identify and address unfinished learning from previous grades. Because the curricular design used bridging standards, Kara was able to look for resources aligned to prior grades to support the prerequisite skills and knowledge her students needed for their current grade stating, "If you want to reteach anything, let's say I'm teaching second grade this year, so I'm going to reteach this first grade skill before I teach the second grade skill." Bridging standards were an important element of LTs for both Jana and Kara. This focus on underlying mathematical concepts allowed them to see a progression of complexity and shift along the trajectory as needed to find the appropriate teaching resources for their students.

The LT-based curricular structure supported the teacher leaders in reframing how they view student learning. Sienna, an instructional coach, found that the LT structure helped teachers see their students' learning as a progression and consider next steps rather than visualizing a gap between students' current understanding and the "final goal of the standard". Kara had a similar epiphany in her second grade classroom. She found that the LT structure allowed her to focus on conceptual understanding in her second grade students' work rather than simply quantifying the number of incorrect answers, stating "When you go to grade something or check it over, I'm not necessarily looking at 'Oh, they got 15 out of 20. They're missing a bunch.' …I'm really zoning and honing in on what patterns I can find. I feel like that's what this cycle has taught me is that there are patterns in student work." Kara's attention to patterns in her students' work allowed her to identify their position on the LT and plan targeted instruction to support their learning. Kara pinpointed her students' understanding of double-digit addition and made appropriate instructional decisions, stating "We just started teaching double digit addition. I know they know their facts, but they aren't getting regrouping. Okay, well then that's what they need. We don't need to work on the basics. It's not that they can't add… They can add single digit numbers, so how can I break that down when we go to add something more complicated than that." Likewise, Rebecca suggested that the LT approach helps lessons to be more effective because they focus on where students are and their precise needs. The teaching resources were cognitively demanding and grounded in LT research, thus equipping Kara to look beyond the number of incorrect responses, focusing instead on student understanding and instructional decisions.

Naomi found the open-ended formative assessment questions to be valuable for revealing students' prior knowledge. On a formative assessment question asking for a number less and greater than 153, Naomi observed, "that open-ended part of those two questions shows their thinking. Like who's thinking one more or one less. For what's greater than 153, one of my kids wrote 582… That gave them a way to show other numbers they know, not just one more, one less." From the formative assessment, Naomi realized that her students needed more experience with numbers beyond the typical hundreds chart and chose the Mystery Number routine to broaden their exposure. In addition, she appreciated how an open-ended task allowed her to see the edges of student thinking with students naming different magnitudes of numbers that fit the criteria for the Mystery Number.

In addressing research question 2a) "How does involvement in this design project influence teacher designers' future work?", we found that teachers noted the ease of usability as a "grab and go resource". Teachers reported an advantage that all the materials were conveniently organized on a website, saying, "It is truly a one stop shop it really, really is" and "since we were already going to teach this anyways, it's easy and convenient, it is a one stop shop." However, this also led to a disconnect between the original project goal of sourcing pedagogically rich resources and instant implementation of those resources. In several instances, teacher designers initially liked the material, but struggled with implementation. Sienna said, "I really struggled with [the task], to the point where I actually had to create my own little recording sheet." She described a conceptual hands on lesson using base ten blocks, which required monitoring and listening to students working in the moment. The recording sheet helped her keep track and assess various student strategies. While teacher designers attributed successful implementation to the convenience of the materials, that convenience led to a perception that a "one stop shop" did not require the same amount of lesson planning.

With the disrupted instruction due to COVID-19, Jana noticed "gaps" in students' understanding, but found these resources useful for understanding and redefining the specific nature of the “gaps” using strength-based language. For example, Jana states, “third graders who have first grade gaps and second grade gaps, this LT has been really important for [teachers] and really useful.” Similarly Mia transitioned from “gaps” to strength-based language by noticing that teachers were more focused on student work rather than “just number correct.” Mia also reported increased student confidence saying, “It was a huge difference just within a week's time and [the students] felt more confident. The kids were not afraid of being wrong anymore, and they were very comfortable with being able to manipulate and do the work.” Mia also reported teacher confidence saying, “The teachers were like blown away that our kids can actually do this.” This transition from “gaps” to strength-based language was particularly important as teachers implemented previous and current grade standards. Jana explained that “A lot of our teachers are trying to become experts in [prior grade] areas that they're not used to being an expert and having the LT and activities at their fingertips has been super helpful so far” because it helps teachers to “think more about the next steps that the students need.”

In addressing research question 2b) “In what ways did teacher designers’ deep dive into LTs translate into their use of LTBI in coaching or leading school districts?”, we found that mathematics specialists and school-based coaches used these resources as a type of professional development to look at LTs using student strengths. Sienna described this as, “looking at more about what is the next step, the thing that [students] need, what is it that our students do know and because of that we're thinking more about the progression of the students, rather than the end.” Mia explained that these resources were especially beneficial after the disrupted learning due to COVID-19 because they “gave me a tool, because I kind of felt as lost as they did. [sic] we've never done this before, we've never faced it… I feel like anybody I've told about it, I feel like it makes so much sense, the fact that it's [strength-based] like this is, this is what they need to come in with versus where they're going.” However, the way the coaches implemented the resources varied. Mia created classroom-ready Google Slides for a routine. “I just kind of developed these little snippets of little activities of Google slides that I would use with the students and the teachers.” Mia also explained the need for creating supplemental resources because “my teachers [sic] are stressed, they are worn out” and by “packaging” the resources, she can share them with teachers and the many long term substitute teachers in her school. Further, these resources have been accepted and distributed at multiple leadership levels from teachers, principals, and even division levels. Leaders at Jana’s school and district were also eager to distribute the modules. She stated, “Our curriculum supervisor at the division level is the one who's really pushing it out for our division [sic] and in our curriculum unit guides that we have for each of our units k-5 with the PDFs from the bridging site have been dropped in so teachers have access to the site. They know about it, they've had to go through video training on it.”

In addressing research question 2c) “How does the focus on strengths-based instruction influence teacher designers’ approach to their instruction?”, we found that teacher designers began to have a more holistic view of student knowledge and they were less overwhelmed. Making the connection to growth mindset, Sienna noted that the teachers she works with were "thinking more about the next steps that the students need rather than 'they can't get to the standard' or 'they're not doing the standard.' It's more about 'this is where they're at, this is the next step they need to know.'" For example, Mia shifted focus from percentage correct to patterns evident in student work, reporting, "there are patterns in student work, and you can often find that they're doing one thing correctly and then they're just misinterpreting whatever next step." This perspective of highlighting strengths reduced teacher stress, because "it just kind of helps it not seem as overwhelming because you can see progress in a student, even though they're not achieving the standard, if you're looking at their strengths and what they do know."

Implication of our Study

This design-based research suggests several practical applications for teachers, coaches, and curriculum developers. Firstly, this research showed that teacher designers' knowledge of LTs supported their instructional decision making. The teacher designers who participated in the professional development program were recruited for their level of experience and recognized mathematics education leadership. And yet, after participating, they felt more equipped to identify and implement rich strength-based lessons. Specifically, teacher designers knew more about LTs and how to interpret student work in order to identify their strengths on the LT.

Secondly, this research showed that teacher designers who were coaches or leaders distributed their understanding about strength-based LTBI on both a small scale, such as coach to teacher, and large scale, such as school or district settings. While the structure of the modules provided support for teachers' understanding of the LT, coaches needed to constantly refer back to the resources until teachers used consistency when implementing them. All leaders reported that the materials were organized, accessible, easy to modify, and invoked discussions about student strengths and differentiated lessons.

Finally, this design research gives insight on the important aspects to consider when designing curricula and resources. The resources in this project were sourced by a recruited group of teacher designers. Through multiple cycles, the teacher designers vetted and implemented the resources, gathered student work, and discussed the implementation to validate the choice of activities. This cycle allowed teacher designers to propose, explore, discuss, and finally edit those resources in order to make them more user-friendly and applicable to the standards. This research demonstrated significant ways that practitioners can engage with learning trajectory research to transform "post pandemic pedagogy" (Ladson-Billing, 2021), reframing the predominant conversation from "educational gaps" to student progress along the LT continuum.

Acknowledgments

The work reported here is supported by the Virginia Department of Education

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Celedón-Pattichis, S., Peters, S. A., Borden, L. L., Males, J. R., Pape, S. J., Chapman, O., Clements, D. H., & Leonard, J. (2018). Asset-Based Approaches to Equitable Mathematics Education Research and Practice. *Journal for Research in Mathematics Education*, *49*(4), 373–389.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. *Improving Access to Mathematics: Diversity and Equity in the Classroom*, 105-117.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6,* 81-89.

Cohen, E. & Lotan, R. & Scarloss, B. & Arellano, A. (1999). Complex instruction: Equity in cooperative learning classrooms. *Theory Into Practice, 38*, 80-86. 10.1080/00405849909543836.

Confrey, J. (2012). Articulating a learning science foundation for learning trajectories in the CCSS-M. In Van Zoerst, L. R. Lo, J. J. & Kratky, J.L. (Eds.). Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology Mathematics Education (pp. 2-20). Kalamazoo, MI. Western Michigan University.

Confrey, J., & Maloney, A. (2010). The construction, refinement, and early validation of the equipartitioning learning trajectory. In Gomez, K., Lyons, L., & Radinsky, J. (Eds.) *Learning in the Disciplines: Proceedings of the 9th International Conference of the Learning Sciences* (ICLS 2010) – Volume 1, Full Papers. International Society of the Learning Sciences: Chicago IL. Retrieved from http://dl.acm.org/citation.cfm?id=1854484.

Gloria Ladson-Billings (2021) I'm here for the hard re-set: Post pandemic pedagogy to preserve our culture. *Equity & Excellence in Education, 54*(1), 68-78, DOI: 10.1080/10665684.2020.1863883

Gresalfi, Melissa & Martin, Taylor & Hand, Victoria & Greeno, James. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics, 70*, 49-70. 10.1007/s10649-008-9141-5.

Knodel, J. (1993) The design and analysis of focus group studies in social science research. In D. Morgen (ed.). *Successful Focus Groups: Advancing the State of the Art* (pp. 35-50). Sage.

Kobett, B. M., & Karp, K.S. (2020). *Strengths-Based Teaching and Learning in Mathematics: Five Teaching Turnarounds for Grades K–6.* Corwin.

Lotan, R. (2003). Group-worthy tasks. *Educational leadership: Journal of the Department of Supervision and Curriculum Development*, *N.E.A., 60,* 72-75.

Moll, L., Amanti, C., Neff, D., & Gonzalez, N.(1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory into Practice, 31*(2), 132–141.

Morgan, D. L. (1997). Conducting and analyzing focus groups. In *Focus groups as qualitative research* (pp. 46-65). SAGE Publications, Inc., https://dx.doi.org/10.4135/9781412984287

Myers, M., Szajin, P., Wilson, P. H., & Edgington, C. (2015). From implicit to explicit: articulating equitable learning trajectories based instruction. *Journal of Urban Mathematics Education, 8*(2), 11–22.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in middle school mathematics: Initiating critical conversations*. Author.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, *6*(2), 91–104. https://doi.org/10.1207/s15327833mtl0602_2

Stake, R. E. (1995) *The art of case study research.* Sage.

Stein, M. K., & Smith, M. (2011). *5 practices for orchestrating productive mathematics discussions*. Reston, VA: National Council of Teachers of Mathematics.

Suh, J. M. & Seshaiyer, P. (2014). Examining teachers' understanding of the mathematical learning progression through vertical articulation during Lesson Study. *Journal of Mathematics Teacher Education*, *18*(3), 217-229.

Suh, J. M., Birkhead, S., Frank, T., Baker, C., Galanti, T., & Seshaiyer, P. (2021). Developing an Asset-Based View of Students' Mathematical Competencies Through Learning Trajectory-Based Lesson Study, *Mathematics Teacher Educator*, *9*(3), 229-245. Retrieved May 29, 2022, from https://pubs.nctm.org/view/journals/mte/9/3/article-p229_1.xml

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156.

Wilson, P. H., Sztajn, P., Edgington, C., & Myers, M. (2015). Teachers' uses of a learning trajectory in student-centered teaching practices. *Journal of Teacher Education, 66*(3), 227–244.

ELEMENTARY MATHEMATICS TEACHERS' ASSEMBLAGE OF NUMBER TALKS: AN INQUIRY GUIDED APPROACH

Hilary R. Tanck
High Point University
htanck@highpoint.edu

Beth L. MacDonald
Illinois State University
blmacd1@ilstu.edu

This inquiry explores elementary mathematics teachers' curriculum work as they integrated a new component, Number Talks, into their existing curriculum assemblages. Number Talks are short 5-15 minute whole group discussions with specific procedures focusing on students' strategies for mental computations. Assemblage theory was used to frame how participants modified Number Talks when incorporating them into their classrooms. Participants described how they integrated Number Talks into their existing curriculum assemblages, collaborated with colleagues creating overlapping assemblages, and deterritorializing Number Talks to change their functionality. We discuss the implications of participants changes to Number Talks and the adjustments participants made to their existing curriculum assemblages.

Keywords: Curriculum, Elementary Education.

Number Talks began in the 1990's and have increased in popularity informing professional development, books, and educational blogs (https://www.mec-math.org/number-talks/). The effects of Number Talks, however, are largely under investigated (Matney et al., 2020). Number Talks are teacher facilitated discussions centered on students' mathematical strategies for solving computational tasks (Parrish & Dominick, 2016). These conversations typically last between five and 15 minutes. Number Talks follow specific procedures in which students solve a task mentally, share solutions, and describe their strategies in a whole group discussion. Due to the focus on students' ideas and strategies, Number Talks alter the role of the teacher moving from a traditional teacher-directed mathematics classroom to a more student directed classroom (Brown, Stein, & Forman, 1996) requiring significant changes to practice for most mathematics teachers (Cobb & Jackson, 2011). As teachers' professional knowledge expands and new curriculum resources are introduced teachers', curriculum work also changes (Gueudet & Trouche, 2009). This study aims to examine teachers' curriculum work as they integrated Number Talks into their mathematics classroom. *The following research question guided the inquiry: how do elementary mathematics teachers assemble Number Talks?*

Theoretical Perspective

Linear or mechanistic frameworks of curriculum are restrictive and reductive and have dominated the conceptualization of curriculum in mathematics education (Fleener, 2002). Moreover, curriculum defined as a bounded series of learning goals isolated from the learning environment is static (Doll, 1993; Grumet, 1988). While researchers (e.g., Remillard, 2005; McDuffie et al., 2018) acknowledge teachers implement curricula materials in unique ways and Cai's (2014) leveled framework considers what occurs in the classroom (i.e., intended curriculum) findings have done little to influence how curriculum is conceptualized. This study embraces the temporality, connectivity, and in process conceptualization of curriculum advocated by Grumet (1988).

Deleuze and Guattari's (1987) theorization of assemblages provided a space for teachers' curriculum work to be considered in process. Deleuze and Guattari described an assemblage as a collection of heterogenous components organized to perform some function. Assemblages are temporary and transform quantitatively and qualitatively as components are added or removed. For example, a teacher enacting a Number Talk may use Number Talks procedures, allow students to select the numbers for the mathematical task, and add a checking your solution component to conclude the discussion in which students use sticky notes on the board to verify their answer. The additions and modifications to Number Talks changes the functionality of this curriculum. Each component works together for a moment in time creating an assemblage. Moreover, an assemblage is not simply a product, but rather the coming into existence or the need to organize, reorganize, and/or construct. As the assemblage shifts or sharpens boundaries map out an assemblage's territory (i.e., territorialization). Conversely, as changes occur, or new components are introduced an assemblage blurs or fragments and deterritorializes.

Methods

Part of a larger Convergent Mixed Methods study (Creswell & Plano-Clark, 2011), this paper explains results from a qualitative analysis of teacher participant interviews, examining ways teachers planned, adapted, and assembled Number Talk curriculum with their already enacted district-driven curriculum.

Participants, Setting, Procedures, and Instruments

Seventeen elementary teachers—teaching 4^{th}, 5^{th}, and 6^{th} grade mathematics—from Duval County School District in the Western United States participated in the interview stage of the study during the fall of 2021. Four of the participants were enacting Number Talks for the second year, while 13 introduced Number Talks into their classrooms for the first time. Each participant engaged in one 15-minute semi-structured interview via Zoom. Interviews focused on teachers' planning process and how Number Talks were being implemented in their classrooms. The interview included questions, such as: (1) what role do Number Talks play in your mathematics classroom?; (2) how do you select number talks?; (3) what adjustments if any, did you make to incorporate Number Talks into your classroom? The researchers created memos in the form of written notes during and following the interviews.

Analysis

An open-coding scheme was used to analyze the memos. Codes were condensed into categories (e.g., integration of Number Talks and curriculum, number sense, and Number Talks procedures). While this process offered a level of analysis, it left the data segregated. A different method of analysis was needed to create opportunities for entanglement. Taking a writing approach to analysis Markham's (2012) bricolage-style of writing (i.e., layered text) or the bringing together of various writing piece was taken up. We initially engaged in analytic memo writing, summarizing the interviews as a collection. Then layered writing was used to bring together interview memos from each participant to create a composite piece (see Figure 1). The layered writing approach created a grass like structure as new concepts and ideas sprouted between previous text.

opportunities to justify their solutions as when they were engaged in their district provided curriculum materials. Rachel integrates her exciting curriculum and NT. The CMI is similar to NT in that the students engage in similar patterns of discourse and they often begin with student-to-student discussion (Amanda). Some teachers used their pre-existing curriculum as a jumping off point to select number talks and incorporate them into their mathematics classrooms. I don't take it from iReady [pre-existing curriculum]. I add on to what I know what to add on with – using prime factors to use factor trees. How each factor has its own fingerprint (Holly). Would be to adjust them as our curriculum changes. As we move on to fractions, we can consider operations with those fractions (Samantha). Additionally, some teachers followed the NT book and progressed through strategies based on the book. Going to the book to see what the next strategy is (Taylor).

Figure 1: Layered writing excerpt

Preliminary Findings

Through the leveled writing analysis tracings or patterns emerged. Participants described the integration of Number Talks into their existing curriculum assemblages, collaborating with colleagues creating overlapping assemblages, and deterritorializing Number Talks to change their functionality. The subsequent section explores these tracings.

Integrating Number Talks into Existing Curriculum Assemblages

Introducing a new component, Number Talks, into participants existing curriculum assemblages detteritorialized, or disrupted, their assemblages. The boundaries of their curriculum assemblages and daily schedules blurred as participants made adjustments. Participants stated one of the biggest challenges was finding time for the Number Talks. Taylor described this challenge, "I had to find time. Where was I going to find the time where it would be effective for students?" Some participants eliminated components of their existing curriculum assemblage. Madison replaced a component, a warm-up worksheet, with Number Talks, "I used to do a warm-up worksheet. I have gotten rid of the worksheets and my spiral review. This is really strange because I used to give 10 problems and now just give one problem." Other participants modified their curriculum assemblages. Amanda addressed the issue of time by changing her morning routine using the Number Talk as bell work. Lana on the other hand moved her math lesson to a different part of the day to allow for an extended math lesson. To accommodate time constraints, participants used Number Talks at varying frequencies (e.g., Jessica used Number Talks every day, Rachael tries to do them two or three times a week).

Participants' conceptualization of how Number Talks fit into their curriculum assemblages ranged from siloed events to integrated components. Participants who conceptualized Number Talks as siloed events either separated Number Talks from the mathematics lesson enacting them at a different point in the day or used Number Talks before the mathematics lessons but saw them as independent events. Stephanie took up a siloed approach, "I do it [Number Talks] first thing in the morning when the students come in. It does not tie into the math lesson." Lana also held a siloed conceptualization. Although Number Talks were taught at the same time as the mathematics lesson—immediately prior to the lesson—they were unconnected events. Kenny on the other hand conceptualized Number Talks as integral components of the mathematics lesson. Kenny used Number Talks to start class and help students get into a mathematical mindset—he described Number talks as a safe place for students to share ideas and lead into the mathematical content of the day. Some participants conceptualized Number Talks as an integrated part of the curriculum but were not yet able to match their conceptualization to their teaching practice. Integrating new components is a process and Dana said she is currently doing Number Talks unrelated to the lesson; however, she wants to eventually tie them into class.

Overlapping Assemblages

Participants' curriculum assemblages overlapped as they collaborated with colleagues and students. Many of the participants planned Number Talks with their team or other colleagues in their building. Participants noted working with colleagues when struggling to get students to try a specific strategy. For example, Alexa was unable to get her students to use a regrouping strategy. She worked with her team to change mathematical tasks like 68-13 to 63-18 to promote the use of a regrouping strategy. Participants' curriculum assemblages also overlapped when they brought student strategies from a colleague's class and used it in their own number talk to demonstrate a specific strategy.

In addition to working with colleagues to develop Number Talks, participants both explicitly and implicitly built Number Talks with their students. Participants (e.g., Marissa and Taylor) implicitly worked with students to adjust pacing based on students' responses to previous Number Talks. Amanda planed with students providing them opportunities to select the numbers for the Number Talk.

Teachers Deterritorialization of Number Talks

As participants integrated Number Talks into their classrooms, they made modifications in order for the Number Talks to function in their classroom. Participants interpreted Number Talk procedures in their own way and adjusted to match their teaching style. Some participants tried to follow the Number Talk procedures as intended. Jessica used dot talks when introducing her class to the Number Talk procedures of hand signals, giving solutions, and sharing strategies. Alexa altered students' physical space moving students to the floor to decrease distractions and ensure students were working the task in their heads instead of writing out their process. Other participants modified Number Talk procedures. Carolyn provided students with sticky notes to record their ideas. COVID-19 also presented challenges participants needed to address and pushed on participants' Number Talk assemblages. For example, Alexa created "calling bodies" or remote students assigned to in person student who communicate via phone call or zoom discussions.

Discussion and Conclusion

Findings elicit questions about how Number Talks are integrated into existing curriculum assemblages. Framing teachers' work with Number Talks using assemblage theory provided opportunities to examine participants' work as temporal and changing with their integration, overlapping components, and modifications. In efforts to accommodate a new component, Number Talks, into their existing curriculum assemblage participants added and removed elements from both their pre-existing curriculum and to Number Talks. Continued research is needed to understand teachers' curriculum work with Number Talks over time. Additionally, future research could consider how teachers' curriculum work with Number Talks impacts students' agency and number sense development.

References

Brown, C. A., Stein, M. K., & Forman, E.A. (1996). Assisting teachers and students to reform the mathematics classroom. *Educational Studies in Mathematics, 31*(1), 63-93.

Cai, J. (2014). Searching for evidence of curricular effect on the teaching and learning of mathematics: Some insights from the LieCal project. *Mathematics Education Research Journal, 26*, 811-831.

Cobb, P., & Jackson, K. (2011). Towards an empirically grounded theory of action for improving the quality of mathematics teaching at scale. *Mathematics Teacher Education and Development,* 13(1), 6-33.

Doll, W.E.J. (1993). *A post-modern perspective on curriculum.* New York, NY: Teachers College Press.

Fleener, M. J. (2002). *Curriculum dynamics: Recreating heart.* New York, NY: Peter Lang.
Grumet, M. R. (1988) *Bitter milk: Women and teaching.* Amherst: University of Massachusetts Press.
Gueudet, G. 7 Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71,* 199-218.
McDuffie, A. R., Choppin, J., Drake, C., Davis, J.D., & Brown, J. (2018) Middle school teachers' differing perceptions and use of curriculum materials and the common core. *Journal of Mathematics Teacher Education. 21,* 545-577.
Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black Holes of Research on Instructional Practice: The Case of Number Talks. *Investigations in mathematics Learning, 12*(4), 246-260.
Markham, A. (2012). Fabrication as ethical practice: Qualitative inquiry in ambiguous internet contexts, *Information, Communication and Society, 15(3)*, 334-353.
Parrish, S., & Dominick, A. (2016). *Number talks: Fractions, decimals, and percentages.* Math Solutions.
Remillard, J. T. (2005) Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211-246.
St. Pierre, E.A. (2018). Writing post qualitative inquiry. *Qualitative Inquiry,* 24, 603-608.

HOW ALGEBRA TEACHERS' PERCEPTIONS OF THE ILLUSTRATIVE MATHEMATICS CURRICULUM EVOLVED OVER THREE YEARS

Mollie H. Appelgate
Iowa State University
mollie@iastate.edu

Ji Yeong I
Iowa State University
jiyeongi@iastate.edu

Anne Foegen
Iowa State University
afoegen@iastate.edu

Betsy Araujo Grando
Iowa State University
betsya@iastate.edu

Keywords: Curriculum, High School Education

Background

The instructional materials that mathematics teachers use impact student learning (Remillard et al., 2014; Lloyd et al., 2017). However, purchasing instructional materials are expensive, hence districts are turning to open source materials such as Illustrative Mathematics (IM). Unlike many other free online resources for instructional materials (e.g. Pinterest (Shapiro et al., 2019)), IM is a comprehensive curriculum with a scope and sequence and is aligned with Common Core content and practice standards (Ed Reports, 2022). The high school IM curriculum became available in 2019 and the elementary curriculum in 2021, so little research has been done on teacher perceptions of the curriculum. This poster aims to address this by sharing teacher perceptions of the IM curriculum using data from teacher interviews from year 1 and year 3 of their implementation of IM.

Methods

All Algebra 1 teachers throughout an urban district began using IM in 2019-2020 (year 1) and it continued to be the primary instructional resource provided by the district in 2021-2022 (year 3). We interviewed 22 Algebra 1 teachers in year 1 and 13 in year 3 about their perceptions of the IM curriculum using an interview protocol. The protocol ranged from open-ended questions, such as "What do you think about the Illustrative Mathematics curriculum?" to more specific, e.g. "How does IM support your ELL students?" The interview transcripts were read to uncover themes that emerged, common themes were grouped when appropriate, and then the interviews were reread to code for the emergent themes. Finally, we recorded how many teachers expressed an idea represented by each emergent theme.

Findings

Some themes stayed the same from year 1 to year 3. For example, over 50% of the teachers in both years noted the benefit of IM's use of real-world contexts for Algebra 1 topics. Over 50% both years also noted a challenge in not enough time for teaching everything. Other themes changed over time. In year 1, teachers appreciated having ready-to-use materials (in previous years they had to create their own teaching resources), however, in year 3, teachers were more likely say the appreciated the design of the curriculum, for example, how it spiraled the content.

Conclusion

As the districts move to open-source instructional materials such as IM it is important to understand how teachers are perceiving the curriculum so that curriculum designers, researchers, and district decision makers can use the information to better support teachers.

References

Ed Reports. (2022). Kendall Hunt's Illustrative Mathematics Traditional (2019). Edreports.org. https://www.edreports.org/reports/overview/kendall-hunts-illustrative-mathematics-traditional-2019

Lloyd, G. M., Cai, J., & Tarr, J. E. (2017). Issues in curriculum studies: Evidence-based insights and future directions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 824-852). Reston, VA: National Council of Teachers of Mathematics.

Remillard, J. T., Harris, B., & Agodini, R. (2014). The influence of curriculum material design on opportunities for student learning. *ZDM—International Journal on Mathematics Education, 46*(5), 735–749.

Shapiro, E. J., Sawyer, A. G., Dick, L. K., & Wismer, T. (2019). Just what online resources are elementary mathematics teachers using?. *Contemporary Issues in Technology and Teacher Education*, *19*(4), 670-686.

ANALYZING THE LEVELS OF COGNITIVE DEMAND OF TASKS IN MATHEMATICS TEXTBOOKS AND UNIVERSITY ENTRANCE EXAMS

Seyedehkhadijeh Azimi Asmaroud

Keywords: University entrance exams; Text book; Levels of cognitive demand

The University Entrance Exam (UEE), a comprehensive 4-5 hours multiple-choice exam known as Konkour, is a centralized nationwide test for high school graduates to gain admission to higher education in Iran. These UEEs have been the sole criterion for students' admission to universities for more than four decades (Salehi & Yunus, 2012). As students' performance in the UEE is a critical factor influencing their future careers, it is essential that UEEs, as assessment tools, be aligned with school curriculums.

This study was part of a larger study that compared the type of the mathematics questions in two series of high school mathematics textbooks, old-series (2016-2018) and new-series (2019 - 2021), with the type of mathematics questions used in the UEEs in the aforementioned periods. The underlying rationale for focusing on the textbooks used in these two periods was that these textbooks are used nationwide and reflect a good picture of the nature of tasks implemented by teachers to prepare students for UEEs. In this study, the result of the study about geometry questions will be discussed.

This study draws from a framework developed by Smith and Stein (1998) about the Levels of Cognitive Demands (LCD) of tasks. This framework categorizes mathematical tasks according to the level of cognitive demands in two categories: a) low-level, including Memorization (LM) and Procedures Without Connections (LP), and b) high-level, including Procedures With Connections (HP) and Doing Mathematics (HDM). Questions related to topics in ratio and proportion, angle relationships in circles, Thales theorem, similarities and its applications, polygons, and area from 10th and 11th-grade textbooks (305 tasks), and also questions related to the same topics in UEEs (34 questions) were coded based on this framework. To ensure the validity of the study, I asked a colleague (Creswell & Miller, 2000) to do coding independently for random questions from UEEs and two textbooks series, then we compared assigned codes.

The findings of the study indicate that the levels of the tasks related to geometry concepts in the old textbook series were predominantly at HP and LP levels (respectively 38% and 36%). The majority of tasks in the new textbook series were at the HP level (49%). There were more HDM tasks in the new textbook series (31 % in comparison to 18% in the old book series). There were not any questions at the LM level in UEEs in both periods. The percentage of questions in UEEs in HP level was much higher in comparison to the two textbook series (53% in 2016 - 2018 UEEs and 66% in 2019- 2021 UEEs). The percentages of questions at the HDM level were 21% in 2016 - 2018 UEEs and 17% in 2019 - 2021 UEEs (tables including more detailed information will be presented in the poster).

Research showed that throughout the classroom sessions LCD of tasks stay the same or decline (Stein, Grover, & Henningsen, 2001). Thus, considering the findings of the study that shows on average the LCD in UEEs are higher than the LCD of tasks provided in both textbook series, it seems that including more tasks in higher LCD is required in the school's curriculum.

References

Elton, L. R. B., & Laurillard, D. M. (1979). Trends in research on student learning. Studies in Higher Education, 4, 87–102.

Creswell, J. W., & Miller, D. L. (2000). Determining validity in qualitative inquiry. Theory into practice, 39(3), 124-

130.
Salehi, H., & Yunus, M. M. (2012). University entrance exam in Iran: A bridge to a dam. Journal of Applied Science Research, 8(12), 1005-1008.
Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. Mathematics teaching in the middle school, 3(4), 268-275.
Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. S. (2001). Implementing standards-based mathematics instruction. Book Reviews, 33(3), 99-101.

CHANGING POPULATIONS: USING THE PSMS WITH TEACHERS

Jonathan D. Bostic	Timothy Folger	Gabriel Matney
Bowling Green State Univ.	Bowling Green State Univ.	Bowling Green State Univ.
bosticj@bgsu.edu	tdfolge@bgsu.edu	gmatney@bgsu.edu

Toni A. May	Kristin Koskey	Gregory Stone
Drexel University	Drexel University	MetriKs Amérique
tas365@drexel.edu	kk3436@drexel.edu	gregory@metriks.com

Keywords: Assessment

Using a test for a purpose it was not intended for can promote misleading results and interpretations, potentially leading to negative consequences from testing (AERA et al., 2014). For example, a mathematics test designed for use with grade 7 students is likely inappropriate for use with grade 3 students. There may be cases when a test can be used with a population related to the intended one; however, validity evidence and claims must be examined. We explored the use of student measures with preservice teachers (PSTs) in a teacher-education context. The present study intends to spark a discussion about using some student measures with teachers. The Problem-solving Measures (PSMs) were developed for use with grades 3-8 students. They measure students' problem-solving performance within the context of the Common Core State Standards for Mathematics (CCSSI, 2010; see Bostic & Sondergeld, 2015; Bostic et al., 2017; Bostic et al., 2021). After their construction, the developers wondered: *If students were expected to engage successfully on the PSMs, then might future grades 3-8 teachers also demonstrate proficiency?*

Methods

Data came from three sources: (a) an expert panel content review, (b) Rasch (1980) modeling of PSM scores, and (c) consequences from testing data from PSTs and PSM administrators. 178 PSTs from a Midwest university completed the PSMs. They came from two teacher education programs: grades K-5 or grades 4-9. PSMs 3-8 were completed in their program's first-year and again in the fourth-year. The intended use for the PSMs was formative and for program evaluation. They were informed that results did not impact course grades. Content and consequences data were gathered from mathematics content and mathematics education instructors. Qualitative data were analyzed using thematic analysis (Miles et al., 2016). Quantitative data were analyzed with WINSTEPS© (Linacre, 2019).

Findings & Discussion

Content experts felt items were appropriate for use with PSTs and connected with content from their classes. PSMs 3-8 fit the Rasch model indicating good psychometric quality. Finally, thematic analysis of consequences data indicated that the PSMs felt no different than a unit test and offered formative data to adapt instruction. Collectively, these findings help to inform the potential uses of the PSMs, a student measure, for use with a related population, PSTs.

Acknowledgments

Ideas in this manuscript stem from grant-funded research by the National Science Foundation (e.g., NSF #1702646, 1720661; 2100988, 2101026). Any opinions, findings,

conclusions, or recommendations expressed by the authors do not necessarily reflect the views of NSF.

References

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education. (2014). *Standards for educational and psychological testing.* Washington, DC: AERA.

Bostic, J., Sondergeld, T, Matney, G., Stone, G., & Hicks, T. (2021). Gathering response process data for a problem-solving measure through whole-class think alouds. *Applied Measurement in Education, 34*(1), 46-60. *DOI: 10.1080.08957347.2020.1835913*

Bostic, J., Sondergeld, T., Folger, T. & Kruse, L. (2017). PSM7 and PSM8: Validating two problem-solving measures. *Journal of Applied Measurement, 18*(2), 151-162.

Bostic, J., & Sondergeld, T. (2015). Measuring sixth-grade students' problem solving: Validating an instrument addressing the mathematics Common Core. *School Science and Mathematics Journal, 115, 281-291*.

Common Core State Standards Initiative. (2010). Common core standards for mathematics. http://www.corestandards.org/Math/

Miles, M., Huberman, A.M., Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook* (3rd Ed.), SAGE Publications.

Rasch, G. (1980). *Probabilistic models for some intelligence and attainment tests* (expanded ed.). Chicago, IL: University of Chicago Press.

Linacre, J. M. (2019). *WINSTEPS®* (Version 4.4.5) [Computer software]. Beaverton, Oregon. https://www.winsteps.com/winsteps.htm

A CHARACTERIZATION OF COLLEGE ALGEBRA ASSESSMENT DURING THE TRANSITION TO EMERGENCY REMOTE TEACHING

Ash Demian
Texas State University
afd9@txstate.edu

Keywords: Assessment, Emergency Remote Teaching, College Algebra

Introduction and the Purpose of the Study

The COVID-19 pandemic led to a global lockdown and compelled the institutions to shift to emergency remote teaching. This transition and the lack of knowledge of using technology affected teaching practices, including assessment. Confronted with new contexts for assessment and threats to validity caused by ineffective proctoring, many instructors had to rethink how to evaluate student progress. This study investigates the common characteristics of college algebra assessment in six dimensions and determines any changes during emergency remote teaching. In addition, the analysis compares the college algebra instructors' views about the purpose of evaluation. Finally, the study tested the efficacy of a new tool that instructors can use to analyze their assessments.

Research Questions

What are the characteristics of the exams given by college algebra professors? What are the college algebra teachers' beliefs about the purpose of assessment? How did these characteristics change during ERT, and what changes persisted after returning to normal teaching? What factors determined which changes faculty chose to retain?

Study Framework

The study framework consists of six dimensions. Cognitive demand, first introduced by Bloom (1956) and revised by Tallman et al. (2016), contains seven hierarchal levels starting at *remember* at the base and *create* at the top. Three additional dimensions, item format, task representation, and solicited solution, were adapted from Tallman et al. (2016). Additionally, the framework contains two new dimensions developed for study: computational demand and verbal demand.

Methodology and Results

Data Collection

Participants of this study are 30-40 college algebra instructors from two- and four-years institutions in Texas that have taught college algebra before, during, and after the COVID-19 pandemic. The study uses three data sources: mid-term and final tests to determine the characteristics of college algebra tests, online surveys to determine the instructors' beliefs about the purpose of assessment and focus-group meetings on Zoom with a subset of instructors.

Results

The preliminary results of twenty tests showed a low level of cognitive demand where 65% of the items were level 2 (recall and apply procedures). On the other hand, a focus group meeting showed that the instructors emphasized higher levels of cognitive demand. Furthermore, 65% of the items are multiple-choice, and 80% are symbolic.

References

Bloom, B. S. (1956). Taxonomy of educational objectives. Vol. 1: Cognitive domain. *New York: McKay, 20*, 24.

Bloom, B. S. (1984). Taxonomy of educational objectives (1956). *The classification of educational goals, Handbook I Cognitive Domain*.

Tallman, M. A., Carlson, M. P., Bressoud, D. M., & Pearson, M. (2016). A characterization of calculus I final exams in US colleges and universities. *International Journal of Research in Undergraduate Mathematics Education, 2*(1), 105-133.

.

EXPLORING DISSONANCE AND HARMONY IN ELEMENTARY MATHEMATICS TEACHERS' CURRICULAR USE, AUTONOMY, DECISION-MAKING, AND COHERENCE

Kristin Doherty
Michigan State University
kdoherty@msu.edu

Mona Baniahmadi
Duquesne University
baniahmadim@duq.edu

Bima Sapkota
Purdue University
bsapkota@purdue.edu

Emmanuel K. Adjei
Purdue University
eadjei@purdue.edu

Jon Brown
University of Arizona
jebrown@email.arizona.edu

Jill A. Newton
Purdue University
janewton@purdue.edu

Marcy B. Wood
University of Arizona
mbwood@arizona.edu

Amy M. Olson
Duquesne University
olsona@duq.edu

Corey Drake
Michigan University
cdrake@msu.edu

We surveyed 524 elementary teachers from 46 states about their mathematics curricular decision-making during the COVID-19 pandemic (Giorgio-Doherty et al., 2021). Building on findings from this study, we designed protocols for individual and focus group interviews with teachers from the same school to further explore the complexity of teachers' curricular use revealed in the survey. Here, we report findings from these interviews. The research questions guiding our study were: (1) How do teachers create coherence across their curricular resources? (2) How and why do teachers select or evaluate and adapt curricular materials to teach mathematics? and (3) How do teachers perceive their curricular autonomy? We found most teachers were using multiple curricular materials (some mandated and some not, ranging up to 11 different sources of materials) to plan and teach mathematics. We found teachers frequently focused on content (e.g., connections to key concepts, standards, or lesson objectives) as the key focus of creating coherence between materials for students and for lesson planning. Content coherence also seemed to drive differentiation. Most teachers reported differentiating more than usual to address larger learning gaps across students in their classes due to the pandemic. Teachers stated a preference for curricular materials (like *IXL*) which allowed them to find tasks on particular content and assign students practice problems related to that content at grade level and above/below. Teachers varied in how they responded to different solution strategies presented by different materials. Some teachers provided coherence by directing students to rewrite directions so they use the same solution strategy as in prior lessons from other materials; other teachers felt students experienced better connections to content as they made sense of different solution strategies from different materials. Several teachers used *TeachersPayTeachers* (TPT), with some reporting materials on TPT engage students better than materials from their primary curriculum, do not need modifications, or better meet their instructional preferences. We found teachers who reported high levels of enjoyment and confidence in teaching mathematics made the most adaptations to curricular materials and most often designed their own. Many teachers reported adapting curriculum to be more engaging to students. This included adding visuals or enlarging text size and creating opportunities for problem-based and hands-on learning. Some reported adapting or creating materials to make connections to the world or be more culturally relevant. Overall, teachers reported high levels of curricular autonomy for all curricular materials, including mandated materials (median score of 7 out of 10) and non-mandated or suggested materials (all scoring 10 out of 10).

References

Giorgio-Doherty, K., Baniahmadi, M., Newton, J., Olson, A. M., Ferguson, K., Sammons, K., Wood, M. B., & Drake. C. (2021). COVID and curriculum: Elementary teachers report on the challenges of teaching and learning mathematics remotely. *Journal of Multicultural Affairs, 6*(2). https://scholarworks.sfasu.edu/jma/vol6/iss2/3

DIMENSIONS OF CARE: A DIFFERENT APPROACH TO ANALYZE TEACHERS' INTERACTION WITH THE CURRICULUM

Ana-Maria Haiduc
Purdue University
ahaiduc@purdue.edu

Literature Review

Teachers' refined thinking about how students learn mathematics is visible in their curriculum decisions when planning. They make decisions influenced by many factors: the design of the materials, teachers' knowledge, skills, beliefs, goals, and context (Remillard et al., 2009). In addition to these factors, I argue that mathematical care influences teachers' decisions. The mathematical care component is conceptualized in this research as a driven factor for attending to students' cognitive processes of learning mathematics and continuously adapting the mathematical activities based on students' specific needs. These two actions align with the first two phases that characterize the notion of a caring relationship largely discussed by Noddings (2013): engrossment (receptive to the learners' needs), and motivational displacement (respond with help to the identified needs). Responding to every student's mathematical needs means making modifications, adaptations, and accommodations to the curriculum (Remillard et al., 2014; Stein et al., 2007). For example, adaptations of the mathematical activities in relation to students' mathematical needs could be in the form of incremental challenges for the students who gave up learning mathematics (cultivating a growth mindset set (Dweck, (2008)) or higher cognitive demanding tasks for students with an increased interest in mathematics (Benbow, et al. 1992), warm-up activities to supply motivation (Williams, 1984) or providing a visual context for English learners (Slavit & Ernst-Slavit, 2007). According to Noddings (2013), students' responses to these actions represent the third component that completes a caring relationship: recognition. When teachers "act with special regard for the particular person in a concrete situation" (Noddings, 2013, p. 24) they demonstrate care for students. Students who enter into a mathematical caring relation "have a sense of being seen by the teacher" (Hackenberg, 2005, p.47), and that is a feeling of being listened to, with ideas valued, and, perhaps, understood.

Methodology

The paper's contribution lies in exploring teachers' rationale when making decisions regarding the mathematical activities they select and the changes they make. For example, designing tasks in relation to students' ways of operating mathematically (Hackenberg, 2005) or accommodations for the students with learning disabilities that do not reduce the level of engagement (Barr et al., 2021) represent dimensions of care related to the students' cognition. Providing recreational problems to show the fun side of mathematics (Cooney, 1985) represents a dimension of care for getting students excited when doing mathematics. Selecting tasks that give students opportunities to show their intelligence in various ways while the teacher is positioning students as competent by coding a variety of skills, mathematical and non-mathematical, disrupting the hierarchy of intellect and giving voice to all students (Louie, 2017), represents a dimension of care for equity. Thus, this research constructs dimensions of care by analyzing episodes from literature that link needs, decisions, and feelings.

Research Question

The research question is how mathematical care can be conceptualized as a factor that influences teachers' curricular decisions?

References

Barr, F., & Mavropoulou, S. (2021). Curriculum accommodations in mathematics instruction for adolescents with mild intellectual disability educated in inclusive classrooms. *International Journal of Disability, Development and Education, 68*(2), 270- 286. https://doi.org/10.1080/1034912X.2019.1684457

Benbow, C. P., Argo, T. A., & Glass, L. W. (1992). Meeting the needs of the gifted in rural areas through acceleration. *Gifted Child Today Magazine*, *15*(2), 15-19.

Cooney, T. (1985). A beginning teacher's view of problem solving. *Journal for Research in Mathematics Education, 16*(5), 324–336. https://doi.org/10.5951/jresematheduc.16.5.0324

Hackenberg, A. (2005). A model of mathematical learning and caring relations. *For the Learning of Mathematics*, *25*(1), 45-51. https://www.jstor.org/stable/40248486

Louie, N. (2017). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics, 97*(1), 55-69. https://doi.org/10.1007/s10649-017-9775-2

Noddings, Nel (2013), *Caring: A Relational Approach to Ethics and Moral Education*, University of California Press.

Remillard, J. T., Herbel-Eisenmann, B. A., & Lloyd, G. M. (2009). *Mathematics teachers at work: connecting curriculum materials and classroom instruction.* New York & London: Routledge.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM - Mathematics Education, 46*(5), 705-718. https://doi.org/10.1007/s11858-014-0600-4

Slavit, D., & Ernst-Slavit, G. (2007). Teaching mathematics and English to English language learners simultaneously. *Middle School Journal*, *39*(2), 4-11. https://doi.org/10.1080/00940771.2007.11461618

Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. *Second handbook of research on mathematics teaching and learning*, *1*(1), 319-370.

Williams, D. E. (1984). Warm-ups—Keys to effective mathematics lessons. *The Arithmetic Teacher*, *32*(1), 40–43. http://www.jstor.org/stable/41193949

EXAMINING ALGEBRA TEACHERS' UNIT PLANNING PROCESSES

Jaepil Han
University of Missouri
jaepilhan@mail.missouri.edu

Keywords: Curriculum, High School Education, Instructional Activities and Practices

There is growing agreement that effective mathematics instruction starts from teachers' careful and purposeful planning (NCTM, 2014). However, teachers' planning often only includes teachers' decision making *before* instruction or it only consists of *selecting* tasks and *organizing* the ways they will enact the tasks with students (Kilpatrick et al., 2001). Because planning individual lessons typically comes after teachers have already established a larger sequence of lessons, it is important to examine how teachers plan at the larger curriculum level. Despite the existence of different curriculum level planning, little attention has been given to teacher planning on larger curriculum levels (e.g., unit planning, course planning). To this end, the purpose of this study was to answer the question: How are the processes of experienced Algebra I teachers' unit planning similar with and/or different from one another?

As part of a larger dissertation study on experienced algebra teachers' unit planning, I analyzed the processes (e.g., practices, sequences) of 5 experienced Algebra I teachers from school districts of varying sizes across the nation, who actively designing curriculum units came. By adapting Roche et al.'s (2014) *teacher planning framework*, I identified the teaching practices that the teachers enacted in unit planning. I then created diagrams of the teachers' unit planning processes (e.g., unit planning timings and sequences) using *logic models* (Yin, 2014).

The experienced teachers engaged with a set of seven unit planning practices and the sequencing of these practices followed a general sequence across the teachers—a) *checking and examining curriculum documents*, b) *setting learning goals*, c) *mapping out lessons and assessments*, d) *planning lessons*, e) *planning assessments*, f) *modifying lessons and assessments*, and g) *evaluating implemented unit plans*. To plan units, the teachers engaged with the practices not only before units but also *during* and even *after* units. Although they enacted the majority of the unit planning practices before unit, they continued their unit planning during unit as they modified the unit pacing or assessments based on what they actually taught and students' progress towards the unit-end goals. After every unit, the teachers evaluated their implemented unit plans through self-reflection or student feedback regarding the unit pacing and difficulty. Thus, the experienced teachers' unit planning consisted of the teaching practices for *planning*, *implementing*, and *reflecting* unit plans.

The present study expanded the traditional conception of teacher's unit planning to that including not only planning but also enacting and reflecting. This expanded conception aligns with Sherin and Drake's (2009) conception of teachers' curriculum use before, during, and after instruction. Thus, this finding noted teachers' continuous engagement with planning and the reflective aspects of planning, which two were often missing from the body of literature on teacher planning. The findings shed light on experienced algebra teachers' planning processes of larger curriculum units. In addition, it provides a better understanding of the nature of teachers' unit planning and a snapshot of how teachers aim to help students learn outside of the classroom environment, which potentially inform us of supports that we may offer teachers to promote alternative approaches to or sequences of the unit planning practices.

References

NCTM. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academies Press.

Roche, A., Clarke, D. M., Clarke, D. J., & Sullivan, P. (2014). Primary teachers' written unit plans in mathematics and their perceptions of essential elements of these. *Mathematics Education Research Journal*, *26*(4), 853–870.

Sherin, M. G., & Drake, C. (2009). Curriculum strategy framework: Investigating patterns in teachers' use of a reform-based elementary mathematics curriculum. *Journal of Curriculum Studies*, *41*(4), 467–500.

Yin, R. K. (2014). Case Study Research Design and Methods (5th ed.). Los Angeles, CA: SAGE Publications, Inc.

FEATURES OF HIGH-QUALITY MATHEMATICS MATERIALS: VARIATIONS IN APPROPRIATENESS FOR DIFFERENT GROUPS OF STUDENTS

Erica N. Mason
University of Illinois, Urbana-Champaign
enmason@illinois.edu

Camille Griffin
University of Illinois, Urbana-Champaign
cgriffi5@illinois.edu

Keywords: curriculum, teacher beliefs, students with disabilities

The instructional materials available to teachers relate to the types of learning opportunities afforded to students (Hiebert & Grouws, 2007). As school districts undergo ambitious mathematics reform, various stakeholders are brought together to make key decisions related to the instructional materials provided and endorsed by the district. The various stakeholders that constitute a district's curriculum committee likely bring various perspectives about the form and function of features of instructional materials they consider high-quality (e.g., Saxe et al., 1999). Further, stakeholders likely also hold varying views about for whom those high-quality forms are "appropriate" (e.g., Jackson et al., 2017). The purpose of this project was to gain insight about the features of instructional materials committee members viewed as "high-quality," specifically (1) the degree to which there was coherence in the form and function of those features across committee members and (2) committee members' views about students' mathematical capabilities in relation to those features.

This work was conducted across a year-long research-practice partnership between a group of special education researchers and one school district. The district was undergoing ambitious mathematics reform in light of persistent course failure at the high school, thus prompting district leaders to make strides toward vertical alignment. A mathematics curriculum committee ($N = 25$) was formed and consisted of district-level coordinators, building-level administrators, instructional coaches, general education teachers, special education teachers, early childhood teachers, dual-language program teachers, all of whom served students across early childhood and through Grade 12. All committee members were invited to participate in two interviews about their experiences being on the committee and their views about what constituted high-quality instructional materials. Thirteen (52%) of the committee members consented to participate in the first round of interviews, which were conducted in November and December 2021. The majority of interviews were conducted via Zoom (n =10), although a few committee members preferred to meet in person. We used a semi-structured interview protocol, and all interviews were audio recorded and later transcribed.

To address the second articulated purpose, preliminary findings suggest members of the curriculum committee viewed some features as appropriate for all students (e.g., "[enrichment activities] can work for everyone."), some features as specifically not appropriate for some students (e.g., "[materials that allow for multiple ways of solving] can be difficult and challenging [for] students who, maybe their brains process in a different way, or they process at a different rate."), and some features that were both appropriate and not appropriate for particular groups. A second round of interviews will be conducted in April and May 2022. Practically, results from this study could be used to inform the ongoing work of this committee as the district moves into the piloting phase of their reform effort. Beyond this school district, findings from this work could reveal critical, but perhaps overlooked, components of ambitious reform, including the need to uncover committee members' views about students' capabilities and consider the relation of such views to decisions about instructional materials.

References

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester, Jr. (Ed.), Second handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics (pp. 371–404). National Council of Teachers of Mathematics.

Jackson, K., Gibbons, L., & Sharpe, C. J. (2017). Teachers' views of students' mathematical capabilities: Challenges and possibilities for ambitious reform. Teachers College Record, 119(7), 1–43.

Saxe, G. B., Gearhart, M., Franke, M. L., Howard, S., & Crockett, M. (1999). Teachers' shifting assessment practices in the context of educational reform in mathematics. Teaching and Teacher Education, 15, 85–105. 10.1016/S0742-051X(98)00032-8

A COMPARATIVE STUDY OF TRIGONOMETRY STANDARDS BETWEEN GHANA, SOUTH AFRICA, THE UNITED STATES, AND ZAMBIA

Rose Mbewe
Purdue University
rmbewe@purdue.edu

Amanda Huffman
Purdue University
huffma33@purdue.edu

Emmanuel Adjei
Purdue University
eadjei@purdue.edu

Keywords: Standards, Curriculum, Policy, Comparative.

Over the past two decades, increasing demand for quality education has been the focus of many educators in the United States who have introduced the standards-based education movement (Ravitch, 1995). Countries worldwide present standards as national curricula (Schmidt et al., 1997). Regardless of their title, such curriculum or standards define the vision for what is important for the nation's children to learn in their schooling. Research shows that there is a connection between mathematics standards and the mathematics performance of the students (Akkoc, 2008). Furthermore, research on students' achievement in mathematics attributes achievement disparities to the curriculum and argues that different curriculum standards may produce different results (Cai, 2004). This comparative standards analysis of mathematics standards addresses one secondary mathematics topic, trigonometry of four countries as we investigated similarities and differences in learning expectations for students from Ghana, South Africa, the United States' proxy Common Core State Standards for Mathematics (CCSSM), and Zambia. Stigler and Perry (2014) emphasized that better understanding of one's own culture gained through comparative studies leads researchers and educators to a more explicit understanding of their own implicit theories about how children learn mathematics. Without comparison, we tend not to question our own traditional teaching practices, and we may not even be aware of the choices we have made in constructing the educational process. We employed Webb's (2007) Depth of Knowledge (DOK) to analyze cognitive expectations to draw similarities and differences. Analyzing curriculum standards offers insight into the recommended level at which trigonometry should be taught.

Preliminary findings show the CCSSM has more strands compared to Ghana, South Africa, and Zambia. Findings suggest that all the four countries' standards have similar trigonometric concepts although the standards vary across DOK levels. For example, all four countries have standards related to drawing trigonometric functions. However, as seen in Table 2, Ghana and Zambia's standards contain a lower-level verb (e.g., to draw) while the CCSSM's and South Africa's standards emphasize a higher-level verb (e.g., to model). We noted that the CCSSM and South Africa did not have any standards at level 1 (e.g., recall and reproduction) while Zambia had 3 and Ghana had 1. Likewise, compared with Ghana and Zambia, the CCSSM was more likely to emphasize higher-order thinking skills at level 3 and level 4 with 11 standards out of 14, while South Africa had 9 standards at higher levels (i.e., level 3, level 4) out of 14. Ghana had 2 standards at levels 3 and 4 out of 8 total standards, while Zambia had 5 standards at levels 3 and 4 out of 14 total standards. Our qualitative analysis showed that all four countries study trigonometric ratios in right-angled triangles and use right-angled triangles to solve problems. The CCSSM were not written to a particular order of grade while the South Africa, Ghana, and Zambia syllabuses are tailored to grade levels, for example, the South African syllabus has trigonometry run throughout the senior secondary level from grade 10 through to grade 12 while the Zambian syllabus has trigonometry taught only in grade 11. Future studies could extend this study by analyzing additional curriculum resources such as textbooks.

References

Akkoc, H. (2008). Pre-service mathematics teachers' concept images of radian. *International Journal of Mathematical Education in Science & Technology, 39*(7), 857-878. https://doi.org/10.1080/00207390802054458

Cai, J. (2004). Why do U.S. and Chinese students think differently in mathematical problem-solving? Exploring the impact of early algebra learning and teachers' beliefs. *Journal of Mathematical Behavior, 23*, 135–167.

Conley, D. T. (2014). *The common core state standards: Insight into their development.* Council of Chief State School Officers.

Department of Basic Education (2011). Curriculum and Assessment Policy Statement Grades 10-12 Mathematics. http://www.education.gov.za

Fi, C. (2003). *Preservice secondary school mathematics teachers' knowledge of trigonometry: Subject matter content knowledge, pedagogical content knowledge, & envisioned pedagogy*. [Unpublished doctoral dissertation]. The University of Iowa.

Hess, K. (2013). *A guide for using Webb's depth of knowledge with common core state standards*. The Common Core Institute, Center for College and Career Readiness.

Ministry of Education (2010). Ghana Teaching Syllabus for Core Mathematics (Senior High School).

National Governor's Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO]. (2010). Common core state standards for mathematics. Authors.

Ministry of Education. (2013b). Zambia Education Framework. Curriculum Development Centre.

Ravitch, D, (1995). National Standards in American Education. Washington, D.C.: Brookings Institution.

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards: The new US intended curriculum. *Educational Researcher*, *40*(3), 103-116. https://doi.org/10.3102/0013189X11405038

Schmidt, W. H., Wang, H. C., & McKnight, C. C. (2005). Curriculum coherence: An examination of US mathematics and science content standards from an international perspective. *Journal of Curriculum Studies*, *37*(5), 525-559. https://doi.org/10.1080/0022027042000294682

Standards in Your State. (2022). Common Core State Standards Initiative: Preparing America's Students For College & Career. http://www.corestandards.org/standards-in-your-state/#:~:text=Forty%2Done%20states%2C%20the%20District,to%20adopt%20their%20academic%20standards

Stigler, J. W., & Perry, M. (1990). Mathematics learning in Japanese, Chinese and American classrooms. In J. W. Stigler, R. A. Shweder, & G. Herdt (Eds.), *Cultural psychology: Essays on comparative human development* (pp. 328–353). Cambridge

Tran, D., Reys, B. J., Teuscher, D., Dingman, S., & Kasmer, L. (2016). Analysis of curriculum standards: An important research area. *Journal for Research in Mathematics Education*, *47*(2), 118-133. https://doi.org/10.5951/jresematheduc.47.2.0118.

Tuna, A. (2013). The effect of the 5E learning cycle model in teaching trigonometry on students' academic achievement and the permanence of their knowledge. *International Journal on New Trends in Education and Their Implications*, *4*(1), 73-87

Webb, N. L. (2007). Issues related to judging the alignment of curriculum standards and assessments. *Applied Measurement in Education*, *20*(1), 7-25. https://doi.org/10.1080/08957340709336728

AN ANALYSIS OF TOPICS INCLUDED IN INTRODUCTORY COLLEGE MATHEMATICS COURSES

Josh Naddor
North Carolina State University
jnaddor@ncsu.edu

Keywords: Curriculum, Undergraduate Education

Introduction

Numerous organizations have provided recommendations for what mathematics K-12 students should learn (National Council of Teachers of Mathematics, National Governors Association Center for Best Practices & Council of Chief State School Officers) and for what preservice mathematics teachers and current mathematics majors should learn (AMTE, 2017; CBMS, 2012; Saxe & Braddy, 2015). There are also recommendations for what the mathematics students should study prior to calculus (Cohen, 1995). Cohen (1995) made recommendations for what mathematics content students should study prior to calculus, which included number sense, symbolism and algebra, geometry, function, discrete mathematics, probability, and statistics, as well as mathematical proof. In a more recent publication, Saxe & Braddy (2015) recommend updating curricula to also address real world connections and affective dimensions of students' learning of mathematics. The purpose of the current study is to examine the content included in college mathematics courses taken prior to calculus in order to better understand how universities are implementing the recommendations about which content to include. Examples of such courses include college algebra, trigonometry, and quantitative reasoning.

Data for this study were 76 course syllabi from 50 public research-intensive universities in the southeastern United States. Each syllabus was analyzed to note the topics covered. These topics were then grouped into categories and tallied using a frequency table. The number of unique courses that contain content related to a specific topic are shown in Table 1.

Table 1. Frequencies of mathematical topics in course syllabi

Financial Mathematics	**40**	**Logic**	**26**	**Set Theory**	**19**
Probability	**39**	**Statistics**	**34**	**Math in the Real World**	**40**
Graph Theory	**18**	**Voting Theory**	**15**	**Combinatorics**	**16**
Algebra	**25**	**Operations Research**	**24**	**Game Theory**	**10**
Number Theory	**14**	**Geometry/Topology**	**16**	**Linear Algebra**	**10**

The topics with the most occurrences were Financial Mathematics (40), Math in the Real World (40), Probability (39), and Statistics (34). Conversely, the subjects with the fewest occurrences were Number Theory (14), Game Theory (10), and Linear Algebra (10). The average number of unique topics was approximately 4.645 topics per class. Several classes contained only one topic while others contained up to eleven unique topics. The topics that occurred most frequently seem to have real-world connections, as recommended by Saxe & Braddy (2015) and Cohen (1995). Additional data need to be collected to better understand affective dimensions related to learning mathematics and how these courses prepare students to appreciate the beauty of mathematics, use technology efficiently, and communicate effectively.

References

Association of Mathematics Teacher Educators. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Cohen, D. (1995). *Crossroads in mathematics: Standards for introductory college mathematics before calculus.* American Mathematical Association of Two Year Colleges. https://eric.ed.gov/?id=ED386231.

Conference Board of the Mathematical Sciences, (2012), The Mathematical Education of Teachers II, Providence, RI, and Washington, DC: American Mathematical Society and Mathematical Association of America.

Saxe, K., & Braddy, L. (2015). A common vision for undergraduate mathematical sciences programs in 2025. Retrieved from http://www.maa.org/sites/default/files/pdf/CommonVisionFinal.pdf

FACILITATING MULTILINGUAL LEARNERS' FRACTION DIVISION PROBLEM SOLVING: IMPACT STUDY FINDINGS

Johannah Nikula	Jill Neumayer DePiper	David Bamat
Education Development Center	WestEd	Research for Action
jnikula@edc.org	jdepiper@wested.org	davidbamat@gmail.com

Keywords: Middle School Education; Problem Solving; Equity, Inclusion, and Diversity

Purpose and Theoretical Framework

This study examines whether a focus on diagramming and language can facilitate mathematical problem solving for multilingual learners (MLs)—i.e., those identified as English learners. To meet MLs' strengths and needs in mathematics, we posit that three interdependent focus areas must be integrated: an asset-based approach towards MLs (e.g., NASEM, 2018; de Araujo et al., 2018; Moschkovich, 2002); student engagement with rigorous mathematics (e.g., Moschkovich, 2013) facilitated by the use of mathematical diagrams (e.g., Stylianou & Silver, 2004; Woodward et al., 2012; Driscoll et al., 2012); and language-focused instructional strategies (Celedón-Pattichis & Ramirez, 2012; Lee et al., 2013; Driscoll et al., 2016; Baker et al., 2014).

Methods

We designed and studied a grade 6 fraction division unit with supports for MLs such as language strategies (e.g., the 3 Reads, structured pairs work, sentence starters) and student use of diagrams (e.g., number lines, area models). We conducted a cluster randomized trial; this poster focuses on: *How did participation in these lessons impact students' diagramming in their fraction division problem solving?* Teachers were randomly assigned to use "business-as-usual" fraction division units (control) or our fraction division unit (treatment). Students completed a pre/post fraction division assessment, which was scored for *Diagramming* based on their representations of quantities and relationships for each item. The analytic sample included data from 246 treatment students (12 teachers) and 229 control students (11 teachers). We estimated the treatment effect of our unit on students' Diagramming scores and modeled the relationship between treatment and the Diagramming outcome through a series of multilevel regression models with students nested within teachers.

Results and Implications

When analyzing Diagramming scores, the best fitting model, controlling for prior diagramming ability and fraction understanding, ML status, and attendance, indicated a statistically significant adjusted mean difference, representing a medium-to-large effect ($b = .571$, $p < .01$). Evidence of a positive treatment effect, and no statistically significant interaction between ML status and condition, suggests that MLs made similar growth as their monolingual peers. By providing empirical evidence about lessons that foster MLs' use of diagrams in problem solving, this study responds to the lack of empirical research with attention to MLs' strengths (de Araujo et al., 2018) and has implications for instruction. This was an initial phase of a larger project; continued work will investigate MLs' perspectives on lesson components.

Acknowledgments

The research reported here was supported by the Institute of Education Sciences, U.S. Department of Education, through R305A170297 to EDC. Opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education

References

Baker, S., Lesaux, N., Jayanthi, M., Dimino, J., Proctor, C. P., Morris, J., Gersten, R., Haymond, K., Kieffer, M. J., Linan-Thompson, S., & Newman-Gonchar, R. (2014). *Teaching academic content and literacy to English learners in elementary and middle school* (NCEE 2014-4012). Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Retrieved from the NCEE website: http://ies.ed.gov/ncee/wwc/publications_reviews.aspx.

Celedón-Pattichis, S., & Ramirez, N. G. (Eds.). (2012). *Beyond good teaching: Advancing mathematics education for ELLs*. Reston, VA: National Council of Teachers of Mathematics.

de Araujo, Z., Roberts, S. A., Wiley, C., & Zahner, W. (2018). English learners in K–12 mathematics education: A review of the literature. *Review of Educational Research, 88*(6), 1–41.

Driscoll, M., Heck, D., & Malzahn, K. (2012). Knowledge for teaching English language learners mathematics. In S. Celedón-Pattichis & N. Ramirez (Eds.), *Beyond good teaching: Advancing mathematics education for ELLs* (pp. 163–181). National Council of Teachers of Mathematics.

Driscoll, M., Nikula, J., & DePiper, J. N. (2016). *Mathematical Thinking and Communication: Access for English Learners.* Portsmouth, NH: Heinemann.

Lee, O., Quinn, H., & Valdés, G. (2013). Science and language for English language learners in relation to Next Generation Science Standards and with implications for Common Core State Standards for English language arts and mathematics. *Educational Researcher*, *42*(4), 223–233.

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking & Learning*, *4*(2&3), 189–212.

Moschkovich, J. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education, 6*(1). 45–57.

National Academies of Sciences, Engineering, and Medicine (NASEM). (2018). *English learners in STEM subjects: Transforming classrooms, schools, and lives.* Washington, DC: The National Academies Press. Retrieved from https://doi.org/10.17226/25182

Stylianou, D. A., & Silver, E. A. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, *6*(4), 353–387.

Woodward, J., Beckmann, S., Driscoll, M., Franke, M., Herzig, P., Jitendra, A., Koedinger, K. R., & Ogbuehi, P. (2012). *Improving mathematical problem solving in grades 4 through 8: A practice guide* (NCEE 2012-4055). National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from http://ies.ed.gov/ncee/wwc/publications_reviews.aspx#pubsearch/.

SPATIALIZING THE EARLY MATHEMATICS CURRICULUM

Robyn K. Pinilla
Southern Methodist University
rpinilla@smu.edu

Keywords: Geometry and Spatial Reasoning, Early Childhood Education, Instructional Activities and Practices, Curriculum

Children develop early spatial reasoning skills, including geometry and measurement, through play from a young age (Huinker et al., 2020), yet early childhood and lower elementary math curricula and teaching aim primarily at numeracy (Bruce et al., 2012). While spatial reasoning supports learning in other mathematical domains (Drefs & D'Amour, 2014; Farmer et al., 2013), early childhood teachers are more prepared to teach numeracy and generally dedicate little instructional time to spatial reasoning (Clements & Sarama, 2011; Moss et al., 2015). Davis and colleagues (2015) called for the radical spatialization of mathematics curricula as a potential solution, with spatial reasoning explicitly taught to support better mathematical reasoning.

The development and malleability of children's spatial reasoning skills are well researched (Sarama & Clements, 2009; Uttal et al., 2013), but few studies investigate what can support teachers of young children enacting spatial reasoning instruction in the classroom (Moss et al., 2015; Woolcott et al., 2020). This study aims to answer the following research question: How can curriculum and related professional development support early childhood teachers to enact spatial reasoning instructional practices?

Drawing on situated learning theory (Lave & Wenger, 1991) and a sociocultural approach to mathematics teacher learning (Horn & Garner, 2022), I respond to Davis and colleagues' (2015) call through a more incremental spatialization of the curriculum (National Research Council, 2006) at a research-practice partnership school.

Methods

This in-progress design research study (Cobb et al., 2003) occurs within a neighborhood STEM school at which the pre-k through 1st grades will open initially in fall 2022. By working with kindergarten and 1st grade teachers, I aim to illustrate the iterative design of a spatialization framework and innovative instructional practices. The school's inquiry learning framework and open-source curriculum use present an opportunity to develop standards-aligned spatializations with incoming teachers. The initial framework to spatialize instructional routines will be reviewed based on teachers' feedback, as elicited through ongoing professional development and observed teaching enactments. These processes support the iterative design of principles to support early childhood teachers' incorporation of spatial reasoning into their teaching practices. Data collected will include researcher-generated artifacts, observation records, and curriculum documents. Data collection and analysis will occur concurrently, with multiple data sources used to triangulate findings for enhanced trustworthiness (Merriam, 1998) and transferability (Lincoln & Guba, 1985) of emergent principles.

Results and Implications

Results of this design research will include information about the tools developed, their curricular alignment, and the iterative steps to articulating a framework to support early childhood teachers in enacting spatial reasoning teaching. This study serves as the first step toward a radical spatialization (Davis et al., 2015) of mathematics curricula by illustrating ways to integrate spatial reasoning instruction in early childhood classrooms.

References

Bruce, C., Moss, J. & Ross, J. (2012). *Survey of JK to grade 2 teachers in Ontario Canada: Report to the literacy and numeracy secretariat of the ministry of education*. http://www.pme38.com/wp-content/uploads/2014/05/RF-Sinclair-et-al.pdf.

Clements, D. & Sarama, J. (2011). Early childhood teacher education: the case of geometry. *Journal of Mathematics Teacher Education*, *14*(2), 133–148. https://doi.org/10.1007/s10857-011-9173-0

Davis, B., Okamoto, Y., & Whiteley, W. (2015). Spatializing school mathematics. In B. Davis and the Spatial Reasoning Study Group (Eds.), *Spatial reasoning in the early years* (pp. 149-160). Routledge.

Drefs, M., & D'Amour, L. (2014). *The application of ambiguous figures to mathematics: In search of the spatial components of number* [Conference proceedings]. Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education. Vancouver, BC.

Farmer, G., Verdine, B., Lucca, K., Davies, T., Dempsey, R., Hirsh-Pasek, K., & Golinkoff, R. M. (2013). *Putting the pieces together: Spatial skills at age 3 predict to spatial and math performance at age 5* [Paper presentation]. The Society for Research in child development. Seattle.

Horn, I., & Garner, B. (2022). Teacher learning of ambitious and equitable mathematics instruction: A sociocultural approach (1st ed.). Routledge. https://doi.org/10.4324/9781003182214

Huinker, D., Yeh, C., & Marshall, A. M. (2020). *Catalyzing Change in Early Childhood and Elementary Mathematics: Initiating Critical Conversations*. National Council of Teachers of Mathematics.

Lave J., & Wegner, E. (1991) *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry.* SAGE.

Merriam, S. B. (1998). *Qualitative Research and Case Study Applications in Education. Revised and Expanded from" Case Study Research in Education."*. Jossey-Bass Publishers.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese Lesson Study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: a case study. *ZDM*, *47*(3), 377-390. https://doi.org/10.1007/s11858-015-0679-2

National Research Council. (2006). *Learning to think spatially: GIS as a support system in the K-12 curriculum*. The National Academies Press. https://doi.org/10.17226/11019.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.

Uttal, D. H., Meadow, N. G., Tipton, E., Hand, L. L., Alden, A. R., Warren, C., & Newcombe, N. S. (2013). The malleability of spatial skills: a meta-analysis of training studies. *Psychological Bulletin*, *139*(2), 352. https://doi.org/10.1037/a0028446

Woolcott, G., Le Tran, T., Mulligan, J., Davis, B., & Mitchelmore, M. (2020). Towards a framework for spatial reasoning and primary mathematics learning: an analytical synthesis of intervention studies. *Mathematics education research journal*, 1-31. https://doi.org/10.1007/s13394-020-00318-x

NOTICING STUDENTS' MATHEMATICAL STRENGTHS IN WRITTEN WORK AND VERBAL EXPLANATIONS

Lisa Skultety
University of Central Arkansas
lskultety@uca.edu

Keywords: Preservice Teacher Education, Teacher Noticing, Instructional Activities and Practices, Assessment

For teachers to successfully enact responsive teaching (Richards & Robertson, 2016) and incorporate students' thinking into their instructional choices, they must first attend to and interpret students' mathematical thinking in their classrooms (Jacobs, Lamb, & Pilipp, 2010). This set of skills, referred to as *noticing students' mathematical thinking*, is particularly challenging for novice teachers (Star, Lynch, & Perova, 2011). While research has highlighted the importance of noticing students' mathematical thinking in math instruction, the discussion often fails to contextualize their noticing in light of equity and power in the classroom. It is necessary to consider that teachers' instructional choices are not solely based on the mathematics that a teacher notices in students' thinking, but also how teachers view students' thinking as valid (Louie, 2018). This can be particularly problematic since teachers, especially novice teachers, have difficulty valuing students' thinking that differs from their approach or solution to a problem and commonly only assign competence to students' strategies that match their own (Bannister et al., 2018). Jilk (2016) argues that the culture of education often trains teachers to notice students' thinking in light of whether their solution is correct or incorrect. This often results in a deficit view of students' thinking where teachers are continually noticing how students' thinking is "wrong." While identifying errors is not necessarily problematic in and of itself and is indeed a necessary component of noticing students' thinking, when a teacher can only see a student's mistakes, the teacher can easily miss valid mathematical thinking.

In this poster, I present my findings to the question, *what mathematical strengths do elementary preservice teachers notice in students' written work and verbal explanations?* I chose to focus on verbal and written work because students' mathematical understanding is most often assessed in these modes. Sixty-two elementary preservice teachers responded to ten activities of students' thinking in a mathematics content course. Five of the activities showcased students' written work and the remaining five activities were videos of students verbally explaining their mathematical thinking. Participants were asked to name what they thought was smart in each student's work and why. I coded participants' responses to questions referencing students' mathematical strengths according to the depth of their analysis and their justification of the strength. Findings demonstrated that overall, preservice teachers were able to name strengths in students' mathematical thinking more frequently in the video verbal explanations, and that those strengths were of deeper substance than analyses of students' written work. Furthermore, in the activities, all of the students with written work arrived at the correct solutions, but the preservice teachers were frequently unable to recognize the novel strategy as correct. These findings highlight the challenge of classroom assessments that are based entirely on students' written work, as preservice and novice teachers may not be able to move beyond "correct" or "incorrect" views of assessment, might not notice correct strategies that deviate from their own thinking, or be able to recognize strengths in all students' mathematical work.

References

Bannister, N., Kalinec-Craig, C., Bowen, D., & Crespo, S. (2018). Learning to Notice and Name Students' Mathematical Strengths: A Digital Experience. Journal of Technology and Teacher Education, 26(1), 13-31.

Jilk, L. M. (2016). Supporting teacher noticing of students' mathematical strengths. *Mathematics Teacher Educator*, *4*(2), 188–199.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169-202.

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, *97*(1), 55-69.

Richards, J. & Robertson, A.D. (2016). *A review of the research on responsive teaching in science and mathematics.* In Robertson, A. D., R. E. Scherr & D. Hammer (Eds.), *Responsive teaching in science and mathematics* (pp. 36-55). New York, NY: Routledge.

Star, J.R., Lynch, K., Perova, N. (2011). Using video to improve preservice mathematics teachers' abilities to attend to classroom features: A replication study. In M.G. Sherin, V.R. Jacobs & R.A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 223- 238). New York, NY: Routledge.

Chapter 3:

Early Algebra, Algebraic Thinking, and Function

ELEMENTARY TEACHERS' SHIFT FROM ARITHMETIC TO FUNCTIONAL THINKING THROUGH PROFESSIONAL DEVELOPMENT

Alysia Goyer
University of Nevada, Reno
agoyer@nevada.unr.edu

Tejvir Grewall
University of Nevada, Reno
tgrewall@unr.edu

Sierra Gil
University of Nevada, Reno
sgil2013@gmail.com

Teruni Lamberg
University of Nevada, Reno
Terunil@unr.edu

Elementary In-service teachers participated in professional development related to algebraic thinking. A pre and posttest was administered. The analysis of how teachers conceptualized and solved a problem involved functional thinking shifted from the pre and posttest. Initially teachers solved the problem using additive thinking involving arithmetic. After professional development, teachers shifted to using functional thinking to solve the same problem. Implications for teacher knowledge and professional development are described.

Keywords: Algebraic Thinking, Professional Development, Teacher Knowledge

Students learn algebraic thinking starting in elementary school where they learn to recognize patterns, understand that the equal sign represents a relationship and express their mathematical ideas through representations. As students reach middle grades, they transition to learning about expressions, and functional thinking. This means that early elementary teachers lay the foundation for later development of understanding of formal algebra (Hohesee, 2017). Most researchers have extensively studied how students think algebraically and what teachers should focus on when teaching algebraic thinking (Jacobs et al., 2007). However, research is lacking on teachers' conceptions of algebraic concepts (Demonty et al., 2017; Warren et al, 2016). Warren et al. (2016) highlights the importance of research related to teacher knowledge since it has a direct influence on teachers' ability to make decisions to support their student learning. This study documents shifts in teachers' conceptions of algebraic thinking as they participated in teacher professional development on algebraic thinking. Specifically we investigated: How did professional development shift K-8 teachers' thinking involving algebraic relationships?

Theoretical Framework

There are three strands of algebra: building generalizations of arithmetic, generalizing toward the idea of a function, and modeling (Kaput et al., 2017). The transition from arithmetic to algebraic thinking involves the ability to notice patterns and relationships among operations as opposed to simply calculating the answer. This process involves engaging in reasoning and sense making as it relates to noticing patterns and relationships (Blanton & Kaput, 2005; Swafford & Langrall, 2000). In middle school students are formally introduced to expressions, equations, and functions. Therefore, they start using variables to represent unknown quantities, changing quantities in expressions and exploring functional thinking (CCSSM, 2012). In addition, they start comparing equivalent expressions based on properties (Kieran, 2004).

Input-output relationships are key in functional thinking, exploring the relationships between variables and their patterns (Scharfenberger & Frazee, 2020) that model quantitative situations algebraically which can be generalized (Driscoll, 2001; Obara, 2019). An input-output table is a tool to represent the relationship by isolating pertinent information in a problem (Store, 2018).

Common misconceptions involve generalizing too quickly, pattern-spotting can remain trivial, and students can generalize about the wrong properties (Driscoll, 1999). Russell et al. (2011), state that to support generalization, teachers should develop routines of noticing, articulating, and investigation. In order to support functional thinking, teachers should engage in physical or conceptual activities that involve relationships between one or two of the variables, keep a record of the quantities, identify the pattern, and coordinate representations of the identified pattern relationship (Russel et al., 2017; Smith, 2017).

Schulman's (1986) work on pedagogical content knowledge (PCK) has paved the way for various branches of mathematics (Ball et al., 2008), yet PCK research is comparatively scarce with algebraic thinking (Store, 2018). There is an interdependence between professional development (PD), teachers' content knowledge and PCK which can ultimately change classroom discourse and practices around algebraic thinking (Nathan & Koellner, 2007). Therefore, this study explored teachers' conception of algebraic and functional thinking in a pre and posttest question on algebraic and functional thinking.

Methods

A PD focusing on algebraic thinking was offered to K-8 elementary teachers as part of a week-long summer institute. The teachers were from a rural school district in a western state. The summer institute focused on having teachers' model and solve similar kinds of problems, and focused on having teachers explore functional relationships and writing expressions. A pre-test (N=20) and post (N=27) was administered, with the analysis of this paper focusing on a single questions, specifically part b) of the questions that required the participants to create an expression.

Tables for a wedding are being set up outside. The seat 8 people as shown below:

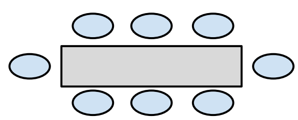


Figure 1

a) How many people will 2 tables seat if the shorter end is placed together?

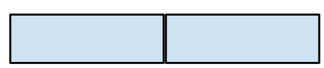


Figure 2

b) Create an expression for how many people n tables will seat.

The PD focused on teacher noticing. The teachers explored a similar problem to the one they solved on the pre-test. Teachers were asked to describe what they noticed. The teachers then expanded on their original table problem to investigate what would change if the tables were pushed together vertically as opposed to horizontally. Figure 3 shows one group of teachers' recorded strategies to generalize the patterns they noticed.

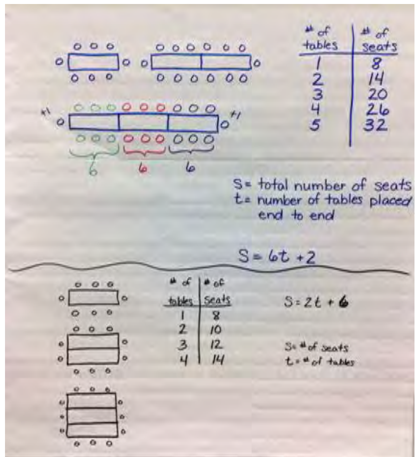


Figure 3

The PD Leader additionally probed the teachers to make sense between the differences between the *6t* and *2t* between the two problems. She asks the teachers to identify what is changing and how it is changing, and how that would be represented in a table and how it be represented graphically. She then redirected the teachers to explain how that relates back to the table model, and what is staying the same. A posttest with a similar problem was administered at the end of the week.

Data Analysis. An inductive coding scheme (Saldaña & Omasta, 2018) was created as follows (see Figure 4). We defined variation as the change between the number of seats for each table(s) added.

Interrater Reliability: Analysis coding of the pre and posttest was conducted blindly between two coders and achieved an initial intercoder reliability of 93.6% using percentage agreement outlined by Syed & Nelson (2015). However, these initial differences were discussed and resolved to achieve full agreement.

Code	Example from Pre-Tests	Description
Noticing Variation & Constant	$6n+2$	Subject was able noticed both constants and variation to generalize
Noticing Only Variation	$6n$	Subject only noticed variation to generalize
Generalized Too Quickly for Variation	$n \times 7$ or $8n$	Subject created a variation generalization that applied to only singular term
Difficulty in Identifying Independent Dependent Variables	$n = 2x - 2$ where x = # of people at one table	Subject was unable to generalize due to an incomplete understanding of independent/dependent variables
Other	$p = n-(2n)$	Other generalizations

Figure 4

Results

The pre-test demonstrates high variability in the difficulties displayed by the participants (see Figure 5). Only 40% accurately noticed both the variation and the constant, and provided a correct expression. This increases drastically in the post-test, with approximately 85% of participants noticing the variation and the constant and providing a correct expression. Additionally 7% of the post-test participants noticed only the variation, showing that there was a dramatic increase in noticing variation overall between the pre and posttests. There were no participants during the post-test that generalized too quickly with an incorrect change, dropping to 0% from 40% on the pre-test. There was one participant in both the pre-test and the post-test that had difficulty identifying independent and dependent variables.

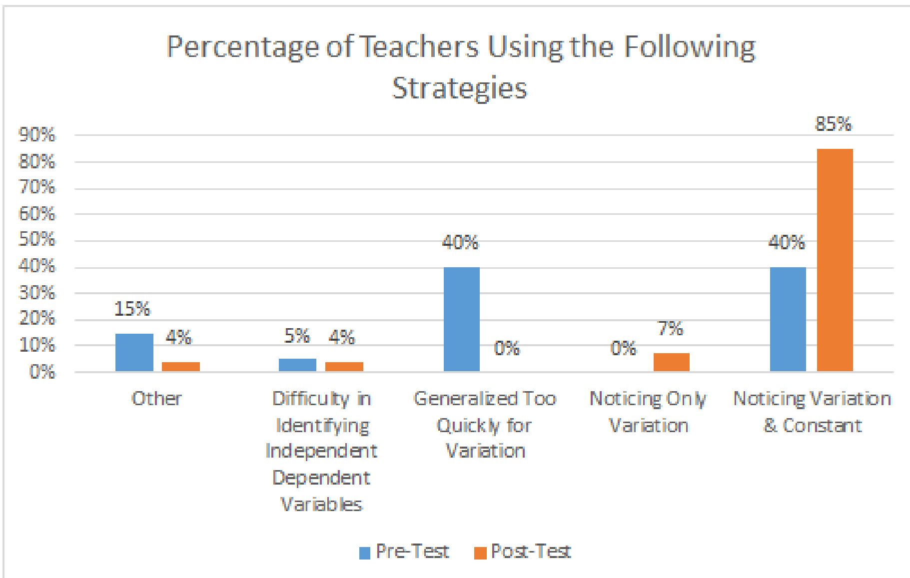


Figure 5

Discussion

The most significant change made to teachers' algebraic thinking was building in the categories of noticing both variation and the constant. This shows a shift away from arithmetic thinking to functional thinking. During the professional development teachers made connections to patterns they noticed, tabular representations, to the algebraic expressions. These connections strengthened the teachers' understanding in identifying variation and constants in their patterns and expressions. However, this illustrates that teachers, just like students, can initially fall into common generalizability pitfalls when analyzing patterns. Teachers can generalize too quickly, generalize about the wrong properties, or have difficulty in understanding the independent and dependent variables as it relates to functional relationships. PD's need to provide opportunities to identify teachers' misconceptions as well as the opportunity to practice noticing, globalizing, and extending their own algebraic generalizations. If teachers are given the opportunity to develop and refine their own algebraic habits, it will reinforce their PCK (Nathan & Koellner, 2007; Shulman, 1986) which then influences their classroom discourse which will encourage them to model and support students' thinking as they develop their own algebraic sense-making.

References

Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. Journal for research in mathematics education, 36(5), 412-446.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). Thinking mathematically. Portsmouth, NH: Heinemann.

Demonty, Vlassis, J., & Fagnant, A. (2018). Algebraic thinking, pattern activities and knowledge for teaching at the transition between primary and secondary school. Educational Studies in Mathematics, 99(1), 1–19. https://doi.org/10.1007/s10649-018-9820-9

Driscoll, M. (1999). Fostering algebraic thinking: A guide for teachers, grades 6-10. Portsmouth, N.H.: Heinemann.

Driscoll, M. (2001). The fostering of algebraic thinking toolkit. Introduction and analyzing written student work. Portsmouth, NH: Heinemann.

Herbst, P., & Chazan, D. (2006). Producing a viable story of geometry instruction: What kind of representation calls forth teachers' practical rationality? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez. (Eds.), Proceedings of the 28th North American Group for the Psychology of Mathematics Education Conference (Vol. 2, pp. 213–220). Mérida, México: UPN.

Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. Journal of Mathematics Teacher Education, 20(3), 231-257.

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.). (2017). Algebra in the early grades. Routledge.

Nathan, M. J., & Koellner, K. (2007). A framework for understanding and cultivating the transition from arithmetic to algebraic reasoning. Mathematical Thinking and Learning, 9(3), 179-192.

National Council of Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Obara, S. (2019). Pre-service teachers exploring the role of pattern-based reasoning in the context of algebraic thinking, EURASIA Journal of Mathematics, Science and Technology Education, 15(11). https://doi.org/10.293333/ejmste/109262

Russell, S. J., Schifter, D., & Bastable, V. (2011). Connecting arithmetic to algebra. Portsmouth, NH, Heinamnn.

Saldaña, J. & Omasta, M. (2018). Qualitative research: Analyzing life. Thousand Oaks, CA: SAGE Publications.

Scharfenberger, A.R., & Frazee, L.M. (2020). Elementary algebraic thinking with patterns in two variables. In: Sacristan, A.I., Cortes-Zavala, J.C. & Ruiz-Arias, P.M. (Eds). Mathematics Education Across Cultures: Proceedings of the 42nd-Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico. https://doi.org/10.512/pmena.42.2020

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

Smith, E. (2017). Representational Thinking as a Framework for Introducing Functions in the Elementary Curriculum. In Algebra in the early grades (pp. 133-160). Routledge.

Store, J.C. (2018). Grounded theory of productive practices for algebraic thinking. Investigations in Mathematics Learning, 10(1), 9-32. https://doi.org/10.1080/19477503.2017.1375353

Swafford, J. O., & Langrall, C. W. (2000). Grade 6 students' pre instructional use of equations to describe and represent problem situations. Journal for Research in Mathematics Education, 31(1), 89-112.

Syed, M., & Nelson, S. C. (2015). Guidelines for establishing reliability when coding narrative data. Emerging Adulthood, 3(6), 375-387.

Warren, E., Trigueros, M., & Ursini, S. (2016). Research on the learning and teaching of algebra. In A. Gutiérrez, G. Leder, & P. Boero (Eds.), The second handbook of research on the psychology of mathematics education. The journey continues (pp. 73–108). Rotterdam: Sense.Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices. Reston, VA: National Council of Teachers of Mathematics.

CONFERENCING TO SUPPORT MATHEMATICAL ARGUMENT WITH MULTILINGUAL STUDENTS

EL USO DE CONFERENCIAS PARA APOYAR ALUMNOS MULTILINGUËS CON ARGUMENTOS MATEMÁTICOS

Salvador Huitzilopochtli
University of California- Santa Cruz
shuitzil@ucsc.edu

Judit N. Moschkovich
University of California-Santa Cruz
jmoschko@ucsc.edu

This classroom design study sought to support multilingual, middle-school students learning mathematical argument with writing practices (leveraging audience, revising, and conferencing). The study was set in a rural, Northern California, dual-immersion school with over 96% Latinx students and over 80% low-income. We used a proof schemes (Harel & Sowder, 1998) framework to examine revisions that students made during mathematical conferences. Findings show that some students used conferencing as an opportunity to revise (or generate arguments) while other students used conferencing as an opportunity to revise procedures or their use of formal terms. Findings support the notion that language should not serve as a gatekeeper that prevents multilingual students from accessing rigorous mathematics.

Keywords: Reasoning and Proof, Culturally Relevant Pedagogy, Middle School Education

Mathematical arguments and proof are difficult for students of mathematics to learn (Healy & Hoyles, 2000). Part of the difficulty is found in the need for students to learn a new *basis of belief* that is neither "natural" (ECEMS, 2011) nor appropriate in settings outside of mathematical activity. Hence, most students arrive to the mathematics classroom and approach mathematical argument with little more than a few examples as "proof" of mathematical claims (Knuth, Choppin, & Bieda, 2009; Harel & Sowder, 1998).

This research presents preliminary findings from a larger dissertation study that sought to support multilingual students' development of mathematical arguments by conferencing on their written responses when asked to justify mathematical claims. This brief research report elaborates findings from the use of mathematical conferences to support students to revise their arguments and was guided by the following question: How did the practice of conferencing support students to revise mathematical arguments?

Conceptual Framework

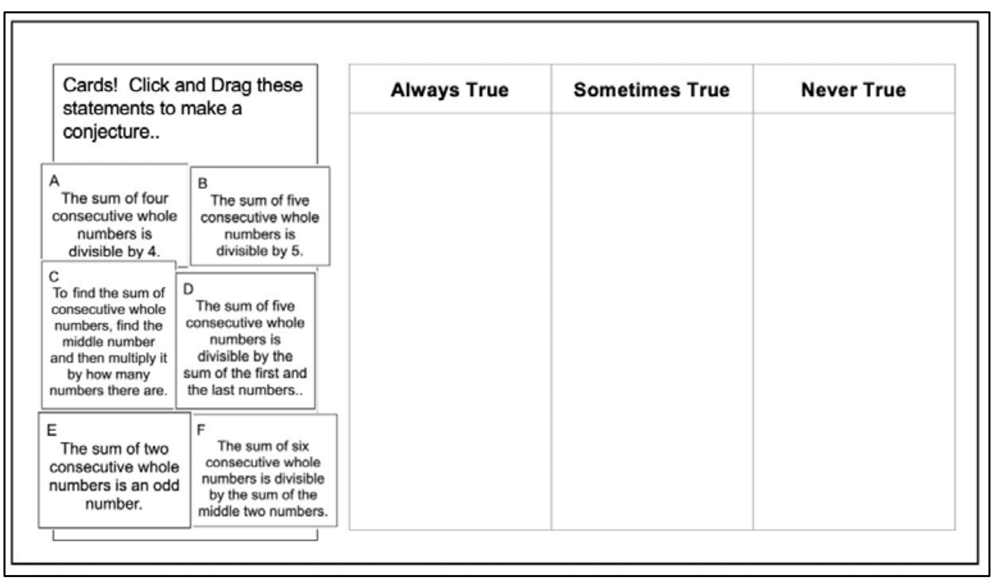
This research is framed by several overarching concepts. First, we used a sociocultural view of mathematics learning and mathematical writing (Moschkovich 2002). We assumed these activities to be discursive and framed by knowledge and practices where meaning is situated and negotiated (Moschkovich, 2015). We focused on the practice of *justifying,* which involves students using informal mathematical arguments to construct mathematical certainty about generalizations (Smith, 2008; Stephens et al., 2017). We used the *proof schemes* framework (Harel & Sowder, 1998) to analyze arguments created by students. In this framework, students could make use of three classes of proof schemes: external conviction, empirical, and analytical. External conviction proof schemes rely on ritual, authoritarian, or symbolic arguments to determine what is true. Empirical proof schemes rely on facts or the senses to construct certainty. Analytic proof schemes seek to settle a claim in general and rely on the structure of mathematical knowledge and logic to convince.

Methods

This research represents a part of a larger classroom design study that used audience and conferencing to support students learning mathematical argument. The study was set in “Esperanza Elementary School” (pseudonym); a rural, Northern California, dual immersion school with 96% Latinx students (over 80% of which are low-income). The study involved a unit of instruction on mathematical argument with three lessons that featured a Convince Form and two rounds of conferences. The Convince Form was designed to leverage audience as it supported the process of justifying by prompting students to ‘convince themselves’, ‘convince a friend’, and then ‘convince a skeptic’ of their mathematical claim. This research brief focuses on two tasks: 1) the *Happy Numbers* task (Dietiker, Baldinger, Kassarjian, & Shreve, 2013), and 2) the *Consecutive Sums* task (MARS, 2015). ‘Happy’ numbers are found by separating the digits of a number, squaring each digit, and then adding the resulting products. If the resulting sum is ‘1’, then the number is ‘Happy’. If the sum is not ‘1’, then the process repeats until it results in a sum of ‘1’ or a sum repeats (in which case, the number is ‘not Happy’). See Figure 1 for prompts related to Task 1. Task 2 involved the Consecutive Sums task. Students were prompted to qualify each of six mathematical conjectures as “Always”, “Sometimes”, or “Never” true and then justify their choice (see Figure 2). Each task was modified to accommodate remote data collection during the ongoing COVID-19 pandemic. We used case studies of a few representative students to illustrate patterns in the data. Author 1 (Huitzilopochtli) conducted the interviews.

A. There are 17 Happy Numbers (from 1 – 99). Find as many as you can.
B. Find five 3-digit happy numbers.
C. Eva found out that 478 is a happy number.
   i. What other numbers *must* be happy numbers?
   ii. How do you know?
   iii. Find at least 10 new happy numbers.

Figure 1: Task 1—Happy Numbers (Dietiker, et al., 2013)



Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

Figure 2: Task 2—Consecutive Sums (MARS, 2015)

Results

Some students used conferencing as an opportunity to revise (or generate) arguments while other students also used conferencing as an opportunity to revise procedures and formal terms.

Revising Arguments

Eight of the 12 conferences involved students revising arguments: three fifths of the students on Task 1 and five sevenths of the students on Task 2. We review the case of "Nina" to illustrate revisions made for Tasks 1.

Prior to the conference, Nina submitted a list of examples in response to Task 1, Part A. Nina did not respond to Parts B or C. During the conference, I prompted Nina to look for patterns in her list of examples. She noticed that some of the 'happy' numbers had the same digits (e.g., 68 and 86). When I asked how that worked, Nina replied, "Because you're… square the individual numbers and… yeah… so, you're gonna get the same, like, sequence for the… yeah." (Transcript turn 19 – 22). Afterwards, I asked Nina to use her idea to find three-digit 'happy' numbers (Part B). Nina said that "you can get the same digits but put a zero in between or at the end" (Transcript turn 28). I asked her why it works, and Nina said, "Because when you square zero, you 're gonna get zero, so it doesn't make any change in the final answer" (Transcript turn 34). At that point, I asked Nina to use her ideas to find which numbers *must* be happy numbers if 478 is a 'happy' number (Part C). Nina correctly determined 4,780, 4,078, and 8,740 were all 'happy' without the need for calculation (Transcript turns 40 – 42). This example shows how Nina used examples to develop the claim that you can use a zero digit to generate novel 'happy' numbers. While her use of the identity property of addition (i.e., $a + 0 = a$) is implicit, Nina is poised to make a general claim and justify it with data (in the form of examples) and warrants (in the form of mathematical properties). Nina began the interview with a list of examples but no claim about how to generate more 'happy' numbers. Then, she engaged in some inductive reasoning as she generated an empirical argument that a zero digit can be used to make novel 'happy' numbers.

Revising Procedures and Formal Terms

Six of the twelve conferences involved students revising either procedures or formal terms; four students revised procedures and five clarified formal terms (three of which revised both). We will review the case of "Yolanda" who revised both.

In response to Task 2, Yolanda qualified all of the claims by placing cards but did not justify any of the choices. During the conference, Yolanda chose to discuss how she qualified the claim that "To find the sum of consecutive whole numbers, find the middle number and then multiply it by how many numbers there are" (Conjecture C; see Figure 3). Yolanda benefitted from support with the procedures for testing the conjecture using examples and using the term "consecutive" with the intended meaning.

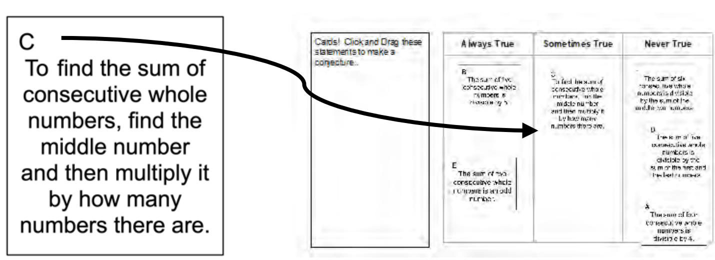


Figure 3: Yolanda's claim is Conjecture C is sometimes true.

I then asked her if she can test the claim, to which she responded, "I don't know" (Transcript turn 42). I picked an example for her to test, 3 + 4 + 5, and she responded by simply adding them to get 12 (Transcript turn 44). I supported her to test the conjecture by walking her through the steps to determine whether multiplying the middle number by the number of addends equals the sum (turns 49 – 58). Then I asked her to come up with her own example to test and she suggested "2 + 4 + 6" (turn 64). Yolanda refined her working definition of what "consecutive" means. Her initial suggestion to test "2 + 4 + 6" (turn 64) used consecutive *even* numbers (i.e., not consecutive whole numbers). My clarification supported her work on the task, and Yolanda later used the intended meaning for 'consecutive' on her own. I asked Yolanda to test her own example and she suggested '9 + 10 + 11 + 12'. Then, she made the following observation:

> Because it's four and if you—you could put four in a group so then it would be two and two. But then, there wouldn't be a middle number. Or, at least, there can't be. And the middle numbers could be 10 and 11, but it would have to be one. (Transcript turn 138)

Yolanda suggested that there is no whole number in the "middle" of a set of four numbers. Before the end of the conference, I invited Yolanda to "say more" about when the conjecture does and does not work. Yolanda responded, "Because if you do 4, I don't think it'll be able to work but if you do more, it could possibly work. Or if you do less" (turn 160). I asked if "3 + 4" worked and Yolanda said, "That can't work" (turn 162) because "it's only two and there would at least have to be three" (turn 164). To conclude, we reviewed what we had done, and I asked her to summarize when she thought the conjecture worked. In the chat, she responded, "it works when there is [sic] 3 and 5+ numbers but it won't work if it's 4 or 2 numbers" (chat file 12/7/2020). With support, Yolanda revised the procedures to test whether an example satisfied the conjecture and her use of the word "consecutive". She began the conference without a justifying scheme and with support is reasoning empirically. Moreover, Yolanda is positioned to further define cases when Conjecture C is true and when it is not.

Discussion and Conclusion

The purpose of this study was to use writing-related practices (leveraging audience, revising, and conferencing) to support multilingual middle school students learning mathematical argument. We found that students used the conferences to revise (or generate) arguments, procedures, and their use of formal terms (such as "consecutive"). These findings suggest that conferencing has potential as a practice to support all students learning mathematical argument, but especially multilingual students to ensure that they understand feedback and receive support to revise their arguments and use of formal terms. An implication of this study is that an iterative approach to justifying might support students to generate more general arguments. The implications of this study extend more broadly because formal terms, such as those common in 'academic language', are often seen as a gatekeeper for multilingual students (NASEM, 2018) and can function as a 'symbolic border' (Faltis & Valdés, 2016). Conferencing might support access to more rigorous mathematics for multilingual students.

Acknowledgments

We would like to acknowledge the students of Esperanza Elementary School and "Ms. Garcia" for their participation. We would also like to thank [funding source] for their support.

References

Dietiker, L., Baldinger, E., Kassarjian, M. and Shreve, B. (2013). *Core Connections 3*. V(5). Elk Grove, CA: CPM Educational Program.

Education Committee of the European Mathematical Society—ECEMS (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *EMS Newsletter*, 82, 50–53.

Faltis, C. J., & Valdés, G. (2016). Preparing teachers for teaching in and advocating for linguistically diverse classrooms: A vade mecum for teacher educators. *Handbook of research on teaching*, 549-592.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *American Mathematical Society*, 7, 234-283.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for research in mathematics education*, 396-428.

Knuth, E., Choppin, J., & Bieda, K. (2009). Middle school students' production of mathematical justifications. In D.A. Stylianou, M.L. Blanton, & E.J. Knuth's (Eds.). *Teaching and learning proof across the grades: A K-16 perspective*. New York, NY: National Council of Teachers of Mathematics/ Routledge.

Mathematics Assessment Resource Service (2015) Evaluating Statements: Consecutive Sums. Mathematics Assessment Project Classroom Challenges: A formative assessment lesson. Available at: http://map.mathshell.org

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical thinking and learning*, 4(2-3), 189-212.

Moschkovich, J. N. (2015). Academic literacy in mathematics for English Learners. *The Journal of Mathematical Behavior*, 40, 43-62.

National Academies of Sciences, Engineering, and Medicine (2018) *English Learners in STEM Subjects: Transforming Classrooms, Schools, and Lives*. Washington, DC: The National Academies Press. https://doi.org/10.17226/25182.

Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum (133-160). In J.J. Kaput, D.W. Carraher, & M.L. Blanton's (Eds.) *Algebra in the early grades*. New York, NY: Routledge.

Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. *Compendium for research in mathematics education*. National Council of Teachers of Mathematics: Reston.

QUANTITATIVE REASONING, DEDUCTIVE REASONING, AND OPERATING WITH ALGEBRAIC SYMBOLS: A NOVEL FRAMEWORK

Teo Paoletti
University of Delaware
teop@udel.edu

Charles Hohensee
University of Delaware
hohensee@udel.edu

Allison Gantt
University of Delaware
agantt@udel.edu

In algebra, operating with symbols meaningfully depends upon a student's quantitative and deductive reasoning. Students can engage in these types of reasoning in different combinations, which has not yet been well conceptualized by mathematics education research. In this report, we present a framework that describes different combinations of how students use quantitative reasoning, deductive reasoning, and/or operating with algebraic symbols. We describe and illustrate each combination using examples from extant literature.

Keywords: Algebra and Algebraic Thinking, Quantitative Reasoning, Deductive Reasoning

Although the subject of algebra involves a wide array of reasoning and skills, one common feature (and source of difficulty) involves operation with algebraic symbols (i.e., variables, expressions, equations, and inequalities) to solve problems (Kaput, 2008; Kieran, 2004; Steffe & Izsák, 2002; Stephens et al., 2017). Previous research has found that even when students can operate with algebraic symbols efficiently, they may not maintain meanings that support them in deductively understanding why they can manipulate algebraic symbols or how symbols are applicable to quantitative situations (e.g., Blanco & Garrote, 2007; Didiş et al., 2011; Hohensee, 2017). We present a framework to support researchers to make sense of students' activity on tasks that may involve intersections of their quantitative reasoning, deductive reasoning, and operating with symbols. Our framework characterizes ways students may employ quantitative and/or deductive reasoning with or without operating with algebraic symbols. We use examples from the extant literature to describe each type of reasoning.

Quantitative Reasoning and Deductive Reasoning

Smith and Thompson (2008) define quantitative reasoning as when individuals conceive of and reason about the relationships between quantities. Quantitative and algebraic reasoning are closely intertwined. For example, Steffe and Izsák (2002) contended that algebraic reasoning entails "quantitative reasoning about constant and varying unknowns" (p. 1164). Similarly, Smith and Thompson (2008) argued that "quantitative reasoning provides conceptual content for powerful forms of representation and manipulation in algebra" (p. 100). However, they also note that, for many students, algebra is reduced "to a set of rituals involving strings of symbols and rules for rewriting them" (ibid, p. 96). Researchers (e.g., Thompson & Thompson, 1995) have conjectured that such an understanding of algebra may stem from a lack of opportunities to construct and reason about relationships between quantities.

By comparing several definitions of deductive reasoning (e.g., Ayalon & Even, 2008; Morris, 1995; Rivera & Becker, 2007; Stylianides & Stylianides, 2008), we synthesized features of deductive reasoning. First, such reasoning starts with given information students must presume is true. Then, students must use this information to reach conclusions they understand are logically true (e.g., do not require empirical verification). Addressing the relationship between deductive reasoning and algebra, Morris (1995) characterized *algebraic deductive reasoning* as applying "rules of transformation [which] include . . . operations specific to algebraic reasoning such as the ability to manipulate algebraic symbols" (p. 9). For our purposes, algebraic deductive reasoning entails manipulating symbols and applying rules in ways the reasoner understands lead to "logically necessary" conclusions (Stylianides & Stylianides, 2008, p. 865).

A Framework to Characterize Students' Quantitative and Deductive Reasoning in Algebra

In Figure 1, we present six regions representing different ways students may leverage quantitative and/or deductive reasoning with or without operating with symbols in algebra. Because our focus is the domain of algebra, we did not include a region entailing deductive reasoning that is absent of quantitative reasoning and operating with symbols.

We do not intend for our framework to categorize students but rather, students' reasoning on particular tasks around specific mathematical content. Further, depending on the context, a student may not *need* to reason in certain ways. Hence, when we say a student does not provide evidence of certain reasoning in a particular context, we are not implying the student cannot reason in those ways. We acknowledge the student may just not have perceived a need to engage in such reasoning. Relatedly, our goal was for the framework to characterize reasoning, not correctness.

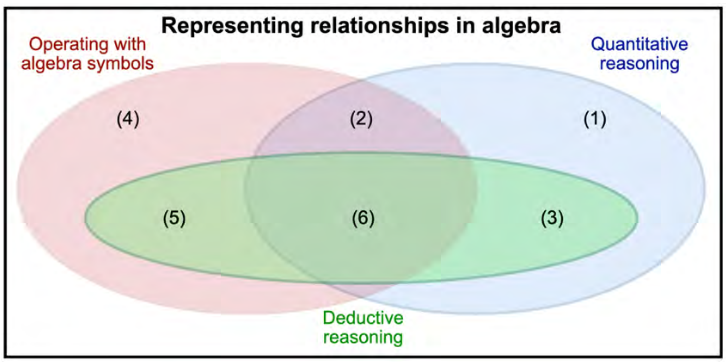


Figure 1: A framework characterizing students' reasoning.

Region 1: Reasoning Quantitatively without Reasoning Deductively or with Symbols

Region 1 reasoning consists of reasoning quantitatively in ways that do not entail reasoning deductively or algebra symbols. In fact, when students first make sense of a novel situation, their initial reasoning may be solely quantitative. For example, Ellis et al. (accepted) described how a 4-year-old student, Mario, addressed the *Faucet Task*. The task prompted students to engage with two quantities: the amount and the temperature of water coming out of a faucet. Mario was also encouraged to consider how these two quantities varied as the cold and hot knobs were turned on or off. With both knobs halfway on, Mario predicted that if the cold knob was turned all the way on, there would be "less water and colder." He then tested his prediction to determine that although the water was colder, there was more water. As he engaged in several scenarios, he began to coordinate changes in water temperature and amount of water as each knob was turned. Critically for our purposes, Mario reasoned about the relationship between turning knobs and the amounts of the quantities *without* the use of symbols to represent his thinking. Further, his reasoning was inductive, relying on testing conjectures, rather than deductive. Therefore, we interpret his reasoning as in Region 1 (i.e., quantitative reasoning only).

Region 2: Reasoning Quantitatively with Symbols without Reasoning Deductively

Region 2 reasoning entails reasoning quantitatively and representing quantitative relationships with algebraic symbols but without reasoning deductively. There are numerous examples of such reasoning in the extant literature (e.g., Dougherty, 2008; Izsák, 2003; Stevens, 2020, in press). For example, Paoletti et al. (2021) described how middle school students used algebraic inequality and equality statements to represent when one quantity was greater than, less than, or equal to a second quantity (e.g., by writing $P < T$, $P > T$, and $P = T$ to represent a moving car's distance from Philadelphia as being less than, greater than, or equal to its distance from Trenton). The students reasoned quantitatively to represent the relationships with algebraic symbols but relied on empirical verification throughout by observing locations in the situation, thereby providing no evidence of deductive reasoning. For this reason, we interpret this as an example of reasoning in Region 2 (i.e., quantitative and with symbols, but not deductive).

Region 3: Reasoning Deductively and Quantitatively but without Algebra Symbols

Region 3 reasoning entails reasoning deductively and quantitatively but without employing algebraic symbols. Hackenberg (2010) exemplifies such reasoning. In this paper, a student, Deborah, reasons about the following problem: "The Lizards' car travels 1/2 of a meter. That's 3/4 the distance the Cobras' car went. Can you make how far the Cobras' car went [in the digital environment] and tell how far it went?" Describing Deborah's activity, Hackenberg stated:

> Deborah immediately copied the unit meter she had made, partitioned it into two equal parts, and partitioned the first part into three equal parts. She pulled out one of those parts and repeated it to make a 4-part bar, which she called four-sixths of a meter. (p. 417)

We note that, Deborah was reasoning quantitatively about the situation as she represented and related the various distances. Further, Deborah relied on the diagram to reason deductively about the relationships involved in the situation. Specifically, her activity suggested she assumed the givens (i.e., that Lizards' car travels 1/2 m and that this is 3/4ths the distance Cobras' car travels); she reasoned that 3/4ths the distance Cobras' car travels is 3/6 m; and she came to the logical conclusion that the total distance Cobras' car travels was 4/6 m. Finally, notice that Deborah did not use algebraic symbols. Thus, we categorize such reasoning as in Region 3.

Regions 4 and 5: Operating with Symbols Non-Quantitatively with or without Reasoning Deductively

Regions 4 and 5 both entail operating with symbols without explicit focus on measurable attributes of objects (i.e., non-quantitative). The difference between Regions 4 and 5 is whether the reasoning is deductive or not. Because students' normatively correct algebraic work could reflect reasoning in either region, using non-normative examples can help differentiate between students' reasoning in the two regions. To highlight this, we describe Regions 4 and 5 in tandem.

Didiş et al. (2011) explored students' reasoning about quadratic equations in ways that highlight the distinction between Regions 4 and 5. Although most students could provide solutions to equations of the form $(x - a)(x - b) = 0$, their evaluations of hypothetical student work suggested that only some were engaging in what we identify as deductive reasoning. The hypothetical work showed a quadratic equation (i.e. $x^2 - 14x + 24 = 3$ where both the quadratic expression and constant had been factored (i.e. $x^2 - 14x + 24 = (x - 12)(x - 2)$ and $3 = 3 \cdot 1$). The hypothetical work arrived at a solution by setting one factor of each expression equal to a factor of the other (i.e., $x - 12 = 3$ and $x - 2 = 1$). Students who argued that the solution was correct may not have maintained meanings for their process based on the zero-product property, which provides the logical underpinnings of this solution (i.e., Region 4). In contrast, we conjecture a student who argued the hypothetical solution is incorrect had assessed that the hypothetical solution did not lead to "logically necessary" conclusions (i.e., Region 5).

As the students made no references to quantities (which is unsurprising given the prompt's abstract context) and they were operating with algebraic symbols, we infer the reasoning is indicative of Regions 4 and 5. These examples highlight how it can often be unclear if a student's procedures entail deductive reasoning when examining normatively correct solutions (e.g., correct solutions to $(x - a)(x - b) = 0$). By responding to unconventional hypothetical student work, we are able to observe potential differences in students' processes as entailing deductive reasoning (Region 5) or not entailing deductive reasoning (Region 4)

Region 6: Reasoning Quantitatively and Deductively while Operating with Symbols

Region 6 reasoning requires both deductive and quantitative reasoning while operating with algebraic symbols. Hall et al. (1989) provided an example of a student engaging in such reasoning on the following problem: "George rode out of town on the bus at an average speed of 24 miles per hour and walked back at an average speed of 3 miles per hour. How far did he go if he was gone for six hours?" (p. 249). Reflective of reasoning quantitatively, the student created several equations which had an explicit focus on measurable attributes (e.g., "walking distance = (3 miles/hr)($6 - x$ hours)") and relationships between measurable attributes (e.g., "bus distance = walking distance"). Further, the student used equations to relate two equivalent distances (e.g., $24x = 18 - 3x$), then reasoned deductively to determine the time, in hours, the bus traveled ("bus distance = 16 miles"). The student then used this value to eventually determine a solution to the problem. Therefore, we classified this reasoning as in Region 6.

Discussion

Our goal in this paper is to present a novel framework that allows researchers to engage in a nuanced analysis of the intersections between students' quantitative reasoning, deductive reasoning, and their operating with algebraic symbols in a particular task or context. Specifically, we theorize that students' reasoning can fit within (at least) six regions when reasoning quantitative and/or deductively while operating or not operating with algebra symbols. The fact that we located examples in the literature of reasoning that align with each region highlights the potential efficacy of this framework to characterize students' reasoning.

Based on our personal experiences and extant research (e.g., Hohensee, 2017), we conjecture most students leave K-12 with ways to operate with symbols but with insufficient understanding about how deductive logic justifies their operations on symbols or how their procedures connect to quantities (Regions 4 and 5). Also, when students are encouraged to reason about quantities, we conjecture quantities are often only considered when setting up equations (Region 2) but then their work with the equations largely ignores the quantities under consideration (moving to Regions 4 or 5) as in the example in Figure 4.

Hence, there is a need to design and research instructional activities that support students in developing meanings that enable them to reason in ways that would fit within Region 6. There are indications that connecting pictorial and algebraic representations can be particularly productive for students (e.g., Bartel et al., 2021; Chu et al., 2017). Research exploring these activities has the potential to address the need to better understand and improve students' algebraic thinking and learning.

References

Ayalon, M., & Even, R. (2008). Deductive reasoning: In the eye of the beholder. *Educational Studies in Mathematics*, *69*(3), 235–247. https://doi.org/10.1007/s10649-008-9136-2

Bartel, A. N., Silla, E. M., Vest, N. A., Nagashima, T., Aleven, V., & Alibali, M. W. (2021). Reasoning about equations with tape diagrams: insights from math teachers and college students. In E. de Vries, Y. Hod, & J. Ahn (Eds.), *Proceedings of the 15th International Conference of the Learning Sciences-ICLS 2021* (pp. 685–688). Bochum, Germany: International Society of the Learning Sciences.

Blanco, L. J., & Garrote, M. (2007). Difficulties in learning inequalities in students of the first year of pre-university education in Spain. *Eurasia Journal of mathematics, science and technology education, 3*(3), 221–229.

https://doi.org/10.12973/ejmste/75401
Chu, J., Rittle-Johnson, B., & Fyfe, E. R. (2017). Diagrams benefit symbolic problem-solving. *British Journal of Educational Psychology, 87*(2), 273–287. https://doi.org/10.1111/bjep.12149
Didiş, M. G., Baş, S., & Erbaş, A. (2011). Students' reasoning in quadratic equations with one unknown. *The Seventh Congress of the European Society for Research in Mathematics Education* (CERME-7), (pp. 479–489). Rzeszów, Poland.
Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 389–412). New York, NY: Taylor and Francis.
Ellerton, N. F., & Clements, M. K. (2011). Prospective middle-school mathematics teachers' knowledge of equations and inequalities. In *Early algebraization* (pp. 379–408). In J. Cai, & E. Knuth (Eds.), *Early Algebraization. Advances in Mathematics Education*. Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-17735-4_21
Ellis, A. B. (2011). Middle school algebra from a functional perspective: A conceptual analysis of quadratic functions. In L. R. Wiest, & T. Lamberg, (Eds.), *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 79–87). Reno, NV; PME-NA. https://files.eric.ed.gov/fulltext/ED585949.pdf
Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. *The Journal of Mathematical Behavior, 39*, 135–155. https://doi.org/10.1016/j.jmathb.2015.06.004
Ellis, A.B., Paoletti, T., & Lockwood, E. (In press) Empirical and Reflective Abstraction. *Genetic Epistemology in Mathematics Education*.
Gamoran, A., & Hannigan, E. C. (2000). Algebra for everyone? Benefits of college-preparatory mathematics for students with diverse abilities in early secondary school. *Educational Evaluation and Policy Analysis, 22*(3), 241–254. https://doi.org/10.3102/01623737022003241
Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432. https://doi.org/10.1080/07370008.2010.511565
Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education, 46*(2), 196–243. https://doi.org/10.5951/jresematheduc.46.2.0196
Hackenberg, A. J., & Lee, M. Y. (2016). Students' distributive reasoning with fractions and unknowns. *Educational Studies in Mathematics*, *93*(2), 245–263.https://doi.org/10.1007/s10649-016-9704-9
Hall, R., Kibler, D., Wenger, E., & Truxaw, C. (1989). Exploring the episodic structure of algebra story problem solving. *Cognition and Instruction*, *6*(3), 223–283. https://doi.org/10.1207/s1532690xci0603_2
Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. *Journal of Mathematics Teacher Education, 20*(3)*,* 231–257. https://doi.org/10.1007/s10857-015-9324-9
Izsák, A. (2003). "We want a statement that Is always true": Criteria for good algebraic representations and the development of modeling knowledge. *Journal for Research in Mathematics Education, 34*(3), 191–227. https://doi.org/10.2307/30034778
Kaput, J. J. (2008). What is algebra? What is algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–18). New York, NY: Taylor and Francis.
Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, *8*(1), 139–151. https://www.researchgate.net/profile/Carolyn-Kieran-2/publication/228526202_Algebraic_thinking_in_the_early_grades_What_is_it/links/55895b8408ae2affe714d428/Algebraic-thinking-in-the-early-grades-What-is-it.pdf
Ko, Y. Y., & Knuth, E. J. (2013). Validating proofs and counterexamples across content domains: Practices of importance for mathematics majors. *Journal of Mathematical Behavior*, *32*(1), 20–35. http://dx.doi.org/10.1016/j.jmathb.2012.09.003
Koedinger, K. R., & Nathan, M. J. (2007). The real story behind story problems: Effects of representations on quantitative reasoning. *The Journal of the Learning Sciences*, *13*(2), 129–164. https://doi.org/10.1207/s15327809jls1302_1
Linchevski, L., & Sfard, A. (1991). Rules without reasons as processes without objects-the case of equations and inequalities. In F. Furinghetti (Ed.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 2, pp. 317-324). Assisi, Italy: PME.
McFarland, J., Hussar, B., Zhang, J., Wang, X., Wang, K., Hein, S., ... & Barmer, A. (2019). *The condition of education 2019* (NCES 2019-144). Washington, DC: National Center for Education Statistics. https://files.eric.ed.gov/fulltext/ED594978.pdf

Morris, A. K. (1995). Development of algebraic reasoning in children and adolescents: Cultural, curricular; and age-related effects. Unpublished doctoral dissertation, The Ohio State University. https://www.proquest.com/docview/304223666?pq-origsite=gscholar&fromopenview=true

Moses, R. P., & Cobb, C. (2001). Organizing algebra: The need to voice a demand. *Social Policy, 31*(4), 4–4.

Paoletti, T., Stevens, I. E. & Vishnubhotla, M. (2021) Comparative and restrictive inequalities. *The Journal of Mathematical Behavior*, 63. https://doi.org/https://doi.org/10.1016/j.jmathb.2021.100895

RAND Mathematics Study Panel Report. (2003). *Mathematical proficiency for all students: A strategic research and development program in mathematics education*. Santa Monica, CA: RAND. https://www.rand.org/content/dam/rand/pubs/monograph_reports/MR1643/RAND_MR1643.pdf

Rivera, F. D., & Becker, J. R. (2007). Abduction–induction (generalization) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behavior*, *26*(2), 140–155. https://doi.org/10.1016/j.jmathb.2007.05.001

Smith, J., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Lawrence Erlbaum Associates.

Steffe, L. P. (1991). Operations that generate quantity. *Journal of Learning and Individual Differences, 3*(1), 61–82. https://doi.org/10.1016/1041-6080(91)90004-K

Steffe, L. P., & Izsák, A. (2002). Pre-service middle-school teachers' construction of linear equation concepts through quantitative reasoning. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 1163–1172). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 386–420). NCTM.

Stevens, I. E. (2020). "Solving versus Relating": Pre-service teachers' conflicting images of formulas and dynamic contexts. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 1223–1227). Cinvestav /AMIUTEM / PME-NA. https://doi.org/10.51272/pmena.42.2020-192

Stevens, I. E. (in press). "A=2πrh is the surface area for a cylinder": Figurative and operative thought with formulas. *Proceedings of the Twenty-Fourth Annual Conference on Research in Undergraduate Mathematics Education*. Boston, MA.

Stylianides, G. J., & Stylianides, A. J. (2008). Proof in school mathematics: Insights from psychological research into students' ability for deductive reasoning. *Mathematical Thinking and Learning*, *10*(2), 103–133. https://doi.org/10.1080/10986060701854425

Thompson, A. G., & Thompson, P. W. (1995). A cognitive perspective on the mathematical preparation of teachers: The case of algebra. In C. B. Lacampagne, W. Blair, & J. Kaput (Eds.), *The Algebra initiative colloquium* (pp. 95–116). U. S. Government Printing Office.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sépulveda (Eds.), *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 31–49). PME.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33–57).

Vaiyavutjamai, P., & Clements, M. (2006). Effects of classroom instruction on student performance on, and understanding of, linear equations and linear inequalities. *Mathematical Thinking and Learning, 8*(2), 113–147. https://doi.org/10.1207/s15327833mtl0802_2

von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning. Studies in Mathematics Education Series: 6*. London, United Kingodm; Falmer Press.

EARLY ALGEBRA MOMENTS: OPPORTUNITIES TO FOSTER STUDENTS' ALGEBRAIC THINKING

MOMENTOS DE ÁLGEBRA ELEMENTAL: OPORTUNIDADES PARA FORMENTAR DEL RAZONAMIENTO ALGEBRAICO

Ingrid Ristroph
The University of Texas at Austin
ingrid.ristroph@utexas.edu

Eric Knuth
The University of Texas at Austin
eric.knuth@austin.utexas.edu

Boram Lee
The University of Texas at Austin
boram_lee@utexas.edu

Despina Stylianou
The City College of New York
dstylianou@ccny.cuny.edu

Hangil Kim
The University of Texas at Austin
hangil_kim@utexas.edu

Bethany Miller
The University of Texas at Austin
b.miller3511@utexas.edu

We explore early algebra moments, that is, the opportunities that arise during the course of classroom instruction that a teacher could potentially capitalize upon to address important early algebra ideas or practices. We categorize early algebra moments as anticipated (based on the intended lesson) or spontaneous (not anticipated and arise "in the moment"). In this paper, we report on the analysis of sixteen video-taped observations of Grade 5 teachers' implementation of the same early algebra lesson and illustrate the results with contrasting exemplars of the two types of moments. In addition, we also highlight an episode of a spontaneous moment that capitalizes on collective argumentation. Finally, we discuss implications for future research, professional development, and student learning associated with the early algebra moments.

Keywords: Algebra and algebraic thinking, elementary school education, teacher noticing, classroom discourse

Blanton and Kaput (2003) described elementary teachers' algebra "eyes and ears" in terms of their ability "to spot opportunities for algebraic thinking" (p. 76). In this paper, we expand on this idea and explore what we call *early algebra moments*, that is, the opportunities that arise during the course of classroom instruction that a teacher could potentially capitalize upon to address important early algebra ideas or practices. We categorize early algebra moments as anticipated or spontaneous moments; in the former case, these are moments that are reasonable to expect based on the intended lesson (and that are also further delineated as taken up or as not taken up by the teacher), and in the latter case, these are moments that were not anticipated and that arise "in the moment."

Background of work

The larger project, from which this study draws, examined the impact of an early algebra intervention for Grades 3-5. The intervention consisted of 18 one-hour lessons for each grade level, and was developed using a learning progressions approach (Authors, 2018). The lessons were taught over the course of the school year for each grade level. The aim of the lessons was to develop students' abilities to generalize, represent, justify, and reason with mathematical structure and relationships across a variety of mathematical contexts. The lessons focused on several big ideas of early algebra: (a) mathematical equivalence; (b) generalized arithmetic; (c) functional thinking; and (d) variable (Kaput, 2008). This study draws from and expands upon a series of studies that examined the impact of this intervention in Grades 3-5 (Author, 2015) comparing the performance of students who participated in the intervention with the performance of students who received their district's regularly-planned curriculum (Author et al., 2015; Author et al., 2017; Author et al., 2018).

Conceptualizing Early Algebra Moments and Their Enactment

We value teachers as co-producers of knowledge, experts, and informants to research (Bromme, 2014; Kieran, Krainer, & Shaughnessy, 2012; Sowder et al., 2005). The relationship between research and practice is bidirectional; research and practice communities have much to contribute to each other's work (Arbaugh, et al., 2010). The professional knowledge of teachers can be characterized as a complex cognitive set of skills honed through formal training and years of practical experiences. This expertise shapes the ability to notice, interpret, and respond to students often within a series of complex classroom events requiring immediate, improvisational teacher reactions (Borko & Livingston, 1989; Shulman, 1987). We ask, "What are examples of teachers' improvisational implementation of early algebra lessons?" Through our examination of teachers' implementation of early algebra lessons, we seek to learn not only about the ways in which they engage (or not) with anticipated early algebra moments, but also the ways in which spontaneous (or unanticipated) moments arise during their enactment of these lessons. This latter case, in particular, serves to highlight the unexpected and often creative ways in which teachers implemented the lessons, and as such, also serves to inform and refine research and professional development in early algebra—a domain of importance to students' future mathematics learning as well as a domain that presents instructional challenges for elementary school teachers (Hohensee, 2015).

Researchers have documented critical moments within a mathematics lesson that afford teachers an opportunity to notice and act in order to further student reasoning and understanding (e.g., Fennema et al., 1996; Leatham et al., 2015; Stockero & Van Zoest, 2013). The specific language used to describe these moments during instruction varies among researchers: "teachable moments", "mathematically significant pedagogical opportunities", "crucial mathematic hinge moments", and "pivotal teaching moments". A commonality of such moments is that they are largely unplanned or unanticipated and thus require teachers to notice them in the moment. Specifically, within early algebra, teachers can capitalize on such moments to draw connections to algebraic thinking and foster practices that promote algebraic thinking (Blanton & Kaput, 2003). Blanton and Kaput argued that these moments are incorporated into classroom teaching on a regular basis and often spontaneously. To illustrate the categories of early algebra moments, consider the following task presented to Grade 3 students: Is the number sentence, 34+10=44+9, true or false? In this case, we expected *anticipated moments* associated with an operational view of the equal sign (i.e., the number sentence is true since 34+10=44) and a computational view (i.e., find the sum of both sides and compare). We also identified a *spontaneous moment* when the teacher drew, and then referred to, separate rectangles around the expressions on each side of the number sentence as a means to visually highlight a relational view of the equal sign (i.e., an equivalence relation between two quantities).

While moments in which teachers note "interesting student algebraic thinking" in video-taped observations of secondary algebra classrooms have been studied (Walkoe, 2015), there is a dearth of such work for the domain of early algebra. We hope to add to the body of early algebra research by detailing these critical moments that afford opportunities to foster students' algebraic thinking. Furthermore, there is a lack of empirical analysis of observations of multiple teachers teaching the same early algebra lesson. Our corpus of data provides a unique opportunity to document and analyze the ways in which teachers capitalized, or did not capitalize, on opportunities to engage students in algebraic thinking.

Methods

In the following section we first describe the data used in the study and the data analysis process. In particular, we describe our coding method and illustrate the coding with representative algebra moments from the focus lesson.

Setting and Participants

This study involved 16 Grade 5 teachers in three school districts located in a southeastern US state. The teachers varied in their years of teaching experience from relatively new teachers to veteran teachers. The districts were diverse in terms of race and socioeconomic status, as well as location (rural, suburban, and urban).

Lesson

Students were taught the early algebra intervention by their classroom teachers as part of regular mathematics instruction. This study focuses on one of the lessons, roughly one hour in duration, implemented by each of the sixteen Grade 5 teachers. There were three parts to the lesson: a warm-up, a main activity, and a concluding discussion that focused on reviewing the main activity's key concepts. For the warm-up of this lesson, students were asked to construct a graph to model the relationship of selling twice as many strawberry ice cream bars as chocolate ice cream bars. The main activity of the lesson focused on the distributive property, and students expressed the shaded, white, and total areas of the rectangular figure (see Figure 1) using variables with expressions, equations, and inequalities. The goal of the main

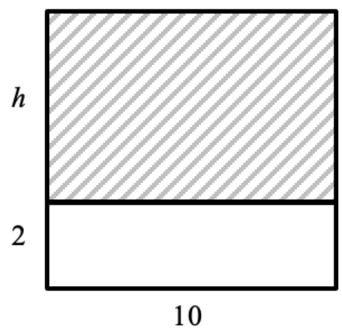


Figure 1: A rectangular model of the distributive property.

activity was to build students' experiences with the distributive property through writing expressions, equations, and inequalities. Students were asked to consider the various rectangular areas to write an inequality that relates the grey rectangular area to the white rectangular area. They were then tasked to write an expression for the total area (i.e., $10(h+2)$ or $10h + 20$). Lastly, students were asked to solve for h if the total area of the composite figure was 100 square units. The concluding activity for the lesson centered on the students determining the difference in area for a rectangle that is $7h$ square units and a rectangle that is $3h$ square units, as well as on the different roles variables play in equations and expressions.

Data Analysis

In order to identify and characterize algebra moments in the observations of the lesson, our coding had two phases. First, prior to viewing the video observations, the research team reviewed the intended lesson, both student and teacher facing materials, and identified a set of 13 anticipated moments for the lesson. An *anticipated moment* is defined as an expected response based on the intended lesson that supports students' understanding of an algebraic concept or fosters an algebraic practice. To illustrate an example of an anticipated moment, the first part of the main lesson prompts students to represent the areas of a rectangle partitioned into two parts, both with length 10 units and widths h and 2 units (see Figure 1). The anticipated moment for this part of the lesson is "variable expressions are written and shared such as $10h$".

Through the course of our viewing of the video observations, in contrast to anticipated moments, we identified *spontaneous moments*. We define spontaneous moments as moments that were not anticipated based on the intended lesson, and that arise "in the moment" to support students' understanding of an algebraic concept or foster an algebraic practice. To illustrate an example of a spontaneous moment, in response to the aforementioned prompt to represent the areas of a partitioned rectangle into two parts (see Figure 1), a spontaneous moment was coded as such when a student objected to the lesson prompt stating "We can't represent the area of the grey rectangle since we don't know what that variable (h) is equal to." A teacher's response to the student was "Do we have to know what the answer is? What h is? …Or can I just leave it as h?" While the printed lesson material for teachers addresses common student misconceptions and details potential student responses, the student's discomfort or resistance to the lack of closure with the variable expression ($10h$) of the area (*cf.* Arcavi, 2005) was not an intended (or anticipated) part of the lesson.

We further delineated each moment as either *taken up* or as *not taken up* by the teacher. A moment taken up requires the teacher to notice and capitalize on the opportunity in a way that supports students' understanding of an algebraic concept or fosters an algebraic practice. In contrast, a moment not taken up is when a teacher does not address an algebra moment (whether intentionally or not). Note that an opportunity not taken up could also occur for algebra moments that, if taken up, would have been coded as spontaneous. In this case, such *not taken up* spontaneous moments were identified by the research team when observing an unanticipated opportunity that arose in the course of instruction but was not addressed by the teacher. For example, in response to writing an expression for the area of the rectangle, a student claims that h must be equal to 8 since h is the eighth letter of the alphabet. The teacher acknowledges "that works out, huh?" but does not address this variable misconception.

The coding process began with the coding of four observations by our team of six coders. After successively meeting after each coded observation to refine our anticipated moments and what constitutes spontaneous moments, we then coded the remaining twelve observations as two teams of three coders. We coded individually and then met to compare and discussed our codes. The intercoder reliability was high for both anticipated moments (approximately 95%) and spontaneous moments (approximately 92%).

Results

The lesson had 13 opportunities for anticipated early algebra moments, and across the 16 teachers' enactment of the lesson, there were 208 total opportunities. Of those, there were 145 instances (70%) in which anticipated moments were taken up, and 63 instances (30%) in which anticipated moments were not taken up. In the former case, we are not making distinctions here with regard to the "quality" with which a teacher took up a particular moment (we illustrate such distinctions below), only that they attended to the algebra moment. In this latter case, the moments that were not taken up most often were a result of teachers not addressing those parts of the lesson (for example, a teacher pressed for time may have skipped parts of the lesson). For spontaneous moments, there were 56 moments identified with 43 of those moments (77%) taken up and 13 of those moments (23%) not taken up. See Figure 2 for a summary of algebra moments counts across the 16 teachers. Of the thirteen possible anticipated moments for each lesson, the average number of anticipated moments taken by each teacher is about 9 while the average number of anticipated moments not taken up is about 4. There was an average of 2.7 spontaneous moments taken by each teacher, and an average of 0.8 spontaneous moments were not taken up by teachers.

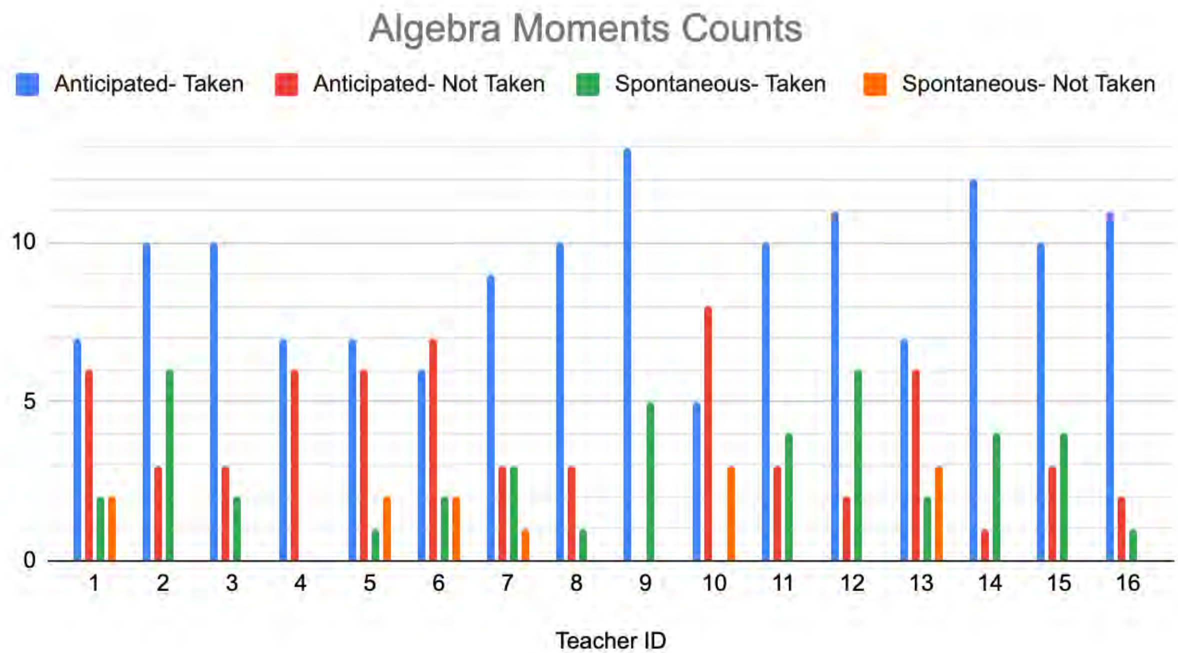


Figure 2: Counts of Anticipated and Spontaneous Moments, Missed and Taken, Across 16 teachers

Taking a closer look at the early algebra moments, we now present several excerpts to illustrate the variety of ways in which the teachers engaged with anticipated and spontaneous moments. The excerpts also serve to highlight qualitative differences in the nature of teachers' enactment around these algebra moments. The following excerpt is representative of teachers' enactment of the anticipated moment associated with describing an inequality that relates the grey rectangular area to the white rectangular area in Figure 1 (i.e., $10h > 20$).

Episode 1: Anticipated Moment, Taken Up, Teacher 11

A student stands at the front of the classroom after writing "$10 \cdot h > 10 \cdot 2$" on the white board.

Teacher: Alright, tell us what you wrote.

Student: Ten times h is greater than 10 times 2.

Teacher: (*to the class*) Does everybody see how that is correct? If you look on your picture, you see that the area of the grey rectangle is larger than the white one. Does that make sense?

Of the 16 observed classrooms, every teacher addressed this anticipated moment. However, we also identified a number of spontaneous moments that arose as students engaged with this task. The following spontaneous moment example occurred when a teacher took up her students' claims that h could represent different values, which would then result in different inequalities.

Although one might expect this to be part of the lesson, the lesson as intended did not suggest that teachers address this point.

Episode 2: Spontaneous Moment, Taken Up, Teacher 2

Teacher	I hear some people…some really good conversations. Leah, can you finish what you were saying about the areas of the white and the grey?
Leah	So, we said that the white part represents the 10×2 …equals 20. And we think that one is greater because …Well, you don't know what *h* represents.
Teacher	Absolutely. Can I get some agreement, disagreement? Maybe someone who agrees that the white (area) is greater? Someone who disagrees, okay? What do you say Aliyah and Sydney?
Aliyah	We agree because 10 times 2 is 20, and 10 times *h* is…um, unknown.
Other students	Wait!…(*inaudible classroom discussion*)
Teacher	All right so 20 is greater than… we don't know (*teacher writes "$20 > 10\times h$" on the board*). Charlene?
Charlene	I kinda agree but I kinda disagree because we don't know what $10 \times h$ is …we don't know what *h* is— maybe it's a 100, but maybe it's 15, so it could go both ways. We don't really know which one is bigger.
Eric	*h* can be any number. It's a variable.
Unknown student	But you don't have *h* right now!
Teacher	Okay. Vincent, what are you thinking?
Vincent	Because in the graph (*picture, Figure 1*) the *h* is more bigger than the 2.
Teacher	In this specific picture? The *h* is bigger than the 2? (*gesturing to a projected Figure 1)* …Based on the picture, we could make an assumption…so then $20 < 10h$?
Unknown student	Well, you didn't say that— you didn't say "based on the picture".
Teacher	Right, right… Well, what number could *h* be that would make it smaller than 20?
Students	(*overlapping talk*) One… Zero!
Teacher	So, if it's zero, it (*gesturing to* $10 \times h$ *in "$20 > 10\times h$"*) would be zero.
Unknown student	Also, one.
Teacher	Yep, I heard a couple of people say "one". If it's one, then it would be 10.
Leah	So, which is right?

Unknown student	We don't know yet.
Teacher	I think it's really cool— it depends on your argument.

In this excerpt, the teacher skillfully takes up a student's claim that relies on the meaning of a variable as an unspecified quantity. The teacher revoiced the claim to the whole class and solicited the reasoning of others. A lively class-wide discussion ensues with students engaged collaboratively as they made and justified other claims using algebraic reasoning about the cases of the variable. The teacher noticed and made productive improvisational decisions to elicit student thinking about an important algebraic idea (i.e., that value of the variable, in this case h, is unknown).

A commonly taken up spontaneous moment that surfaced across the observations during the main activity as well as during the concluding summary activity of the lesson involved the use of substitution to evaluate the equality between two expressions (not suggested as potential student approach in the teacher-facing lesson materials). In the former case, students suggested substituting $h = 8$ into $10(h + 2)$ and $10h + 20$ to determine if the two ways to express the area of the larger rectangle were equivalent. While in the latter case, students' explanations regarding the difference in the areas between two rectangles, one with area $7h$ and the other with area $4h$, involved substituting a numerical value into the equation $7h – 3h = 4h$ to evaluate the truth value of the equation. In both cases, the teachers took up students' suggestions by either performing the substitution on the board or having a student perform and evaluate the substitution.

There were also a significant proportion (23%) of spontaneous early algebra moments (as identified by the research team) that were not taken up by teachers. For example, several teachers did not take up opportunities to engage students in discussion about their desire to assign a specific value to the variable h before an expression for the total area of the two rectangular areas could be determined. In one teacher's classroom, several students expressed a need to physically measure with a ruler or an improvised ruler (e.g., repeated measurement with the width of their finger, tick marks on a paper) to determine a numeric value for h. One student, using a pencil as a ruler, insisted that the length of h was half of 10 (based on the figure itself) and thus thought that $h = 5$. The teacher's response was "okay, then plug that in for your h". Here the teacher did not address the meaning associated with using a variable in the area expression, and the lesson materials did not anticipate that students might express the need to assign a specific value to h in order to write an expression for the areas of the rectangles.

Discussion

Algebra moments that surface in the implementation of early algebra lessons that the research team anticipated, indeed occurred, and were addressed by the majority of teachers (not a surprising result if teachers simply followed the lesson). Similarly, those spontaneous algebra moments that the research team did not anticipate, were also addressed by the majority of teachers. These latter findings are both somewhat surprising given often negative characterizations of elementary school teachers' mathematics knowledge, and promising with respect to teachers' abilities to engage their students with early algebra ideas and practices. In short, at least for these sixteen teachers, their early algebra "eyes and ears" were relatively well developed. Yet, for those early algebra moments not taken up—both anticipated and spontaneous—the current study offers little insight into why these moments were not taken up or missed. In some cases, there may certainly be reasonable professional explanations supporting a teacher's decision to not take up a moment including practical limitations such as time and classroom interruptions. In other cases, it may be that the moment was indeed missed, that is, a teacher did not notice the opportunity as it arose during the course of instruction—cases that speak to the need for professional development if teachers are expected to meaningfully engage their students in early algebra.

There are also important characteristics of algebraic moments that, although not reported on here, are a part of our ongoing study, namely, the quality of teachers' instructional practice around early algebra moments. It is clear as we view the lesson observations that not every algebra moment is addressed in exactly the same way, and that there are differences in the ways in which teachers address these moments as well as in what students may learn as a result. For example, the warm up for this lesson prompted students to create a coordinate graph to represent the relationship between two co-varying quantities—one which was increasing at a rate twice that of the other quantity. One teacher facilitated a classroom discussion with her students in which they first predicted what such a graphical representation would look like prior to students generating graphs themselves, and then shared and compared the student-generated graphs. In contrast, another teacher led a teacher-centered episode in which she generated a table of the two co-varying quantities for students to copy, and subsequently led the creation of the coordinate graphical representation of the relationship with little to no student input.

Conclusion

In this paper we introduced the idea of early algebra moments, both those moments that arise as anticipated from an examination of a lesson and those moments that arise spontaneously in the course of instruction. By examining the same lesson implemented by sixteen teachers, we are able to see the variety of ways in which teachers engage with these moments (or, in some cases, do not engage), highlighting not only the diversity of approaches, but also the quality of instruction around these moments. Given the space constraints, we primarily focused on illustrating several examples of anticipated and spontaneous moments, These results offer promise with respect to teachers' implementation of early algebra in the elementary classroom as well as highlight the need to support teachers' efforts to develop their early algebra eyes and ears. Moreover, the results offer insights for understanding how the early algebra curricular materials support teachers in anticipating students' algebraic thinking and how teachers' spontaneous responses to moments can further students algebraic thinking. By studying and spotlighting teachers' facilitation of these moments we hope to amplify high quality teaching of early algebra. Additionally, understanding algebraic moments that were not taken up in classrooms could serve to hone professional development around noticing and attending to students' algebraic thinking.

Acknowledgement

The research reported here was supported in part by the National Science Foundation under Award DRL-XXXXXXX. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

References

Arbaugh, F., Herbel-Eisenmann, B., Ramirez, N., Knuth, E., Kranendonk, H., Reed Quander, J. (2010). Linking research and practice: The NCTM Research Agenda Conference report. Reston, VA: National Council of Teachers of Mathematics.

Arcavi, A. (2005). Developing and Using Symbol Sense in Mathematics. *For the Learning of Mathematics, 25(2), 42-47.*

Blanton, M. L., & Kaput, J. J. (2003). Developing Elementary Teachers' Algebra Eyes and Ears. Teaching Children Mathematics TCM*,* 10(*2*), 70-77.

Borko, H., & Livingston, C. (1989). Cognition and Improvisation: Differences in Mathematics Instruction by Expert and Novice Teachers. *American Educational Research Journal, 26*(4), 473–498. https://doi.org/10.3102/00028312026004473

Bromme, R. (2014). *Der Lehrer als Experte: Zur Psychologie des professionellen Wissens* (Vol. 7). Waxmann Verlag.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Epson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*, 403–434.

Hohensee, C. (2017). Preparing elementary prospective teachers to teach early algebra. *Journal of Mathematics Teacher Education*, 20(3), 231-257. doi: 10.1007/s10857-015-9324-9

Kieran C., Krainer K., Shaughnessy J.M. (2012). Linking Research to Practice: Teachers as Key Stakeholders in Mathematics Education Research. In: Clements M., Bishop A., Keitel C., Kilpatrick J., Leung F. (eds) *Third International Handbook of Mathematics Education*. Springer International Handbooks of Education, vol 27. Springer, New York, NY. https://doi.org/10.1007/978-1-4614-4684-2_12

Leatham, K., Peterson, B., Stockero, S., & Van Zoest, L. (2015). Conceptualizing Mathematically Significant Pedagogical Opportunities to Build on Student Thinking. *Journal for Research in Mathematics Education, 46*(1), 88-124. doi:10.5951/jresematheduc.46.1.0088

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57(1*), 1-23.

Stockero, S., & Zoest, L. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. *Journal of Mathematics Teacher Education, 16*(2), 125–147.

Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education, 18*, 523–550. doi:10.1007/s10857-014-9289-0

PROGRESSIONS IN GRADES K–1 STUDENTS' UNDERSTANDING OF PARITY ARGUMENTS

Maria Blanton
TERC
Maria_Blanton@terc.edu

Ingrid Ristroph
University of Texas at Austin
Ingrid.Ristroph@utexas.edu

Angela Murphy Gardiner
TERC
angela_gardiner@terc.edu

Ana Stephens
University of Wisconsin-Madison
acstephens@wisc.edu

Rena Stroud
Merrimack College
stroudr@merrimack.edu

Eric Knuth
University of Texas at Austin
eric.knuth@austin.utexas.edu

Understanding how young learners come to construct viable mathematical arguments about general claims is a critical objective in early algebra research. The study reported here characterizes empirically developed progressions in Grades K–1 students' thinking about arguments concerning sums of evens and odds. Data are drawn from classroom lessons of an early algebra instructional sequence and interviews conducted at four timepoints during the implementation of the sequence. Overall, students transitioned from unfamiliarity with the concepts of even or odd prior to instruction in Kindergarten to making valid parity arguments at the conclusion of instruction in Grade 1. Results of this study align with other research that shows young learners can develop viable arguments to justify mathematical generalizations.

Keywords: Reasoning and proof; algebra and algebraic thinking; learning trajectories and progressions; early childhood education

Purpose of the Study

Research increasingly supports that engaging in proving in developmentally appropriate ways in the elementary grades can deepen students' conceptual understanding of mathematics as a sense-making activity (Carpenter et al., 2003; Stylianides & Ball, 2008; Van Ness & Maher, 2019). Moreover, when elementary students are supported through instruction, they can learn to use deductive—rather than empirical—reasoning to build mathematical arguments (Stylianides & Stylianides, 2008). However, intervention studies are needed to develop a "fine-grained conceptualization" (Stylianides, 2007b, p. 18) of appropriate forms of proving in early grades and to understand how curriculum can support students' construction of viable arguments.

The study reported here responds to this call by identifying learning progressions (Clements & Sarama, 2004) in how children construct mathematical arguments as they are taught an early algebra instructional sequence in Kindergarten (hereafter, Grade K) and Grade 1. The study focuses on the following question: *What levels of thinking do Grades K–1 students exhibit in their understanding of arguments about sums of evens and odds as they advance through an early algebra instructional sequence?*

Perspective

The study reported here is part of a suite of projects in which we use a learning progressions approach (Clements & Sarama, 2004) to build an effective early algebra intervention across Grades K–5 and to identify progressions in students' thinking as they advance through the instructional sequence that forms the intervention. We organize early algebra around the practices of generalizing, representing, justifying, and reasoning with mathematical structure and relationships (Blanton et al., 2018). The focus on children's mathematical arguments here is a natural part of our early algebra research. In particular, in this study we are interested in justifying as a practice of building arguments for claims about general relationships, which can elevate the role of argumentation in the elementary grades.

Methods

Our research design involved the use of classroom teaching experiments (CTEs) with individual interviews (Cobb & Steffe, 1983) to identify progressions in children's thinking. Because CTEs incorporate instructional design with the ongoing analysis of classroom data, they serve as an important mechanism for developing empirically based conjectures about progressions in thinking as students advance through an instructional sequence (Clements et al., 2007; Lesh & Lehrer, 2000).

The K–1 early algebra instructional sequence in the study reported here consisted of eighteen 30-minute lessons for each grade. A subset of lessons within the instructional sequence (3 in Grade K, 2 in Grade 1) addressed the development of parity arguments. Lessons initially focused on preliminary concepts such as "pair," "even number," and "odd number," then transitioned to the development of representation-based arguments (Schifter, 2009) about the parity of the sum of two even numbers, two odd numbers, and then an even and an odd number. Additional lessons (8 in Grade K, 3 in Grade 1) reviewed parity concepts in lesson warm-ups.

Participants

Two Grade K classrooms (Year 1) and two Grade 1 classrooms (Year 2) in one school in the Northeastern US participated in the study. Effort was made to keep the initial Grade K cohort intact in Grade 1. Table 1 shows the number of participants by grade level. The school district's demographics consisted of 10% students of color and 16% students categorized as low SES.

Table 1. Participants by Grade Level

Grade	Total No. of Participants	Participated in Grade K only	Participated in Grade 1 only	Participated in Grades K–1
K	48	22		26
1	48		22	26

Data Collection

Grade-level early algebra lessons were taught approximately once per week during each school year by a member of the research team. All lessons were videotaped, and lessons or portions of lessons (e.g., lesson warm-ups) related to even and odd concepts were identified for analysis. A subset of students was selected at each grade for semi-structured, 30-minute, individual pre/post interviews. Classroom teachers helped identify students who fit a diverse academic range. An early math diagnostic assessment (EMDA™) was administered to students at the beginning of Grade K to further ensure academic diversity. In all, 17 students in Grades K–1 participated in video-taped interviews about parity concepts. Twenty-four full or partial classroom lessons and 40 interviews were analyzed.

Data Analysis

We used a grounded theory approach (Strauss & Corbin, 1990) that focused on the identification of progressions in students' thinking around the concepts of pair, parity of numbers, and arguments about parity of sums. Interview data were analyzed independently by three team members to identify preliminary codes (levels of thinking) for our core concepts. Theoretical memos (Glaser, 1998) were constructed to provide supporting evidence for the codes, or levels. Agreement among coders was determined by comparing coding decisions and negotiating any discrepancies around early codes/levels. New codes were identified as warranted and data were re-analyzed until subsequent coding did not change our emerging models. Video of classroom data were then analyzed for confirming or disconfirming evidence of emerging codes (levels). The levels-as-coding schemes were then organized based on our empirical findings vis-à-vis canonical understandings of the mathematics attempted (Battista, 2004).

Results

In Table 2, we report our findings on levels in students' thinking about the most complex interview task—parity arguments about the sum of arbitrary even and/or odd numbers. We then share some observations from our study.

Table 2. Parity Arguments for the Sum of Two Arbitrary Even/Odd Numbers

	Non-Structural Reasoning	
Level 1 *Empirical*	Students suggest a strategy that uses or implies the use of testing cases to determine parity for an arbitrary sum.	Student provides an argument based on empirical experience (counting): "If you add even and odd it's usually even because every time I count that, it's usually even. One time it wasn't."
	Structural Reasoning	
Level 2 *Generic Number*	Students use a pairs strategy applied to generic numbers to reason about parity of the sum. A pairs strategy involved finding if a number represented through cubes (for example) could be separated into pairs without a cube left over.	When asked if the sum of "a really big odd number" and a "really big even number" is even or odd, a student gave the following response: "Odd, because let's say you did it with [12 and 15]. Fifteen has a leftover and 12 doesn't have a leftover, so it can't combine." He indicates that by "can't combine" he means there would still be a leftover.
Level 3 *Arbitrary Number*	Without referencing any numbers, students use a pairs strategy to reason about parity of the sum.	When asked to explain why any even plus any odd is odd, a student argues, "Because an odd number has a leftover and an even number doesn't have a leftover and if you combine an odd number with an even number, it equals an odd number," confirming that there would still be a leftover.

There are important similarities between the levels of thinking students exhibited and taxonomies reported elsewhere. Empirical reasoning (Level 1), by which students examined randomly chosen cases to establish a conjecture's truth, reflects *naïve empiricism* (Balacheff, 1988) or an *empirical (inductive) proof scheme* (Harel & Sowder, 2007). In the example provided, the student reflects on prior experiences in "counting" (adding) particular even and odd numbers as the basis of her argument. Schifter (2009) similarly characterizes this type of argument as *inference from instances* in her analysis of Grade 3 students' parity arguments. We see Level 2 as consistent with the notion of *generic example* (Balacheff, 1988) or *deductive (transformational) proof scheme* (Harel & Sowder, 1998) because of the use of a specific number as a generic placeholder for a general number in students' arguments. Schifter (2009) refers to this form of argument as *reasoning from representation or story context*. Finally, Level

3 reflects the construct of *thought experiment* (Balacheff, 1988), where actions are dissociated from specific examples. With young children, these actions are based on "the language of the everyday (Balacheff, 1988, p. 228)" and not the transformation of formal symbolic expressions. Level 3 thinking seems to also reflect the emergence of *structure in thought* (Harel & Soto, 2017) in that students were able to present a sequence of arguments without the need to manipulate physical, visual, or symbolic representations. Our work extends that reported elsewhere in that our focus is on the genesis of these ideas at the start of formal schooling (Grades K–1). Identifying similarities in students' thinking across K–16 grade domains can help us better understand how to structure curriculum that connects and builds ideas over time.

We found students' ability to develop representation-based parity arguments (rather than empirical arguments) to be surprisingly robust. For example, at pre-interview Grade K students were unfamiliar with the concepts of pair and even and odd numbers, but by Grade K post-interview all students were able to define even and odd using a pairs strategy and many students routinely used a pairs strategy to reason about the parity of numbers represented in concrete, visual, and abstract forms. Moreover, although Grade K students were not asked to build representation-based parity arguments at pre-interview (given that they were unfamiliar with the concepts "even" and "odd"), by Grade 1 pre-interview no students used an empirical argument to justify why the sum of an even and an odd would be odd. Six out of 10 students were able to correctly use a structural argument involving a pairs strategy (three students could not build either type of argument; one student was not asked this question).

Although these students were capable of producing structural arguments (Levels 2–3), we do not claim that they yet appreciated the power of the generality of their arguments, nor did they yet understand the logical proof structure underlying their arguments. In truth, older students struggle with this as well (Stylianou et al., 2015). At this early, pre-symbolic point in their understanding, students' thinking was more intuitive than anticipatory or intentional. However, a goal of early algebra is to help students build on these intuitive ways of thinking so that their understanding can deepen over time. For young children to engage in Level 3 thinking, even in informal, non-symbolic ways, is a critical starting point.

Conclusion

The *Common Core State Standards* (NGA Center & CCSSO, 2010) maintains that elementary students should be able to "construct viable arguments and critique the reasoning of others." There is a vital need, however, for research-based, curricular pathways by which this goal can be met (Bieda et al., 2014), as well as studies that detail how young learners' argumentation progresses as they advance through such pathways (Stylianides, 2007a). The study reported here is intended to help address this by characterizing the emergence of students' understanding of viable parity arguments from an early algebra instructional sequence Understanding how young leaners develop robust ways of thinking and arguing mathematically can not only help avoid situations where students develop "a conception of proof in the elementary school that has to be undone or unlearned in high school (p. 4, Stylianides, 2007b)," but can also help realize the ambitious learning standards advocated in current reforms.

Acknowledgments

The research reported here was supported by the US Department of Education under IES Award # R305A170378. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the US Department of Education.

References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In Pimm, D. (Ed.), Mathematics, teachers, and children (pp. 216-235). London: Hodder & Stoughton.

Battista, M. (2004). Applying cognition-based assessment to elementary school students' development of understanding of area and volume measurement. Mathematical Thinking and Learning, 6(2), 185–204. doi:10.1207/s15327833mtl0602_6

Bieda, K. N., Ji, X., Drwencke, J., & Picard, A. (2014). Reasoning-and-proving opportunities in elementary mathematics textbooks. International Journal of Educational Research, 64, 71–80.

Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A., Stroud, R., Fonger, N., & Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), Teaching and learning algebraic thinking with 5- to 12-year-olds: The global evolution of an emerging field of research and practice (pp. 27-49). Cham, Switzerland: Springer International Publishing.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). Thinking mathematically: Integrating arithmetic and algebra in elementary school. Portsmouth, NH: Heinemann.

Clements, D. H. & Sarama, J. (2004). Learning trajectories in mathematics education. Mathematical Thinking and Learning, 6(2), 81–89. doi:10.1207/s15327833mtl0602_1

Clements, D. H., Sarama, J., & Barrett, J. E. (2007). A longitudinal examination of children's developing knowledge of measurement. Normal, IL and Buffalo, NY: National Science Foundation.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14(2), 83–94. DOI: 10.2307/748576

Glaser, B. G. (1998). Doing grounded theory: Issues and discussions. Mill Valley, CA: Sociology Press.

Harel, G., & Soto, O. (2017). Structural reasoning. International Journal for Research in Undergraduate Mathematics Education, 3, 225–242. doi:10.1007/s40753-016-0041-2

Harel, G., & Sowder, L. (1998). Students proof schemes: Results from exploratory studies. In A. H. Schoenfeld, J. Kaput, & E. Dubinsky (Eds.), Research in collegiate mathematics education III. Providence, RI: American Mathematical Society.

Harel, G., & Sowder, L. (2007). Toward a comprehensive perspective on proof. In F. Lester (Ed.), Second handbook of research on mathematics teaching and learning. Greenwich, CT: Information Age Publishing.

Lesh, R. & Lehrer, R. (1999). Iterative refinement cycles for videotape analyses of conceptual change. In A. Kelly & R. Lesh (Eds.), Handbook of research in mathematics education (pp. 665-708). Hillsdale, NJ: Lawrence Erlbaum.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Council of Chief State School Officers. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Schifter, D. (2009). Representation-based proof in the elementary grades. In D. Stylianou, M. Blanton & E. Knuth (Eds.), Teaching and learning proof across grades: A K-16 Perspective (pp. 71-86). New York, NY: Routledge.

Strauss, A.L., & Corbin, J. (1990). Basics of Qualitative Research: Grounded Theory Procedures and Techniques. Los Angeles, CA: Sage.

Stylianides, A. J. (2007a). Proof and proving in school mathematics. Journal for Research in Mathematics Education, 38, 289–321.

Stylianides, A. J. (2007b). The notion of proof in the context of elementary school mathematics. Educational Studies in Mathematics, 65, 1–20.

Stylianides, A. J., & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. Journal of mathematics teacher education, 11(4), 307-332.

Stylianides, G. J., & Stylianides, A. J. (2008). Proof in school mathematics: Insights from psychological research into students' ability for deductive reasoning. Mathematical Thinking and Learning, 10(2), 103–133.

Stylianou, D. & Blanton, M. & Rotou, O. (2015). Undergraduate students' proof conceptions: Relationships between understanding, beliefs, and classroom experiences with learning proof. International Journal for Research in Undergraduate Mathematics Education, 1 (1), 91-134.

Van Ness, C. K., & Maher, C. A. (2019). Analysis of the argumentation of nine-year-olds engaged in discourse about comparing fraction models. Journal of Mathematical Behavior, 53, 13–41.

AN EXPLORATION OF HOW COLLEGE STUDENTS THINK ABOUT PARENTHESES IN THE CONTEXT OF ALGEBRAIC SYNTAX

Claire Wladis
BMCC/CUNY,
CUNY Graduate Center
cwladis@bmcc.cuny.edu

Ben Sencindiver
CUNY Graduate Center
bsencindiver@gc.cuny.edu

Kathleen Offenholley
BMCC/CUNY
koffenholley@bmcc.cuny.edu

In this paper we explore how college students across different courses appeared to interpret the meaning of parentheses or brackets in the context of algebraic syntax. This work was influenced by theories of computational vs structural thinking, and also considered the extent to which students' definitions, computational work, and explanations appeared to be consistent with specific normative definitions of parentheses. In analyzing student work, several categories of students' conceptions emerged, which may be helpful in diagnosing which conceptions may be more productive or problematic as students progress through algebra. For students who appear to conceptualize parentheses as a cue to non-normative procedures, several categories of procedures were found, which could have implications for instruction.

Keywords: algebra and algebraic thinking; cognition

Parentheses, or brackets, play a critical role in symbolic mathematics. However, how students think about parentheses in various mathematical contexts is significantly understudied. While some studies examine how parentheses may or may not cue the correct order of operations during arithmetic tasks in primary school or whether adding extra parentheses helps students to better "see" algebraic structure (e.g., Gunnarsson et al., 2016), there appears to be no systematic research which explores which kinds of meanings students have for brackets or parentheses, particularly in a wider array of content domains which include substantial symbolic algebra. In this paper we aim to address this gap. Though extensive data collection with college students in a wide range of classes, we explore students' meanings for parentheses and generate several common categories of conceptions which students may hold.

Literature Review

Research into student's use of parentheses in algebra has been limited, and what does exist has focused primarily on what students do with parentheses when calculating rather than how students conceptualize parentheses. In the context of structure sense, Hoch and Dreyfus (2004) found that secondary students tend to use structural approaches to solve algebraic equations more when an equation uses more parentheses than when less parentheses are present. In interviews, students' use of parentheses were mixed, where some students approached problems by first 'opening' parentheses, while others preferred maintaining parentheses, finding the symbol helpful and saying "with parentheses, it's easier to see" (p.3-54). Similar findings have been reported with middle school students as well (e.g., Linchevski & Livneh, 1999), where some students at times operated on parentheses within expressions as if they are not present (Gunnarsson et al., 2016), while others operated in markedly different ways when working with expressions within parentheses (Banerjee & Subramaniam, 2005). Given that the presence of parentheses may change one's view of a mathematical expression, we think it is worthwhile to explore students' meanings of parentheses.

Theoretical Framework

Tall and Vinner (1981) describe an individual's understanding of a concept in terms of their concept image, or the "total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Within this structure, they describe one's *personal* concept definition as the "form of words that a learner uses for [their] own explanation of [their] (evoked) concept image" (p. 152). One's personal concept definition is idiosyncratic to the learner, and is contrasted with the *formal* concept definition, which is "a concept definition which is accepted by the mathematical community at large" (p.152). In the examples that follow, when we refer to a student's definition, we are referring to their personal concept definition, although our model is a second order model, as we do not have direct access to student's personal concept images or definitions.

In the context of this paper, we consider the role that parentheses play in the syntax of algebraic expressions and equations in which the symbols are used to indicate an operation which is prioritized over other adjacent operations; we call this the ***grouping role*** of parentheses. This role of parentheses is pervasive throughout symbolic representations across domains, but we focus on algebra examples in this paper. We note that this grouping role is distinct from other roles which parentheses may play, such as denoting intervals, ordered pairs, sets, etc.

Students may conceptualize parentheses from either a computational or a structural view, just as they may conceptualize algebraic syntax more generally as computational[1] or structural (Sfard, 1995). In a ***computational view***, students conceptualize parentheses as cuing a particular calculation or procedure, which may be normative or non-normative. In a ***structural view***, students conceptualize parentheses as demarcating a particular unified sub-expression which can be treated as an object in and of itself. For example, a student with a computational view may normatively "see" the expression $2(5+3)$ as indicating that one should first add $5+3$ and then multiply 2 by the result. A student with a structural view is able to "see" this also as 2 times whatever the result of $5+3$ might be, without actually performing the addition of $5+3$ and replacing it with the single number 8 first; instead the substring $(5+3)$ can be seen as an object itself. In an arithmetic example such as this one, the structural view appears less important, but in an algebraic expression like $2(x+3)$, the affordances of a structural view suddenly becomes more apparent and, at times, necessary, for example, for many cases of u-substitution in upper level classes where one must have a structural view of certain substrings as objects.

Edwards and Ward (2004) distinguished between two sorts of definitions: those that are created through experiences with the term (an *extracted* definition), and those that are expressed in an explicit well-defined way (a *stipulated* definition). When students have a computational view of parentheses, this may be stipulated and normative or non-normative (which may have been extracted from their experiences with common types of tasks in which parentheses occur, rather than taken from stipulated normative definitions). For some students, parentheses may only cue the normative set of procedures stipulated by the order of operations, while for others, parentheses may cue a particular procedure, regardless of whether that procedure is appropriate (e.g., taking whatever is outside the parentheses and multiplying it by each "thing" inside the parentheses, even when the structure of the algebraic expression with which they are working is not in line with the distributive property). We summarize this in the Figure 1, where we see learners' conceptions varying along a continuum, where conceptions to the right are more generalizable and transferable to a wider range of problems than those to the left. We note that an individual's conceptions of parentheses are not fixed and can further vary back and forth along

1 Sfard (1995) refers to 'operational' views, but we use the term computational here.

this continuum, even within a single problem.

Extracted computational view: Learners conceptualize brackets as a cue to a procedure, but that procedure is not consistent in all contexts with the stipulated order of operations and other symbolic conventions (e.g., to multiply regardless of actual operations).	Stipulated normative computational view: Learners are able to conceptualize brackets as a cue to follow the stipulated order of operations (i.e., simplify what is inside the innermost brackets first).	Structural view: Learners are able to conceptualize brackets as demarcating a unified sub-expression within a larger expression or equation, which can be thought of as an object in and of itself. This may be the reification of the process of the stipulated order of operations.

Figure 1: Types of Conceptions of Parentheses

Methods

Data for this study includes students' written responses to open-ended questions and transcripts from stimulated recall interviews about similar multiple-choice questions. Open-ended responses were collected from 124 students at an urban community college in 18 different courses, from developmental elementary algebra (similar to Algebra I in high school) to linear algebra. Open-ended questions were about what parentheses means to the students (both in and out of context), as well as algebra problems where students were asked to simplify expressions that included parentheses. Students' open-ended responses were analyzed using thematic analysis (Braun & Clarke, 2006). This analysis was influenced by an initial theoretical stance which was particularly attuned to noticing similarities in patterns of students' responses with theory on extracted and stipulated definitions (Edwards & Ward, 2004) and computational and structural (Sfard, 1995) conceptions. Analysis of students' work led to a more nuanced emergent coding scheme of students' responses for definitions of parentheses. We select students' work to illustrate the resulting coding scheme in this work and draw on interview transcripts to further expound on these patterns.

Results

In this section we present several vignettes to illustrate the different ways in which students appeared to be conceptualizing parentheses in the context of algebra expressions and equations. We note that categorizations of student work here are second order models: we cannot actually know what a student is thinking—we only categorize what they conveyed through their work.

Different non-normative computational conceptions of parentheses

It was common for parentheses to cue various non-normative procedures, though they tended to fall mostly into one of three categories: "Multiplication", "do first", and "ease of reading".

"Multiplication" conception of parentheses. The most common response among students at all levels when asked what parentheses mean in mathematics was "multiplication". We note that technically parentheses do not indicate multiplication; they simply co-occur often with concatenation, which actually indicates multiplication. However, during data collection in both interviews and open-ended questions, we found that students rarely recognized this distinction. This may be an extracted definition taken from their many experiences with tasks in which parentheses and multiplication co-occur, or it may even sometimes be inaccurately stipulated by instructors (we have observed instructors in classrooms using phrases like "parentheses mean to multiply"). So while not technically mathematically normative, this view can be a very rational reaction to existing instructional practices. However, this conception may lead to incorrect computational results when the operation next to the parentheses is not multiplication, as well as a fundamental misunderstanding of the role that parentheses are intended to play in algebraic expressions and equations. Consider, for example, the work from Alpha (see Figure 2) who was enrolled in an elementary algebra course (a non-credit course similar in content to Algebra I).

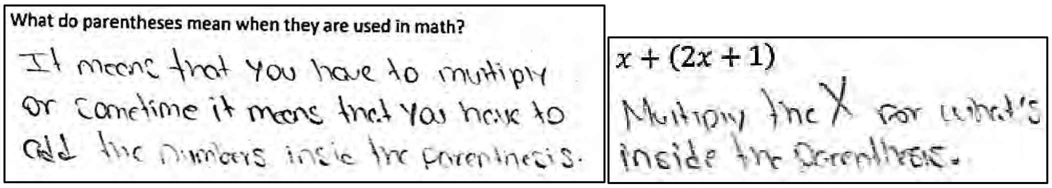


Figure 2. Alpha's Parentheses Definition and Evoked Interpretation in Computation

Alpha's definition of parentheses (Figure 2, left) shows two meanings. Firstly, they tell us that parentheses tell us to multiply; but they also say that parentheses tell us that we have to add the numbers inside the parentheses, which we understand to be a kind of "do first" interpretation of parentheses (see next section). Alpha further provides an example of how the interpretation of parentheses as multiplication can be problematic during computation on the right in Figure 2, where they multiply even though addition is the operation between x and $(2x + 1)$.

"Do first" conception of parentheses. The second meaning Alpha's definition (Figure 2, left) may be referring to is a potentially normative definition of parentheses-- it is not completely clear from their wording, but they may be referring to the precedence of parentheses in the order of operations. We describe this thinking more in the next section, but focus now on some ways in which students applied this definition non-normatively. We begin by considering the work of a student whom we will call Gamma, who was enrolled in a course for students training to be elementary school teachers (prerequisites for this course include elementary and intermediate algebra, but not precalculus). In this excerpt, Gamma is attempting to simplify $3 - 2 \cdot (8 - 3^2)$.

> So, I try in my head, so they break it down, PEMDAS, so parentheses first, so I did like eight minus three is five and I know that's going to leave me with five squared. So, I just left that as it is and looked over here and seen what has five to the second exponent. And realized, you know, okay that's the same because that would be my next step is to solve the parentheses…. Because PEMDAS I solved what's in the parentheses first and then looked at the exponents and then that's pretty much how I saw it.

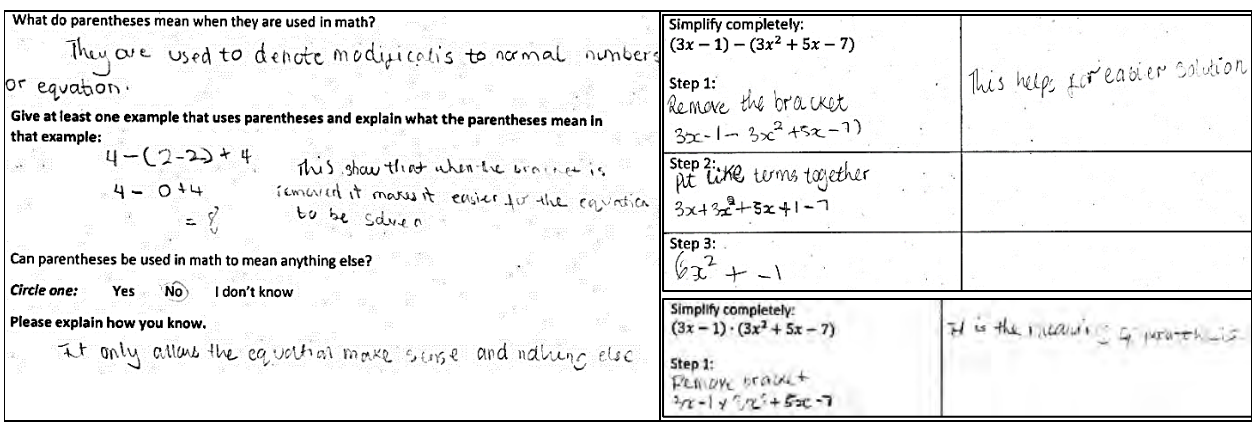
Gamma appears to be thinking of the parentheses as something which cues work inside them to be done first, and because exponents come after parentheses, they do $8 - 3$ first. We classify this as an extracted view, because their notion of "doing the parentheses first" is cuing an incorrect extracted definition of the order of operations, in which things inside the parentheses are combined together in order to "get rid of" the parentheses before doing the exponent that is inside the parentheses itself. Here Gamma appears to be interpreting the parentheses as a cue to a particular (incorrect) computational action.

In other work, Gamma is able to use this conception of parentheses to simplify an algebraic expression correctly, although they still use a computational approach. In the next excerpt, they were asked about what was being multiplied by 2 in the expression $2 \cdot (3x - 5)$. We note that even though the question was intended to prompt an object approach (by thinking of $(3x - 5)$ as the object being multiplied by 2), Gamma still gives a computational explanation focused on "solving", even though that is not what has been asked.

When I think of parentheses it's something that has to be done first. In this particular problem, I feel like you have to distribute because you have x there. So, it's like you can't solve $3x$ minus five. I don't think you could just like get an answer from that. You have to solve what's inside of the parentheses. So, what's in the parentheses is $3x$ minus five. So, in order for me to solve that I must distribute the two that's outside into that equation. I just think of PEMDAS, you have got to do parentheses first, and then exponents, and down the line. So, I just look at a question, I know I have to do something with the parentheses first.

Gamma's notion of parentheses here appears to be a "do first" conception—they appear to be correctly connecting their notion of the meaning of the parentheses to the stipulated order of operations. However, it is not completely clear whether this "do first" notion is fully well-defined and in line with stipulated definitions. Would Gamma distribute in other problems inappropriately, for example if the two were an exponent of the parentheses instead of a coefficient? We cannot be sure since Gamma was not interviewed about such questions. During data collection, we saw many students who seemed to convey a "do first" definition of brackets/parentheses, but who performed computational work that violated the order of operations in some cases and not in others. This suggests that we may want to be more attentive to how students interpret the "do first" conceptualization of parentheses as they move from arithmetic expressions (in which what is inside the parentheses can be simplified) to algebra (where what is inside the parentheses may not be able to be simplified).

"Ease of reading" conception of parentheses. Now we consider some examples in which students appear to have interpreted parentheses as unnecessary symbols which can be removed; sometimes they justify this by explaining that parentheses are just there to make expressions or equations "easier to read". We note that this is related to the "bracket ignoring" procedure identified by Gunnarsson, et al. (2016) with middle school students. First we consider work from an elementary algebra student whom we call Tau (Figure 3).


What do parentheses mean when they are used in math?

They are used to denote modificatis to normal numbers or equation.

Give at least one example that uses parentheses and explain what the parentheses mean in that example:

$4-(2-2)+4$
$4-0+4$
$=8$

This show that when the bracket is removed it makes it easier for the equation to be solved

Can parentheses be used in math to mean anything else?

Circle one: Yes No I don't know

Please explain how you know.

It only allows the equation make sense and nothing else

Simplify completely:
$(3x-1)-(3x^2+5x-7)$

Step 1:
Remove the bracket
$3x-1-3x^2+5x-7)$

This helps for easier solution

Step 2:
Put like terms together
$3x+3x^2+5x+1-7$

Step 3:
$6x^2+-1$

Simplify completely:
$(3x-1)\cdot(3x^2+5x-7)$

Step 1:
Remove bracket
$3x-1 \times 3x^2+5x-7$

It is the meaning of parenthesis

Figure 3. Tau's Definition of Parentheses and Computational work

Tau appears to conceptualize parentheses as something that makes it "easier for the equation to be solved" or "allows the equation to make sense". It is not entirely clear what they mean in this case—for example, in the arithmetic example that they give in Figure 3, their calculations appear to be correct. They "remove" the parentheses in that case by simplifying what is inside them first. However, this seems not to generalize in the same way to algebra cases. When we look at how they dealt with parentheses in practice when solving standard algebra tasks, we see that their first step was often to remove the parentheses before proceeding (see right of Figure 3), similarly to erasing the parenthetical symbols. Tau even justify their work by writing "Remove the bracket. This helps with easier solution" and "Remove bracket…It is the meaning of parentheses." We suspect that Tau has extracted from their prior mathematical experiences that it is desirable to find valid mathematical ways to "get rid of the bracket", this process involves simply removing the bracket, and parentheses are superfluous.

This may also be related to the "do first" conception of parentheses, as students may feel that they must remove the parentheses in order to do any computation with what is inside. Some students may remove the parentheses using a valid transformation (e.g., the distributive property, or expanding multiplication of two polynomials) and other students may remove the parentheses using an invalid transformation, like Tau. This approach was not limited only to elementary algebra students. In Figure 4, we see similar work from a Calculus I student, Zeta.

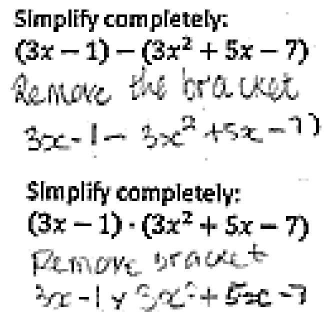

Simplify completely:
$(3x-1)-(3x^2+5x-7)$
Remove the bracket
$3x-1-3x^2+5x-7)$
Simplify completely:
$(3x-1)\cdot(3x^2+5x-7)$
Remove bracket
$3x-1 \times 3x^2+5x-7$

Figure 4. Zeta's Work in which They Remove Parentheses Arbitrarily

Combinations of different extracted and stipulated computational views.

So far we have seen some examples of different types of non-normative extracted conceptions of parentheses, but there were also students who showed a mix of normative and non-normative meanings for parentheses. One such example comes from a student, Epsilon, enrolled in the second semester of a one-year math course for future elementary school teachers (the prerequisites for the first semester of the course were elementary and intermediate algebra, but not precalculus). In an interview, Epsilon was asked which part was being subtracted, or taken away, in the expression $3x^2 - 2(x^2 + 1)$:

Interviewer:	What do the parentheses mean here in this expression?
Epsilon:	Multiplication.
Interviewer:	Do you know what the order of operations is?
Epsilon:	PEMDAS.
Interviewer:	How does the order of operations help you to understand what is being subtracted in this expression?
Epsilon:	So first, you have to deal with the parentheses. So, then that means that everything with the parentheses is like together in a box.

Epsilon, like many students, thinks of multiplication first when they are asked what parentheses mean. When asked directly about how the order of operations relates to what they are doing in the problem, they exhibit the "do first" notion of parentheses (we can't tell from this context whether it is normative or not), pointing out that "first, you have to deal with the parentheses". But they immediately follow that up with a structural grouping interpretation of parentheses, saying "that everything with the parentheses is like together in a box". It is unclear to us how Epsilon is interpreting parentheses to mean multiplication here, but when they cite a "do first" conception of parentheses they immediately link it to a grouping conception, where the "do first" conception is a justification ("then that means") for the grouping conception. While we cannot be certain, we suspect that Epsilon has reified the order-of-operations process into the grouping object, which would explain how their "do first" and grouping conceptions are linked.

Some students may use the language "parentheses mean multiplication", but then go on to only describe a correct alternative meaning, at least in the context of Algebra I course content; this may be a case in which they are incorrectly describing the meaning of the parentheses symbol, but are correctly performing computation with parentheses, so while logically incorrect, this imprecision of language may be less problematic in this case. For example, consider the next excerpt from Epsilon, in which they discuss their meanings of parentheses when asked whether $\sqrt{(3xy)^2}$ and $\left(\sqrt{3xy}\right)^2$ represent the same operations on the same things in the same order:

> [Parentheses] don't really mean like multiplication per say in this instance. So, in the first one the parentheses is the expression of $3xy$ to the second power. So, it's distributing the- Well, in both instance it's distributing the second power, but it's just where it's placed. I'm sorry, I don't know if I'm making any sense, but because of where the parentheses are placed it just changes the meaning.
>
> So, here [pointing to first expression] you have to deal with the parentheses first. So, in the first equation, you have to distribute the second power to the $3xy$ and then take the square root. And then in the second equation, you have to take the square root first of $3xy$ and then square it and those would result in different answers probably.

In this explanation, Epsilon says that parentheses mean multiplication, but then they say that this is not necessarily the correct interpretation in this case. Here instead they treat the parentheses as a cue to distribute the exponent. Their work is correct, but it is unclear if this notion of parentheses cuing distributing might lead them to distribute the exponent inappropriately in other cases (e.g., if instead of $3xy$ in the parentheses, there were a multi-term expression). As with Gamma, we had no questions on the set of questions on which Epsilon was interviewed which would allow us to see if Epsilon might distribute incorrectly in other cases. So while Epsilon's work here appears to be based on stipulated definitions, and on the surface appears to be less extracted and less problematic than Alpha's multiplication conception in terms of how it may impact their computational work, we cannot be sure whether it is actually completely grounded in standard stipulated definitions or whether their approach here might prove more problematic on other questions. Compared to Alpha, Epsilon at least recognizes that parentheses do not always "mean" multiplication, and therefore Epsilon may be more receptive to rethinking their initial statement that parentheses themselves "mean" multiplication. However, more detailed analysis of Epsilon's thinking is necessary if we are to understand whether it aligns with normative meanings for parentheses and is just being expressed in an ill-defined way, or if it actually conflicts with normative meanings.

We note also that Epsilon's explanations in this excerpt are entirely computational: e.g., they describe the process of "distributing" the square to $3xy$. This is a common pattern where students appear to rely on the notion of distributing rather than thinking of the whole $3xy$ as a single object being squared. There is not necessarily anything wrong with the way that Epsilon has explained this; however, being able to conceptualize $(3xy)$ as an object itself might be essential in other contexts, so it might be important to further assess the extent to which Epsilon is able to do this when needed. For example, we note that Epsilon struggles to explain their thinking here; it is possible that if they were able to describe this using a structural view (e.g., "in the first expression only $3xy$ is being squared, but in the second expression $\sqrt{3xy}$ is being squared"), it might be easier for them to provide a justification for their thinking.

Grouping view of parentheses

We now consider one more example in which a student appears to be conceptualizing parentheses structurally as a grouping mechanism. The following excerpt is a Calculus I student Theta's response after being asked what the result would be of substituting $2y$ in for x into the expression $2x^2 - 7x + 3$.

> If you're putting something in for something else, I always usually keep parentheses around it to make sure that I maintain whatever structure the original function had... especially in this one [the first term] because x is being multiplied, and squared, the parentheses really make a difference because if we don't have them, we could get a different, the wrong answer…. you don't need [the parentheses in $-7(2y)$] as long as you make sure you multiply the -7 by 2.

Theta mentions structure explicitly and explains how removing the parentheses gives the expression a different syntactic meaning, which in the case of $2(2y)^2$ will result in a "different result" but in the case of $7(2y)$ will not yield a different result, seemingly because of the order of operations (although they do not use the word "order of operations" explicitly). Theta's explanation shows how they appear to have linked their structural view of the parentheses to their conception of the order of operations, suggesting that it is a reification of this process.

Implications and Conclusion

This paper is an initial exploration of how students may conceptualize brackets and parentheses and adds to the literature by providing hypothetical models of what students may attribute to parentheses or brackets. We make no recommendations about how students should be taught about parentheses, however we think that these students' responses do suggest that they extract many non-normative meanings of parentheses in algebraic syntax and this may impact students' computational work. This suggests that it may be important for instructors and curricula writers to think carefully about whether their language or examples may be encouraging students to think of parentheses as "meaning multiplication", as an instruction to "do something" (where what should be done and why may be ill-defined or non-normative), or as something that is always superfluous. We suspect that simple phrases like "do the parentheses first" are being interpreted by students in a number of ways and may need to be articulated more clearly to students. It may be helpful to provide a variety of examples where parentheses are and are not extraneous to support students in coming to more normative understanding of the role that parentheses are intended to play in algebraic expressions and equations. In addition, it may be important for future research to explore in more detail how a learner comes to reify the normative "do first" view of parentheses into a grouping conception of parentheses, in which they are able to "see" the substring inside the parentheses as a unified subexpression. Thus, further research which explores the prevalence of these various views among different student groups as well as its relationship to computational work could help to improve instruction.

Acknowledgments

The research reported in this paper was supported by the National Science Foundation (grant no. DUE-1760491). The opinions expressed here are those of the authors and do not represent those of the granting agency

References

Banerjee, R., & Subramaniam, K. (2005). Developing procedure and structure sense of arithmetic expressions. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 121-128. Melbourne: PME.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *32*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Edwards, B. S., & Ward, M. B. (2004). Surprises from mathematics education research: Student (mis) use of mathematical definitions. *The American Mathematical Monthly*, 111(5), 411-424.

Gunnarsson, R., Sönnerhed, W. W., & Hernell, B. (2016). Does it help to use mathematically superfluous brackets when teaching the rules for the order of operations? *Educational Studies in Mathematics*, *92*(1), 91–105. https://doi.org/10.1007/s10649-015-9667-2

Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 49–55). PME.

Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, *40*(2), 173–196. https://doi.org/10.1023/A:1003606308064

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *The Journal of Mathematical Behavior*, *14*(1), 15-39.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169. https://doi.org/10.1007/BF00305619

USES OF THE EQUAL SIGN AND EQUATION TYPES IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS

Daniel K. Siebert
Brigham Young University
dsiebert@mathed.byu.edu

Chelsea Dickson
Brigham Young University
chelsea.toler@gmail.com

Research suggests that students' difficulties in studying algebraic topics in middle school can be remedied at least in part by teaching students to use a relational meaning for the equal sign to reason about equations. However, little empirical research has been done to investigate what meanings for the equal sign and equation types are common in middle school mathematics. This study examines two series of 7th and 8th grade mathematics textbooks to identify what equal sign meanings and equation types are being used in middle school mathematics. Three meanings for the equal sign were used in all four textbooks, and each equation type was typically associated with only one meaning of the equal sign. The results imply that students need to develop three different meanings for the equal sign to succeed in middle school mathematics, and that recognizing equation types can help indicate which meaning of the equal sign is being used.

Keywords: Algebra and Algebraic Thinking, Middle School Education

One of the challenges that students face as they transition from elementary to middle school and begin studying topics in algebra is that they must learn to interpret the equal sign differently from its typical use in arithmetic. From an early age, children's predominant meaning for the equal sign is *operational*—they see the equal sign as an indicator that they should "do something" (Blanton et al., 2018; Kieran, 1981). Blanton et al. (2018) found that kindergarten students typically have this operational meaning for the equal sign even before they have received formal mathematics instruction. Furthermore, the kindergarteners in their study continued to use the operational meaning even after eight weeks of instruction designed to help them develop the relational meaning of the equal sign, namely that the equal sign indicates equivalence. Researchers have shown that the operational meaning for the equal sign persists through elementary school and is still prevalent among middle school students (Knuth et al., 2008; Knuth et al., 2005). Because the operational meaning leads to errors in reasoning about equations and other middle school topics (Kieran, 1981; Knuth et al., 2005; Matz, 1982), students' success in learning middle school mathematics is dependent upon them developing the relational meaning of the equal sign.

While the literature clearly demonstrates the importance of a relational meaning for the equal sign, research has been less clear concerning what constitutes a complete or expert understanding of the equal sign. Research on K-12 students' understanding of the equal sign suggests that students need to replace the operational meaning with the relational meaning (Blanton et al., 2018; Carpenter et al., 2003; Knuth et al., 2008), inadvertently implying that the relational meaning of the equal sign is sufficient by itself. In contrast, scholars who have studied or discussed advanced understandings of the equal sign do not treat the operational meaning pejoratively; rather, the operational and relational meanings are considered to be but two of the multiple, legitimate ways to interpret the equal sign (Jones & Pratt, 2011; Prediger, 2010). Seo and Ginsburg (2003) go as far as to note that when children read an equation in the form of $a + b = \square$ using an operational meaning, their interpretation is correct. Little empirical research exists regarding which of the multiple meanings of the equal sign are commonly used in middle school mathematics, where students begin to study algebraic topics in depth. Also, scholars have yet to examine how students know when to use which meanings. The purpose of this research study is to examine the meanings of the equal sign that are used in 7^{th} and 8^{th} grade mathematics textbooks to understand which meanings are commonly used and to examine the contexts in which they are used for characteristics that indicate which meaning of the equal sign is appropriate.

Background

Meanings for the Equal Sign

The research literature suggests that there are three unique meanings for the equal sign: operational, relational, and assignment. The *operational* meaning (Carpenter et al., 2003; Kieran, 1981; Prediger, 2010) is used to signify that an operation has been performed on one side of the equal sign (typically the left) and that the other side contains the result of that operation. For example, when the equal sign is given the operational meaning, the equation $2 + 4 = 6$ means that 4 was added to 2 and that 6 is the result of this addition. The *relational* meaning (Carpenter et al., 2003; Kieran, 1981; Prediger, 2010) indicates a claim that the expressions on either side of the equal sign are equivalent. For example, when the equal sign is given the relational meaning, the equation $2 + 4 = 6$ means that $2 + 4$ and 6 are equivalent, i.e., have the same numeric value. The *assignment* meaning (referred to as *specification* by Prediger, 2010) is used to assign a number or algebraic expression as the value or rule for a symbol or function. For example, when the equal sign is given the assignment meaning, the equation $r = 5$ is interpreted as indicating that r is assigned the value of 5 in equations or expressions containing r, such as the equation $P = 2\pi r$. Similarly, the equal sign in the equation $f(x) = x + 2$ is interpretated as f being assigned the rule, "add 2 to the input." Jones and Pratt (2011) have suggested a fourth meaning for the equal sign, namely *substitution*, which they define as indicating that the expression on one side of an equation can be replaced by the expression on the other side in other equations that contain the first expression. However, we interpret substitution as an equation operator, not a new meaning of the equal sign, and believe that both the relational and assignment meanings can indicate that substitution is a valid equation operation.

Research that has studied students' understanding of the equal sign has largely focused on the operational and relational meanings, often with the intent to explore how students can change their operational conception of the equal sign to a relational conceptualization (Blanton et al., 2018; Carpenter et al., 2003; Knuth et al., 2005). While we acknowledge the importance of helping students develop a relational meaning, we believe this research focus has caused scholars to overlook the crucial role that all three meanings play in secondary school mathematics. In particular, research is needed to understand how the three different meanings for the equal sign are used in middle school mathematics, where students are first required to regularly switch between meanings as they engage in algebraic activity. Equally important is research that identifies contextual clues that students can use to decipher which meanings of the equal sign are appropriate in different contexts.

Equation Types

Given that it is likely that multiple meanings of the equal sign are used in middle school mathematics, then it is important to understand the conditions under which each meaning should be used. These conditions likely involve the type of equation that is being used. Past researchers have sorted algebraic equations based on the quantities for which they are true and to what purposes the equations serve. Drawing across the literature, equations can be separated into three classes: tautologies, constraint equations, and specifications (Kieran, 1981; Matz, 1982; Prediger, 2010). *Tautologies* are equations that are true for large sets of numbers. They include transformations that show algebraic or arithmetic manipulations (e.g., $2(3x - 1) = 6x - 2$) and identities that illustrate properties of real numbers (e.g., $a + b = b + a$). *Constraint equations* are equations for which the expressions on each side of the equal sign are not equal for all numbers, such as $6x = 4x + 2$. *Specifications* are equations that are used to define the value of a variable or the rule for a function, such as $x = 3$ and $f(x) = -x + 7$.

The research on equation types is limited. Most of the research has focused on constraint equations, and researchers have found that many students erroneously use an operational meaning for the equal sign when first exposed to constraint equations (Knuth et al., 2008; Stephens et al., 2013). Students' understanding of tautologies and specifications have received little attention, and no empirical studies have examined which meanings of the equal sign are used when making sense of the various equation types. Research is also missing regarding which equation types are common in middle school mathematics.

To address the lack of empirical research on equal sign meanings and equation types in middle school mathematics, we conducted an analysis of two 7th and 8th grade mathematics textbook series to answer the following research questions:

1. Which meanings for the equal sign and equation types are used in middle grades mathematics textbooks?
2. Which meanings of the equal sign are used for each equation type in middle grades mathematics textbooks?

Method

Textbooks

Two curricula were chosen for this study: *Connected Mathematics 3* (CMP3, Lappan et al., 2014) and *Eureka Math* (2015). CMP3 is the most recent edition of a middle school mathematics curriculum created by mathematics education faculty at Michigan State University and funded by NSF. Eureka Math is a K-12 mathematics curriculum developed through collaboration between teachers and mathematics experts and is available for free through EngageNY.org. According to a recent Rand report (Opfer et al., 2016), both curricula are widely used; of teachers surveyed in states that have adopted the Common Core State Standards for Mathematics (CCSSM, NGA & CCSSO, 2010), 34% of the teachers reported using CMP3 and 34% reported using Eureka Math. Of the curricula surveyed in the report, Eureka Math was judged to be most closely aligned with the CCSSM, and CMP3 was judged to be the second most closely aligned. Both curricula take an integrated approach to mathematics topics and use a task-based instructional approach.

To sample the two different curricula, both teacher and student editions of grade 7 and 8 mathematics were considered. The student editions of these textbooks included only the in-class tasks for the instruction, brief summary sections about the mathematics targeted by the tasks, and problem sets for homework. In contrast, the teacher editions contained more detailed explanations of the mathematics being taught and solutions to the tasks, as well as all of the content included in the student edition. In order to explore a wide range of uses of the equal sign, we chose to analyze the sections in the grade 7 textbooks that were aligned with the CCSSM standards clusters *ratios and proportions* (7.RP), *expressions and equations* (7.EE), and *geometry* (7.G), and the sections in the grade 8 textbooks that were aligned with the standards clusters *expressions and equations* (8.EE), *functions* (8.F), and *geometry* (8.G). Homework problems were not considered in the analysis, as we anticipated that the types of uses of the equal sign that would appear in the homework problem sets would likely be well represented in the instructional parts of each topic section.

Analysis

We began our analysis by creating descriptions for a list of equation types and subtypes that we encountered in the literature. We then individually coded a few sections from both curricula, compared our coding, identified and resolved differences, and updated our codes and descriptions, sometimes adding new codes to our list or combining multiple codes into a single code. Each time we coded, we would first go through the materials and identify all the equations, and then go back and code each equation for meaning of the equal sign and equation type. We repeated this process of coding a small set of new material and comparing our codes multiple times, each time returning to previously coded data and updating our coding to reflect the most recent changes in our codes and descriptions. After 4 iterations, we reached a level of 93% interrater reliability with the Eureka curriculum. It took us 5 iterations to reach an 88% interrater reliability for the CMP3 curriculum. After achieving these high levels of interrater reliability, we continued to code separately and compare coding, analyzing approximately 90% of the data this way. The remaining 10% of the data was coded by the second author only.

We depended on our extensive experience in mathematics and our experiences using textbooks as students and teachers to make decisions about which meanings for the equal sign were being used. Some equations could be read with multiple meanings, but we coded only the meaning that best fit the context. For example, the equation $2(3x + 2) = 6x + 4$ that appeared in a problem solution (Eureka Math 8, Module 4, p. 45) could potentially be interpreted from both an operational and relational meaning. However, because the purpose of the equation was to find the result of distributing the 2 across the $3x$ and 2, and not to justify that $2(3x + 2)$ was equivalent to $6x + 4$, we coded this equation as operational. We found that each equation from the data could be coded as a single equation type and involving a single meaning of the equal sign, and that this coding maintained the integrity of the data.

After coding all of the data, we looked across our coded data for patterns between meanings for the equal sign and equation types. We also looked for variations in the equations coded for each equation type to create even clearer, more robust descriptions. We present our results next.

Results

In our analysis of equal sign meanings, we coded 11,149 equations. Overall, the relational, operational, and assignments meanings were used in 79.3%, 14.0%, and 6.7% of the equations, respectively. The frequencies for the three meanings of the equal sign separated by curriculum, year, and edition are reported in Table 1. The most common meaning for the equal sign was the relational meaning, which was used for over 80% of the equations in CMP3 7 and 8 and in Eureka Math 8. The only textbook data for which the relational meaning was not the most common was the Eureka Math 7 Student Edition; because of the heavy use of specification equations in the geometry unit, the assignment meaning was the most common meaning for the equal sign in the data from this textbook. The second most used meaning varied across curriculum, year, and edition. Teacher editions tended to use the operational meaning more than student editions. This was largely due to task solutions being included in teacher editions but not student editions. While there were large variations in the frequencies for the three meanings across textbooks, all three meanings appeared in each textbook. While we looked for additional meanings for the equal sign beside the three meanings described in the literature, no other meanings for the equal sign were encountered in the data.

Although we found all three meanings for the equal sign in every textbook, we did not find an explicit discussion in either curriculum concerning the multiple meanings for the equal sign.

Rather, changes in the meaning of the equal sign across problems, tasks, lessons, and units were not explicitly acknowledged nor identified as being a potential source of difficulty for students.

Table 1: Frequencies of Equal Sign Meanings

Meaning of the Equal Sign	Eureka Math 7 Combined	Eureka Math 7 Teacher	Eureka Math 7 Student	CMP3-7 Combined	CMP3-7 Teacher	CMP3-7 Student
Relational	65.7%	68.2%	40.3%	83.2%	81.1%	92.7%
Operational	26.8%	28.9%	5.2%	8.2%	9.6%	2.0%
Assignment	7.5%	2.9%	54.5%	8.6%	9.3%	5.3%

Meaning of the Equal Sign	Eureka Math 8 Combined	Eureka Math 8 Teacher	Eureka Math 8 Student	CMP3-8 Combined	CMP3-8 Teacher	CMP3-8 Student
Relational	83.8%	83.6%	85.3%	82.5%	80.6%	90.5%
Operational	11.2%	12.3%	2.5%	8.4%	9.1%	5.6%
Assignment	5.0%	4.1%	12.2%	9.1%	10.3%	3.9%

In our analysis of the relationship between meanings for the equal sign and equation types, we found clear patterns in our data. See Table 2 for the equation types we found in the data and their associated meanings for the equal sign. Except for the equation type *transformation*, each equation type was exclusively associated with a single meaning of the equal sign. Recall that transformations are equations that show a numeric or algebraic manipulation has been performed. Transformations were used two ways in the textbooks: to record a series of operations and their results, and to justify the equivalence of two expressions. These two purposes are associated with the operational and relational meanings, respectively. While the transformation equations themselves did not provide cues as to which meaning of the equal sign was to be used, the contexts in which the transformation equations were used did; in particular, the textbook authors used words like "because" or "since" to mark transformation equations that were meant to be read with the relational meaning of the equal sign and serve as evidence in support of an argument.

Our analysis of equations led to the creation of subclasses of equations that seem to be missing from the literature. An important distinction made during our analysis was to separate constraint equations into three subclasses: *constraint equations with parameters* (CEP), used to describe the general form of a family of functions or relations (e.g., $y = kx$, Eureka Math 7, Module 1, p. 81); *constraint equations with variables* (CEV), used to represent functions or relations between covarying quantities (e.g., $2x + 3y = 9$, Eureka Math 8, Module 4, p. 308); and *constraint equations with an unknown* (CEU), used to find or test solutions for an unknown value (e.g., $x + 5 + 7x + 5 = 98$, Eureka Math 7, Module 3, p. 135). The distinction between constraint equation subtypes is helpful not only because each subtype serves different purposes, but also because subtypes were frequently transformed into other subtypes during mathematical activity. For example, we noted that the authors often converted CEPs to CEVs by replacing parameters with numeric values, converted CEVs into CEUs by substituting in a value for one of the covarying quantities, and converted CEUs into numeric identities and non-identities (i.e., numeric equations that are either correct or incorrect, respectively) by substituting values for the unknowns to test if those values were solutions. For students to follow the ongoing mathematical activity, they must perceive that the change from one equation subtype to the another retains some of the key characteristics of the original equation, and thus preserves some form of equivalence between the two equations.

Table 2: Equation Types Arranged According to Associated Meaning of the Equal Sign

Meaning of Equal Sign	Equation Types	Purpose
Operational	• Transformation Equations	to perform an operation and record the result
Assignment	• Specification Equations	to assign a value or expression to a variable with intent to substitute
	• Restriction Specification Equations	to restrict the values that can be assigned to a variable
Relational	• Constraint Equations with Parameters	to provide a template for a family of functions or relations
	• Constraint Equations with Variables	to specify the relationship between two covarying quantities so that specific, paired values for the two quantities can be found
	• Constraint Equations with One Unknown	to specify constraints that determine which values an unknown can have
	• Formal Identity	to express a general mathematical property or principle
	• Contextual Identity	to specify the relationship between quantities in a specific context
	• Numeric Identity	to justify that a result is correct
	• Numeric Non-identity	to justify that a number was in fact *not* a solution
	• Transformation Equations	to justify the equivalence of two expressions
	• Unit Identity	to specify the relationship between two units
	• Result Equations	to declare a resulting value or solution that has been found by completing a computation or measurement

Another subclass of equations encountered in our analysis was the *result equation*. A result equation has the form *variable* = *numeric value*, with the purpose of presenting the result of computation or measurement performed somewhere else in the lesson. For example, "p = 90" was used to report the result of having added up the lengths of the sides of a rectangle to find the perimeter, and "$AB = 4$" was used to report the result of having counted the squares on a grid to measure the length of $\overline{AB}$. Because result equations report the result of a computation or measurement, they involve a relational meaning rather than an assignment meaning for the equal sign. Thus, result equations differ from specification equations in that the value of the variable in a result equation is derived from mathematical activity, while the value of the variable in a specification equation is an assigned value that is to be used for substitution later in the problem.

Discussion

Our study confirms that the three meanings for the equal sign documented in the literature appear in middle school mathematics curricula. This result is important because it suggests that for students to be successful in middle school mathematics, they must develop fluency with multiple meanings of the equal sign. Rather than helping students replace their operational meaning for the equal sign with a relational meaning, teachers instead need to help students develop an understanding of all three meanings and the contexts in which they should be used. Based on the curricula we analyzed, an explicit discussion of the different meanings of the equal sign may largely be missing from middle school mathematics textbooks. By explicitly teaching the three meanings of the equal sign and modeling expert reasoning with the equal sign, teachers can help students recognize crucial ways of reasoning in middle school mathematics.

We anticipate that some researchers might object to legitimizing the use of the operational meaning, since a reliance solely on the operational meaning has been linked with errors in solving equations of the form $a + b = \_\_ + d$ (Carpenter et al., 2003) and algebraic constraint equations (Knuth et al., 2006). After all, any equation that is interpreted with an operational meaning can be meaningfully reinterpreted with a relational meaning. However, we noticed that while reading equations in our study that we did in fact read some of them using only an operational meaning, and while we could reinterpret those equations using a relational meaning, it required extra cognitive work. We anticipate that the use of the operational meaning in algebra not only makes clearer the purpose of some equations, but also allows readers to reduce cognitive load as they construct meanings for those equations. For example, if someone writes $2(3x + 2) = 6x + 4$ to show that she performed the distributive property, reading the equation operationally to identify which operation has been formed and whether it was performed correctly matches the intent of the author. To also require the reader to add a relational meaning, interpreting the equal sign to indicate that "$2(3x + 2)$ and $6x + 4$ will yield the same value for all values of x," requires extra cognitive work. For experienced readers, identifying the operation that was performed and checking that it was performed correctly is sufficient, because this correctness *implies* a relational meaning. Such an implications does not mean that the reader must stop to construct the relational meaning; rather, the reader knows that a relational meaning *could be* constructed because a legitimate operation was performed correctly. This allows the reader to interpret the equation meaningfully while engaging in less cognitive work.

In addition to explicit instruction on the three meanings of the equal sign, we also think that students can benefit from the explicit teaching of equation types. An important result of the study is that equation types are largely associated with a single meaning of the equal sign. This close association between meanings and equation types suggests that recognizing equation types can help students determine when a particular meaning for the equal sign should be used. However, students will likely have to learn to attend to additional cues from the contexts in which equations are used, particularly when reasoning about transformation equations, because transformation equations can be read using either an operational or relational meaning for the equal sign.

Because all three meanings for the equal sign are present in middle school mathematics, research on students' understanding of the equal sign may benefit from adopting a new perspective. Currently, students' difficulties with the equal sign are often attributed to the need for students to replace their arithmetic, operational meaning of the equal sign with an algebraic, relational meaning (Carpenter et al., 2003; Kieran, 1981; Knuth et al., 2008). However, a more accurate formulation of the problem may be that students' difficulties are caused by their need to acquire new meanings for the equal sign and relate those meanings to their existing operational understanding. This reformulation of the problem suggests new directions for research, such as how students learn to reason with the assignment meaning or extend their operational meaning from numeric to algebraic equations. Research efforts that focus on understanding how students develop multiple meanings for the equal sign and coordinate those meanings when reasoning with equations is essential if the field is to make progress in helping students develop a rich, flexible understanding of the equal sign.

References

Blanton, M., Otálora, Y., Brizuela, B. M., Gardiner, A. M., Sawrey, K. B., Gibbins, A., & Kim, Y. (2018). Exploring kindergarten students' early understandings of the equal sign. *Mathematical Thinking and Learning, 20*, 167-201. https://doi.org/10.1080/10986065.2018.1474534

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Heinemann.

Eureka Math. (2015). Great Minds. https://greatminds.org/math

Jones, I., & Pratt, D. (2011). A substituting meaning for the equals sign in arithmetic notating tasks. *Journal for Research in Mathematics Education, 43*(1), 2-33.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326.

Knuth, E. J., Alibali, M. W., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2008). The importance of equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School, 13*(9), 514-519.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2005). Middle school students' understanding of core algebraic concepts: Equivalence and variable. *ZDM, 37*(1), 68-76.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

Lappan, G., Friel, S. N., Fey, J. T., & Phillips, E. D. (2014). *Connected Mathematics 3*. Pearson.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems* (pp. 25-50). Academic Press.

National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. NGA Center and CCSSO. http://www.corestandards.org

Opfer, V. D., Kaufman, J. H., & Thompson, L. E. (2016). *Implementation of K-12 State Standards for Mathematics and English Language Arts and Literacy: Findings from the American Teacher Panel.* R. Corporation. https://www.rand.org/pubs/research_reports/RR1529-1.html

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education, 13*(1), 73-93.

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence…": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise* (pp. 161-187). Lawrence Erlbaum Associates.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior, 32*, 173-182.

TEACHER LANGUAGE AND GESTURE IN AN INTERVENTION FOCUSED ON DEVELOPING KINDERGARTENERS' UNDERSTANDINGS OF THE EQUAL SIGN

Yewon Sung
California State University-San Bernardino
yewon.sung@csusb.edu

Ana Stephens
University of Wisconsin-Madison
acstephens@wisc.edu

Ranza Veltri Torres
University of Wisconsin-Madison
rveltri@wisc.edu

Susanne Strachota
Tufts University
susanne.strachota@tufts.edu

Maria Blanton
TERC
maria_blanton@terc.edu

Angela Gardiner
TERC
angela_gardiner@terc.edu

Rena Stroud
Merrimack College
stroudr@merrimack.edu

Eric Knuth
University of Texas at Austin
eric.knuth@austin.utexas.edu

This research reports on the teacher language and gesture that contributed to shifts in thinking about the equal sign and equations observed in twenty kindergarteners who took part in an early algebra intervention. Our analysis revealed ways in which the teacher used language and gesture to support students in moving from describing and working with the equal sign operationally (i.e., as a signal to compute) to describing the symbol as indicating the equivalence of two amounts and successfully working with equations of various forms. We detail four kinds of language and two kinds of gesture specifically related to mathematical equivalence that we believe contributed to students' growth.

Keywords: early algebraic thinking; mathematical content early years (Grades K-2); teaching and classroom practice

Understanding the equivalence relationship denoted by the equal sign is foundational to algebraic thinking (e.g., Baroody & Ginsburg, 1983; Carpenter et al., 2003; Kieran, 1981; NCTM, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). Unfortunately, many students struggle to understand the equal sign as a relational symbol indicating the "sameness" or interchangeability of the two sides of an equation. Such students often hold instead an operational view of the equal sign as a stimulus to compute. This operational view manifests itself in students' struggles to make sense of equations in forms other than $a + b = c$. This view is problematic, as students with an operational view of the equal sign experience more difficulty solving equations when they reach middle school than those with a relational view (Knuth et al., 2006).

In Sung et al. (2021), we reported shifts in equal sign understanding observed in 20 kindergarten students who participated in an early algebra intervention focused in part on developing students' understanding of equivalence. For example, we found that students grew in their ability to provide a relational definition of the equal sign (from one student prior to the intervention to nine students post-intervention). We also observed growth in the kinds of equations that students accepted as valid. For example, prior to the intervention, only nine students accepted 5 = 2 + 3 as a valid equation, only three accepted 2 = 2, and only six accepted 4 + 2 = 3 + 3. At the conclusion of the intervention, these numbers grew to 20, 17, and 13, respectively. For more detail about the measures used and results from the study of student growth in understanding, see Sung et al. (2021).

Future studies will be needed to tease apart the influence of the particular tasks with which students were asked to engage and other aspects of their classroom experiences on their growth in understanding. In this paper, we focus on features of the teacher's practice–in particular, her use of language and gesture around mathematical equivalence–that we believe may have contributed to the shifts we observed in students' understanding of the equal sign. We first review literature on teaching practices—particularly, concerning language and gesture—that have been shown to support students' learning about mathematical equivalence and that informed our analysis of classroom observations. We next describe our research methods including the intervention in which students participated. Finally, we describe the teacher language and gesture observed in the intervention classroom that we believe contributed to students' growth.

Teacher Moves that Support Student Understanding of Mathematical Equivalence

While the selection of appropriate tasks and tools is critical to developing students' mathematical understandings and has been the focus of much of our prior work (e.g., Stephens et al., 2013; Stephens et al., 2020), the ways in which teachers implement instruction is equally critical. For example, teachers should establish a classroom culture in which students' strategies (rather than merely correct answers) are valued (e.g., Carpenter et al., 2015; Hiebert et al., 1997), engage students in meaningful classroom discussion (e.g., Ghousseini, 2015), and ensure that every student is a valued and contributing member of the classroom community (e.g., Skinner et al., 2019). There are many possible foci when observing the practice of a mathematics teacher. We focus here on practices that might be particularly significant in developing students' understandings of mathematical equivalence. Specifically, we analyze classroom data using language and gesture as our conceptual frames, investigating how teachers' precise use of language and gesture can support students' understanding of equivalence.

Language

The quantity and quality of the mathematics language to which students are exposed during instruction has implications for their mathematics learning as early as preschool (Klibanoff et al., 2006). The Common Core's Standards for Mathematical Practice (NGA & CCSSO, 2010) state that mathematically proficient students "use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately." Also important are the ways in which this symbol is used and described by teachers.

Hill et al. (2008) identified teachers' use of language as a critical factor in the quality of mathematics instruction. They focused in particular on "the density of accurate mathematical language in instruction, the use of language to clearly convey mathematical ideas, [and] any explicit discussion of the use of mathematical language" (Hill et al., 2008, p. 437). Included in this focus on mathematical language in Hill et al.'s coding of classroom instruction is the use of "general language including analogies, metaphors, and stories used to convey mathematical concepts" (p. 510) and teachers' explicit attention to defining terms, showing how to use them, and pointing out specific labels or names.

Gesture

It has been shown across multiple disciplines and age groups that students learn better when spoken instruction is accompanied by gesture (Goldin-Meadow & Singer, 2003; Macedonia et al., 2011; Ping & Goldin-Meadow, 2008; Singer & Goldin-Meadow, 2005; Valenzeno et al., 2003). Alibali and Nathan (2007) identified three broad categories of gesture that they found supported sixth-grade students' learning about modeling with algebraic equations. *Pointing* *gestures* were defined as gestures used to indicate objects, locations, inscriptions, or students, most often produced with fingers or hands. *Representational gestures* were those involving a handshape or motion of the hand or arm representing an object, action, concept or relation. Finally, *writing gestures* were defined as writing that the teacher produced while speaking. For example, a writing gesture might occur when a teacher underlines a mathematical inscription on the board while describing it with speech. Alibali and Nathan describe gestures as "essential to teachers' ability to conduct their practice" (p. 362).

Congdon et al. (2017) argue for the importance of students experiencing language and gesture at the same time when learning mathematical concepts. The third-grade students in their study were more successful learning to solve mathematical equivalence problems such as 7 + 8 + 5 = ___ + 5 when they experienced lessons involving simultaneous as opposed to sequential language and gesture. Congdon et al. (2017) argue that "gesture capitalizes on its synchrony with speech to promote learning that lasts and can be generalized" (p. 65). Wakefield et al. (2018) argue that the benefits of gesture go beyond directing attention to include its ability to add content to speech. In the case of algebraic equations, this can include using gestures to foster students' ability to see underlying structure in equations (Goldin-Meadow et al., 2009).

While acknowledging that the use of particular tasks and tools coupled with "good teaching" assisted students in making the shifts in equal sign understanding we observed in Sung et al. (2021), here we specifically investigate the teacher's use of language and gesture related to mathematical equivalence to consider how these practices may have contributed to the growth we observed. Our research question is: *How does the teacher use language and gesture to emphasize the relational meaning of the equal sign and encourage acceptance of various equation forms?*

Method

Participants

This study included two classrooms of kindergarten students in the Northeastern U.S. (n = 45) who took part in an early algebra intervention.

The Intervention

The early algebra intervention consisted of 18 lessons of approximately 30 minutes each. This intervention built on a previously implemented early algebra intervention that was found to be effective for developing students' algebraic thinking in Grades 3–5 (Blanton et al., 2019). Of the eighteen lessons, seven focused specifically on developing students' understandings of mathematical equivalence and fluency with equations (Lessons 7, 8, 9, 10, 11, 13, and 14; see Table 1). These lessons engaged students in tasks that have been found to be supportive of developing a relational view of the equal sign (e.g., true/false and missing value equations). Many of these lessons also included activities with balance scales and Unifix® cubes, which provide critical links between concrete and abstract representations of equations and have been found to mediate students' relational understandings of the equal sign (e.g., Alibali, 1999; Araya et al., 2010; Fyfe et al., 2015; Linchevski & Herscovics, 1996; Stephens et al., 2020).

We began this series of seven lessons by having students compare quantities of objects on a pan balance to develop the meaning of balance, introducing the equal sign, and modeling the state of the concrete scales with equations of the form $a = a$ (Lesson 7). Number balances were introduced to students in Lesson 8 and were used to represent equations with no operations (for example, 4 = 4), then equations with operations only to the right of the equal sign (for example, 7 = 3 + 4). These activities were designed to encourage students to work with equations in a variety of forms. In subsequent lessons, students explored decompositions of numbers and ways to represent these with equations (Lesson 9 and 10; as in Blanton et al., 2018) as well as true-false and missing value equations of the forms $a = a$ and $c = a + b$ (Lessons 11, 13, and 14). See Table 1 for a summary of the seven lessons.

Table 1: Summary of Seven Lessons Focused on Mathematical Equivalence

7. Is it Balanced? (Part 1) • Use a pan balance to explore equivalence and inequality • Write inequality statements of the form $a < b$ and equations of the form $a = a$ to represent the relationship shown on a pan balance • Generalize based on experiences with the pan balance (e.g., “To balance the scale, you have to have the same number of cubes on both sides.”)
8. Is it Balanced? (Part 2) • Use a number balance to explore equivalence • Write equations of the form $a = a$ and $c = a + b$ to represent the relationship shown on a number balance
9. Decomposing Numbers • Decompose a set of cubes into two groups in multiple ways and represent with pictures (e.g., show 8 as two groups of 4; 5 and 3; 6 and 2; 7 and 1; 8 and 0)
10. Modeling Decomposition with Equations • Represent the decompositions from Lesson 9 with equations of the form $c = a + b$ (e.g., express “8 is the same as 3 and 5” as “8 = 3 + 5”)
11. Is this True? • Evaluate equations of the form $a = a$ *and* $c = a + b$ as true or false
13. What’s the Missing Number? • Solve missing value equations of the forms $a + b = c$, $a = a$, and $c = a + b$
14. Adding Zero • Write an equation to represent a situation exemplifying the Additive Identity • Identify as true or false and find the missing value in equations exemplifying the Additive Identity (e.g., 2 = 2 + 0, 3 + __ = 3) • Express the Additive Identity in words and with a variable equation

Data Collection and Analysis

Classroom video data were collected during all lessons. We focused our analysis on one of the two classrooms and within this classroom on the seven lessons that addressed mathematical equivalence. These seven lessons were transcribed. Three team members individually watched one lesson per week while highlighting and annotating the transcript with anything they felt was worthy of further discussion. Weekly meetings initially involved the sharing of different perspectives as we worked toward the goal of more specific questions to guide our viewings of the classroom video and eventually a stable coding scheme. We ultimately settled on a coding scheme that focused on the teacher’s use of language around equations and the meaning of the equal sign during whole-group discussion. Repeated viewings led us to notice the use of gesture around two important aspects of equations—the idea of balance, and the notion of the sides of an equation—that were also integrated into our initial coding scheme.

Once an initial coding scheme was in place, three team members individually coded one classroom video and then compared results, discussed challenges, and refined the scheme for content and clarity. The project team continued this process with subsequent lesson videos, using an approach consistent with Syed and Nelson's (2015) recommendations for establishing reliability in qualitative research in ways that emphasize rigorous iterative and collaborative coding scheme development and group consensus. The final coding scheme, describing the language and gesture used by the teacher to refer to features of equations or the meaning of the equal sign during whole-class discussion, is shown in Figure 1.

Language Codes

ISA ("Is the same as"): Teacher verbally links "=" with "is the same as." Often happens when writing or speaking an equation and replacing "=" with "is the same as" or "is the same amount as" or "is the same number as" (e.g., "5 is the same as 2 + 3" while writing 5 = 2 + 3).
DEF (Definition): Teacher explicitly states or uses the definition of the equal sign, often stating that the equal sign means the two sides of the equation are the same number or the same amount (e.g., "This equation is true because the equal sign means the two sides are the same amount"; "We can use an equal sign because the amounts are the same"; "We can't use the equal sign because we don't have the same amount on both sides").
BAL (Balance): Teacher speaks of equations as being "balanced" or not balanced or links the idea of a balanced (or not balanced) scale to an equation (e.g., "This equation is true because the equation is balanced"; "What number will balance this equation?").
FORM (Equation form): Teacher explicitly addresses the form of an equation (e.g., "Is it ok to write equations like this?"; "This equation is not backwards"; "Equations don't need a plus sign").

Gesture codes

GBAL (Balance gesture): Teacher uses hands or arms to make a balance gesture when discussing the equal sign or a particular equation (e.g., holding arms out and saying "it balances" when discussing why an equation is true). Often gets coded along with BAL.
GSIDES (Equation sides gesture): Teacher emphasizes the sides of equations by pointing to the sides, waving her hand over the sides, or writing "V" underneath the side(s) to indicate grouping while talking about "sides." Can apply even when only two numbers/expressions are written and not (yet) the equal sign. Can also apply if "sides" are mentioned *while* the teacher is writing one or more equation sides on the board.

Figure 1: Teacher Language and Gesture Concerning Mathematical Equivalence

Results

Table 2 shows the frequency of the teacher's use of language and gesture related to mathematical equivalence across the seven lessons focused in this area. In what follows, we discuss each of the lessons and provide an overview of the language and gesture we observed in that lesson. We also provide examples of each language and gesture code.

Table 2: Frequency of the Teacher's Use of Language and Gesture Related to Mathematical Equivalence by Lesson

Lessons	7	8	9	10	11	13	14	Total
Language Codes								
ISA ("Is the same as")	0	15	0	11	2	0	0	28
DEF (Definition)	1	6	0	4	4	2	0	17
BAL (Balance)	0	1	0	0	6	8	11	26
FORM (Equation form)	0	1	0	3	2	0	0	6
Gesture Codes								
GBAL (Balance gesture)	0	0	0	2	4	4	1	11
GSIDES (Equation sides gesture)	0	13	0	9	6	3	7	38

Recall that in Lesson 7, students chose among the symbols <, >, and = to describe relationships shown on a balance scale. It was in this context that the teacher first stated, "The equal sign means [the two numbers are] the same" (DEF). In Lesson 8, as students spent more time writing equations to represent relationships shown on balance scales, the use of the equal sign increased, as did language around its meaning. In fact, the number of times the teacher explicitly stated or used the definition of the equal sign (DEF), replaced the "=" with "is the same as" (ISA), and explicitly gestured to the sides of an equation (GSIDES) were greatest in this lesson. For example, as she wrote "11 + 3 = 14" on the board, the teacher said, "11 plus 3 is the same as 14." Another time, when the teacher pointed to each side of the equation 3 = 3 on the board, she said, "I have 3 on this side [pointing to left "3"], I have 3 on that side [pointing to right "3"]." Lesson 8 also provided the teacher an opportunity to address with students the various forms an equation can take (FORM). At the conclusion of the lesson, she stated:

> Check out all the different equations we wrote today. We wrote… 3 + 1 = 1 + 3, with two digits on both sides [pointing to the equation as a whole]. We wrote 7 = 7, one digit on both sides [pointing to the equation as a whole]. We wrote 7 = 5 + 2, one digit on one side [pointing to the left side of the equation], two digits on the other [pointing to the right side of the equation] .... We can write equations in all sorts of different ways. All sorts of forms.

In Lesson 9, students used Unifix® cubes to represent different ways to decompose the number 8 but did not work with equations or the equal sign. Lesson 10 started with the teacher asking students for examples of equations. Multiple students offered examples in the form $a + b = c$. In an effort to emphasize the meaning of the equal sign and eventually encourage students to rely on this meaning to propose other equation forms, the teacher returned to the "is the same as" language used frequently in Lesson 8 to replace "=" as well as gesture to emphasize the distinct sides of equations written on the board. This use of "is the same as" language and gesturing to sides of equations again often happened together. At one point, the teacher discussed the student-proposed equation 5 + 5 = 10 written on the board as follows:

> So, let's check this out. 5 plus 5 is the same as 10 [pointing to each element of equation while reading and replacing "=" with "is the same as"] because I have 10 on this side [using "^" notation and writing "10" above "5 + 5" on left side] and I have 10 on this side [pointing to "10" on right side].

The teacher later proposed the equation 5 = 4 + 1 for discussion. A student expressed some discomfort with this equation, saying she would rather it look more like the previously discussed 5 + 5 = 10. The teacher responded by addressing the class:

> I think it [pointing to 5 = 4 + 1 on the board] fits all the rules of the equation. Let's check it out. It has an equal sign, right? So, that is good [makes a thumbs up gesture]. We have five on this side [pointing to "5" on the right side], right? Do we have five on this side [pointing to "4 + 1" on the right side and drawing "v" shape and "5" below "4 + 1"]?

The teacher explicitly discussed the forms an equation can take, with the goal of expanding the range of equations students would accept as valid by emphasizing the meaning of the equal sign.

Lessons 11, 13, and 14 engaged students in evaluating equations as true or false and solving equations. Throughout these lessons, the teacher referred to equations as "balanced" or not balanced. Sometimes she linked the idea of a balanced (or not balanced) scale to an equation (BAL). She also used representational gestures (Alibali & Nathan, 2007) to represent the idea of balance (GBAL). For example, in Lesson 11 the teacher referred to the equation 3 = 1 + 2 as balanced while simultaneously holding her hands out and gesturing like a balance.

The focus of each lesson seemed to influence the type of language or gesture that was implemented. For example, early lessons focused on introducing the meaning of the equal sign included frequent use of "is the same as" (ISA) and explicit definitions (DEF). As the focus of the lessons shifted towards using these concepts to evaluate and solve equations, we saw an increase in the teacher's discussion of equations as "balanced" (BAL) and her use of a balance gesture when discussing equations (GBAL). Across the seven lessons that addressed mathematical equivalence, we observed 26 instances of the BAL code, 17 instances of the DEF code, 6 instances of the FORM code, 28 instances of the ISA code, 11 instances of the GBAL code and 38 instances of the GSIDES code (see Table 2).

Discussion

Our teacher-researcher used language and gesture to focus students' attention on the meaning of the equal sign and, relatedly, to encourage them to expand the range of equation forms they would accept. As we reflect on our findings, three points come to the fore.

First, when we consider the teacher's use of language related to the equal sign, we find that this language is varied and serves varied purposes. Early in the series of lessons (in particular, Lesson 8, when students were writing equations to represent balanced scales), the teacher frequently defined the equal sign (DEF) and reiterated its meaning by replacing "=" with "is the same as" (ISA) in speech. This very precise use of language reflected the teacher's explicit attention to pointing out critical meanings for students and clearly conveying mathematical ideas (Hill et al., 2008). This "is the same as" language resurfaced in Lesson 10, when students were again writing equations, this time to model decompositions, and the teacher spoke these equations aloud. In later lessons (in particular, Lessons 11, 13, and 14), when students were evaluating equations as true or false and solving missing value equations, the teacher spoke of equations as being "balanced" (BAL) to help students develop a relational view of equivalence.

This use of “balance” reflects Hill et al.’s (2008) attention to the use of analogies and metaphors to help students construct mathematical meaning. Finally, while it occurred less frequently than the other language codes, the teacher at times explicitly referred to the forms of equations (FORM), highlighting that those equations do not have to follow the familiar $a + b = c$ form.

Second, we found two main gestures used by the teacher that promoted students’ understanding of mathematical equivalence. The first was a representational gesture (Alibali & Nathan, 2007) in which the teacher extended her hands or her entire arms to mimic a balancing scale (GBAL). This occurred most frequently in lessons 11 and 13, when students were evaluating or solving equations. The teacher’s balance gestures served to remind students of the link between a balancing scale and a balancing equation. The second gesture we observed very often was a pointing gesture (Alibali & Nathan, 2007) indicating the sides of an equation (GSIDES). Indeed, this was the most prevalent language or gesture code, occurring 38 times across the seven lessons. It might be assumed that it is obvious what the “sides” of an equation are. However, Rittle-Johnson and Alibali (1999) found that fewer than one third of the fourth- and fifth-grade students in their study of procedural and conceptual knowledge correctly identified the “sides” of a given equation, suggesting that gestural emphasis is in fact needed. It may be that emphasizing “sides” of equations helps students see the equal sign as the “center” or balance point of the equation.

Third, we found a particular language-gesture combination—ISA + GSIDES—occurred rather frequently in Lessons 8 and 10. Recall that these were lessons in which students wrote equations to represent balanced scales (Lesson 8) or decompositions of numbers (Lesson 10). This combination occurred when the teacher “talked through” an equation by pointing to and speaking of the equation’s sides while replacing “=” with “is the same as.” This simultaneous use of speech and gesture supports Congdon et al.’s (2017) assertion regarding the power of synchronizing these two modalities in instruction. The teacher used this language-gesture combination when discussing equations of the form $a + b = c$, $a = a$, and $c = a + b$. This combination of language and gesture explicitly emphasized the relational meaning of the equal sign and the fact that, because the equal sign indicates sameness, multiple equation forms are possible.

Conclusion

Early algebra is crucial to bridging students’ informal intuitions about structure and relationships to more formal ways of reasoning and representing mathematical ideas. Our findings shed light on teacher language and gestures that may support students in developing a relational understanding of the equal sign, a crucial aspect of algebraic thinking. Our work also strengthens the argument that the synchronous use of speech and gesture can promote learning.

Acknowledgments

The research reported here was supported in part by the US Department of Education under IES Award #R305A170378. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the US Department of Education.

References

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology, 35*, 127–145. doi.org/10.1037/0012-1649.35.1.127

Alibali, M. W., & Nathan, M. J. (2007). Teachers’ gestures as a means of scaffolding students’ understanding: Evidence from an early algebra lesson. In R. Goldman, R. Pea, B. Barron, & S. J. Derry (Eds.), *Video research in the learning sciences* (pp. 349–365). Routledge.

Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, M. A., Lacourly, N., ... & Dartnell, P. (2010). The effect of analogies on learning to solve algebraic equations. *Pedagogies: An International Journal, 5*(3), 216–232. doi-org.ezproxy.library.wisc.edu/10.1080/1554480X.2010.486160

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the" equals" sign. *The Elementary School Journal, 84*(2), 199–212. doi.org/10.1086/461356

Blanton, M., Otalora Sevilla, Y., Brizuela, B., Gardiner, A., Sawrey, K., & Gibbons, A. (2018). Exploring kindergarten students’ early understandings of the equal sign. *Mathematical Thinking and Learning, 20*(3), 167–201. doi-org.ezproxy.library.wisc.edu/10.1080/10986065.2018.1474534

Blanton, M., Stroud, R., Stephens, A., Gardiner, A. M., Stylianou, D. A., Knuth, E., Isler-Baykal, I., & Strachota, S. (2019). Does early algebra matter? The effectiveness of an early algebra intervention in grades 3 to 5. *American Educational Research Journal, 56*(5), 1930–1972. doi.org/10.3102/0002831219832301

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children’s mathematics: Cognitively guided instruction*. Heinemann.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically*. Heinemann.

Congdon, E. L., Novack, M. A., Brooks, N., Hemani-Lopez, N., O’Keefe, L., & Goldin-Meadow, S. (2017). Better together: Simultaneous presentation of speech and gesture in math instruction supports generalization and retention. *Learning and Instruction, 50*, 65–74. doi.org/10.1016/j.learninstruc.2017.03.005

Fyfe, E. R., McNeil, N. M., & Borjas, S. (2015). Benefits of “concreteness fading” for children's mathematics understanding. *Learning and Instruction, 35*, 104–120. doi.org/10.1016/j.learninstruc.2014.10.004

Ghousseini, H. (2015). Core practices and problems of practice in learning to lead classroom discussions. *The Elementary School Journal, 115*(3), 334–357. doi.org/10.1086/680053

Goldin-Meadow, S., Cook, S. W., & Mitchell, Z. A. (2009). Gesturing gives children new ideas about math. *Psychological Science, 20*(3), 267–272. doi.org/10.1111/j.1467-9280.2009.02297.x

Goldin-Meadow, S., & Singer, M. A. (2003). From children's hands to adults' ears: Gesture's role in the learning process. *Developmental Psychology, 39*, 509–520. doi.org/10.1037/0012-1649.39.3.509

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Heinemann.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430–511. doi.org/10.1080/07370000802177235

Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.). (2008). *Algebra in the early grades*. Lawrence Erlbaum/Taylor & Francis Group & National Council of Teachers of Mathematics. doi.org/10.4324/9781315097435

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326. doi.org/10.1007/BF00311062

Klibanoff, R. S., Levine, S. C., Huttenlocher, J., Vasilyeva, M., & Hedges, L. V. (2006). Preschool children's mathematical knowledge: The effect of teacher "math talk." *Developmental Psychology, 42*(1), 59–69. doi.org/10.1037/0012-1649.42.1.59

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297–312. doi.org/10.2307/30034852

Linchevski, L., & Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics, 30*(1), 39–65. doi.org/10.1007/BF00163752

Macedonia, M., Müller, K., & Friederici, A. D. (2011). The impact of iconic gestures on foreign language word learning and its neural substrate. *Human Brain Mapping, 32*, 982–998. doi.org/10.1002/hbm.21084

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_ Math%20Standards.pdf

Ping, R. M., & Goldin-Meadow, S. (2008). Hands in the air: Using ungrounded iconic gestures to teach children conservation of quantity. *Developmental Psychology, 44*, 1277–1287. doi.org/10.1037/0012-1649.44.5.1277

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*(1), 175–189. doi.org/10.1037/0022-0663.91.1.175

Singer, M. & Goldin-Meadow, S. (2005). Children learn when their teacher's gestures and speech differ. *Psychological Science, 16*(2), 85–89. doi-org.ezproxy.library.wisc.edu/10.1111/j.0956-7976.2005.00786

Skinner, A., Louie, N., & Baldinger, E. M. (2019). Learning to see students' mathematical strengths. *Teaching Children Mathematics, 25*(6), 339–344. doi.org/10.5951/teacchilmath.25.6.0338

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M. & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior, 32*, 173–182. doi.org/10.1016/j.jmathb.2013.02.001

Stephens, A., Sung, Y., Strachota, S., Veltri Torres, R., Morton, K., Murphy Gardiner, A., & Stroud, R. (2020). The role of balance scales in supporting productive thinking about equations among diverse learners. *Mathematical Thinking and Learning*. doi.org/10.1080/10986065.2020.1793055

Sung, Y., Stephens, A., Veltri Torres, R., Strachota, S., Blanton, M., Gardiner, A. M., Knuth, E., & Stroud, R. (2021). What language and gesture support kindergarteners' developing understanding of the equal sign? Manuscript under review.

Syed, M., & Nelson, S. C. (2015). Guidelines for establishing reliability when coding narrative data. *Emerging Adulthood, 3*(6), 375–387. doi.org/10.1177/2167696815587648

Valenzeno, L., Alibali, M. A., & Klatzky, R. (2003). Teachers' gestures facilitate students' learning: A lesson in symmetry. *Contemporary Educational Psychology, 28*(2), 187–204. doi.org/10.1016/S0361-476X(02)00007-3

Wakefield, E. M., Novack, M. A., Congdon, E. L., Franconeri, S., & Goldin-Meadow, S. (2018). Gesture helps learners learn, but not merely by guiding their visual attention. *Developmental Science, 21*(6), e1266. doi.org/10.1111/desc.12664

DEVELOPING MEANINGS FOR GRAPHS FROM NUMBER LINES: THE CASE OF MIKE

Halil I. Tasova
California State University San Bernardino
halil.tasova@csusb.edu

In this study, I report on developmental shifts of a middle school student's (Mike) graphing activity as I implement an instructional sequence that emphasizes quantitative and covariational reasoning. The results suggest that representing quantities' magnitudes as varying length of directed bars on two parallel lines, forming a new space by making those lines orthogonal, and generating a point by joining those bars were an integral part of Mike developing productive meanings for graphing quantities in a Cartesian plane.

Keywords: Algebra and Algebraic Thinking, Cognition, Mathematical Representations

Constructing and interpreting graphs is a middle school mathematics topic that provides an opportunity to foster powerful learning (Leinhardt et al., 1990). Students, however, experience a number of challenges (e.g., conceiving graphs as picture of situation, event phenomena, or the literal motion of an object; see Johnson et al., 2020, for a summary of challenges) in interpreting and making sense of graphs that ultimately affect their learning of many topics in algebra and calculus (e.g., Clement, 1989; Hattikudur et al., 2012; Moore & Thompson, 2015). In particular, middle school students encounter many challenges when transitioning from discrete graphs to continuous graphs as they are influenced by their past experience with bar graphs and line plots introduced in K–5 education (Boote & Boote, 2017). A promising way to support students in developing productive meanings for graphs is to emphasize a graph as an emergent trace of how two quantities vary simultaneously (i.e., emergent shape thinking; Moore, 2021). Numerous researchers have investigated students' ability of graphing coordinates (e.g., Ponte, 1984; Shaw et al., 1983), formulas (e.g., Hattikudur et al., 2012), and motion (e.g., Botzer & Yerushalmy, 2008; Radford et al., 2009; Nemirovsky et al., 1998), but far fewer researchers have focused on investigating how students construct graphs as emergent traces of quantities' covariation (e.g., Frank, 2016; 2017; Tasova, 2021). Even fewer researchers have investigated the sources of students' capacities to create a point as a simultaneous representation of two quantities' values so that they can construct a graph as an emergent trace of those values' covariation (e.g., Tasova & Moore, 2020; Thompson et al., 2017). Thus, I investigate the following questions: What ways of thinking do students engage in graphing activities intended to emphasize quantitative reasoning? What ways of reasoning are involved in students developing productive meanings for graphs?

Theoretical Framework: Quantitative and Covariational Reasoning

This study focuses on students' graphing activities involved in reasoning with relationships between quantities in real-world situations. I use *quantity* to refer to a conceptual entity an individual constructs as a measurable attribute of an object (Thompson, 2011). When engaging in dynamic contexts, students can use *covariational reasoning*, which entails attending to how one quantity varies in relation to the other (Thompson & Carlson, 2017). In this study, I demonstrate ways in which students make sense of quantitative relationships in dynamic events and in graphs by reasoning with quantities' *magnitudes* (i.e., the quantitative size of an object's measurable attribute) independent of numerical values (i.e., the result of measuring that attribute). Researchers have argued for the powerful role of reasoning with magnitudes in supporting students understanding of quantitative relationships (e.g., Thompson et al., 2014).

A relationship between two quantities is often represented in a Cartesian plane formed by two perpendicular number lines creating a two-dimensional (2D) space. The idea of representing quantities' values or magnitudes on number lines is often taken for granted by researchers when discussing students' construction of coordinate graphs (Lee et al., 2018). This is not surprising given that the focus of instruction is given to bar graphs and pointwise readings in elementary and secondary education. The researchers (Earnest, 2015; Schliemann et al., 2013) who have studied the relationship between the conception of number lines and coordinate systems have focused on individual points on the number lines and the intervals between them. Their focus did not give attention to representing quantities' magnitudes and continuously changing quantities. Thus, in this study, I extend this previous work by investigating the nature and extend of students' ability to represent varying quantities' magnitudes on number lines, and how those abilities influence their meanings of graphs.

Methods

Participants

This study is situated within a larger study that examined two sixth-grade students' (Mike and Naya) and four seventh-grade students' (Ella, Dave, Marc, and Zane) graphing activities in a teaching experiment (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. This study focuses on Mike's meanings for graphs and his developmental shift of those meanings over the teaching experiment.

Data Sources and Analysis

Mike participated in 14 teaching sessions each of which last for approximately one hour. Data sources included video and transcripts that captured the participants' exact words, gestures, and drawings. Students' written work, screen recordings from a tablet device, and notes taken by research members who participated in the sessions were also collected as data sources. I conducted a conceptual analysis in order to understand students' verbal explanations and actions and develop viable models of their mathematics (Steffe & Thompson, 2000). My analysis relied on the generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to develop working models of Mike's thinking based on his observable and audible behaviors.

Tasks

Before conducting the teaching experiment, I developed an initial sequence of tasks with a dynamic geometry software and displayed on a tablet device (for digital versions of all the tasks, go to https://www.geogebra.org/m/szscxzd8). In Downtown Athens Bike Task (DABT), I present the students with a map of Downtown Athens highlighting a straight road, two places located near the road (i.e., the Arch and Canon; see Figure 1a), and a bike on this road. I asked students to graph the relationship between the bike's distance from Arch (DfA) and distance from Cannon (DfC) as the bike moves at a constant speed back and forth along the road.

In the Crow Task (CT), I present with the same map in addition to a movable crow (see Figure 3). I also presented a Cartesian plane with the horizontal axis labeled as DfC and vertical axis labeled as DfA. Students can move the crow freely while observing how the corresponding black point in the plane moves (see https://youtu.be/5JgrPAuG15Y).

In Matching Game Task (MGT; Figure 2a), I present the students with the same map highlighting a straight road (i.e., College Avenue). Arch and Cannon are located on the road, and a bike rides along this road. I also present a dynamic tool where students engage with quantities' magnitudes represented by directed bars placed on empty number lines (also called magnitude lines). The directed bars can be varied in length (see https://youtu.be/knrN7XCbyOY). In this part of the task, the bars and the bike in the situation are not synced, and I ask students to adjust the length of the bars on the magnitude lines to match with various bike positions in the map. I aim to help students understand that the length of both bars should be constrained in a way that they simultaneously represent the bike's DfA and DfC. I asked students to place tick marks as a way to *record* those lengths on the magnitude lines before I change the bike's location to another state. I conjectured that this tool might help students when they move to 2D space to represent two quantities by a single point in a coordinate plane.

Results

Mike's Graph in Downtown Athens Bike Task

I asked Mike to sketch a graph to represent the relationship between the bike's DfA and DfC. The reader can see Mike's graph in Figure 1b, which consisted of a line upward from left to right. Mike first conceived the *entire* vertical and horizontal axes of the plane as the psychical Cannon and Arch, respectively. The axes did not represent magnitude or number lines, but instead were referents to the objects themselves. Mike then drew a line upward from left to right to represent "where the bike travels." He added tick marks and dots on his line graph "to represent like where the bike could be." For example, when the bike is at a location 1 in the map (i.e., DfA and DfC are each at their minimum), he said "it [the bike] would go right here [*pointing to the tick mark on his graph near 1 in Figure 1b*]."

Figure 1: (a) DABT and (b) Mike's graph (numbered labels are added for the reader)

Mike also imagined moving the bike on the plane according to the variation of its DfA and DfC. He conceived DfC and DfA as being represented by vertical and horizontal segments drawn on the plane, respectively. He said "this distance [*moving his finger up and down over the vertical line segment that he drew on the plane*] is the distance from the Cannon and this distance [*moving his finger over the horizontal line segment that he drew on the plane*] is the distance from the Arch." The vertical segments on the plane represented the bike's DfC because Mike assimilated the horizontal axis as a reference ray that he measured bike's DfC from. Similarly, the horizontal segments on the plane represented the bike's DfA because he assimilated the vertical axis as a refence ray he measured the bike's DfA from. Increasing length of the horizontal and vertical segments on the plane indicated an increase in the bike's DfA and DfC. Moreover, Mike understood that the length of horizontal segment is shorter than the length of the vertical segment for each point on the graph as the bike's DfA is always less than the bike's DfC on the map. Thus, although Mike assimilated his graph as "where the bike travels," Mike's graph was not the bike's path *as it is* seen in the map; it was not an iconic translation. Mike's meaning of the points included determining quantitative features of the bike in the situation (i.e., its DfA and its DfC) and he ensured they preserved those quantitative properties on the plane.

I consider Mike's graphing activity as a different way of graphing relationships because he engaged in quantitative operations to make sense of the results of his activity and produce a graph to his satisfaction. Specifically, he represented the quantities' magnitudes in the space by committing to two frames of reference (i.e., the axes), and then placing a point where those magnitudes meet on the path he produced. Tasova and Moore (2020) termed this meaning of points as a *spatial-quantitative multiplicative object*, which involves an individual envisioning point on the plane as a *location/object* that also entails *quantitative* properties. This meaning of points on the space is different than representing quantities' magnitudes *on* the axis and creating a point by taking two orthogonal magnitudes along the axis and creating projections. In the next activity, I provided an opportunity for Mike to develop such a meaning of a point.

Mike's Activity in Matching Game Task

I designed MGT (Figure 2a) in order to engage Mike to (i) represent two quantities' magnitudes on parallel magnitude lines and then transition to (ii) representing two quantities' magnitudes as *a single point* by making the magnitude lines orthogonal and projecting the magnitudes on the plane. I conjectured that Mike's engagement with MGT could promote an understanding of points for Mike as an abstract object in relation to representing two quantities' magnitudes rather than representing a physical object that moves on the plane. I let Mike to change the bike's location on the path and observe how the bars were moving on the magnitude lines (see https://youtu.be/6Q3pSDhXEMs). I asked Mike what the red and blue bars might represent. He responded that the blue bar represents the bike's DfA, and the red bar represents the bike's DfC. When asked to show how he knew, he located the bike near Cannon on the map and said "well, the bike's distance from Arch right now [*measuring the length of the blue bar on the magnitude line with his fingers*] is this much [*translating his fingers—by keeping the same distance between them—and placing it on the map*]". He also added "it [*pointing to the bike on the map*] is right by the Cannon, so, it [the red bar] is not there [*pointing to the zero point on the magnitude line labeled 'Distance from Cannon'*]."

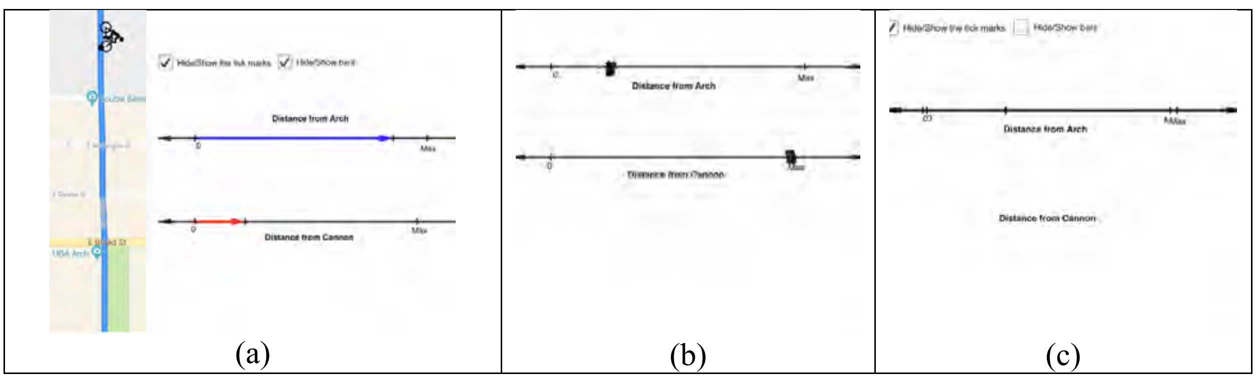


Figure 2: (a) MGT (b) Mike's tick marks on the magnitude lines representing the bike's DfA and DfC, and (c) Mike's activity seeking to create a single point

I also hid the red and blue bars on the magnitude lines. Then, I asked Mike to place tick marks on the magnitude lines to show where the head of each bar would be on the magnitude lines for different locations of the bike on the map. For example, I located the bike at the bottom of the map and Mike successfully placed tick marks on each magnitude line accordingly (see Figure 2b). When placing the tick mark for the bike's DfA, he engaged in the same measurement activity with his fingers—measuring the distances with his fingers on the map and translating his fingers onto the magnitude lines by keeping the same distance between them. Moreover, he stated that "tick marks show where the end of the bars would be." I conjectured that this meaning of tick marks might help Mike to *record* the variation of the bar on the magnitude line since the varying bars do not leave trace or a mark as they move on the magnitude lines. In turn, I conjecture that this activity might help Mike when we move to 2D space to record the relationship between two varying quantities on each axis of the plane.

With Mike doing so successfully and indicating that he conceived of the tick marks in relation to quantities' magnitudes, I decided to transition to the next part of MGT where I asked Mike if he knew a way to represent the bike's DfA and DfC by *a single point* instead of the two tick marks that he placed on two parallel magnitude lines. I also informed him that he can move and rotate those lines freely on the tablet screen. Mike was not able to create a single point to represent two quantities' magnitudes, except for one case where the bike's DfA and DfC had the same length (see Figure 2c). He moved the magnitude lines on top of each other by matching the zero and max points. He then located the bike in the middle between Arch and Cannon on the map and pointed out that the tick marks representing the bike's DfA and DfC were matched on the magnitude lines. Referring to the tick mark, he said "it represents both the distance from Arch and the Cannon because when the bike is right there [*pointing to the bike located in between Arch and Cannon*], they are both the same distance from the Cannon and the Arch."

He could not find a way to create a single point that simultaneously represented other states of the bike's DfA and DfC. He needed to construct a new space that was different than the one-dimensional (1D) space in a way that he could satisfy the simultaneity for all states of the bike's DfA and DfC. This was a need, from my perspective, why we need a 2D space. I didn't have evidence that Mike perceived such a need at the moment as he was satisfied with his activity for one occasion of the bike. This suggested that I should provide other opportunities for Mike that could afford him to structure the space in a way that he could see additional features that is not available in 1D space. Thus, I decided to move to CT where I provided him a coordinate system as a given space. My goal was to provide Mike with additional figurative material that might afford him to structure the space in a way that is compatible with Cartesian plane.

Mike's Activity in Crow Task

Mike was able to move the crow freely while observing how the corresponding black point on the plane moved (see https://youtu.be/mIpOiSuq0mE). When asked to explain what the black point might represent on the plane, he conceived the dot as the crow. However, he was not sure how the black dot moved according to how the crow flied on the map. He said, "I guess [the dot is] the crow, but I still don't understand this" meaning that he did not know how the black dot moved on the plane as the crow moved on the map. Since I knew he was able to reason with quantities, I wanted to draw his attention to the crow's DfA and DfC in the situation when the crow was not flying (i.e., static moment). I asked Mike to show how he imagined the crow's DfA and DfC on the map when the crow was located next to Georgia Theatre (see Figure 3a, left). He drew two segments to show the crow's DfA and DfC on the map (Figure 3a, left). I then asked him to place tick marks on the axis of the plane to represent the crow's DfA and DfC. Recall that, in MGT, Mike was able to transform (i.e., dis-embed and represent) the magnitude of the bike's DfA and DfC from situation to the parallel magnitude lines by ensuring to preserve their length by using his fingers. He successfully placed tick marks on each magnitude line to represent the bike's DfA and DfC (see Figure 2b). Since he never assimilated what I perceive to be the axes of the coordinate plane as the magnitude lines, I did not expect him to insert tick marks on each axis normatively in his first attempt. However, I still desired to ask this question because this could draw his attention to the axes of the plane in later attempts. My goal was to help him to make connection between his activity of inserting tick marks on two parallel magnitude lines and inserting tick marks on the axis of the coordinate system.

He first assimilated the black dot on the plane as the crow stating “so, um, and this is where the crow is.” Mike then plotted two tick marks on each axis where he thought where the physical Arch and Cannon were on the axes (Figure 3a, right). He said “these tick marks are where the Cannon and Arch is, I think.” Mike then represented the crow’s DfA as the distance from the Arch on the axis and the crow’s DfC as distance from the Cannon on the other axis. He drew segments from these marks to the black dot on the plane explaining “these [s*liding his index finger over the segments he drew on the plane*] are the two distances. I mean, I don’t know” (see Figure 3a, right; the computer-generated tick marks on each axis were not available at the moment). This suggested that his meaning of the point at the moment included envisioning the point as a location/object by coordinating quantitative features of the object on the plane, which is the same meaning that Mike had in his previous graphing activity in DABT (see Figure 1).

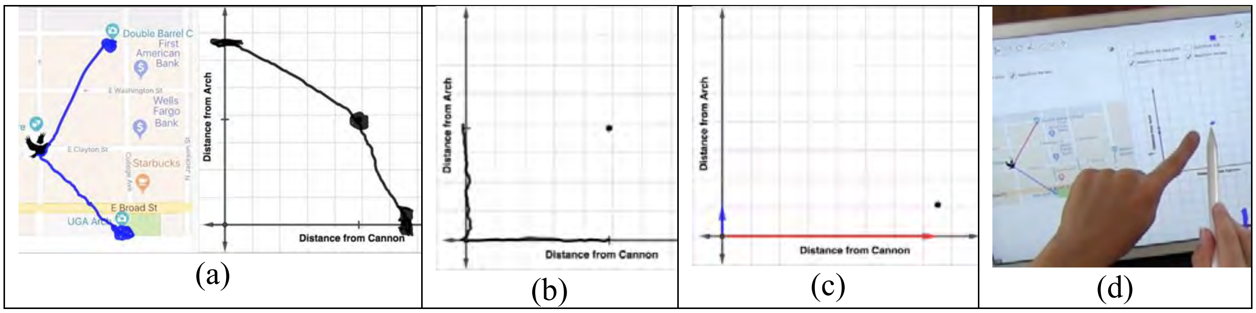


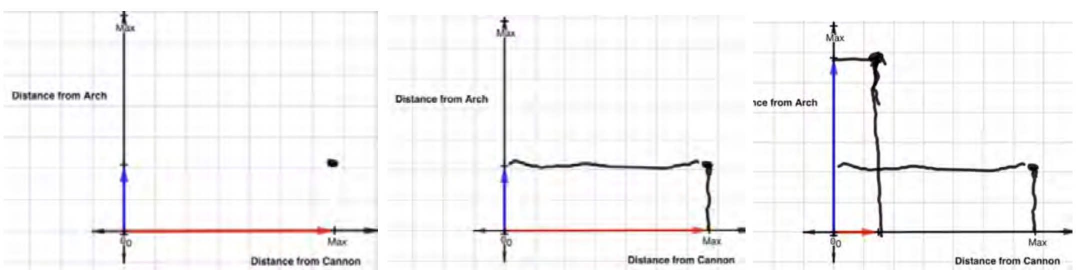
Figure 3: (a) Mike’s activity in CT, (b) Mike’s segments on each axis, (c) Bars in CT, and (d) Mike’s physical action of joining the projection of the two magnitudes on the plane

Then, I showed the computer-generated tick marks on each axis of the coordinate plane (see the additional tick marks on each axis in Figure 3a, right). My goal was to provide additional figurative material for Mike that could afford him to conceive the axis as the magnitude lines. Mike immediately surprised by the locations of the tick marks on the axes. He questioned “why are they there” and he could not resolve this perturbation at the moment. Since I knew Mike had a meaning for tick marks on the parallel magnitude lines in relation to quantities’ magnitudes in MGT, I decided to ask Mike to reflect on his previous activity where he placed tick marks on the magnitude lines in MGT. I asked Mike what the tick marks meant to him in the previous activity. He said, “end of the bar, like where the bar ended.” I then asked him where these bars might be on the plane in CT. He began erasing his work on the plane in Figure 3a (right). He stated that “based on, um, the little tick marks, um, like maybe there is two bars, maybe like there is two lengths from the origin that go to these two [*referring to the tick marks*].” He then drew the horizontal and vertical bars on axes (see Figure 3b). When asked to describe what these bars represented, he responded “the distances from the Cannon and Arch.”

In order to get more insights into his meaning of bars on the axis, I showed the blue and red bars on the map and on the axes (see https://youtu.be/oFGjq9UgQNA?t=11). I asked Mike to move the crow on the map and observe what was going on with the bars on each axis of the plane. As he moved the crow on the map, Mike determined that the bars on the axes and the bars on the map “are the same.” He elaborated by saying “lines are getting longer and shorter, like the lines here [*pointing to the bars in the map*] or the bars are equal to the bars like [inaudible].” For example, he moved the crow right next to the Arch on the map and determined the length of the blue bar was too small (see Figure 3c). He said “it [the crow] is right next to the Arch [on the map] and the blue bar is really small.” This activity provided evidence that Mike conceived the bars on the axes of the plane in relation to the quantities’ magnitudes. Thus, I decided to investigate his meaning of a point on the plane to see if and how Mike could join two magnitudes and generate a point that represented both. Then, I hid the black dot on the plane keeping the blue and red bar on each axis visually available and asked Mike where the black dot would be on the plane. Mike plotted a point on the plane by intersecting the projections of the quantities' magnitudes that were represented on each axis of the plane (see his joining gesture in Figure 3d) and his meaning of the point included "the crow's distance from Cannon and Arch."

Mike's Activity in Matching Game Task (Second Attempt)

I next directed Mike to the previous task (i.e., MGT) and asked that he create a single point to represent the bike's DfA and DfC simultaneously given two parallel magnitude lines (see Figure 2a). I asked Mike if this task and CT are related. Mike said "maybe, if we make this [*pointing to the parallel magnitude lines*] a *x* and *y*-axis thing." Then, Mike rotated one of the magnitude lines and placed it perpendicular to the other magnitude line by matching the zero points in both lines (see Figure 4a). He then plotted a point where the projection of the red and blue bars intersected on the plane. I asked him how he knew this point showed both the bike's DfA and DfC, he drew the vertical and horizontal segments (see Figure 4b) and said "it [*referring to the vertical segment*] is the bike's distance from the Arch and it [*referring to the horizontal segment*] is the bike's distance from the Arch." Mike also successfully created a single point to represent the bike's DfA and DfC for another location of the bike (see Figure 4c).



(a) (b) (c)

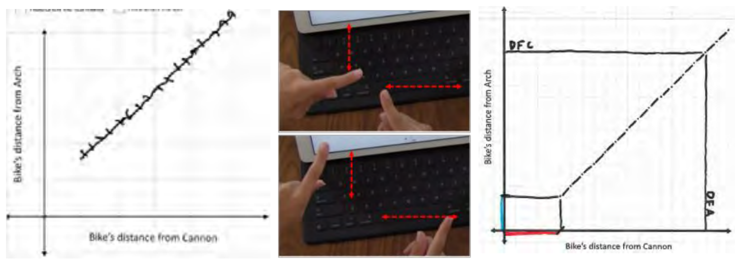
Figure 4: (a) Mike's orthogonal magnitude lines and his single point representing both quantities, (b and c) Mike's segments on the plane to show the magnitudes

Mike's Final Graph in Downtown Athens Bike Task

In DABT, I played the bike animation (see Figure 1a) and asked Mike to draw a representation showing how the bike's DfA and DfC varied. Mike began his graphing activity by drawing a straight line upward from left to right on the tablet screen (Figure 5a). Mike also drew his graph on paper as a dotted and dashed straight line upward from left to right (see Figure 5c, the additional horizontal and vertical segments were not drawn yet). When asked to illustrate how his graph showed the relationship between DfA and DfC, he moved his right index finger horizontally back and forth in the air and moved his left index finger up and down vertically in the air (see the red arrows an indication of the path for his fingers in Figure 5b). His movement of the fingers suggested that he might be imagining the varying bars in relation to his graph on the plane. I asked him to elaborate on his finger movements. He explained,

> When we moved them, we moved them like [*making the same gestures in Figure 5b over the vertical and horizontal axis of the plane*], the two lines go like [*joining his fingers on the graph*] and it crosses these little tick marks [*pointing to the tick marks on his graph by sliding his finger on his graph from left to right*].

From his activity, I infer that Mike imagined the projection of quantities' magnitudes on the axes meeting on the plane along with the variation of these bars creating the graph. In order to get more insights into his meaning of the graph on paper, I asked Mike to show how his graph on paper showed the bike's DfA and DfC for certain states of the bike in the situation. For example, I asked Mike how his graph showed the moment where the bike was located at the right side and at the middle of the path on the map. Mike drew horizontal and vertical segments on the plane representing the bike's DfA and DfC to show these two moments (see Figure 5c). The horizontal segment represented the bike's DfC because it was the projection of the red bar on the horizontal axis. Similarly, the vertical segment on the plane represented the bike's DfA because it was the projection of the blue bar on the vertical axis. I infer that the way he represented quantities' magnitudes was now compatible with representing a point in a canonical Cartesian plane.



(a) (b) (c)

Figure 5: (a) Mike's final draft in DABT on the tablet, (b) Mike's finger movement in the air, and (c) Re-drawing his final draft on paper.

Discussion

In this study, I illustrated Mike's meanings for graphs and the development of those meanings from the teaching experiment as I implemented an instructional sequence that I developed in order to support him in developing emergent shape thinking—a productive and covariational meaning for graphs from quantitative reasoning perspective (Moore, 2021). In particular, I aimed at promoting meanings of (i) representing quantities' magnitudes as varying, directed bars on parallel magnitude lines, (ii) coordinate system as a two-dimensional space constituted by two orthogonal magnitude lines, (iii) points in coordinate systems as a simultaneous representation of two quantities' magnitudes, and (iv) graphs as an emergent trace of that point representing the quantities' covariation.

Mike initially assimilated the points in the plane in relation to the physical objects that appear in the situation, and his meanings for points were based in quantitative properties. I consider Mike's initial representational activity as illustrating an alternative meaning of a coordinate system and coordinate points. I found that being able to represent the quantities magnitudes as varying bars on the magnitude lines was a critical development in the teaching experiment as it laid a foundation for students to re-organize the space and construct a Cartesian coordinate system. In his subsequent activity in MGT after his engagement in CT, Mike successfully manipulated the parallel magnitude lines to create a single point on a Cartesian plane. This re-organization of the space was an important moment for him to be able to use the plane in order to create a single point that simultaneously represented two quantities magnitudes. Furthermore, representing quantities on magnitude lines played a significant role in his development as he subsequently and frequently referred to the variation of quantities' magnitudes represented *on the axes* when explaining how his graphs showed certain covariational relationship on the plane.

Acknowledgments

This material is based upon work supported by NSF under Grant No. DRL-1350342 and DUE-1920538.

References

Boote, S. K., & Boote, D. N. (2017). Leaping from Discrete to Continuous Independent Variables. *Elementary School Journal, 117*(3), 455–484.

Botzer, G. & Yerushalmy, M. (2008). Embodied Semiotic Activities and Their Role in the Construction of Mathematical Meaning of Motion Graphs. *International Journal of Computers for Mathematical Learning, 13*(2), 111.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Clement, J. (1989). The concept of variation and misconceptions in Cartesian graphing. *Focus on Learning Problems in Mathematics, 1*(1-2), 77-87.

Corbin, J. M., & Strauss, A. (2008). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.). Thousand Oaks, CA: Sage.

Earnest, D. (2015). From number lines to graphs in the coordinate plane: Investigating problem solving across mathematical representations. *Cognition and Instruction, 33*(1).

Frank, K. M. (2016a). *Students' conceptualizations and representations of how two quantities change together.* Paper presented at the 19th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education, San Diego, CA.

Frank, K. (2016b). Plotting points: Implications of "over and up" on students' covariational reasoning. In Wood, M.B., Turner, E. E., Civil, M., & Eli, J.A. (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 573-580. Tucson, AZ: The University of Arizona

Frank, K. M. (2017). *Examining the development of students' covariational reasoning in the context of graphing.* (Unpublished doctoral dissertation), Arizona State University, Tempe, AZ.

Hattikudur, S., Prather, R. W., Asquith, P., Alibali, M. W., Knuth, E., & Nathan, M. (2012). Constructing graphical representations: middle schoolers' intuitions and developing knowledge about slope and y-intercept. *School Science and Mathematics, 112*(4), 230–240. https://doi.org/10.1111/j.1949-8594.2012.00138.x

Johnson, H. L., McClintock, E., & Gardner, A. (2020). Opportunities for reasoning: Digital task design to promote students' conceptions of graphs as representing relationships between quantities. *Digital Experiences in Mathematics Education*.

Lee, H. Y., Hardison, H., & Paoletti, T. (2018). Uses of coordinate systems: A conceptual analysis with pedagogical implications. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), P*roceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1307–1314). Greenville, SC: University of South Carolina & Clemson University.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64.

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. I. Engelke, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Pittsburgh, PA: West Virginia University.

Moore, K. C. (2021). Graphical Shape Thinking and Transfer. In: Hohensee, C., Lobato, J. (eds) *Transfer of Learning. Research in Mathematics Education*. Springer, Cham.

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119–172.

Ponte, J. (1984). *Functional reasoning and the interpretation of Cartesian graphs.* (Unpublished doctoral dissertation), University of Georgia, Athens, Georgia.

Radford, L., Edwards, L. & Arzarello, F. (2009). Introduction: beyond words. *Educational Studies in Mathematics, 70*, 91–95. https://doi.org/10.1007/s10649-008-9172-y

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Schliemann, A.D., Carraher, D.W., & Caddle, M. (2013). From Seeing Points to Seeing Intervals in Number Lines and Graphs. in B. Brizuela & B. Gravel (Eds.) *Show me what you know: Exploring Representations across STEM disciplines*. Teachers College Press.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillside, NJ.

Tasova, H. I., & Moore, K. C. (2020). Framework for representing a multiplicative object in the context of graphing. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 210–219). Cinvestav/PME-NA

Tasova, H. I. (2021). Developing middle school students' meanings for constructing graphs through reasoning quantitatively (Doctoral dissertation). https://esploro.libs.uga.edu/esploro/outputs/9949391256202959

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W., Carlson, M. P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: A hypothesis about foundational reasoning abilities in algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S. Belbase (Eds.), *Epistemic algebraic students: Emerging models of students' algebraic knowing* (pp. 1-24). Laramie, WY: University of Wyoming.

Thompson, P. W., Hatfield, N. J., Yoon, H., Joshua, S., & Byereley, C. (2017). Covariational reasoning among U.S. and South Korean secondary mathematics teachers. *Journal of Mathematical Behavior, 48*, 95–111.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium of research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

STUDENT REASONING IN DYNAMIC SITUATIONS: SPATIAL PROXIMITY REASONING

Halil I. Tasova
California State University San Bernardino
halil.tasova@csusb.edu

In this paper, I introduce a mode of reasoning, which I call spatial proximity reasoning, that entails coordinating the (co)variation of an object's degree of proximity (i.e., closeness or nearness) to other objects (i.e., reference objects) in a dynamic real-world situation. I compare and contrast spatial proximity reasoning with quantitative (co)variational reasoning. I also present the implications of spatial proximity reasoning on students' graphing activities in one- and two-dimensional spaces.

Keywords: Algebra and Algebraic Thinking, Cognition, Mathematical Representations.

This paper reports on a study addressing the following research questions: (a) How does spatial proximity reasoning differ from quantitative covariational reasoning? (b) What are the implications of spatial proximity reasoning for students' graphing activities? In the findings below, I illustrate that Zane who engaged in spatial proximity reasoning assimilated what I perceived to be the representation of a quantity's magnitude on a number line in a certain way that was different from Ella who engaged in quantitative covariation. Zane imagined the physical objects on the number line whereas Ella represented the quantity's magnitude on the number line isolated and decomposed from its physical context. In turn, their representational activity in two-dimensional space differed as an implication of their reasoning.

Background

This study focuses on students' reasoning when they engage in dynamic real-world situations that involve quantities varying (from my perspective). I use *quantity* to refer to a conceptual entity an individual constructs as a measurable attribute of an object (Thompson, 2011). When engaging in dynamic contexts, students can use *covariational reasoning*, which entails attending to how one quantity varies in relation to the other (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Different from students' quantitative covariational reasoning, I identified a new mode of reasoning, which I call spatial proximity reasoning. I define students' spatial proximity reasoning as coordinating the (co)variation of an object's degree of proximity (i.e., closeness or nearness) to other objects (i.e., reference objects) in a dynamic situation (e.g., the bike is getting closer to or farther from Arch). Note that students do not engage in coordinating the magnitudes or values of quantities, instead they coordinate the *nearness* between two objects without attending to the measurable attributes of these objects. For example, I observed students conceiving "the bike's distance from Arch" not as a measurable attribute of the bike, instead as the bike's degree of proximity to Arch. Indication of this type of reasoning might include having students using the linguistic adverbs "closer", "farther" when there is no evidence of students imagining quantified attributes of an object (e.g., the values or magnitudes of the bike's distance form Arch getting increase or decrease). I have often observed that student's image of variation regarding the bike's proximity to Arch included two separate objects that were getting close to or farther from each other, rather than including a varying length of a segment. Based on the analysis of my teaching experiments, I present evidence of spatial proximity reasoning and identify implications for students' representational activities (e.g., imagining the physical objects on the number line or on the coordinate plane).

Methods

This study is situated within a larger study that examined two sixth-grade and four seventh-grade students' graphing activities in a teaching experiment (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. In this study, I focus on two students' data, Ella and Zane (seventh-graders) because their reasoning and graphing activities were distinct, and worth documenting and contrasting. Ella and Zane (as a pair) participated in 6 teaching sessions each of which last for approximately one-hour.

Data sources included video and transcripts of each session that captured their exact words, gestures, and drawings. I conducted a conceptual analysis in order to understand their verbal explanations and actions and develop viable models of their mathematics (Steffe & Thompson, 2000). My analysis relied on generative and axial methods (Corbin & Strauss, 2008), and it was guided by an attempt to develop working models of their thinking.

Before conducting the teaching experiment, I developed an initial sequence of tasks each of which was designed with a dynamic geometry software and displayed on a tablet device (see https://www.geogebra.org/m/uyy2kxz8 for digital versions of the tasks presented in this paper). In *Downtown Athens Bike Task* (DABT), I present the students with a map of Downtown Athens highlighting a straight road (i.e., Clayton St.; Figure 1a, left). I asked students to graph the relationship between the bike's distance from Arch (DfA) and distance from Cannon (DfC) as the bike moves at a constant speed back and forth along the road. I also designed numerous tasks where students engage with quantities' magnitudes represented by varying length of directed bars placed on empty number lines (also called *magnitude lines*, see Figure 1a, right). The length of directed bars on the magnitude lines vary according to the bike's movement in the map. Note that I call the line "empty number line" in order to emphasize magnitude reasoning as opposed to numerical or value reasoning.

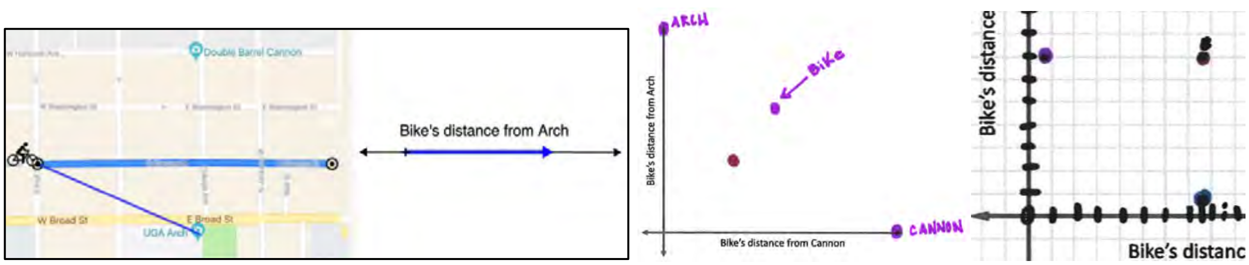
Results

In this section, I illustrate and contrast Ella and Zane's activity where they were asked to interpret the varying bar on a magnitude line (see Figure 1a, or https://youtu.be/xMa9sNMxRIo). While moving the bike to the right from its position seen in Figure 1a, I drew students' attention to the fact that the right end side of blue bar on the magnitude line was moving to the left (indicating the bike's DfA was decreasing). Then, I asked Zane and Ella to explain why that was happening. Referring to the map, Zane claimed, "the bike is getting closer to Arch," which was compatible with how he described the variation of the bike's DfA in the previous activities using the linguistic adverbs "closer to" and "farther from." I did not have any evidence that Zane coordinated the bike's DfA as the bike's measurable attributes. When I explicitly asked him to visually show how he saw bike's DfA on the map, he wrote "Far" at each end side of the bike's path on the map where the bike was far from Arch. This suggested that Zane's image of variation included coordinating the variation of the bike's proximity to Arch.

Then, by pointing to the blue bar on the magnitude line (Figure 1a, right) and tracing the pen over the line from right to left, he added it "is gonna get closer to right here [*pointing to the zero point on the magnitude line*], which is Arch." Zane conceived Arch and bike as physically placed on the left and right end side of the blue bar, respectively, and he thus perceived blue bar's change to be compatible with his spatial proximity reasoning because the bike moved closer to the Arch *along* the line, with Arch playing the role of reference object. Zane abstracted the two objects (Arch and bike) and their proximity when engaging in the situation. Thus, he imagined these physical objects in his representational activity too.

Differing from Zane, Ella determined that the bike's DfA is decreasing while moving the bike to the right in the map. She explained "it [*pointing to the blue bar*] is gonna get smaller because distance is smaller on the number line too." Moreover, Ella labeled the starting point as "zero", whereas Zane conceived the same point on the magnitude line as "Arch." Her activity suggested that Ella conceived the length of the blue bar on the map and on the magnitude line as a representation of the bike's DfA. I conjecture that Ella's activity was an implication of her quantitative variational reasoning (i.e., the bike's DfA is decreasing) whereas Zane's activity was an implication of his spatial proximity reasoning (i.e., the bike is getting closer to Arch).

In order to understand how Zane understood Ella's response and focus on "distance", I asked Zane to re-voice Ella. Zane's explanation included his original reasoning (i.e., imagining the Arch and the bike physically on the line getting closer and farther). Thus, Zane assimilated Ella's answer to his meaning because they were compatible with respect to the behavior of figurative material. Since Zane's meaning included spatial proximity (i.e., nearness) between two places, it made sense to imagine Arch in place of the origin in one-dimension because the value of the bike's DfA was zero at that point. However, their meanings and reasoning have different implications when we move to two-dimensional systems, which I illustrate as follows.



(a) (b) (c)

Figure 1: (a) Representing the bike's DfA on the map and on the magnitude line, (b) Zane's graph, and (c) Ella's graph

In DABT, I asked them to represent the relationship between the bike's DfA and DfC on a given paper with a coordinate plane. By the time I asked this question, the picture of the situation and the parallel magnitude lines on the tablet screen was visually available to them. Recall that Zane conceived the objects from the situation (i.e., Arch and the bike) on the magnitude line, and he thus conceived the varying bar as illustrating the bike was getting closer to Arch. In two-dimensional space, and via conceiving Arch and Cannon on the axes implied by the labels, Zane attended to the spatial proximity between objects by coordinating the bike's location in the plane (see his graph in DABT in Figure 1b). To justify his graph, he rotated the plane counterclockwise in order to match with perceptual feature of the map. That is, he rotated the plane in a way that Cannon and Arch are at the top and bottom of the plane, respectively, and the bike moved horizontally in the plane, just like it did on the map.

Although Zane's form of reasoning is productive in terms of completing the goal of the activity as he perceived it (i.e., showing where the bike is in the plane), this creates a problem in the normative Cartesian that is based on two directed distances (i.e., horizontal and vertical). On the other hand, Ella conceived the length of the bar on the magnitude line as a proxy for the quantity that she conceived in the situation (i.e., the bike's DfA). In doing so, she conceived a constrain regarding how to represent the variation of a quantity on a magnitude line (e.g., only left and right on a horizontal line). Thus, she organized the space accordingly in later activities when considering two-dimensional space (see her graph in DABT in Figure 1c). Ella referred to the axes of the plane as "number line" and plotted tick marks on each axis in conjunction with tick marks plotted on the magnitude lines on the tablet screen. She added dots near certain tick marks on each axis (see the dots in Figure 1c) to represent a certain state of bike's DfA and DfC. Then, she plotted a point in the plane "where those two [*tracing the pen in the air from the dots on each axis to the dot on the plane horizontally and vertically, respectively*] would meet up if they have like a little line." This provided evidence that Ella was able to assimilate the axes of the plane in relation to the magnitude lines and, in turn, she was able to imagine the magnitudes varying on the axes of the coordinate plane. Since it has a significant impact on students' graphing activity, we, as a researcher or teacher, should pay attention to the types of reasoning students might demonstrate when they graph in dynamic situations.

Discussion

In this study, I illustrated various ways in which students reasoned in dynamic situations (i.e., quantitative covariational reasoning and spatial proximity reasoning). These different ways of conceiving dynamic situations suggest that it is really important for us, as researchers and teachers, to understand how students conceptualize the objects' varying attributes in the situation. We should not assume that the variation that students might conceive is the same as *given* in the dynamic situation. Thompson and Carlson (2017) pointed out that researchers might conceive the varying quantities in a situation; however, it might not be the case for students. In their extensive literature review, Leinhardt et al. (1990) reported that researchers often thought that variation has something to do with the situation only as opposed to acknowledging the importance of conceptualizing the variation in students' mind. Thus, merely offering dynamic situations to students does not imply that students will engage in quantitative covariational reasoning. We need to understand how students might conceive dynamic situations and how they might imagine and represent the quantities' magnitudes vary.

I believe the distinction I made in this study could provide a lens for researchers to investigate students' graphing activity especially in contexts in which a *motion* occurs. Since many embodied activities often involve graphing with a motion sensor technology, students could either coordinate an object's distance from the motion detector or coordinate the object's proximity to the motion detector when creating or interpreting a graph. Researchers (e.g., Hobson, 2017; Nemirovsky et al., 1998) reported that students could produce a normative graph (or interpret a graph normatively) as a representation of the motion although they don't attend to the quantities in the situation, such as distance between the object and the reference point (e.g., motion detector). For example, when engaging in creating graphs generated by a motion detector representing how a handheld button's distance from the detector varied over time, Nemirovsky et al. (1998) reported that students determined "the line on the screen becomes higher when one moves the button away from the tower and lower when one moves the button toward it" (p. 122). Nemirovsky et al. interpreted that, for the students in their study, distance between the button and the tower (i.e., motion detector) was not related to meters, instead it "concerned the possibility and impossibility of creating certain graphical responses" (p. 164), which is related to spatial proximity reasoning. They noted that distance "is about the impossibility of certain actions that would be possible if the objects were nearby, not a matter of meters" (p. 164). Therefore, I believe the construct, spatial proximity reasoning, can be used as an additional lens when investigating students' representation of motion as a graph (e.g., distance-time graphs).

Acknowledgments

This material is based upon work supported by NSF under Grant No. DRL-1350342 and DUE-1920538.

References

Corbin, J. M., & Strauss, A. (2008). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (3rd ed.). Thousand Oaks, CA: Sage.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Hobson, N.L.F. (2017). *Quantities and covariation: an inquiry into the reasoning of experts engaged in graphically representing dynamic situations* (Master's thesis). https://esploro.libs.uga.edu/esploro/outputs/9949334635302959

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*(1), 1–64. Retrieved from http://www.eric.ed.gov/ERICWebPortal/detail?accno=EJ414297

Nemirovsky, R., Tierney, C., & Wright, T. (1998). Body motion and graphing. *Cognition and Instruction, 16*(2), 119–172.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkings, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), *Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298-303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Hillside, NJ.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33-57). Laramie, WY: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium of research in mathematics education*. Reston, VA: National Council of Teachers of Mathematics.

VARIABLE TYPES IN MIDDLE SCHOOL MATHEMATICS CURRICULA

Daniel K. Siebert
Brigham Young University
dsiebert@mathed.byu.edu

Ashlyn Rounds
Brigham Young University
ashrounds888@gmail.com

Keywords: Algebra and Algebraic Reasoning, Middle School Education, Mathematical Representations

One of the main practices that separates arithmetic and algebraic reasoning is the use of variables (i.e., literal symbols). Drawing upon decades of research, Lucariello et al. (2014) noted that secondary school students struggle to make sense of and use variables correctly, making errors like ignoring variables in algebraic expressions, treating variables as labels for objects, or assuming all variables refer to specific unknowns. At least part of students' confusion regarding variables is caused by the multiple meanings associated with variables in algebra (Blanton et al., 2011; Philipp, 1992; Usiskin, 1988). We refer to the meanings and accompanying uses for variables as *variable types*. Examples of variable types include *generalized number* (for expressing properties), *unknown number* (for solving an equations), and *parameter* (for describing a family of functions or problem types). Usiskin (1988) highlighted the complexity of deciphering variable type by not only identifying many variable types, but also by showing that the variable type associated with a literal symbol can change during algebraic activity, even within a single problem. While scholars have identified and described many variable types, little empirical research has been conducted to determine which variable types students encounter in secondary school mathematics, particularly in middle school where the frequent use of variables is first encountered.

In this study, we investigated the variable types used in three popular US 7-8 grades mathematics curricula: *Eureka Math* (2015), *Connected Mathematics 3* (Lappan et al., 2014), and *Glencoe Math* (2015). We sampled 3 units from each grade of each textbook series. For each textbook, we selected 2 units related to the expression and equations strand and 1 unit related to the geometry strand of the Common Core State Standards for Mathematics (NGA & CCSSO, 2010). We started our textbook analysis using a coding scheme consisting of the variable types described by Usiskin (1988) and Blanton et al. (2011). We both participated in coding all 3 textbook series, and began coding individually when interrater reliability exceeded 90% on each textbook series. Original codes were modified and new codes added as needed to fit the data.

Our analysis led to the identification of 8 main variable types. All 8 types were observed in both grades of all 3 of the textbook series. The most widely used variable types were *unknowns*, *placeholders*, and *objects*. All lessons used at least 2 variable types, and some involved as many as 5 types. Adding to the complexity of variable use was the frequent need to reinterpret a literal symbol without explicit acknowledgement by textbook authors that the type had changed. For example, the textbook authors would use variables as *placeholders* to represent relationships between quantities in a story problem, reinterpret them as *unknowns* at the start of the equation solving process, treat them as *objects* when performing symbolic manipulation, and then rethink of them as *knowns* to check the answer. Our results indicate that a sophisticated understanding of variable types and their accompanying uses is necessary to make sense of algebraic expressions and equations even in the early weeks of a 7^{th} grade course. These findings suggest a strong need for explicit instruction on variable type and use to enable students to meaningfully create, make sense of, and operate on variables in algebra.

References

Blanton, M., Levi, L., Crites, T., Dougherty, B. J., & Zbiek, R. M. (2011). *Developing essential understanding of algebraic thinking for teaching mathematics in grades 3-5*. National Council of Teachers of Mathematics.

Eureka Math. (2015). Great Minds. https://greatminds.org/math

Glencoe Math. (2015). McGraw Hill.

Lappan, G., Friel, S. N., Fey, J. T., & Phillips, E. D. (2014). *Connected Mathematics 3*. Pearson.

Lucariello, J., Tine, M. T., & Ganley, C. M. (2014). A formative assessment of students' algebraic variable misconceptions. *Journal of Mathematical Behavior, 33*, 30-41. https://doi.org/http://dx.doi.org/10.1016/j.jmathb.2013.09.001

National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. NGA Center and CCSSO. http://www.corestandards.org

Philipp, R. A. (1992). The many uses of algebraic variables. *Mathematics Teacher, 85*(7), 557-561.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. Coxford (Ed.), *Ideas of algebra: K-12* (pp. 8-19). NCTM.

Chapter 4:
Equity and Justice

LEVERAGING EQUITY AND CIVIC EMPATHY THROUGH COMMUNITY-BASED MATHEMATICAL MODELING

Julia M. Aguirre
University of Washington Tacoma
jaguirre@uw.edu

Jennifer Suh
George Mason University
jsuh4@gmu.edu

Holly Tate
George Mason University
htate2@gmu.edu

Mary Alice Carlson
Montana State University
marycarlson5@montana.edu

Elizabeth Fulton
Montana State University
elizabeth.fulton@montana.edu

Erin E. Turner
University of Arizona
eturner@arizona.edu

This theoretical paper describes how Community-based Mathematical Modeling can advance equity and cultivate civic empathy in elementary school settings. We provide a framework for community-based mathematical modeling instruction consisting of five goals: facilitating connections, fostering engagement, promoting rigor, cultivating civic empathy, and elevating justice. We illustrate how these goals work together to advance equity and cultivate civic empathy through classroom vignettes of community-based modeling lessons. Through this theoretical synthesis, implications for community-based mathematical modeling instruction will be discussed.

Keywords: Equity, Inclusion and Diversity, Modeling, Elementary Education, Culturally Relevant Pedagogy, Professional Development

Mathematical Modeling as an Equity and Empathy Lever

Mathematical modeling (MM) is an iterative cyclical process that uses mathematics to make-sense of and analyze real-world situations. MM includes posing the real-world problem, identifying important quantities and making assumptions to define variables, which then leads to developing a mathematical model, analyzing and assessing the solution, and refining the model (Garfunkel & Montgomery, 2019). MM nurtures 21st century skills of creativity, critical thinking, communication, and collaboration (Suh & Seshaiyer, 2013). It is an area of mathematics that offers children opportunities to grow mathematically and connect mathematics to their lived experiences, families, and communities. There has been a well-established literature on mathematical modeling. However, the centering of equity and empathy remains underemphasized (see Anhalt et al., 2018; Aguirre et al, 2019; Suh et al, 2018; Turner et al, 2022; Cirillo et al., 2016; for notable exceptions). We argue that MM is a humanizing endeavor that authentically connects to the real world, starting with ill-defined, often messy community-based problems and providing opportunities for students to develop empathy and compassion toward other people, living things and our planet (Lee et al., 2021). By showcasing children's diverse perspectives, curiosities, and mathematical and experiential strengths as resources for learning, mathematical modeling is a powerful *lever for equity and civic empathy* in the elementary mathematics classroom.

In our work, equity means that: All students **in light of their humanity** – personal experiences, backgrounds, histories, languages, physical and emotional well-being – must have the opportunity and support to learn rich mathematics that fosters meaning-making, empowers decision-making, critiques, challenges, and transforms inequities/injustices. It promotes equitable access, attainment, and advancement for all students (Aguirre, 2009; Aguirre et al, 2013).

In terms of empathy, we draw on the foundational work of Freire (1970) on critical consciousness and recent report from the National Academy of Education (Lee et al., 2021) for civic reasoning and discourse to deepen children's empathy with others who may have different perspectives. Civic empathy- "working to see the world from another person's perspective—can help us to overcome some of the impediments to listening and can improve our ability to relate to each other civically" (Lee et al., 2021, p. 39). Mirra's (2018) concept of *critical* civic empathy is useful because she differentiates individual empathy, or simply seeing another's perspective, from critical civic empathy, which includes seeing another's perspective while also engaging in structural analysis of power and privilege to take civic action for social transformation. We lean on this concept of critical civic empathy as we focus on community-based mathematical modeling. There is growing consensus that elementary grade students can engage in community-based mathematical modeling. Researchers specifically highlight how students draw on their lived experiences to support mathematizing activity (Albarracín & Gorgorió, 2020). Turner et al (2011) found that when children mathematized situations rooted in their communities, what the authors referred to as *community mathematization*, children considered the real-world implications of their decisions and models.

Recent research confirms how teachers foster this connection through *community-based* MM tasks and routines (CMM). Turner et al (2022) document how elementary teachers designed community-based modeling tasks centering social and environmental justice such as helping a community-center order food supplies to feed unsheltered families and determining plastic waste generated from the school cafeteria. Aguirre et al (under review) designed an instructional routine called Mathematizing-the-World routines (MWR) that leverages community mathematization. The MWR focuses on three prompts that are similar to those in the introductory act of Three Act Tasks (Lomax et al., 2017). The first question, "What do you notice?" strengthens students' observation skills. The second question, "What do you wonder?," builds on students' curiosities and strengthens problem-posing skills. The third question, "What questions can be solved using mathematics?" draws students' attention to problem-posing with mathematics. To foster empathy and connection while mathematical problem-posing, the teacher offers additional prompts including: Why is this question/issue important? "Why should we care about this question/issue?" "Who else might care about this question/issue?" "Why might it be important to them?" (Arnold et al, 2022). For each prompt, teachers record student responses on a class chart. In some cases, the class identifies a specific mathematical question to investigate further. The teacher may also use this routine to launch a CMM task.

This theoretical paper describes how CMM can advance equity and cultivate civic empathy in elementary school settings. We continue to push the field of MM and teacher education to center on equity, empathy, and justice; opening up space for young children to be critical decision makers and community change agents (Aguirre et al, 2019; Anhalt et al., 2018; Cirillo et al., 2016). We draw on current and past MM projects that advance equity and strengthen teaching in elementary settings. Through this work we identified 5 instructional goals that frame how CMM leverages equity and civic empathy: *facilitating connections, fostering engagement, promoting rigor, cultivating civic empathy, and elevating justice.* To illustrate these goals in action we present two elementary classroom vignettes: a MWR on analyzing racial representation in school libraries and an environmental justice modeling task on water quality. To ground our work, we propose a theoretical framework to leverage equity and empathy with CMM.

Theoretical Framework to Leverage Equity and Empathy in Modeling

Below, we identify five instructional goals in CMM instruction that advances equity and empathy in elementary classrooms: ***facilitating connections, fostering engagement, promoting rigor, cultivating civic empathy and elevating justice*** (see Figure 1). Each goal is grounded in the literature and preliminary findings from our projects.

Goal 1: Facilitate Connections

Through CMM children can connect mathematics to themselves, their peers, families, communities, and to the world. CMM acts as a mirror and a lens for students and their teachers. The emphasis on deliberately connecting to children's lived experiences promotes multiple opportunities for children to see themselves in the mathematics (mirror) and use mathematics to analyze the world around them (lens). CMM connects children with each other in order to share ideas and experiences. By building multiple connections to self, family, community, other subjects, and the world, CMM humanizes mathematics teaching and learning (Suh et al, 2018; Gutierrez, 2018).

Goal 2: Foster Equitable Engagement

CMM activities open up access to meaningful complex problem solving for children with varied mathematical, cultural, and linguistic backgrounds. Modeling lessons incorporate a variety of participation structures to elicit diverse contributions, support peer collaboration, and minimize status issues in the classroom (Featherstone et al., 2011). CMM tasks are also group-worthy tasks that have multiple entry points and solution paths for students with varied mathematical strengths to work collectively and solve (Horn, 2006). We have also seen that modeling activities build math stamina and perseverance. In our project, teachers reported more children engage in mathematics for longer periods of time with modeling activities (Turner et al, 2021).

Goal 3: Promote Rigor

CMM activities are high cognitive demand activities that emphasize observation, analysis, argumentation, and critique (Smith & Stein, 1998). All strands of mathematical proficiency can be accessed through modeling activities: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions (National Research Council, 2001). Historically, one way children have been marginalized in mathematics is by denying them access to rich and rigorous mathematics. Through racist tracking practices, challenging enrichment opportunities are often reserved for children with a specific label like "gifted" or "highly capable". However, denying access to rich complex tasks that foster curiosity and problem posing is a form of dehumanization (Aguirre, 2009; Freire, 1970). CMM opens up new avenues to engage more children in high cognitive demand (rigorous) activities.

Goal 4: Cultivate Civic Empathy

Learning for Justice (https://www.learningforjustice.org/) provides educators with a set of social justice standards (Learning for Justice, n.d.) as a road map for anti-bias education with four critical domains—identity, diversity, justice and action. In the diversity domain, one anchor standard states-*Students will respond to diversity by building empathy, respect, understanding and connection.* To unpack what building empathy means, we lean on Mirra's (2018) work on critical civic empathy–moving beyond oneself and into the perspective of another person and then to take civic action. Mirra (2021) challenges educators to consider the power of empathy- "If my experience of seeking to empathize with another person's challenges does not encourage me to heal the roots of those challenges–through community action, voting, protesting—then what good is that empathy, right?" (p. 12). CMM has the transformative potential (Jemal, 2017) to develop a level of critical civic empathy that leads to action to transform contextual factors that may be perpetuating inequitable conditions and that are necessary for change.

Goal 5: Elevate Justice

CMM provides opportunities to examine issues of fairness, access, power, and civic engagement in our homes, communities, and schools. CMM and teaching mathematics for social justice share three important features: both engage students in ill-defined (messy) problems for which there can be multiple and equally valid approaches; both leverage real-world knowledge that students bring to the classroom; both raise students' interest in mathematics by supporting them to better understand their world (Cirillo et al., 2016). Children have questions about the world; they have ideas about what it means for a situation to be fair. They have heard adults talk about societal topics such as healthcare, government sanctioned violence, racism, as well as food, water, and air quality. Some experience economic hardship while others look for ways to relieve it. Some experience natural disasters such as hurricanes, earthquakes, and fires. And most recently, we have all been touched by the COVID19 pandemic. CMM can provide a space for students and their teachers to engage in these justice-focused discussions.

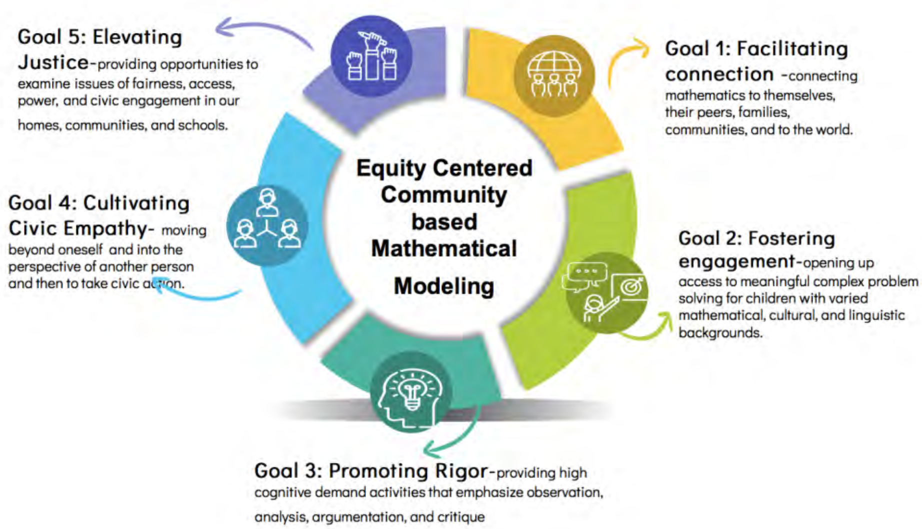


Figure 1: Theoretical Framework to Leverage Equity and Empathy in Modeling

Community-based Mathematical Modeling Case Vignettes

Vignette #1 MWR - Exploring representation of class/school libraries

The first vignette provides a picture of practice from a team of elementary teachers who wanted to integrate MM to prompt deep conversation around the important social issues of ethnic representation in library collections and to learn about race as a part of identity. They launched the unit with a MWR, where a picture prompts (see Figure 2) students to notice and wonder about a phenomenon. In order to mathematize representation of race in classroom libraries, students examined a numberless graphical representation. As students noticed and wondered about the real-world statistics the mathematician was communicating, their conversation suggested awareness of injustice. Students perceived this as an issue of fairness, connecting the availability of main character representation to the disparate number of mirrors for different races and the characters' facial expressions. Empathetic comments emerged such as, "The kids are frowning because there’s a lot of books about White people and not a lot about Brown people,” and “The Latinx girl is shocked, the mirror is broken. This is not fair.”

What do you notice?	What do you wonder?
-The kids are frowning because there's a lot of books about White people and not a lot about Brown people -Back then, they said White people were better so they didn't write as many books about other people. -The LatinX girl is shocked. -The mirror is broken. -It's not fair.	-[POC] are probably wondering why there aren't more books about them. -Why isn't it fair? -Why are they not getting the same attention as the White boy? -Why is the bear's mirror bigger than the people?

(Data on books by and about Black, Indigenous and People of Color published for children and teens compiled by the Cooperative Children’s Book Center, School of Education, University of Wisconsin-Madison)

Figure 2: Numberless Graph Depicting Characters From Diverse Backgrounds Used in Data Talk

This data talk compelled third graders to take inventory of their classroom libraries, framing an investigation to see if their personal race was represented in the main character of books (mirror of representation) as well as if they could see another's racial representation among the books (window to representation of others) (Sim-Bishop, 1990). The analysis of their classroom library involved collective decisions from the students about how to sort the books, what racial categories to use, and how to determine the race of the main character. The inventory revealed the underrepresentation of diverse main characters and led to a powerful modeling task to make their library more fairly represent their classroom demographics. The class created a model that determined how many books of diverse characters should be purchased and wrote a letter to the principal asking for funds to make their library collection more representative of the class. This vignette showcases how the CMM had a transformative potential (Jemal, 2017) to develop a level of consciousness around race representations in their library that led to action to get more books for their library.

The absence of a child’s culture and race from classroom libraries is problematic as they receive a message that their school does not think their culture is important enough to feature in the library. This library representation modeling task **facilitated connections** by giving students ownership in how they would solve the problem of underrepresentation. Having students identify books that reflected their culture and race provided an opportunity for students to make connections to their identities. It **fostered engagement** through a group-worthy task with multiple entry points for students to participate in mathematical and democratic decision making. Grade appropriate **rigor was promoted** through important data analysis, representation, and mathematical decision-making for justice. It also **cultivated civic empathy**, first with feelings of sadness and outrage at the realization of injustice, then through community pride and healing in taking action to purchase books that affirmed their own representation as well as their peers'. Taking action **elevated students’ sense of justice**, and reinforced how they can use mathematics to address inequities and transform their community into a better place.

Vignette #2 Safe Water for Schools

This vignette focuses on students learning about the importance of access to clean water through studying the Flint water crisis and engaging in a math modeling task about safe water for their school community. To **facilitate connections**, teachers launched the CMM task by building context about the water crisis in Flint, Michigan. Students read short articles and viewed images of corroded pipes and brown water in bottles. Teachers shared information about how lead is toxic to the human body, especially for children's brains. Next, teachers engaged in an MWR routine with images of water fountains covered with plastic and signs that said "do not drink" and an image of two elementary students taking a large five-gallon jug of water from a supply room. The MWR provided space to **foster engagement** as different students voiced their noticings, wonderings, and personal connections and emotions about the situation. Figure 3 depicts the wonderings from a 5th grade class. The starred questions are questions that could be answered with math. Students then pursued a modeling task that asked them to imagine themselves in this situation, **cultivating civic empathy** as they determined how many large jugs of water they would need for their class for one day, week or month.

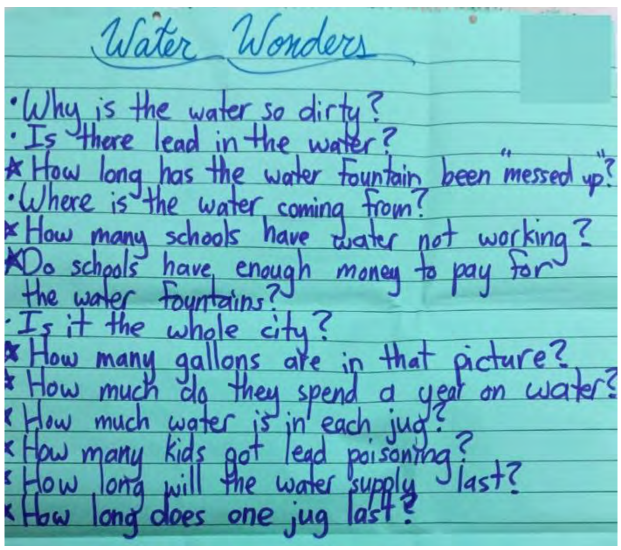


Figure 3: MWR Chart

In Ms. G's 4th grade class, students were living this reality. Their classroom sink/water fountain had recently been shut off due to lead found in the water. They had dispensers with large jugs of water in the hallway shared with other classes. Students identified two important quantities to take into consideration. They needed to figure out how many people would be drinking and how much they would drink per day. Students wrestled with who to include, for example just students; students and teacher; absent students or guests. To help students make decisions about the amount of water to drink, teachers provided a data table that gave ranges of daily water needs (in cups or ounces) by age and grade level. Teachers also shared that the large jugs were 5 gallons and there are 16 cups in 1 gallon. Students worked together in groups to make decisions that took into consideration how long they spent at school, and if people brought their own water bottles. Students' solutions varied with some groups deciding they would need 2 large jugs, 3 large jugs or 4 large jugs. After several students shared their solutions, Ms. G invited students to think about how other students could use their models if a similar situation arose in their school. In other words, to generalize.

Ms. G: So, we've all done our presentations, but we didn't get a chance to talk about what directions would you give another kid. If you were telling a kid in another class how to make a plan for how much water to order, what are some things we could say we could generalize? We can say, that is generally true, that is consistent? So, if another kid from another class wanted to make that plan for the water, what are some things we would tell them that they could do? Max?

Max: I would probably say first start off to find how many kids are in your class. And how often do they drink water? And, ask for their age and just go from there.

Ms. G: Okay so knowing their age. Does it matter that one team said 7 cups because their age was young on the chart and another team said 10 cups? Does it matter or is that okay to make a decision like that?

Many Students: Yeah - Make a decision

Ms. G: So, you make your decision and then you go with it to do the math. What else can we generalize?...

Max: One thing is to define, like, how long are you going to do it. Are you going to do it for a day, or are you going to do it for a week?

Ms. G: Ok. Amari, you have a thought?

Amari: Yeah, how do you know if all the kids are going to show up for school?

Ms. G: Yes, some kids could be absent. Those are other things. So, when you plan how many students, you might assume that somebody is going to be absent.

Ms. G. **promoted rigor** by asking students to generalize, as this communicates that students are capable of complex mathematical work. Ms. G supported students in this rigorous practice by reminding students of the components of their own models, and by revoicing the ideas that students shared. The CMM lesson **facilitated connections** as it was relevant to Ms. G's students; they found lead in the water at their school, and students were genuinely concerned about access to safe water. Finally, Ms. G **advanced equity and civic empathy** by inviting students to connect the specific situation of the task to broader issues of social and environmental justice. By generalizing their solutions, students had opportunities to cultivate empathy, and elevate justice for their own and other's communities.

Theory Synthesis and Implications for Teaching CMM for Equity and Empathy

These two vignettes illustrated how community-based mathematical modeling promotes **five equity goals** for mathematical modeling instruction in elementary settings (*facilitate connections, foster engagement, promote rigor, cultivate empathy and elevate justice*). Next, we discuss these theoretical perspectives on mathematical modeling as a lever for equity and civic empathy in relation to the phases of the mathematical modeling process.

In the first phase, *Making Sense of the Problem*, CMM affords opportunities for teachers to select meaningful contexts that connect to student experiences and issues of fairness, access, representation and justice. In both of our vignettes, elementary students were engaged in *local* community-based problems -- issues of equitable representations of classroom library collections and access to clean water - that also align with broader initiatives to reduce inequity and improve the health and lives of global citizens(https://sdgs.un.org/goals). We make these parallels to global issues to demonstrate how community-based math modeling has transformative potential (Jemal, 2017) for even young mathematical modelers to bring awareness of inequities, and to experience critical civic empathy to take action in powerful ways in their community. By providing images, videos, realia, and graphs to support students' engagement in discussion, we can position students' funds of knowledge and experiences as resources to pose problems.

In the second phase, *Identify Important Quantities*, CMM afforded opportunities to listen to students' ideas and experiences with a sense of curiosity. Modelers make assumptions and decisions when defining variables and determine what they consider to be important which impact the models they create. This is also where data can present evidence of inequities as the library representation modeling task revealed a disproportionate number of books in one racial category, as compared to the others. Older students might examine inequities in COVID 19 pandemic data such as the disproportionate mortality rate for African American and Latinx communities. These important quantities drive the need to better understand underlying issues.

In the third phase, *Build and Operate on a Model*, CMM affords students opportunities to use diverse tools, strategies and representations to build and operate on models and to recognize the strengths in other students' strategies. Teachers honor the ways that students draw on mathematical understandings and funds of knowledge in their model building work. This phase is what inquiry-based learning and "doing mathematics" looks like in terms of promoting rigor and engagement in a cognitively demanding task (Smith & Stein, 1998). This type of learning disrupts the misguided belief that children need to master the basics before engaging in rich and complex math which often happens in schools with historically marginalized student groups causing harm and differential learning experiences (NCTM, 2020).

In the fourth phase, *Analyze and Interpret Models*, CMM affords opportunities for students to collaborate, refine, and consider multiple models by comparing and contrasting strategies and to identify how decisions and assumptions impact solutions. This is a crucial component of perspective taking that is highlighted in the social justice standards (Learning for Justice, n.d.) and in the civic reasoning and discourse literature (Lee et al., 2021). Encouraging students to revise and refine models *foster engagement.*

In the final phase, *Validate and Generalize Models*, CMM encourages students to consider how the components of their own models would work or need to change in a new situation. The Safe Water task enabled students to consider situations similar to the Flint Water Crisis. And, in fact, one classroom recently had their water shut off because lead was found. Inviting students to generalize models to help others in similar situations cultivates civic empathy and empowers action to change the inequities discovered through mathematical analysis in the CMM task.

We situate this work in the midst of the 'double pandemic' (Brennan, 2020) which refers to the COVID-19 interruptions as well as the heightened awareness of systemic racism. Aligned to the PMENA 44's theme, *Critical Dissonance and Resonant Harmony*, we share this theoretical paper to engage in conversation to move beyond the "dissonance" and inequities in our world and consider how Community-based Mathematical Modeling can be a "post pandemic pedagogy" (Ladson-Billings, 2021) to advance equity and cultivate civic empathy in school mathematics. Our hope is that the framework offers a pathway for harmony and humanizing children's learning experiences in the elementary mathematics classroom.

Acknowledgments

The work reported here is supported by a grant from the National Science Foundation (Grants 2010269, 2008997, 2010202, 2010178). We would like to thank all the courageous educators, children and families who continue to impress upon us the importance of making our world a more just and humanizing place.

References

Aguirre, J. (2009) Privileging mathematics and equity in teacher education: Framework, counter-resistance strategies and reflections from a Latina mathematics educator. In B. Greer, S. Mukhopadhyay, S. Nelson-Barber, and A. Powell (Eds). *Culturally responsive mathematics education*. (pp. 295-319). New York: Routledge.

Aguirre, J.M., Anhalt, C., Cortez, R., Turner, E.E., & Simi-Muller, K., (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice. *Mathematics Teacher Educator*. 7(2) 7-26.

Aguirre, J. Mayfield-Ingram, K., & Martin, D. (2013). *The Impact of Identity in K-8 Mathematics Learning and Teaching: Rethinking Equity-based Practices.* Reston, VA: National Council of Teachers of Mathematics.

Aguirre, J.M., Turner, E.E., McVicar, E., Roth McDuffie, A., Foote, M.Q., & Carll, E. (under review). *Mathematizing the world routine: Leveraging children's multiple mathematical knowledge bases and curiosities in the elementary classroom*.

Albarracin, L., & Gorgorio, N. (2020). Mathematical modeling projects oriented towards social impact as generators of learning opportunities: A case study. *Education Sciences, 8*(11), 1-20. DOI:10.3390/math8112034

Anhalt, C., Cortez, R. & Been Bennett, A. (2018). The emergence of mathematical modeling competencies: An investigation of prospective secondary mathematics teachers. *Mathematical Thinking and Learning International Journal,* 20(3) 1-20. DOI10.1080/10986065.2018.1474532

Arnold, E.G., Burroughs, E.A., Carlson, M.A., Fulton, E.W., & Wickstrom, M.H. (2022) *Becoming a teacher of mathematical modeling: K-Grade 5*. Reston: National Council of Teachers of Mathematics.

Brennan, B. (June 3, 2020). America is facing a 'double pandemic': coronavirus COVID-19 and racism https://www.abc.net.au/news/2020-06-04/america-is-sick-with-a-double-pandemic-coronavirus-and-racisim/12315452

Cirillo, M., Pelesko, J., Felton-Koestler, M., & Rubel, L. (2016). Perspectives on modeling in school mathematics. In C. R. Hirsch & A. R. Roth McDuffie (Eds.), *Mathematical Modeling and Modeling Mathematics* (pp. 3–16). National Council of Teachers of Mathematics.

Cirillo, M., Bartell, T. G., & Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A. Roth Mcduffie (Eds.), *Annual perspectives in mathematics education: Mathematical modeling and modeling with mathematics* (pp. 87-96). National Council of Teachers of Mathematics.

Cooperative Children's Book Center. (2021, April 16). *Books by and/or about Black, Indigenous, and People of Color (all years).* Cooperative Children's Book Center University of Wisconsin-Madison. https://ccbc.education.wisc.edu/literature-resources/ccbc-diversity-statistics/books-by-about-poc-fnn/

Featherstone, H., Crespo, S., Jilk, L.M., Oslund, J., Parks, A., Wood, M. (2011). *Smarter together! Collaboration and equity in the elementary math classroom*. Reston: National Council of Teachers of Mathematics.

Freire, P. (1970). *Pedagogy of the oppressed*. Bloomsbury.

Garfunkel, S. & Montgomery, M., (2019) GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education, Second Edition, COMAP and SIAM, Philadelphia

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In *Equity in discourse for mathematics education* (pp. 17-33). Springer.

Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. Goffney, R. Gutierrez, & M. Boston (Eds.), *Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1-10). National Council of Teachers of Mathematics.

Horn, I. S. (2006). Lessons learned from detracked mathematics departments. *Theory Into Practice*, *45*(1), 72-81.

Jemal, A. (2017). Critical consciousness: A critique and critical analysis of the literature. *The Urban Review*, *49*(4), 602–626.

Ladson-Billings, G. (2021) I'm Here for the Hard Re-Set: Post Pandemic Pedagogy to Preserve Our Culture, *Equity & Excellence in Education, 54*:1, 68-78, DOI: 10.1080/10665684.2020.1863883

Learning for Justice. (n.d.) *Social justice standards.* Learning for justice. https://www.learningforjustice.org/frameworks/social-justice-standards

Lee, C.D., White, G., & Dong, D. (Eds.). (2021). *Educating for civic reasoning and discourse*. National Academy of Education.

Lomax, K., Alfonzo, K., Dietz, S., Kleyman, E., & Kazemi, E. (2017). Trying three-act tasks with primary students. *Teaching Children Mathematics, 24*(2), 112-119.

Mirra, N. (2018). *Educating for empathy: Literacy learning and civic engagement*. Teachers College Press.

Mirra, N. & Puntel, C. (2021). Can Empathy Meet the Challenges of Now? Meditation and Dialogue Voices From the Middle; *Urbana 29*(1), 18-23.

National Research Council. (2001) Adding it Up: Helping Children Learn Mathematics. Washington, DC: The National Academies Press.
Sims-Bishop, R. (1990). Mirrors, windows, and sliding glass doors. Perspectives, 1(3), ix–xi. Retrieved September 17, 2020, from http://www.rif.org/us/literacy-resources/multicultural/mirrors-windows-and-sliding-glass-doors.h
Smith, M. S. & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School,* 3, 344–50.
Suh, J. M., Burke, L., Britton, K., Matson, K., Ferguson, L., Jamieson, S., & Seshaiyer, P. (2018). Every Penny Counts: Promoting Community Engagement to Engage Students in Mathematical Modeling. In R. Gutierrez & Goffney, I. (Eds.), *Annual Perspectives in Mathematics Education:* Rehumanizing Mathematics for Students who are Black, Indigenous, and/or Latin@. (pp. 63-78). Reston, VA: National Council of Teachers of Mathematics.
Suh, J., & Seshaiyer, P. (2013). Informing Practice: Mathematical Practices That Promote Twenty-First Century Skills. *Mathematics Teaching in the Middle School*, *19*(3), 132-137.
Turner, E. E., Bennett, A. B., Granillo, M., Ponnuru, N., Roth Mcduffie, A., Foote, M. Q., Aguirre, J.M. & McVicar, E. (2022). Authenticity of elementary teacher designed and implemented mathematical modeling tasks. *Mathematical Thinking and Learning*, 1-24.
Turner, E.E., Roth McDuffie, A., Aguirre, J.M., Foote, M.Q., Chappelle, C., Bennett, A, Granillo, M., Ponnuru, N. (2021). Upcycling plastic bags to make jump ropes: Elementary students leverage experiences and knowledge as they engage in a relevant community-oriented mathematical modeling task. In J. Suh, Wickstrom M. H., & English L. (Eds.). *Exploring the Nature of Mathematical Modeling in the Early Grades*. Springer.
Turner, E.E., Varley Gutierrez, M. and Díez-Palomar, J. (2011) Latino/a bilingual elementary students pose and investigate problems grounded in community settings. In Moschkovich, J.N, Téllez, T., & Civil, M. (Eds) *Latina/os and Mathematics Education: Research on Learning and Teaching in Classrooms and Communities* United Kingdom: Information Age Publishing, pp. 149–174.
United Nations. (2018). *The 2030 agenda and the sustainable development goals*. Retrieved from https://sdgs.un.org/goals

DISCOURSES OF JUSTICE: CONNECTING VISIONS AND PRACTICES TO IDENTIFY AREAS FOR FUTURE RESEARCH AND TEACHING

Megan Brunner
Oregon State University
brunnerm@oregonstate.edu

This systematic review aims to identify the Discourses (Gee, 2000) invoked regarding justice in PK-12 mathematics education literature, linking visions and practice. The three Discourses of Justice presented in this manuscript draw upon different visions of justice, where the differences arise through the proposed locus of change – the individual (Empowerment), the institution (Transformation), and ideologies around purposes of education (Democracy). However, the Discourses also share similarities across the associated teaching practices for each. There are differences in usage of these Discourses across the literature, which present opportunities for innovations in future research and teaching toward a more just math education system.

Keywords: Social justice; systemic change; instructional vision

Social justice in mathematics education has been a long-standing and evolving conversation among researchers (Gutierrez, 2002, 2013; Ladson-Billings, 1995, 2021; Martin, 2019; Secada, 1994), teacher educators (Aguirre, et al., 2019; Bartell, 2013; Felton-Koestler, 2019; Wager & Stinson, 2012), and teachers and administrators (Gutstein, 2006, 2013; NCSM & TODOS, 2016). Each of these groups of stakeholders has wondered, what does it mean to do social justice work in mathematics education? How does one teach mathematics, prepare teachers, or conduct research in a way that serves social justice goals?

In response, many have raised the need for identifying classroom practices that can support more just mathematics teaching (Aguirre & Zavala, 2013, Bartell, et al., 2017, Garii & Rule, 2009). Others have talked about the need for teachers to develop critical consciousness and reflective capacity to better attune to the dynamics of power and privilege in their classrooms (Esmonde, 2014, Gutstein, 2006, Harper, 2019, Kokka, 2019, Ladson Billings, 2021, Stinson, 2014). As the field engages in these conversations, a variety of terminology is invoked to problematize the state of mathematics education and identify focal points for change. These differing perspectives can lead to challenges in understanding the outcomes of possible instructional innovations; across the literature, different implications for practice may be aligned to a variety of perceptions of what justice entails (Chubbuck & Zembylas, 2008) Researchers, teacher educators, and teachers look to the literature for insight and inspiration on next steps towards justice. Thus, it is important to be able to parse what members of the field perceive as justice and the connected actions they suggest to make progress towards those aims.

Parsing perceptions of justice and their associated practices can occur by examining the framings researchers link together when discussing justice in mathematics education. Some mathematics education researchers have called out the ways different framings operate to communicate the values, beliefs, and perspectives one might hold and how these perceptions shape their actions, such as Louie's (2017) exploration of a culture of exclusion in a mathematics department. In this study, I use the term "vision" to describe the multifaceted set of values, beliefs, goals, and priorities that guide one's exploration of justice. Many researchers claim that teachers' classroom practice is entangled with their visions of justice (Adiredja & Louie, 2020; Chubbuck & Zembylas, 2008; Gutierrez, 2002; Horn, 2007). Consider that the problem perceived by a researcher influences the focus of their study, the data they collect, and the ways they interpret that data; the problem articulated by a teacher educator shapes the conversations they facilitate and the facets of instruction they focus on in their teaching; the purpose teachers perceive for shifting their instruction influences the practices they enact and the ways they interact with other members of their institutions. Making visions of justice and the connected practices and actions explicit provides a foundation to unpack the ways researchers, teacher educators, and teachers work towards the goal of a more just mathematics education. Explicitly articulating a vision of justice is challenging, yet without the acknowledgment of problems of injustice one perceives, the instructional choices one makes can become confused or misaligned with their goals toward justice.

This study is a systematic literature review of K-12 mathematics education research regarding justice. First, I overview foundational literature on justice in K-12 mathematics education. Then, I articulate the theoretical underpinnings of Discourses as a way to connect visions of justice and the actions taken to achieve that vision (Gee, 2000, 2008). I discuss my methods for selecting relevant literature and the coding process I applied, through which I found three different Discourses of Justice in K-12 mathematics education research. I present each Discourse of Justice and detail the ways it is invoked across the research base. Finally, I propose implications for research and teaching based on the current status of Discourses of Justice in the field.

Theoretical Background

The ways that people use language and other forms of communication build on those that come before them, which serves to construct sets of meaning or interpretation. Gee (2000, 2008) identifies these sets of meaning as Discourses, where invoking a Discourse can identify one as a certain type of person. Invoking a Discourse involves not just language (written or spoken) but also "…language, other symbolic expressions, and 'artifacts', of thinking, feeling, believing, valuing, and acting" (Gee, 1996, p. 131). These features are considered discourses, or "stretches of language in action" (Gee, 1996). Regarding the research literature, discourses are the ideas and practices captured in stretches of writing. The ways these discourses are used to make an argument across a manuscript draws upon (and simultaneously adds to) particular sets of meaning or framing (Discourses). Invoking a certain Discourse can help position the researcher towards a certain audience or align their work with other researchers in the field.

Visions of and Practices Towards Justice

Discourses can be recognized through the beliefs, values, and goals one insinuates, and the actions, or practices, one engages in towards those goals (Gee, 2000, 2008). In this study, I consider the beliefs, values, and goals one holds about the future of a more just mathematics education as their "vision of justice" (Hytten & Bettez, 2011; Picower, 2012). A vision of justice can encompass the reasons justice is needed (i.e., the injustice being addressed), the things justice should provide or lead to, and the key features of justice that are seen as necessary and important. Visions of justice refer to the overarching perspectives one holds for what education should look like in a more just world. Visions of justice include macro-level considerations about value systems and perspectives that one considers regarding mathematics education. These values and beliefs are communicated through micro-level interactions as one takes action to achieve their goals (Ryve, 2011). The other component of Discourses is the practices that align with certain beliefs, values, and goals. Practices include the habitual actions of instruction and the pedagogical strategies one might engage in to achieve a teaching goal (Lampert, 2010). Practices may be of a variety of grain sizes, but they are action-oriented and implementable in contextually situated ways. Practices may not be aligned with any one vision of justice. The combination of visions of justice and the practices one sees as serving that vision are what constitutes a Discourse.

Conveying Meaning via D/discourses

D/discourses (Gee, 1996, 2000) coordinate the ways language is used to construct and negotiate meaning, where language involves not just speech and text but also actions, practices, and ways of interacting. Discourses are sustained through the cultural, political, and institutional recognition that happens within interactions as a result of the ways individuals portray and perceive themselves and others (Gee, 2000). That is, as individuals interact with others, they invoke parts of Discourses that give insight into the meanings they are conveying and the ways they want to be perceived.

Discourses may be invoked in a few ways. First, we can *revoice* language stretches associated with a particular Discourse. Revoicing serves to align intended meanings with the histories and prior uses of that phrasing (Bakhtin, 1981). In research, revoicing occurs through citations or quotations to support an argument; in teaching, revoicing may come through discussing certain pedagogical resources or goals an educator holds for a lesson or task. Framing one's work alongside others who use similar language in their research and teaching raises a set of meanings from which to interpret their current argument. Another way Discourses might be invoked is through similar cues of meaning explicated through new stretches of language (e.g., talk and action), otherwise thought of as *refracting* (Volosinov, 1973). Refracting is the reauthoring of a Discourse into a new context. The power of refracting is important to acknowledge, as discourses are enacted within interactions; individuals need to act in a way that draws upon available Discourses but fits the context in a given moment.

Discourses are macro-level frames for sense-making that get recognized in interaction through combinations of practices, behaviors, values, and tools, among other features. Drawing upon certain Discourses can support utilizing specific tools or practices to achieve objectives. Discourses link perceptions of situations to the actions one may take toward an objective; in pursuing a goal of a more just mathematics education system, the Discourses available to an actor will shape the perceptions of justice they hold and the actions they see as necessary to achieve it. Likewise, holding a certain vision of justice may influence the practices that one sees as accessible at a given moment. This study argues that connecting visions of justice and the practices that achieve those visions will help researchers, teacher educators, and teachers more clearly communicate their goals for education regarding justice.

Research Questions

Identifying the Discourses that are available across the field of K-12 mathematics education research and teaching can be challenging - Discourses are culturally recognized and acknowledged, so they can operate implicitly as well as explicitly in one's actions. However, knowing how researchers, teacher educators, and teachers are invoking Discourses to situate their work of transforming the mathematics education system is necessary to understand their scope and intentions. Further, recognizing different framings of justice (against the perception of injustices currently existing in the mathematics education system) is necessary to continue aligning, innovating, and imagining the actions that are possible to take in creating change and a more just system. Thus, this study explores the questions:

1. What are the Discourses of Justice that exist in the literature on justice in PK-12 mathematics education?

2. How does the literature on justice in PK-12 mathematics education invoke Discourses of Justice? What implications does this have for future research and teaching?

Methodology

Systematic literature reviews provide a synthesis of the research base on a topic, to present arguments for new perspectives or provide insight for future research (Petticrew & Roberts, 2008). This systematic literature review serves to synthesize research on justice in K-12 mathematics education and to present an argument for using Discourses of Justice as an organizational and analytical lens in future research and practice.

Data Collection: Identification of Literature

To identify literature relevant to the construction of Discourses of Justice, I set a series of selection criteria. I searched the Education Resources Information Center (ERIC) and Google Scholar to look across general education and mathematics databases for manuscripts published after 2000 regarding "mathematics" AND "justice" to yield 409 and 420 initial results, respectively. Subsequent rounds of criteria were applied to the initial results to narrow the focus and applicability of selected papers to answer the research question (Petticrew & Roberts, 2008; Yolcu, 2019). Manuscripts that did not explicitly discuss mathematics *and* justice in these three areas were excluded. Then, I skimmed the entire body of the remaining manuscripts for definitions of justice and explicit focus on teaching mathematics in grades PK-12. Texts that did not make an explicit theorization or definition of justice were excluded. In this stage, I also excluded manuscripts that did not explicitly connect to PK-12 mathematics education in terms of research focus or participants. Finally, I read each of the remaining manuscripts in full, excluding papers from the same author or group of authors which leveraged the same articulation of justice. In my reading of the data set, I kept notes on additional texts that were regularly cited about social justice research but had not been identified in my original search results; I reviewed each of these texts using the exclusion criteria laid out above, and 5 additional manuscripts passed each stage and were added to the data set. Thus, the data set for this study consisted of 70 total manuscripts focused on PK-12 mathematics education and justice.

Data Analysis: Segmenting and Coding for Discourses of Justice

Once demographic information was identified for all manuscripts in the data set, I segmented each text into sections. Segmenting the manuscripts provides a narrower focus for coding and allows for patterns to arise around how Discourses are invoked throughout manuscripts in the field. The three-section descriptions I used to chunk each manuscript were *Problem Setting*, *Theoretical Framing*, and *Findings & Discussion*. While these sections are aligned with some headers of manuscripts, every paper uses a unique organization for its argument. I only coded the sections of the manuscript that were specific to justice and mathematics education.

I read each manuscript section multiple times and highlighted phrases or sentences that provided answers to each of the analytic questions, color coding for each (Table 1).

Table 1: Analytic Questions for Eliciting Discourses

Analytic Questions	Contribution to Eliciting Discourses
Q1. Why is "justice" important?	articulates the problem needing to be solved
Q2. What will "justice" provide?	evidence of the action that should occur
Q3. Who decides what "justice" is?	identifies the stakeholders and responsible actors

Q4. What are the key elements of "justice"?	clarifies the focus and values of the Discourse
Q5. How is "justice" assessed or achieved?	explains what tools and practices will get utilized
Q6. What implications for "justice" are reported?	reports challenges and insights for tools and practices

Adapted from Churchward & Willis (2019)

As this open coding process was completed for each manuscript, I kept memos to track common themes and patterns I noticed in the ways authors discussed justice (Auerbach & Silverstein, 2006). I regularly re-read all the excerpted text across all coded manuscripts for each analytic question to identify new themes in the data or notice outliers in the framing of justice used across manuscripts. Once all 70 manuscripts were coded and entered into the spreadsheet, I created themes of the Discourses of Justice out of my memo-ed patterns. I then re-read the manuscripts and coded the manuscript sections with the Discourse of Justice descriptions. Instances where the text excerpts from coding did not align with the descriptions of the Discourses led to revision across the themes, until all Discourses were able to be applied to the coded data without outliers (Auerbach & Silverstein, 2006).

The resulting three themes, or Discourses, represent sets of meaning that encompassed answers to all six analytic questions. These Discourses stand apart from one another due to the focus placed on what needs to occur to achieve justice in mathematics education. The Discourse of Justice as Empowerment centers on the empowerment of individuals in pursuing justice; the Discourse of Justice as Transformation focuses on taking action to challenge systems, structures, and policies at an institutional level; and the Discourse of Justice as Democracy identifies a need for ideological change to truly achieve justice in mathematics education.

Findings: Three Discourses of Justice

The three Discourses of Justice (DoJ) presented in this manuscript draw upon different visions of justice, where the differences arise through the proposed locus of change. However, the Discourses also share similarities across the associated teaching practices for each. I discuss trends for how each DoJ is invoked throughout arguments in manuscripts in K-12 mathematics education. I connect each trend to potential reasons for their existence and propose future areas of exploration and innovation for research and teaching.

A Discourse of Justice as Empowerment

The Discourse of Justice as Empowerment (DoJ-E) gains its name based on its focus on the empowerment of individuals via mathematics education. In this frame, justice is achieved through individuals becoming more empowered. this occurs through developing students' (a) mathematical power (e.g. Gutstein, 2003; Frankenstein, 2013; Nicol et al., 2019; Voss & Rickards, 2016); (b) participation, agency, and identity (e.g. Aguirre, et al., 2013; Hand, 2012; Planas & Civil, 2009), (c) humanity and ethical awareness, (e.g. Atweh & Brady, 2009; Boylan, 2009; Nava et al., 2019; Register, et al., 2020) and (4) awareness of the power and role of mathematics in structuring the world (e.g. Brelias, 2015; Felton-Koestler, 2017; Gutstein, 2016).

Patterns of Empowerment. The DoJ-E centers on the improvement of individual learning opportunities and personal growth in awareness and agency. This Discourse was the most commonly invoked of the three, with 100 percent of analyzed papers containing at least one section that referenced this Discourse (Table 2). Only 13 papers did not cite a notion of justice based on the DoJ-E in the problem setting section. Almost every single manuscript analyzed in this study leveraged a theoretical background (64 of 70) or presented findings or implications (68 of 70) that invoked a DoJ-E. Further, this Discourse was consistently invoked throughout manuscripts, with 56 out of 70 papers drawing upon a DoJ-E in every section. This implies that many researchers in mathematics education are envisioning justice as empowerment from their initial conceptualizations of problems for study, through their articulation of justice and the theories that guide their study and return to notions of empowerment in their presentation and interpretations of findings.

Table 2: Discourse of Justice Across Manuscript Sections[1]

	Problem Setting	Theoretical Framing	Findings & Discussion	At Least One Section[2]	All Sections[3]
Empowerment	57	64	68	70	56
Transformation	31	43	32	51	17
Democracy	19	14	17	28	7

[1]n=70 total manuscripts reviewed
[2]number of papers that invoked the Discourse of Justice in at least one section of the manuscript
[3]number of papers that invoked the Discourse of Justice in every section of the manuscript

This focus on individual empowerment draws from the research base, which has a strong focus on identity, participation, and practices as evidence of learning. In 2000, Lerman invited the sociocultural turn in mathematics education, followed by Gutiérrez's (2013) call to take up sociopolitical perspectives in teaching and research. Research from these perspectives regarding teaching practice and social justice naturally involves some focus on impacting individuals. Researchers, teacher educators, and teachers alike must focus on supporting all students' growth as mathematicians and as people in the pursuit of justice. True systemic justice cannot happen without individual empowerment and changes within interpersonal interactions (Gutiérrez, 2009). It is challenging to do social justice work (Gregson, 2013; Gutstein, 2003) and discussing specific practices and principles that impact learning and classroom interactions can seem more manageable (Bartell, et al., 2017), as well as provide more opportunities for evidence of change. However, researchers and practitioners must be able to connect practice to the broader visions for justice and the mechanisms for change they are working through; without this explication, the impact of these practices may not address structural issues past those at the micro-level of classroom interactions.

A Discourse of Justice as Transformation

The Discourse of Justice as Transformation (DoJ-T) resonates closely with ideas from foundational critical theorists and mathematics educators who discuss justice such as Freire (1993) and Gutstein (2003, 2006). Researchers who discuss notions of justice using the DoJ-T focus on taking action to challenge and transform unjust systems, structures, and policies. These researchers move past building awareness of inequities or providing access for success within inequitable systems to discuss actions to shift the systems themselves. This Discourse is invoked as researchers frame goals of social justice teaching towards transforming (a) the norms and practices of the discipline of mathematics (e.g., Felton-Koestler, 2019; Hughes & Laura, 2018; Povey, 2002, (b) structures and policies in schools (e.g., Kokka, 2015; Raygoza, 2020; Rands, 2013), and (c) societal organization and systems (e.g., Gutstein, 2013; Leonard & Moore, 2014; Martin et al., 2010).

Patterns of Transformation. The DoJ-T was regularly evoked across literature on justice in K-12 mathematics education. This Discourse encompassed conceptions of justice that focus on the disruption of unjust systems, structures, and policies – and acting to right those wrongs. Fifty-one manuscripts cited a DoJ-T at least once across possible sections (Table 2). Of these instances, 43 manuscripts invoked a DoJ-T in the Theoretical Framing sections (84%). Some of these manuscripts elaborated a DoJ-T from their initial conceptualization of a problem through to their results or implications (n=17). Of those 43, 10 manuscripts invoke a DoJ-T only in their theoretical framework, and not in any other section of their argument. This pattern shows that DoJ-T is most commonly referenced through explication of theoretical foundations and is less likely to be drawn upon in setting up studies or interpreting findings.

The predominance of DoJ-T in the Theoretical Framework sections of manuscripts, whether in conjunction with other DoJs or on its own, speaks to the power of envisioning structural change in mathematics education. Much of the foundation for this power can be traced back to critical theorists in mathematics education, such as Freire (1993), Frankenstein (1983, 1990), and Gutstein (2003, 2006). These theorists all identify some component of transforming society and schooling in their conceptions of justice. As researchers, teacher educators, and teachers build on these foundational ideas of justice in math education, they are revoicing notions of transformation. The revoicing of DoJ-T consists of the vision of justice and fair systems, as well as the call to action. Less common in this revoicing is a variety of practices and implications of what this call to action might entail. The literature emphasizes the use of real-world, open-ended tasks within which students construct alternative solutions that rectify inequities, which are still challenging to incorporate effectively (Bartell, 2013; Gregson, 2013; Gutstein, 2003, 2016; Harrison, 2015). However, because of the hesitation to turn “justice” into a set of teaching practices – instead choosing to talk about justice as a sliding signifier (Larnell, et al., 2016; Stinson & Wager, 2012) – it is challenging to communicate how to act towards transformation in their environments. Transformation is a lofty vision, and the steps to achieve it are not clearly defined. Future research should support the linkages between micro-level interactions and structural transformations necessary for achieving system-wide justice.

A Discourse of Justice as Democracy

The Discourse of Justice as Democracy (DoJ-D) attends to a broader ideological motivation for pursuing justice. The ideological purpose of general education discussed concerning social justice work is to prepare well-developed and successful persons to sustain society. This vision of justice entails pursuing democratic societies constructed and maintained by civically engaged persons (Apple, 1992). Achieving justice involves the organization of teaching and learning to serve these democratic ideals. In mathematics education scholarship, researchers who discuss justice through a DoJ-D identify ideals of preparing members of society who take up (a) democratic participation and environments (e.g., Frankenstein, 1990; Panthi, et al., 2018; Reagan, et al., 2011) and (b) citizenship and civic engagement (e.g., Bond & Chernoff, 2015; Kokka, 2019; Ndlovu, 2011; Tanase & Lucey, 2017).

Patterns of Democracy. The DoJ-D was the least likely of the three Discourses to be invoked across the research base. The DoJ-D was found in 28 unique manuscripts (Table 2). Papers often leveraged notions of justice as democracy when constructing their problems for research (n=19) or in discussing implications of their arguments (n=17). A further 14 papers mentioned democracy in their theorization of justice. However, only 7 papers invoked a DoJ-D across every section, which implies that 13 of the 28 total articles referencing democracy only utilized a DoJ-D in one section and did not carry this thread consistently across their argument. I see this trend of DoJ-D as evidence that phrases like “democracy,” “citizenship,” or “civic/political engagement” are connected to broader ideologies around the role of education in preparing students to be meaningful participants in society. This ideology is useful to justify the attention to social justice in mathematics, but there is a lack of connection to clear theorizations and aligned practices to achieve it.

The role of democracy and citizenship as central principles of teachers' pedagogy requires critical thinking and analysis, not just the incorporation of specific teaching practices into classrooms. The DoJ-D lacks consistent refraction in manuscripts such that it is challenging to understand how one should work toward goals of democratic justice. Many manuscripts do not as clearly link instructional practices to visions of justice through the lens of DoJ-D (e.g., Turhan Turkkan & Karakus, 2018), or they identify instructional practices that overlap with those suggested in other DoJs. Ladson-Billings (1995) notes that citizenship can be achieved best by the practice of critically analyzing societal injustices, which connects DoJ-D to Transformation. Other researchers echo this call, articulating that TMfSJ practices, which emphasize transformation, are essential to creating a democratic society (e.g., Gutstein, 2003; Reagan, et al., 2011). Register and colleagues (2020) theorize about the role of mathematics literacy in public policy and decision-making for the democratic preservation of society; however, when discussing implications for teaching, these authors recommend practices around identity work and participation, rigorous mathematics, and other forms of empowerment. Other manuscripts simplify their calls for democracy by building democratic participation through group work and complex tasks (Kokka, 2015) or equitable turns of talk (Hung, 2015), similar to strategies aligning with the DoJ-E. Researchers need to explicate connections they see between the DoJs or provide more context to facilitate discussion on how teaching practices might be different in action when attending to different visions of justice.

Conclusion: A Call to Action

This study is a systematic literature view of the ways justice is discussed in research on K-12 mathematics education. It provides a theoretical lens of Discourses (Gee, 2000, 2008) to connect visions of what justice provides with the actions necessary to achieve it. The three DoJs that arose from the analysis of the literature represent different articulations of justice around different mechanisms of change (the individual, the institution, and ideologies, respectively). This manuscript posits that these DoJs are invoked inconsistently across research arguments, with some Discourses more fully connected and articulated than others. The patterns of use led to the identification of areas for future scholarship and exploration. The purpose of establishing this lens is not to prioritize one DoJ over another, nor is it to critique the existing research's current innovations towards constructing a more just mathematics education system. Rather, this perspective provides an opportunity for intentional reflection in scholarship on justice in mathematics education and identifies areas for further exploration and clarification of how to move, together, towards a more just system.

Acknowledgments

Many thanks are extended to Dr. Rebekah Elliott for their ongoing support around the ideas expressed in this paper.

References

Adiredja, A. P., & Louie, N. (2020). Untangling the web of deficit discourses in mathematics education. For the Learning of Mathematics, 40(1), 42–46. https://doi.org/10.1080/07370008.2019.1677664

Aguirre, J. M., Anhalt, C. O., Cortez, R., Turner, E. E., & Simic-Muller, K. (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice: The Flint water task. Mathematics Teacher Educator, 7(2), 7–26.

Aguirre, J. M., & del Rosario Zavala, M. (2013). Making culturally responsive mathematics teaching explicit: A lesson analysis tool. Pedagogies: An International Journal, 8(2), 163–190. https://doi.org/10.1080/1554480X.2013.768518
Atweh, B., & Brady, K. (2009). Socially response-able mathematics education: Implications of an ethical approach. EURASIA Journal of Mathematics, Science & Technology Education, 5(3), 267–276.
Auerbach, C. F., & Silverstein, L. B. (2003). Qualitative data: An introduction to coding and analysis. New York University Press.
Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. Journal for Research in Mathematics Education, 44(1), 129–163. https://doi.org/10.5951/jresematheduc.44.1.0129
Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the Standards for Mathematical Practice. Journal for Research in Mathematics Education, 48(1), 7. https://doi.org/10.5951/jresematheduc.48.1.0007
Bond, G., & Chernoff, E. J. (2015). Mathematics and social justice: A symbiotic pedagogy. Journal of Urban Mathematics Education, 8(1), 24–30.
Boylan, M. (2009). Engaging with issues of emotionality in mathematics teacher education for social justice. Journal of Mathematics Teacher Education, 12(6), 427–443. https://doi.org/10.1007/s10857-009-9117-0
Brelias, A. (2015). Mathematics for what? High school students reflect on mathematics as a tool for social inquiry. Democracy & Education, 23(1), 11.
Chubbuck, S. M., & Zembylas, M. (2008). The emotional ambivalence of socially just teaching: A case study of a novice urban schoolteacher. American Educational Research Journal, 45(2), 274–318. https://doi.org/10.3102/0002831207311586
Churchward, P., & Willis, J. (2019). The pursuit of teacher quality: Identifying some of the multiple discourses of quality that impact the work of teacher educators. Asia-Pacific Journal of Teacher Education, 47(3), 251–264. https://doi.org/10.1080/1359866X.2018.1555792
Esmonde, I. (2014). “Nobody’s rich and nobody’s poor … it sounds good, but it’s actually not”: Affluent students learning mathematics and social justice. Journal of the Learning Sciences, 23(3), 348–391. https://doi.org/10.1080/10508406.2013.847371
Felton-Koestler, M. D. (2017). Mathematics education as sociopolitical: Prospective teachers’ views of the what, who, and how. Journal of Mathematics Teacher Education, 20(1), 49–74. https://doi.org/10.1007/s10857-015-9315-x
Felton-Koestler, M. D. (2019). “Children know more than I think they do”: The evolution of one teacher’s views about equitable mathematics teaching. Journal of Mathematics Teacher Education, 22(2), 153–177. https://doi.org/10.1007/s10857-017-9384-0
Frankenstein, M. (1983). Critical mathematics education: An application of Paulo Freire’s epistemology. The Journal of Education, 165(4), 315–339.
Frankenstein, M. (1990). Incorporating Race, Gender, and Class Issues into a Critical Mathematics Literacy Curriculum. The Journal of Negro Education, 59(3), 336–347. https://doi.org/10.2307/2295568
Freire, P. (1970). Pedagogy of the oppressed (M. B. Ramos, Trans.; 30th anniversary ed). Continuum International Publishing Group.
Garii, B., & Rule, A. C. (2009). Integrating social justice with mathematics and science: An analysis of student teacher lessons. Teaching and Teacher Education: An International Journal of Research and Studies, 25(3), 490–499. https://doi.org/10.1016/j.tate.2008.11.003
Gee, J. P. (2000). Identity as an analytic lens for research in education. Review of Research in Education, 25, 99. https://doi.org/10.2307/1167322
Gee, J. P. (2008). Social linguistics and literacies: Ideology in discourses (3rd ed.). Routledge.
Gregson, S. A. (2013). Negotiating social justice teaching: One full-time teacher’s practice viewed from the trenches. Journal for Research in Mathematics Education, 44(1), 164–198.
Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. Mathematical Thinking and Learning, 4(2–3), 145–187. https://doi.org/10.1207/S15327833MTL04023_4
Gutiérrez, R. (2009). Framing equity: Helping students “play the game” and “change the game.” Teaching for Excellence and Equity in Mathematics, 1(1), 5–7.
Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. Journal for Research in Mathematics Education, 44(1), 37–68.
Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. Journal for

Research in Mathematics Education, 34(1), 37–73. JSTOR. https://doi.org/10.2307/30034699
Gutstein, E. (2006). “The real world as we have seen it”: Latino/a parents’ voices on teaching mathematics for social justice. Mathematical Thinking & Learning: An International Journal, 8(3), 331–358. https://doi.org/10.1207/s15327833mtl0803_7
Gutstein, E. (2013). Whose community is this? Mathematics of neighborhood displacement. Rethinking Schools, 27(3), 11–17.
Gutstein, E. (2016). “Our issues, our people—Math as our weapon”: Critical mathematics in a Chicago neighborhood high school. Journal for Research in Mathematics Education, 47(5), 454–504.
Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. Educational Studies in Mathematics, 80(1–2), 233–247. https://doi.org/10.1007/s10649-012-9387-9
Harper, F. K. (2019). A qualitative metasynthesis of teaching mathematics for social justice in action: Pitfalls and promises of practice. Journal for Research in Mathematics Education, 50(3), 268–310.
Harrison, L. (2015). Teaching social justice through mathematics: A self-study of bridging theory to practice. Middle Grades Review, 1(1), 13.
Horn, I. S. (2007). Fast kids, slow kids, lazy kids: Framing the mismatch problem in mathematics teachers’ conversations. The Journal of the Learning Sciences, 16(1), 37–79.
Hughes, A., & Laura, R. (2018). The contribution of aboriginal epistemologies to mathematics education in Australia: Exploring the silences. Educational Philosophy and Theory, 50(4), 338–348. https://doi.org/10.1080/00131857.2017.1359782
Hung, M. (2015). Talking circles promote equitable discourse. The Mathematics Teacher, 109(4), 256–260. https://doi.org/10.5951/mathteacher.109.4.0256
Hytten, K., & Bettez, S. C. (2011). Understanding education for social justice. The Journal of Educational Foundations, 25(1/2), 7.
Kokka, K. (2015). Addressing dilemmas of social justice mathematics instruction through collaboration of students, educators, and researchers. Educational Considerations, 43(1), 13–21.
Kokka, K. (2019). Healing-informed social justice mathematics: Promoting students’ sociopolitical consciousness and well-being in mathematics class. Urban Education, 54(9), 1179–1209. https://doi.org/10.1177/0042085918806947
Kokka, K. (2020). Social justice pedagogy for whom? Developing privileged students’ critical mathematics consciousness. Urban Review: Issues and Ideas in Public Education, 52(4), 778–803. https://doi.org/10.1007/s11256-020-00578-8
Ladson-Billings, G. (1995). But that’s just good teaching! The case for culturally relevant pedagogy. Theory into Practice, 34(3), 159–165.
Ladson-Billings, G. (2021). Does that count? How mathematics education can support justice-focused anti-racist teaching and learning. Journal of Urban Mathematics Education, 14(1B), 1–5.
Lampert, M. (2010). Learning teaching in, from, and for practice: What do we mean? Journal of Teacher Education, 61(1–2), 21–34. https://doi.org/10.1177/0022487109347321
Larnell, G. V., Bullock, E. C., & Jett, C. C. (2016). Rethinking teaching and learning mathematics for social justice from a critical race perspective. Journal of Education, 196(1), 19–29. https://doi.org/10.1177/002205741619600104
Lerman, S. (2000). The social turn in mathematics education research. In J. Boaler (Ed.), Multiple Perspectives on Mathematics Teaching and Learning (pp. 19–44). Greenwood Publishing Group.
Leonard, J., & Moore, C. M. (2014). Learning to enact social justice pedagogy in mathematics classrooms. Action in Teacher Education, 36(1), 76–95. https://doi.org/10.1080/01626620.2013.861371
Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. Journal for Research in Mathematics Education, 48(5), 488. https://doi.org/10.5951/jresematheduc.48.5.0488
Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. Race, Ethnicity and Education, 22(4), 459–478. https://doi.org/10.1080/13613324.2019.1592833
Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. Journal of Urban Mathematics Education, 3(2), 13.
National Council of Supervisors of Mathematics & TODOS: Mathematics for ALL. (2016). Mathematics education through the lens of social justice: Acknowledgement, actions, and accountability. mathedleadership.org
Nava, I., Park, J., Dockterman, D., Kawasaki, J., Schweig, J., Quartz, K. H., & Martinez, J. F. (2019). Measuring teaching quality of secondary mathematics and science residents: A classroom observation framework. Journal of Teacher Education, 70(2), 139–154. https://doi.org/10.1177/0022487118755699

Ndlovu, M. C. (2011). Re-envisioning the scholarship of engagement: Lessons from a university-school partnership project for mathematics and science teaching. South African Journal of Higher Education, 25(7), 1397–1415.
Nicol, C., Bragg, L. A., Radzimski, V., Yaro, K., Chen, A., & Amoah, E. (2019). Learning to teach the M in/for STEM for social justice. ZDM: The International Journal on Mathematics Education, 51(6), 1005–1016. https://doi.org/10.1007/s11858-019-01065-5
Nolan, K. (2009). Mathematics in and through social justice: Another misunderstood marriage? Journal of Mathematics Teacher Education, 12(3), 205–216. https://doi.org/10.1007/s10857-009-9111-6
Panthi, R. K., Luitel, B. C., & Belbase, S. (2018). Teachers' perception of social justice in mathematics classrooms. REDIMAT - Journal of Research in Mathematics Education, 7(1), 7–37.
Picower, B. (2012). Teacher activism: Enacting a vision for social justice. Equity & Excellence in Education, 45(4), 561–574. https://doi.org/10.1080/10665684.2012.717848
Planas, N., & Civil, M. (2009). Working with mathematics teachers and immigrant students: An empowerment perspective. Journal of Mathematics Teacher Education, 12(6), 391–409. https://doi.org/10.1007/s10857-009-9116-1
Povey, H. (2002). Promoting social justice in and through the mathematics curriculum: Exploring the connections with epistemologies of mathematics. Mathematics Education Research Journal, 14(3), 190–201.
Rands, K. (2013). Supporting transgender and gender-nonconforming youth through teaching mathematics for social justice. Journal of LGBT Youth, 10(1–2), 106–126. https://doi.org/10.1080/19361653.2012.717813
Raygoza, M. C. (2020). Counting the experiences and beliefs of secondary teachers striving to teach mathematics for social justice in urban schools. Urban Education, 55(8–9), 1142–1171. https://doi.org/10.1177/0042085916672289
Reagan, E. M., Pedulla, J. J., Jong, C., Cannady, M., & Cochran-Smith, M. (2011). Measuring practices of teaching for social justice in elementary mathematics classrooms. Educational Research Quarterly, 34(3), 15–39.
Register, J. T., Pugalenthi, P., & Stephan, M. (2020). Designing for ethical reasoning in mathematics [and STEM] education. Electronic Journal for Research in Science & Mathematics Education, 24(2), 1411–157.
Ryve, A. (2011). Discourse research in mathematics education: A critical evaluation of 108 journal articles. Journal for Research in Mathematics Education, 42(2), 167–199.
Stinson, D. W. (2014). Teaching mathematics for social justice: An ethical and moral imperative? Journal of Urban Mathematics Education, 7(2), 1–5.
Tanase, M. F., & Lucey, T. A. (2017). Pre-service teachers' awareness of interdisciplinary connections: Mathematics, financial literacy, and social justice issues. Investigations in Mathematics Learning, 9(1), 2–18. https://doi.org/10.1080/19477503.2016.1245027
Turhan Turkkan, B., & Karakus, M. (2018). The opinions of middle school mathematics teachers on the integration of mathematics courses and social issues. European Journal of Educational Research, 7(2), 397–406.
Voss, R., & Rickards, T. (2016). Using social justice pedagogies to improve student numeracy in secondary school education. Journal of Education and Practice, 7(15), 40–47.
Wager, A. A., & Stinson, D. W. (Eds.). (2012). Teaching mathematics for social justice: Conversations with educators. National Council of Teachers of Mathematics.
Yolcu, A. (2019). Research on equitable mathematics teaching practices: Insights into its divergences and convergences. Review of Education, 7(3), 701–730. https://doi.org/10.1002/rev3.3163

EQUITABLE TEACHING PRACTICES: DEVELOPING EMERGENT BILINGUALS' POSITIVE MATHEMATICAL IDENTITIES

PRÁCTICAS EQUITATIVAS DE ENSEÑANZA: DESARROLLANDO IDENTIDADES MATEMÁTICAS POSITIVAS EN LOS BILINGÜES EMERGENTES

Libni B. Castellón
University of Wyoming
lcastel1@uwyo.edu

Richard Kitchen
University of Wyoming
rkitchen@uwyo.edu

Karla Matute
University of Wyoming
kmatute@uwyo.edu

This paper focuses on Emergent bilinguals (EBs) who traditionally face unequal opportunities to learn mathematics, harming their identities. The purpose of this paper is to illustrate how a fifth-grade teacher cultivated the development of her EBs' mathematical identities by giving them opportunities to participate in cognitively demanding activities. Drawing on a conception of mathematical identity as something that changes in response to different situations, we illustrate how a fifth-grade teacher positively impacted her students' mathematical identities. The results reveal that when teachers use instructional strategies such as distributing mathematical authority, positioning students as mathematically capable, and incorporating students' languages as a resource for instruction, their EBs have multiple opportunities to build positive mathematical identities.

Keywords: Equity, Inclusion, and Diversity, Instructional Activities and Practices, Problem Solving.

Most teachers in the United States have or will have Emergent Bilinguals (EBs) in their classrooms, and few are well prepared to teach them (Samson & Collins, 2012). In this paper, the term EBs is used as equivalent of the term English Language Learners (ELs) because it has a more positive connotation of students language abilities; It acknowledges students proficiency in one language who are becoming proficiency in another language (de Araujo et al., 2018). EBs face unequal opportunities to learn mathematics, reflected in their limited access to advanced mathematical courses and experienced and qualified teachers (Flores, 2007; Samson & Collins, 2012). One approach to address the opportunity gap is to provide EBs with meaningful opportunities to learn mathematics (Moschkovich, 2013). Providing equitable opportunities to learn mathematics for EBs entails increasing their access to teachers who focus on developing their reasoning and use language as a means to enhance instruction (Moschkovich, 2013). An outcome of such instruction is positive mathematical identities for EBs (Bartell et al., 2017).

This paper explores how "Ms. Wilson" implemented equitable teaching practices to support students to develop positive mathematical identities. Equitable teaching practices allow teachers to provide all students with opportunities to participate in cognitively demanding activities that enhance their mathematical identities (Moschkovich, 2013). The equitable teaching practices portrayed in this study were implemented to support every student, particularly those historically marginalized by race, class, ethnicity, culture, and language. The research question we explore here is: How do the opportunities provided to EBs during the implementation of equitable teaching practices help students develop positive mathematical identities?

Equitable Mathematics Teaching Practices

Equitable teaching practices in mathematics are defined as teachers providing all students with opportunities to participate in high cognitively demanding activities that respond to students' needs and enhance their mathematical identities (Moschkovich, 2013). Equitable mathematics teaching practices develop students' conceptual understanding, mathematical reasoning, and mathematical identities through discourse by integrating students' previous knowledge, cultures, and languages (Moschkovich, 2013). Bartell et al. (2017) identified nine equitable mathematics teaching practices: Draw on students' funds of knowledge, establish classroom norms for participation, position students as capable, monitor how students position each other, attend explicitly to race and culture, recognize multiple forms of discourse and language as a resource, press for academic success, attend to students' mathematical thinking, and support the development of a sociopolitical disposition. In this study, we highlight three of these practices as characteristics of Ms. Wilsons' classroom that positively impacted her students' mathematical identities: (1) Positioning students as capable, (2) Recognizing multiple forms of discourse and language as a resource, and (3) Establishing classroom norms for participation.

Teachers' instructional practices determine the degree to which students develop confidence in themselves and see them as mathematical doers (Aguirre et al., 2013). Teachers can support students' development of positive mathematical identities by believing in their students' capabilities, allowing students to use diverse strategies and forms of communication, giving students a voice, and monitoring students' identities development (Allen & Schnell, 2016). Students' positive mathematical identities are developed when their peers or teachers recognize their mathematical contributions (Wood et al., 2019). Unfortunately, EBs are more likely to face low expectations and be stereotyped, harming their mathematical identities (Flores, 2007). Positioning students as mathematical capable is promoted when teachers move away from stereotyping students based on their cultural backgrounds, select challenging curricula, and share mathematical authority with students (Bartell et al., 2017). Classrooms in which teachers hold high expectations for their students result in students having more opportunities to learn challenging mathematics (Flores, 2007). Teachers provide students with opportunities to communicate their ideas, justify their reasoning, make sense of others' ideas, and transfer their knowledge to other contexts and concepts (Banes et al., 2018; Moschkovich, 2013). Moreover, teachers consider students' cultural and linguistic differences as strengths (Bartell et al., 2017).

As part of their instruction, teachers should choose high cognitively demanding tasks and maintain or enhance that demand during classroom implementation. Cognitively demanding tasks have the potential to develop students' conceptual understanding through oral and written explanations that involve pictures, gestures, diagrams, and formal and informal language (Wood et al., 2019). In a classroom where promoting mathematical reasoning is the goal, tasks must provide opportunities for all students to engage in thinking, make comparisons, establish relationships, make conjectures, validate, and justify their thinking (Vale et al., 2017). Teachers of EBs must provide mathematically and linguistic rich tasks that allow all students to collaborate, interact, think, talk, reason, and use their first language (McGraw & Rubinstein-Ávila, 2009). Instruction of EBs needs to incorporate processes such as explaining, comparing, conjecturing, generalizing, justifying, and exemplifying, referred to as the language of reasoning (Bragg et al., 2016). Teachers should use questions, sentence frames, and prompts to support students in communicating their reasoning (Coleman & Goldenberg, 2009). Tasks serve as resources for positioning students as mathematic doers and consequently enhance their mathematical identities.

Equitable teaching practices for EBs support developing mathematical reasoning by implementing mathematical activities that allow students to use multiple resources to understand, solve, and communicate their mathematical ideas (Moschkovich, 2013). Teachers of EBs need to emphasize the development of everyday and academic English language and focus on culturally diverse and inclusive practices (Samson & Collins, 2012). Incorporating students' first language and cultures creates a school philosophy where home and community are integrated, facilitating access to the curriculum, reducing stress, and maintaining self-esteem (Necochea & Cline, 2000). Teachers are encouraged to use students' first languages for support when possible. For example, prior to a lesson, teachers might preview the content for the students in their primary language and identify vocabulary and different representations that may help students in the communication process (Coleman & Goldenberg, 2009). Additionally, translanguaging, identification of cognates (e.g., equation and ecuación), and use of words in context are strategies teachers can use to help students make meaning of new languages (Coleman & Goldenberg, 2009). Translanguating goes beyond language switching and includes all the linguistic resources to communicate based on social and cultural practices (García & Wei, 2014).

In general, students benefit from teachers establishing sociomathematical norms for participation as an equitable teaching practice (Bartell et al., 2017). Sociomathematical norms go beyond regular classroom norms and are specific norms in mathematics that regulate discussion, encourage students to justify their mathematical ideas, make their reasoning clear, and improve their reasoning skills (Yackel & Cobb, 1996). Sociomathematical norms influence the effectiveness of instruction by providing an adequate environment to support student engagement. In particular, establishing sociomathematical norms helps build a community by setting up and guiding discussions so that all students have a voice and their participation is valued (Bartell et al., 2017). As students become familiar with the accepted norms, they understand their role and the role of others in the classroom (Yackel & Cobb, 1996). In equitable classrooms, students have consistent opportunities to share their reasoning and to assess the reasoning of others; they become a mathematical authority in the classroom (Cobb et al., 2009).

Mathematical Identity

Research on mathematical identity considers students' persistence and interest in mathematics, their motivation to learn mathematics (Cobb et al., 2009), and the relationship between learning and cultural and social issues that influence the learning environment (Cobb & Hodge, 2007). Teachers' instructional choices determine the impact that instruction will have on students and the degree to which they develop positive mathematical identities (Aguirre et al., 2013). As part of instruction, teachers should support students to develop confidence and see themselves as powerful doers of mathematics. Teachers can accomplish this goal by grouping students to work collaboratively, establishing high expectations for students, and creating environments where teachers are facilitators of classroom activities (Bartell et al., 2017). Allen and Schnell (2016) share some strategies that teachers can use to support students' development of positive mathematical identities. These strategies include knowing and believing in the students, reconceptualizing mathematical success (e.g., giving value to the use of different solution strategies instead of already defined procedures), prioritizing students' voices, and monitoring identity formation. Teachers should know and believe in their students' capabilities. In short, "viewing student attributes as assets rather than deficits" (Allen & Schnell, 2016, p. 401). When students believe in one another as problem solvers, they are able to value the positive contributions of their peers.

Meanwhile, teachers can use instructional strategies such as representing new ideas in diverse ways, asking purposeful questions, listening to others, revoicing and paraphrasing students' reasoning, discussing an idea before writing it down, establishing connections, and communicating mathematics ideas using different forms of communication (Moschkovich, 2013). Teachers can prioritize student voice by giving students opportunities to discuss their ideas in small groups and then in the whole class. The incorporation of formative assessments can support teachers to assess student's readiness and understanding of the language and the content of the lesson, as well as tools to monitor identity formation (Allen & Schnell, 2016; Hakuta, 2014).

Beyond these strategies, distributing classroom authority and incorporating students' languages and cultures as resources during instruction are equitable teaching practices (Moschkovich, 2015) that support students' burgeoning mathematical identities. Using language as a resource allows EBs to discuss and share their mathematical reasoning and understand mathematics. An important part of such instruction involves teachers creating a learning environment where students' roles and norms for participation are well defined; specifically, students understand that their ideas are important and need to be publicly shared (Yackel & Cobb, 1996). Additionally, teachers need to promote respectful relationships among students and position them as mathematically capable (Bartell et al., 2017).

Theoretical Framework

The theoretical framework for this study is socio-constructivism, which integrates cognitive, constructivist, and socio-cultural theories to make sense of the different teaching practices and examine students' mathematical reasoning, languages, and identities (Shepard, 2000). Cognitive theories provide the means to look at students' thinking structures as they develop their solutions and specific forms of reasoning (in this case, algebraic reasoning). Constructivist theories guide our analysis of social constructions of knowledge through discourse by looking at classroom interactions. Finally, socio-cultural theories inform our interpretation of students' prior knowledge, experiences, and languages in developing their mathematical identities (Cobb & Hodge, 2007).

Research Methodology

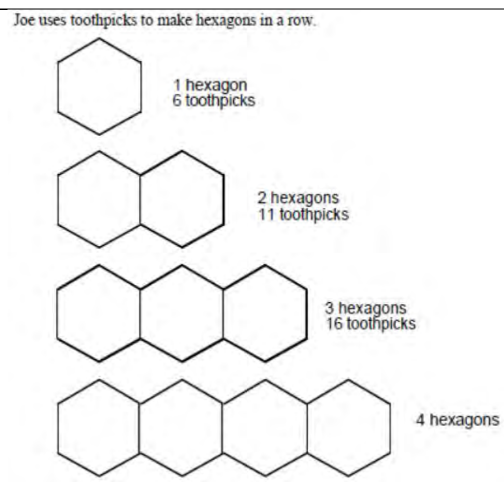
During the 2019-2020 academic year, we gathered detailed evidence of a teacher's and students' experiences during problem-solving lessons (Creswell & Poth, 2018). An explorative case study helped us identify the research question and the procedures to refine our intervention (Yin, 2014). A fifth-grade teacher, "Ms. Wilson" who worked in an urban elementary school, and our research team, collaborated three times to develop the "Discursive Mathematics Protocol," an instructional protocol designed to support the development of students' mathematical reasoning and language competencies (see Kitchen et al., 2020, for additional information about the DMP). The DMP builds on Pólya's (1945/1986) iconic problem-solving heuristic and incorporates research-based "language practices" and essential teaching practices borrowed from the *Principles to actions: Ensuring mathematical success for all* (NCTM, 2014).

This paper examines how Ms. Wilson's implementation of the equitable teaching practices with the aid of the DMP contributed to developing her students' mathematical identities. Ms. Wilson is a bilingual (English-Spanish) White woman certified as an English as a Second Language (ESL) teacher. When this study was undertaken, Ms. Wilson had taught for five years, attending a significant number of EBs because of her ESL expertise. For each problem-solving lesson, the research team engaged Ms. Wilson in a planning session conducted via Zoom prior to the lesson and in a debriefing session after the lesson. The focus of the planning lessons was on the language and cognitive demand of the task, possible strategies and challenges, and the formulation of questions to elicit students' thinking and participation. The debriefing sessions serve as a space for reflection and improvement. The teacher and the team discuss what went well in the class and what can be improved to support students' participation and development of mathematical reasoning and language. During the problem-solving lessons, the three research team members and Ms. Wilson followed a co-teaching approach (Cook & Friend, 1995) to teach the lesson (roles and amount of teaching vary across lessons). Team teaching served as a strategy to directly support Ms. Wilson to plan, deliver, and reflect on the problem-solving lessons.

We gathered information to gain insights about the equitable teaching practices and how the different opportunities provided to EBs supported them to develop positive mathematical identities. We collected data from two problem-solving lessons that included students' work, videotapes of interactions among students and Ms. Wilson and just the students, and videotapes of Ms. Wilson carrying out each lesson. In addition, we interviewed Ms. Wilson to learn directly from her about the teaching practices she implemented to support EBs in developing positive mathematical identities.

We interpreted videotapes, student work samples, and the interview conducted with Ms. Wilson using interpretative methods (Creswell & Poth, 2018). The emergent themes summarize Ms. Wilson's beliefs and teaching practices during problem-solving lessons that supported students' development of positive mathematical identities. In the findings, we discuss each indicator by examining Ms. Wilson's narratives, her teaching practices, and her students' work.

For the problem-solving lessons illustrated here, the "Hexagons in a Row" task (See Figure 1) was implemented. In the first, second, and fourth questions, students start by finding near generalizations (patterns that can be found by drawing or making a short table) and then directly apply the derived functional relationship to find a far generalization (patterns that require to find arithmetic or algebraic expressions) (Callejo & Zapatera, 2017). In questions 3 and 5, students apply the inverse of the functional relationship derived (reverse thinking) (Callejo & Zapatera, 2017) to find the number of hexagons made with a given number of toothpicks.



1. How many toothpicks does Joe need to make 5 hexagons? Explain how you figured it out.
2. How many toothpicks does Joe need to make 12 hexagons? Explain how you figured it out.
3. Joe has 76 toothpicks. How many hexagons in a row can he make? Explain how you figured it out.

Extension questions

4. How many toothpicks does Joe need to make 100 hexagons? Explain how you figured it out.
5. Joe has 1001 toothpicks. How many hexagons in a row can he make? Explain how you figured it out.

Adapted from Mathematical Assessment Resources Service
https://www.insidemathematics.org/sites/default/files/materials/hexagons%20in%20a%20row.pdf

Figure 1: Hexagons in a Row task

Research Findings

We identified three major themes related to equitable teaching practices enacted in Ms. Wilson's classroom during the mathematics lesson to support EBs' development of positive mathematical identities: Sharing mathematical authority, positioning students as mathematically capable, and incorporating students' languages as resources for instruction. Table 1 summarizes the different indicators we identified in the data. All the vignettes are instances of EBs' oral participation in the class or written work.

Table 1: Themes Derived for Mathematical Identity

Theme/	Indicator	Practice code
Sharing mathematical authority	Students had consistent opportunities to develop and articulate their solutions to tasks and assess the solutions of others (Cobb et al., 2009)	Exchange and assessment of ideas
	Students acknowledge the importance of making their ideas public shared (Bartell et al., 2017).	Making ideas public
Positioning Students as mathematically capable	Teachers believe their students are mathematically capable (Bartell et al., 2017).	Teacher's beliefs
	Choose and implement high cognitive demand tasks.	Tasks
Incorporating students' languages as resources	Incorporate of students' languages as resources during instruction (Moschkovich, 2015)	Language
	Review vocabulary and develop mathematical language (Banes et al., 2018).	Vocabulary
	Engage students in mathematical discourse (Banes et al., 2018).	Discourse

Ms. Wilson Shared her Mathematical Authority. She distributed the authority in the classroom, letting students share their ideas and assess the ideas of others. She engaged students in productive discussions that included asking purposeful questions, using and connecting multiple representations, interpreting diagrams, and discussing ideas in small groups and whole-class discussions. Throughout, Ms. Wilson encouraged students to build arguments and to reject or accept the strategies or solutions presented in the class. For example, three students shared the expressions they use to calculate the number of toothpicks to make 12 hexagons. Students discussed the similarities, differences, and validity of those expressions to answer the problem. Most of the students could indicate what each number in the expressions represented (number of hexagons, number of toothpicks to make one hexagon, and number of touching sides) and applied those expressions to respond to questions in the task. Laura described the three different expressions provided for her classmates giving the arguments for their validation (expression 1: 12 x 5 + 1, expression 2: 6 x 12 -11, expression 3: 5 x 11 + 6)

> Me and Sandra said that for the first one, 12 times five plus one; Ryan was looking at it like they were all five [the number of sides in a hexagon]. And then he saw that one of them has six sides while the others have five, so he had to add that one to show that one is six [has six sides]. The second one, six times 12 minus 11, shows that it was easier for them to think of them [the sides] as all six and then because only one of them was six and there was 12 hexagons, they needed to subtract 11 because one still has six [sides]. [For] five times 11 plus six; we thought of it as the two different groups, the groups of the fives and the one six, and so it just depends on how you group them.

Laura's explanation showed a clear understanding of the different strategies. She acknowledged her peers' ideas and tried to make sense of those ideas. Students presented their solutions, while

Ms. Wilson asked them to indicate the meaning of each term in their expressions. We can see that through this exchange, Laura was able to summarize the different strategies indicating how many sides were counted, how the hexagons were grouped, and why some expressions included addition and others subtraction. Laura was able to connect multiple ideas and restate others' thinking. In Ms. Wilson's class, students were constantly asked to make their ideas public and assess others' ideas to engage them in learning mathematics and as a means to demonstrate that their ideas were important contributions. Students in Ms. Wilson's classroom valued all the mathematical ideas presented to solve the task. Ms. Wilson let students decide when a strategy or solution was wrong.

Positioning Students as Mathematically Capable. Ms. Wilson's high expectations were reflected in her students' positive attitudes and participation in the class. She believed in students' capacities to solve challenging tasks. In the following excerpt, Ms. Wilson talked about the hexagons in a row task and the possibilities it offers to the students.

> … when I was thinking about [choosing a task] I was trying to think more in line with the content that I was doing, but I think this one was broader, and so it allowed for more thinking and problem-solving, which is something I would like to be focusing on more…how does this [content of the task] connect to a bigger mathematical idea or what can you take away from this and use later because I feel like there's so much good in math and the hexagons task that maybe some of them [the students] will say I can look for these different patterns next time.

When implementing cognitively demanding tasks, Ms. Wilson was challenged by some of her students who had negative attitudes toward mathematics or lower expectations for themselves. Some of her EBs were not confident about their abilities. We asked her how she dealt with EBs when they did not participate in the class and how she helped them realize that they could solve the task.

> Students like Samuel [an EB student], he definitely wants teacher direction first, even if he does understand. He wants to like to clarify everything with me. So I'm working on it with him of like, we can talk through it, but I'm not going to tell you what to do. And same with today, he really likes to just know that he's doing the right thing. And so, for them, it's just like building confidence.

Building students' confidence increased their participation and helped them develop positive mathematical identities. Ms. Wilson created a learning environment that empowered her EBs. She constantly compared their work to the work of mathematicians. At the conclusion of the problem-solving lesson, she stated the following to the whole class:

> What does complete [a task] look like? It is having an answer for each question, showing your work for each question, and then explaining your thinking. Each place [questions in the task] has explain how you figure it out. You will not be able to communicate with them [the researchers] after this task is over, right? So, you have to put it in writing. Great mathematicians put their ideas into writing.

Ms. Wilson had high expectations for her students and prioritized their voices. Students talked about mathematics, compared their solutions, tried other strategies, and assessed their thinking and other students' thinking.

Incorporating Students' Languages as Resources in Instruction. Ms. Wilson recognized and valued students' backgrounds as a resource to learn mathematics (Aguirre et al., 2013). Her acknowledgment of students' languages helped her EBs develop positive mathematical identities. As a whole class, they discussed the meaning of some words, used the words in context, and put the problem in their own words. Mathematical discourse played an important role in supporting students to understand the task and share their ideas using multiple forms of communication. Students in Ms. Wilson's class mainly increased their vocabulary by talking. Gestures were widely used to describe mathematical patterns during which students pointed at different parts of the diagram or showed how to count the toothpicks and the shared sides. Gestures combined with speech helped students assess their thinking by noticing disagreement between the description provided and the elements pointed to in the figure. One EB, Samuel, explained that there were four shared sides in four hexagons in a row. He used the diagram as an aid to explain his understanding of the task. When counting and pointing to the shared sides, Samuel realized that there were just three of them. The use of gestures supported him and other students to communicate their ideas. Ana, another EB, used her pencil and the figure representing four hexagons to explain how she found the number of toothpicks needed to make five hexagons:

> What I did was I got 6 as like a whole, so there is 6 right there [circle the first hexagon and write down 6], and then there's 1, 2, 3, 4, 5 [she outlined the second hexagon using the pencil] and then there's 5 [counting the third hexagon], and then there's 5 here and then is 5, 5, 5, and since there are five hexagons is another hexagon right there [sketching the fifth hexagon] so then 5 times 4, because is 4 right then plus 6 because is the hexagon here [She pointed out the first hexagon in figure 4] and then 26 which is the answer.

Consistently communicating her ideas and combining linguistic resources (gestures, diagrams, language) helped Ana and other students to assess their solutions and become more confident about their ideas.

Discussions and implications

In this study, we explored how the implementation of equitable teaching practices provided EBs with opportunities to develop positive mathematical identities. Ms. Wilson's implementation of equitable teaching practices, such as opportunities to solve cognitively demanding tasks, use multiple forms of discourse and language, position students as mathematically capable, and share mathematical authority, helped her students develop positive mathematical identities. She valued the ideas of her students and held high expectations. The excerpts show that representations, discourse, gestures, and language served as resources for students to communicate their reasoning, increasing their opportunities to participate in class to share and assess their ideas (Banes et al., 2018; Cobb et al., 2009). Students like Laura recognized multiple solution strategies and were able to justify the validity, similarities, and differences between them. Students in Ms. Wilson's class had the authority to evaluate others' thinking and were positioned as mathematical capable. Their linguistic approaches were seen as strengths instead of deficiencies contributing to the development of positive mathematical identities (Bartell et al., 2017).

Incorporating equitable teaching practices in the mathematics classroom benefits all students, increasing participation and enhancing students' mathematical identities (Cobb et al., 2009; Yolcu, 2019). This study demonstrates that incorporating equitable teaching practices is beneficial for EBs to develop positive mathematical identities. Specifically, our analysis of equitable teaching practices shows the benefits of providing opportunities for EBs to simultaneously develop mathematical reasoning and language competencies.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). The Impact of Identity in K-8 Mathematics Learning and Teaching: Rethinking Equity-Based Practices. In *The Impact of Identity in K-8 Mathematics Learning and Teaching: Rethinking Equity-Based Practices*. The National Council of Teachers of Mathematics, Inc. https://pubs.nctm.org/view/book/9780873538565/9780873538565.xml

Allen, K., & Schnell, K. (2016). Developing mathematics identity. *Mathematics Teaching in the Middle School*, *21*(7), 398–405. https://doi.org/10.5951/mathteacmiddscho.21.7.0398

Banes, L. C., Ambrose, R. C., Bayley, R., Restani, R. M., & Martin, H. A. (2018). Mathematical classroom discussion as an equitable practice: Effects on elementary English learners' performance. *Journal of Language, Identity & Education*, *17*(6), 416–433. https://doi.org/10.1080/15348458.2018.1500464

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a Framework for Research Linking Equitable Teaching With the Standards for Mathematical Practice. *Journal for Research in Mathematics Education*, *48*(1), 7–21. https://doi.org/10.5951/jresematheduc.48.1.0007

Bragg, L. A., Herbert, S., Loong, E. Y., Vale, C., & Widjaja, W. (2016). Primary teachers notice the impact of language on children's mathematical reasoning. *Mathematics Education Research Journal; Dordrecht*, *28*(4), 523–544. http://dx.doi.org/10.1007/s13394-016-0178-y

Callejo, M. L., & Zapatera, A. (2017). Prospective primary teachers' noticing of students' understanding of pattern generalization. *Journal of Mathematics Teacher Education*, *20*(4), 309–333. https://doi.org/10.1007/s10857-016-9343-1

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An Interpretive Scheme for Analyzing the Identities That Students Develop in Mathematics Classrooms. *Journal for Research in Mathematics Education*, *40*(1), 40–68.

Cobb, P., & Hodge, L. L. (2007). Culture, identity, and equity in the mathematics classroom. In N. S. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 159–172). Teachers College Press.

Coleman, R., & Goldenberg, C. (2009). What does research say about effective practices for English learners? *Kappa Delta Pi Record*, *46*(1), 10–16. https://doi.org/10.1080/00228958.2009.10516683

Cook, L., & Friend, M. (1995). Co-Teaching: Guidelines for Creating Effective Practices. *Focus on Exceptional Children*, *28*. https://doi.org/10.17161/fec.v28i3.6852

Creswell, J. W., & Poth, C. N. (2018). *Qualitative inquiry and research design choosing among five approaches* (4th ed.). SAGE publications, Inc.

de Araujo, Z., Roberts, S. A., Willey, C., & Zahner, W. (2018). English learners in K–12 mathematics education: A review of the literature. *Review of Educational Research*, *88*(6), 879–919. https://doi.org/10.3102/0034654318798093

Flores, A. (2007). Examining disparities in mathematics education: Achievement gap or opportunity gap? *The High School Journal*, *91*(1), 29–42.

García, O., & Wei, L. (2014). Translanguaging and Education. In O. García & L. Wei (Eds.), *Translanguaging: Language, Bilingualism and Education* (pp. 63–77). Palgrave Macmillan UK. https://doi.org/10.1057/9781137385765_5

Hakuta, K. (2014). Assessment of Content and Language in Light of the New Standards: Challenges and Opportunities for English Language Learners. *The Journal of Negro Education; Washington, 83*(4), 433–441. https://doi.org/10.7709/jnegroeducation.83.4.0433

Kitchen, R., Castellón, L. B., & Matute, K. (2020). Supporting English learner students' development of the mathematics register through an instructional intervention. *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, 1993–2000. https://doi.org/10.51272/pmena.42.2020-335

McGraw, R., & Rubinstein-Ávila, E. (2009). Middle school immigrant students developing mathematical reasoning in Spanish and English. *Bilingual Research Journal*, *31*(1–2), 147–173. https://doi.org/10.1080/15235880802640656

Moschkovich, J. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education*, *6*(1), 45–57.

Moschkovich, J. (2015). Academic literacy in mathematics for English learners. *The Journal of Mathematical Behavior*, *40*, 43–62. https://doi.org/10.1016/j.jmathb.2015.01.005

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Author.

Necochea, J., & Cline, Z. (2000). Effective Educational Practices for English Language Learners within Mainstream Settings. *Race Ethnicity and Education*, *3*(3), 317–332. https://doi.org/10.1080/713693040
Pólya, G. (1945/1986). *How to solve it: A new aspect of mathematical method* (J. H. Conway, Ed.; Expanded Princeton Science Library ed). Princeton University Press.
Samson, J. F., & Collins, B. A. (2012). *Preparing all teachers to meet the needs of English language learners*. 29.
Shepard, L. A. (2000). The role of assessment in a learning culture. *Educational Researcher*, *29*(7), 4–14. https://doi.org/10.2307/1176145
Vale, C., Bragg, L. A., Widjaja, W., Herbert, S., & Loong, E. Y.-K. (2017). Children's mathematical reasoning: Opportunities for developing understanding and creative thinking. *Australian Primary Mathematics Classroom*, *22*(1), 3-. Gale In Context: Science.
Wood, M. B., Sheldon, J., Felton-Koestler, M. D., Oslund, J., Parks, A. N., Crespo, S., & Featherstone, H. (2019). 8 Teaching Moves Supporting Equitable Participation. *Teaching Children Mathematics*, *25*(4), 218–223. https://doi.org/10.5951/teacchilmath.25.4.0218
Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, *27*(4), 458–477. JSTOR. https://doi.org/10.2307/749877
Yin, R. K. (2014). *Case study research: Design and methods* (5 edition). SAGE.
Yolcu, A. (2019). Research on equitable mathematics teaching practices: Insights into its divergences and convergences. *Review of Education*, *7*(3), 701–730. https://doi.org/10.1002/rev3.3163
Zahner, W., Velazquez, G., Moschkovich, J., Vahey, P., & Lara-Meloy, T. (2012). Mathematics teaching practices with technology that support conceptual understanding for Latino/a students. *The Journal of Mathematical Behavior*, *31*, 431–446. https://doi.org/10.1016/j.jmathb.2012.06.002

ELICITING YOUTH DIGITAL MATHEMATICS STORIES: THE IMPACT OF A DIGITAL MATHEMATICS STORYTELLING SUMMER CAMP EXPERIENCE

Theodore Chao
The Ohio State University
chao.160@osu.edu

Melissa Adams-Corral
California State University, Stanislaus
madamscorral@csustan.edu

Ayse Ozturk
The Ohio State University
ozturk.25@osu.edu

Ho-Chieh Lin
The Ohio State University
lin.2532@osu.edu

Angga Hidayat
The Ohio State University
hidayat.8@osu.edu

This research report analyzes the process of engaging 18 youth in urban emergent communities to enact Digital Mathematics Storytelling to explore their mathematics identities. The youth, in grades 7-11, engaged in the process of crafting and sharing their digital mathematics stories within two week long summer camps. Using a Participant and (Re)design Research Methodology, the research team explored how the constructs of Digital Storytelling, Mathematics Identity, and Storytelling can help us better know how to craft experiences that connect to youth knowledge.

Keywords: Equity, Inclusion, and Diversity; Design Experiments; Informal Education; Technology

Objectives

During the COVID-19 pandemic, many urban emergent school districts[1] in the United States of America, already dealing with racial and socioeconomic segregation, were overwhelmed with how to support the health and safety of their students and communities, while also responsible for their students' academic learning. Because of this context, research projects based within school settings had to either switch to a completely remote model or completely disconnect from the school context.

Our research team delt with the dissonance of our urban emergent partner schools having to engage in a much more stressful version of teaching by enacting a community-based approach, creating our own external summer camp experiences to invite youth to work with and co-create ways to enact Digital Mathematics Storytelling together with us without having to burden school and teaching partners. In particular, our summer camps hoped to create community-oriented spaces for thriving mathematics conversations, particularly for youth from urban emergent communities, after a year of remote learning that isolated many young people from their school communities. While our other work focuses on the actual stories and learning that young people enacted when engaging in Digital Mathematics Storytelling, this paper focuses on the ways our research team were able to plan, augment, and enact two week-long summer camps to best serve and learn from 18 youth, aged 12-17, in the wake of the COVID-19 pandemic. The objectives of this research study involved: (1) developing a protocol for Digital Mathematics Storytelling through Participatory (Re)design Research and (2) understanding how youth might see the construct of storytelling as connected to their own mathematics identities.

[1] The term urban emergent signifies communities in large cities, but not as large as metropolitan areas such as New York or Los Angeles. These urban emergent communities, however, encounter the same scarcity of resources and historical issues of segregation (Milner, 2012).

Theoretical Perspectives

Non-white children in urban emergent communities are often positioned in ways that do not allow them to craft their own mathematics identity. In particular, while students in urban emergent communities live mathematically rich lives, they are provided few opportunities to connect their out-of-school mathematical knowledge to in-school mathematics tasks that are often based on white, middle-class lifestyles they cannot relate to (Civil, 2009; Nasir & de Royston, 2013). For instance, when a student shares about the intricate proportional reasoning she witnesses in the kitchen as her mother and aunts prepare multiple dishes in unison, she positions them not only as holders of culinary knowledge, but also of mathematics knowledge. However, scaling up this approach of connecting out-of-school mathematics knowledge to in-school mathematics learning has been difficult because (1) family and community mathematics often operates socially, involving storytelling and group problem solving, while school mathematics is often positioned as internal knowledge measured individually (D'Ambrosio, 1985; González et al., 2001; Powell & Frankenstein, 1997), and (2) exploring students' home and community mathematics knowledges asks for even more labor from already overworked teachers.

Community Funds of Knowledge

One approach to address this issue of connecting out-of-school mathematics to in-school mathematics is through *funds of knowledge*, which honors families, communities, and the knowledge they bring to classroom mathematics. This community-based approach to mathematics teaching recognizes that "the historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual's functioning and well-being" are mathematically rich resources (Moll et al., 1992, p. 133). A child's motivation to learn mathematics, thereby increases based on the increase in connections they make between two types of mathematics: (1) the mathematics they see in their families and communities and (2) the mathematics they experience in the classroom (Aguirre et al., 2013; Civil, 2016; González et al., 2001).

Within most urban and urban emergent communities, mathematics knowledge exists within social discourses such as storytelling (Turner et al., 2009). However, in the mathematics classroom, mathematics is treated and assessed as individual knowledge with little social interaction (González et al., 2001). For instance, Turner et al. (2012) detailed how well-meaning elementary teachers who were not from a Latinx community created a contrived mathematics activity around buying and selling *raspados* (shaved ices) at a community stand, but they did not ask children nor community members to share how mathematics conversations actually take place at the *raspado* stand. Surface-level attempts like this create harm by tokenizing children's cultures and imposing foreign values. In this case, the teachers imposed a value of capitalistic efficiency–getting the most *raspados* for your money, rather than connecting to community values (Bright, 2016).

To be successful, a community funds of knowledge approach to mathematics teaching must invite family and community members into the learning experience (Moll et al., 1992). These experiences help mathematics educators become part of the communities they serve, helping them understand how mathematics knowledge exists outside of Eurocentric ways of knowing (D'Ambrosio, 1985; Powell & Frankenstein, 1997), how mathematical thinking is situated within local contexts (Lave, 1988), and how to merge the classroom and community spaces to form a third space that honors all practices, knowledges, and beliefs (Gutiérrez et al., 1999). However, reaching this level of awareness is particularly difficult as it essentially asks teachers to become ethnographic researchers (Oughton, 2010). This approach also has a tendency to prioritize teachers' cultural capital, since the teacher is often involved in translating the community knowledge for the classroom (Oughton, 2010; Turner & Drake, 2016).

Digital Storytelling

Another approach to connecting out-of-school mathematics to in-school mathematics involves the ancient art of storytelling. When students in urban emergent communities tell narratives about their out-of-school mathematical experiences, they position themselves and their communities as mathematical. They tap into the power of authorship to counter stigmas that discourage them from engaging in mathematics (Aguirre et al., 2013; Love, 2014; Özpinar et al., 2017). In a world where video and image-sharing platforms, such as Instagram and TikTok allow for routine sharing of personal stories (Rideout, 2017; The Associated Press-NORC Center for Public Affairs Research, 2017), a strong case exists for utilizing these emerging digital literacies. Therefore, this study's aim involved developing and exploring how Digital Mathematics Storytelling, a mechanism in which students use videos, photographs, and audio to craft and share mathematically-rich narratives from their families and communities, might help connect out-of-school mathematics to in-school mathematics in a non-school-based setting.

Digital storytelling involves using images, voice, and music to tell a story in the form of a short one to five-minute videos (Lambert, 2013). Storytelling is the primary mechanism for sharing important knowledge between generations in nearly all cultures–the very act of storytelling can define a community, a history, and a shared knowledge base (Prusak et al., 2012). Furthermore, children from marginalized communities often tell counter-stories that push against deficit-focused stereotypes, using these stories to explore and strengthen their identities and connect more deeply to their communities (Delgado, 1989; Duncan, 2006; Matias & Grosland, 2016).

The transformative moment in a digital storytelling cycle happens within the listening-focused small group space called the *storycircle*, when storytellers share their stories-in-progress to each other (Lambert, 2013). This storycircle forges a collaborative learning space where participants see the value of their cultural and community identity (de Jager et al., 2017). It is within the sharing of and retelling of stories that mathematics identities are formed as children begin to see themselves as part of their mathematics narrative (Aguirre et al., 2013; Chao, 2014; Langer-Osuna, 2015; Sfard & Prusak, 2005). Often, structured story prompts or visual organizers can help children manage the complexity in their stories (Ohler, 2006), so that children do not simply document what is happening, but craft a narrative that involves conflict, growth, and change in order to strengthen their ties to the community. Simply put, Digital Mathematics Storytelling is not just documenting what one sees, but involves crafting a narrative that centers one's humanity, growth, conflict, and soul.

Figure 1 shows our initial framework of a digital mathematics storytelling experience, which features *two* storycircles, the first involving self-reflection and the second involving sharing drafts for feedback from peers and adults. In this example, a 4th-grade child initially wants to tell a story about how she uses mathematics in the kitchen, a classic connection to home mathematics. In her first storycircle, as she thinks about how to tell her story and realizes that she does not see much measuring and adherence to recipes at home, but rather sees her mother and aunt cooking through approximation and tasting. In her second storycircle, she shares her emerging narrative with peers and uses their feedback to assemble video and audio clips to create a finished video to share at a culminating community screening. This research paper troubles our initial Digital Mathematics Storytelling framework by exploring the research question: *During* *the COVID-19 pandemic, when engaging in classroom-based research became too burdensome for teachers in urban emergent communities, what happens when enacting and building a Digital Mathematics Storytelling workshop for youth, particularly in connection to the construct of mathematics identity?*

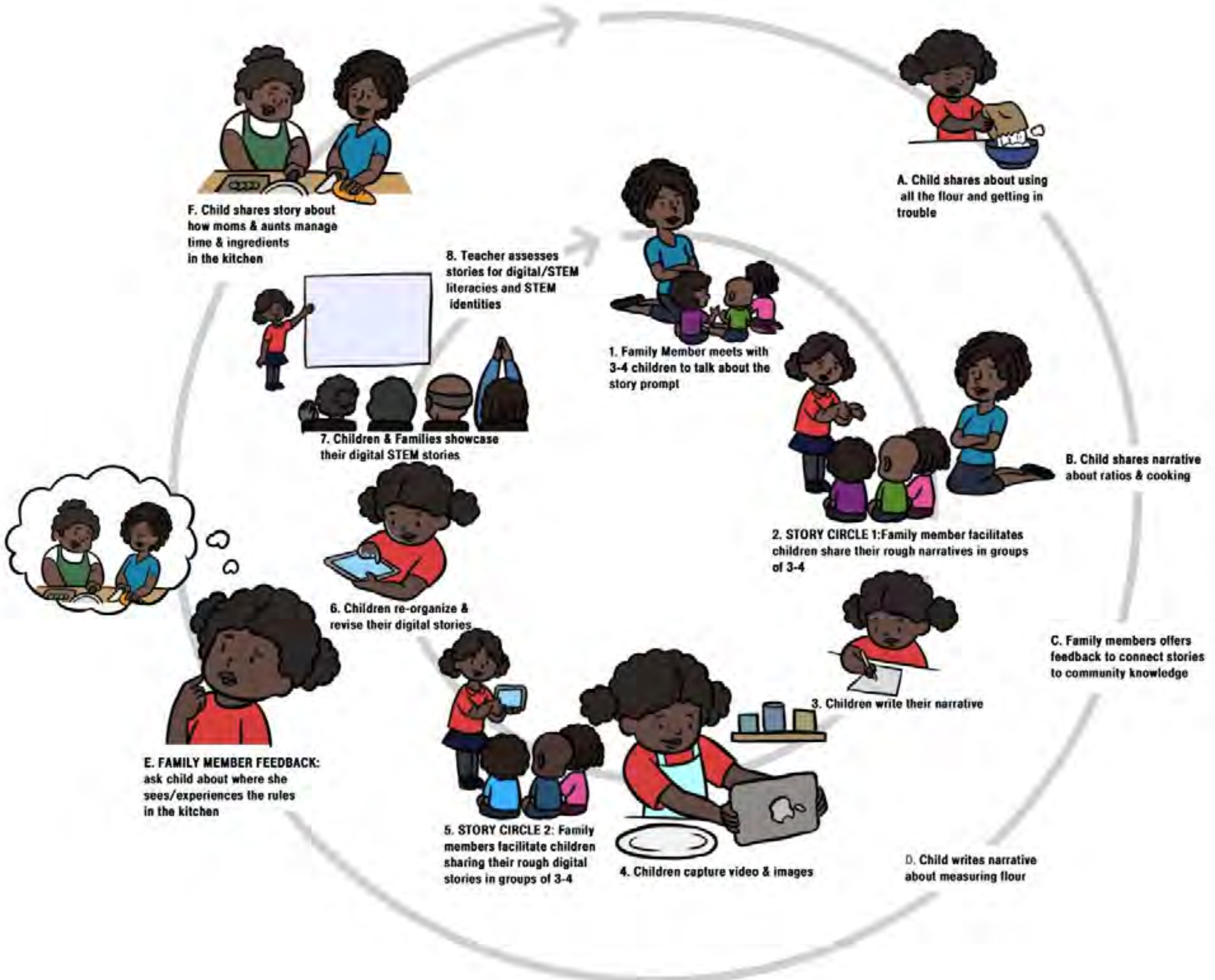


Figure 1. The Initial Storycircle Framework, which shows how a child's mathematics story evolves through two storycircles of reflection, sharing, and revision.

Research Design and Methodology

The research team enacted two week-long digital mathematics storytelling camps involving 18 students, aged 12-17, in grades 7-11. The students came from a variety of school districts and ethnic backgrounds and were recruited heavily from the researchers' own networks. The camps were hosted at a university-owned community innovation and outreach center in the Midwestern region of the United States of America. Because the camp was hosted in a facility operated by this university, all participants had to show proof of COVID-19 vaccination, which potentially influenced the pool of recruited participants.

Participatory and (Re)design Research

Researching technology that values participants' voices invites a Participatory Design Research methodology, which builds upon design-based research, but positions all participants as integral to designing the research goals and aims (Bang & Vossoughi, 2016). One of the main critiques, however, of design-based research (Cobb et al., 2003; The Design-Based Research Collective, 2003) is that, while very suitable for technology-driven educational research, the researchers still hold the authority to make research decisions (Engeström, 2011). Therefore, we attempted to enact a modified (Re)design Research method, to allow participants to not just engage in the enactment, but also help us with the very design of the Digital Mathematics Storytelling research framework itself.

By utilizing feedback from students who participate in the Digital Mathematics Storytelling camp, we used this developmental study to focus on developing protocols of how to implement Digital Mathematics Storytelling for middle and high-school aged students in urban emergent communities. Additionally, we explored the ways that Digital Mathematics Storytelling impacted and allowed insight into students' evolving mathematics and storytelling identities.

Research Measures

Pre and Post-Interviews. Over the course of the first two days of the Digital Mathematics Storytelling camp, a member of the research team met with each student to talk about the student's identities around mathematics and digital storytelling in a 5-to-10-minute interview. Then, on the last day of the camp, each student engaged in a similar 5-to-10-minute post-intervention interview with a research team member. These interviews also asked students about what they would like to change about the camp, which allowed us as facilitators to alter our daily structures to respond to what the students felt could better help them become Digital Mathematics Storytellers..

Analysis of Digital Mathematics Stories. The final Digital Mathematics Stories that the students submitted as their "final project", as well as the digital artifacts (videos, images, photographs, text comments) that the students used in their process to create their stories, were also analyzed. In this paper, we focus less on the content of these stories and more on what they unveil about the ways the camp and the protocols of Digital Mathematics Storytelling engaged the students.

Reflective Analysis. Over the course of the two camps, the students engaged in the developing Digital Mathematics Story curriculum. At the beginning of the first camp, this curriculum consisted of four loose modules, each involving a combination of whole-group, small-group, and individual activities. The modules were designed to take about three hours to enact, which was the length of each day of the camp. This initial Digital Mathematics Story curriculum was created and maintained by the research team and based heavily on the existing Digital Storytelling modules created over the past 30 years by StoryCenter (Lambert, 2013). Every afternoon, after the students had left the camp, the research team met for an hour to discuss how the day went and how to alter the next day's plans, with the only caveat being that the camp should culminate in the students creating individual 3–5-minute Digital Mathematics Story videos to present on the last day of the camp. These conversations were recorded and served as an artifact of how we engaged in continual tweaking and augmentation of the daily activities to best engage the students.

Analysis

The three measures, as detailed in Table 1, show: (1) The Pre and Post-Interviews, (2) analysis of the Digital Mathematics Stories, and (3) the recordings and notes of the daily reflections to document how we changed the curriculum. First, in terms of exploring ways to enacting a Digital Mathematics Storytelling protocol, we used a Participatory (Re)design Research Framework (Bang & Vossoughi, 2016; de Jager et al., 2017; Lambert, 2013). Second, to attend to student's mathematics identities as connected to other narrative and other social identities (Aguirre et al., 2013; Langer-Osuna & Nasir, 2016; Sfard & Prusak, 2005).

Table 1: Alignment of the research objectives, data collection, and analysis methods

Research Objective	Data	Analysis
Enactment of a Digital Mathematics Storytelling Experience	Pre/Post Interviews Students' Stories Reflective Conversations	Participatory (Re)design (Bang & Vossoughi, 2016; The Design-Based Research Collective, 2003)
Mathematics & Storytelling Identitites	Pre/Post Interviews Students' Stories	Identity as Narrative (Sfard & Prusak, 2005)

Results

Evolution of the Digital Mathematics Storytelling Enactment

The initial digital mathematics storytelling modules consisted of a warmup game to start the day, two initial activities to help get students comfortable with making and editing videos, and four daily modules focusing on students making videos that explored their identities as a (1) storyteller, (2) digital storyteller, (3) mathematician, and (4) digital mathematics storyteller. Additionally, each morning started off with a community building game. Each of these modules focused on exploring activities around those identities such as learning to tell a story in three acts or how a close-up shot conveys a different tone than a wide-angle shot.

In the early feedback from the students and our own reflective conversations, we noticed that (1) students gravitated towards group tasks and wanted to engage in almost all activities in small groups as opposed to individually and (2) students who signed up to attend this camp thought that this would primarily be a mathematics camp and thereby were hoping to have more opportunities to engage in mathematics. Both early observations led to modifications to our protocol.

Community Activities

This summer camp occurred in 2021. Every single student in our camp had spent the prior year in a remote or virtual learning environment and had missed opportunities to work with other peers in a physical setting. So, we decided to augment all the modules so that they did not emphasize individual videos and contributions, and to use the majority of our camp time to focus on community and group activities and games. This was a move that seemed appreciated, as almost every one of the post-interviews mentioned that the best part of the camp was the games that students played to get to know each other and build camaraderie. In hindsight, we even realized that the end goal of having each student create their own individual digital mathematics story took away from this emphasis on community as it situated digital storytelling as an individual endeavor. In the post-interviews, all but three of the students mentioned that they had much more fun working on the group video projects as opposed to making their own individual videos. And when students were asked to create individual video stories, two students never finished their videos, but were willing contributors to the group video stories. Figure 2 shows an example of the ways the students engaged in creating videos together in small groups.

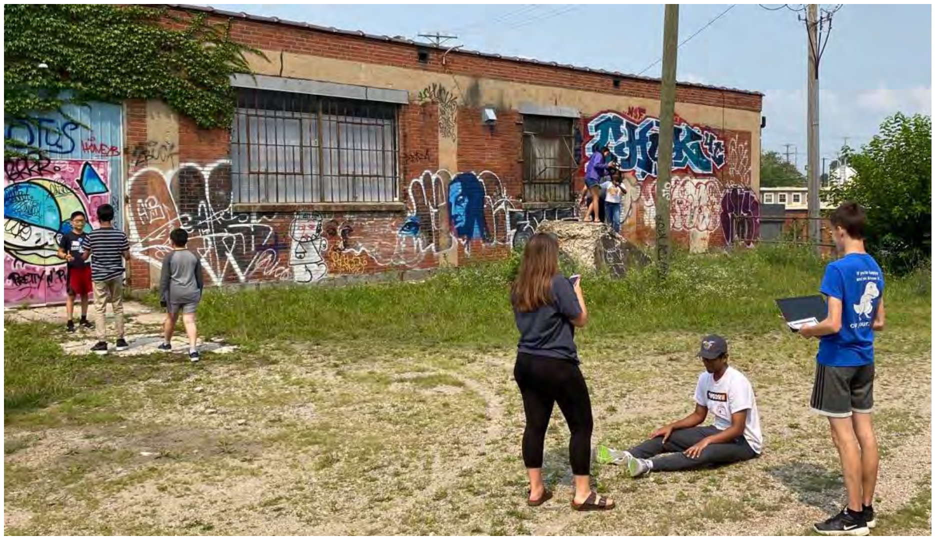


Figure 2: One finding was that students engaged much more with group video storytelling, as shown here, as opposed to making individual videos.

Mathematics Identity

Because so many of the students wanted to engage in mathematics, the research team decided to stop focusing so heavily on telling stories and actually engage the students in an augmented mathematics circle, opening up space for students to join groups to solve open-ended, challenging mathematics tasks. Figure 3 shows a group of students analyzing and discussing each other's strategy to an open-ended mathematics task. In the majority of the post-interviews, students mentioned how much fun it was to engage in this open ended, inquiry-based mathematics discussion, particularly because it was so different than the mathematics they were used to doing at school.

Figure 3: Students were surprised that this was more a video making camp and not a mathematics camp, so we gave them an opportunity to engage in group mathematics problem solving

Storytelling Identity

Additionally, students expressed to us that they found engaging in visual storytelling and learning about storytelling to be quite new and challenging. The majority of students who attended the camp shared that they had experience making videos in the past, but that these were videos for class projects or talking head videos. In our emphasis on understanding the various aspects of video filmmaking, such as exploring the differences in wide, medium, close-up, and extreme close-up shots or how to diagram a story into an exposition, rising action, climax, falling action, and conclusion, we realized that a generational divide existed between what we, as educational researchers thought storytelling was as compared to our students. We crafted our modules to explore storytelling from unpacking popular commercial films from *Pixar* or *Marvel Studios*. But what we found was that, while these pop-culture references were recognized by the students, our students' funds of knowledge around videomaking and storytelling revolved more around a new style of visual communication: *the YouTube video*. Again and again, our students wanted to make videos where they spoke directly to the camera, use graphics and editing features that drew visual emphasis onto particular objects or close-ups, and contemplate how to engage the viewer *after* their video was viewed.

Discussion

Overall, our digital mathematics storytelling camps opened a space for us to learn directly from our participants about how to enact and support the process of digital mathematics storytelling, as connected to emerging mathematics and storytelling identities. As with all (re)design research, what we sought was not a solution to a problem, but to better understand a process, the process of connecting storytelling to mathematics using digital video technology. First, we found that our students in urban emergent communities craved opportunities to engage with each other, both as mathematicians and storytellers. We interpret this to mean that as a field, the mathematics experiences our students encounter are too heavily based in individual accomplishment and competition. Second, we learned that our own interpretation of digital storytelling needs to be updated for the modern world, in which our own students' savviness with the language of *TikTok* and *YouTube* enabled them to revise what digital storytelling is and looks like. And finally, we all struggled in connecting storytelling to mathematics. A week-long camp is not enough time to do away with the years of focusing on mathematics as solving short, single-answer problems and seeing and understanding the ways narrative structures can be used to explore mathematical thinking. Our work is ambitious. But through this experience, we found a glimmer of hope that a digital mathematics storytelling experience can start re-orientating a student as to what mathematics is, how it connects to personal experiences, and how it is the product of one's own narrative.

Acknowledgments

We are thankful to our participants and their families for sharing their time, their communities, and their very personal stories and experiences during these Digital Mathematics Storytelling camps. This work is supported by NSF, Project #1943208.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. National Council of Teachers of Mathematics. http://www.nctm.org/catalog/product.aspx?ID=14119

Bang, M., & Vossoughi, S. (2016). Participatory Design Research and Educational Justice: Studying Learning and Relations Within Social Change Making. Cognition and Instruction, 34(3), 173–193. https://doi.org/10.1080/07370008.2016.1181879

Bright, A. (2016). The Problem with Story Problems. Rethinking Schools Publishers, 30(4). https://www.rethinkingschools.org/articles/the-problem-with-story-problems

Chao, T. (2014). Photo-Elicitation/Photovoice interviews to study mathematics teacher identity. In J. Cai, J. Middleton, & L. Van Zoest (Eds.), Current research in mathematics teacher education: Contributions by PME-NA researchers (pp. 93–113). Springer.

Civil, M. (2009). A reflection on my work with Latino parents and mathematics. Teaching for Excellence and Equity In Mathematics, 1, 9–13.

Civil, M. (2016). STEM learning research through a funds of knowledge lens. Cultural Studies of Science Education, 11(1), 41–59. https://doi.org/10.1007/s11422-014-9648-2

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9–13.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. For the Learning of Mathematics, 5(1), 44–48.

de Jager, A., Fogarty, A., Tewson, A., Lenette, C., & Boydell, K. (2017). Digital Storytelling in Research: A Systematic Review. The Qualitative Report, 22(10), 2548–2582.

Delgado, R. (1989). Storytelling for oppositionists and others: A plea for narrative. Michigan Law Review, 87(8), 2411–2441.

Duncan, G. A. (2006). Critical Race ethnography in education: Narrative, inequality and the problem of epistemology. In A. D. Dixson & C. K. Rousseau (Eds.), Critical Race Theory in education: All God's children got a song (pp. 191–212). Routledge.

Engeström, Y. (2011). From design experiments to formative interventions. Theory & Psychology, 21(5), 598–628. https://doi.org/10.1177/0959354311419252

González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging Funds of Distributed Knowledge: Creating Zones of Practices in Mathematics. Journal of Education for Students Placed at Risk (JESPAR), 6(1–2), 115–132. https://doi.org/10.1207/S15327671ESPR0601-2_7

Gutiérrez, K. D., Baquedano-López, P., & Tejeda, C. (1999). Rethinking diversity: Hybridity and hybrid language practices in the third space. Mind, Culture, and Activity, 6(4), 286–303. https://doi.org/10.1080/10749039909524733

Lambert, J. (2013). Digital storytelling: Capturing lives, creating community. Routledge.

Langer-Osuna, J. M. (2015). From getting "fired" to becoming a collaborator: A case of the coconstruction of identity and engagement in a project-based mathematics classroom. Journal of the Learning Sciences, 24(1), 53–92.

Langer-Osuna, J. M., & Nasir, N. S. (2016). Rehumanizing the "other" race, culture, and identity in education research. Review of Research in Education, 40(1), 723–743.

Lave, J. (1988). Cognition in practice: Mind, mathematics and culture in everyday life. Cambridge University Press.

Love, B. (2014). Urban storytelling: How storyboarding, moviemaking, and Hip-Hop-based education can promote students' critical voice. English Journal, 103(5), 53–58.

Matias, C. E., & Grosland, T. J. (2016). Digital Storytelling as Racial Justice Digital Hopes for Deconstructing Whiteness in Teacher Education. Journal of Teacher Education, 67(2), 152–164. https://doi.org/10.1177/0022487115624493

Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of Knowledge for Teaching: Using a Qualitative Approach to Connect Homes and Classrooms. Theory Into Practice, 31(2), 132–141.

Nasir, N. S., & de Royston, M. M. (2013). Power, identity, and mathematical practices outside and inside school. Journal for Research in Mathematics Education, 44(1), 264–287.

Ohler, J. (2006). The world of Digital Storytelling. Educational Leadership, 63(4), 44–47.

Oughton, H. (2010). Funds of knowledge—A conceptual critique. Studies in the Education of Adults, 42(1), 63–78.

Özpinar, İ., Gökçe, S., & Yenmez, A. A. (2017). Effects of digital storytelling in mathematics instruction on academic achievement and examination of teacher-student opinions on the process. Journal of Education and Training Studies, 5(10), 137. https://doi.org/10.11114/jets.v5i10.2595

Powell, A. B., & Frankenstein, M. (1997). Ethnomathematics: Challenging Eurocentrism in mathematics education. SUNY Press.

Prusak, L., Groh, K., Denning, S., & Brown, J. S. (2012). Storytelling in organizations. Routledge.

Rideout, V. (2017). The Common Sense census: Media use by kids age zero to eight (pp. 1–58). Common Sense Media. https://www.commonsensemedia.org/sites/default/files/uploads/research/csm_zerotoeight_fullreport_release_2.pdf

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. Educational Researcher, 34(4), 14–22.

The Associated Press-NORC Center for Public Affairs Research. (2017). Instagram and Snapchat are Most Popular Social Networks for Teens; Black Teens are Most Active on Social Media, Messaging Apps. University of Chicago. http://apnorc.org:80/projects/Pages/HTML%20Reports/instagram-and-snapchat-are-most-popular-social-networks-for-teens.aspx

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. Educational Researcher, 5–8.

Turner, E., & Drake, C. (2016). A Review of Research on Prospective Teachers' Learning About Children's Mathematical Thinking and Cultural Funds of Knowledge. Journal of Teacher Education, 67(1), 32–46. https://doi.org/10.1177/0022487115597476

Turner, E., Drake, C., McDuffie, A. R., Aguirre, J., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. Journal of Mathematics Teacher Education, 15(1), 67–82.

Turner, E., Gutiérrez, M. V., Simic-Muller, K., & Díez-Palomar, J. (2009). "Everything is math in the whole world": Integrating critical and community knowledge in authentic mathematical investigations with elementary Latina/o students. Mathematical Thinking and Learning, 11(3), 136–157.

WHITENESS IN FEARMONGERING TOWARDS MATHEMATICS EDUCATION REFORM

Eric Cordero-Siy	Michael Lolkus	Frances K. Harper
University of Wisconsin-Madison	Purdue University-West Lafayette	University of Tennessee-Knoxville
epsiy@wisc.edu	mlolkus@purdue.edu	francesharper@utk.edu

In this paper, we analyze a widely shared article from a prominent conservative media outlet that positions current mathematics education reform as something to be feared. We characterize how the text upholds whiteness and positions actors attempting to accomplish reform as bad actors. This work has implications in how mathematics education researchers can understand, approach, and respond to the current backlash towards reform and beyond.

Keywords: Social Justice; Equity, Inclusion, and Diversity; Policy

Recent reform efforts to center issues of equity and social justice in mathematics classrooms have been under fire from the loudest sectors of right-wing media. In its current form, reform efforts have centered around the inclusion of social justice discussions in mathematics classrooms or as some policy-makers frame it, "critical race theory" with some states explicitly banning the topic and setting up "tip lines" to report teachers who teach anything that could be misconstrued as teaching "critical race theory." The hysteria around incorporating social justice issues in mathematics classrooms has been increasing alongside recent attacks on mathematics education researchers. For instance, critics of the proposed new California mathematics framework argued that integrating issues of racial justice in mathematics will water down rigorous mathematics and harm society because "with fewer people who know math well, how are we going to build bridges, launch rockets or advance technologically?" (Evers, 2021). Even though the number of vocal critics of the current mathematics education reform (MER) are increasing, there has been little work to understand the nature of the backlash and how it is growing. It is essential to understand the machinations of the backlash because it is through understanding the tools of backlash, we can contextualize meanings and diffuse their potency (Rubel & McCloskey, 2019). In this paper, we ask, "How is the current MER movement towards racial and social justice framed in conservative media?" Using critical discourse analysis (CDA) and theories of whiteness, we examine a National Review (NR) article on the current MER. We demonstrate how this text upholds whiteness by the ways it constructs MER and its actors.

Theoretical Framework

We examine the resistance to current MER efforts toward racial and social justice in mathematics education through the lens of whiteness. Whiteness, "the ideology that maintains White supremacy, valuing one racial group over others" (Battey & Leyva, 2016, p. 50), operates in mathematical spaces to maintain the status quo, across three dimensions of *institutional*, *labor*, and *identity*. Here, *institutional* refers to neutral approaches to teaching mathematics, privileging of some voices and histories, and recognizing who has power in mathematical spaces; *labor* recognizes the ways in which emotional, behavioral, and cognitive contributions of marginalized individuals are undervalued; *identity* recognizes ways in which people who identify as white or with white ideals are granted unearned privilege and unquestioned validity. Furthermore, in (mathematics) education, "those who benefit from whiteness hoard real property to gain intellectual property" (Bullock, 2017, p. 633).

We also adopted Okun's (2021) lens of white supremacy culture to consider ways in which power is maintained. We are interested in four of the 18 elements: *fear*, *one right way*, *either-or and the binary*, and *power hoarding*. Okun shares *fear* is the primary strategy for sustaining white supremacy culture by undergirding all other elements, and "promised safety is false because it is always based on the abuse and misuse of power" (p. 7). *One right way* adheres to the belief that there is one right way to do things, and those who do not conform must have something wrong with them. An *either-or and binary* perspective limits the recognizing possibilities of both/and, and attempts to simplify complexity. Finally, *power hoarding* is used in white supremacy to instill the belief that there is limited power to go around.

Data, Methods, and Analysis

We use CDA to "explore connections between educational practices and social contexts" (Mullet, 2018, p. 117) and how power structures are maintained (Oughton, 2007). We explored one text from the NR entitled "The Folly of 'Woke Math'" (Spivak, 2021). To analyze the text, we used Mullet's (2018) CDA framework. Mullet outlines CDA in stages, yet allowing for flexibility to "best fit the scope and goals of the research problem.... [and] textual analysis is also left to the analyst." In this case, we use whiteness constructs to interpret the data. Moreover, Mullet cautions analysis may "further the researcher's own ideological agenda, rather than the agenda of the disempowered" (p. 123). To this end, we are mathematics education researchers and teacher educators (two of whom are white) with a commitment to racial and social justice.

Based on the framework, we selected a discourse to examine (i.e., the backlash to decenter social and racial justice in MER), selected and explored the background of a text, identified overarching themes, analyzed external (i.e., Discourses, ideologies) and internal (i.e., power, positionalities) relations, and interpreted the themes as manifestations of whiteness. Although literature on whiteness helped sensitize us to code for themes, concepts from the literature on whiteness were used to interpret the themes rather than dictate the themes.

We selected the NR text because the backlash against current MER is fueled by accessible, widely shared articles such as this text. NR is credited for defining conservatism in the modern American era (Hart, 2006; Sivek, 2008; Bogus, 2011) with many conservatives crediting NR as formative in their understanding of politics (Del Visco, 2019). In their 2021 media kit, NR received 25 million page views, produced 75,000 units for print, and communicated with 788,000 newsletter registrants and 1.4 million followers on social media. They also identify their audience as 86% male, 14% female, and a household income of $187,000 (they do not specify the measure of central tendency). Of their readers, 84% "took action as a result of reading NR." The text we analyzed (i.e., Spivak, 2021), has been shared on social media 1,678 times according to sharescore.com. By examining sources with a strong ideological identity, we show the constructed identities and how they operate within an ideological bound (Del Visco, 2019).

Results & Discussion

In Spivak's (2021) article, NR constructs mathematics in ways aligning with how researchers have operationalized whiteness in mathematics education. They also construct and position current MER as internally inconsistent, dangerous, and growing. This results in a fear-based image of the current MER. In our elaboration, we use quotations for direct lifts from the text.

Naming the Paradigm: Backlash to a Threat to whiteness

Throughout the text, the author elaborates on "woke math," which we call *the paradigm*—white supremacy culture is upheld by long-standing conceptions of mathematics (e.g., focus on answers, using correct methods). In opposition, the author positions mathematics as objective:

> It is an undeniable truth that there are correct and incorrect answers in basic math. There is no white math, or black math. There is only math. Americans, particularly our black and Hispanic students, are falling behind because, instead of finding better ways to teach, progressive educators debase math.

By naming mathematics as apolitical and absolute truth, as refuted by the paradigm, the author adheres to perspectives of mathematics that have been, and continue to, uphold whiteness (i.e., *institutional*; Battey & Leyva, 2016). Namely, the paradigm focuses on "getting the right answer, using the right method, or believing some students are more capable than others is white supremacy." The author's positioning of mathematics maintains white supremacy through claiming there is one right way to engage with mathematics (Okun, 2021), and using mathematics to sort and rank students based on perceived ability (TODOS, 2020).

Further, the author frames the paradigm as a threat by positioning mathematics as removed from ongoing conversations of equity and justice. For instance, the author comments on "the idiocy of having math teachers lead discussions on social justice instead of teaching black children how to do math." By upholding binaries of practice (Lolkus et al., 2022), or an *either-or* mentality (Okun, 2021), that justice-oriented teaching does not or cannot also support rigorous mathematics, the author perpetuates whiteness. The author also positions the paradigm as inconsistent because, although it supports Black and Brown children, they are "falling behind" because of a de-emphasis on better teaching and that those who espouse the paradigm "claim the mantle of protector have become the oppressors." They also argued the paradigm eliminates practices supporting "needs of black children, [thus] they do not believe black children can learn math." White supremacists used this tactic to remedy "past and current racism by asserting that the plea for justice is racist itself" (Rubel & McCloskey, 2019, p. 117).

Ultimately, the author's critique of and resistance to the paradigm demonstrates how white supremacy upholds and maintains the order and structure of the status quo (Giroux, 2021). By calling out examples of the paradigm as threatening mathematics institutional systems that have historically benefitted white folks, the backlash upholds whiteness.

Fearmongering: A Mechanism to Preserve whiteness

Fearmongering, or attempts to "use fear to divide and conquer, always in the service of profit and power for a few at the expense of the many" (Okun, 2021, p. 7) is a part of white supremacy culture. Much like attempts to create fear of the 'other' through harsh language and labels (e.g., Zimmer, 2019), the author of the text describes the paradigm as unsuitable due to being "dangerous," "fiction," "idiocy," "cringe-inducing," and "gobbledygook." Exemplars of the danger of the paradigm are laid out when the author reinforces mathematics needing nothing more than correct methods by recounting the loss of the Mars Climate Orbiter, the failure to track and intercept an Iraqi Scud missile, and deaths associated with errors in medication doses.

Aligned with safety, the author instills fear by illustrating what happens when the othered (i.e., minoritized) students are included. They argued ending standardized and proficiency exams, currently used for international competition and college admittance (i.e., *institutional*; Battey & Leyva, 2016), would "pre-select outcomes by race" and "adjust the scores for poor students." They also see tracking as "hold[ing] back gifted children" and not "permitting gifted students to take advanced courses" and limiting "better students to advance," "no one learns, [thus] outcomes are equitable." Again, they promote whiteness and preserve white supremacy by positioning some students as superior to others (i.e., *identity*; Battey & Leyva, 2016) and for Black and Brown students to learn, white students must not (i.e., *power hoarding*; Okun, 2021).

The author describes the paradigm as something to be feared by illustrating a society marred with death and "substantial property damage" with an educational system that assesses based on "skin pigmentation;" thus, positioning the paradigm as something to be avoided at all costs.

Bad actors: Orchestrating the paradigm. As an extension of the fearmongering, the author elucidates the paradigm as a growing movement with powerful orchestrators. Throughout the text, they introduce two powerful actors, "progressive" mathematics educators and their funders.

First, the author positioned a group of "progressive" mathematics education researchers (i.e., Deborah Ball and Rochelle Gutiérrez) who espouse and promote the paradigm. The author casts them as actors not to be taken seriously by using emotional verbiage when describing their actions such as "complains," "groans," "fretting," and "overstates." We take this framing to discredit their ideas, a mechanism usually directed towards female scholars (Cameron, 2022). Moreover, the emotional language is also placed upon their work. For instance, they write, "Gutiérrez worries that algebra and geometry perpetuate privilege," and "Ball complains that math is a 'harbor for whiteness.'" Thus, the author positions them and their work as not to be taken seriously because it "sound[s] like parody."

The author named powerful institutions and their initiatives as one of the driving forces behind the dissemination of the paradigm. They note the Gates Foundation's involvement as a funder of Ball's project, TeachingWorks, and positions it as "ready to profit." They identify Education Trust-West as one of the funders for the toolkit, *A Pathway to Equitable Math Instruction*. They position Education Trust-West closely to the Obama administration by identifying the organization's president, John B. King Jr., Obama's secretary of education from 2016-2017. The inclusion of both Obama and Gates positions them as the orchestrators of the paradigm. It is clear through past articles in NR how they position both Obama and Gates as the bogeymen of the left (Smith, 2021; McLaughlin, 2021) and, based on the intended audience of the text, would bring less credibility to the paradigm. Moreover, the author argued that these actors may be a reason why the paradigm is "gaining traction" and "strongly endorsed by educators, leading mathematics organizations, and policy-makers." As part of the fearmongering, the author creates an image of the paradigm not as a static curricular feature, but as a flawed and emotionally-based growing initiative backed by powerful orchestrators.

Conclusions and Implications

We demonstrated how NR constructs mathematics as steeped in whiteness and how they reject whiteness as a feature of status quo mathematics by positioning this as a paradigm to be feared. They illustrate a world where, if the paradigm is fully implemented, it would have adverse consequences to its readers' lives. They then amplify this fear by portraying the paradigm as a growing phenomenon with the help of several powerful actors and institutes.

This work has implications on understanding backlash to MER. Perhaps, whiteness is a feature of resistance to MER. In that, NR is not unique; it echoes past critics that do not call out whiteness explicitly in the way NR does. For instance, in Tom Lehrer's song New Math, he derides making sense of algorithms in favor of getting the correct answer. The critics of the late 90's MER argued reform would deteriorate the "economic and social well-being of our nation" because MER "favors weaker students, [and] the top students being shortchanged" (Wu, 1997, p. 951). Tactics have also persevered. Critics of the 90's standards positioned MER as unreliable and inconsistent because of "extremely bad" and "fuzzy" mathematics (Schoenfeld, 2004). It is through understanding the nature of the backlash that mathematics educators may see how whiteness rebuilds the status quo. In turn, educators and allies can begin to respond to the loudest voices suppressing those who fight for more inclusive and just mathematics education.

References

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education, 9*(2), 49–80. https://doi.org/10.21423/jume-v9i2a294

Bogus, C.T. (2013). *Buckley: William F. Buckley Jr. and the rise of American conservatism*. Bloomsbury Press.

Bullock, E. (2017). Only STEM can save us? Examining race, place, and STEM education as property. *Educational Studies, 53*(6), 628–641. https://doi.org/10.1080/00131946.2017.1369082

Cameron, D. (2022). Women, civility, and the language of politics: Realities and representations. *The Political Quarterly, 93*(1), 25-41. https://onlinelibrary.wiley.com/doi/pdf/10.1111/1467-923X.13050

Del Visco, S. (2019) Yellow peril, red scare: Race and communism in *National Review. Ethnic and Racial Studies*, *42*(4), 626–644, https://doi.org/10.1080/01419870.2017.1409900

Giroux, H. (2021). The plague of white supremacy in the age of fascist politics. *Fast Capitalism, 18*(1), 83-97. https://doi.org/10.32855/fcapital.202101.008

Hart, J. (2006). *The making of the American conservative mind: National Review and its times.* Intercollegiate Studies Institute.

Lolkus, M., Cordero-Siy, E., & Harper, F. K. (2022). Gatekeeping in mathematics for social and racial justice: Reflections on a conversation among colleagues [Manuscript submitted for publication]. Curriculum & Instruction, Purdue University.

McLaughlin, D. (2021, December 21). *The cruel irony of our recent presidential choices.* National Review. https://www.nationalreview.com/corner/the-cruel-irony-of-our-recent-presidential-choices/

Mullet, D. R. (2018). A general critical discourse analysis framework for educational research. *Journal of Advanced Academics, 29*(2), 116–142. https://doi.org/10.1177/1932202X18758260

Okun, T. (2021, May). *White supremacy culture - Still here.* White Supremacy Culture. https://drive.google.com/file/d/1XR_7M_9qa64zZ00_JyFVTAjmjVU-uSz8/view

Oughton, H. (2007). Constructing the 'ideal learner': a critical discourse analysis of the adult numeracy core curriculum. *Research in Post-Compulsory Education, 12*(2), 259–275. https://doi.org/10.1080/13596740701387536

Rubel, L., & McCloskey, A.V. (2019). The "soft bigotry of low expectations" and its role in maintaining white supremacy through mathematics education. *Occasional Paper Series, 41,* 113–127.

Schoenfeld, A. H. (2004). The math wars. *Educational Policy, 18*(1), 253–286. https://doi.org/10.1177/0895904803260042

Smith, W. J. (2021). *Bill Gates's plan to destroy animal agriculture.* National Review. https://www.nationalreview.com/corner/bill-gatess-plan-to-destroy-animal-agriculture/

Sivek, S.C. (2008). Editing conservatism: How National Review magazine framed and mobilized a political movement. *Mass Communication and Society*, *(11)*3, 248–274. https://doi.org/10.1080/15205430701791030

Spivak, K.M. (2021, September 16). *The folly of 'woke' math.* National Review. https://www.nationalreview.com/2021/09/the-folly-of-woke-math/

TODOS: Mathematics for All. (2020). *The mo(ve)ment to prioritize antiracist mathematics: Planning for this and every school year*. https://www.todos-math.org/assets/The%20Movement%20to%20Prioritize%20Antiracist%20Mathematics%20Ed%20by%20TODOS%20June%202020.edited.pdf

Wu, H. (1997). The mathematics education reform: Why you should be concerned and what you can do. *The American Mathematical Monthly, 104*(10), 946–954. https://doi.org/10.2307/2974477

Zimmer, B. (2019, August 6). *Where does Trump's 'invasion' rhetoric come from? The racist language used by the president—and the alleged El Paso shooter—to describe immigrants originated on the West Coast more than a century ago.* The Atlantic. https://www.theatlantic.com/entertainment/archive/2019/08/trump-immigrant-invasion-language-origins/595579/

THE WORLDS THAT TRANSCRIPTS HIDE IN MATHEMATICS EDUCATION RESEARCH

Higinio Dominguez
Michigan State University
higinio@msu.edu

Sofía Abreu
Michigan State University
abreusof@msu.edu

This brief research report showcases an alternative to transcript analysis that halts the process of reducing participants' expressive worlds to an orderly regime of words. Drawing from a philosophical conversation with grade-five students on the concept of space, the paper advances a methodological/analytical perspective that recalls transcripts and reissues them as liminal spaces of possibility while interrogating the limitations of conventional approaches to transcript analyses for generating new knowledge. Several philosophies inform our contribution, including Indigenous Knowledges, Feminisms, New Materialisms, and Posthumanism. The paper describes the process of constructing a transcript and coming to see it beyond its static, linear nature. The findings suggest that the encounter of researcher with data constitutes an opportunity to re-encounter possibility, embrace generosity, and engage imagination in research.

Keywords: research methods; elementary school education; equity, inclusion and diversity.

Objectives and Research Question

Many educational research practices promote a distancing of the researcher from the participants by reducing their world of experiences to codes stamped on transcripts or transcript excerpts that interrupt the continuity and vibrancy of a human experience. Arguments supporting these practices herald the objectivity (e.g., the case of coding) or representativity (e.g., the case of excerpts) of knowledge. Good research, then, must generate knowledge that skews the redundant details that belong in the imperfect material/subjective experiences of humans.

Our motivation for writing this paper grew out of our concern for the dissonance between philosophies that openly and honestly recognize how human experience is uncertain yet alive and vibrant, and the widely accepted analytic approaches that remove both uncertainty and vibrancy from our human affairs. Our concern has slowly grown into this opportunity to provide a specific example of how we as researcher-graduate student have come to *re*cognize—to know again, but differently (Dominguez, 2014)—our encounter with data. The purpose of this short research report is to share with our mathematics education research community this specific example that illustrates (a) a methodological/analytical critique on how language territorializes methodologies and analytic approaches in mathematics education research, (b) an alternative that we see as trespassing the static lines of text on transcripts as a transgressively liberating methodology that takes the research back to worlds filled with possibility and philosophy, and (c) a theoretical perspective that looks more into the future of knowledge, the possible that has been sequestered, and a reinvigorating vision of the relationship between researchers and data, including how we encounter data in our research. Accordingly, our research question is: What possible worlds (of ideas, relationships, and resonances) emerge when researchers halt the process of reducing everything to language and instead re-enter the space between words and images as a way of reinvigorating the researcher-data encounter? What opportunities to advance the learning of mathematics can we envision from these generous researcher-data encounters?

Philosophical Perspective

Every line in a transcript and every frame in a video includes both the actual (what was realized) and the virtual (not in the sense of unreal, but in the sense of unrealized, what remained as a possibility). In Spanish, “lo que se quedó en el tintero” (what remained in the inkwell).

Figure 1: Lines of flight as reductionist process is halted

This drawing (see Figure 1), for example, expresses all at once the reductionist process of researcher-data encounter that we halted right at the space between video and transcript, where tensions produced lines of flight (vibrant dark area moving in multiple directions).

This new way of thinking about the researcher-data encounter is minoritarian when compared with more commonly accepted views (Deleuze & Guattari, 1987). As Lenz Taguchi (2012) explains, “The expression ‘becoming minoritarian’ defines the very process of escape from ‘majoritarian’ norms, subject positions, and habits of mind and practice” (p. 267). Reinvigorating the researcher-data encounter aligns well with the idea of a living theory (Whitehead, 2009), for we see images, sounds, movements, materials, bodies, and the concept of space as a living, vibrant assemblage (Bennet, 2010; Cajete, 2000). In that living time-space-matter (Barad, 2007), agentivity emerged not from individual humans—the traditional perspective from the humanities—but from the assemblage of forces, intensities, and energies coming together (Little Bear, 2000; Mikulan & Sinclair, 2017).

Our current technology for producing data is human- and language-centric, with a focus on sequentially recording the actual while neglecting the virtual. Yet, virtuality (Barad, 2012; Deleuze, 1995) is as important as what becomes actualized (Whitehead, 1978). Virtuality is not completely invisible, though, for it tends to awaken gestures (Châtelet, 2000). Returning to a neglected virtuality as our methodological/analytic approach does, “requires another kind of thinking […] using all of our bodily faculties and our imaginary” (Lenz Taguchi, 2012, p. 267).

Drawing reconnects us to virtuality, by freeing the researcher-data encounter from its linear cage to an all-over relationality of a drawing surface (Sousanis, 2015). The possibilities become endlessly rhizomatic, and the vibrant indeterminacy of spatial, temporal, and bodily boundaries become more palpable (Abreu, in press). Drawing can become a generative node of transformation where the virtual is particularly vibrant since the actual and the virtual are in intense tension as uncertainty “está en la mera punta del lápiz”—is right at the tip of the pencil. A drawing is never ‘finished’ as it continues to be an active force that transforms and is transformed by the world after the drawing encounter (de Freitas & Sinclair, 2012).

The material agentivity of drawing brings into focus the role of multiple expressivities that emerge from spaces of tension in which multiple materialities look unapologetically into the researcher’s eyes (and other senses). Barad (2007) reminds us, “Neither discursive practices nor material phenomena are ontologically or epistemologically prior. Neither can be explained in terms of the other. Neither is reducible to the other. Neither has privileged status in determining the other. Neither is articulated or articulable in the absence of the other; matter and meaning are mutually articulated” (p. 152). We therefore envision our analytic approach as “a ‘meeting place,’ where different data, each with their own trajectories, come together with the researcher to create something new” (Taylor, 2013, p. 814).

Modes of Inquiry

Consistent with the idea of possibility as emerging from uncertainty (and vice versa), our analytical approach emerged not as pre-planned but rather from conditions of uncertainty as follows. The first author had been collaborating with one teacher for about two years as part of his community engaged scholarship. One morning, he received the usual phone text from the teacher, this time asking him to help her teach how to plot ordered pairs. Rather than turning this opportunity into a procedure-oriented lesson, the first author decided to engage the grade-five students into a philosophical discussion of space. An influential idea that was brought into this discussion was that “the body does not move into space and time, it creates space and time: There is no space and time before movement” (Manning, 2007, p. xiii). With a small plane made with popsicle sticks hanging from a piece of thread and three cardboard Cartesian planes, the philosophical discussion attracted the students’ moving bodies, gestures, drawings, imagination, and language into an assemblage that was videorecorded by one graduate research assistant.

Although term “data” comes from the Latin datum = given (Brinkmann, 2014), data are never generously given to researchers; we produce data. We find this recognition helpful in terms of sensitizing us to the idea of the researcher-data encounters. One of these key encounters occurred when we watched the video followed by the transcript production and reading. We thus felt the need to have responsive and responsible encounters with data. Consequently, in our analysis we assumed that the words in the transcripts and the moving images on the video were as agentive as the participants and researchers. As such, we needed to co-respond with these data, as in the correspondence of people exchanging letters (Ingold, 2013; Roth, 2016).

Results…Or rather Possibilities

Figure 2: Panels from *Young Philosophers*

Our results, or rather possibilities, come from a comics book we produced titled, *Young Philosophers: What Can They Teach Us about the Concept of Space?* Drawings in this book emerged saturated with *possibilities*, creating a *generous* flow of ideas that harmoniously connected with other ideas, while harboring *imagination* at every trace of the pencil and use of the space on the page (see Figure 2). We allowed, neither video nor the transcript alone to influence our analytic departure, our line of flight (Deleuze, 1995); instead, we sought the energy of the tension between the two. The energy we describe here can also be understood by envisioning a zigzag movement between the words in the transcript and the moving images on the video. As an example, on the page at left, we trespass the fixity of the words "…everything is kind of living…" by illustrating this idea as shown on the bottom part of the page. These illustrations—not the words on the transcript; not the images on the video—took us back to philosophical ideas about space (e.g., Cajete, 2000; Kimmerer, 2013; Little Bear, 2000; Manning, 2007). The generosity of these possibilities is in how our analytic approach invites the virtual to be at par with the actual. We refused to further reduce words into codes; instead, we re-encounter the words with a recognition of their agentivity to revive our imagination with images that, being also agentive, mobilized ideas that had been virtual.

Although it may be tempting to see our comic book as a final result, we prefer to see *in* it a space of possibility, a living artefact whose life is sustained as it passes across diverse hands, assemblages, and imaginations. At each passing, an expansion of the space that connects the actual with the possible occurs. The following is one moment filled with possibilities. Unlike traditional expectations to generalize findings, our hope is to generalize possibilities. Is what you will read also possible in your own context? In your own research? In your own practice?

Discussion and Takeaway

Transcript analysis answers the question: What did actually happen here? In contrast, our analytic approach, with its focus on halting reductionism, addresses the question: What could have happened here? Not content with an exclusive focus on the actual, our approach brings the researchers into a reinvigorating encounter with the data by considering spaces of tension; in this case, the space between video and transcript. Attention to this space allowed us to vivify the virtual through the continuous energy and imagination of drawing. Through drawing, we were able to connect students' philosophical ideas with the ideas of other philosophers. These possibilities emerged from our analysis in the form of a line of flight (Deleuze, 1995). We see in these lines of flight a more generous reading (Spence, 2011) of the worlds that transcripts tend to reduce to the actualized. For us, an important takeaway is how generous readings can contribute to address issues of equity, diversity, and inclusivity in our research.

Our methodological/analytic approach draws on feminist new materialisms and Indigenous Knowledges. Collectively, these philosophies look more into the future and the possible (Grosz, 2010) than more traditional theories that are interested in confirming a kind of knowledge that is assumed to pre-exist, which we see as a way of disdaining the future. Our interest in the virtual is motivated by our love for the possible; analyses that are closed and deterministic exclude not only the possible but our faculty to imagine it. We wonder how many students and their ability to imagine the possible have been excluded by these kinds of ungenerous readings of their worlds.

Finally, the "wheels" of transcript analysis continue to turn, and we wonder to what extent are they turning away from complex realities and more toward what Popkewitz (2004) calls the fabricated fictions of educational research. Our analytic approach intends to halt this wheel, to redirect it, not to an objective or finished reality, but to the potentiality that is part of every reality. We see the space between video and transcript as filled with tensions that we can use as catalysts for what is possible. Video, transcript, and coding are all complicit in reducing human experience. Reducing the human experience is dehumanizing. We are concerned with humanizing researchers (Tuck & Yang, 2014) first, so we all can see the limitations of current technologies when placed vis-à-vis force such as possibility, imagination, and generosity. Given the transformative effect this methodological/analytic approach has had on the emergent career of a mathematics education graduate student, Sofía, and the mid-career of a mathematics education researcher, Higinio, we believe the approach has the potential to re-engage researchers more generally to conduct more humanizing research in mathematics education.

Acknowledgment

This material is based upon work supported by the National Science Foundation under Grant 2055322. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Abreu, S. (in press). Possible (re)configurings of mathematics and mathematics education through drawing. *Journal for Theoretical and Marginal Mathematics Education*.

Barad, K. (2007). *Meeting the universe halfway*. *Quantum physics and the entanglement of matter and meaning*. Duke University Press.

Barad, K. (2012). On touching: The inhuman that therefore I am. *Differences: A Journal of Cultural Feminist Studies, 23*(3), 206-223.

Bennett, J. (2010). *Vibrant matter: A political ecology of things*. Duke University Press.

Brinkmann, S. (2014). Doing without data. *Qualitative Inquiry, 20*(6), 720-725.

Cajete, G. (2000). *Native science: Natural laws of interdependence*. Clear Light Publishers.

Châtelet, G. (2000). *Figuring space: Philosophy, mathematics and physics*. (Trans. Robert Shore and Muriel Zagha). Kluwer Academic Publishers.

de Freitas, E., & Sinclair, N. (2012). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*, *80*(1/2), 133–152.

Deleuze, G. (1995). *Difference and repetition*. Columbia University Press.

Dominguez, H. (2014). Reconocer: Recognizing resources with English learners in mathematics. In T. G. Bartell & A. Flores (Eds.) *TODOS Research Monograph 3: Embracing resources of children, families, communities, and cultures in mathematics learning* (pp. 65-79). TODOS, Mathematics for All.

Grosz, E. (2010). The untimeliness of feminist theory. *NORA—Nordic Journal of Feminist and Gender Research, 18*(1), 48-51.

Ingold, T. (2013). *Making: Anthropology, archaeology, art and architecture*. Routledge.

Kimmerer, R. W. (2013). *Braiding sweetgrass: Indigenous wisdom, scientific knowledge and the teachings of plants.* Milkweed.

Lenz Taguchi, H. (2012). A diffractive and Deleuzian approach to analysing interview data. *Feminist Theory, 13*(3), 265-281.

Little Bear, L. (2000). Jagged worldviews colliding. In M. Battiste (Ed.), *Reclaiming Indigenous voice and vision* (pp. 77-85). UBC Press.

Manning, E. (2007). *Politics of touch: Sense, movement, sovereignty*. University of Minnesota Press.

Mikulan, P., & Sinclair, N. (2017). Thinking mathematics pedagogy stratigraphically in the Anthropocene. *Philosophy of Mathematics Education Journal, 32*, 1-13.

Popkewitz, T. (2004). The alchemy of the mathematics curriculum: Inscriptions and the fabrication of the child. *American Educational Research Journal, 41*(1), 3-34.

Roth, W. M. (2016). Growing-making mathematics: a dynamic perspective on people, materials, and movement in classrooms. *Educational Studies in Mathematics, 93*, 87-103.

Spence, L. K. (2021). Deconstructing generous reading: Revising a writing assessment. *Critical Questions in Education, 12*(2), 100-113.

Sousanis, N. (2015). *Unflattening.* Harvard University Press.

Taylor, L. (2013). The case as space: Implications of relational thinking for methodology and method. *Qualitative Inquiry, 19*(10), 807-817.

Tuck, E., & Yang, K. W. (2014). R-words: Refusing research. In D. Paris & M. T. Winn (Eds.), *Humanizing research: Decolonizing qualitative inquiry with youth and communities* (pp. 223-248). Sage Publications.

Whitehead, A. N. (1978). *Process and reality*. The Free Press.

Whitehead, J. (2009). Generating living theory and understanding in action research studies. *Action Research, 7*(1), 85-99.

AN EXPLORATORY ACTION RESEARCH STUDY OF SOCIAL JUSTICE MATHEMATICS IN UNDERGRADUATE PRECALCULUS

Abigail Erskine
Syracuse University
aberskin@syr.edu

Brian Odiwuor
Syracuse University
bodiwuor@syr.edu

Nicole L. Fonger
Syracuse University
nfonger@syr.edu

In this exploratory action research study we investigated teaching mathematics for social justice in undergraduate mathematics. We asked: How if at all do undergraduate students consider the importance of local social justice issues as embedded in the context of a precalculus course assignment?; and As instructors, how might feedback from students' experiences of a social justice mathematics lab inform our future teaching and research practices? We discuss emerging findings from our initial analyses of student surveys. We also elaborate how these findings will inform our efforts to link research and practice at the intersection of supporting students' math cognition and exploring social justice issues through teaching mathematics for social justice.

Keywords: Precalculus, Problem Solving, Social Justice, Culturally Relevant Pedagogy.

Social justice is, ultimately, about understanding the macro systemic power dynamics that perpetuate the conditions in which we live. Social Justice Mathematics (SJM) educators explicitly aim to develop students' sociopolitical consciousness in addition to teaching mathematics content (Gutstein 2006; Gutiérrez 2013). This bivalent approach to justice emphasizes that both redistributive justice, or equitable distribution of fundamental resources, and recognition justice, or respect for human dignity and diversity with all groups having a voice, are necessary to achieve social justice (Basok et al 2006; Gonzalez, 2009; Bartell, 2013). SJM may also be referred to as critical mathematics orteaching math for social justice (Kokka, 2015).

The purpose of this exploratory action research study is to traverse the terrain of incorporating research on SJM teaching into our classrooms as mathematics educators at the university level. The main problem we aimed to address in this research was to move from theory to practice in employing conceptual orientations toward SJM in our own teaching and research. Our research was guided by the following questions: How if at all do undergraduate students consider the importance of local social justice issues as embedded in the context of a precalculus course assignment? As instructors, how might feedback from students' experiences of a SJM lab inform our future teaching and research practices? In this paper we discuss emerging findings from our initial analyses of students' surveys. We also elaborate how these findings will inform our future efforts to link research and practice at the intersection of supporting students' mathematical literacies and exploring social justice issues.

Conceptual and Theoretical Framework

Gutstein (2006) and Kokka (2020) posit two sets of goals in teaching mathematics for social justice. First are the *mathematics pedagogical goals* which aim to support students in gaining mathematical power to read the world, succeed academically, and conceive of mathematics as a useful tool. Second are the *social justice pedagogical goals*—reading the world with mathematics and being able to take actions to change the world. SJM involves developing positive cultural and social identities among students to coexist in the current diverse society where values from both dominant cultures and non-dominant cultures are preserved.

We draw on Freire's (1970) notion of 'praxis' to argue that students should use mathematics to investigate and challenge injustices and inequalities relating to their own lives and wider society. Praxis can be understood as "reflection and action upon the world in order to transform it" (Freire, 1970, p.36). The goal is to recognize and draw upon learners' lived experiences to emphasize the cultural relevance of mathematics (Aguirre et al., 2013) and develop a critical understanding of the nature of mathematics and its position and status within education and society. In this approach, students can play a role in providing solutions to injustices that surround us. As argued in Freire (1970) it is not enough for people to study the world only, there is also a need to act towards creating a more just world. To accomplish such a goal, students must gain a deeper understanding of the conditions of their lives and the social dynamics of their real world. To incorporate social justice issues into mathematics teaching, tasks can be designed to improve students' mathematical thinking which will help them to become aware of the social injustices that occur within society at large and in their own lives at the same time increasing their mathematical understanding.

Methods

We employed one cycle of action research (Segal, 2009) to explore the phenomena of integrating social justice issues into mathematics lessons and course assignments for undergraduate precalculus students. This cycle of action research included systematic planning, engagement with students, and reflection on our research-based teaching intervention.

Participants and Teaching Context

This research was conducted at a university in the Northeastern U.S. All students enrolled in an undergraduate precalculus course in Fall 2021 completed a problem solving lab within the first four weeks of the semester. Of this larger pool of students, 16 students chose to complete a survey about their experiences during the last 2 weeks of the semester.

The pre-calculus course is designed to prepare students for success in the study of calculus. The core focus of the course is to develop representational fluency in using functions to model relationships between quantities. Throughout the semester, students engage in applied problem solving in collaborative group settings using graphing technologies.

Authors' Statements of Researchers Positionality

As argued in D'Ambrosio et al. 's (2013) and Holmes's (2020), the researcher's positionality influences the design of research in terms of how participants are recruited, the kind of data collected, and how that data are analyzed. I (Abigail) recognize myself as an African female, a graduate student, and an advocate for educational equality. I understand what it means to be in a classroom with people from diverse backgrounds. I believe in an educational setting where every student has the right to be as inclusive as possible in the classroom. I was a lead instructor for Precalculus in the Fall of 2021 and supervised students who participated in this study. I (Brian) identify as African, a father, STEM educator, researcher, and Ph.D. student and subscribe to liberalism(Bell, 2014; Charvet, 2018) as political and moral philosophy. As a man, I recognize that there are privileges that I may have because of my gender, the same way I recognize the alienations I have experienced as a person of color residing in the United States. This has motivated me to advocate for critical consciousness or conscientização (Freire,1970) in the spaces I participate in. As a white female mathematics education researcher and faculty member (Nicole), I seek to explore both research and practice focused on SJM to advance a more meaningful understanding of mathematics for a racially diverse group of students. My interest in SJM is largely motivated by a personal journey of unlearning racism and whiteness ideologies as they intersect with my work as a mathematics educator. During Fall 2021 I was the course supervisor for all undergraduate precalculus courses that study participants completed. In this role, I set the syllabus and introduced the problem solving lab that is the focus of this research study.

Problem Solving Lab

Fonger designed the problem-solving lab by adopting a set of tasks that a high school mathematics teacher had developed over the past 4 years (Keech, Routhouska, & Fonger, in press). Erskine was among the group of instructors who gave this task to precalculus students at while they were learning about linear function and change. In general, students investigated trends in the historical population of residents in a city and the role of building a federally funded highway system that impacted population decline in that city. The mathematical focus of this lab was to model and make predictions of population data over time using linear equations. Prior to completing the mathematics tasks of the problem-solving lab, students read news articles and a zine (a mini-magazine) to inform their understanding of the social issues in this lab related to redlining, the razing of a predominantly Black neighborhood, and the impact of highways on population decline in this city. We intended to support students' understanding of how government decisions negatively impacted many Black residents of the city. Figure re 1 introduces the lab.

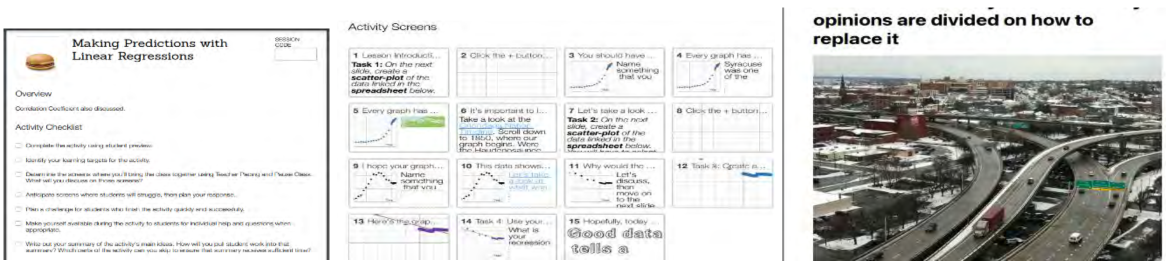


Figure 1: Screenshots of the problem solving lab, activities, and related texts.

Research Instruments and Data Collection

We used two primary instruments to collect data: students' written responses to the problem-solving lab, and a post-lab survey. We designed these instruments to obtain information about students' experiences of the mathematics lab with a focus on relating their classroom mathematical experiences to real world situations and social justice issues. The problem solving lab write up was in the form of paragraph responses designed to address both the mathematics (e.g., write the equation of your "line of best fit") and the social issues (e.g., what does the highway have to do with population decline in this city?). The survey consisted of 10 questions and students responded on a 5-point Likert-type scale asking them to (1) Strongly Agree, (2) Somewhat agree, (3) Neither agree nor disagree, (4) Somewhat disagree, or (5) Strongly Disagree.

Data Analysis

Following Morse (2009), we adopted a mixed methods approach wherein the qualitative component (collection of student problem solving lab) preceded and informed the design and delivery of a quantitative component (the survey of student experiences). At the time of this writing, our analyses are ongoing. We focus this brief report on our analysis of the survey responses. We plan to share qualitative analyses of students' lab responses when presenting this research.

Results and Discussion

Recall the two research questions that guided this study: How if at all do undergraduate students consider the importance of local social justice issues as embedded in the context of a precalculus course assignment?; As instructors, how might feedback from students' experiences of a social justice mathematics lab inform our future teaching and research practices? To the first question, we share a summary of student responses to the survey questions in Table 1.

Table 1: Students' experiences with social justice mathematics lab

	N	Strongly disagree	Somewhat disagree	Neither agree nor disagree	Somewhat agree	Strongly agree.
The Problem Solving Lab supported me to read and interpret a scatter plot relating two quantities (city population, time since 1850)	16	6%	0%	13%	50%	31%
The Problem Solving Lab supported me to model local data with a linear function and interpret the meaning of the data with respect to the context.	16	0%	6%	19%	31%	44%
l The Problem Solving Lab supported me to learn about possible reasons the city population declined after the year 1950	16	0%	0%	13%	19%	69%
The Problem Solving Lab supported me to believe that the construction of the highway is an important issue for city residents	16	0%	0%	19%	19%	63%
The Problem Solving Lab supported me to affect change on social justice issues in the city	16	0%	0%	31%	38%	31%
The Problem Solving Lab supported me to identify inequities for city residents	16	0%	0%	19%	25%	56%
The Problem Solving Lab supported me to appreciate the connection between math and problem-solving skills.	16	0%	6%	13%	25%	56%
The Problem-Solving Lab is relevant to my life experiences.	16	0%	19%	19%	38%	25%
The Problem-Solving Lab is relevant to what I learn in math class.	16	13%	13%	19%	31%	25%

From Table 1, we see that over 60% of the participants either agree or strongly agreed to the survey questions. The percentage of student who disagree or strongly disagree was 6% or 0% on 7 out of the 10 questions. We can infer that majority of the students (>70%) can relate mathematical content learnt in class to social issues that revolve around them. This analysis also shows that the social issues incorporated in the pre-calculus lab seemed to support students' awareness and consciousness of being able to identify inequities for city residents.

These data inform our response to the second question as well. It is troubling that 26% of responses disagreed with the idea that the problem solving lab is relevant to what I learned in math class. Moreover, 38% of respondents either disagree or are neutral about the relevance of this lab to their life experiences. As instructors, we had designed the problem solving where lab to connect to issues in the same city where the university classrooms are located. We have used this feedback to inform and revised iterations of the problem solving lab. For example, we have added content to the lab that explicitly connects to the *place* of the social justice issues. Given that many students on the university campus have not physically visited the site of the social injustice, we also introduced additional videos and visualizations as resources to connect to students' experiences in potentially more meaningful ways. We would also like to move toward designing opportunities for SJM that not only mathematizes the social justice issues, but pot position students to address the issues.

In future iterations of this lab, we will take greater care to help students see explicit links between the lab and course goals.

References

Aguirre, J. M., & del Rosario Zavala, M. (2013). Making culturally responsive mathematics teaching explicit: A lesson analysis tool. Pedagogies: *An international journal*, 8(2), 163-190.

Bartell, T. G. (2013). "Learning to Teach Mathematics for Social Justice: Negotiating Social Justice and Mathematical Goals." *Journal for Research in Mathematics Education* 44(1): 129-163.

Basok, T., Ilcan, S., & Noonan, J. (2006). Citizenship, human rights, and social justice. *Citizenship studies*, *10*(3), 267-273.

Bell, D. (2014). What is liberalism? Political Theory, 42(6), 682–715.

D'Ambrosio, B., Frankenstein, M., Gutiérrez, R., Kastberg, S., Martin, D. B., Moschkovich, J., ... & Barnes, D. (2013). Positioning oneself in mathematics education research: JRME Equity Special Issue Editorial Panel. *Journal for Research in Mathematics Education*, *44*(1), 11-22.

Charvet, J. (2018). Liberalism: The Basics (1st ed.). Routledge.

Freire, P. (1970). Pedagogy of the Oppressed. London: Penguin Random House.

Gonzalez, L. (2009). "Teaching Mathematics for Social Justice: Reflections on a Community of Practice for Urban High School Mathematics Teachers." Journal of Urban Mathematics Education 2(1).

Gutierrez, R. (2013). "The Sociopolitical Turn in Mathematics Education." *Journal for Research in Mathematics Education* 44(1): 37-68.

Gutstein, Eric. 2006. Reading and writing the world with mathematics: Toward a pedagogy for social justice. Taylor & Francis (2006).

Holmes, A. G. D. (2020). Researcher Positionality--A Consideration of Its Influence and Place in Qualitative Research--A New Researcher Guide. *Shanlax International Journal of Education*, *8*(4), 1-10.

Keech, K., Routhouska, B., Fonger, N. L. (in press). People, place and population prediction. *Mathematics Teacher Learning and Teaching.*

Kokka, K. (2015). Addressing Dilemmas of Social Justice Mathematics Instruction through Collaboration of Students, Educators, and Researchers. *Educational Considerations*, *43*(1), 13-21.

Kokka, K. (2020). Social justice pedagogy for whom? Developing privileged students' critical mathematics consciousness. *The Urban Review*, *52*(4), 778-803.

Morse, J. (2009). Procedures and practice of mixed method design: maintaining control, rigor, and complexity. In *Procedures and Practice of Mixed Method Design* (pp. 340-352).

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

OBSCURING RACIAL INEQUITIES THROUGH GENDER-FOCUSED DISCOURSE: AN ANALYSIS OF STUDENT PARTICIPATION IN A CALCULUS CLASS

Heather Fink
University of California, Berkeley
hfink@berkeley.edu

Inequities in STEM participation for people of color and women are documented extensively, often highlighting issues of representation and/or achievement. This study adds to the literature by looking at participation inequities in a high school calculus class. Equity is conceptualized as a fair distribution of opportunities for students to engage in rich mathematical experiences. Analysis of participants' dialog revealed talk about student participation focused primarily on gender patterns, even when pertaining to only one race-gender group of students (i.e., White males). Analysis of whole-class participation showed White dominance superseded male dominance; White males had the most robust opportunities to participate followed by White females, females of color, and males of color. Findings suggest participants' gender-focused discourse obscured racial inequities in classroom participation.

Keywords: Gender; Equity, Inclusion, and Diversity; Classroom Discourse

Decades of research document inequities related to sexism and racism in STEM (science, technology, engineering, and mathematics) participation (e.g., Martin, 2009; Wang & Degol, 2017). Often participation studies focus on issues of representation (e.g., Cheryan, et. al., 2017) or achievement (e.g., Oates, 2009). In contrast, this study looked at participation inequities as they played out through interactions in a 12^{th}-grade calculus class over one distance learning semester. The focus of this study was on how classroom participants (teacher, student teacher, and students) made sense of students' participation with respect to race and/or gender and how whole-class participation patterns corresponded to race and/or gender.

Equity is defined as the fair distribution of opportunities for students to engage in rich mathematical experiences, supporting deep disciplinary understandings and positive mathematical identities (Esmonde, 2009; Schoenfeld, 2014). Students' identities are conceptualized as complex and multi-dimensional. In the words of Crenshaw (1991), "Because women of color experience racism in ways not always the same as those experienced by men of color and sexism in ways not always parallel to experiences of white women, antiracism and feminism are limited, even on their own terms" (p. 1252). Racism and sexism extend into classroom spaces where students' opportunities to participate and learn mathematics are shaped in racialized and gendered ways (Esmonde & Langer-Osuna, 2013; Gholson & Martin, 2019; Sengupta-Irving & Vossoughi, 2019).

This study aims to provide a nuanced understanding of students' varied experiences in a high school calculus class "by making visible what has been obscure and bringing to the center what has been marginalized" (Bullock, 2018, p.123). Research questions are: 1) How did participants articulate participation issues related to race and/or gender? 2) How were whole-class contributions distributed across students in the class?

Theoretical and Conceptual Framing

Teaching and learning are considered sociopolitical endeavors through which power is distributed, either perpetuating persistent inequities or challenging existing hierarchies and injustices (Gutiérrez, 2013; Philip et. al., 2017; Valero, 2018). Based on sociocultural theories, learning is defined as changes in students' participation in collective classroom practices (Lave & Wenger, 1991; Vygotsky, 1978) and is a function of students' opportunities to participate (Gresalfi, et. al., 2009). Inequities are created and perpetuated through the construction of disparate and stratified opportunities for students to engage in meaningful learning experiences (Esmonde & Langer-Osuna, 2013; Shah & Crespo, 2018). Over time, patterns of participation play out through interactions between teachers, students, and mathematical tasks (Clarke, 2004), and these interactions are racialized and gendered forms of experience (Leyva, et. al., 2021; Martin, 2006).

Methods

Data Collection

Participants. Participants included one math teacher (White female), one student teacher (Mexican male), one researcher (White female) and 12th-grade students in the teacher's 1st period (Pd.1) AB Calculus class and 2nd period (Pd.2) AB Calculus class. The public high school used distance learning during the 2020-21 school year. Due to the pandemic, students took only three courses at a time, resulting in fewer lessons per course than is typical. Race and gender identities are self-stated for participant interviewees. Racial identities included Black, Chinese, Filipino, Iranian-American, Mexican, Mixed, White, and others. Gender identities included female and male. I have selected identity groups intentionally and I acknowledge they are imperfect. Non-White identifying students are grouped as "Students of color" based on how students talked about themselves related to their peers. (Guadalupe said, "it's different with PoC girls and the White girls in class…") I have chosen to capitalize "White" to draw attention to the power this classification holds, intending to highlight the non-neutrality of this label (Ewing, 2020).

Participant Discourse Data. Table 1 displays descriptions for the four data types included in the race-gender discourse analysis: lessons, lesson debriefs, one-on-one participant interviews, and informal conversations. Table 2 shows the race and gender of participant interviewees.

Table 1: Data Used for Discourse Analysis

Source Type	With Whom?	Quantity	Total Hours
Lessons	Pd.1 class	45 observations (semester 2)	45
	Pd.2 class	47 observations (semester 2)	47
Lesson Debrief Sessions	teacher & student teacher	30 Pd.1 debrief sessions	2.0
		45 Pd.2 debrief sessions	7.7
Participant Interviews	teacher	1 teacher, 4 interviews	3.6
	student teacher	1 student teacher, 3 interviews	1.9
	Pd.1 students	9 students, 16 interviews	9.4
	Pd.2 students	8 students, 11 interviews	6.0
Informal Conversations	teacher & student teacher	8 recorded Zoom conversations	1.8
		12 email chains	n/a

Table 2: Interviewee Identities

	People of Color	White
Female	Students-5	Teacher-1 Students-4
Male	Student Teacher-1 Students-5	Students-3

All lessons, lesson debriefs, and participant interviews were video-recorded through Zoom and took place during the spring semester (Jan - June 2021). Most video-recorded informal conversations and email exchanges took place during this timeframe as well; however, several informal exchanges between the researcher and teacher occurred during the fall semester.

Student Contribution Data. This paper uses observations of whole-class discussions that occurred during 30 "regular" semester 2 Calculus lessons (Pd. 1); optional "help" lessons were excluded from the analysis (Table 3). In addition, Pd. 2 calculus lessons were excluded from this analysis due to space, though both classes are included in the race-gender discourse analysis.

Table 3: Description of Data Used for Contribution Analysis

	Pd. 1 Calculus
# of School Days (Semester 2)	48
# EQUIP Whole-Class Observations	40*
# of "Regular" Lesson Obs	**30**
# of Optional "Help" Lesson Obs	10

* 5 days - groupwork only; 2 days - testing, 1 day - researcher absence

Whole-class discussion data was collected using EQUIP (Equity QUantified In Participation) (Reinholz & Shah, 2018), an observation tool used by researchers to track user-defined discourse dimensions at the student contribution level. The discourse dimensions in EQUIP relevant to this paper are *Contribution solicitation method* (i.e., cold-called, encouraged, volunteered) and *Contribution type* (i.e., responded to a teacher question, asked a question, shared solution, responded to student question, declined to answer, offered a comment, identified mistake, read out loud, shared screen). Every time a student contributed during a whole-class discussion, the contribution was logged using EQUIP and selections were made for each discourse dimension.

Data Analysis

Participant Discourse Analysis. All explicit references to gender and/or race made by participant interviewees (from Pd. 1 & Pd. 2) during the spring semester 2021 were identified from video; several conversations between the teacher and researcher at the end of the fall semester 2020 were also included. All references were sorted into one of four topics (Table 4).

Table 4: Reference Topics with Examples

Topic	References to…	Example
Participation	how categories of students participated during class	White boys dominated class conversations
Representation	the relative number of participants in activities, locations, professions	lack of female students in advanced math courses
Beyond this Class	patterns extending to students' community or society generally	lower expectations for females of color doing math
Self-Reflection	participants' first-hand experiences	pressure to do well and prove assumptions wrong

Once sorted by general topic, references were grouped into specific observations, accounting for different wording but the same meaning. For example, the comments "I noticed the classroom is very male-dominate" and "the boys were talking too much" were both coded as the same participation issue (i.e., boys dominated class conversations). These observations were then sorted according to identity category (e.g., gender, race-gender, race-class) depending on how participants framed their observations. For example, if a participant referred to students' gender and race identities, the observation was assigned *race-gender* as the identity category. References were also coded depending on which participant made the observation.

Whole-Class Contributions. Student contributions were exported from EQUIP to excel and merged with student gender and race data (self-stated when available, otherwise teacher ascribed). Data were aggregated by lesson, student, and class. Quantities were first tabulated by gender (i.e., male, female) and then by race-gender groups (i.e., female students of color, White female students, male students of color, White male students). Only student contributions from Pd.1 (not Pd.2) were included in this paper due to space limitations.

Findings

Findings are organized according to the two research questions. The first section reports how classroom participants (the teacher, student teacher, researcher, focal students) articulated participation issues related to gender and/or race. The second section reports how whole-class contributions were distributed across gender-race groups.

Participation Issues as Articulated by Classroom Participants

To understand how participants conceived participation "issues," we take a closer look at the references related to classroom participation that were positioned by participants as less-than-ideal. Table 5 shows specific classroom participation issues by identity category, including which participant(s) made the references.

Table 5: Race and/or Gender Participation Issues by Participant

	Observations of... Classroom Participation Issues	# of References Total	By Whom?
GENDER (30)	Boys dominated class conversations	12	T-6, ST-2, S-4
	Girls did not speak up	7	T-6, S-1
	Boys did not collaborate as well as girls in small-groups	7	T-4, S-3
	Student comment-"Wednesday help sessions are for girls"	3	ST-2, S-1
	Boys were sought out as math experts more than girls	1	T-1
RACE-GENDER (6)	White boys dominated class conversations	6	T-3, ST-2, R-1
RACE (1)	Students had different opportunities to participate (White vs. students of color)	1	R-1
RACE - CLASS (2)	Affluent White kids spoke about expensive ski vacations during class	2	T-2

KEY

T	Calculus teacher
ST	Student teacher
S	Student(s) in the class
R	Researcher

Almost all observations of participation issues referenced gender in some way, either gender-alone or the intersection of gender and race (36 out of 39, 92%). The most common participation issue articulated was that boys dominated class conversations, identified by the teacher, student teacher, researcher, and four students. There were 18 references to this issue; 12 referred to boys in general and 6 referred to White boys. This issue was talked about as a gender issue (boy dominance) twice as often as a race-gender issue (White boy dominance), even though the students to whom participants were referring were all White males. The four students who mentioned this issue were students of color (3 female, 1 male) and all described the issue without mentioning race. However, it is possible students *thought* about the issue as race-related but chose to *speak* about it as gender-related. On the other hand, the adults all spoke about imbalanced participation as a race-gender issue at least once. The teacher spoke about this issue sometimes as a gender issue and sometimes as a race-gender issue, as did the student teacher. The researcher made one reference to the issue, referring to it as "White boy" dominance.

Another often-mentioned participation issue was that girls in the class did not speak up enough. The teacher mentioned this issue 6 different times, expressing frustration with the situation. During a lesson debrief, the teacher shared, "I'm really annoyed with [name 1] and [name 2] cuz they are the starlets in this class. They get everything right. They would have been in BC Calculus, but they decided not to bother. And they're not contributing to the class." The issue was also brought up once by a White female student who shared, "For the really hard problems, there are 3 girls who get it and they're the only girls in the class- or not the only girls, the only *students* in the entire class who get it, but they don't really want to speak up and explain it or talk about it, which I've found really interesting." The three students to whom the teacher and student are referring are all female students of color; however, the issue was identified as a gender issue, not a race-gender issue. The teacher and student's confusion regarding why students were not speaking up could have been connected to presumed similarities in gender identity, but they were not accounting for differences in racial identities. The White female teacher and White female student may not have known what it was like to be a female *of color* in a *White* male-dominated math class. In support of this theory, here is one example of how race shaped experiences for one female student of color in this class:

> Within calculus and the girls, I feel like we should be supporting each other more than some of us do… it's different with PoC girls and the White girls in the class… It's a little bit more hostile, [White girl] is very hostile toward me. And so, I'll be like, 'Oh, okay.' I don't want to overstep. And with [PoC girl] it's not like that… I understand the PoC girls more and so they're a bit more supportive… a lot of times [White girls] are a little bit more reserved about their work, which is fine. You don't have to help us or share your work with us, but we don't necessarily act that way towards them… it's not the same. It's just different.

There was only one reference to a participation issue related to race-only; that reference was made by the researcher. After completing a preliminary class observation at the end of the fall semester, the researcher shared with the teacher her observation that only one student of color had opportunities during the class period to participate in competent mathematical ways (e.g., answer how/why questions). The only other race-related participation issue raised was by the teacher in reference to two affluent White students talking about their ski vacations.

Student Contributions during Whole-Class Discussions

Contributions were quantified by totaling the number of contributions made by Pd. 1 students during regular lesson whole-class discussions over the entire semester. There were 28 students enrolled in this course who made a total of 767 contributions during the 30 observed lessons.

Given participants' gender-focused discourse shown in the previous section, contributions were initially examined by gender. Table 6 shows total contributions, mean contributions, and contribution solicitations broken down by gender (female / male). Values in the table are color-coded depending on how the percent of contributions compares to the percent of female and male students in the class. If the percent of contributions made by a group is more than 3% higher than the percent of population represented by that group, then table values are green. If the contribution percent is within 3% of the population percent, then values are yellow. If the contribution percent is more than 3% lower than the population percent, then values are red. The same color-coding scheme is used in all tables in this section.

Table 6: Student Contributions by Gender

	Total # of Students	Total Contributions (Semester 2)	Mean Cont. per Student	Contribution Solicitations		
				Cold-Called	Encouraged	Volunteered
ALL Students	28	767	27.4	218	44	505
Female	8	181	22.6	74	21	86
	29%	24%		34%	48%	17%
Male	20	586	29.3	144	23	419
	71%	76%		66%	52%	83%

KEY	Difference from Population %
Green	3% or more higher
Yellow	+/- 3%
Red	3% or more lower

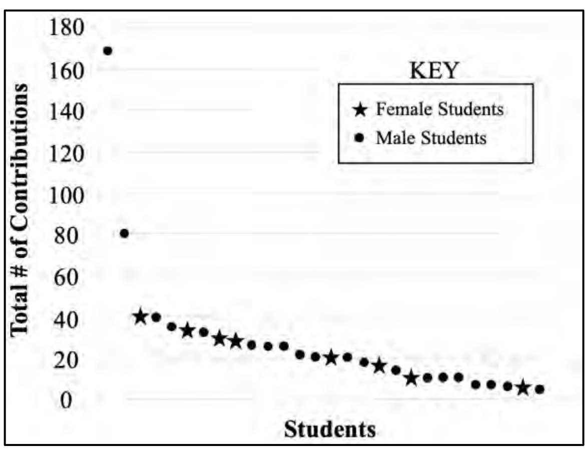
Female students made 24% of all whole-class discussion contributions averaging 22.6 contributions per student, which is 5% lower than what would be expected given perfect alignment between contributions and population representation. The difference is notable but not extreme. Looking at solicitation method highlights bigger differences in participation patterns between female and male students. Specifically, 83% of voluntary contributions were made by male students, and the teacher encouraged roughly the same number of contributions from female and male students. In addition, the teacher cold-called on male students twice as often as female students. Considering females represented 29% of students, the number of voluntary contributions was considerably lower than expected for females, whereas the numbers of cold-called and encouraged contributions were higher. Table 7 shows contributions made, again broken down by gender, by participation type (e.g., # of responses to a teacher question, # of questions asked, # of times a student declined to answer a question).

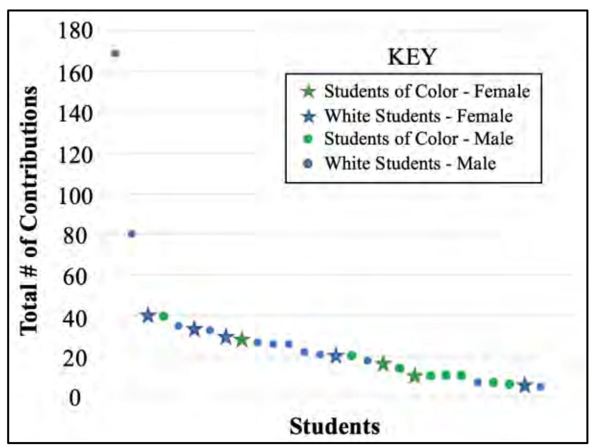
Table 7: Student Contributions by Participation Type and Gender

		Total # of Students	Responded to Teacher Q	Asked a Q	Shared Solution	Responded to Student Q	Declined to Answer	Offered a Comment	Identified Mistake	Read Out Loud	Shared Screen
ALL Students		28	482	132	45	30	29	25	12	8	4
Female		8	106	14	18	10	13	6	6	4	4
		29%	22%	11%	40%	33%	45%	24%	50%	50%	100%
Male		20	376	118	27	20	16	19	6	4	0
		71%	78%	89%	60%	67%	55%	76%	50%	50%	0%

Data indicate females (compared to males) were somewhat less likely to offer unsolicited comments or respond to teacher questions and much less likely to ask questions. They were also somewhat more likely to respond to student questions and much more likely to share problem solutions, decline to answer questions, identify mistakes, read out loud, and share their screen. There was a mix of seemingly productive and unproductive forms of participation by female students. Taken together, data suggest female students were less likely than male students to participate in academically risky ways. One explanation is that females shared problem solutions and identified peers' mistakes when they felt confident with content; in contrast, they declined to answer teacher questions and neglected to ask their own questions when they were more unsure.

To get a better sense of how contributions were distributed among individual students, total contributions per student were sorted from high to low and then plotted (Figures 1a & 1b).





Figures 1a & 1b: Contributions per Student by Gender (1a) & Race-Gender (1b)

Figure 1a shows two outlier male students (dots), one with 169 contributions and one with 80. Female students (stars) seem to be spread evenly across the class, with females taking 4 out of the top 9 contributor positions. Based on Figure 1a, one might conclude inequitable participation in this class could be attributed to the dominant males; presumably, if these two students stepped back allowing others to contribute, then all would be ok. However, adding race to the analysis tells a notably different story. Figure 1b reveals 12 out of the top 14 class contributors were White (86%), including the two dominant males. Although females seemed to be distributed evenly across the spectrum of class contributors, students of color (green symbols) were not. This finding prompted analyses of contributions by race-gender groups (Tables 8 & 9).

Table 8: Student Contributions by Participation Type and Race-Gender

	Students of Color						White Students					
	Total # of Students	Total Contributions (Semester 2)	Mean Cont. per Student	Contribution Solicitations			Total # of Students	Total Contributions (Semester 2)	Mean Cont. per Student	Contribution Solicitations		
				Cold-Called	Encouraged	Volunteered				Cold-Called	Encouraged	Volunteered
Female	3 11%	54 7%	18	21 10%	6 14%	27 5%	5 18%	127 17%	25.4	53 24%	15 34%	59 12%
Male	8 29%	117 15%	14.6	53 24%	5 11%	59 12%	12 43%	469 61%	39.1	91 42%	18 41%	360 71%

Table 9: Student Contributions by Participation Type and Race-Gender

	Total # of Students	Responded to Teacher Q	Asked a Q	Shared Solution	Responded to Student Q	Declined to Answer	Offered a Comment	Identified Mistake	Read Out Loud	Shared Screen
ALL Students	28	482	132	45	30	29	25	12	8	4
Students of Color										
Female	3 11%	25 5%	7 5%	7 16%	3 10%	4 14%	1 4%	2 17%	2 25%	3 75%
Male	8 29%	74 15%	24 18%	7 16%	3 10%	8 28%	0 0%	1 8%	0 0%	0 0%
White Students										
Female	5 18%	81 17%	7 5%	11 24%	7 23%	9 31%	5 20%	4 33%	2 25%	1 25%
Male	12 43%	302 63%	94 71%	20 44%	17 57%	8 28%	19 76%	5 42%	4 50%	0 0%

Findings from this analysis reveal that participation inequities in this class extended beyond gender groups. Male students did not dominate class contributions; *White* male students dominated. In fact, male students of color experienced the fewest opportunities to participate across all contribution categories compared to the other gender-race groups. Male students of color made 117 contributions, only half of what would be expected given their representation in the class (15% of contributions vs. 29% of population). They were the least likely group to share problem solutions, respond to student questions, or offer unsolicited comments. None of these three types of participation were required by the teacher; the teacher asked for volunteers to share homework solutions for extra credit and if no one volunteered, the teacher did it. The teacher also answered students' questions if no student volunteered to answer them, and unsolicited comments were shared without any explicit invitation and therefore not required. The only participation type for which male students of color were within the expected range was declining to answer questions. They declined to answer their fair share of questions, but this is the one contribution category for which higher is not better.

Female students of color experienced the second fewest opportunities to participate. The three group members accounted for 54 contributions (7% of total contributions vs. 11% of population) averaging 18 contributions per student, slightly higher than male students of color (14.6 per student), but lower than White females (25.4 per student) and White males (39.1 per student). Female students of color were cold-called and encouraged to contribute as expected, while White female students were cold-called and encouraged *more often* than expected. In addition, White female students responded to teacher questions and offered unsolicited comments as expected, while female students of color contributed in these ways *less often* than expected. White female students were not that different from their White male peers with most contribution frequencies falling within the expected range or above. Consistent with the initial analysis, the exceptions seem to be related to risk-averse participation and seem to be true regardless of race; specifically, females (in general) were less likely to ask questions, more likely to decline to answer, and less likely to offer voluntary contributions. With 469 total contributions (61% of total contributions vs. 43% of population) White male students dominated whole-class participation. The two outlier White male students certainly impacted the contribution numbers, but they were not the only White males contributing; nine out of the top 14 class contributors were White males. The only two participation types with lower-than-expected rates for White males were declining to answer questions and sharing screens. Four times during the semester the teacher asked, “Can someone please share the task prompt on their screen?” and all four times females responded to this request; three of those four times were female students of color.

Discussion

Participants spent much more time talking about classroom participation issues related to gender than race. Even issues that pertained only to White male students or female students of color were repeatedly framed with respect to gender alone. Differences in whole-class participation by gender did indeed exist but the analysis of gender-race groups revealed a more nuanced, and consequentially different, participation story. Findings revealed *White* dominance superseded *male* dominance; White male students had the most opportunities to participate, followed by White female students, female students of color, and finally male students of color.

Taken together, findings suggest participants’ gender-focused discourse obscured racial inequities in classroom participation. Gender-focused discourse drew attention to discrepancies in male / female participation, but differences in the participation of White students and students of color were not noticed, or at least not talked about. The only participation issue mentioned related to race (and not gender) was an issue I raised before the start of the spring semester. After doing a trial run with EQUIP, I shared with the teacher that only one student of color had been given opportunities during that class period to contribute in “mathematically competent” ways. The teacher acknowledged the importance of this observation at the time. Yet, neither one of us (both White women) referenced this (or any other) race-related participation issue again. It was not that I noticed issues and chose not to mention them; I did not see racial inequities in spring semester participation until I conducted the gender-race analysis of student contributions.

Questions remain as to how our identities as White women may have clouded our ability to talk about and notice racial inequities. However, the student teacher, a man of color, did not mention race-related participation issues either. It is very possible that students of color noticed racial inequities but chose not to talk about them. Despite the seemingly strong relationships I built with students that year, the fact remained that I was a White woman researcher asking students of color to share with me their personal observations. Students shared a lot, but I cannot assume they shared everything they noticed or experienced.

While attending to gender inequity is extremely important, attending to gender alone is not enough. Discourse focused exclusively on gender obscures the intersectional experiences of numerous other marginalized groups of students, including male students of color, transgender and queer students, and students labelled with “disabilities.” In addition, attending only to gender reifies gender as a binary, which further marginalizes students who identify as neither male nor female. Moving forward, an intersectional approach to mathematics education research is of utmost importance. Topics worth exploring further include: how racial and gender-based inequities interact with inequities related to virtual classroom spaces, how gender and race shape participants’ noticing, how classroom participation varies for groups of students across different contexts, and how the experiences of individual students within the same identity groups compare to one another.

References

Bullock, E. C. (2018). Intersectional Analysis in Critical Mathematics Education Research: A Response to Figure Hiding. *Review of Research in Education*, *42*(1), 122–145.

Cheryan, S., Ziegler, S. A., Montoya, A. K., & Jiang, L. (2017). Why are some STEM fields more gender balanced than others?. *Psychological bulletin*, *143*(1), 1.

Clarke, D. (2004). Patterns of participation in the mathematics classroom. *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, *2*, 231–238.

Crenshaw, K. (1991). Mapping the Margins: Intersectionality, Identity Politics, and Violence against Women of Color. *Stanford Law Review*, *43*(6), 1241.

Eslami-Rasekh, Z., & Valizadeh, K. (2004). Classroom Activities Viewed from Different Perspectives: Learners' Voice and Teachers' Voice. *TESL-EJ*, *8*(3).

Esmonde, I. (2009). Ideas and Identities: Supporting Equity in Cooperative Mathematics Learning. *Review of Educational Research*, *79*(2), 1008–1043.

Esmonde, I., & Langer-Osuna, J. M. (2013). Power in Numbers: Student Participation in Mathematical Discussions in Heterogeneous Spaces. *Journal for Research in Mathematics Education*, 44(1), 288–315.

Ewing, E. L. (2020). I'm a Black Scholar Who Studies Race. Here's Why I Capitalize 'White.'" Medium. July 2, 2020.

Gholson, M. L., & Martin, D. B. (2019). Blackgirl face: racialized and gendered performativity in mathematical contexts. *ZDM - Mathematics Education*, 51(3), 391–404.

Gresalfi, M., Martin, T., Hand, V., & Greeno, J. (2009). Constructing competence: An analysis of student participation in the activity systems of mathematics classrooms. *Educational Studies in Mathematics*, *70*(1), 49–70.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for research in mathematics education*, 44(1), 37-68.

Lave, J., & Wenger, E. (1991). *Situated Learning: Legitimate Peripheral Participation*. Cambridge University Press.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing Racialized and Gendered Mechanisms of Undergraduate Precalculus and Calculus Classroom Instruction. *Cognition and Instruction*, 39(1), 1–34.

Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning*, *8*(3), 197-229.

Martin, D. B. (2009). Researching race in mathematics education. *Teachers College Record, 111*(2), 295-338.

Oates, G. L. S. C. (2009). An empirical test of five prominent explanations for the black–white academic performance gap. *Social Psychology of Education*, *12*(4), 415-441.

Philip, T. M., Jurow, A. S., Vossoughi, S., Bang, M., & Zavala, M. (2017). The Learning Sciences in a New Era of U.S. Nationalism. *Cognition and Instruction*, *35*(2), 91–102.

Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education*, *49*(2), 140-177.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? A story of research and practice, productively intertwined. *Educational Researcher*, *43*(8), 404-412.

Sengupta-Irving, T., & Vossoughi, S. (2019). Not in their name: re-interpreting discourses of STEM learning through the subjective experiences of minoritized girls. *Race Ethnicity and Education*, *22*(4), 479–501.

Shah, N., & Crespo, S. (2018). Cultural narratives and status hierarchies: Tools for identifying and disrupting inequity in mathematics classroom interaction. In *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 23-37). Brill Sense.

Valero, P. (2018). Political Perspectives in Mathematics Education. *Encyclopedia of Mathematics Education*, (1987), 1–4. https://doi.org/10.1007/978-3-319-77487-9_126-4

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Mental Processes.* Cambridge, MA: Harvard University Press.

Wang, M. T., & Degol, J. L. (2017). Gender gap in science, technology, engineering, and mathematics (STEM): Current knowledge, implications for practice, policy, and future directions. *Educational psychology review*, 29(1), 119-140.

BLACK STUDENTS' TENSIONS WHEN FREEDOM DREAMING ABOUT AN IDEAL MATHEMATICS EDUCATION IN RURAL APPALACHIA

Sean P. Freeland	Matthew P. Campbell
West Virginia University	West Virginia University
sfreelan@mix.wvu.edu	mpcampbell@mail.wvu.edu

This preliminary work is from a larger study on the experiences of Black students learning mathematics in rural Appalachia. Following previous findings, this piece analyzes a series of interviews with Black students from West Virginia who were asked to imagine their ideal mathematics education through the lens of critiquing injustices. These freedom dreams were intended to develop features of a new mathematics education; however, there were tensions present in their re-imagination. They were often constrained by dominant notions of achievement and race in mathematics education. These tensions and student voices need to be taken into consideration in reform and change in mathematics education in rural Appalachia and beyond.

Keywords: Equity, Inclusion, and Diversity; Social Justice; Systemic Change

Past movements in mathematics education have had de-humanizing effects on students, particularly Black students and other students of color. Success in mathematics education has been defined by achievement on standardized tests and a limited conception of what it means to do mathematics (Martin, 2019). Mathematics education policy has contributed to this de-humanization through standardization (Martin, 2015), including through the standards movement of the past three decades (e.g., CCSSO, 2010; NCTM, 1989, 2000). This de-humanization also sidesteps systemic racial issues in mathematics education, instead focusing on ways to "help" or remediate the "low" performance of poor or Black students (Bullock, 2019).

Recent efforts to pursue equity in mathematics education have focused on humanizing mathematics and the experiences of students engaging in school mathematics (i.e., Gutstein, 2006; Rubel et al., 2016, Su, 2020). However, much of the focus on educational equity in mathematics education attends primarily to traditional notions of achievement (i.e. grades, standardized test scores) and access to high quality mathematics. As Gutierrez (2012) states, this only partially addresses the concept of equity and the de-humanization of mathematics and students by focusing only on dominant structures while failing to emphasize critical issues of power and identity issues (Gutierrez, 2012). Current notions of equity are generally uncritical and avoid human and social issues. They maintain the status quo in mathematics education, failing to re-think the field, especially for Black students. This allows mathematics education to remain as a White institutional space into which Black students are "invited" (Bullock, 2019). A more human mathematics education is possible by re-thinking and critiquing current embedded practices and structures and by considering the voices of students, particularly Black students.

In this brief report, we highlight preliminary analyses of a series of interviews with Black students from West Virginia to re-imagine a new mathematics education, highlighting the ideas and tensions that shape their perspectives. This research contributes to efforts to humanize mathematics education and center student voices by situating the work in a context that is under-researched—rural, central Appalachia—where issues of race are traditionally backgrounded (Cabell, 1985; Hayden, 2002). Students were asked to re-imagine a mathematics education that focuses on identity, power and racial issues. This was an attempt at a "freedom dream" (Love, 2019) of what mathematics education can be through the lens of critiquing injustice and harnessing the power of Appalachians and their communities. We enter this work as White, cisgender male mathematics educators who live and work in rural Appalachia. We acknowledge our Whiteness and the benefits we receive from the status quo in mathematics education, and we believe in the power of students in Appalachia, particularly Black students, and use this work to elevate their voices in this re-imagining.

Freedom Dreaming/Black Liberatory Mathematics

In order to make mathematics education truly equitable and human, particularly for Black students, and to make it an instrument for social justice, we need to begin truly re-imagining what it can be. Freedom dreams are imaginations that are focused on challenging injustices in education and society (Love, 2019). Freedom dreams are, however, grounded in reality. They are not "whimsical, unattainable daydreams, they are critical and imaginative dreams of collective resistance" (Love, 2019, p. 101). The de-humanization of the educational system, in particular mathematics education, is so ingrained that to change it requires imaginative resistance to the oppressive structures that exist including standardized achievement tests and linear teaching based on memorization and procedures. This is especially necessary as Black students have experienced various forms of violence in their participation in mathematics education. They have long been invited to the space, but only if they apply to the norms of mainstream mathematics education (Martin, et al., 2019). Freedom dreaming can move towards a Black Liberatory Mathematics education that moves away from the notions of access, equity, and inclusion which often fall short. This is a way to re-think mathematics in a way that values Black students, their families, their communities, and their cultures (Martin et al., 2019).

Rural Appalachia as the Context of the Study

As hooks (2009) states, there are competing cultures in rural Appalachia: "the world of mainstream white supremacist capitalist power and the world of defiant anarchy that championed freedom for everyone" (p. 11). The goal of freedom dreaming is to tap into the second culture that is found in so many communities across Appalachia and to upend the first dominant culture. This can have an impact not only on Appalachian communities, but it can build mathematics communities that challenge the status quo. Rural Appalachians, and particularly Black people in the region, face health, environmental, educational, and other issues imposed on them by a White capitalistic society (Catte, 2018). A new mathematics education can work to dismantle the structures in mathematics education and society that contribute to these disparities. To begin to make progress on this vision, we pursue the following research questions: (1) What do Black students in rural Appalachia consider an ideal mathematics education when they freedom dream through the lens of critiques of injustice and community-based solutions? (2) What tensions exist for students between this re-imagination and traditional notions of success rooted in de-humanization?

Methods and Data Analysis

Data for this study come from a larger study exploring Black students' experiences learning mathematics in rural Appalachia. Participants were six high school students from across West Virginia who identify as Black and who took part in a statewide after-school mathematics and science program focused on mentoring students to enter and succeed in STEM-based degree programs. The first author was previously an assistant director of the program, which provided initial mentoring relationships with the participants. In the program, students conduct community-based research projects, thus providing experience in addressing injustice in their communities. They also are admitted to the program based on traditional academic achievement, and they must maintain a certain grade point average to remain in the program.

The first author conducted initial one-on-one and focus group interviews with the students followed by individual follow-up interviews. Each interview was approximately sixty minutes in length. Following previous findings (Freeland, 2021), this work is focused on a re-imagination of mathematics education by Black students who are subjected to de-humanizing mathematics education in the region. Students were asked multiple questions particular to this research. First, they were asked to describe their ideal mathematics education. Next, they were asked to consider how their work in communities and with community-based research could be applied to mathematics classes. Lastly, students were asked what they would want mathematics teachers to know about teaching mathematics to Black students in rural Appalachia.

Data analysis occurred over multiple phases via open coding (Miles et al., 2019). The first stage involved coding students' responses when asked to freedom dream and finding examples of a re-thinking or re-imagining of mathematics education. While the goal was to move from codes to features of a re-imagined mathematics education, what emerged were tensions present in their freedom dreams. To explore the tensions that were present, codes were further refined to include examples of humanization and de-humanization. Also, we drew on Rubel's (2017) work discussing "changing the game" and "playing the game" of mathematics education, providing another lens to explore the conflicts that students experienced when freedom dreaming. This data focuses on three students from the larger study: Letitia, Zahara, and Yara. Their freedom dreams represent a preliminary view of the complexity of re-thinking mathematics education and the tensions students feel when doing so.

Preliminary Findings

Tensions Associated with Achievement in Mathematics Education

Students' freedom dreams re-imagine the way that classroom interactions occur and re-thinks traditional classroom structures. However, these dreams always have an eye towards traditional achievement through exams, standardized testing, and high grades. For example, Letitia mentioned a way for students to work consistently with their classmates in groups of two or more. However, this found its way to a discussion of assessment as she imagined that students could complete tests together:

> We can have like two people at a table or little area. And they can work together on their stuff and be like partners for the rest of the year. Assigned partners and stuff and they help each other even when they're doing a test. Because it can be like on paper. And they're both trying to solve it.

Letitia also wants a re-imagining of grades, but she still sees the need for them. She said, "I would still keep grades, just a different type of grading based on how you understand it. But maybe not like A B C D…" However, Letitia strives greatly for high grades even though it causes her stress. She said, "I want to have good grades because I wanted to graduate with honors. I would get stressed out if I got a bad grade. I'll have like a breakdown." Similarly, Yara feels the pressure and de-humanization of grading policies while still seeing value in them. When asked if her re-imagined mathematics education would need grades, she said:

> I don't know because I like that I have good grades. Like it makes me feel good that I'm doing good and stuff. But also, I feel like I'm stress a lot about my grades. I'm always checking my grades constantly. Constantly.

In contrast to Letitia's ideas about collective learning, Zahara imagines a freer mathematics education that relies on the individual to move at their own pace in learning. She saw a video about a high school in Canada where "you learned at your own pace." She said, "There weren't really grades you just learned what you need to learn, and you just go past it whenever you learn it and just keep going." However, she also realized that teachers must prepare them for the ACT and SAT, so although they could learn whatever they could individually, the next teacher would need to "pick up where they left off." The traditional ideas of mathematics education as linear and achievement based remained strong.

Tensions Associated with Race in Rural Appalachia

The main tension for students associated with race was the feelings of wanting to be treated equally to their White peers while also having their race and Blackness recognized and honored. This is especially important in the mostly White, rural spaces these students live and learn. This is highlighted by Zahara who said, "It would be best for the teacher to treat all of them, every student, the same because if you're the only Black person you don't want to feel different more than you already do." Letitia agrees as she believes that many people doubt Black students' intelligence or mathematics abilities:

> I want them to know that you're not dumb. 'Cause people have this I guess idea that Black people aren't smart. And not really nice … or smart. I think people should just ease up on that. And treat everybody the same not any different.

This feeling of wanting to be treated exactly the same contrasts with their desire for their Blackness to be acknowledged and for people to be treated appropriately. Zahara said, "You know just treat everyone like you should be treating them in the first place." She also expressed her pride in being Black: "We didn't ask to be Black, but I'm proud to be Black. But that doesn't mean we have to be talked to or treated different like we're not as smart as other kids." Letitia sums up this tension when discussing the equal treatment of White and Black students in rural Appalachia. She said, "I think (treating everybody the same) that's different than just like saying you're the same like treatin' you White. You know, treat me the same, but you know that I'm Black and not White. And that, this is different."

Discussion and Conclusion

Black students were able to freedom dream and re-imagine a more human mathematics education. However, they were often constrained by the deeply ingrained structures including testing and grading that have defined mathematics education. They are also constrained by the Whiteness that exists in their context. Based on these students' experiences (Author, 2021), they often dislike mathematics education and view it as a series of de-humanizing series of memorizations and procedures. A re-imagining needs to occur for Black students in rural Appalachia to live in their full humanity; however, the tensions they exhibit are very real and need to be taken into consideration and honored. Their perspectives as Black students in rural Appalachia must be a part of any reform and change in mathematics classrooms.

While the students interviewed for this study seem unfulfilled in their de-humanizing mathematics education—seen as a means to an end for college or career—those same endpoints constrain the opportunities to imagine what else is possible. This provides an important starting point for future work with students and teachers to work towards a full re-imagining of a more human mathematics education focused on community, particularly in this context. These students' voices also provide an important voice now and moving forward in the discussion of mathematics education and other systems in rural Appalachia and beyond.

References

Bullock, E. C. (2019). Mathematics curriculum reform as racial remediation: A historical counter-story. In J. Davis & C. C. Jett (Eds.), *Critical race theory in mathematics education* (pp. 75-97). Routledge.

Cabell, E. J. (1985). Black invisibility and racism in Appalachia: An informal survey. In W. H. Turner & E. J. Cabell (Eds.), *Blacks in Appalachia* (pp. 3-10). University Press of Kentucky.

Catte, E. (2018). *What you are getting wrong about Appalachia.* Belt Publishing.

Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* CCSSO.

Freeland, S. P. (2021). Black students' mathematics identities in rural Appalachia. In Olanoff, D., Johnson, K., Spitzer, S. M. (Eds.). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Philadelphia, PA.

Gutierrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Primm (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies* (pp. 17-33). Springer.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Towards a pedagogy for social justice.* Routledge.

Hayden, Jr., W. (2002). In search of justice: White privilege in Appalachia. *Journal of Appalachian Studies, 8*1, 120-131.

hooks, b. (2009). *Belonging: A culture of place.* Routledge.

Love, B. L. (2019). *We want to do more than survive: Abolitionist teaching and the pursuit of educational freedom.* Beacon Press.

Martin, D. B. (2015). The collective Black and "Principles to Action." *Journal of Urban Mathematics Education, 8*(1), 17-23.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education, 22*(4), 459-478.

Martin, D. B., Price, P. G., & Moore, R. (2019). Refusing systemic violence against Black children: Toward a Black Liberatory Mathematics Education. In J. Davis & C. C. Jett (Eds.), *Critical race theory in mathematics education* (pp. 32-55). Routledge.

Miles, M. B., Huberman, A. M., Saldana, J. (2018). *Qualitative data analysis: A methods sourcebook.* SAGE.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards.* NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. NCTM.

Rubel, L. H. (2017). Equity-directed instructional practices: Beyond the dominant perspective. *Journal of Urban Mathematics Education, 10*(2), 66-105.

Rubel, L., Lim, V., Hall-Wieckert, M., & Sullivan, M. (2016). Teaching mathematics for spatial justice: An investigation of the lottery. *Cognition & Instruction, 34*(1), 1-26.

Su, F. (2020). *Mathematics for human flourishing.* Yale University Press.

PRINCIPAL'S STORYLINES ABOUT LANGUAGES IN MATHEMATICS CLASS

Annica Andersson
University of South-Eastern Norway
Annica.Andersson@usn.no

Beth Herbel-Eisenmann
Michigan State University
bhe@msu.edu

Hilja Lisa Huru
Artic University of Norway
hilja.huru@uit.no

David Wagner
University of New Brunswick
dwagner@unb.ca

Keywords: equity, inclusion and diversity, First nations and Indigenous cultures, social justice

Principals play an important role in shaping, encouraging and supporting mathematics teachers in their daily work. They also have the responsibility for teachers and students' welfare. A survey of almost 1000 district decision-makers in the U.S, however, found that these administrators tend to make decisions about structuring students' opportunities to learn based on principles of equality rather than principles of equity (Herbel-Eisenmann, et al., 2018). Bound with, or with support of, regulating policy, principals move their school life forward day by day, being responsible for students' opportunities to learn in all subjects, including mathematics. Conversations and interviews with principals in this study focused on understanding the complexities of storylines—or grand narratives—related to organizing mathematics education for Indigenous and (rather) newly migrated students in Norwegian schools. This paper addresses the storylines about these minoritized youth, mathematics and languages.

Our research goal is to understand a range of "available" storylines (Wagner et al., 2019) to inform our longitudinal, participatory research project with teachers, administrators, community members, youth, and families. We are motivated by the fact that the opportunities that students from minoritized groups have for learning mathematics are often diminished as compared to their majority group peers due to systemic and institutional issues (Barwell et al., 2007; Civil, 2012; Le Roux, 2016; Meaney et al., 2011; Valoyes-Chávez & Martin, 2016). Over time, our partnership will develop strength-based pedagogies based on positive, asset-based storylines. In this paper, we analyze in-depth interviews with nine Norwegian principals and share some of their complex storylines about minoritized student groups, their languages and their learning of mathematics.

Positioning Theory and Storylines

The key construct we use for this paper, the idea of storylines or "lived stories for which told stories already exist" (Harré, 2012, p. 198), comes from positioning theory. As Davies and Harré (1999) point out, the multiple storylines at play "are organized through conversations around various poles, such as events, characters, and moral dilemmas. Cultural stereotypes like nurse/patient, conductor/orchestra, mother/son may be called on as a resource. It is important to remember that these cultural resources may be understood differently by different people." (p. 39). Storylines are negotiable; they are reciprocal and contingent (Wagner & Herbel-Eisenmann, 2009). Storylines also can occur at a range of scales (Herbel-Eisenmann, et al., 2015). Although positioning theory has been used in mathematics education research for over a decade, there is typically less focus on storylines than on positionings. Because storylines indicate what is expected in an interaction and can shape the potential fluid roles or positions that are made available to people, they are important to consider in relationship to mathematics teaching and learning.

Methodology

Context

This research is part of the Norwegian Research Council's FINNUT-granted project MIM: Mathematics Education in Indigenous and Migrational contexts: Storylines, Cultures and Strength-Based Pedagogies, (see https://www.usn.no/mim), a collaboration drawing on participatory approaches to investigate educational possibilities and desires, in times of societal changes and movements. Although we focus on the Norwegian context, we recognize that these kinds of societal changes and movements impact many countries throughout the world. With these changes and movements of people, language diversity may be the most obvious challenge in mathematics classrooms, but this reality also connects to cultural differences and conventional characteristics of the discipline. Indigenous communities have experienced linguistic and other challenges for decades as a result of colonization. In Norwegian contexts, several groups with a different origin than Norwegian are recognized, often with a reference to a connection to a (former) nation state. These groups are mainly seen mentioned from the 1970s and onwards when migrant workers from Pakistan, Turkey, and Morocco started to arrive in Norway. More recently immigration also comprises people from other European countries and from conflict areas in, for example, Asia and Africa (Reisel, Hermansen & Kindt, 2019). In addition to these, there are peoples/nations of Norway (without their own nation state) that appear in the contexts we are interested in; the Kven and Sami peoples belonging to the northern part of Norway. The schools in which we have partnerships are in both the northern and southern parts of Norway. Principals in northern Norway are expected to provide mathematics education in Sami, and sometimes in endangered languages such as Kven. In some of the further north communities, and in the principals' schools in the south, there are a high number of migrated youth who have arrived in the last decade and hence are not (yet) proficient in the main language of instruction, Norwegian.

Methods

Between August 2021 and now, we have been working together with people in five schools. As part of this work we did In-depth interviews with nine principals in our collaborating schools in Norway. Each interview involved two people from our university-based team and one school principal and took place on Zoom. The interviews were recorded, then transcribed in Norwegian and were also translated into English. We independently read each interview multiple times and took notes on the range of ways Indigenous and (newly) migrated students, families, and teachers were positioned in the episodes and tried to identify the kinds of storylines that were at play. We then met and shared the storylines we had identified and evidence for that storyline and, after discussion, came to consensus on a set of storylines to report. As Wagner and Herbel-Eisenmann (2009) have pointed out, there is not a 'correct' way to name a storyline because each participant's perspective may be different. Thus, we drew on the context of the interview, information we knew about the school and community, and the educational, historical, and political context of the specific country to consider how to name the storylines.

Results

In the data material from conversations with principals focusing on Norwegian minoritized youth and their mathematics education, we found a number of storylines, Here we share two larger intertwining storylines, with both supporting and contradicting "smaller" storylines in the conversations. We found that the main storylines were similar from schools in different geographical and socioeconomic areas in Norway. How the principals developed and talked about the storylines, however, differed geographically, with an emphasis on the particular communities in their areas. In this section, we elaborate on the two overarching storylines: "Mother tongue teachers are important resources” and “Students don’t (yet) have the basic language skills to build their mathematics language” in separate sections while acknowledging that they intertwine and sometimes overlap each other.

Mother tongue teachers are important resources

The overarching storyline that all principals address is their concerns for giving students the best opportunities to learn mathematics. The principals talked about how they strive to find support for students who do not (yet) know the language of instruction (Norwegian) or try to ensure teachers can teach or support in the classrooms in languages other than Norwegian. Depending on the geographical area, however, the principal’s way of talking about this storyline differed.

In the Northern context, the principals are obliged to offer education in the minoritised languages, as Sami and Kven. In addition, a number of the parents from indigenous communities want to protect and revitalize their languages and hence wants these languages to be the language of instruction for their children in schools. There are also a number of parents, however, who want their children to learn and know mathematics in Norwegian because this will support the students’ further university studies. In addition, it is very difficult to find teachers who can teach mathematics in Sami and/or Kven. Teachers who are both educated in mathematics teaching, and know the Sami language are rare, if they even exist. The challenges are complex and contradictory.

In southern Norway, there are a higher number of minoritised student groups due to recent migrations. A high number of youth came to Norway as refugees, or in families who arrived in Norway for work. These students are usually offered to participate in a “mottagsklasse”, a “welcome/arrival class”, in schools where these exist. In these classes, they together get accustomed to the school, learn Norwegian and learn subjects such as mathematics even if they are of different ages and are on different learning levels in the subjects. After one year, the students are dispersed in, what is described as “ordinary” classrooms where children are in the same age groups despite knowledge levels. Principals do not talk about these “welcome/arrival classrooms” as ordinary: they are “special” classrooms, in some schools, where the students go one year.

A storyline about the latest arrived students is that it is hard for them to learn mathematics because they do not (yet) know the language of instruction, and the teachers—mainly Norwegian—do not know the students’ languages. As one principal said, “they get lost in translation and it goes both ways”. The principals talked about wanting to employ a higher number of mathematics teachers with migrational background, to teach in these classes. Another option they suggested would be to have mother tongue teachers present in the classrooms.

Students don’t (yet) have the basic language skills to build their mathematics language

This storyline is shared by principals both in the north and south of Norway, however, with different explanations. The storyline addresses minoritised students who do not have sufficient language skills either in their mother tongue language or in Norwegian. These students may be second or third generation migrants, who heard and talked Norwegian only in day care (for example kindergarten and preschool) and then during school hours only. At home, they speak a “modified”, or “adjusted” mother tongue language, mostly lost due to parents, grandparents and/or siblings not being fully exposed to this language while living in Norway over time. The principals talk about these students as having extra challenges as they do not have enough language skills to build a mathematical language in either of their languages.

In the north, the challenges with this storyline are experienced in the same vein, however, the reasons for the storyline are different. The reasons here concern either endangered languages (for example Kven) but also generations that have lost their indigenous languages due to being forced to speak Norwegian during “fornorsksningsperioden”. In other words, families have “lost their heart language” (mistat hjärtespråket sitt) (Broch Johansen, 2020). The situations are complex, and we here give two examples: If the children have been exposed to, for example, Sami, during kindergarten, pre-school and the early years, and/or at home, then they might not know the basic mathematical concepts in Norwegian when they move to secondary school where they switch language to Norwegian as language of instruction. Or, conversely, a student may have attended an early education in Norwegian, and then moved to a secondary school where they may learn mathematics in Sami. Or they were exposed to the indigenous languages in the early years of schooling, but at home they speak Norwegian only. In these cases and according to this storyline, as a minoritised youth, they get caught in the structures and do not have sufficient language skills in any language to build their mathematical language skills. For the principals, the impact and complexities of this storyline becomes an ethical dilemma.

Discussion

We are sharing here our early interpretations about the interviews with principals. In the presentation we will have translated transcripts to share, which will help us illustrate the complexities of these storylines. We recognize that some of these storylines also appear in first attempts to gather international data related to understanding school policies and principals' acknowledgement of difference and strategies that assist or deny access to schools for all children. For example, Goddard (2007) reported that “the issue of adequate funding for interpreters is present in many countries, as is the practice—albeit contested by some—of using children to interpret for their parents” (p. 4). Providing financial resources for mother tongue teachers in schools would be a systemic approach to supporting language for newly arrive immigrants and Indigenous students. Alternatively, resources could be provided that would educate a higher number of teachers with migrational backgrounds to become educators.

In Sweden, Johansson, et al. (2007) reported mixed messages from principals and teachers about the kinds of responses and attitudes to the increasing school diversity. Alongside valuing diversity for how it enriched the school community, for example, was associations with problems of “language difficulties, social programs, support programmes…[and needs of] special support” (p. 26). One of the complexities for us relates to our desire to avoid perpetuating deficit framing of students and cultures. When we identify storylines that pervade education discourse, deficit-framed storylines exist. We struggle to find ways of reporting on these storylines to research and professional education communities while making it clear that we see them as problematic. We also acknowledge that other storylines we have found have centered on individuals rather than systems or structures that need to change (Andersson, et al., 2021; Ryan, et al., 2021). An additional complexity in our work relates to the pandemic and its influence on our research plan. Although we planned to foreground the experiences of minoritized children in Norwegian mathematics classrooms, the pandemic's restrictions on face-to-face interaction forced us to talk with principals before talking with students. We consider this arrangement less than ideal for research drawing on participatory methodology that focuses on classroom opportunities.

Acknowledgments

This research is funded by the Norwegian Research Council's FINNUT-granted project MIM: Mathematics Education in Indigenous and Migrational contexts: Storylines, Cultures and Strength-Based Pedagogies. See https://www.usn.no/mim

References

Andersson, A., Ryan, U., Herbel-Eisenmann, B., Huru, H.L., Wagner, D. (2021, October). Storylines in news media

texts: A focus on mathematics education and minoritized groups. Research paper at the *43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Philadelphia, PA.
Barwell, R., Barton, B., & Setati, M. (2007). Multilingual issues in mathematics education: Introduction. *Educational Studies in Mathematics, 64*(2), 113–119.
Broch Johansen, S. (2020). *Brev til Kommisjonen*. Kárášjohka: Samisk Litteratursenter AS.
Civil, M. (2012). Opportunities to learn in mathematics education: Insights from research with “non-dominant” communities. *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 43–59). Taipei: PME.
Davies, B. & Harré, R. (1990). Positioning: the discursive production of selves. *Journal for the Theory of Social Behaviour, 20*(1), 43–63.
Goddard, J.T. (2007). School leadership and equity: results from an international comparative study. *School Leadership & Management, 27*(1), 1-5. DOI: 10.1080/13632430601092230
Harré, R. (2012). Positioning theory: Moral dimensions of social-cultural psychology. In J. Valsiner (Ed.), *The Oxford handbook of culture and psychology* (pp. 191–206). NY: Oxford University Press.
Herbel-Eisenmann, B., Keazer, L., & Traynor, A. (2018). District decision-makers’ considerations of equity and equality related to students’ opportunities to learn algebra. *Teachers College Record, 120*(9), 1-38.
Herbel-Eisenmann, B., Wagner, D., Johnson, K., Suh, H. & Figueras, H. (2015). Positioning in mathematics education: Revelations on an imported theory. *Educational Studies in Mathematics, 89*(2), 185–204.
Johansson, O., Davis, A., & Geijer, L. (2007). A perspective on diversity, equality and equity in Swedish schools. *School Leadership & Management, 27*(1), 21-33.
Le Roux, K. (2016). Re-imagining mathematics and mathematics education for equity and social justice in the changing South African university. *Critical Studies in Teaching and Learning, 4*(2), 45–67.
Meaney, T., Trinick, T., & Fairhall, U. (2011). *Collaborating to meet language challenges in indigenous mathematics classrooms* (Vol. 52). Springer Science & Business Media.
Reisel L., Hermansen A.S., Kindt M.T. (2019) Norway: Ethnic (In)equality in a Social-Democratic Welfare State. In: Stevens P., Dworkin A. (eds) *The Palgrave handbook of race and ethnic inequalities in education*. (pp. 843–884). Palgrave Macmillan, Cham.
Ryan, U., Andersson, A., Herbel-Eisenmann, B., Huru, H.L., Wagner, D. (2021, September). “Minoritised mathematics students are motivated by gratitude”—an analysis of storylines in Norwegian public media. *11th Annual Mathematics Education & Society* conference, Austria.
Valoyes-Chávez, L., & Martin, D. B. (2016). Exploring racism inside and outside the mathematics classroom in two different contexts: Colombia and USA. Intercultural Education, 27(4), 363–376.
Wagner, D., Andersson, A. & Herbel-Eisenmann, B. (2019). Available positions, identities and discourses in mathematics classrooms. Proceedings of the Tenth International Mathematics Education and Society Conference, Hyderabad, India.
Wagner, D., & Herbel-Eisenmann, B. (2009). Re-mythologizing mathematics through attention to classroom positioning. *Educational Studies in Mathematics, 72*(1), 1–15.

ANALYTICAL PROCESSES FOR MEASURING EQUITABLE NOTICING IN MATHEMATICS

PROCESOS ANALÍTICOS PARA MEDIR LA MIRADA PROFESIONAL Y EQUIDAD EN MATEMÁTICAS

Cindy Jong
University of Kentucky
cindy.jong@uky.edu

Jonathan Thomas
University of Kentucky
jonathan.thomas1@uky.edu

Walker Mask
University of Kentucky
walker.mask@uky.edu

Molly Fisher
University of Kentucky
molly.fisher@uky.edu

Edna Schack
Morehead State University
edna.schack@gmail.com

In this paper, we provide two analytical methods for measuring the intersecting constructs of equity and professional noticing, as defined by Jacobs et al. (2010). One is a scoring rubric utilizing Gutiérrez's (2009) four dimensions of equity to include access, achievement, identity, and power, along the dominant and critical axes. The second is a flowchart to examine asset, deficit, or neutral language in professional noticing responses about students' mathematical thinking. We discuss the affordances and limitations of both methods, along with challenges for measuring equitable noticing in meaningful ways. Preliminary findings have shown a positive increase in the dominant axis where responses elaborate upon the dimensions of access and achievement.

Keywords: Teacher Noticing; Equity, Inclusion, and Diversity; Research Methods

Perspectives

There is emerging interest in studying equity in conjunction with professional noticing (Jong, 2017; Louie et al., 2021). For example, both Kalinec-Craig (2017) and Hand (2012) have examined student positioning (and, by extension, power) in the context of professional noticing. Specifically, Hand's construction of *taking up space* does not simply "comprise [students] participation in classroom mathematical practices, but rather is about being able to contribute to the classroom community that is aligned with who one sees herself as becoming" and the extent to which teachers' in-the-moment decision-making enlarges or constrains such student contributions through manifestations of culturally dominant ideologies (p. 237). Such connections of noticing and equity are consistent with portrayals of professional noticing as contested and political space (Lefstein & Snell, 2011; Louie, 2018). Regarding the pedagogical activity that might productively influence such spaces, van Es et al. (2017) posited a number of practices (i.e., make norms explicit for doing mathematics, support students in developing mathematical identities) and associated foci for professional noticing that they describe as *noticing for equity*. Building on this work, Louie et al. (2021) recently developed a sociopolitical framework for conceptualizing anti-deficit teacher noticing, which aligns closely to the flowcharts we will present.

The aim of this brief report is to describe two analytical methods for measuring equitable noticing, which is a burgeoning area of research (Jong, et al., 2021). Several scholars have developed frameworks for equity or social justice in mathematics (Goffney et al., 2018; Gutiérrez, 2009; Stinson & Wager, 2012), which have been applied to research in mathematics education. However, limited studies have shown analytical methods that are generalizable to multiple research settings within equitable noticing. Jackson et al. (2018) describe a coding dictionary they created in the form of a matrix that uses Gutiérrez's (2009) four dimensions of equity as columns on top and three levels of noticing as rows, which mirror the components of professional noticing (Jacobs et al., 2010). They then provide sample text based on preservice teachers' responses to vignettes about equitable access to mathematics, which is helpful to understand how they operationalized the varying dimensions of equity across levels of noticing. In our project, the depth of responses to the varying dimensions of equity were also taken into consideration.

Methodology

Context of Project

The research team was awarded an NSF grant to develop modules that integrate equity and noticing for mathematics methods courses. A total of eight modules were developed and implemented in mathematics methods courses for elementary teacher candidates during fall 2020 and spring 2021. The two frameworks woven into the modules are Gutiérrez's (2009) four dimensions of equity and Jacobs, Lamb, and Philipp's (2010) professional noticing of children's mathematical thinking, including attending, interpreting, and deciding. To measure improvement in teacher candidates' professional noticing skills and deepen understanding of equitable mathematics teaching, we designed a video-based instrument that is administered pre- and post-methods course. After viewing a 74-second clip of a second-grade classroom with racially diverse students participating in a number talk to solve the open number sentence of 10 + 10 = __ + 5, teacher candidates respond to five prompts. Three prompts relate to that which they attended, interpreted, and decided about the children's mathematical thinking in the video. And, the following two additional prompts relate to equitable noticing. Q5: "Describe how equity relates to this classroom scenario in different ways." Q6: "If you were the classroom teacher, how might you strengthen equity in this classroom scenario?" Here, we focus on our analysis of the noticing equity responses which is framed by the work of Gutiérrez (2009, see figure 1).

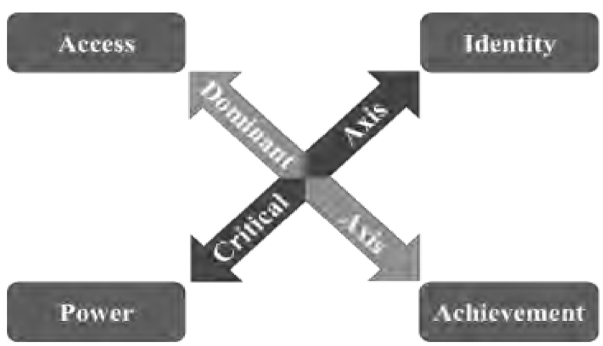


Figure 1: Four Dimensions of Equity (Gutiérrez, 2009)

Gutiérrez describes four dimensions categorized into two axes: the dominant axis (access and achievement) and critical axis (identity and power). We developed a scoring rubric ranging from 0-4 for each axis of equity (see figure 2). Alongside the scoring rubric, we created a codebook that includes codes for each dimension of equity to help us determine which dimension is mentioned and whether the response elaborates on a dimension while also grounding it in the video of the classroom scenario. While we acknowledge the limitations in this approach, we also think there is great potential for our scoring rubric to be applied to the analysis of additional mathematics classroom video clips to further research the intersection of noticing and equity. In addition, we used a flowchart developed for another equitable noticing project to code each response based on whether the framing of the language was oriented toward one of the following: asset, deficit, neutral, or both (Thomas, et al., 2020). However, in this paper we do not include results of these categories.

0	1	2	3	4
incorrect	"OR" Attend to (mention) either dimension of the axis	"AND" Attend to (mention) both dimensions of the axis	**"OR" Attend to either dimension of axis with interpretation/productive response along the axis**	**"AND" Attend to both dimensions of axis with interpretation/productive response along the axis**

***Must be connected to classroom scenario to move beyond level 2.**

Figure 2: Coding Rubric for the Dominant and Critical Axis

Each response was given a score for the dominant and critical axes. Take this attending response as an example. *"The teacher doesn't seem to call on many girls. This could be a classroom dynamic they struggle with, or a bias on the teacher's part. This relates to equity because she isn't giving each child equal access to the problem. Only certain students have the floor to give their answers, while the others sit around looking bored. It also connects to the demographics of the classroom. Which students need extra scaffolding and help while other can get the correct answer quickly."* For the dominant axis, this would score a 3 because it provided an interpretation of the access dimension by elaborating on it. For the critical axis, it would score a 4 because it elaborated on the identity dimension with gender then acknowledged power by who has a voice to share their answers (see figure 3). We created a spreadsheet with a Boolean Function to score responses depending on whether it elaborated on one of the dimensions on the axis as the "OR" (for the score of 3) or included both dimensions within the axis and elaborated one of them for the "AND" (score of 4). The spreadsheet also had a summarizing tab that highlighted discrepancies between the two scorers to discuss and negotiate.

Figure 3: Google Spreadsheet with Boolean Function for Scoring each Axis

In our scoring, we also wanted to flag any responses that discussed systemic or structural inequities and potentially score those as a 5, but none included that in the equitable deciding responses for Q6 where it was more of a possibility. In addition, we wanted to flag responses that were deficit oriented, incorrect, or unproductive. For example, the following response speaks to access, but is incorrect and unproductive: *"Students did not understand the problem and should be grouped to make sure they do."* Because of the nod to access, this response was scored a 1, but also given a negative code due to the deficit orientation. Using both scoring mechanisms (dimension and orientation) for each response assured another step toward reliability.

Results

Given our limited sample size (n=34) of matched responses from pre to post, we cannot make substantial claims based on statistically significant differences (see Table 1). However, we noticed that for Question 6 related to the dominant axis, there was an increase in responses that mention access and achievement. What is more encouraging is that post-responses appear to be more complex and include rationales. They also noticed that only boys were selected in the video and recommended in Q6 that girls should also be selected to elevate their voices, and that students should be given time to share their ideas with a partner before a whole group discussion.

Table 1: Paired T-test Results

Pre/Post Responses	Change in Mean	Std. Dev.	Std. Error Mean	t	Sig. (2-tailed)
Q5 Dominant Score	1	0.75593	0.2672	3.742	0.007
Q5 Critical Score	0.125	1.45774	0.5153	0.243	0.815
Q6 Dominant Score	0.625	0.74402	0.2630	2.376	0.049
Q6 Critical Score	0.375	0.74402	0.2630	1.426	0.197

Discussion

We are beginning to see some patterns emerge in the way preservice teachers notice equitable teaching practices in mathematics. It is encouraging that there was some growth detected in the dominant axis of equity related to decisions in Q6. However, we also acknowledge the many limitations of the implementation of the modules during the pandemic year that included a small sample size. It might also be the case that the video-based assessment and analytical process did not allow for more variation in the dimensions of equity to be fully explored. In fall 2021, there were more sites that implemented the modules with over 200 preservice teachers. Thus, we aim to better examine potential changes and relationships among the dimensions of equitable noticing in this next round of coding.

Video-based measures have been widely used in noticing because they serve as an approximation of practice (Jacobs, 2017). However, the video only shows a snapshot of a classroom interaction and may not provide a full picture of the classroom environment or dynamics. Expanding the type of classroom artifacts to include vignettes, videos, and student work might capture a more valid measure on preservice teachers' equitable noticing in mathematics that could also have greater variation across dimensions of equity.

Acknowledgments

This research was supported by a grant from the IUSE Award #1914810. The opinions expressed herein are those of the authors and do not necessarily reflect views of the National Science Foundation.

References

Goffney, I., Gutiérrez, R., & Boston, M. (Eds.), (2018). *Annual Perspectives in Mathematics Education 2018: Rehumanizing Mathematics for Black, Indigenous, and Latinx Students.* NCTM: Reston, VA.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game.". Teaching for

Excellence and Equity in Mathematics, 1(1), 4-8.
Gutiérrez, R. (2018). Introduction: The need to rehumanize mathematics. In I. Goffney., R. Gutiérrez, & M. Boston (Eds.), *Annual Perspectives in Mathematics Education 2018: Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1-12). NCTM: Reston, VA.
Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. Educational Studies in Mathematics, 80, 233-247.
Jackson, C., Taylor, C. E., & Buchheister, K. (2018). Seeing mathematics through different eyes: An equitable approach to use with prospective teachers. In Toward equity and social justice in mathematics education (pp. 263-285). Springer, Cham.
Jacobs, V. R. (2017). Complexities in measuring teacher noticing: Commentary. In Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks (pp. 273-279). New York, NY: Springer.
Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41, 169-202.
Jong, C., Schack, E.O., Thomas, J., Fisher, M.H., & Dueber, D. (2021). What role does professional noticing play? Examining connections with affect and mathematical knowledge for teaching among preservice teachers. ZDM Mathematics Education, 53(1), 151-164.
Jong, C. (2017). Extending equitable teaching practices in teacher noticing: Commentary. In Schack, E. O., Fisher, M. H., & Wilhelm, J.A. (Eds.). Teacher Noticing: Bridging and Broadening Perspectives, Contexts, and Frameworks (pp. 207-214). New York, NY: Springer.
Kalinec-Craig, C. (2017). Everything matters: Mexican-American prospective elementary teachers noticing issues of status and participation while learning to teach mathematics. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks (pp. 215-229). New York, NY: Springer.
Lefstein, A. & Snell, J. (2011). Professional vision and the politics of teacher learning. Teacher and Teacher Education, 27, 505-514.
Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. Educational Studies of Mathematics, 97, 55-69.
Louie, N., Adiredja, A. P., & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: the FAIR framework for anti-deficit noticing. ZDM–Mathematics Education, 1-13.
Stinson, D. W., & Wager, A. (2012). A sojourn into the empowering uncertainties of teaching and learning mathematics for social change. Teaching mathematics for social justice: Conversations with educators, 1(1), 3-18.
Thomas, J., Marzilli, T., Sawyer, B., Jong, C., Schack, E.O., & Fisher, M.H. (2020). Investigating the Manifestations of Bias in Professional Noticing of Mathematical Thinking among Preservice Teachers. Journal of Mathematics Education at Teachers College, (11)1, 1-11.
van Es, E., Hand, V., & Mercado, J. (2017). Making visible the relationship between teachers' noticing for equity and equitable teaching practice. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks (pp. 251-272). New York, NY: Springer.

USING ANTI-BIAS PICTURE BOOKS TO ELICIT SOCIAL JUSTICE MATHEMATICS LESSONS IN EARLY CHILDHOOD TEACHER EDUCATION

Bailey Kaufman-Ridout
Fordham University
bkaufman12@fordham.edu

Alesia Mickle Moldavan
Georgia Southern University
amoldavan@georgiasouthern.edu

This study examines early childhood preservice teachers' experiences using anti-bias picture books to design mathematics lessons addressing issues of social justice during a children's literature course. With reference to the Teaching Mathematics for Social Justice framework (Gutstein, 2006) and the Learning for Justice (2016) social justice standards, preservice teachers designed lessons to address issues of identity, diversity, justice, and action. Findings indicate that the preservice teachers were more likely to target social justice pedagogical goals focused on identity and diversity over justice and action. Additionally, the preservice teachers experienced increased confidence when working with such lessons. Recommendations are made that preservice teachers in early childhood education have explicit opportunities to engage in such tasks to develop their justice-related mathematical consciousness and professional practice.

Keywords: Preservice Teacher Education, Social Justice, Early Childhood Education, Instructional Activities and Practices

Early childhood educators provide some of the first experiences for young learners to engage with mathematical thinking (National Research Council, 2009). Thus, early childhood preservice teachers (PTs) need the opportunity to engage in explicit tasks focused on mathematics in order to become well-rounded early childhood educators (National Council of Teachers of Mathematics, 2013). An increasing priority in the field of early childhood education (ECE) is creating equitable instruction that is reflective of the world outside the classroom (Alanís et al., 2021). With content only lessons, students can experience a disconnect between the skills they learn in the classroom and real-world applications. The intersection of mathematics and social justice is recognized as Teaching Mathematics for Social Justice (TMfSJ) and aims to use mathematics as a tool to confront systemic challenges within and outside the classroom (Gutstein, 2006). When establishing goals for TMfSJ lessons, teachers need to consider both mathematics and social justice standards. Learning for Justice (2016) developed social justice standards that organize anti-bias standards within four domains: Identity, Diversity, Justice, and Action (IDJA). These standards feature grade-level outcomes that are applicable to young learners and can be used in ECE to target social justice ideas.

While the current literature examining TMfSJ lessons is primarily focused on mathematical concepts found in secondary settings, mathematics in ECE settings can benefit from the goals of TMfSJ. Although the literature on TMfSJ is broad, Ward (2020) demonstrates that TMfSJ can be used in ECE to create opportunities for action targeting social justice through child-led decision-making. Current studies (e.g., Bartell, 2013; Myers, 2019) regarding PTs experiences with TMfSJ are more commonly associated with secondary education, with very few looking at the experiences and beliefs of early childhood PTs. This study examined the research question: What are PTs' experiences engaging in an anti-bias picture book task to develop their skills and confidence to design mathematics lessons targeting social justice issues?

Theoretical Framework

Rooted in Freire's (1970) concept of critical education and Frankenstein's (1983) application of the theory to mathematics, TMfSJ aims to empower students to investigate and challenge societal inequities through the development of their critical consciousness and subsequent application of mathematical skills (Berry et al., 2020). This empowerment occurs through the prioritizing of both mathematical pedagogical goals and social justice pedagogical goals (Gutstein, 2006). One noted tension of TMfSJ is how the success of students or teachers engaging with both pedagogical goals is determined (Kokka, 2015). One variable noted in contributing to success in achieving both goals is the ability to cater instructional time to both goals (Gregson, 2013; Gutstein, 2006). Learning to balance both mathematical and social justice concepts and goals is a skill that is developed in TMfSJ practices.

For purposes of this work, we consider the theoretical lens of Berry et al. (2020)'s social justice mathematics framework that incorporates six elements detailing the progression of the social and mathematical practices that lead to action and public products. We position the elements into two stages, which we deem as progression markers, to describe where PTs are on their journey of "justice-related mathematical consciousness". We note the first stage as "beginning of justice-related mathematical consciousness" to address the preliminary progress toward equitable teaching practices that consider authentic and challenging questions or concerns to generate social and mathematical understanding. The second stage described as "emergent and continuing justice-related mathematical consciousness" continues beyond those questions or concerns to create a social and mathematical investigation that requires reflection and action, resulting in a tangible outcome. We posit that students who created an action plan were more inclined to put their plan into action, thereby having success within their targeted pedagogical goals (Kokka, 2015).

Research Methods

We used a qualitative case study (Yin, 2014) across two asynchronous sections of a children's literature course at a rural, public two-year college in southwestern United States. Consenting participants included 35 PTs who self-identified as female (n=33), male (n=1), and gender fluid (n=1) seeking early childhood teaching certification. Their demographics could be further described as Hispanic/Latinx (n=20), White (n=6), Black (n=3), Asian (n=2), American Indian/Alaskan Native (n=1), Biracial (n=1), Egyptian-American (n=1), and other (n=1).

Task Design

Students worked through asynchronous, scaffolded modules to complete this task. The first module encompassed resources from DREME (2022) to explore the use of picture books to teach early mathematics as well as review of the Learning for Justice (2016) anti-bias framework and Ward's (2020) research. Next, the PTs were tasked with selecting a relevant anti-bias picture book with consideration for how the books embodied standards from one of the anti-bias IDJA domains. They also considered which early mathematics standard each book could target, such as spatial relationships, number sense, classification, measurement, and patterning (Carpenter et al., 2017). Using their selected standards, PTs then wrote an early mathematics lesson targeting social justice in reference to their anti-bias picture book. The final action required PTs to reflect on their experience incorporating social justice into a mathematics lesson.

Data Sources

A variety of data sources were collected including pre- and post-surveys, artifacts (lesson plans), and reflection responses. Before accessing course material, PTs completed a pre-survey and answered questions relating to confidence incorporating themes of social justice into lesson planning. All participants submitted lesson plans that included both the mathematics and social justice standards that informed the lesson, background information about their selected anti-bias picture book, and a step-by-step description of the lesson. After completing the lesson plan, participants answered reflection questions about their specific lesson and responded to the post-survey that addressed their confidence incorporating themes of social justice into lesson planning after having attempted this anti-bias picture book task.

Data Analysis

We initially looked at which social justice domains the participants selected standards from and if certain books were chosen for specific domains. We also noted which early mathematics skills were selected. In addition to analyzing which standards were selected, we used descriptive, in vivo, and values coding (Miles et al., 2018) to discover what themes or aspects of identity (e.g., race, gender, ethnicity, socioeconomic status) each task and respective book centered in the context of the social justice standards. We then coded for themes within the reflective responses regarding how their mathematics task and selected anti-bias picture book encourages advocacy and action, reflects diversity as an asset, and promotes social justice mathematical thinking.

Findings

Most of the PTs (n=23; 65.71%) selected Identity and Diversity standards rather than Justice and Action standards (n=8; 22.86%), with the remaining (n=4; 11.43%) not selecting a standard. Figure 1 shows the associated issue within each domain that PTs targeted using their selected texts. For example, the domains of Identity, Justice, and Action included race as a focused issue in the lessons; however, race was not a focus in the Diversity domain. While it is good practice to support Identity and Diversity within ECE, the TMfSJ framework encourages more critical thought through inclusion of standards from the Justice and Action domains (Berry et al., 2020). The most common mathematical concept selected by PTs (n=19; 54.28%) was number sense. Others included patterning (n=5; 14.29%), measurement (n=5; 14.29%), classification (n=5; 14.29%), and spatial relationships (n=1; 2.85%).

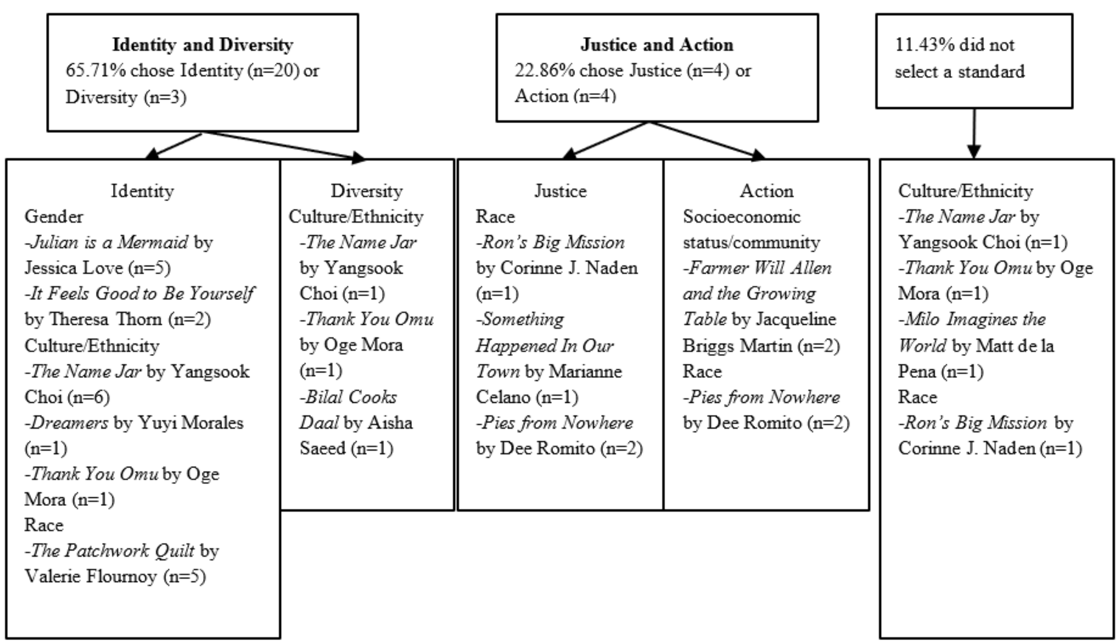


Figure 1: PT's Book Selections by Social Justice Domain

It was apparent that the PTs who selected anti-bias books within the Identity and Diversity domains created lessons that leaned heavily on the social justice goals with less focus on the mathematical goals. Findings revealed that there were fewer lessons that seemed to integrate the two pedagogical goals in a cohesive manner. One lesson plan that succeeded in balancing the pedagogical goals used an Action standard and targeted number sense. The PT used the book *Farmer Will Allen and the Growing Table* by Jacqueline Briggs Martin to explore concepts of community leaders, food deserts, and food insecurity to organize a class action plan to address food insecurity through a classroom food drive. The class demonstrated one-to-one correspondence and cardinality by counting the collected food items. A degree of mathematical consciousness was required for this PT to create a TMfSJ lesson plan that included an Action plan resulting in a tangible product for the community. When identifying a PT's stage in their journey of justice-related mathematical consciousness, it is necessary to examine the ability, or inability, that a PT has in balancing mathematical and social justice goals.

As noted in Figure 2, the PTs' mean confidence levels with social justice lessons increased after participating in the task. Furthermore, an interesting dichotomy arose in the reflective questions. For instance, two PTs commented that their text and lesson promotes social justice mathematical thinking by emphasizing how all individuals should have fair opportunities to engage. Another PT stated, "some of our peers have had the opportunity to visit some of these places and others haven't, which highlights the fact that life is easier for some people and harder for others and reasons are not always fair." These three PTs recognize that fair opportunities are needed; however, the third PT makes deliberate action plans in their lesson to critique systemic injustices, thereby demonstrating an "emergent and continuing justice-related mathematical consciousness". PTs need exposure to pedagogies that prioritize social justice, such as TMfSJ, in order to gain the confidence to create and implement these lessons into their future practice.

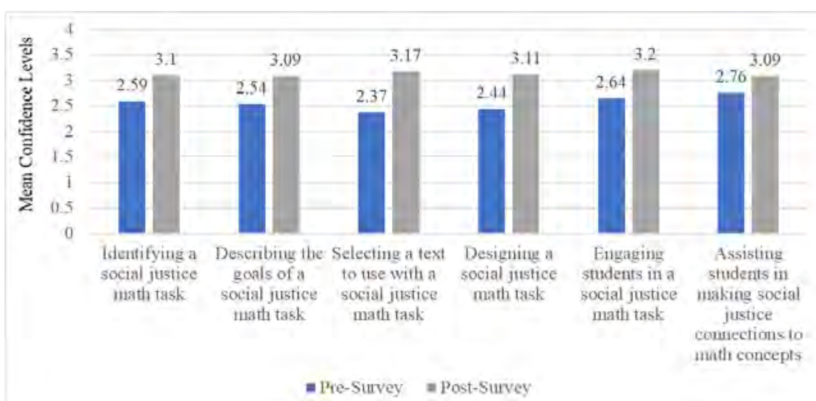


Figure 2: PT's Mean Confidence Levels with Social Justice Lessons

Conclusion

The application of TMfSJ is needed not only in the early childhood classroom but also teacher education programs. Positioning this work in generalist courses, in addition to mathematics methods courses, is important especially given how early childhood PTs lack opportunities to dive deeper into mathematical concepts and issues (Bates et al., 2013). TMfSJ gives PTs the ability to use mathematics as a tool to critically analyze the world around them, which will expand lessons past the Learning for Justice (2016) domains of Identity and Diversity and into Justice and Action (Bartell, 2013; Ward, 2020). Recognizing that PTs are at various stages of their justice-related mathematical consciousness means that scaffolded support is important for success when engaging with TMfSJ in the early childhood setting.

References

Alanis, I., Iruka, I. U., & Friedman, S. (2021). *Advancing equity and embracing diversity in early childhood education: Elevating voices and actions.* National Association for the Education of Young Children.

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, *44*(1), 129–163.

Bates, A. B., Latham, N. I., & Kim, J. (2013). Do I have to teach math? Early childhood pre-service teachers' fears of teaching mathematics. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, *5*, 1–10.

Berry, R. Q., III., Conway, B. M., IV., Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice*. Corwin Press.

Carpenter, T. P., Franke, M. L., Johnson, N. C., Turrou, A. C., & Wager, A. A. (2017). *Young children's mathematics: Cognitively guided instruction in early childhood education*. Heinemann.

DREME. (2022). *Overview: How to use picture books*. https://prek-math-te.stanford.edu/overview/how-use-picture-books

Frankenstein, M. (1983). Critical mathematics education: An application of Paulo Freire's epistemology. *The Journal of Education*, *165*(4), 315–339.

Freire, P. (1970). *Pedagogy of the oppressed*. Continuum.

Gregson, S. A. (2013). Negotiating social justice teaching: One full-time teacher's practice viewed from the trenches. *Journal for Research in Mathematics Education*, *44*(1), 164–198.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. Routledge.

Kokka, K. (2015). Addressing dilemmas of social justice mathematics instruction through collaboration of students, educators, and researchers. *Educational Considerations*, *43*(1), 13–21.

Learning for Justice. (2016). *Social justice standards: The teaching tolerance anti-bias framework.* https://www.learningforjustice.org/sites/default/files/2020-09/TT-Social-Justice-Standards

Miles, M. B., Huberman, A. M., & Saldaña, J. (2018). *Qualitative data analysis: A methods sourcebook*. SAGE.

Myers, M. (2019). Learning to teach mathematics for social justice: Helping preservice teachers connect critical topics to standards. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1087–1095). St. Louis, MO.

National Council of Teachers of Mathematics. (2013). *Mathematics in early childhood learning: A position of the National Council of Teachers of Mathematics*. National Council of Teachers of Mathematics.

National Research Council. (2009). *Mathematics learning in early childhood: Paths toward excellence and equity.* National Academy Press.

Ward, J. (2020). Exploring playground access with mathematics. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(11), 887–894.

Yin, R. K. (2014). *Case study research: Design and methods* (5th ed.). SAGE.

REDESIGNING MATHEMATICS FOR THE MARGINS: EMPATHY AS A DISRUPTOR OF DEFICIT THINKING

Rachel Lambert
University of California
Santa Barbara
rlambert@ucsb.edu

Kara Imm
Hunter College, New
York, NY
karaimm@gmail.com

A persistent problem in mathematics education has been deficit thinking by teachers about the mathematical potential of students with disabilities and students who have been marginalized in mathematics. This paper reports analysis of findings of a six-week virtual class in the summer of 2020 for 45 mathematics and special educators which integrated Universal Design for Learning (UDL) with design thinking, focusing on the teaching and learning of mathematics. Our approach grounds UDL in empathy, beginning design with developing understanding of student perspective through Empathy Interviews. In focus group conversations nine months late, participating classroom teachers reported that empathy and Empathy Interviews were the most important long-term takeaway of the course, and that empathy became a touchstone in their teaching practice. Empathy seemed to disrupt deficit thinking for these teachers.

Keywords: Equity and Diversity; Students with Disabilities; Teacher Beliefs; Affect, Emotion, Beliefs, and Attitudes.

A fundamental issue with including students with disabilities into inquiry-based mathematics is not the difficulties those students might have, but deficit thinking about the students held by teachers (Lambert, 2018). This deficit thinking is a part of the larger stigma and oppression of disabled people, but also a specific result of decades of traditional special education questioning whether of these students to profit from inquiry-based teaching (Lambert & Tan, 2020). Disrupting deficit thinking in teachers is necessary to create opportunities. But how do we as math educators and researchers disrupt deficit thinking? What kinds of practices can engage teachers in on-going development away from deficit thinking?

This brief report discusses positive findings from a professional development course for math and special educators on Universal Design for Learning through a Design Thinking lens. In our 6-session course, we explored UDL for math, as well as rethinking disability through frameworks such as disability rights and neurodiversity. Groups of educators worked together in Design Teams to identify a problem of practice around inclusion and accessibility, to learn more about the users' perspectives through empathy interviews, and to engage in a cycle of Design Thinking to develop a prototype for a new curricula, system, space or routine. Data sources for our research included class video recordings, transcripts and artifacts, as well as pre- and post-surveys.

Our course goals included empathy, specifically as an element of design thinking.

RQ1: How did participants who took our online course on UDL and Design Thinking in Mathematics take up elements of the class in their practice during the 2020-2021 school year?

RQ2: How did participants conceptualize empathy and empathy interviews? How did they operatize empathy interviews in their classrooms after the course?

Literature Review

Instead of locating disability within individual students, In UDL, disability is located in inaccessible classrooms, curriculum, and spaces (Meyer et al., 2014). The potential of UDL rests in its power to redesign these classrooms, curriculum, and systems to work better for students with disabilities. We take a critical approach of UDL through a disability studies approach (Domage 2017). We have integrated Design Thinking for Educators into our approach to UDL (Lambert et al., 2021). Design Thinking among teachers is seen as a collaborative, human-centered problem-solving approach which starts by developing empathy and understanding for the user's experiences (IDEO 2012). Empathy, as suggested by proponents of design thinking, attempts to integrate two larger ideas: perspective-taking and cultivating sense of care. Cultivating empathy for a user is to more clearly understand how that user lives in the world --- how they experience it and make sense of it. This requires a designer to put aside their own experiences, beliefs and ideas and work to view from another's point of view. Within design thinking for educators, teachers are positioned to appreciate and make sense of their students' lived experiences, so that they can later design for and with them.

In our use of Design Thinking we highlighted the importance of the empathy interview, in which the designer elicits a series of stories from the user to better understand their experiences. Empathy interviews are meant to be informal conversations between designers and users that uncover the users' experiences in a particular setting, their feelings about those experiences and ultimately, their unmet needs within that context. The goal, in a way, is to understand a person's thoughts, emotions, and motivations, so that we can design for them. When leading an interview the strategy is typically to get the user to tell stories, because stories allow designers to better make sense of how a person sees the world.

Because the empathy interview can be rich, detailed and complex, it is often the case that designers work in teams to better capture the important ideas and moments. Often the detailed notes from an interview are synthesized into a few broad categories and codified into an *empathy map*. The making of empathy maps allows the designers to move out of the empathy phase of designing and into the define phase, where two goals emerge: to develop a deep understanding of the user and develop an actionable problem statement, or point of view. Additionally, the visual aspect of an empathy map itself (not a narrative statement but a colorful retelling) reflects a common mindset of designers: showing, not telling. While not specific to mathematics education, empathy has been found to be a critical feature in teaching. Teachers' ability to empathize with their students shapes the teachers' response to problem behaviors (Wink et al. 2021). One meta-analysis of student teacher relationships found that empathy by the teacher was the strongest predictor of positive outcomes for students (Cornelius-White, 2007).

Findings from survey data (Lambert et al., 2021) indicate that the course resulted in shifts in participants' conceptualization of UDL as a set of guidelines to an active design process. Our initial survey findings also indicate that our centering of empathy became a guiding principle for participants' understanding of designing for disability. We wondered how classroom teachers who participated in the summer course would take up these ideas during a year of pandemic teaching.

Methods

We used a design research teaching experiment methodology (Cobb et al., 2003) for this study, beginning with our theory of action and using multiple, iterative methods to understand participant's learning. This paper focuses on data from a focus group of classroom teachers held in April 2021, 9 months after the completion of the course, recruited with an email sent to all course participants. The four participants in the focus group all identified as women, 3 as white, 1 as Latina, and 1 as disabled. Two were high school special educators, and two were elementary school general education teachers at a dual language school. All teachers had students with disabilities in their classes during the 2020-2021 school year, with the majority of all their students being Latinx.

Data Sources and Analysis

The focus group was semi-structured, with questions designed collectively by the research team. Two members of the research team participated in the focus group, with one facilitating and one listening for follow-up questions. The first set of questions explored experiences teaching in the 20-21 school year, then UDL and Design Thinking. After the focus group, a member of the research team transcribed the interview and wrote an initial memo on the findings, as they related to each of the research questions. All members of the research team then conducted in vivo coding of the transcript. We analyzed these codes and came to consensus on codes and themes.

Results

An important context in these results is the modality of teaching. These teachers are describing their work in the 2020-2021 school year in California, primarily in districts that were conducting school through virtual distance learning. Their students, including emergent bilingual and disabled students in elementary and high school, were taking classes over Zoom. Only one teacher, Bella, a secondary teacher, was teaching hybrid, with students in person and over Zoom each week. Learning over Zoom created (or perhaps heightened) the importance of building relationships with students (Lambert et al., 2020), particularly students with disabilities.

We found that classroom teachers described the work of the course as extremely useful for the challenging teaching of 2020-2021, particularly taking up empathy as a core aspect of their practice. When asked the general question, "What did you take away from the course," the first and all following respondents focused on the concept of empathy and the specific tool of empathy interviews. We found three themes in their understanding of empathy and its role in their teaching. First, all participants reported using empathy interviews extensively in order to design for students' actual needs and to build relationships. Second, empathy appeared to interrupt deficit thinking in designing and reflecting on mathematics activities. Finally, empathy was used as a way to understand larger themes in their work, including racial equity and inequalities around the pandemic.

Empathy Interviews

Both secondary special education math teachers described themselves as "case managers" and both did empathy interviews with all or most of the students on their caseload. Zelda, a secondary special educator who taught math, responded to our question ("What did you take away from the course, if anything") by first centering empathy as a core value in her work, "Empathy is absolutely like, just keeping that at the heart of my practice." She then described how she used Empathy Interviews with all her students. Both elementary inclusive teachers also did a version of Empathy Interviews. One interviewed all her students at the beginning and then the end of the year, over Zoom. The other elementary math teacher also did Empathy Interviews with all of her students, as well as with each family. The focus was on developing understanding of the student's perspective and experience learning mathematics, with the teacher framed as listener. The teachers discussed the importance of building relationships through these interviews, of learning more about the student's experiences. This knowledge, particularly of the student’s past experiences learning mathematics, was described as invaluable, and played a central role in the teacher’s design of their courses that year.

Shifts away from deficit thinking

Teachers described that the course effected a shift away from deficit thinking and towards more complex, strengths-based understanding of students. They reported that this shift was linked to empathy. Discussing a shift away from deficit thinking, Bella said, “I think empathy has a lot to do with that too, because once we put ourselves in the place of a child, with a disability . . . it has to change our thinking. So I think empathy is a gateway for that.” Zelda noted that the course “revealed to me some of my deficit thinking that I wasn’t aware was necessarily there.” She described a math activity with her students over Zoom that seemed to go poorly. She narrates her thought process, first how she immediately jumped to the idea that that kind of activity was too complex for her students. Then, she narratives how she interrupts that thought,

> I can catch that thought and be like, no it's not that they can't do it, it's like – what, do we need to try again? Does [the student] need more time? You know, and also asking them, like, ‘How is this going?’ So I think, for me, I feel like the course has hugely impacted my practice.

Empathy appears to be useful in helping teachers disrupt their habits of assessing students as knowing, or not knowing, or capable or not capable, and instead to see learning as a process. And to see design of instruction as necessarily co-design with students.

Larger themes related to empathy

Another application of empathy was to note when it was lacking in other teachers during the pandemic. Zelda noted that for her, as a case manager of high school students with disabilities, she brough to the math department’s attention the high number of students with disabilities who were failing math over Zoom. She reports that the math department was “rigid” and lacked empathy. She describes a feeling a “rage” towards other educators who do not seem to connect with the students.

> it's – there's been moments of rage, I'm not gonna lie …there’s such a lack of empathy, right? There's such a lack of empathy with – and rigidity. Like if there's no other time for you to like rethink your grading like – this isn't going to do it for you? What’s gonna do it? But I do think sometimes in crisis actually people like stick harder to their guns, you know they hold to their guns they're like No, this is how I do it.

Empathy seems for these educators to link the core values of social justice with relationships, and a more relational approach to the teacher student relationship.

Significance

We call attention to how a focus on empathy, coupled with analytic tools for putting empathy into practice, was important in shifting deficit thinking about students with disabilities, and shifting teachers to more flexible mathematics teaching. While empathy was an important part of our design, we did not expect it to be the primary driver of change for teachers. We certainly did not expect all four of the teachers in our focus group to have used Empathy Interviews extensively in their challenging classroom work in 2020-2021. They reported, however, how central empathy was to their work in challenging ties. In future work, we seek to better understand links for racial equity as well as disability equity. As mentioned by a participant in the course in the surveys “empathy really is like, it's like the center of the pandemic, for me, and also the racial justice work that has needed to happen and it's like cracked open in this new way.

References

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

Cornelius-White, J. (2007). Learner-centered teacher-student relationships are effective: A meta-analysis. *Review of Educational Research*, *77*(1), 113-143.

Dolmage, J. T. (2017). *Academic ableism: Disability and higher education*. University of Michigan Press.

IDEO. (2012). *Design thinking for educators*. http://designthinkingforeducators.com/.

Lambert, R. (2018). “Indefensible, illogical, and unsupported”; Countering deficit mythologies about the potential of students with Learning Disabilities in mathematics. *Education Sciences*, *8*(2), 72. https://doi.org/10.3390/educsci8020072

Lambert, R., Imm, K., Schuck, R., Choi, S., & McNiff, A. (2021). “UDL Is the What, Design Thinking Is the How:” Designing for Differentiation in Mathematics. *Mathematics Teacher Education and Development*, *23*(3), 54–77.

Lambert, R., Sugita, T., Yeh, C., Hunt, J. H., & Brophy, S. (2020). Documenting increased participation of a student with autism in the standards for mathematical practice. *Journal of Educational Psychology*, *112*(3), 494–513. https://doi.org/10.1037/edu0000425

Lambert, R., & Tan, P. (2020). Does disability matter in mathematics educational research? A critical comparison of research on students with and without disabilities. *Mathematics Education Research Journal*, *35*, 5–35. https://doi.org/10.1007/s13394-019-00299-6

Meyer, A., Rose, D. H., & Gordon, D. T. (2014). *Universal design for learning: Theory and practice*. CAST Professional Publishing. http://udltheorypractice.cast.org/

Wink, M. N., LaRusso, M. D., & Smith, R. L. (2021). Teacher empathy and students with problem behaviors: Examining teachers' perceptions, responses, relationships, and burnout. *Psychology in the Schools*.

"I WAS PLAYING INTO WHITE SUPREMACY": PRESERVICE TEACHERS' NARRATIVES POSITIONING WHITENESS, MATHEMATICS, AND THEMSELVES

Alexa W.C. Lee-Hassan
Learning Sciences Research Institute, University of Illinois at Chicago
aleeha2@uic.edu

Preservice teachers need to develop political conocimiento—experiential political knowledge—to be able to effectively advocate for their students. This includes developing a complex understanding of their own positionality with respect to race and mathematics. This brief report, part of a larger study, uses interviews of three preservice teachers describing their participation in a math content course to explore the diversity in their developing political conocimiento. Responses ranged from considering race irrelevant due to the quasi-anonymity of online instruction to drawing connections between competitiveness and White supremacy.

Keywords: Preservice Teacher Education, Social Justice, Affect, Emotion, Beliefs, and Attitudes, and Equity, Inclusion, and Diversity

In spring of 2020 Arundhati Roy described the potential for the COVID 19 pandemic to serve as a catalyst for a fundamental reorganization of our global society. The Movement for Black Lives and the so-called "racial reckoning" that followed seemed like important steps forward, but many changes were shallow, short-lived, and overwhelmed by the subsequent White-supremacist backlash (Holloway, 2021). Part of this backlash manifested as laws related to teaching, with 36 states restricting education about racism, bias, Critical Race Theory, and related topics (Stout & Wilburn, 2022). In contrast, 17 states have recently passed laws expanding requirements for teachers to address such topics. Teacher educators who work with preservice teachers (PSTs) need to acknowledge the divisive and high-stakes nature of the teaching environment into which our PSTs will graduate. We must help our PSTs develop the political understanding they will need to successfully navigate this environment and advocate for themselves and their students. Fostering the necessary learning requires addressing issues of race, justice, and equity in all aspects of teacher preparation programs (Garii & Appova, 2013; Gutiérrez, 2013; McDonald & Zeichner, 2009; Nieto, 2000; Wiedeman, 2002; Xenofontos et al., 2020), including math content courses (Lee-Hassan, 2021; Felton-Koestler, 2020). This brief report explores the developing political knowledge that PSTs in an urban education program demonstrate as they reflect on their participation in a racially diverse math content course.

Theoretical Framework

Gutiérrez (2012; 2013; 2017) describes the political knowledge that teachers need as political conocimiento for teaching mathematics (PCTM). This framework highlights the need to integrate political knowledge with mathematical content knowledge, pedagogical knowledge, and knowledge developed with students and communities. Gutiérrez draws conocimiento from the work of Anzaldúa (2002), describing a situated form of knowledge that develops through experience in community and entails the need for solidarity and praxis. Relationships and context are critical to the development of conocimiento, so PCTM is inherently context-dependent. Part of PCTM involves teachers' developing understanding of connections between mathematics, math education, and Whiteness (Gutiérrez, 2017), which can include mathematics' privileged role in society and a dominant culture of mathematics valuing hierarchies, competition, and a particular version of intelligence. In this report I propose that racially diverse, collaborative math content courses are spaces where PSTs' PCTM can manifest as they navigate and reflect on their relationships and participation. I focus on the question: How can PCTM related to race manifest through PSTs' reflections on class participation in a racially diverse math content course?

Methods

Context and Participants

The data analyzed in this report were drawn from a larger study exploring the development of PCTM in a math content course for elementary math concentrators and secondary math PSTs. The course was initially designed to develop a conceptual, function-based understanding of algebra. The larger study modified the curriculum to incorporate explorations of data from the local school district and reflections and discussions about PSTs' personal experiences to foster their developing PCTM. Tasks highlighted the relevance of race and racism in the contexts we analyzed and encouraged PSTs to share background and ideas from other courses, but connections between race, racism, White supremacy and mathematics were not explored as explicitly as might happen in an anti-racist math methods course. During the study the course used remote, synchronous instruction. There were 36 PSTs in the course with a racial distribution that approximated the university overall (no racial majority). After the course, 3 students consented to participate in the interviews analyzed in this paper. All names are pseudonyms.

- Emily: White woman pursuing her masters in secondary mathematics education
- Jessica: White woman pursuing an undergraduate degree in elementary education with a mathematics endorsement
- Anjali: South Asian woman pursuing an undergraduate degree in elementary education with mathematics and special education endorsements

The low participation rate was likely due to the sensitive nature of the course and the high-stress context of the pandemic. The analyses in this report are intended to illustrate a range of potential manifestations of PCTM, not to represent PSTs' overall experience in the course.

Positionality

I am a White, agender individual who attended suburban public schools, and whose work in urban elementary schools and universities has been grounded in critical perspectives and a focus on equity and social justice. I held a dual role as teacher/researcher for this project, so my positionality influenced all stages of the research (D'Ambrosio et al., 2013). Most relevant to the current analyses, my design and facilitation of the course included multiple prompts for PSTs to reflect on their course participation, frank discussions of race and racism, and emotional check-ins and discussions of mental health. All three participants identified the course's supportive environment as important for their comfort level, a comfort level that probably influenced their decision to engage in the interviews. Our shared race and my history of discussing race in class likely made Jessica and Emily more open in their interviews. My critical perspectives likely influenced what the PSTs felt comfortable sharing in the interviews, and certainly informed what aspects of their responses I identified as significant during my analyses.

Data Sources and Analyses

The data from this paper came from semi-structured interviews that occurred the summer after the course ended. Interviews were approximately 75 minutes long and included questions about PSTs' beliefs and experiences related to social justice, mathematics, and links between the two. This report focuses on responses to two questions about participation during the course.

1. We talked a lot at the beginning of the semester about trying to balance different people's participation in class discussions. Tell me about your thinking during those conversations.
2. Did you consider race at all when thinking about participation and class relationships?

Responses were analyzed using narrative analysis. Analyses focused on a) who/what PSTs included in their narratives and b) how PSTs positioned the individuals and entities they included with respect to each other. This paper focuses on how PSTs positioned themselves, Whiteness, and mathematics. These positionings offer preliminary insights into their developing PCTM.

Findings and Implications

All three participants described the challenges entailed in engaging in class discussion using Zoom. Both Emily and Jessica described struggling with the impulse to talk too much during class discussion, and when I asked explicitly about the role of race they both described their Whiteness as relevant to that struggle. The ways in which they narrated those struggles, however, highlighted differences in their developing PCTM.

Jessica framed her high levels of participation as a personality trait, describing herself as "a very wordy person…I bond with conversation." She described herself as wanting to engage in discussion with her peers because it's "how [she] build[s] relationships" and empathy. She highlighted that her current desire to build empathy contrasted with her experiences growing up. When asked about race, Jessica acknowledged that

> When it boils down to it I am a white woman. Um, whether or not I view myself as a white woman in every single sentence that I say, that is how I'm perceived, because that is who I am…It has an effect [laughs] on everything I say if someone knows that I'm white. Especially people of color. If they know I'm white and they're hearing me it plays into everything. It just has an effect on it. Um, and so I don't wanna overspeak in general. (Jessica)

In this quote Jessica positioned her race as a personal characteristic that is primarily relevant based on how other people—particularly people of color—perceive her. She had the declarative knowledge that her Whiteness matters, and she connected it with a concern about overspeaking, but her descriptions of why Whiteness matters were somewhat vague, alluding to other people's perceptions and a desire for everyone to have "space to speak." She did not connect her reflections about empathy with her discussion of race.

Emily, in contrast, described herself as having "a complex about being the best student"—an explicitly negative and active characteristic. When asked about race she explicitly connected that complex to "playing into white supremacy." She explained how even though she thought of herself as "hav[ing] done a lot of work on [her]self, and… done a lot of work in the world," in the context of this math class she found herself enacting toxic levels of competitiveness that she associated with Whiteness and settler colonialism. In her words:

> [S]ome of this shit is buried deep, you know? And...it can come out in—It came out in, like, ways that I was not proud of, honestly. You know? And, yeah. I think that, I think I did—I thought about it A LOT. I thought about it a lot in terms of how I was using my Whiteness, my age, my education, and all of that—all of my privilege, right? To, like, dominate a class. [laughs] It was sad when you put it that way. [half laughs] I guess what I'm trying to communicate is, like, I think I'm just doing my best, but that's what behind it, right? (Emily)

Here Emily identified a toxic characteristic of White supremacy that played a role in her behavior. She acknowledged its roots in her upbringing, but also acknowledged the ways in which it harmed her classmates and the class community and described her efforts to do better. She also reflected on the challenges of balancing participation through collective effort using a pedagogical lens that considered the relevance of mathematics in her experience.

> [S]aying, like, "Can we have a collective class?" or "Can we have a collaborative classroom." You know. So even, like, trying—even if it doesn't come out perfectly…know, because that's not a revolutionary concept in, like, you know, a seminar, right? Like, if you're in, like, a, you know, women's studies seminar or whatever [laughs] like, that's not, you know—they're probably going to try to do it like that, right? [laughs] Um...but in a math class, you know, that's, like, a whole nother—that's not something that usually is on the table. And so I think even putting it on the table was something that really made me reconsider and think deeply about,,,what does, like, an equitable class look like, you know? (Emily)

Here Emily alternated between positioning herself as a student and as a teacher. She compared the political implications of the pedagogical choices made in the course and cold-calling techniques she had been instructed to use at her current teaching job, and specified the ways in which the context of teaching mathematics influenced the pedagogical and political implications.

Anjali, in contrast, said that she did not think that race played a significant role in the dynamics of class discussions. From her perspective the fact that class was on Zoom and many students kept their videos off and mainly communicated via chat provided a degree of anonymity that "was nice because, like, nobody could judge you." She felt that overtalking was "just dependent on how the student was as a whole." She did, however, describe a lot of anxiety around how classmates or group members might be judging her class participation. In particular, she highlighted Emily's apparent competence as connected to her insecurity.

> [Emily] knew what she was doing. I was like, "Wow. I feel very dumb compared to this." And so, er, like, I tried very hard, like, that's why I would private message you questions, because I was very nervous to ask in front of people, because they're like, "Wow. She doesn't get this. Wow. Like, she should really know what's happening." And obviously I'm sure other people were in the same boat, but at that time—because I also didn't know you as well as I did nearly at the end of the semester—it was more like, just scared—being scared. (Anjali)

Here Anjali shows the other perspective of the domination that Emily described. Anjali positions herself as anxious because of the potential or perceived judgement of her classmates. It is important to note that while Anjali did not describe race as relevant to the classroom dynamics, in other parts of her interview she explicitly positioned herself as "a person of color" and described why she believed that positionality would give her better insight and help her build relationships when teaching students of color compared with White teachers.

These preliminary analyses illustrate some of the ways in which, given opportunities for reflection and discussion, PSTs' experiences in a math content course can support and illuminate their developing PCTM. As Emily explicitly noted, even though she had thought about Whiteness and competitiveness in other contexts, reflecting on her practice in an ongoing math class put things in a different perspective. The political nature of math teaching and learning (Gutiérrez, 2017) makes participation in math content courses a potentially valuable space for PST reflection as part of teacher preparation programs addressing race, equity, and justice.

References

Anzaldúa, G. (2002). now let us shift . . . the path of conocimiento . . . inner work, public acts. In G. Anzaldúa & A. Keating (Eds.), *this bridge we call home* (pp. 540-578). New York, NY: Routledge.

D'Ambrosio, B., Frankenstein, M., Gutiérrez, R., Kastberg, S., Martin, D. B., Moschovich, J., Taylor, E., & Barnes, D. (2013). Positioning oneself in mathematics education research. *Journal for Research in Mathematics Education*, 44(1), 11-22.

Felton-Koestler, M. D. (2020). Teaching Sociopolitical Issues in Mathematics Teacher Preparation: What do Mathematics Teacher Educators Need to Know?. *The Mathematics Enthusiast*, *17*(2), 435-468.

Garii, B., & Appova, A. (2013). Crossing the great divide: Teacher candidates, mathematics, and social justice. *Teaching and teacher education*, *34*, 198-213.

Gutiérrez, R. (2012). Embracing Nepantla: Rethinking "Knowledge" and its Use in Mathematics Teaching. REDIMAT - *Journal of Research in Mathematics Education*, 1 (1), 29-56. doi: 10.4471/redimat.2012.02

Gutiérrez, R. (2013). Why (urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education*, *6*(2), 7-19.

Gutiérrez, R. (2017). Political conocimiento for teaching mathematics: Why teachers need it and how to develop it. In S. Kastberg, A. M. Tyminski, A. Lischka, & W. Sanchez (Eds.), *Building support for scholarly practices in mathematics methods* (pp. 11-38). Charlotte, NC: Information Age Publishing.

Holloway, K. (2021, July 19). Our 'racial rckoning' is turning out to be a white lie: Black demands for full citizenship are being treated as entitlement and calls for racial accountability redefined as white persecution. The Nation. https://www.thenation.com/article/society/black-lives-matter-backlash-2/

Lee-Hassan, A. W. C. (2021, October). Forms of Nascent Political Conocimiento Leveraged in a Content Course for Preservice Elementary Teachers. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Productive Struggle: Persevering Through Challenges: Proceedings of the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Philadelphia, PA. October 14-17, 2021.

McDonald, M. & Zeichner, K. M. (2009). Social justice teacher education. In W. Ayers, T. M. Quinn, & D. Stovall (Eds.), *Handbook of social justice in education* (pp. 595-610). Routledge.

Nieto, S. (2000). Placing equity front and center: Some thoughts on transforming teacher education for a new century. *Journal of teacher education*, *51*(3), 180-187.

Stout, C. & Wilburn, T. (2022, Feb. 1). CRRT map: Efforts to restrict teaching racism and bias have multiplied across the U.S. Chalkbeat.org. https://www.chalkbeat.org/22525983/map-critical-race-theory-legislation-teaching-racism

Xenofontos, C., Fraser, S., Priestley, A., & Priestley, M. (2020). Mathematics teachers and social justice: a systematic review of empirical studies. *Oxford Review of Education*, 1-17.

FACULTY AND STUDENT PERCEPTIONS OF INSTRUCTIONAL SERVINGNESS IN GATEWAY MATHEMATICS COURSES AT A HISPANIC-SERVING INSTITUTION

Luis A. Leyva
Vanderbilt University
luis.a.leyva@vanderbilt.edu

Nicollette D. Mitchell
Vanderbilt University
nicollette.d.mitchell@vanderbilt.edu

R. Taylor McNeill
Vanderbilt University
reagin.t.mcneill@vanderbilt.edu

Martha H. Byrne
Sonoma State University
byrnema@sonoma.edu

Ben Ford
Sonoma State University
ben.ford@sonoma.edu

Lorely A. Chávez
Vanderbilt University
lorely.a.chavez@vanderbilt.edu

Enrique M. Abreu-Ramos
Vanderbilt University
enrique.m.abreu-ramos@Vanderbilt.Edu

Research exploring how Hispanic-Serving Institutions (HSIs) serve Latin STEM students has largely focused on features of organizational structures (e.g., support programs), but minimally examined instruction and classroom experiences. This is an important gap to fill, especially in gateway mathematics courses, where faculty relationships and quality of instruction impact Latin* students' persistence and identities in STEM. To advance such research, this report presents findings from an analysis of how perspectives from HSI mathematics faculty and students about instruction in introductory statistics converged and diverged in terms of serving Latin* populations. We present two illustrative cases of dissonant and resonant perspectives on serving Latin* students through instruction that frames mathematical ability expansively (e.g., not limited to being fast or correct). We conclude with research and practice implications.*

Keywords: Equity, Inclusion & Diversity; Undergraduate Education; Data Analysis & Statistics

Purpose of the Study

Institutions are designated as Hispanic-Serving Institutions (HSIs) if at least 25 percent of enrolled undergraduates are Latin*[1]. Despite receiving funding to increase Latin* postsecondary persistence, HSIs often lack visions of serving Latin* students and thus may not advance equitable outcomes or affirming support (Garcia, 2020). Thus, racialized and intersectional oppression among Latin* STEM students (Leyva, 2016; McGee, 2016; Rodriguez et al., 2017) are not necessarily disrupted in HSIs (Contreras Aguirre et al., 2020; Valenzuela, 2020).

Instructional quality in gateway courses, including calculus and statistics, is a major contributor to Black and Latin* attrition in STEM (Dadgar et al., 2021; Larsen et al., 2016). Although research that examines racially-equitable features of gateway mathematics instruction is emerging (Hagman, 2021; Leyva, McNeill, et al., 2021), an explicit focus on equity for Latin* students has not been adopted. Such inquiry can build a theory of racially- and culturally-affirming instruction in gateway courses to strengthen Latin* students' persistence and sense of belonging in STEM, along with enhancing HSIs' efforts to serve Latin* STEM students.

To fill this research gap, our report presents an analysis from a larger study of a HSI mathematics department's reform of instructional and organizational practices to better serve

[1] The asterisk in Latin* creates space for fluidity in gender identities among Latin American people. Latin* responds to (mis)use of Latinx, a term reserved for Latin American gender-nonconforming peoples (Salinas & Lozano, 2019).

Latin* students. We looked across faculty and student perceptions of instruction for fostering equitable opportunities of classroom participation among Latin* students in gateway courses. The following research question, aligned with the PME-NA 44 theme, is addressed: *What are forms of dissonance and resonance across faculty and student perceptions of serving Latin* students through instruction in gateway mathematics courses?* We raise implications for improving ways to serve Latin* students in mathematics departments based on our findings.

Theoretical Perspective & Relevant Literature

A guiding perspective for our analysis is *servingness*, a framework for developing culturally-enhancing experiences at HSIs for Latin* students across various dimensions, including outcomes, experiences, organizational structures, and external forces (Garcia et al., 2019). Prior research on serving Latin* STEM students in HSIs has focused on organizational features that promote academic persistence, such as course design (Chang & Chen, 2020; Meling et al., 2020), departmental policies (e.g., course placements; Burn et al., 2019), and support programs (Cruz et al., 2019). This work has importantly depicted structural interventions' advancement of equitable outcomes for course success and STEM persistence. However, these outcome-focused analyses left implicit how organizational features impacted Latin* students' racialized experiences specifically in STEM. Instruction in STEM courses was also not central in analyses. Extending this work, the present analysis uses servingness as a lens to examine HSI faculty members' and students' perceptions of instruction in gateway courses as reinforcing or disrupting racialized outcomes (e.g., grades) and experiences (e.g., cultural validation). Our analysis also considered how departmental structures (e.g., professional development) for fostering culturally-responsive learning and resisting forces of white supremacy shaped servingness in mathematics instruction.

Faculty advance servingness at HSIs through instruction and student support that disrupt racialized oppression to promote Latin* students' positive identity development and sense of belonging (Ching, 2019). However, it remains unclear the extent to which STEM faculty at HSIs leverage awareness of Latin* students' cultural backgrounds and racialized experiences to inform pedagogy. For example, in a study of HSI faculty's racial attitudes, Garcia and colleagues (2020) found that faculty held low levels of colorblind attitudes, indicating racial consciousness. However, findings also showed that STEM faculty demonstrated more colorblind attitudes than faculty in other disciplines, even when controlling for differing racial and gender demographics. Colorblind faculty attitudes in mathematics and STEM more broadly leave stereotypes of ability, inequities of access, and other racialized forces unchallenged (Bensimon et al., 2019; McCoy et al., 2015; McNeill et al., 2022), which can reinforce deficit views and thus limit instructional servingness (Chase et al., 2013; Ching, 2019). The impacts of faculty racial attitudes, including orientations to servingness through STEM instruction in HSIs as well as Latin* students' classroom experiences, have been underexplored. Our study fills this gap by triangulating faculty and student perspectives to explore the efficacy of gateway course instruction in servingness.

Methods

Study Context and Participants

Our present analysis comes from a larger study exploring the effectiveness of an ongoing, equity-oriented professional development (PD) in the mathematics department at Sonoma State University (SSU). SSU is a medium-sized, public university in California that recently received the HSI designation. In 2021, enrolled undergraduate students at SSU were approximately 45% white, 35% Latin*, 7% two or more races, 5% Asian, 2% Black or African American, and 6% some other race. Faculty PD began in summer 2021, and data collection started in fall 2021. The two-year faculty PD aims to develop culturally-responsive institutional practices, including classroom instruction and student support, to better serve Latin* students.

The research team (a collaboration between two faculty involved in the PD and educational researchers at Vanderbilt University) designed a study to look across instructors' and Latin* students' perspectives on instructional servingness in gateway courses. Twelve mathematics faculty PD participants agreed to participate in the study. Over half of faculty participants served as gateway course instructors. Latin* student participants were recruited from gateway courses by sending an e-mail message to all enrolled students with a flyer containing a URL to express interest and completing classroom visits (in-person and virtual) to provide a study overview. Instructors were not informed if any of their students served as study participants and vice versa. To the extent possible, we purposefully sampled from students who expressed interest to have multiple voices from each course section as well as to ensure variation in race-gender identities.

Data Collection

Researchers at Vanderbilt (1 faculty, 2 Ph.D. students, 1 graduate student, and 1 undergraduate student) led data collection. SSU team members also served as study participants and did not assume data collection responsibilities. Using event journaling methodology (Leyva, Quea, et al., 2021), one data source was student and faculty participants' journaling of events from instruction in gateway courses perceived as marginalizing or supportive for Latin* students. Journal entries were submitted using an online form that prompted participants for a description of events and reflection on why events were perceived in these ways. Students journaled about their course experiences as learners, and faculty journaled about their practices as instructors. Journaling was ongoing throughout the fall semester without any required number of entries.

Near the end of the semester, each participant completed a 90-minute, semi-structured individual interview on Zoom. Interviews were audiotaped and transcribed. To the extent possible, we matched interviewers and participants by race and gender as an attempt to build comfort with discussing issues of structural oppression. The first half of the interview explored what serving Latin* students meant to participants in gateway mathematics courses and in general. The second half of the interview centered around two prompts of instructional events that reflected emergent themes of servingness from journaling. Each prompt was written as a composite of journaled events. One prompt was written from a faculty perspective and addressed issues of student support. The second prompt, which is central to the analysis presented in this research report, was written from a student perspective and focused on classroom participation:

> I was asked to share my solution for a homework problem during class. I was unsure if I solved the problem correctly. I was worried about presenting an incorrect solution in front of the instructor and the rest of the class. My instructor acknowledged that my final answer was incorrect and guided me through steps for solving the problem. The instructor also mentioned that perfection was not the expectation for success and how students can learn from errors. This interaction made me feel less pressure about having to always be correct and encouraged me to more readily share my thinking during class.

For each prompt, participants were asked about the instructional event's frequency of occurrence and potential for serving Latin* students as well as recommended changes in gateway courses for more racially-equitable participation or support. All participants were prompted for specific examples from their gateway courses to support their reasoning about servingness.

Data Analysis and Positionality

Vanderbilt research team members de-identified data prior to sharing with SSU research team members for analysis. Information that explicitly or implicitly revealed participants' identities (e.g., names, students' backgrounds, faculty's professional histories) was redacted. For the present research report, we completed an analysis of data specific to participants involved in courses that contained both faculty and student participants. Our analysis looked across data for 4 faculty participants and 6 Latin* first-year student participants associated with calculus and statistics courses. In addition to journaling, we analyzed interview responses to questions about the meaning of serving Latin* students in and beyond gateway mathematics instruction, along with responses to the prompt about classroom participation presented above.

A research team member from each university coded each faculty participant's data. Two Vanderbilt team members coded each student participant's data since de-identification was still in progress. To the extent possible, at least one coder for student data identified as Latin* to have an insider perspective for data analysis. Team members independently and inductively coded data to flag instances when participants raised instructional features and departmental structures perceived as advancing or limiting servingness in gateway courses, including attention to Latin* students' social identities and cultural backgrounds. When individual coding was complete, one coder from each pair synthesized the two sets of codes for each participant's data into themes on instructional servingness. Themes were exchanged and discussed during weekly team meetings. To address our research question about dissonance and resonance across HSI faculty and student perspectives on servingness, the research team identified points of convergence and divergence in how faculty members and students in the same class perceived servingness in instruction.

Our research team approached the present analysis with critical reflexivity. The team consists of two Latino cisgender men, a Latina cisgender woman, a Black cisgender woman, a white transmasculine person, a white cisgender woman, and a white cisgender man. We brought awareness of how our varying forms of privilege and oppression influence our inquiry on instructional servingness in mathematics at HSIs. The team resisted deficit engagement with participants' reflections as mathematics learners or educators and constantly recognized how gateway mathematics instruction is situated in broader systems of social power. Interviewers and coders bracketed their lived experiences in engaging with participants' reflections to avoid analytically distorting their perspectives, all while approaching the study with a lens of criticality to interrogate structures that limit servingness in STEM contexts across HSIs. Our infusion of multiple faculty members' and Latin* students' voices in the findings captures complexities of instructional servingness in gateway mathematics and avoids portraying it in essentializing ways.

Findings

We present findings centered on an emergent theme of instructional servingness – namely, expansive framing of mathematical ability to increase Latin* students' classroom participation. This framing resists dominant views of correctness and speed as indicating innateness of ability, which do not account for educational inequalities that limit Latin* students' opportunities to develop such skills. Two cases of coupling HSI faculty and student perspectives were most illustrative of this theme. One coupling includes faculty member Dina (white woman[2]) and her students, Rosalinda (Mexican female) and Christian (Mexican male). The second pairing includes faculty member Michael (white male) and his students, Lisandra (Chicana female) and Nereyda (Salvadoran female). These couplings were specific to introductory statistics courses that enrolled exclusively first-year students and predominantly Latin* students. We organize our findings in two sections that each presents a coupling of faculty-student perspectives to highlight dissonance and resonance in reflections on instructional servingness through framing ability.

[2] Participants' identities are reported as they were self-described during interviews.

"My Teacher Likes to Tell Us That It's Okay to Ask for Help and If We Get Something Wrong, It's Not the End of the World. It Just Means That We're Actually Learning."

Serving Latin* students for faculty participant, Dina (white female), meant "need[ing] to do what it takes for all [her] students to succeed" regardless of race and other social differences, which included expanding opportunities for classroom participation. Dina perceived Latin* students' participation as often inhibited by narrow constructions of mathematical ability in K-12 education that frame being correct and quick as essential to being successful.

Interviewer:	How common do you think it is for Hispanic and Latinx students to be concerned about being incorrect or being unwilling or unable to share?
Dina:	It's very common… I would say, 'I'm afraid I have the wrong answer' is very common.... The way they're [Latin* students] prepared in math makes them feel stupid… We've been taught that math is fast.

To disrupt limited views of mathematical ability and broaden space for Latin* students' participation, Dina asks students to share prior experiences with mathematics and emphasizes that "math is really hard" to reassure them that struggle does not reflect inability. She interpreted her practice, along with the faculty behavior featured in the interview prompt, as bringing relief to Latin* students about being negatively judged when sharing their thinking. Dina approached instruction with awareness of potential linguistic barriers to classroom participation among Latin* students, especially in statistics where terms (e.g., bias) have specific meanings that differ from everyday use, "If English is not your first language, that's already a huge piece of baggage… You are at risk of having a language gap… I need to pay a lot of attention to… [students] being able to use and understand and pronounce." Dina's perspectives convey awareness of how narrow ideas of mathematical ability collide with Latin* students' experiences of educational inequities. Her awareness led to instruction that framed ability expansively and practices that clarified mathematical concepts in order to increase Latin* student participation.

Dina's view of serving Latin* students through instruction resonates with Rosalinda's reflection on her experience as a Mexican female in her class. In a journal entry, Rosalinda wrote that Dina's instruction made her "more confident in math… [and] capable in math" after having several high school mathematics teachers who undermined Latin* students' ability and dismissed her contributions. She experienced Dina's practice of framing errors as entry points for learning as reducing her fears about being wrong and encouraging her to readily participate.

Rosalinda:	Math isn't my strongest subject and… I have failed a few classes because my teachers didn't know how to serve me or other students like me. And so I like my class [with Dina]… My teacher likes to tell us that it's okay to ask for help and if we get something wrong, it's not the end of the world… It just means that we're actually learning rather than feeling bad. Because in the past, when I would get something wrong, it felt the worst.
[…]	
Interviewer:	How comfortable do you feel as a Latina student to participate?
Rosalinda:	I'm slowly getting more comfortable when it comes to going up to the board… Every time my professor, before we present, she'll be like, 'If you have any questions, even if it's a small question, feel free to ask. If we don't ask, she'll be like, 'It's okay if you get it wrong, I'll help guide you.'

Dina’s instruction served to increase Rosalinda’s access to classroom participation as a Mexican female with a history of racial oppression in K-12 mathematics, which captures resonance in faculty and student views on servingness through an expansive framing of ability.

Christian’s reflection on his classroom experience as a Mexican male student largely resonated with Dina’s view on serving Latin* students and aligns with Rosalinda’s experiences in the classroom. He likened the faculty behavior in the interview prompt to Dina’s comments during instruction for encouraging participation, “This sounds very much like my professor who has [said] things [like] ‘Mathematicians don’t always do the problems right away. They take their time’… It definitely encouraged me to not rush and take my time in math.” Christian perceived Dina’s instruction removing pressures of solving problems quickly to demonstrate his mathematical ability, which he commonly experienced prior to college. Such classroom practices brought Christian to feel like Dina “made [him] like math… [and] feel involved all the time.”

However, Christian also noted limitations on his access to participation despite Dina’s expansive framing of mathematical ability. He sometimes felt his participation was constrained by his command of the English language. Christian described “get[ting] a bit anxious sometimes for presenting [his] math solutions” out of concerns about any embarrassingly incorrect phrasing or pronunciation, “Some words, I can’t pronounce them correctly, so I struggle a bit getting my phrasing out there.” Although Christian journaled about how Dina strengthened his mathematical understanding through explanations of vocabulary, classroom participation remained less accessible to him due to risks of classmates’ negative judgment about how he communicates his ideas as an emergent bilingual. Dissonance between Dina’s and Christian’s perspectives convey how servingness through instruction that resists narrow views of mathematical ability must account for varying levels of comfort with verbal communication, especially among Latin* students developing command of the English language on top of mathematical terminology.

“When I Asked My Teacher for Help After Class and He Stayed Until I Understood the Problem, It Was A Little Bit of a Shocker For Me! I Thought I Would Be Perceived as the Stereotype That Hispanic Students are Lazy… [and] They Don't Understand Anything.”

Faculty participant, Michael (white male), reflected on growth in his development of instructional practices that foster servingness, “It's a huge part of my job right now, going through that change that's happening at our school [becoming an HSI]... Redefining my job around my students... I'm just like in preschool with the whole thing.” Michael positioned himself as constantly trying to learn more about Latin* students’ backgrounds and needs, “The way I’m trying to do it is just trying to collect more, and more, and more information, and think about how we can serve based on the information.” While Michael recognized the value of understanding Latin* students as a group, he also felt tensions about how such knowledge may shape assumptions and activate stereotypes when interacting with students, “I want to get my head around that, but I don't want to look at a student and think I know what's going on.”

Like Dina, Michael noted fear of being wrong as a barrier to Latin* student participation due to the equating of ability with correctness and speed in K-12 mathematics. He shared instructional comments about leveraging wrong answers as learning opportunities to allay such fears and frame ability expansively, “If it’s [a student’s answer] wrong, I’m going to actually be happier… You’re not the only person making a mistake, first of all, and then it’s going to give us a discussion to have.’” At the same time, Michael grappled with concerns about how his social distance from Latin* students as a white male, on top of students’ histories as K-12 mathematics learners, shaped skepticism toward his instructional messages, “They’ve [Latin* students] had dozens of teachers… And I’m just one guy they don’t know, who doesn’t look like them… So it’s hard for me to make provocative statements like this about how to approach this [mathematics learning] differently.” He reflected on the promise of connecting his students in statistics classes with former Latin* students who can “bridg[e] that culture, and race, and age gap” and disrupt current students’ distrust about his expansive views of mathematical ability.

Lisandra’s (Chicana female) reflection on her classroom experience largely exhibited dissonance and also somewhat resonated with Michael’s conceptions of servingness. Although Lisandra observed instructional practices with an expansive view of mathematical ability (e.g., attending to student struggles), she did not feel comfortable sharing incorrect thinking or exhibiting struggle with content in the statistics classroom, “It doesn't feel like, ‘Oh, we can be wrong because it's alright; we're learning.’ It doesn't feel like that. It feels like, ‘Oh, you have to be right.” Reflecting on how Latin* students are commonly worried about being wrong, she shared, “It’s been one of the many reasons why I haven’t spoken up in any class, because I’m not sure and I don’t want to embarrass myself... So I just keep to myself… I don’t want to sound dumb.” Lisandra’s concerns about being negatively judged for her intellect as a Latina student prompted her to hold back on participating in her statistics class, which resonated with Michael’s concerns about Latin* students’ skepticism toward his framing of mathematical ability.

Lisandra also described how her inhibited participation, despite Michael’s encouraging instructional behaviors, stemmed from a lack of community in the classroom. Servingness, for Lisandra, was forming part of a community that understood her as a first-generation Chicana female where she can “fully feel comfortable and safe to reach out” when needing support. The lack of community in her statistics classroom made her feel isolated and that her needs were not recognized, “I feel on my own… The teacher goes around and everything… but I feel like in the classroom, it doesn’t feel so much as a community… I don’t feel represented.” Lisandra, as a result, was often “not feeling safe in the environment enough to speak” in terms of sharing ideas and seeking help. Dissonance between Michael’s and Lisandra’s perspectives convey how servingness through instruction with an expansive framing of mathematical ability can fall short in increasing Latin* students’ access to participation. Lisandra was missing a foundation of safety and community to feel comfortable taking risks in her participation as a Chicana female.

Nereyda’s reflections as a Salvadoran female were both resonant and dissonant with her instructor Michael’s perspectives on servingness anchored in an expansive view of mathematical ability. Unlike Lisandra, Nereyda perceived her classroom experience, including Michael’s openness to student struggle and creation of a non-judgmental learning environment, as aligned with her views of servingness in mathematics. She related the faculty behavior in the interview prompt to Michael’s instruction, “He's constantly telling us that there's no need for perfection. ‘Do not worry about getting the right answer. It's just the process of learning and showing me what you have learned.’” She shared instances of seeking and receiving Michael’s support, which she perceived as especially impactful for Latin* students who often “don’t speak up because they don’t think they’re going to get the help they need” due to stereotypes of being dumb or lazy. Nereyda journaled about such support from Michael after class that disrupted stereotyping influences, which often limit Latin* students’ access to classroom participation.

> When I asked my teacher for help after class and he stayed until I understood the problem, it was a little bit of a shocker for me! I thought I would be perceived as the stereotype that Hispanic students are lazy in their study [sic] and they don't understand anything, but it made me feel heard and seen [and] that it's okay to not get things from the get-go. I now don't want to hide out.. I'm eager to go to class and keep making mistakes so I can continue getting help from my professor.

Nereyda's reflections of relief from pressures of sharing correct mathematical thinking and concealing academic struggles, thus, capture resonance with Michael's perspectives of serving Latin* students through instruction that challenges narrow constructions of ability.

Despite perceiving an overall sense of servingness through Michael's instructional practices of welcoming students' contributions and questions, Nereyda still experienced moments of discomfort with participation as a Salvadoran female due to stereotypes of ability.

> There's that stereotype that women aren't really good at math and they just don't understand. I feel like that impacts me in the classroom sometimes. I just get kind of intimidated… As well as with race, I feel like they [faculty] think that Hispanic students… need a little bit more attention… [and] can't capture the full event [referring to instruction] and it's not really like that. I journaled about it. It's the stereotype.

Nereyda particularly felt intimidated during groupwork to seek support and share her ideas. She observed a racialized pattern in self-sorted groups with "white students working together and… Latinx students just sitting out on those sidelines." When working with female and Latin* peers, either the group or Nereyda herself suppressed asking Michael for help, "In that space, I feel a little iffy or uncomfortable… He [Michael] goes around and helps people, but I feel like sometimes, some days, I can't speak up." Nereyda felt that further opening the statistics classroom space where "you share your identity with the class and when your classmates share their identity with you…creates a safe place of trust" that could alleviate racialized-gendered tensions with classroom participation. Therefore, dissonance is also evident across Michael's and Nereyda's perspectives on servingness. Michael approached instruction with commitment to resist stereotypes about Latin* students and increase participation through an expansive framing of mathematical ability. While Nereyda valued the reduced risks of seeking help and sharing her thinking, the unchallenged stereotypes of ability in her class often left participation inaccessible.

Discussion and Implications

Our findings reveal complexities of instructional servingness in gateway mathematics courses. Resonance in HSI faculty and student perspectives captured how instruction with an expansive framing of mathematical ability alleviated pressures of being quick and correct, which increased opportunities of classroom participation and support. At the same time, dissonance across faculty and student perspectives conveyed how these instructional practices were not always responsive to Latin* students' social positions (e.g., emergent bilinguals' language-based support needs, navigating racial and gender stereotypes of ability). Extending research on racial attitudes among HSI faculty (e.g., Garcia et al., 2020), our study shows how faculty consciousness of Latin* students' racialized experiences in mathematics that informed expansive framing of ability did not necessarily translate to experiences of servingness. Latin* students' perspectives suggest how instructional servingness that ensures equitable classroom participation requires explicitly confronting stereotypes of ability and building identity-affirming community.

Future work can extend our analysis by further unpacking Latin* identities to explore how within-group differences (e.g., nationality, gender and sexuality) can further inform servingness in HSI mathematics classrooms. We also call for research that explores how features of instructional servingness shift in upper-level mathematics and across other HSI contexts. In terms of practice, we call for HSI mathematics faculty to use instructional feedback from students to explore dissonance and resonance between classroom practices and Latin* students' needs. Mathematics faculty can also engage in observations of instruction with departmental colleagues and educational researchers to dissect practices that advance or constrain servingness.

References

Bensimon, E., Dowd, A., Stanton-Salazar, R., & Dávila, B. (2019). The role of institutional agents in providing institutional support to Latinx students in STEM. Review of Higher Education, 42(4), 1689–1721.

Burn, H., Zamani-Gallaher, E., Mesa, V., & Wood, J. L. (2019). Transitioning STEM learners to calculus: Findings from a national survey of mathematics chairs in two-year colleges by Hispanic-Serving institutional designation, MathAMATYC Educator, 10(2), 5-13.

Chang, C., & Chen, Z. (2020). Math course redesign in a private four-year Hispanic-Serving institute to address diverse equitable and inclusive issues. arXiv preprint arXiv:2006.14792.

Chase, M. M., Bensimon, E. M., Shieh, L. T., Jones, T., & Dowd, A. C. (2013). Constraints and opportunities for practitioner agency in STEM programs in Hispanic serving community colleges. In R. T. Palmer & J. L. Wood (Eds.), Community colleges and STEM (pp. 192-212). Routledge.

Ching, C. D. (2019). Supporting Latinx students in Hispanic-Serving Institutions: An exploration of faculty perceptions and actions. Journal of Latinos and Education, 21(1), 39-58.

Contreras Aguirre, H. C., Gonzalez, E., & Banda, R. M. (2020). Latina college students' experiences in STEM at Hispanic-serving institutions: Framed within Latino critical race theory. International Journal of Qualitative Studies in Education, 33(8), 810-823.

Cruz, C., Rajpal, G., Lecocke, M., Martines, I., & Lurie, A. (2021). Peer coaching program development: A framework of first-year Latina/o student persistence pursuing STEM pathways at a Hispanic Serving Institution. Journal of Hispanic Higher Education, 20(4), 365-384.

Dadgar, M., Buck, D., & Burdman, P. (2021). Solving for equity: Design and implementation of new postsecondary math pathways. Just Equations.

García, G. A. (2020). Hispanic Serving Institutions (HSIs) in practice: Defining" servingness" at HSIs. Information Age Publishing.

García, G. A., Koren, E. R., & Cuellar, M. G. (2020). Assessing color-neutral racial attitudes of faculty at Hispanic-Serving Institutions. AERA Open, 6(3).

García, G. A., Núñez, A. M., & Sansone, V. A. (2019). Toward a multidimensional conceptual framework for understanding "servingness" in Hispanic-serving institutions: A synthesis of the research. Review of Educational Research, 89(5), 745-784.

Hagman, J. E. (2021). The eighth characteristic for successful calculus programs: Diversity, equity, & inclusion practices. PRIMUS: Problems Resources, and Issues in Mathematics Undergraduate Studies, 31(1), 70-90.

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2016). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 526-550). NCTM.

Leyva, L. A. (2016). An intersectional analysis of Latin@ college women's counter-stories in mathematics. Journal of Urban Mathematics Education, 9(2), 81-121.

Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. The Journal of Higher Education, 92(5), 784-814.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. Cognition & Instruction, 39(1), 1-34.

McCoy, D., Winkle-Wagner, R., & Luedke, C. (2015). Colorblind mentoring? Exploring white faculty mentoring of students of color. Journal of Diversity in Higher Education, 8(4), 225.

McGee, E. O. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture?. American Educational Research Journal, 53(6), 1626-1662.

McNeill, R. T., Leyva, L. A., & Marshall, B. L. (2022). "They're just students. There's no clear distinction": An analysis of colorblind, gender-neutral faculty discourses on undergraduate calculus instruction. The Journal of Learning Sciences.

Meling, V. B., Kupczynski, L., Mundy, M. A., & Green, M. E. (2012). The role of supplemental instruction in success and retention in math courses at a Hispanic-Serving Institution. Business Education Innovation Journal, 4(2), 20-31.

Rodriguez, S., Cunningham, K., & Jordan, A. (2019). STEM identity development for Latinas: The role of self-and outside recognition. Journal of Hispanic Higher Education, 18(3), 254-272.

Salinas, C., & Lozano, A. (2019). Mapping and recontextualizing the evolution of the term Latinx: An environmental scanning in higher education. Journal of Latinos and Education, 18(4), 302–315.

Valenzuela, A. (2020). STEM diversity and student Latina/o resilience: A reflection. International Journal of Qualitative Studies in Education, 33(8), 898-904.

MATHEMATICS TEACHER EDUCATORS' REFLECTIONS ON THEIR OWN PRACTICE OF EQUITY AND JUSTICE WITH TEACHER CANDIDATES

Nirmala Naresh
University of North Texas
Nirmala.naresh@unt.edu

Colleen M Eddy
University of North Texas
colleen.eddy@unt.edu

Karisma Morton
University of North Texas
Karisma.morton@unt.edu

Abstract. We are three mathematics teacher educators (MTEs) at a Hispanic serving institution that predominantly teach mathematics education courses for elementary and secondary teacher candidates (TCs). In this paper, we present findings from a collaborative self-study in which we used dimensions of the Rehumanizing Mathematics (RM) framework (Gutiérrez, 2018), to situate and integrate our work across mathematics content and methods courses. Our diverse cultural and academic backgrounds and lived experiences profoundly influence our professional practices, particularly our work with teacher candidates. We intentionally utilize counter storytelling as a way to describe how our respective work with TCs aligns with particular dimensions in the RM framework. Our findings indicate that our collaborative self-reflections have positively influenced our individual pedagogical practices. We have recognized through our collaborations that we have much to learn from one another to both inform and strengthen our practice as MTEs in ways that put forth a collective commitment to rehumanizing mathematics education for our students and the children they will ultimately teach.

Keywords: Culturally Relevant Pedagogy, Equity, Inclusion, and Diversity, Ethnomathematics, Systemic Change.

Introduction

Author A. As a woman of color, international scholar, a non-native English speaker, and a citizen of a formerly colonized nation, I have experienced firsthand how a conventional math curriculum, implemented in a traditional school setting has become a tool that disadvantages marginalized students, damages their self-esteem, and strips them of agency and identity. I use ethnomathematics as a tool to help learners (re) claim ownership of mathematics, acknowledge cultural funds of knowledge and establish harmony between academic mathematics and out-of-school mathematics.

Author B. I am a privileged white heterosexual female and follower of Jesus Christ and know that I hold a place of privilege and power as a mathematics teacher educator. I take ownership for my conscious and unconscious acts of racism. I realize that my position and power inherently can be a deterrent to BIPOC students unless I provide space for them to bring their whole selves to the classroom and be critical of the social and socio-mathematical norms that continue to be perpetuated in a traditional mathematics classroom.

Author C. As a former mathematics teacher of Black and brown high school students, I have witnessed how cumulative systemic inequalities in mathematics learning and teaching impact many student outcomes including affective outcomes like confidence and math identity, as well as achievement outcomes such as enrollment in advanced math courses, college, and interest in STEM majors. I have had to address their expressions of not seeing themselves in the curriculum or even feeling unseen by their own teachers.

We begin by sharing excerpts that reflect our positionality and its impact on our work as Mathematics Teacher Educators. In this paper, we describe how we disrupt the dominance of near-universal conventional mathematics (NUC) (Bishop, 1990) in mathematics content and methods courses for teacher candidates (TCs). Research reported in this paper is part of a larger project in which we engaged K-8 TCs at a Hispanic-serving Institution in experiences that potentially broadened their perceptions of and vision for mathematics education. Stimulating discussion and action is the goal around this question:

> How do we, as Mathematics Teacher Educators, attempt to rehumanize mathematics in our content and methods courses?

We intend for this collaboration and communication to result in tangible ideas that will add to the current Mathematics Teacher Educator knowledge based on promoting more humanizing practices in mathematics teacher education.

Theoretical Foundations

The philosophical questions, Mathematics for Whom? How? and What? are being repeatedly asked as it dictates what mathematics and whose mathematics is taught and how it is taught (Aguirre, Mayfield-Ingram, & Martin, 2013). Around the world, most schooling dehumanizes mathematics by claiming that mathematics is neutral, and the culture and identities students bring to the classroom should be lost to achieve academically (Paris & Alim, 2017). The research fields of situated cognition (Lave & Wenger, 1991), ethnomathematics (D'Ambrosio, 1985), culturally responsive mathematics education (Gay, 2000, 2018) and critical mathematics education (Frankenstein, 1983; Gutstein, 2006; Skovsmose, 1985) provide the theoretical foundations, research knowledge base, and practical methods to address the central research question.

Adopting these perspectives allows us, as researchers, to challenge dominant and traditionally told narratives about teaching mathematics and about learners in a mathematics classroom. From a practical standpoint, these theoretical constructs enable us to investigate meaningful pedagogical approaches that will empower and transform learners. For a truly transformative mathematics education, we must continue to engage with and support teachers and TCs who are in a position to address the philosophical questions.

As MTEs, we aspire that our instruction will prepare competent and caring teachers who will serve as an advocate for each learner. To make this a reality, we must provide seamlessly structured learning experiences that will enable our TCs to engage in a critical reflection of their beliefs about mathematics and teaching. Equally important is to identify those MTE practices that dehumanize mathematics teaching and learning. Most of the focus for rehumanizing mathematics focuses on what teachers and TCs can do in their classrooms but there has been much less on the practice of MTEs' practice for rehumanizing mathematics. Thus, from an MTE vantage point, we embrace the chosen theoretical perspectives to find commonalities in our work with TCs, creating counter spaces that refine and redefine our goals for mathematics teacher education.

Theoretical Framework

We use rehumanizing mathematics (RM) (Gutiérrez, 2018) as a lens to engage in a critical inquiry of our "selves", and to position and discuss our work. We are drawn to this framework because it provides powerful affordances for weaving funds of knowledge (Gonzalez, Moll &

Amanti, 2005), culturally responsive mathematics (Gay, 2010), ethnomathematics (D'Ambrosio, 1985), and critical mathematics education (Frankenstein, 1983; Gutstein, 2006; Skovsmose, 1985) into the fabric of our scholarship.

This framework includes eight dimensions (Gutiérrez, 2018) (See Figure 1), three of which are most relevant to the work discussed in this paper. *Cultures/ histories* glorifies ethnomathematics as a lens to counter eurocentricism, acknowledge and valorize contributions from learners' funds of knowledge, and attend to the history and evolution of mathematical ideas through non- dominant civilizations and communities. A second facet, *windows and mirrors* allows freedom of access to learners and "see themselves in the [mathematics] curriculum" (Gutiérrez, p. 5). This allows an MTE to provide a space for TCs to explore their relationship with mathematics. TCs draw on their own mathematics learning experiences to investigate their traditionally held beliefs about the teaching and learning of mathematics. *Positioning/participation*, the third feature that we highlight, enables an MTE to recognize the order of class systems in mathematics learning spaces in and out of classrooms and find ways to shift authority and share the space with learners.

Figure 1: Eight dimensions of Rehumanizing Mathematics (Gutierrez, 2018)

Methodology

We adopt a collaborative self-study (Bullock & Sater, 2017; Lovin et.al., 2012) methodology to frame and situate our work. As Loughran (2004) posits, "There is no one way, or correct way, of doing self-study. Rather, how a self-study might be 'done' depends on what is sought to be better understood" (p. 15). As we engaged in this self-study, we noted similarities in our approaches to present mathematics as a human activity and parallels in our efforts to rehumanize mathematics with our teacher candidates. We engage in a deep reflection of our roles as MTEs in relation to our "selves" and in relation to others. Table 1 provides a general overview of our teaching practices.

Table 1: Author information

Authors	Department	Courses Taught
A	Mathematics	Elementary & Secondary Math content courses
B	Teacher Education	Secondary STEM Methods course
C	Teacher Education	Elementary Math Methods course

Three characteristics define our self-study – collaborative inquiry, contextual knowledge, and a commitment for improving our practice. The traditional research methods of data collection and data analyses are a misfit in our chosen methodology. Instead, we intentionally use a counter storytelling method (Alemán, 2017) to present narratives informed by our personal and professional lived experiences. This method derives from the interpretivist tradition (Walsham, 1993) where researchers advance their work by showcasing "multiple perspectives of stories and who tells the stories," (Creswell, 2007, p. 24). The similarities that we noted in our work led to intriguing stories that we are compelled to share. Our personal and academic backgrounds and lived experiences profoundly influence our dispositions, values, habits of mind, and scholarship and we begin each story by stating our positionality. Next, we describe how our work fits along a particular dimension of the RM framework. The narratives are bound by a common commitment to providing humanizing mathematics learning experiences for our teacher candidates.

Our Stories

Author A's Narrative (cultures/histories)

I am a Mathematics Teacher Educator, and I subscribe to socio-cultural and critical theories of learning. I am also a learner enculturated in a nation subject to centuries of invasion, tyranny, and colonization. This resulted in the suppression of the rich history of Indian mathematics which distorted the philosophy of mathematics and how it is taught and learned (Raju, 2004). My lived experiences and teaching memoirs with diverse learners in K-12 settings and higher education contexts across two continents has illuminated how the near universal conventional curriculum has dominated academic mathematics, treating "mathematical knowledge as being uncontestable, objective, and disassociated from experience, history, and cultures" (Barton, 2008). Math learning experiences in such settings perpetuated among many learners, a narrow perception of and an aversion towards mathematics. Thus, I care deeply about issues of inclusion in mathematics classrooms, particularly for historically marginalized students.

My research interests primarily lie in the field of ethnomathematics which espouses connections between mathematics, culture, and society. From this perspective, I position and discuss my work along the *culture/histories* dimension of the RM framework because of its explicit acknowledgement and attention to the foundational principles of ethnomathematics. I mostly teach mathematics content courses where I have witnessed firsthand how many TCs, particularly those that are marginalized, feel disenfranchised in a math classroom that subscribes to a conventional, near universal math curriculum. Future teachers can learn how to "enhance" and "restore dignity" (D'Ambrosio, 2002) to learners in their classroom by experiencing it themselves. Thus, I deliberately anchor coursework to the principles of ethnomathematics and components of a culturally responsive mathematics education (Gay, 2000) to portray mathematics as a human activity (Bishop, 1990). These constructs also provide the necessary pedagogical reinforcements and equips me with empathy, consideration, and skills necessary to appreciate all individuals that comprise my classroom community and to see them as capable of learning and doing mathematics.

In many courses that I teach, coursework comprises practical and research components (Naresh & Kasmer, 2018). As part of the practical component, each week, TCs participate in at least one content exploration activity set in a social/cultural/historical/political context. For instance, as part of content explorations, I designed a pattern investigation activity based on the tradition of Kolam, an artform that has been practiced for many centuries by the women of the household in Tamilnadu (Ascher, 2002). By highlighting connections between mathematics and my own culture and by placing the spotlight on the mathematics of Kolam-drawing women, we develop an appreciation of the diversity and respect of cultural heritages, thus supporting the belief that all people are capable of doing mathematics. The research component requires TCs to complete a mathematics and culture project in which they partner with a community member to identify out-of-school activities that have a potential to transform into rich mathematical tasks (Naresh, Goodson-Espy & Poling, 2017). TCs use communal knowledge to frame problems that are central to their lived experiences, classical knowledge to develop mathematical competencies, and critical knowledge to gain a comprehensive understanding of the sociopolitical context for the problem (Gutstein, 2006). We adopt a socio-critical modeling approach in conjunction with Gutstein's 3-C framework that provides an "emancipatory perspective" [that] leads to a critical understanding of the surrounding world" (Kaiser & Sriraman, 2006, p. 304).

Author B's Narrative (participation/positioning)

My conscious journey to transform my instructional space to be culturally relevant (LadsonBillings, 1995) began while teaching a secondary STEM methods course about six years ago. A critical review and reflection of my own teaching revealed that I did not know the K-12 mathematics and sciences experiences and identities of the teacher candidates (Aguirre et al, 2013). During this time, I became conscious and acknowledged my privilege for identifying as a white heterosexual female and follower of Jesus Christ (DiAngelo, 2018). As a result, I hold a place of privilege and power as a mathematics teacher educator. For this reason, I chose the *participation/positioning* dimension of rehumanizing mathematics (Gutiérrez, 2018).

First, I needed to acknowledge my own upbringing in a single parent non-Christian home where I experienced both poverty and wealth. At home, my mom and maternal grandmother shared stories of our Indigenous heritage, but we have no record or connections to our Indigenous tribe. At school, I did not share my heritage but sought to fit in. My mathematical experiences were initially impeded by my dyslexia. In eighth grade was when I first recall experiencing agency in mathematics. Now I seek to support teacher candidates in giving agency and power to the students they will teach. My position and power inherently can be a deterrent to BIPOC teacher candidates unless I provide space for them to bring their whole selves to the classroom and be critical of the social and socio-mathematical norms that continue to be perpetuated in a traditional mathematics classroom. Next, are examples of activities in STEM and mathematics methods courses.

In the secondary STEM methods course, equity and diversity became a thread throughout the course rather than finite topics covered in the semester. Virtual anonymous discussions embedded in the course beginning with candidates writing their STEM autobiography and other discussion such as how their culture was an advantage and disadvantage. The anonymous post to each other gave teacher candidates a voice and space to share what they would not share with one another. In the middle grades mathematics methods course, teacher candidates chose to lead in *Number Talks* (Parrish & Dominick, 2016; Sun, Baldinger, & Humphreys, 2018) or keeping a log of what occurred in class. Everyone signed up once for a particular class for posting to the class Twitter about what they saw as worthwhile for sharing with the teacher community. My thinking was that by including Number Talks as part of our weekly routine, the position of teacher candidates would be us rather than me and them. In the elementary mathematics methods course, teacher candidates also lead Number Talks (Parrish & Dominick, 2016), but rotated in Quick Draw (Wheatley, 2007), and mathematics literacy during the semester too. Instead of Twitter, students reflect weekly through the course webpage on what was the highlight of their learning.

What I continue to find difficult is that I am ultimately seen as the person with authority/power who gives grades that account toward their progress in the teacher preparation program. Ideally, I would like the work we do together to be intrinsically motivating rather than what is needed for the 'A'. By having a component of the grading system for weekly engagement, I can capture in the moment and reflect on the participation and positioning of teacher candidates.

Author C's Narrative (mirrors/windows)

I am an Afro-Caribbean mathematics teacher educator with a positive mathematics identity that was developed largely due to the influence and encouragement of K-12 teachers who always made me feel capable in mathematics. My positionality towards my work with elementary preservice teachers stems from my converging identities as a former high school mathematics teacher at a Title I school, mother to two elementary aged Black boys, and mathematician. In my research I investigate the nuanced ways that inequity occurs in the mathematics learning of children from racially minoritized groups. Furthermore, in my work with PSTs, I am most interested in facilitating the evolution of their critical consciousness (Ladson-Billings, 1995) and investigating the varied ways that this evolution can occur. Due to systemic injustices, including racism, that have been illuminated during the global pandemic, it is especially important that preservice teachers have opportunities to be self-reflective and explore sensitive topics that may have impacted their own mathematics learning experiences and those of the children they will teach (Gay & Kirkland, 2003).

Through my research, as well as my lived experiences, I am aware of how cumulative systemic inequalities in mathematics learning and teaching impact many student outcomes including affective outcomes like confidence and math identity (Riegle-Crumb, Morton, Nguyen, & Dasgupta, 2019) as well as achievement outcomes such as enrollment in advanced math courses, college, and interest in STEM majors (Madkins & Morton, 2021). I am always striving to center the lived experiences of my TCs (especially those of Color) in my elementary math methods courses as a means to illuminate, confront and expose past traumas in mathematics learning in an attempt to interrupt reproduction of those traumas. I value my work with elementary preservice teachers. Through this work I am able to directly influence and contribute to mathematics teacher education that centers equity and antiracism in math methods courses.

Recognizing that our PSTs are the future teachers of racially minoritized children traditionally underserved and marginalized in math classrooms, children much like my own two boys, adds a certain level of urgency to my work. However, I couple this urgency with critical grace (Amidon et al, 2020) recognizing that this process should be an intentional and continuous one.

In my elementary mathematics methods course, I strive to create a space for PSTs to reflect on their own experiences learning mathematics and have explicit discussions about the common inequities that exist in mathematics learning and teaching. When delving into the dimension of *mirrors/windows*, I engage TCs in a number of activities that stem from the math autobiography assignment (Aguirre, Mayfield-Ingram, Martin, 2013) where students tell their mathematical story, recounting their in-school and out-of-school learning experiences with mathematics. As they answer several question prompts, TCs describe specific moments in their elementary mathematics learning. Following this we spend time unpacking those autobiographies through 2 in class activities: word cloud and math corners activity (Ward, 2020). The dimension *mirrors/windows* allows TCs a space for exploring their relationship with mathematics as well as "seeking to understand themselves and others in relationship" (Gutiérrez, 2018). Both activities are intended to illuminate the commonalities and differences in the TCs' mathematics learning experiences and begin to have discussions around those patterns. The RM framework allows us as MTEs to address inequitable opportunities to learn mathematics for children traditionally marginalized in math classrooms by highlighting these same inequities in the learning experiences of those who are becoming teachers themselves, teacher candidates.

Impacts of Our Collective Work on our Individual Practice

In this paper, we shared findings from a collaborative self-study that traced the collective journey of a team of MTEs. We used three dimensions of the RM framework to situate and integrate our work across mathematics content and methods courses. As MTEs, we along with our teacher candidates are the needed change to create more humanizing and socially just contexts for learning and teaching mathematics. Our collaborative self-reflections have begun to deeply influence our individual practices as mathematics teacher educators. We each have particular strengths developed through our lived experiences, both personal and professional, that we draw from in our collaborative work. For instance, while we are all mathematics teacher educators, one of us is the sole MTE faculty in the mathematics department unlike the other two MTEs who are faculty in a teacher education department. This difference presents the opportunity for conversations around the pedagogical challenges commonplace within many mathematics departments that are in direct opposition to the dehumanizing practices that undergird our work. Similarly, our identities as female mathematicians intersected with our racial/ethnic identities further inform our collective understandings around privilege, oppression, and equity in mathematics education as they relate to our own mathematics learning experiences in schools, as well as our individual work with teacher candidates. We have recognized through our collaborations that we have much to learn from one another to both inform and strengthen our practice as MTEs in ways that put forth a collective commitment to rehumanizing mathematics education for our students and the children they will ultimately teach.

Conclusion

Together, we reflect on our own math learning experiences and recall instances when we felt a sense of joy, pride, and belonging, and discover ways to re-experience those in our courses. Mathematics teacher education courses must offer a space for transformative learning experiences "not only for how students understand their abilities in mathematics but also in how they understand their relationship to others in the world" (Staats, 2006, p. 41). Our work with TCs, which we have presented using the lens of RM, has enabled us to challenge myths such as a) meaningful math cannot exist when we engage students in non-traditional math activities, b) math knowledge is mostly generated within the walls of the academy, and c) "just plain folks" are credible sources of knowledge and are able to demonstrate the use of more sophisticated mathematics. This work is rewarding and challenging at once; as a collective, we are presenting a counter-narrative to the dominant perspective that has permeated mathematics education. We believe in the potential this work holds for transforming the landscape of mathematics education and mathematics teacher education. Concomitantly, we must continue to engage in a critical inquiry of our own biases, assumptions, and teaching practices to become advocates for and to provide more empathetic mathematics learning experiences for TCs. In our future work, we intend to be critical friends (Ragoonaden & Bullock, 2016) to one another, utilizing both formal (e.g., required annual peer observations) and informal means to provide feedback to one another in ways that advance our collective agenda.

Our collaboration has resulted in an emerging Community of Practice (CoP) (Lave & Wenger, 1991) within our institution focused on providing more streamlined and humanizing learning experiences for TCs. We hope that this work will spark interest among other likeminded MTEs beyond our institutional context, broaden participation, and strengthen our CoP. In addition, from a research standpoint, it will enable us to a) investigate ways in which we can encompass additional dimensions of RM in our work, b) continue to present counter narratives to empower TCs that are marginalized in mathematics learning spaces and c) contribute to a collective repertoire of knowledge on MTE professional development centered on rehumanizing mathematics.

References

Aguirre, J., Mayfield-Ingram, K., & Martin, D. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. Reston, VA: National Council of Teachers of Mathematics.

Alemán, S. M. A Critical race counterstory: Chicana/o subjectivities vs. Journalism objectivity. Taboo: The Journal of Culture and Education 16(1). https://doi.org/10.31390/taboo.16.1.08

Amidon, J., Marshall, A. M., & Smith, R.(February, 2020). Grace in Learning: A Necessary Condition of Equitable Practices in the Mathematics Classroom. Session at annual conference of the Association for
Mathematics Teacher Educators. Phoenix, AZ

Ascher, M. (2002). The Kolam Tradition: A tradition of figure-drawing in southern India expresses mathematical ideas and has attracted the attention of computer science. American Scientist, 90(1), 56–63. http://www.jstor.org/stable/27857597

Barton, B. (2008). The language of mathematics: Telling mathematical tales. New York, NY: Springer.

Bishop, A. J. (1990). Western mathematics: the secret weapon of cultural imperialism. Race and Class, 32 (2), 5165. Bullock, S. M., & Sator, A. (2018) Developing a Pedagogy of "Making" through Collaborative Self-Study, Studying
Teacher Education, 14(1), 56-70, DOI: 10.1080/17425964.2017.1413342

Creswell, J. W. (2007). Qualitative inquiry and research design: Choosing among five approaches. Thousand Oaks, CA: Sage.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. For the Learning of Mathematics, 5, 44-48.

D'Ambrosio, U. (2002). Ethnomathematics: an overview. Proceedings of the II Congresso Internacional de Etnomatematica, Ouro Preto, Brazil.

DiAngelo, R. (2018). White fragility: Why it's so hard for white people to talk about racism. Beacon Press. Frankenstein, M. (1983). Critical mathematics education: An application of Paulo Freire's epistemology. Journal of
Education, 165(4), 315–339. https://doi.org/10.1177/002205748316500403

Gay, G. (2010). Culturally responsive teaching: Theory, practice, & research. New York: Teachers College Press Gay, G., & Kirkland, K. (2003). Developing cultural critical consciousness and self-reflection in preservice teacher education. Theory into practice, 42(3), 181-187.

Gonzalez, N., Moll, L. & Amanti, C. (2005) Funds of Knowledge: Theorising Practices in Households, Communities, and Classrooms. Mahwah, NJ, Lawrence Erlbaum Associates.
Gutstein, E. (2006). Reading and writing the world with mathematics: Toward a pedagogy for social justice.
Routledge, New York.
Gutiérrez, R. (2018). Introduction. In Goffney, I., Gutiérrez, R., & Boston, M. (Eds.). Rehumanizing mathematics for Black, Indigenous, and Latinx students, (pp. 1-10). National Council of Teachers of Mathematics.
Kaiser, G., & Sriraman, B. (2006). A Global Survey of International Perspectives on Modeling in Mathematics Education. ZDM - The International Journal on Mathematics Education, 38(3), 302-310.
Ladson-Billings, G. (1995). But that's just good teaching! The case for culturally relevant pedagogy. Theory into practice, 34(3), 159-165.
Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. Cambridge: Cambridge University Press.
Loughran, J. J. (2004). A history and context of self-study of teaching and teacher education practices. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), International handbook of self-study of teaching and teacher education practices (pp. 7–39). Dordrecht, The Netherlands: Springer.
Lovin, L. H., Sanchez, W. B., Leatham, K. R., Chauvot, J. B., Kastberg, S. E., & Norton, A. H. (2012). Examining beliefs and practices of self and others: Pivotal points for change and growth for mathematics teacher educators. Studying Teacher Education, 8(1), 51-68. doi:10.1080/17425964.2012.657018.
Madkins, T.C., Morton, K. (2021). Disrupting anti-blackness with young learners in STEM: Strategies for elementary science and mathematics teacher education. Canadian Journal Science, Mathematics and Technology Education, 21, 239–256. https://doi.org/10.1007/s42330-021-00159-1
Morton, K., & Riegle-Crumb, C. (2019). Who gets in? Examining inequality in eighth-grade algebra? Journal for Research in Mathematics Education, 50 (5), 529-544.
Naresh., N., Goodson-Espy, T., & Poling, L. (2017). Using CME to Empower Prospective Teachers (and Students) emerge as Mathematica Modellers. Proceedings of the 9th International Conference of Mathematics, Education, and Society, Volos, Greece.
Naresh, N., & Kasmer, L. (2018). Using ethnomathematics perspective to widen the vision of mathematics teacher education curriculum. In Bartell, T, G., (Ed.), Toward Equity and Social Justice in Mathematics Education, Research in Mathematics Education Series, Springer.
Paris, D., & Alim, H. S. (2017). Culturally sustaining pedagogies: Teaching and learning for justice in a changing world. New York : Teachers College Press.
Parrish, S., & Dominick, A. (2016). Number talks: Fractions, decimals, and percentages. Math Solutions. Ragoonaden, K., & Bullock, S. (2016). Critical friends: The practiced wisdom of professional development. In K.
Ragoonaden & S. Bullock (Eds.), Mindfulness and critical friendship: A new perspective on professional development for educators (pp. 13–32). Lexington Books.
Raju, C. K. (2004). Math wars and the epistemic divide in mathematics. Proceedings of Episteme I, Homi Bhabha Center for Science Education, Mumbai, India.
Riegle-Crumb, C., Morton, K., Nguyen, U., & Dasgupta, N. (2019). Inquiry-based instruction in science and mathematics in middle school classrooms: Examining its association with students' attitudes by gender and race/ethnicity. AERA Open, 5(3).
Skovsmose, O. (1985). Mathematical education versus critical education. Educational Studies in Mathematics. 16, 337--354.
Staats, S. (2006). The case for rich contexts in ethnomathematics lessons. The Journal of Mathematics and Culture, V1(1), 39-56.
Sun, K. L., Baldinger, E. E., & Humphreys, C. (2018). Number talks: Gateway to sense making. The Mathematics Teacher, 112(1), 48-54.
Walsham, G. (2006). Doing interpretive research. European Journal of Informative Systems. 15, 320–330 https://doi.org/10.1057/palgrave.ejis.3000589
Ward, J. (2020). The Road You Traveled: Reexamining PSTs Experiences to Talk About the Sociopolitical Nature of Math. Session at annual conference of the Association for Mathematics Teacher Educators.
Phoenix, AZ.
Wheatley, G. (2007). Quick draw. Mathematics Learning: Bethany Beach, DE, USA.

BEYOND THE WORLDS OF ITS DEFICIT MODELING APPROACHES: SITUATING ETHNOMATHEMATICS RESEARCH WITHIN THE SOCIAL-DESIGN EXPERIMENT INQUIRY

Emmanuel Nti-Asante
University of Massachusetts, Dartmouth
entiasante@umassd.edu

The heritage practices of many communities of color have historically been and continue to be the target of deficit approaches. In Mathematics Education, the existing solutions to this problem have focused on modeling approaches to bridge cultural ways of knowing with the institutionalized forms. This article argues that solutions to deficit approaches in cultural ways of knowing in mathematics education (ethnomathematics) are not simply about modeling bridges. We must bring together and extend these domains' various activities and practices in a forward-looking third space that occurs socially for all learners to voice, personalize, challenge, and negotiate mathematical meanings. Ethnomathematics research holds an intellectual accomplishment that results from one's ability to remediate the mathematical motives in activities done by diverse ethnic groups to include the mathematics used in formal spaces.

Keywords: Sustainability, Ethnomathematics, Culturally Relevant Pedagogies, Informal Education.

Background

As we know, our society—hence, our classrooms—are becoming increasingly diverse in terms of racial, ethnic, cultural, economic, and religious diversity (Colquitt, 2014; Mensah, 2021). However, education environments in such pluralistic societies offer students little purpose (Paris & Alim, 2017) to connect the content, pedagogical, and curricula approaches to the intellectual motives in ethnic practices of students from the non-dominant groups in the same educational environment.

Researchers like D'Ambrosio (1986) proposed ethnomathematics as one of the approaches to identifying the mathematical ideas of culturally distinct people (D'Ambrosio, 1986). The educational implications of ethnomathematics include those cited in Blikstein (2020, p. 117) by Adam et al. (2003). Ethnomathematics is a supplement for students to see mathematics as a response to human needs found in nearly every culture. A springboard for academic mathematics. As a stage in teaching and learning of mathematics that starts from the mathematical world of the child and then moves into other cultures. As support for preparing rich learning situations and activities, an approach that looks at the classroom itself as a situated cultural context and mathematical learning as part of this context.

I build on these educational implications to situate ethnomathematics as a research approach that connects the practices of today's mathematics classroom to that of ethnic practices. This includes connecting the Euclidean geometry in the classroom with the geometric practices in ethnic woodwork practices like measurement and construction in Africa. I conceptualize ethnomathematics as a third-space construct situated within the Freirian and Neo-Vygotskian approaches to intervening in educational activities (Gutiérrez, 2008). This approach allows us to bring together and extend various activities and practices of these domains' (institutionalized Mathematics and ethnomathematics) in a forward-looking third space (Paris, 2012; Guitierez,2008). This third space serves as a cross-cultural space (Nasir et al., 2010) for all learners to remediate by reorganizing and reconstructing (Gutiérrez et al., 2009) the narratives in the form of written texts of ethnic motives of mathematical practices to add to the institutionalized mathematics (Gutiérrez & Jurow, 2016).

Theoretical Framework

In 1977, the late Professor D'Ambrosio Ubiratan concluded that educators usually study the aspects of knowledge and learning in mathematics education (generation of knowledge, its social and intellectual organization, and its diffusion) in isolation from one another. Researchers identified mathematics education research with cognition, epistemology, history, sociology, and other educational disciplines but not culture. Hence, D'Ambrosio coined the term "ethnomathematics" during a presentation for the American Association for the Advancement of Science in 1977. Ethnomathematics (D'Ambrosio, 1986) became the first well-developed attempt to find curricular and theoretical combinations of institutionalized and ethnic motives of mathematics (Blikstein, 2020). Etymologically, D'Ambrosio (1990) defined ethnomathematics in the following way: the prefix "ethno" is today accepted as a broad term that refers to the social-cultural context and, therefore, includes language jargon, codes of behavior, myths, and symbols. The derivation of "mathema" is complex but means explaining, knowing, understanding, and doing activities such as ciphering, measuring, classifying, inferring, and modeling. The suffix "tics" is derived from techné and has the same root as technique. For example, Gerdes (1986) and Mukhopadyay (2008) studied implicit mathematics in basket-weaving techniques. Gilmer (1995) studied mathematics in techniques of braiding African American hairstyles.

Criticisms Against Ethnomathematics

Despite the importance of ethnomathematics, its approaches to bridging cultural ways of knowing (braiding of hair, weaving of baskets in Native American communities) with the institutionalized mathematics (like Euclidean Geometry) have been the center of critiques and, thus, its current research focus. Research in ethnomathematics uses institutional knowledge to model ethnic knowledge (Dowling, 1998; Eglash et al., 2021; Pais, 2011; Rowlands & Carson, 2002; Skovsmose & Renuka, 1997). Some researchers are known to be imposing institutionalized mathematical concepts and practices on ethnic motives of mathematics. Take, for example, an investigation of Navajo weaving by Babbitt et al. (2012, pp.1-2); they posited that:

> a common recurring angle in the weaving patterns was about 30 degrees. When asked about this particular angle, the weaver said it is created by an "up one over one" pattern (up to one weft over one warp). We could see why that would result in a 30-degree angle: because the height to width ratio of each successive stitch was about 1/3. But she then said she could make lots of other patterns (up to one over 2, up to two over 3, etc.).

In the quote above, majority ethno-mathematicians know that the native basket weaver knows not the rubrics of angles. However, the exchange above shows how sometimes we tend to research ethnic motives of mathematics by what Pais (2011) described as squeezing the natives from their otherness (Pais, 2011). This unidirectional approach deprives ethnic groups of their emic approaches to mathematics. Thus ethno mathematicians deprive the ethnic groups of their social and cultural specificity about differences that make mathematical practices unique from their point of view (Rosa & Orey, 2019). This risks reproducing etic approaches leading to colonial, primivitzing views about mathematics in ethnic practices (Dowling, 1998; Eglash et al., 2021; Powell & Frankenstein,1997). Thus, researchers build ethnic mathematical practices from an outsider's view (Rosa & Orey, 2019).

Building from the above, as cited in Lachney et al. (2016, p. 222), Rosa and Orey (2013) use the language of dialectics to describe their work on *ethnomodeling*, aiming for a bidirectional synthesis of emic and etic interpretations. Ethnomodelling, which combines ethnomathematics and mathematical modeling, is an approach that creates a firm foundation that allows for the integration of emic, etic, and dialogic approaches in exploring mathematical knowledge developed by the members of distinct cultural groups (Rosa & Orey, 2019). However, this bidirectional approach had some troubling asymmetry (Lachney et al., 2016). The bidirectional approach mainly assumes that the academic form of mathematics is outside of time and space: final, perfected, ahistorical, and universal (Lachney et al., 2016). Hence, Lachney et al. (2016) argued for a new form of modeling ethnic motives of mathematics with that of institutionalized mathematics. They believed that any new approach to bringing these two systems of knowledge together must show how Western science, mathematics, and computing are also culturally specific (Lachney et al., 2016).

Another troubling asymmetrical approach was Pias and Mesquita's (2013) Urban Boundaries Project and Tedre and Eglash's (2008) ethnocomputing. These projects have parts that failed to correspond to academic mathematics or local mathematics. This one-way approach fails students to learn to appreciate Mathematical achievements from their own and other cultures (D'Ambrosio, 1990). In the urban boundary project, Pias and Mesquita(2013) believed that our society needs to create alternative educational settings to respond to the increasing problem of exclusion faced by so-called minority populations. They think that as long as schools are structured as credit systems (Vinner, 1997), only within non-scholarized settings can a genuine ethnomathematical approach be reached. However, what is the essence of this setting if it has no linkage to the mainstream where students also go to learn? Researchers identified initiatives like the Urban Boundary Projects as a potential cause for some ethnic children's conception that doing Mathematics is as being White (Fryer & Torelli, 2005).

Mathematicians and anthropologists use computational media platforms to address these challenges from initiatives like the Urban Boundary Project (Ethnocomputing, 2017). However, a different project for using this computational media platform expanded its initial foci (Eglash, 2006). It was a rubric for the computational aspects of the computational media platform for Computer Science Education (Tedre & Eglash, 2008). The computational media platform became an amalgamation of both mathematics and computations in cultures for Mathematics and Computer Science Education research. This has resulted in the problem of striking the difference between using the platform for its ethnomathematics purpose or Computer Science Education. Lachney et al. (2016) found that adding computations in cultures to the computational platform for Computer Science Education has disempowered indigenous communities and silenced their mathematical heritage.

Lachney et al. (2016), building on their critique of Rosa and Orey's bidirectional form of modeling and that of the computation media platform, called for a recursive form of modeling. Lachney et al. (2016) concluded that there are bi-directional flows within the indigenous culture and bi-directional flows within the western culture (such as the interaction between people and machines). Hence, bringing those together results in a recursive modeling process, a nesting of flows within flows. (Lachney et al., 2016). These researchers fully fleshed out the recursive modeling in their Adinkra Computing (AC, acronym used in the original study) project. AC was a form of computations in cultures within the computational media (Eglash & Tedre, 2008) in the West African context. Hence, it is not surprising to find that it equally disempowered indigenous communities and silenced their mathematical heritage.

Also, in the recursive modeling, students only moved after computer simulations to physical renderings to repurpose STEM innovation and discovery in the service of indigenous community development (Eglash et al., 2019). This is a form of Rosa and Orey's (2013) bidirectional approach, which Lachney et al. (2016) critiqued as troubling and asymmetrical. In addition to this, recent use of recursive modeling (Eglash et al., 2021) never focused on authentic forms of cultural making (Blikstein, 2020), mathematics, and its ethnomathematics. Everything existed on the Computer. Students tend to lose the realistic pre-planning of executing any artistic pattern from diverse cultural groups. I assert that there is no reason why a mathematical model cannot include the possibility of an indigenous counterpart or conscious reflection on the part of communities who shape their own material conditions from generation to generation (Eglash, 1997).

In line with this, Barbosa (2006) and Rosa and Orey (2015) proposed socio-critical forms of modeling. This is a learning milieu where teachers invite students to take a problem from everyday or other sciences that are not pure mathematics and investigate it concerning reality via mathematics (Barbosa, 2003). However, much of everyday mathematics ignores the diverse ethnic motives of Mathematics. The question to be answered here is whose everyday? Beyond this, ethnomathematics research is more than just looking for modeling approaches; I assert that it is an encompassing and robust socio-critical literary program. Instead of researchers using everyday, I assert we use relevant ethnic mathematical practices. I will discuss this as we go forward.

What Other Options Are Available?

Ethnomathematics reflects the growing consciousness of the existence of mathematics, particularly in a certain way to (sub)cultures (Gerdes, 2001). Ethnomathematics challenge that the development of mathematics is not unilinear (Gerdes, 2001). Hence, we can use broad terms, including, in particular, counting, locating, measuring, designing, playing, and explaining (Bishop, 1988), which emphasize and analyze the influences of socio-cultural factors on the teaching, learning, and development of mathematics (Gerdes, 2001). Institutionalized mathematics is already a composition of Mathematics from the Greeks like geometry, expressions, and algorithms from the Indians, and Europeans like algebra. This characterization makes the techniques and truths of Mathematics a cultural product (Gerdes, 2001). Hence, we can revitalize ethnomathematics research to expand our understanding of the development of mathematics like geometry beyond Greeks to cultures of others like Africans, Native Americans, and many others (NCTM, 2021).

Generally, this research interest in moving "everyday life into the school" (Scribner & Cole, 1973, p. 558), in this case, "the mathematical thoughts in practices of diverse ethnic groups into schools" has a long tradition in neo-Vygotskian approaches. They help intervene in educational activities (see Gutiérrez & Jurow, 2016). This theoretical orientation and educational and social goal inspire the design of educational settings that seek to bring together cultural forms of learning and scientific practices in ways that places them on a more level playing field, that is, without reproducing inequitable hierarchical relations (Gutiérrez & Jurow, 2016). A typical neo-Vygotskian approach is the social-design experiments, leading to socio-critical literacies (Guitterez, 2008; Gutiérrez & Jurow, 2016). In what follows, I describe how I frame ethnomathematics in the social-design experiments and the consequential socio-critical literacies.

Ethnomathematics and Social Design Experiments

Social design experiments (SDEs) are an advanced approach to design research organized around a commitment to transforming the educational and social injustices faced by members of non-dominant communities to promote social equity and learning (Gutiérrez & Jurow, 2016). The injustices toward which social design experiments focus are those that are structural, systemic, and experienced as unjust by non-dominant communities. These include those discussed above. Thus, the silencing and marginalization of the mathematical practices of diverse ethnic groups by researchers in an attempt to bring them into the academic space. This also includes education environments in pluralistic societies that offer students little purpose (Paris & Alim, 2017) to connect the content, pedagogical, and curricular approaches to students' intellectual motives in ethnic practices from the non-dominant groups.

Researchers focus on SDEs is on redressing historical injustices and inequities and the developing of theories focused on the organization of equitable learning opportunities. Hence, situating ethnomathematics research in SDEs is focused on finding solutions to those mentioned above structural and systemic injustices in the pluralistic Mathematics classroom. Researchers achieved this through the SDE methodology combines traditions of design-based research (Design-Based Research Collective, 2003), namely; (a) the need to develop innovative approaches for educational improvement and (b) the need to address theoretical questions about the nature of learning in context (c) the need to study learning in the real world rather than in the laboratory (d) the need to go beyond narrow measures of learning and understand how complex learning ecologies support learning (Gutiérrez & Jurow, 2016, p.2). As well democratizing such forms of design-based inquiry. It seeks to make the design experimentation process a coconstruction between different institutional stakeholders (Cole & The Distributed Literacy Consortium,2006), which leads to social transformations (Gutiérrez & Jurow, 2016). In addition, social design experiments build on the different forms of knowledge and expertise that people develop across the multiple contexts of their lives, recognizing that even in the face of oppression, non-dominant groups have many resources for being resilient (Gutiérrez & Jurow, 2016). Hence, situating ethnomathematics research in SDEs is a means for learners from diverse ethnic backgrounds to find the pluralistic mathematical classroom as a meaningful shared space where they co-construct mathematical meanings from both the academic space and that of their diverse backgrounds. By building on ethnic motives of mathematics, the solutions and the learning that emerges through SDEs have a greater likelihood of achieving sustainability, meaning, and impact. This is contrary to the deficit approaches I outlined above.

Of significance is that SDEs' explicit focus on disrupting educational, structural, and historical inequities through the design of transformative learning activities provides openings for learning, a context of critique for resisting and challenging the conditions that create and sustain inequities, and space for generating their possible solutions (Gutiérrez & Jurow, 2016). With this perspective on equity-oriented design research in mind, SDE researchers ask and offer one solution to this central research question: How can researchers organize design interventions that intentionally leverage the diverse forms of expertise of non-dominant communities so as to create the possibility of more consequential learning opportunities for them? (Gutiérrez & Jurow, 2016, p.4). The answer proposed by Gutiérrez and Jurow (2016) is that the designs at the center of SDE strategically link practices that are valued differentially, like the ethnic motives of mathematics in society, with that of the dominant forms like academic mathematics for learners to reorganize and reconstruct. I describe designing for ethnomathematics through SDEs below.

Designing for Ethnomathematics through SDEs

Designing SDEs begins with teachers creating a repertoire of ethnomathematics testimonios. This well-developed complementary methodology enables one to uncover the mathematical motives (Pias, 2013), like measurement and designing in diverse ethnic practices like basket weaving (Gerdes, 1986). Through cognitive anthropological interviews, teachers video record the basket weaving and hair braiding techniques of Indian and African Americans, respectively. Teachers, students, or Native scholars write the audio from the video recordings into texts of the Native language, English, and many more. This written narrative is for students to use in the pluralistic mathematics classroom. They serve as mediational artifacts. Ethnomathematics testimonios are also syncretic texts, designed to exploit the existing hybridity and help to create particular social environments of development in which students begin to reimagine who they are and what they might be able to accomplish, academically and beyond (Gutiérrez et al., 2009). Developing such ethnomathematics testimonios involves conditional knowledge that involves teachers knowing when to use particular sources of ethnic motives of mathematics and how to evaluate their effectiveness concerning institutionalized mathematics. Ethnomathematics testimonio keeps to Papert's (2000) reminder. Children can learn powerful ideas about the world if we forgo those that schools have politically disempowered, teaching mathematics only through applying formulas or by proposing problems situated in "fake" contexts that fail to be meaningful for pupils.

Ethnomathematics testimonios serve as the fabric of knowledge from cultures (Nasir & Hand, 2006). They are the detailed description of cultures, including pre-planning thoughts and natural language use in cultural practices. Subsequent use of such narratives in diverse contexts might call for translations to eliminate the language barrier. This is possible through the fractionated trading zone metaphor (Collins et al., 2010). Thus, we can see that mathematical thoughts are boundary objects and interactional expertise. They adapt to the local needs and constraints of the several parties employing them (materially and linguistically) yet are robust enough to maintain a common identity across sites. This fractionated trading zone informs the definition of culture in my use of social-design experiments. Culture is both by the material culture largely in the absence of linguistic interchange. Also, culture is the interactional expertise. Students mediate by language mainly in the absence of the material culture from the ethnomathematics testimonio. The literacy in the neo-Vygotskian focus of ethnomathematics reflects that of Guitterez and Jurow (2016) and diSessa (2018). Thus, it is a form of massive intellectual accomplishment as learners go through a grand "remediation" of shifting and expanding fundamental forms of mathematics as they unfold in diverse ethnic groups and their practices to include institutionalized forms of mathematics as universally known and used. The premise for using the ethnomathematics testimonio in SDE follows that of Engeström's re-mediational design. Thus,

> Rather than giving the child just a task, ignoring her interpretation and reconstruction of the task, and observing how she manages, Vygotsky and his colleagues typically gave the child also potentially useful mediating artifacts - tools and signs. With them, the nature of the task could be radically changed. The child's potential capabilities and emerging new psychological formations might be revealed. (Engeström et al., 1986, p. 5)

As in social-design experimentation (Gutiérrez & Jurow, 2016), developing ethnomathematics through SDEs does not aim merely to contact the practices of non-dominant communities and institutions (e.g., schools). Rather, researchers organized ethnic motives of mathematics for students. Students reconstruct and reorganize to engender new forms of knowledge and expertise. This knowledge and expertise embody characteristics of the best of both sets of practices (ethnomathematics and institutionalized mathematics), albeit in new forms. At the same time, it is critical to note that the researcher designed ethnomathematics testimonios to transform the institution to embrace multiple ways of knowing, acting, and valuing. These social design experiments (Vossoughi & Gutiérrez, 2008) are organized around expansive forms of learning, powerful literacies, and hybrid language practices that result from the intercultural exchange and boundary-crossing involved in students' everyday lives like socio-critical literacies.

Socio-Critical Literacies and Ethnomathematics

Socio-critical literacies is a syncretic literacy that emphasizes the development of literacies in which researchers reframe everyday and institutional literacies into powerful literacies oriented toward critical social thought (Gutiérrez, 2008). As Gerdes (2001) characterizes ethnomathematics, socio-critical literacies focus on the cultural dimensions of learning. It takes an approach that resists home and school binaries, formal and informal learning. Instead, it focuses on what takes hold as children and youth move in and across their everyday lives' various settings and contexts (see Tuomi-Gröhn & Engeström, 2003). This approach identifies both possibility and constraint within and across contexts (Cole & Engeström, 1993). From this perspective, we can capitalize on the understanding of the cultural dimensions of learning and development that occur as "people, ideas, and practices of different communities meet, collide, and merge" (Engeström, 2005, p. 46). The socio-critical agenda of ethnomathematics does not see the original ethnic motives of mathematics as a profound source of only intervention (see Vossoughi et al., 2016). It rejects the assumptions that the computational capital of some cultures is hidden (Eglash & Bennett, 2009; Gerdes,1985) and needs a suitable environment before being made available to their owner (Eglash & Bennett, 2009).

The socio-critical forms of ethnomathematics assume that they are untapped. When tapped, they are a deep source of learning in their cultural context, which overlaps with/into other modes/mediums of learning for expansive forms of knowledge. This distinguishes the socio-critical form from most research in recursive modeling of ethnomathematics, which mostly rejects the original forms of learning. The socio-critical literacies conceptualize the notion of repertoires of practice. SDE researchers capture both vertical and horizontal forms of expertise through this repertoire of practices (Gutiérrez & Rogoff, 2003); this includes what students learn in formal learning environments such as schools and what they learn by participating in a range of practices and ethnic motives of mathematics. The socio-critical forms of ethnomathematics conceptualize the idea of a Third Space (Gutiérrez & Roggoff, 2003). This Third Space construct is more than a celebration of the local literacies of students from non-dominant groups; and certainly more than what students can do with assistance or scaffolding; and also more than ahistorical accounts of individual discrete events, literacy practices, and the social interaction within (Gutiérrez & Rogoff, 2003). Instead, it is a transformative space where the potential for an expanded form of learning and the heightening of the development of new knowledge (Gutiérrez & Rogoff, 2003).

Conclusion

The divide between many educational institutions and the students they are supposed to serve grows (Paris, 2017). Students do not feel that their identities are affirmed through the curriculum taught; it is irrelevant, very impractical, and exclusionary to their backgrounds, experiences, and lives (Paris, 2017). In Mathematics Education, a unidirectional approach has been the mode for researchers to connect the cultural ways (ethnomathematics) and institutionalized forms of learning. This seems to be very challenging. As Miller (1992, p.11) posited; how can anyone who is schooled in conventional Western mathematics “see” any form of mathematics other than that which resembles the conventional mathematics with which she is familiar? The answer to this question partially formed the focus of this write up. I assert that although these diverse forms of mathematics might be familiar, the language, terminologies, tools and many others used in one context might be different from the other. It only takes remediation in the form of reorganizing and reconstruction for one to resolve the constradictions between these two systems of knowledge for an expanded form of learning. One way to start with is to write the pre-planning thought and natural language used in ethnic motives of mathematics for learners and teachers to co-construct by adding to that of the institutionalized mathematics.

In this article, I proposed using repertoires of ethnomathematics testimonios through the neo-Vygotskian theories of intervening activities. Learners can reorganize mathematical motives in the making processes of material culture from a diverse ethnic group into institutionalized forms of Mathematics (Vygotsky, 1978). Through this, I assert that we will all become enlightened about the motives of mathematics from other cultures through the neo-Vygotskian foci of ethnomathematics research. This would help us avoid the thought that ethnomathematics research is mainly for people of color and merely turning cultural products into lessons for a white-dominant society. As a PMENA-43 reviewer who rejected my initiatives, among other reasons, stated:

> "…as a community, we need to be careful of turning cultural products into lessons for a White dominant society and be critical of who this research is to research (on Communities of Color) is out to benefit (the communities with whom we are conducting research on/with or the Mathematics classroom where mostly white middle/upper-class student benefit".

I have emphasized that institutionalized mathematics comprises cultures like Greek Geometry, Indian Algebra, American English, and symbols. Hence, mathematics education research can add that of other cultures on the grounds of Mathematics. Untapped Mathematics in other cultures might be the source of our discoveries in the form of advanced mathematical thoughts. I hypothesize that ethnomathematics as a syncretic mathematical activity is a massive intellectual accomplishment that one achieves from the grand remediation of mathematical motives in the activities done by diverse ethnic groups to include institutionalized forms of mathematics. This is a practical approach for all learners to reorganize and reconstruct non-routine Mathematical problems. One approach is designing mathematical activities that would involve all manner of learners to think with at the intersection of cultural presence, embedded knowledge, and the possibility for personal identification (Papert, 1980). Through such activities, we can answer research questions including; (a) how does ethnomathematics testimonios stimulate reflexive discussions among all learners in common cultures of Mathematics about its connectedness with the mathematical motives in other (sub)cultures? (b) How do we create a repertoire of ethnomathematics testimonios from holistic ethnomethodology which is globally accessible to all?

References

Barbosa, J. C. (2006). Mathematical modelling in classroom: A socio-critical and discursive perspective. *ZDM, 38*(3), 293-301.

Barbosa, J. C. (2003).What is mathematical modelling? In S. J. Lamon, W. A. Parker, & S. K. Houston (Eds.), Mathematical modelling: a way of life.

ICTMA11 (pp. 227-234). Chichester: Horwood Publishing

Babbitt, B., Lyles, D., Eglash, R., (2012). "From ethnomathematics to ethnocomputing," In S. Mukhopadhyay & W.-M. Roth (Eds.)., *Alternative*

Forms of Knowing (in) Mathematics, (pp. 205-219). Sense Publishers, .

Bassanezi, R. C. (2002). Ensino-aprendizagem com modelagem matemática. [Teaching-learning with mathematical modelling]. Editora Contexto.

Bishop, A.J., (1988). The Interaction of Mathematics Education with Culture, Culture Dynamics. DOI:

10.1177/92137408800100200.
Blikstein, P. (2020). Cheesemaking emancipation: A critical theory of cultural making. In N. Holbert, M. Berland, & Y. Kafai (Eds.), *Designing*
constructionist futures: The art, theory, and practice of learning designs (pp. 115-126). MIT Press.
Bodrova, E., & Leong, D. J. (2007). Tools of the mind: The Vygotskian approach to early childhood education, 2. Pearson.
Cazden, C., & Leggett, E. (1976). Culturally responsive education: A discussion of LAU remedies, II. Prepared for the U.S. Department of Health,
Education, and Welfare. National Institute of Education.
Cole, M., & Engeström, Y. (1993). A cultural-historical approach to distributed cognition. In G. Salomon (Ed.), *Distributed cognitions: Psychological*
and educational considerations (pp. 1–46). Cambridge University Press.
Colquitt, R. L., (2014). Social justice in mathematics education. PhD dissertation. University of Tennessee,
https://trace.tennessee.edu/utk_graddiss/2686
Collins, H. M., Evans R., & Gorman M.E. (2010). Trading zones and interactional expertise. In: Gorman ME (ed) *Trading zones and interactional expertise: Creating new kinds of collaboration.* (pp 53-70). The MIT Press,
D'Ambrosio, U. (1990). Etnomatemática. [Ethnomathematics]. EditoraÁtica.
D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5,* 44-48.
Dowling, P. (1998) The Sociology of Mathematics Education: Mathematical Myths/ Pedagogic Texts. London, UK: Falmer.
DiSessa, A. A. (2018). Computational literacy and "the big picture" concerning computers in mathematics education. Mathematical Thinking and
Learning, 20(1), 3–31. https://doi.org/10.1080/10986065.2018.1403544.
Duranti, A., & Ochs, E. (1997). Syncretic literacy in a Samoan American family. In L. B. Resnick, R. Säljö, C. Pontecorvo, & B. Burge (Eds.),*Discourse, tools, and reasoning: Essays on situated cognition (pp. 169–202).* Springer.
Eglash, R., Bennett, A., Cooke, L., Babbitt, W., & Lachney, M., (2021).Counter-hegemonic computing: Toward computer science education for value generation and emancipation. *ACM*. (21), 4.https://doi.org/10.1145/3449024
Eglash, R., Lachney, M., Babbitt, W., Bennett, A., Reinhardt, M., & Davis, J. (2019). Decolonizing education with Anishinaabe arcs: generative
STEM as a path to indigenous futurity. *Education Tech Research Dev.* doi:10.1007/s11423-019-09728-6
Eglash, R. & Bennett, A. (2009). Teaching with hidden capital: Agency in children's mathematical explorations of cornrow hairstyle simulations.
Children, Youth, and Environments 19 (1).
Engeström, Y. (1998). Activity theory and individual and social transformation. In Engeström, Y, Miettinen, R., & Punamaki, R.-L. (Eds.),*Perspectives on activity theory* (pp. 19-38). Cambridge University Press.
Ethnocomputing(2017).http://www.composingdigitalmedia.org/s17_dmtp/webs/jik//ethnocomputing.html
Eglash, R., Bennett,A., O'donnell, C., Jennings, S., & Cintorino, M. (2006). Culturally situated design tools: Ethnocomputing from field site to classroom. *American Anthropologist (108*)2: 347-62.
Eglash, R. (1997). Bamana sand divination: recursion in ethnomathematics. *American Anthropologist, 99*(1). 112-122.
Engeström, Y. (2005). Developmental work research: Expanding activity theory in practice (Vol. 12). Lehmanns Media.
Fryer, Roland G., Jr., and Paul, T. (2005). An Empirical Analysis of "Acting White." Available from: http://www.economics.harvard.edu/faculty/fryer/files/fryer_torelli.pdf
Gerdes, P.(1986) On culture, mathematics and curriculum development in Mozambique. In S. Mellin-Olsen, & M. Johnsen Hoines, (Eds.),Mathematics and Culture, a seminar report, Caspar Forlag, Radal, 15-42
Gerdes, P. (2001). Ethnomathematics as a new research field, illustraties by studies of mathematical ideas in African history. In J. J. Saldana (Ed.), *Science and cultural diversity: Filling a gap in the history of science* (pp. 11–36). Sociedad Latinoamericana de Historia de las Ciencias y laTecnolog a.
Gilmer, Gloria. "Mathematical Patterns in African American Hairstyles." The Mathematics Department of The State University of New York at Buffalo, n.d. Web. 19 Mar. 2017.
Gutiérrez, K. D. & Jurow, A. A. (2016): Social design experiments: Toward equity by design. *Journal of the Learning Sciences*, DOI:10.1080/10508406.2016.120454

Gutiérrez, K., Baquedano-Lopez, P., & Tejeda, C. (2009). Rethinking diversity: Hybridity and hybrid language practices in the third space. *Mind,Culture, and Activity, 6*(4), 286-303

Gutiérrez, K. D. (2008). Developing Socio-critical literacy in the third space. *Reading Research Quarterly, 43(*2), 148–164

Gutiérrez, K. D., & Rogoff, B. (2003). Cultural ways of learning: Individual traits or repertoires of practice. *Educational Researcher, 32*(5), 19-25.

Lachney, M., Bennett, A., Appiah, J., & Eglash, R. (2016). Modeling in ethnocomputing: Replacing bidirectional flows with recursive emergence. *International Journal for Research in Mathematics Education 6,* (1) 219–243

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal, 32,* 465–491.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race, Ethnicity, and Education, 22*(4), 459–478.

Mensah, M., F. (2021). Culturally relevant and culturally responsive: Two theories of science practice teaching.*Science and children 58*(4).

Mukhopadhyay, S. (2008). The Decorative Impulse: Ethnomathematics and Tlingit Basketry. In S. Bharath, M. Claus, B. Astrid, F. Viktor, (Eds.)

Proceedings of the 2nd International Symposium on Mathematics and its Connections to the Arts and Sciences (MACAS2). Centre for Science and Mathematics Education, pp.57-74.

Nasir, N. S., & Hand, V. M. (2006). Exploring Sociocultural Perspectives on Race, Culture, and Learning. *Review of Educational Research, 76,*449-475.https://doi.org/10.3102/00346543076004449

National Council of Teaching Mathematics (NCTM).. (December 2021)." The wonder, joy, beauty, and power of Geometry. nctm.link/EenYp.

Papert, S. (2000). What's the Big Idea? Steps toward a pedagogy of idea power. I*BM Systems Journal, 39*,3.

Paris, D., & Alim, H.S. (2017). Culturally sustaining pedagogies: Teaching and learning for justice in a changing world. Teacher's College Press.

Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational Researcher, 41(*3), 93–97.

Pais, A. (2011). Criticisms and contradictions of ethnomathematics. *Educational Studies in Mathematics 76(*2), 209-230.

Pais, A., & Mesquita, M. (2013). Ethnomathematics in non-formal educational settings: the Urban Boundaries project. *Revista Latinoamericana de Etnomatemática, 6*(3), 134-144.

Powell, A., & Frankenstein, M. (1997). Ethnomathematics: Challenging Eurocentrism in mathematics education. State University of New York Press.

Rosa, M., & Orey, D. C. (2019). Ethnomodelling as the translation of diverse cultural mathematical practices. In: B. Sriraman, B. (Ed.), Handbook of the Mathematics of the Arts and Sciences (pp. 1-29). Springer Nature.

Rosa, M., & Orey, D. C. (2015). A trivium curriculum for mathematics based on literacy, matheracy, and technocracy: an ethnomathematics perspective. *ZDM - International Journal on Mathematics Education, 47*(4), 587–598. https://doi.org/10.1007/s11858-015-0688-1

Rowlands, S., & Carson, R. (2002). Where would formal, academic mathematics stand in a curriculum informed by ethnomathematics? A critical review of ethnomathematics. *Educational Studies in Mathematics, 50*, 79–102.

Scribner, S., & Cole, M. (1973, November 9). Cognitive consequences of formal and informal education. Science, 182, 553–559.doi:10.1126/science.182.4112.553

Skovsmose, O., & Vithal, R. (1997). The end of innocence: A critique of 'ethnomathematics. *Educational Studies in Mathematics, 34,* 131–158.

Tedre, M., & Eglash, R. (2008). Ethnocomputing. In M. Fuller (Eds.), Software studies:a lexicon (pp. 92-101). MIT Press.

Tuomi-Gröhn, T., & Engeström, Y., & Young, M. (2003). From transfer to boundary-crossing between school and work as a tool for developing vocational education: An introduction. In T. Tuomi-Gröhn & Y. Engeström (Eds.), Between school and work: New perspectives on transfer and boundary-crossing (pp. 1–15). Pergamon.

Vossoughi, S., Paula, H., and Meng, E. (2016). Making through the lens of culture and power: towards transformative visions for educational equity. *Harvard Educational Review*, 206–32.

Vinner, S. (1997). From intuition to inhibition—mathematics education and other endangered species. In E. Pehkonen (Ed.) Proceedings of the 21th Conference of the International Group for Psychology of Mathematics Education (PME21) (Vol. 1, pp.63–78).Finland.

Vygotsky, L. S. (1978). Mind in society: the development of higher psychological processes.Harvard University Press

TRANSLANGUAGING AS A SOURCE OF MATHEMATICAL SENSE-MAKING PRACTICE DURING PROBLEM-SOLVING

Ayse Ozturk
The Ohio State University, Newark
ozturk.25@osu.edu

This study focuses on how strategic translanguaging occurs when bilingual students engage in a real-life-oriented problem-solving process and communicated their reasoning. Data for this report came from an after-school program for bilingual high school students. Action-research methodology guided data collection from classroom sessions that consisted of problem-solving with quantities. Analysis revealed that translanguaging can be a source of the sense-making process in mathematical problem-solving that leverages bilingual students' linguistic repertoire to combine everyday and mathematical registers while bridging multiple solution pathways.

Keywords: Communication, Equity, Inclusion and Diversity, Problem Solving

Research has shown that bilingual students can use a variety of tools to construct sense-making in mathematical problem-solving, such as community funds of knowledge (Gutiérrez, 2017); cultural resources (Moschkovich, 2015); everyday experience (Domínguez, 2011); and mathematical discourse, gestures, and representations (Garcia, 2017). More recent research has demonstrated that to maximize mathematical sensemaking during communication-based problem-solving, bilingual students strategically use two languages by code-switching or code-alternating (DiNapoli & Morales, Jr., 2021). This paper aims to build on prior research by studying how strategic translanguaging occurs when Turkish-American bilingual students make sense of real-life-based problem-solving processes and communicate their reasoning.

The Conceptual Framework

Drawing on Garcia's (2017) description of translanguaging as a holistic and dynamic meaning-making practice that involves coordinating linguistic and semiotic resources to make sense of concepts and contexts of mathematical problems, we adapted DiNapoli and Morales, Jr.'s (2021) framework for translanguaging mathematical practice (Figure 1) to investigate Turkish-American bilingual students' problem-solving processes:

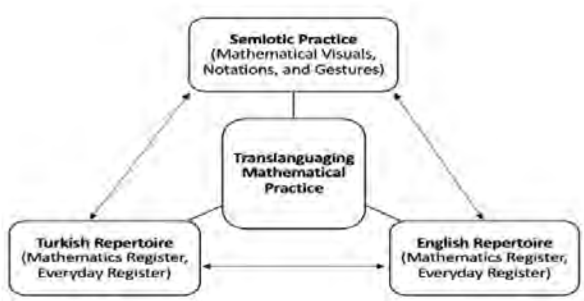


Figure 1: Framework for Translanguaging Mathematical Practice (Adapted from DiNapoli & Morales, Jr., 2021)

Within the framework, bilingual students activate their linguistic repertoire and switch between Turkish and English by utilizing mathematics and an everyday linguistic register. In order to complement mathematical discourse, bilingual students deepen their problem-solving experience with mathematical representations, including visuals, notations, and gestures. The framework guided the data analysis to reach an understanding of students' communication of their mathematical reasoning.

Research Setting and Participants

Data for this paper came from a more extensive research study conducted during a 6-week long after-school program for high school students at a public university in a Midwestern state. The class met twice a week for a total of 10 sessions, or 20 hours. The goal of the research program was to examine how bilingual secondary students call upon their mathematical experiences in real-life problem-solving contexts involving quantities (Ozturk, 2021).

The participants included six 10th-grade Turkish-American students from middle-class socioeconomic backgrounds and first-generation college dreamers. These students were all children of Turkish parents, their home language is Turkish, and their school language is English. Constant code-switching between Turkish and English is the typical way of speaking for all of them. Therefore, the study provided a flexible environment to use the two languages as they engaged in classroom activities. All participants' names given here are pseudonyms.

Methods and Data Collection

Drawing upon an action-research methodology (Mostofo & Zambo, 2015), the after-school classroom sessions consisted of real-life-based problem-solving with quantities. In each session, students solved problems in pairs, and then small groups shared strategies with others to reflect upon diverse ways of reasoning. Students were free to code-switch any time during problem-solving and group presentations, and the researcher communicated with students in two languages. All sessions were videotaped and transcribed. In addition, the students' written work was collected.

Data Analysis, Example Group Dialog, and Written Work

Data for this study consisted of video excerpts in which students were solving problems and communicating their solutions to the class. The criterion for the selection of video excerpts was that they contain students translanguaging while problem-solving. We paid attention to how students used resources such as code-switching, visual-graphic representations, or gestures while developing sense-making of quantities in the problem contexts.

Drawing from a thematic analysis (Braun & Clarke, 2012), the data analysis consisted of four steps: 1) Capturing evidence of students' translanguaging instances; 2) using a conceptual framework (Figure 1) to code student's use of English and Turkish as well as everyday and mathematical registers; 3) elaborating each translanguaging moment in which students supported their communication with non-verbal mathematical representations; 4) using the constant comparative method, we inductively created themes to describe the ways in which evidence of strategic translanguaging occupied different roles during the phases of problem-solving and collective sense-making.

Although analysis is still in process, to illustrate one theme from our data regarding the role of strategic translanguaging in the problem-solving process, we purposely selected a vignette from a group of students that worked on a problem that demonstrated students' interaction while beginning the problem and selecting appropriate mathematical units to continue their solution.

Translanguaging as an entrance of problem-solving while selecting appropriate units. In this theme, students work in pairs in the beginning phase of the problem-solving process. They brainstorm and decide how to make sense of a problem and choose methods of mathematization, abstracting a given situation and representing and manipulating it symbolically. For example, in the Traffic Jam Problem (Figure 2), students were required to perform estimation, relate quantities, and work on units in order to make sense of their interpretations of the calculation results.

> Last Sunday, an accident caused a traffic jam 15 miles long on a straight stretch of a two-lane freeway. How many people do you think were in the traffic jam?

Figure 2: Traffic Jam Problem (Adapted from Illustrative Math, 2016)

While the problem involves simple arithmetic operations, students must make reasonable assumptions about the average length of vehicles and the space between vehicles in the traffic jam. In the excerpt below, Hale and Asya were trying to imagine a 24-kilometer-long traffic jam and discussed how many cars might fit such a traffic jam. Although the other two groups chose to continue with 15 miles as a length unit, this group first converted units from the U.S. customary system to the metric system to do the rest of the calculations.

While Asya was searching for photos (from Google) to determine how long 15 miles was in kilometers and how many cars could fit in a 24 km-long road to visualize the traffic jam scenario in her mind, Hale asked her why 24 km was important to her. Asya immediately switched the code from English to Turkish and explained that it is hard to imagine the length in miles (Figure 3):

Hale:	What are you looking up?
Asya:	24 km. Kac araba sigabilir 24 km'lik trafige diye baktim.
Hale:	Hmmm—neden 24 km ki?
Asya:	Cunku 15 mili hayal edemiyorum, ama 15 mili kilometreye cevirirsek, kafamda yaklasik 24 yada 25 kilometre ne kadar biliyorum.Teyzem ve bizim evimizin arasi kadar bi uzaklik.
Hale:	Oh, I got you. Make sense.

⟷

Hale:	What are you looking up?
Asya:	24 km. I looked up how many cars might fit in 24 km traffic jam.
Hale:	Hmmm—**why 24 km?**
Asya:	Because I **cannot imagine 15 miles**. If we convert 15 miles to km, it's about 24 km. **In my mind,** I know how long **24 or 25 km is; it's like the distance from our house to my aunt's house.**
Hale:	Oh, I got you. **Make sense.**

Figure 3: Transcript and English Translation: Evidence of Translanguaging Practice (Turkish-English)

She preferred to imagine the distance in kilometers because she could relate the distance (approximately 25 km) to her everyday experience. Asya combined everyday and mathematical registers when describing 24 km (close enough to 25 km) as "it's like the distance" between her and her aunt's houses. She rethought the context of the problem, 15 miles, in the metric system because describing and representing 15 miles was not accessible at the beginning. When she explained this unit conversion (from 15 miles to 24 km), she immediately switched code into Turkish. This might have occurred because thinking in the metric system was learned in Turkish before she was introduced to the U.S. customary system; thus, working with metric units stimulated her Turkish repertoire as an intellectual resource for mathematical sense-making.

Converting units from miles to kilometers was a strategic decision because it allowed students to access mathematical ideas through their prior work—making assumptions about the length of vehicles and spaces between them in the traffic jam. Note that communication between

Hale and Asya was a valuable exchange through meaning-making in the problem-solving process. Hale's affirmative comments ("Make sense" and "I got you") about Asya's unit conversion description showed a sign that this group persevered in problem-solving steps as long as both students met at a shared understanding built on collaborative meaning-making efforts.

After Hale's confirmation, Asya started drawing the two-lane freeway and wrote down 15 miles as 24 kilometers using the sign "≈" of approximation (see Figure 4). Their first assumption about an average car length was 4.5 meters. They converted 24 kilometers to meters to more easily divide that into cars 4.5 meters long. In this way, they estimated how many cars would fit in one lane, and multiplying by 2 resulted in approximately 10,600 cars in the traffic jam. Another assumption was that the average number of people in a car during a traffic jam was 2. Students multiplied 2 with 10,600 cars and reached the result of approximately 21, 200 people would be in the traffic jam. Figure 4 illustrates students' first attempt to solve this problem:

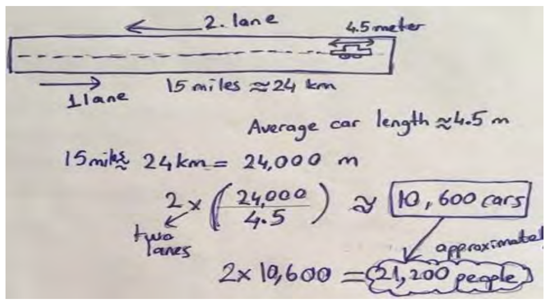


Figure 4: Example Solution for Traffic Jam Problem

When Hale and Asya presented their results to the whole class, other groups asked questions, such as why they made the assumption of 2 people in a car or why they assumed there would be only cars in the traffic. Hale and Asya also realized they had not considered the gaps between the cars, and they assumed the cars were lined up back-to-back. Hale said that even in a traffic jam, there should be a gap between cars. Students made recalculations by considering different scenarios, such as what if there were trucks and motorcycles in the traffic, and how changing vehicle numbers could impact the total number of people. Later, the class focused on how to come up with minimum and maximum numbers of people in the traffic jam if they reconsidered the important factors that could change the results (vehicle types, the length of gaps, etc.)

Preliminary Results and Conclusion

The study focused on understanding how strategic translanguaging occurs when Turkish-American bilingual students make sense of real-life-based problem-solving processes. The preliminary results showed that translanguaging occupied different roles during the phases of problem-solving and collective sense-making. The results revealed that students used translanguaging as an entrance to problem-solving while strategically selecting units that were more accessible to them.

These findings resonate with previous studies (e.g., Garcia, 2017) suggesting that positioning translanguaging as a source for sense-making practice encourages bilingual students to leverage their full linguistic repertoire as well as to combine everyday and mathematical registers. The results also show that strategically selecting appropriate quantities and their units became tools to justify mathematical reasoning and shared understanding among bilingual students. Therefore, strategic translanguaging is a powerful tool for bilingual students to bridge multiple pathways while engaging in problem-solving.

References

Braun, V. & Clarke, V. (2012) Thematic analysis. In Cooper, H. (Ed.), *The Handbook of Research Methods in Psychology*. Washington, DC: American Psychological Association.

DiNapoli, J., & Morales Jr, H. (2021). Translanguaging to Persevere Is Key for Latinx Bilinguals' Mathematical Success. *Journal of Urban Mathematics Education*, *14*(2), 71–104.

García, O. (2017). Translanguaging in schools: Subiendo y bajando, bajando y subiendo as afterword. *Journal of Language, Identity & Education, 16*(4), 256–263.

Gutiérrez, R. (2017). Why mathematics (education) was late to the backlash party: The need for a revolution. *Journal of Urban Mathematics Education, 10*(2), 8–24.Illustrative Mathematics. (2016). HSN-Q.A. Reason quantitively and use units to solve problems. http://tasks.illustrativemathematics.org/content-standards/HSN/Q/A

Moschkovich, J. (2015). Academic literacy in mathematics for English Learners. *The Journal of Mathematical Behavior*, *40*, 43–62.

Mostofo, J., & Zambo, R. (2015). Improving instruction in the mathematics methods classroom through action research. *Educational Action Research*, *23*(4), 497–513.

Ozturk, A. (2021). *Examining the Relationship Between Instructional Practices and Students' Mathematical Modeling Competencies*. The Ohio State University (Doctoral Dissertation).

THE COMPETING EXPECTATIONS OF MOTHERHOOD AND MATHEMATICS: INTERNALIZED GENDERED AND RACIALIZED NARRATIVES WITHIN

Sam Prough
University of Delaware
sprough@udel.edu

I draw attention to the tensions between the expectations of mathematics and motherhood to highlight how mothers positively engage with their children in mathematics, even when that activity is not recognized or granted legitimacy. I ask: what do mothers' experiences in mathematics and with their children say about U.S. society's expectations of mathematics and motherhood? I highlight the experiences of five participants as they reflect on their mathematical experiences and interactions with their children. The findings demonstrate the pervasive nature of gendered binaries and privileged whiteness in the expectations of mothers in both motherhood and mathematics. The constraining expectations of what it means to mother and be involved in mathematics shows the need to shift perspectives of what counts as mothering, as parenting, and what counts as mathematics.

Keywords: gender; equity, inclusion, and diversity

The images of motherhood and mathematics in Western society have long perpetuated expectations that privilege the experiences of certain people, but not others. Expectations surrounding parenting in the U.S. are deeply rooted in gendered binaries and the norms of white, middle-class families (Dillaway & Pare, 2008). Mathematics is framed as a field for white, middle-class men aligning to the norms of masculinity and limiting who can be viewed as mathematical (Hottinger, 2016). Mothers, however, are often expected to support the early education of their children, which can include mathematics (Odenweller & Rittenour, 2017). Although some may argue that early mathematical work is not subjected to similar framing of masculinization the issue lies in the erasure of early mathematical practices as mathematics. Skills such as counting and early numeracy are described as separate from and less rigorous than mathematics (Hacker, 2016), which can further emphasize a divide in what and who counts as doing mathematical work. When a masculine mathematics intersects with the expectations of teaching in motherhood, tensions come to the forefront. A dissonance is clear between the expectations of who can do mathematics and mothers raising children that questions what it means for women who are mothers to support children in mathematical learning. Bringing attention to such dissonance, highlighting the specific tensions in the gendered and racialized components of motherhood and mathematical participation, has the ability to shift societal perspectives of what counts as mothering and what counts as mathematics.

I draw attention to the tensions between the expectations of mathematics and motherhood to highlight the challenges of parents helping their young children learn. I respond to the following research question: What do mothers' experiences in mathematics and with their children say about U.S. society's expectations of both mathematics and motherhood? Weaving together the narratives of five mothers, the findings show persistent challenges with assumptions of whiteness and upheld gender binaries in the roles of motherhood and mathematics. The reflections from the five mothers show the pervasiveness of such expectations from their time in school and its current impact on their activity now with their own children. Connecting the experiences of the participants to the literature on parents' experience in mathematics and the role of motherhood offers opportunities to disrupt notions of what mothers must do in children's early mathematical learning by considering the possibilities for what motherhood and mathematics can look like in different contexts.

Perspectives on Motherhood and Mathematics Expectations

This study expands on the literature that addresses the experiences of mothers in mathematics *and* their resultant interactions with children by considering the interactions between the two bodies of work. Some literature considers the experience of mothers as mathematics learners (Chase, 2018), some considers mothers' beliefs or impact on their children's mathematical performance (e.g., Cannon & Ginsburg, 2008; Tomasetto et al., 2011), and others have considered mothers' mathematical interactions with their children (e.g., Anderson & Anderson, 2018). It is necessary to put the experiences of mothers and their interactions with children in conversation, to understand the complex relationship of expectations put on them in their roles of motherhood and early mathematics teachers for their children. In this section, I describe the literature on the Western expectations of both motherhood and (school) mathematics. Then I consider the tensions between those expectations on a parent being upheld to the expectations of both.

Expectations of Motherhood

Previous literature has shown a continued connection between womanhood and motherhood (e.g., Arendell, 2000; Collins, 2000; Roberts, 1993). In this way, the term 'woman' is often seen as synonymous to 'mother' (Mora, 2006). Historical images and expectations of motherhood often focused on mothers' labor centered at home and fathers' labor out of the home, with clear divides in what mothers and fathers were expected to do for their family. This division of labor and familial expectations for mothers has evolved into present day to include additional responsibilities, such as working to provide other resources to the family while still being the primary parent to raise children (e.g., Duxbury & Higgins, 1991; Rothausen-Vange, 2004). Zhao and colleagues (2011) argued that "traditional thinking assumes that women should do a greater share of household labor and childcare, and mothers may be particularly overloaded compared to fathers and women without children" (p. 725). This traditional thinking persists in ways today that may not look like an expectation that mothers stay at home or homeschool their children, but still supports notions of mothers taking on the greater share of childcare.

Some of the pervasive expectations surrounding motherhood in Western society include a focus on (1) mothers being the exclusive or primary caregiver for their children (Dillaway & Pare, 2008), (2) mothering being a sacrifice of other aspects of a mothers' life (Maher & Saugeres, 2007), and (3) good mothers being recognized by the well-behaved nature of their children (Austin & Carpenter, 2008). Unfortunately, these three pervasive expectations for motherhood do not consider the complexities of what mothers do and how they do it to support their children. For example, families may be in situations where it is not possible for a mother to be the exclusive or primary caregiver, but this should not imply such a person is a bad mother because of that. Analysis of mothers' actions, context, and intentions with their children can illuminate other ways of doing motherhood.

Expectations of (School) Mathematics

What is typically expected in mathematics in school environments centers on students getting the correct answer to a problem and working quickly, where a teacher has the authority to define what student actions and answers count as appropriately mathematical or not (Esmonde et al., 2013). School mathematics is often about memorization and algorithms, even when it aspires to be about understanding. Marta Civil (2002) characterizes the traditional norms of school mathematics as "an overreliance on paper-and-pencil computations with little meaning, clearly formulated problems following prescribed algorithms, and a focus on symbolic manipulation deprived of meaning" (p. 41). Additionally, what is determined as appropriately mathematical or who mathematics is for is based in systems that privilege the norms of white and middle-class men (Rousseau Anderson & Tate, 2016). In the minds of many, the framing of mathematics in school is privileged over other types and styles of mathematical learning, such as what might happen at home (Takeuchi, 2018). Given the images and expectations of what mathematics is, students can develop ideas about themselves and their mathematical ability, taking these understandings beyond school. The expectation of mathematics as a masculine subject, intended for men but not women, has persisted (Hottinger, 2016). Such narratives about a white-centered and masculine mathematics continue to shape students' perspectives of themselves as mathematical as they grow up and become parents.

Dissonance in Expectations for Women and Mothers in Mathematics

The specific masculinization of mathematics provides context for the impact gendered expectations of mathematics has on mothers' experiences in the subject. Past understanding of mathematics and who is capable of mathematical work assumed that women do not or should not be learning and doing mathematics (Hottinger, 2016). Mathematics is a subject associated with masculine characteristics such as working independently and thinking with logic and rationality (Ernest, 2018; Solomon, 2012), but at the same time distancing these characteristics from women and their potential association with mathematics (e.g., Connell, 2010; L.A. Leyva, 2017). In the same way that mathematics highlights a separation between men and masculinity from women and femininity, the roles of parents (mother and father) have also been divided to correspond to what is considered 'appropriate' gender roles. Fathers, who are assumed more aligned to the masculine, are associated with mathematics, working, providing for their family outside of the home (Rothausen-Vange, 2004). Mothers, who are assumed more aligned to the feminine and not expected to do or understand mathematics, with a focus on domestic work that includes raising and teaching young children (Zhao et al., 2011). But the expectations of mothers to support children's early learning as part of domestic work is in tension with the expectation of women's inability to be mathematical. Past studies on mothers' involvement in children's early mathematical learning have not addressed the dissonance in expectations of mothers' roles as early teachers and a masculine mathematics.

Theoretical Framing

While the tension in expectations for mothers and mathematics stems from gender binaries for the actions and images of men and women, I want to consider the multi-layered influence of both gender and race in expectations for being a mother and doing mathematics. I use Collins' (2000) concept of matrices of domination as an analytical tool to critique the influence of expectations in mathematics and motherhood surrounding gender and race. Collins defines a matrix of domination as the organization of intersection oppressions. For example, the tension a mother feels when engaging in mathematics with children is influenced by the expectations of doing mathematics and being a mother that privileges gendered family roles, white and middle-class norms. As Collins (2000) argues, "regardless of the particular intersections involved, structural, disciplinary, hegemonic, and interpersonal domains of power reappear across quite different forms of oppression" (p. 21). The organization of expectations and domains of power of gendered binaries reappear in both motherhood and mathematics. Similarly, the privileging of whiteness reappears in both expectations of being a mother and doing mathematics.

Methods and Modes of Inquiry

I focus the findings for this paper on the experiences of five participants from a larger study about the impact of past experience on mothers' current mathematical interactions with children. The smaller set of participants is intended to give space to their narratives in the context of existing literature on the expectations of mathematics and motherhood but does not represent a generalization for all mothers' (or parents') experiences with parenting or mathematics. The larger study used a narrative inquiry approach to frame the narratives of the mothers to understand parts of their lives and make connections to the narratives of others (Clandinin & Connelly, 2000). Due to limited space, I focus only on short excerpts from their shared stories to draw attention to their discussion of motherhood and mathematics expectations. The study consisted of three interviews, two observations with debriefs, and collected artifacts to support shared stories with each participant. The elements of the study intended to capture participants' past experiences, current stories of mathematical activity, what that activity looked like and confirmation of those stories. I focus on the experiences of five participants: Amanda, Brittany, Courtney, Elizabeth, and Tara[1]. These mothers ranged in age from their late 20s to late 30s, living in rural to urban locations. Table 1 shows more detail about each of the participants.

Table 1: Name and Basic Demographics of Participants

Participant	Children (+Age)	Race	Occupation
Amanda	Grant (2)	Native American	Graduate student
Brittany	Rosie (2), Bobby (4)	White	Part-time music teacher
Courtney	Eva (3)	Black	Teacher
Elizabeth	Talia (2), Luna (7)	White	Stay-at-home parent
Tara	Sasha (1.5), Peter (3.5)	White	Part-time grocery worker

Transcripts from the three interviews and debriefs, as well as field notes from the two observations, were used as data for the study. Analysis focused on the context of participants' stories in response to the literature's expectations of motherhood and mathematics. An initial coding scheme looked for moments in their stories that described (a) emotions in mathematics, (b) positive mathematics engagement, (c) past experience influencing current mathematical interaction, (d) gendered experiences in mathematics and mothering, and (e) racialized experiences in mathematics and mothering. The details from codes a, d, and e informed the deeper analysis for this paper. Nuance in the mothers' feelings, words, and self-definitions received careful attention in coding for moments of mathematical engagement and motherhood activity. This specific attention to detail is meant to parallel the ideas of the work of the feminist narrative inquirer Madison (1993). As such, the transcription and analysis involved careful attention to deeper detail such as when the participants laughed, paused, sighed, or used filler words in sharing their stories.

Findings

What shaped the mothers' interpretations of themselves in their interaction with mathematics as well as motherhood identity was shaped by larger societal narratives of what it meant to do mathematics and be mothers. The mothers noted what counted as mothering or doing mathematics in implicit and explicit ways that highlighted gender binaries and the privileging of

[1] All participant names and the names of their children are pseudonyms.

white norms. The layers of expectations in gender and race, appearing in both conceptions of motherhood and mathematics, aligns with Collins' (2000) matrices of domination, as the organization of intersecting oppressions. In this section, I highlight aspects of the participants' stories around motherhood and mathematics that show common expectations in those areas that privilege gender binaries and white norms as described in previous literature. Although the audience for this work is centered in mathematics education, I specifically foreground the motherhood connections in relation to mathematics as the role of mothering in mathematics education is often ignored or devalued (Jackson & Remillard, 2005).

Motherhood Narratives

All but one participant shared how their mother did all or most of the work in raising them, including supporting their early learning, which shaped an expectation of what they saw was important to do for their children. Across all five mothers was a general expectation to be the primary caregiver for their children. Several participants also closely aligned their identities with that of being first and foremost a mother, reflected in the literature as a sacrifice of self for the sake of motherhood (Garey, 1995; Maher & Saugeres, 2007). Brittany, for example, talked about how she saw and defined herself in the past and now, sharing "the things that I feel like really define, in the past musical theater really defined a lot of my life and now it's mostly, my kids and my family. So sometimes that has been kind of a hard transition" (Interview 3). Brittany explained how she felt she needed to see her children and family as her primary focus, centering her role as mother above other aspects of herself, which used to center music and musical theater. The domestic work of participants' mothers as well as their frequent identification with the role themselves shaped how they interpreted their activity, as a necessity to being identified as mothers first and being the primary caregiver over other partners.

The influence of race on expectations of motherhood were more explicit in the stories of the mothers who identified as mothers of color or who had children of color. As Western society has privileged white norms in motherhood (Dillaway & Pare, 2008), reflections of fitting into such norms or helping their children fit into a world that does not privilege their experiences outside of whiteness. Due to limited space, I highlight the experiences of Elizabeth and Tara in the remainder of this section, both mothers who identified as white but currently parent children of color. In both cases, how they spoke about race related only to the experiences of parenting their children of color, not in the stories of their experiences growing up or how their actions as white mothers would be perceived. Elizabeth saw the challenges of appropriate representation in children's books, with many characters in popular and classic literature reflecting white faces. She saw this as a disservice to her daughter, Talia, who was Hispanic, sharing that she

> was really disappointed at how many books were like white kids in the books…And so, I set all those aside and I'm, we're giving those away [laughs] cause I, I think more and more how important it is for when {Talia} to open a book to see people like her reflected in that book. (Interview 3)

Elizabeth's statements reflect an expectation in motherhood (to help her daughter learn about herself through reading) and the challenges to meet that expectation given the books she has available. The activity of motherhood has an embedded association with whiteness in Western society (Morgan, 2018). In Elizabeth's story, this is reflected in the characters in children's books where there are white characters meant for white mothers to read to their white children. But those characters and expectations do not align to the experiences Elizabeth sees as important for her daughter, Talia.

Tara considered the impact of assumed whiteness for her children's future and what she needed to consider as a parent, raising children in a world that privileges whiteness. In talking about what she hopes for her children as they grow up, Tara shared the following story about potential racial prejudice they may encounter:

> Yeah I'm worried about my kids. Um, because right now they're in the super cute adorable Korean, half Korean/half white stage and they're just angelic and everyone loves them but then when they become teenagers they'll be little shits and then you know they won't have that you know, that draw anymore, and I just hope that they can just be respectful. (Interview 3)

Tara's worries for her two young children reflect how they will be perceived by others. Within her worries are particular actions as a mother she feels she needs to prepare them for such a future with actions and behaviors that fit into a world that privileges whiteness.

In the stories of these participants, the influence of gendered binaries and privileged whiteness informs their actions and thoughts about motherhood. Mothers like Brittany feel a personal responsibility to be the primary caregiver of their children, centering an identity of mother before other options, because of the expectations of women as mothers, nurturers (Austin & Carpenter, 2008). Mothers like Elizabeth and Tara consider what it means to raise children of color in a world that privileges the norms of white children and how to best support them. The next section highlights stories within the experiences of mathematics where assumptions of gendered binaries and whiteness reappear to assert particular expectations on the participants.

Mathematical Narratives

When talking about their experiences with mathematics, the participants highlighted several expectations in alignment with what happens in school mathematics, such as mathematics being a quick mental task, working on problems is done independently, and that such problems have one correct method and solution. Several of the mothers (Amanda, Elizabeth, and Brittany) shared how these characteristics of mathematics did not work for them and drew connections between a lack of mathematical ability to it being a subject meant more for men. Elizabeth, Brittany, and Tara also highlighted how they assumed their partners (identified as men) were just better at mathematics than they were, even though Elizabeth and Tara had extensive experience in the subject. Amanda explicitly identified the gendered expectation of who could or could not do mathematics from her schooling experience and how that shaped her perspective of herself, sharing that:

> I feel like it was very gendered when I was a kid. Like I always assumed there were the smart boys in class. And they were just better at math and that maybe it had something to do with me being a girl that I wasn't good at math. My mom would openly talk about how she wasn't good at math either. And then I felt like all the kids who were the best in my mind at doing math, that they could, like in elementary school they could do the timed tests and get the most answers down on the paper. And um, they didn't seem to have a problem doing algebra, were like, I can think of 1 or 2 guys specifically that I was always in the same classes with and they were always like the good at math guys. (Interview 1)

Amanda relates to expectations of school mathematics as well as the influence of her mother to shape a perspective that she must not be good in mathematics. Mendick (2006) highlights the challenges of identifying with mathematics when the subject is associated with the masculine and gendered binaries position men and fathers as masculine but women and mothers as feminine.

How people are perceived in school and in the mathematics classroom also includes racial representation, where the norms of whiteness are privileged along with characteristics of masculinity in mathematics (Hottinger, 2016). Courtney, who was an excellent student and particularly skilled in mathematics, shared stories of a teacher who constantly doubted her ability to answer questions and know the content, reflecting a similar bias in all his students of color. The assumption of whiteness in being able to learn or do well in school shaped how Courtney thought about her daughter, Eva and what she needed to be successful in school.

> I always think that I have to include the fact that yes you may not look like all of your classmates and that's ok. Um but how can we use that to teach, to teach them more about you? Or more about African Americans or like try to use that as a learning opportunity and allow her to find her own voice … I mean she's still capable of doing almost whatever, she wants or whatever she puts her mind to at least. So I think it will definitely come into play, it's come into play a little bit but, yeah, I think, I, I'm definitely I wanna be more conscious of it because I wasn't when I was a kid and I wish that I had been. (Interview 3)

What Courtney shares reflects the assumed whiteness in both school and mathematics. Her daughter Eva may need to learn to stand up for who she is and how she learns, which Courtney pulls from her own experiences in school as a Black girl.

The experiences of the mothers in mathematics show concerns about the perceptions of who can do mathematics that centers white norms and masculinity. Amanda talks about how she thought of herself as less capable in mathematics because she identified as a girl, enforced by perspectives of her own mother and the actions of students in her class. Amanda later returned to this perspective and considered what this would mean for her soon-to-be-born daughter and not wanting her to get the impression that mathematics also is not for her. Similarly, Courtney connected her own experiences with mathematics and the challenges with being perceived as a good mathematics student as a Black woman and what she felt was important to support her own daughter.

Discussion and Action

The experiences described by the mothers in both their past interactions with mathematics and how they engaged in the present with their children in the subject were influenced by the expectations of what it means to be a mother and to do mathematics in U.S. society. The gendered binaries shaping U.S. society uphold a divide in the roles of mothers and women vs. fathers and men that continue to define rigid expectations for mathematics and motherhood. Mathematics is often aligned with men and masculinity (Hottinger, 2016) and mothering is assumed to be the work of women, aligned with femininity (Odenweller & Rittenour, 2017). In mothering, the participants highlighted the expectation that they (as mothers) should be in charge of raising their children and teaching them. However, in mathematics many participants highlighted assumptions about the subject being more for boys and their perceived lack of ability in comparison to others, particularly their partners. Both instances point to the dissonance in expectations for mothers and doing mathematics; that mothers should help their young children learn but that mothers (often aligned with the identity of women) are lacking in mathematical skill. How are mothers expected to support most or all of children's early learning, which includes mathematics, while simultaneously being viewed as less capable in the subject? The response to such tensions cannot be an acceptance that mothers simply cannot well support early mathematics development.

The tensions in expectations of mathematics and motherhood extends beyond gendered binaries. The privileging and assumptions of whiteness are also a major part of U.S. society's interpretation of what counts as both motherhood and mathematics. Literature has documented the embedded racial expectations in policy and practice, as Marx (2006) states "the privileges attached to Whiteness have been, and continue to be, perpetuated in subtle ways through American institutions and popular culture" (p. 52). In mothering, the participants draw attention to what is available to support children that assumes whiteness (such as Elizabeth's recognition of primarily white characters in children's books). But in mathematics, participants reflected on how white students are assumed more capable and what it might take to support themselves and their children of color to recognize their own ability. Douglas and Attewell (2017) summarize the challenge of viewing some children as more capable than others, stating "the belief that mathematics requires innate talent is widespread in our culture…it rationalizes the notion that only some students will be able to learn school mathematics successfully" (p. 651). These perspectives of whiteness show the expectations of motherhood that support white mothers and their white children. At the same time, the expectations within mathematics make assumptions about who can and cannot do mathematics based on their race. Before any mathematical activity begins, mothers of color and mothers with children of color are already positioned as less capable and with fewer resources that support their experiences as not white.

How the mothers of this study experienced school and reflected on their experiences interacting with children as parents were shaped by gendered binaries of roles for mothers as well as expectations of whiteness in their parenting and participation in mathematics. The structures of expectations in both motherhood and mathematics have maintained activities and beliefs that uphold gendered roles for parents and their activity and privilege the norms of white families. The relationship between the expectations of motherhood and mathematics is more complex than a list of tensions or challenges experienced by individuals who are mothers and also support early mathematical learning. Take, for example, the context and experience of Elizabeth. As a white woman parenting a daughter of color, Elizabeth was concerned for the images and role models her daughter had that looked like her, with many media representations for children were white characters. However, the influence of race did not yet extend to Elizabeth's own experiences growing up.

Future work on mathematics experience and identity for mothers, particularly related to parents, requires a broader perspective on what it means to raise children and what it means to do mathematics. The limited images of what is expected for mothers and what is expected to be successful in mathematics contribute to the participants' narratives of this study; the mothers frame themselves as less mathematically capable while worrying about how their children will take up mathematics or be perceived by others. More can be seen with a similar analysis of a wider array of parents supporting early mathematical learning, such as different family dynamics and different associations with the role of 'motherhood'. Creating broader perspectives of both parenting and mathematics involves parent-centered research, considering both the actions, intent and context of how parents engage with their children in mathematics. Centering and listening closely to the experiences and actions of parents creates an opportunity for mathematics education researchers to understand new ways of mathematical engagement and new possibilities for connecting to children's mathematical learning.

References

Anderson, A., & Anderson, J. (2018). Math-in-context: The types of math preschoolers "do" at home. In I. Elia, J. Mulligan, A. Anderson, A. Baccaglini-Frank, & C. Benz (Eds.), *Contemporary research and perspectives on*

early childhood mathematics education (pp. 183–202). Springer International Publishing. https://doi.org/10.1007/978-3-319-73432-3_10
Arendell, T. (2000). Conceiving and investigating motherhood: The decade's scholarship. *Journal of Marriage and the Family*, *62*, 1192–1207. https://doi.org/10.1111/j.1741-3737.2000.01192.x
Austin, H., & Carpenter, L. (2008). Troubled, troublesome, troubling mothers: The dilemma of difference in women's personal motherhood narratives. *Narrative Inquiry, 18*(2), 378–392. https://doi.org/10.1075/ni.18.1.10aus
Cannon, J., & Ginsburg, H. P. (2008). "Doing the math": Maternal beliefs about early mathematics versus language learning. *Early Education and Development, 19*(2), 238–260. https://doi.org/10.1080/10409280801963913
Chase, E. (2018). Time, third space, and counternarratives of achievement for young mothers in high school. *Educational Studies, 54*(3), 237–252. https://doi.org/10.1080/00131946.2017.1413371
Civil, M. (2002). Culture and mathematics: A community approach. *Journal of Intercultural Studies*, *23*(2), 133–148. https://doi.org/10.1080/07256860220151050A
Clandinin, D. J., & Connelly, F. M. (2000). *Narrative inquiry: Experience and story in qualitative research.* Jossey-Bass. https://www.wiley.com/en-us/Narrative+Inquiry%3A+Experience+and+Story+in+Qualitative+ Research-p-9780787972769
Collins, P. H. (2000). *Black feminist thought.* Routledge. https://doi.org/10.4324/9780203900055
Connell, R. W. (2010). The social organization of masculinity. In C. R. McCann & S.-K. Kim (Eds.), *Feminist theory reader: local and global perspectives* (Second, pp. 232–243). Routledge. https://www.taylorfrancis.com/chapters/mono/10.4324/9781315680675-48/raewyn-connell-social-organization-masculinity-carole-mccann-seung-kyung-kim
Dillaway, H., & Pare, E. (2008). Locating mothers: How cultural debates about stay-at-home versus working mothers define women and home. *Journal of Family Issues*, *29*(4), 437–464. https://doi.org/10.1177/0192513X07310309
Douglas, D., & Attewell, P. (2017). School mathematics as gatekeeper. *The Sociological Quarterly.* https://doi.org/10.1080/00380253.2017.1354733
Duxbury, L. E., & Higgins, C. A. (1991). Gender differences in work-family conflict. *The Journal of Applied Psychology*, *76*(1), 60–74. https://doi.org/10.1037/0021-9010.76.1.60
Ernest, P. (2018). The philosophy of mathematics education: An overview. In P. Ernest (Ed.), The Philosophy of Mathematics Education Today (pp. 13–35). Springer International Publishing. https://doi.org/10.1007/978-3-319-77760-3_2
Esmonde, I., Blair, K.P., Goldman, S., Martin, L., Jimenez, O., & Pea, R. (2013). Math I am: What we learn from stories that people tell about math in their lives. In B. Bevan, P. Bell, R. Stevens, & A. Razfar (Eds.), *LOST Opportunities: Learning in Out-of-School Time* (pp. 7–27). Springer Netherlands. https://doi.org/10.1007/978-94-007-4304-5_2
Garey, A. I. (1995). Constructing motherhood on the night shift: "Working mothers" as "stay-at-home moms." *Qualitative Sociology*, *18*(4), 415–437. https://doi.org/10.1007/BF02404489
Hacker, A. (2016). *The math myth and other STEM delusions.* Literary Ventures, Inc. https://thenewpress.com/books/math-myth
Hottinger, S. N. (2016). *Inventing the mathematician: Gender, race, and our cultural understanding of mathematics*. SUNY Press. https://sunypress.edu/Books/I/Inventing-the-Mathematician
Jackson, K., & Remillard, J. T. (2005). Rethinking parent involvement: African American mothers construct their roles in the mathematics education of their children. *The School Community Journal*, *15*(1), 51–73. https://www.adi.org/journal/ss05/Jackson%20&%20Remillard.pdf
Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 397–433. https://doi.org/10.5951/jresematheduc.48.4.0397
Madison, D. S. (1993). "That was my occupation": Oral narrative, performance, and Black feminist thought. *Text and Performance Quarterly, 13*(3), 213–232. https://doi.org/10.1080/10462939309366051
Maher, J., & Saugeres, L. (2007). To be or not to be a mother?: Women negotiating cultural representations of mothering. *Journal Of Sociology*, *43*(1), 5–21. https://doi.org/10.1177/1440783307073931
Marx, S. (2006). *Revealing the invisible: Confronting passive racism in teacher education.* Routledge. https://www.routledge.com/Revealing-the-Invisible-Confronting-Passive-Racism-in-Teacher-Education/Marx/p/book/9780415953436
Mendick, H. (2006). *Masculinities in Mathematics* (D. Epstein & M. M. an Ghaill (eds.)). Open University Press. http://www.mcgraw-hill.co.uk/html/0335218288.html

Mora, C. (2006). The meaning of womanhood in the neoliberal age: Class and age-based narratives of chilean Women. *Gender Issues*, *23*, 44–61. https://doi.org/10.1007/s12147-000-0022-1
Morgan, D. F. (2018). Visible black motherhood is a revolution. *Biography*, *41*(4), 856–875. https://doi.org/10.1353/bio.2018.0082
Odenweller, K. G., & Rittenour, C. E. (2017). Stereotypes of stay-at-home and working mothers. *Southern Communication Journal*, *82*(2), 57–72. https://doi.org/10.1080/1041794X.2017.1287214
Roberts, D. E. (1993). Racism and patriarchy in the meaning of motherhood. *Pen Law: Legal Scholarship Repository. https://scholarship.law.upenn.edu/faculty_scholarship/595*
Rothausen-Vange, T. J. (2004). Gender: Work-family ideologies and roles. *Organization Management Journal*, *1*(1), 55–60. https://doi.org/10.1057/omj.2004.13
Rousseau Anderson, C., & Tate, W.F. (2016). Toward a sociology of mathematics education: Examining democratic access in U.S. schools. In L.D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 374–394). Routledge. https://www.routledge.com/Handbook-of-International-Research-in-Mathematics-Education/English-Kirshner/p/book/9780415832038
Solomon, Y. (2012). Finding a voice? Narrating the female self in mathematics. *Educational Studies in Mathematics*, *80*(1-2), 171–183. https://doi.org/10.1007/s10649-012-9384-z
Takeuchi, M. A. (2018). Power and identity in immigrant parents' involvement in early years mathematics learning. *Educational Studies in Mathematics*, *97*(1), 39–53. https://doi.org/10.1007/s10649-017-9781-4
Tomasetto, C., Alparone, F. R., & Cadinu, M. (2011). Girls' math performance under stereotype threat: the moderating role of mothers' gender stereotypes. *Developmental Psychology*, *47*(4), 943–949. https://doi.org/10.1037/a0024047
Zhao, J., Settles, B. H., & Sheng, X. (2011). Family-to-work conflict: Gender, equity and workplace policies. *Journal of Comparative Family Studies, 42*(5), 723–738. https://doi.org/10.3138/jcfs.42.5.723

THE INTEGRATION OF MATHEMATICAL AND SOCIAL CONTENT IN SECONDARY LESSONS FOR SOCIAL JUSTICE

Meghan Riling
Vanderbilt University
meghan.riling@vanderbilt.edu

Leslie Dietiker
Boston University
dietiker@bu.edu

Xi Yu
Boston University
xyu@bu.edu

Erin Barno
Boston University
ebarno@bu.edu

In recent years, mathematics lessons addressing social justice have been increasingly published and taught, but little is known about how mathematical and social content unfolds within them. In this study, we interpret five secondary social justice mathematics lessons (SJMLs) as stories and analyze the questions that arise within them. We found that most of these lessons began and ended with a focus on social matters, while shorter mathematical questions arose in the middle of the lessons. However, the lessons varied in terms of what proportion of the questions overall were focused on social or mathematical thinking. Finally, we found that narrative devices were used in several lessons when the focus transitioned from social issues to mathematics, or vice versa. This way of thinking about lessons could enable the design of SJMLs that meaningfully integrate social and mathematical thinking.

Keywords: Social Justice, High School Education, Curriculum

To confront social inequity and systems of oppression, secondary mathematics teachers are increasingly seeking to integrate social justice themes within their lessons. Expanding opportunities for students to engage in a thoughtful analysis of social injustice and its consequences may enable students to become informed citizens (Gutstein, 2006). Moreover, it can enable students to recognize mathematics as a useful tool in better understanding the world, which in turn increases the relevance of mathematics and the motivation for its study (e.g., Darby-Hobbs, 2013). Finally, integrating social justice issues within mathematics lessons can provide an opportunity to (re)position mathematics as a human endeavor as opposed to a set of a priori truths (Ernest, 1998), created in part to solve worldly problems with which students can personally identify (Berry III et al., 2020).

Yet, incorporating social justice topics while teaching mathematical content is a complex goal, which therefore benefits from curricular support. Since most secondary mathematics teachers base instruction on published materials (Banilower et al., 2013, 2018), those who want to include social justice content will likely rely on existing materials. To address this need, a growing collection of social justice mathematics lessons ("SJMLs") has been published. Using published SJMLs enables teachers to explore the benefits of these lessons while developing critical new teaching practices (e.g., managing challenging discussions).

However, there is a potential for none of these benefits to come about, and maybe even for harm to be done, if SJMLs are not well-designed. In particular, SJMLs may poorly balance mathematical and social justice content. A lesson may give students the opportunity to engage deeply in a social justice issue, but treat mathematical content superficially, offering few opportunities for mathematical learning. On the other extreme, a lesson may give students the opportunity to engage deeply in mathematical thinking, but treat social justice content superficially. Such a lesson could limit insight into the world or into recognizing ways to take action and promote justice. We propose that to be beneficial and successful, SJMLs must interweave mathematical and social justice content so that they mutually benefit one another and advance what students know about both (what we refer to as *integration*).

Previous work has demonstrated how a narrative framework can enable different structures of content in lessons to become visible (e.g., Dietiker & Richman, 2021; Richman et al., 2018). This approach draws from literary theory, approaching unfolding mathematical content within a lesson as a story. Similar to a literary story, a *mathematical story* is the ordered sequence of events (such as tasks or discussions) that connect the beginning with its end (Dietiker, 2013, 2015). This conceptual metaphor provides the basis for studying how an unfolding story supports student inquiry, such as how withholding information can stimulate a reader to ask a question or how incremental revelations can enable a reader to piece together information. Thus, with mathematical stories come analytic tools that highlight what questions are raised and how these questions are addressed over the course of the lesson (i.e., analyzing the plot of the unfolding story).

By adapting the framework to include content related to social justice, we can learn whether and how social justice and mathematical content can mutually develop, potentially enabling us to identify lessons that span the spectrum as described earlier. Thus, this study explores how mathematical and social justice content is integrated (or not) in existing high school SJMLs. We begin to address the questions: *When viewed as narrative, what is the nature of SJMLs? Specifically, how is the mathematical and social content integrated?*

Theoretical Framework

Literary theory suggests that the "same story" (e.g., *Wizard of Oz* by L. Frank Baum) can be told in different ways (e.g., a reader learning that Dorothy is asleep the entire time from the start vs. learning this at the end). This difference is attributed to a story's *plot*, which describes how a story enables or constrains what a reader knows at different points of the story. Thus, a *plot*[1] can be described by examining the opportunities a story offers as it unfolds for a reader to recognize they don't know something that they want to know (i.e., ask or adopt a question) and to make progress in learning what they do not yet know (i.e., deducing part of a question). For example, plots that contain questions that span multiple parts of a story (e.g., How will Dorothy get home?) can enable a reader to make connections and anticipate what is to come (Nodelman & Reimer, 2003). As the number of questions a reader engages with increases (e.g., Who is the wizard?), the plot "thickens," creating a potentially interconnected web of questions with which the reader is parsing information and seeking answers. At the same time, in order to maintain a reader's interest, a story needs to regularly enable a reader to make progress on enduring questions and answer some questions relatively quickly. When a story only offers questions that are not answered nor even partially addressed for large portions of a story, a reader may not sense progress and may even forget some of the questions that remain unanswered (Nodelman & Reimer, 2003). Finally, in a plot with many open questions, learning new information may impact what is known by a reader about multiple questions at the same time. For example, when a reader learns that the wizard is not magical, this also reveals that the Tin Man always had a heart and that there may not be a way for Dorothy to return home. Therefore, within a learning context, a *SMJ plot* is a description of the dynamically changing tension between what is known and unknown for a student experiencing a *SMJ story*.

[1] Although the word "plot" has specialized mathematical meaning regarding the graphical representation of data, in this study, we use the term in a literary sense.

Story Arcs within a Lesson Plot

Central to a lesson plot is how questions emerge and are addressed for a student. Similar to the plots of literary stories, lesson plots contain multiple questions that can be answered immediately or remain open for longer portions of the story. How each question opens and is addressed constitutes a *story arc*. Multiple story arcs can be open simultaneously. Within written curriculum materials, story arcs, indicated via questions and other curricular elements (e.g., teacher notes or expected responses), arise that may involve different kinds of thinking. Due to our focus on the integration of mathematical and social content, we are particularly interested in distinguishing whether story arcs involve students engaging with mathematical thinking, thinking about social phenomena, or both.

For the purposes of this study, *mathematical story arcs* are those that expect students to engage in mathematizing, which involves abstraction and pattern-seeking (Ernest, 2018; Gutiérrez, 2017; Su & Jackson, 2020), and using objects (e.g., numbers), or practices (e.g., proving) that have been developed over time through mathematizing. When engaging in mathematizing, those objects and practices are interpreted in terms of their meaning and relationships within mathematical systems. For example, a story arc based on a question about the distance between two buildings would include mathematizing the buildings' locations, as one calculates the distance between their coordinates, but does not require the consideration of the impact on humans who travel between the buildings. Common practices of mathematizing include counting, calculating, and measuring (Gutiérrez 2017, Ernest, 2018).

Likewise, we consider *social story arcs* to be those that require interpreting an object or practice within its social context. For example, a social story arc based on a question about the use of a building would include the consideration of how humans would use the building and what value it might hold for various people, but would not require mathematizing the building's coordinates on a map, even if they are noted superficially for communication purposes. Social values may be material, emotional, political, or spiritual (Gutiérrez 2017, Ernest, 2018).

Hybrid story arcs involve both mathematizing and interpreting a social context. For example, a story arc based on a question about how two proposed locations of a grocery store would impact local residents would be hybrid because answering the question requires both calculating distances between the proposed locations and other buildings (i.e., mathematizing), and comparing the impact of differing distances on potential shoppers (i.e., social thinking). These foci (mathematical or social) may occur simultaneously or in tandem. Although many conceptualizations of mathematical thinking define it as being detached from the material or human world (Ernest, 2018), we hold that it is possible to mathematize while recognizing social connections. Allowing for multiple types of knowing may lead to tensions (Gutiérrez, 2017) or explorations about what each can illuminate about the other.

Methods

To learn more about the nature of the content of SJMLs, we analyzed five lessons as stories. In this section, we briefly explain our processes of lesson selection and analysis.

Lesson selection

To analyze SJMLs that secondary teachers might use with their students,we identified lessons that teachers would have access to, including those that are published online and those published in print. We identified three main sources of SJMLs for high school mathematics topics that contained substantive teacher guidance (see Table 1). To create a set of lessons that would contain variety while being of manageable size for comparison, we selected one or two lessons from each source, aiming to have variety in mathematical topics and social justice foci.

Table 1: Lessons Analyzed

Curricular Source	Lesson	Social Justice Focus	Mathematical Topic
Reflecting the World: A guide to Incorporating Equity in Mathematics Teacher Education ("Reflecting", Felton-Koestler et al., 2017)	*Incarceration Rates*	Increasing incarceration rates in the USA	Modeling and making predictions from data
	Water Bottles for Detroit ("*Water Bottles*")	Access to clean water in Detroit, MI	3D nets, surface area, and volume
High School Mathematics Lessons to Explore, Understand, and Respond to Social Justice ("Responding"; Berry III et al., 2020)	*What's a Fair Living Wage?* ("*Fair Living Wage*")	Living wage	Systems of linear equations
	Bringing Healthy Food Choices to the Desert ("Food Desert")	Access to nutrition	Centers of triangles
Citizen Math ("Citizen", Citizen Math, accessed August 2021)	*Wage War*	Minimum wage	Systems of linear equations

For our analysis, we considered all available student and teacher resources, as well as any online materials referred to in the lessons, such as videos or infographics. In instances in which online materials had been updated since publication, we included the updated online materials.

SJM Plot Analysis

To examine patterns in how SJMLs unfold, we categorized story arcs that arose in the lessons according to our framework of mathematical and social expectations. We used teacher notes, lesson overviews, and student-facing curricular materials to determine how the lesson design expected content to unfold. For example, if the materials described students creating and presenting posters of information, we included this as part of the unfolding information that would enter that SJML story at that point of the lesson. This analysis was performed by researchers in pairs, who gathered periodically to find consensus.

First, to analyze the plot of a SJML, researchers identified the *acts* of the SJM story. The beginning of an act occurs when the focus of the SJM story changes. For example, if a lesson shifts from having students compare wages across the country to having students explain how wages are determined, then researchers would mark a new act.

Next, we identified all questions, both implicit and explicit, that potentially arise throughout each SJML. We included questions that were asked explicitly in the lesson materials, or were suggested implicitly (i.e., researchers anticipated that students would potentially consider the implicit questions in order to accomplish the goals of the SJML). For each question, researchers identified the story arcs by marking, for each question, the act in which it is initially proposed or formulated, how information advances what is known about the question in subsequent acts, and how the answers to questions were disclosed. We also noted whether there are potential moments that could prevent students from making straightforward progress toward an answer to a question. To do so, we borrowed three codes from the *mathematical story framework* (authors, year): *equivocation* (an instance in which students are allowed to make an incorrect assumption), *snare* (an instance in which there is explicit misinformation about a question), and *jamming* (an instance at which point something prevents students from making progress on a question).

We categorized the story arcs of each lesson based on whether they expected students to engage in thinking about mathematics, social contexts, or both. For example, the question "How would we describe the US imprisonment changes on the graph from 1925 to 1970 (stable / increasing / decreasing / fluctuating)?" (*Incarceration Rates*) would be classified as mathematical because it involves mathematizing and can be answered without considering the social context; the text indicates that it expects students only to identify that the graph is increasing. In contrast, the question "What are the benefits and limitations of Design 1 as a water bottle?" (*Water Bottles*) would be categorized as hybrid because students are expected to figure out that a given geometric design packs well for shipping to Detroit, but it is not the most efficient in holding water based on volume and surface area calculations. Finally, "What is minimum wage for tipped employees?" (*Fair Living Wage*) is an example of a social question as it expects students to consider people or aspects of the social world, but not to mathematize.

To compare focus on mathematics and social contexts across the lessons, we also evaluated how much each act of a lesson emphasized mathematics or social topics on a scale from 0-100%: 0% meaning a total emphasis on mathematics, 50% a perfect balance of mathematics and social topics, and 100% a total emphasis on social topics. To calculate these percentages, we assigned values for each question open in an act (0% for each mathematical question, 50% for each hybrid question, and 100% for each social question), found the sum, and divided by how many questions were open.

Findings

Within these lessons, students are expected to engage in both mathematical and social thinking, but the focus shifts between the two across a given lesson. Most of the lessons begin with a focus on social contexts, shift to mathematical phenomena, and then expect students to revisit the social context, perhaps answering questions raised earlier. Sometimes the transition is motivated because questions in one paradigm require inquiry in the other. To explain how these SJMLs integrate mathematical and social justice content, we first share our findings about the type and length of story arcs that compose each SJML. Next, we describe the variation in how the emphasis on different topics changes over the course of the lessons. Finally, we discuss how special narrative characteristics identified in three of the lessons could impact shifts in focus between social and mathematical ideas.

Story Arcs

We identified three distinct patterns in how the story arcs focused student inquiry. To illustrate these patterns, each row of the SJM plot diagrams in Figure 1 shows how a question is raised and addressed throughout a story (i.e., a story arc). The shaded portion of the row indicates in which act (represented by the columns) the corresponding story arc is raised and the extent of the lesson during which the question remains unanswered. The story arcs are colored based on their category (social, hybrid, or mathematical), and the letters in each cell represent a shift in what is known about the question during that act. In *Incarceration Rates* and *Water Bottles*, over two-thirds of the story arcs are mathematical. *Fair Living Wage* has a large concentration of hybrid story arcs, with mathematical and social story arcs interspersed periodically. Finally, in *Food Desert* and *Wage War*, the majority of story arcs are social or hybrid, while mathematical story arcs make up less than half of each lessons' arcs.

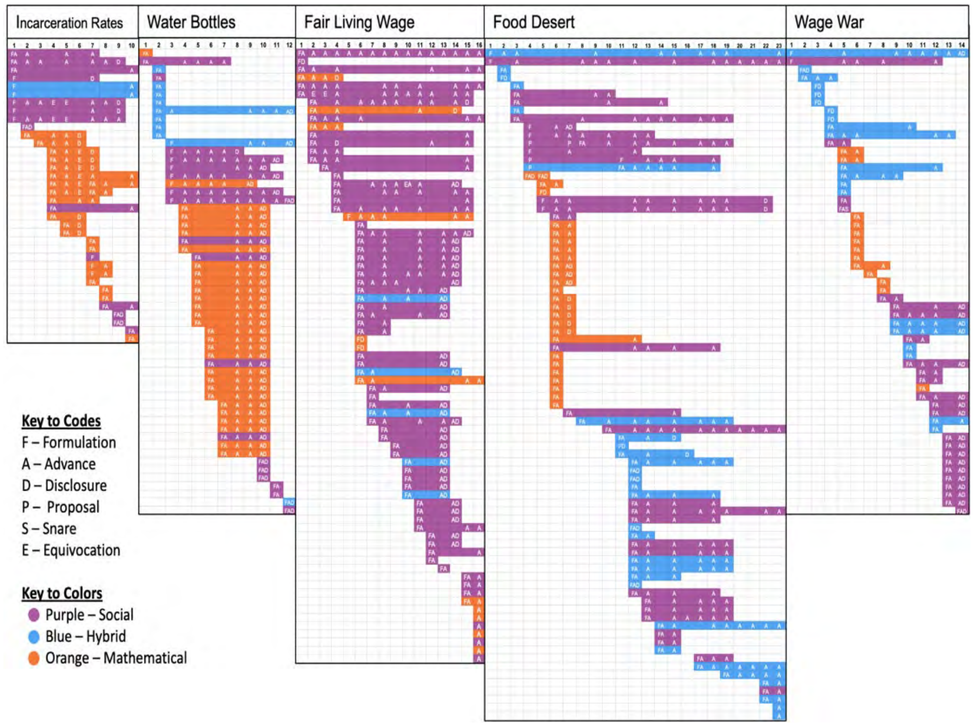


Figure 1: SJM Plots for the SJM lessons. Large version at [blinded].

In all of the lessons, students were invited to consider social phenomena for longer periods of time. All lessons contain overarching social or hybrid story arcs that stayed unanswered across the entire lesson, or all but one act. However, none of the lessons contain mathematical story arcs that spanned such a large portion of the lesson. Although some of the lessons do contain a large proportion of mathematical story arcs (i.e., *Incarceration Rates* and *Water Bottles*), the mathematical story arcs are often short and grouped together within only a few acts. On average, hybrid and social story arcs are longer and mathematical arcs are shorter (see Table 2). In two lessons, *Food Desert* and *Wage War*, mathematical story arcs are particularly short.

Table 2: Average Arc Lengths by Lesson and Type of Question

Lesson	Social	Hybrid	Mathematics	Overall
Incarceration Rates	10.0	5.7	3.1	4.2
Water Bottles	2.9	4.8	5.3	2.8
Fair Living Wage	4.2	7.0	6.4	6.4
Food Desert	4.8	9.5	1.8	4.6
Wage War	3.1	3.4	1.3	1.6

Story Acts

There are also distinct patterns of attention to mathematics vs social contexts that became evident as we compared the lessons (see Table 3). In all of the lessons, there is an early emphasis on social contexts. The emphasis on mathematics increases in the middle of the lessons, and then decreases as questions start to be resolved via the text. However, the degree of emphasis on mathematics and/or social context differs. Some lessons have a greater focus on mathematics (*Incarceration Rates* and *Water Bottles*), while others have a greater emphasis on social and hybrid phenomena throughout (*Fair Living Wage* and *Food Desert*). Another characteristic that distinguishes the lessons is that some shift abruptly between the two foci (note extreme values such as 0% and 100% in *Wage War*), while others are more balanced throughout (as can be seen by many acts with between 40-60% emphasis on social contexts).

Table 3: Focus Maps of the SJMLs (intensity of shading represents the relative degree of focus on mathematizing, where blue represents a focus on social contexts)

Lesson/Acts									0	1	2	3	4	5	6	7	8	9	0	1	2	3
Incarceration Rates	1%	2%	8%	2%	4%	%	7%	0%	0%	4%												
Water Bottles	5%	4%	6%	7%	2%	2%	5%	3%	7%	0%	7%	0%										
Fair Living Wage	8%	2%	1%	9%	5%	4%	0%	8%	0%	5%	2%	0%	3%	2%	7%	7%						
Food Desert	5%	00%	1%	8%	0%	0%	9%	0%	7%	7%	0%	1%	9%	0%	8%	5%	7%	7%	9%	0%	0%	3%	3%
Wage War	0%	00%	00%	2%	3%	1%	%	3%	1%	8%	5%	9%	5%	8%									

Special codes: Equivocations and Snares

Three lessons included equivocations and snares, which have the potential to motivate or curtail the pursuit of understanding of social issues within the lessons.

In *Incarceration Rates*, equivocations within six related mathematical story arcs enable a surprise with the potential to shift the focus of the lesson from mathematics to social contexts. Students are prompted to predict how prison populations will grow, using data from 1925 to 1970 (see Figure 2a). The data is fairly linear, which encourages students to assume that a linear equation would lead to an accurate prediction. Later, students are given additional data, up to 2006, which does not follow the linear trend (see Figure 2b). Instead, it increases at a much faster rate. Students' realization that the new values are much higher than their earlier predictions could provoke students to ask and pursue answers to questions about incarceration in the US that had not yet been raised in this lesson, such as "What caused US imprisonment rates to greatly increase in the 1970s?" The initial withholding of data may cause students to feel more impacted by the drastic difference in imprisonment rate, increasing their felt need to address it.

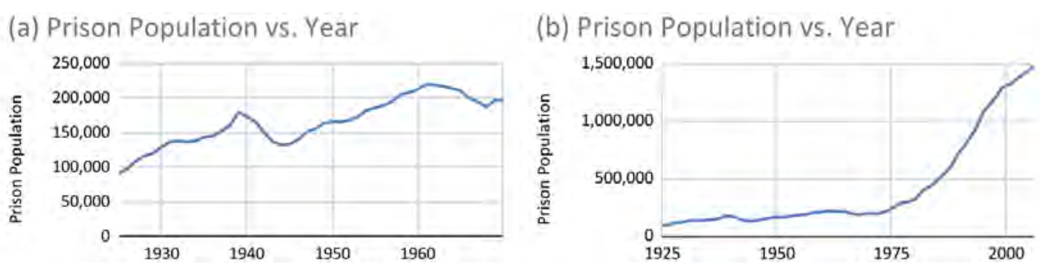


Figure 2: Incarceration rates for (a) years 1925-1970 and (b) years 1925-2006.

In *Fair Living Wage*, an equivocation in a social story arc gives students a reason to pursue mathematical questions. The text asks students to figure out what a suitable standard of living is. A video and infographic present a living wage for each of the 50 states, suggesting that the living wage is the same for every person in a state. However, students later learn that there are many factors that can impact the living wage of every individual (e.g., number of children). This complexity is revealed gradually, through hybrid story arcs. The initial equivocation that a living wage is a straightforward concept makes this complexity unexpected, which in turn can make the inquiry into related social issues (e.g., the racial pay gap in the US) more compelling.

In *Wage War*, a snare within a hybrid story arc demonstrates how mathematics can obscure a social context. The lesson models the relationship between supply (individuals who want a job) and demand (number of employees companies want to hire) of labor for a company with a system of linear equations. Students are prompted to calculate the intersection of the two equations, representing a supposedly ideal wage and number of workers. The lesson asks how changing the minimum wage could impact the labor market, suggesting that the company will hire fewer workers if it pays higher wages. No other factors are introduced for consideration. The simplistic linear function used to model a complex social phenomenon is a misrepresentation, constituting a snare. The snare does not engender further inquiry into the mathematical or social concerns of the lesson. That is, students are not expected to probe any related social or mathematical phenomena in order to identify the misrepresentation. Students who independently identify the model as overly simplistic may come away from the lesson with the impression that mathematics is not useful for interrogating a complex world.

Discussion

Since all lessons analyzed in this study were framed by extended social or hybrid story arcs, mathematical story arcs act as support for investigating social contexts in SJMLs, rather than being meaningfully integrated with social inquiry. Thus, mathematics is largely positioned as a tool worth using to solve social problems. This means that these lessons do not readily offer students the opportunity to engage in extended mathematical thinking, or develop complex mathematical ideas of their own through the exploration of social phenomena. While it is possible that the lessons may introduce new mathematical ideas when enacted in some classrooms, we found no evidence of that intent, and teacher-facing materials offered no support for doing so. We are concerned that the lack of extended mathematical inquiry in these lessons may perpetuate the belief that mathematics is for others to develop, and for students only to use. We wonder: Could SJMLs be designed to have extended mathematical questions, in addition to social ones, so that students can view mathematical inquiry as connecting to the world?

We also note that the design of lessons does not necessarily indicate how it will be enacted in a classroom, which has implications for teachers using these lessons. For example, we identified a trend of using narrative devices, such as withholding information, in order to enable a plot twist later in the lesson, such as that seen in *Incarceration Rates*. This design approach may serve to make these lessons more interesting and engaging for students. However, it also introduces the potential for lessons to fall flat during enactment. What if a lesson's success hinges on students being surprised by something that they already know? This points to the importance of teachers having knowledge of their students that goes beyond their mathematical abilities and classroom behaviors, to include students' prior experience with social justice topics.

Finally, we categorized story arcs based on whether they would include social interpretation, but we did not analyze how the arcs would engage students in thinking about social *justice*. Social justice involves critiquing the status quo or taking action to change the world, which is not necessarily happening just because students are considering questions about social contexts. Future work is needed to analyze how these lessons enable students to engage in the work of social justice and challenge the way things are.

Acknowledgments

This material is based on work funded by Boston University under Grant No. 9090011775.

References

Banilower, E., Smith, P. S., Malzahn, K. A., Plumley, C. L., Gordon, E. M., & Hayes, M. L. (2018). *Report of the 2018 NSSME+* (The National Survey of Science & Mathematics Education). Horizon Research, Inc. http://horizon-research.com/NSSME/2018-nssme/research-products/reports/technical-report

Banilower, E., Smith, P. S., Weiss, I. R., Malzahn, K. A., Campbell, K. M., & Weis, A. M. (2013). *Report of the 2012 national survey of science and mathematics education*. Horizon Research, Inc.

Berry III, R. Q., Conway, B. M., Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice* (First edition). Corwin.

Darby-Hobbs, L. (2013). Responding to a relevance imperative in school science and mathematics: Humanising the curriculum through story. *Research in Science Education*, *43*(1), 77–97. https://doi.org/10.1007/s11165-011-9244-3

Dietiker, L. (2013). Mathematical texts as narrative: Rethinking curriculum. *For the Learning of Mathematics*, *33*(3), 14–19.

Dietiker, L. (2015). Shaping mathematics into compelling stories: A curriculum design heuristic. *Journal of the International Society for Design and Development in Education*. https://open.bu.edu/handle/2144/25609

Dietiker, L., & Richman, A. S. (2021). How textbooks can promote inquiry: Using a narrative framework to investigate the design of mathematical content in a lesson. *Journal for Research in Mathematics Education*, *52*(3), 301–331. https://doi.org/10.5951/jresematheduc-2020-0318

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics* (1st edition). SUNY Press. https://epdf.tips/social-constructivism-as-a-philosophy-of-mathematics.html

Ernest, P. (2018). The ethics of mathematics: Is mathematics harmful? In P. Ernest (Ed.), *The Philosophy of Mathematics Education Today* (pp. 187–216). Springer International Publishing. https://doi.org/10.1007/978-3-319-77760-3_12

Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. *North American chapter of the international group for the psychology of mathematics education*.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. Routledge.

Nodelman, P., & Reimer, M. (2003). *The pleasures of children's literature* (3rd edition). Pearson.

Richman, A. S., Dietiker, L., & Riling, M. (2018). The plot thickens: The aesthetic dimensions of a captivating mathematics lesson. *The Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2018.08.005

Su, F., & Jackson, C. (2020). *Mathematics for human flourishing* (Illustrated edition). Yale University Press.

BUILDING COMMUNITY IN A RESEARCH PROJECT TEAM THROUGH IDENTITY SHARING

Molly Robinson
Portland State University
molrob2@pdx.edu

Eva Thanheiser
Portland State University
evat@pdx.edu

Amanda Sugimoto
Portland State University
asugimo2@pdx.edu

Danny Penner
Portland State University
aubrap@pdx.edu

Simon Byeonguk Han
Portland State University
byeonguk@pdx.edu

Courtney Koestler
Ohio University
koestler@ohio.edu

Mathew D. Felton-Koestler
Ohio University
felton@ohio.edu

Howard E. Bartels
Ohio University
hb134218@ohio.edu

Jennifer A. Thompson
Ohio University
jh859394@ohio.edu

Laura L. Wolfe
Ohio University
wolfel2@ohio.edu

This study presents the use of the Social Identity Wheel (SIW, Social Identity Wheel, 2021), to build community in a newly formed mathematics education research team. The SIW, originally built for use in classrooms, was used to allow each team member to share about themselves which led to the team learning about each other. The research focus of the team is to connect social and political issues to mathematics in elementary classrooms. Reflecting on identity and discussing social and political issues are essential components of this work. Hence, building community is part of fostering safe, productive environments in which to build these tasks. To better understand what building a community might look like, the project team used the Social Identity Wheel to build community within the team and get to know one another.

Keywords: Equity, Inclusion, and Diversity; Social Justice; Professional Development; Research Methods

Research suggests that understanding teacher identities in mathematics professional development (PD) can make PD more impactful. Teacher identities impact how they participate in PD, as well as what knowledge they both take up from the PD and how they implement PD ideas in their classrooms (Battey & Franke, 2008). Wager and Foote (2013) urge that mathematics and equity PD facilitators take teachers' lived experiences into account and view teachers as a resource. Moreover, implementing mathematical tasks that connect to social and political issues requires significant teacher reflection on their students' identities, their own identities, how these identities interact with the task context, and how these identities might interact with each other (Koestler et al., in press). Given this extant scholarship, it is important that professional development communities spend time reflecting on identities.

As part of a larger grant project, our team collaborates with a group of expert teachers in an ongoing PD setting to build mathematical tasks that connect to social and political issues. This work necessarily requires reflecting on identities and discussing social and political issues. To create a safe space for these potentially difficult conversations and to be able to better collaborate with each other and with the collaborating teachers, the project team considered ways that we could build community among ourselves. Particularly, our team wanted to create opportunities for each of us to share about our identities in an effort to get to know each other better. Thus, we decided to use the *Social Identity Wheel* as one way to do so.

The Social Identity Wheel (SIW, *Social Identity Wheel*, 2021), which was developed at the University of Michigan, is one tool that can be used to support sharing about identities. Jacobsen and Mustafa (2019) suggested a similar tool, a Social Identity Map, intended to support researchers in beginning to "think deeply about how their assumptions translate into discussions with participants, influence their understanding of participants' experiences and lives, and how this impacts the way they code, analyze, and interpret findings," (p. 11). The project team wanted to collectively get to know each other better as a community and explore the model of using the SIW with people who do not yet know each other well. We chose the SIW as a vehicle to present ourselves to each other to work towards forming a community.

Theoretical Background

The study of identity in mathematics education has taken many paths: student identity (e.g., Aguirre et al., 2013), mathematics teacher identity (e.g., Battey & Franke, 2008; Wager & Foote, 2013), and researchers' identities (e.g., Glesne, 2011). Bartell and Johnson (2013) argue that researchers should openly talk about identity and their own privileged research positions to avoid paternalism and consider the role of their privilege in their research. Similarly, Foote and Bartell (2011) propose that "life histories provide a particular opportunity to explore researcher positionality that might be used more widely as a support to understanding the relationship among the researcher, the researched, and the research problem," (p. 65). Given that it is important for researchers to share and reflect upon their own identities and how their identities shape and impact their research, then it is arguably just as important for teams of researchers to share and discuss their identities in the context of the research team.

Identity is dynamic, mutable, and socially constructed (Park, 2015; Vygotsky, 1979). Naturally, theories about identity are manifold and complex. Sfard & Prusak (2005) argue that identity does not exist as a tangible object; rather, it is discursive in nature. This means that identities are collectively created and exist in the narratives that are told about individuals and that individuals tell about themselves. These narratives become part of an individual's identity when they are "reifying, endorsable and significant" (Sfard & Prusak, 2005, p. 16). Tangible tools like Social Identity Maps or Social Identity Wheels become a means of eliciting and supporting a person's narration of their identities (Jacobson & Mustafa, 2019). Further, Social Identity Wheels could allow individuals on a research team to narrate their various identities and identify the resources that they draw upon when constructing their narrative. In turn, this storying can become a part of the ongoing discursive construction of identity, particularly within the context of the research team. We treated these narrative identities supported by the SIW tool as a vehicle by which we might get to know one another. In this paper, we examine the research question: *How, if at all, does the SIW tool support the project team in getting to know one another?*

Methods

Social Identity Wheel Presentations

The SIW (*Social Identity Wheel*, 2021) is an oval wheel, with 11 social identity categories listed around the outside: age, religious or spiritual affiliation, race, ethnicity, socio-economic status, gender, sex, sexual orientation, national origin, first language, physical/emtional/developmental (dis)ability. There are five prompts in the center of the oval to help users reflect on their identity, including "identities you think about most often" and “identities that have the greatest effect on how others perceive you.” Each team member created a representation of their own responses to each of the categories on the SIW, placed these in a shared online folder, and narrated their SIW to the group at later team meetings. Questions, comments, and connections from other group members were encouraged after the narrator was done. Time for sharing and asking questions was intentionally not limited and included up to two SIW per meeting. In this paper we treat the initial presentation and the discussion that followed as one unit, and we call this unit the presentation. The mean and median average presentation times were 30 minutes with a range of 17 minutes to 45 minutes. Team members consisted of 4 project PIs, 5 graduate students, and 1 project manager who is also a graduate student. The PIs are Andrew, Bailey, Sandy, and Anna; the graduate students are Augusto, Jeff, Judy, Ellison, and Silvia; and the project manager is Sharon (all names are pseudonyms). Data collected for this study included recordings and transcripts of the presentations of each team member’s SIW. Transcripts were broken into paragraphs by the constructs that they addressed (i.e., race, gender, etc.). Research team members then coded each presentation with the category of the SIW tool that was the focus of the paragraph. Multiple categories were coded if they were part of the same paragraph. Coding was done in MaxQDA. Visuals of each presentation were created based on the coding through MaxQDA.

Survey

After all SIW were presented, we surveyed all team members via Qualtrics to learn how presenting and listening to others presenting their SIW impacted the community formation. Questions included open and closed formats. Sample questions from the survey include:

- How, if at all, has this activity impacted your interactions with the group overall and/or specific group members?
- Were there things that you shared that you wouldn't have shared outside the group? If so, what were they, and why did you choose to share in this context?
- What, if any, were the benefits of having done this activity with our group?

Survey responses were coded using thematic analysis (Braun & Clarke, 2006) by a research team member to identify themes that appeared across multiple responses to each question and across multiple questions. For example, in response to Question B, participants wrote things like, “I am trying to become more open about who I am,” “I felt like I needed to share [about my anxiety] to be honest,” or “I felt like if I was going to convey an accurate representation of my identity… that [my mental health] was a very important piece that could not be left out.” Initial codes for these responses related to representing oneself honestly and authentically. Participants also wrote things like, “[The other group members] shared some vulnerable details, so it felt like it would be unfair NOT to also be vulnerable” in response to Question B, or “As a faculty member on the grant I did feel some responsibility to be a bit more open and illustrative of the kinds of things that could be shared and included,” in response to Question C. Initial codes for these responses related to desire for reciprocity in authentic sharing. Participants also referenced authenticity directly; one participant wrote, “After I shared, I felt like I was more able to authentically show up to the group in future meetings” in response to Question D. Together, responses like these and their initial codes constituted the theme of “authenticity and reciprocity,” which emerged across multiple responses to the same question and across responses to different questions. This example is illustrative of the way in which themes were developed. The final themes are shared in the results section.

Results

Social Identity Wheel Presentations

Though all team members used the SIW to begin their reflection and sharing, the topics discussed during presentations varied. Figure 1 shows the visualizations of the topics discussed for each team member; this provides an overview of some of the major topics that team members talked about. Duration varied across SIW presentations, but their visualizations were resized so that all visualizations are approximately the same size despite any time differences.

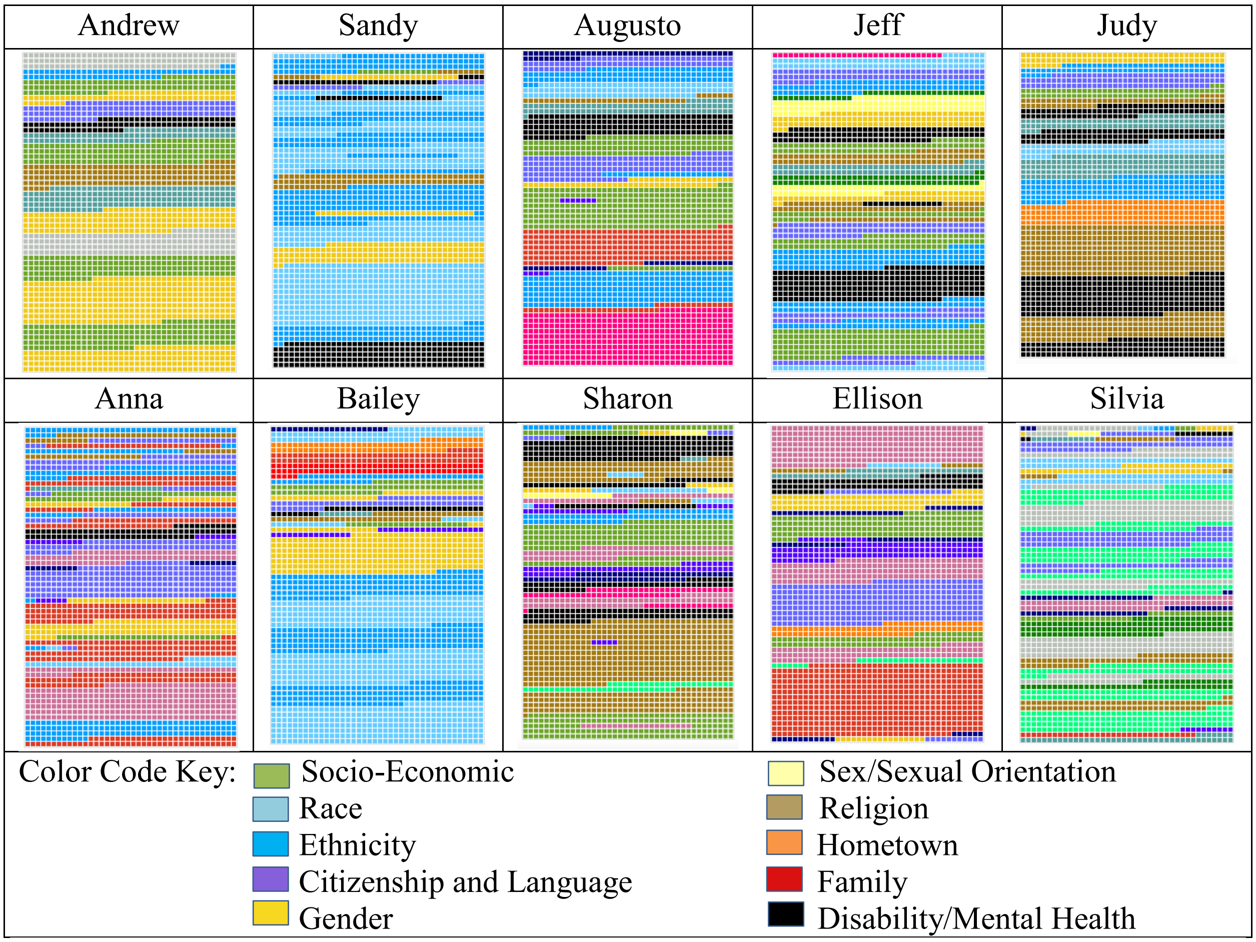


Figure 1: Visualization of topics discussed during team members' presentation of the SIW. *Note* The color code key includes only those codes that come directly from the SIW categories, or that came up across multiple presentations.

Of note is that for half of the research team the themes of disability/mental health (Sandy, Jeff, Judy, Anna, Sharon), ethnicity (Sandy, Augusto, Jeff, Anna, Bailey), and/or socio-economic status (Andrew, Augusto, Jeff, Ellison, Sharon) showed up in the presentations during the discussion portion. Sex and/or gender (Andrew, Sandy, Anna, Jeff, Bailey) was also discussed by 5 team members during the discussion portion of the presentation. Religion was discussed by 4 team members (Sandy, Jeff, Judy, Sharon) during the discussion portion of the presentation. These topics typically do not get discussed in large research group settings but are learned about in more intimate conversations. For most of the participants the presentation and discussion focused mainly on 2 to 4 constructs.

Survey Responses

The theme that appeared most in the survey responses was "Community Building," with 66 occurrences across the 10 participants and 8 open questions. Four of the codes within this theme appeared at least 8 times and across at least 7 team members, with "increased interpersonal knowledge" (IIK) mentioned by all 10 participants, "bonding, community/relationship building" (BCRB) and "authenticity and reciprocity" (AR) mentioned by 8 of the 10 participants, and "teamwork" (TW) mentioned by 7 of the 10 participants. See Table 1 for the code counts.

Table 1: Code Counts for Community Building Theme

Team Member	Code Occurrences				Total
	IIK	BCRP	AR	TW	
Andrew (PI)	1	2	1	1	5
Sandy (PI)	1	1	2	1	5
Augusto	2	0	0	0	2
Jeff	2	0	0	1	3
Judy	1	2	3	1	8
Anna (PI)	1	1	1	0	3
Bailey (PI)	3	3	1	2	9
Sharon	2	3	2	1	8
Ellison	1	3	2	2	8
Silvia	1	2	1	0	4
Number of Team Members with at Least One Occurrence	10	8	8	7	
Total Occurrences Across All Team Members	**15**	**17**	**13**	**9**	

Note **Codes are: increased interpersonal knowledge (IIK), bonding, community/relationship building (BCRB), authenticity and reciprocity (AR), and teamwork (TW).**

Increased Interpersonal Knowledge (IIK) Knowing more about someone does not necessarily mean that a bond or relationship is formed between individuals, so the code IIK is distinct from the BCRB code. However, some occurrences of IIK overlap with BCRB. For instance, Andrew said "I've enjoyed learning about others and I feel that it increases our intimacy and connection." The portion "I've enjoyed learning about others" on its own does not imply building an emotional bond but does emphasize learning about others, and so the response was coded with IIK. Because Andrew continued with "I feel that it increases our intimacy and connection," which indicates building relationship bonds, this response was then also coded with BCRB. Knowing about others and bonding with others *may* overlap, but they do not inherently overlap. Augusto and Jeff were the two group members who did not have any responses directly mentioning bonding with other group members, though they each had two instances of mentioning that they learned about others. Augusto shared that "I think for our group with such diversity, it was important to dive much deeper into our lives and backgrounds, and not just our names and professional credentials or programs and faces." Sharon felt that the activity supported "a deeper contextual understanding of a person's lived experiences." This knowledge of other team members was one benefit of using the SIW that Bailey identified: "I think one benefit is understanding people's backgrounds and identities, and knowing how they position themselves so that I don't have to assume."

Bonding, Community/Relationship Building (BCRB) Responses received the code BCRB when they referred to forming or building relationships, either with other team members or with the group as a whole. Several team members observed that the remote meetings made it hard to bond, and that the activity helped address this problem. Anna expresses it here: "I think it was a great bonding activity learning more about everyone. I feel like it sped up the process of having chats especially since we are on Zoom and I am not sure when and how we would get a chance to get to know one another better." Silvia joined the group for the first time to share her identity wheel, and felt "practically instantly at home with the group; I felt that all the ice had been broken after sharing, and that I was part of the group and less of an outsider." The activity allowed Silvia to quickly feel bonded to the group. It also created an opportunity for conversations that might not have happened otherwise. Sharon noted that she has "had a few one on one conversations with individuals that most likely would not have occurred had the identity wheel activity not taken place." Sandy echoed the connections others found, sharing that she "found resonance with people that I wasn't expecting to find, which was nice." This resonance and connection helped build community within the group. Judy shared that as a result of the activity, she felt "more empathy for the group members after hearing their own backgrounds and vulnerabilities and struggles." Ellison said that one of the benefits of the activity is the team's "ability to connect with and support others in the group making the group more cohesive." The BCRB code was present in all but two of the group members' responses, as seen in Table 1. However, those two team members' responses both showed instances of the code IIK.

Authenticity and Reciprocity (AR) A total of 13 responses were coded with AR, including the excerpts previously shared to describe how the theme was developed. Each response met at least one of these criteria: explicitly mentioned authenticity, indicated wanting to be open and authentic about themselves to the group, indicated sharing details to model authenticity or vulnerability *to* others, or sharing details because authentic sharing was modeled *by* others. Sandy explicitly mentioned authenticity in two of her responses, including this excerpt: "I also think it gave me some ideas of how people see/position themselves so I can integrate that into how I see them can work to have more authentic interactions with them." Sharon also explicitly mentioned authenticity, saying “I feel as though the senior leadership impart equitable practices, which were demonstrated by every person’s authentic participation.” Sharon shared that, though it was challenging, she gave herself “permission to speak and fully participate without regret.” Judy echoed this desire to represent herself authentically to the group with this response: “I wanted to share enough to feel like I was being honest and not hiding anything important.” For Bailey, their connection to other group members impacted their understanding of authentic sharing. Bailey wondered “what some people would think when I got to certain descriptions, especially those who know me well. Like, would [they] wonder why I was holding back?” Several people were encouraged by other SIW presentations to share more details about themselves, as seen in Ellison’s response: “I did return to my wheel after seeing others’ work and felt more comfortable adding to it at that point.” In response to a different question, Ellison also noted that “Because everyone was participating there is some comfort in the reciprocity.” The AR responses signal that team members were influenced by a desire to be honest and authentic, either for themselves or as a responsibility to foster reciprocal sharing with the group.

Teamwork (TW) Seven team members specifically referenced how the activity might impact the research work the team will do for the grant, as seen in the 9 instances of TW. Three of the PIs talked about the grant work generally. Andrew referenced specific goals of the grant, noting that the team’s experience with the SIW “will be helpful in thinking through our project work as we consider how to create lessons and ask teachers and ourselves to reflect on our identities relative to the topics in the lessons.” Sandy referenced the grant work more generally, sharing that we “now have a fuller understanding of each other and how who we are might shape how we approach the work of the grant.” Bailey referenced teamwork in response to two questions. One was more general: “I thought it was a good activity to do as a project team.” In another response, Bailey shared, “I only added in the pronouns in at the end because I thought that it was important to share since we would be working together for the next 3 years.” Three of the graduate students referenced teamwork interactions in their TW responses. Jeff shared that the SIW activity helped him “have a better sense of what should I be careful or attend to when I interact with other group members in general.” Ellison explicitly mentioned her graduate student role in the group related to teamwork: “I also feel more comfortable sharing my ideas, needs, and concerns since I feel like I have an identity beyond ‘graduate student’ in the group.” On a similar note, Judy shared “I feel emotionally safer in the group, so I feel like I can participate more and make mistakes and accept critiques easier.” Her response goes on to mention how the SIW activity might support navigating difficult conversations: “Before the activity, I felt like there were some ‘hot’ moments where maybe people misunderstood someone or felt challenged. It seems like we will be able to deal with those more directly as a group now.” Responses coded with TW talked about teamwork in reference to specific grant goals, working together as a team, and group interactions.

Conclusion

The activity with the SIWs supported the project team in getting to know one another and fostered bonding and building community. Every team member’s response was coded with at least two of the Community Building codes. Though not explicitly elicited with the survey questions, there were 9 instances in responses that referenced working together as a project team, as seen in the instances of teamwork. Every team member’s response showed at least two instances of either bonding, community/relationship building or increased interpersonal knowledge, which suggests the activity *did* support the team in getting to know each other at a deeper level.

The identity wheel allowed for the feel of more intimate discussions in the larger research group (over Zoom), as evidenced by discussions of disability/mental health, ethnicity, gender, and religion during the discussion portion of many presentations. Some team members described the activity as speeding up the getting to know you process. Anna shared "I feel like it sped up the process of having chats especially since we are on Zoom and I am not sure when and how we would get a chance to get to know one another better." This suggests that the SIW tool might accelerate community building in groups, particularly those that meet remotely and have fewer opportunities for casual, spontaneous conversations. It is also important to note that no one activity is sufficient for building community. Activities like this one can begin to facilitate community building, but community building requires sustained effort.

While this was implemented remotely, this tool was originally designed for in-person use. Research teams wanting to get to know one another and build community could possibly see similar results, but we do not claim that our experience is generalizable to other groups and settings. Rather, our goal is to contribute to ongoing work regarding researcher identity and positionality and how it can be explicitly shared. It is important to note that there may have been some status issues at play in our group. In fact, 4 of the responses mentioned power issues as a potential drawback or pressure that participants might experience, regardless of whether or not the respondent felt that pressure themselves. For example, all the graduate students have one of the PIs as their advisor. These power dynamics could have influenced the types of SIW sharing individuals engaged in, as well as the questions or comments they felt comfortable sharing. Group norms and feelings of safety will not be the same in every group. While our team emphasized several times that members need only share what they are comfortable sharing, some may have still felt pressure to share. Sharing identity stories requires vulnerability on the part of the sharer. Hence, careful emphasis needs to be put on establishing group norms. While many of our team describe that this experience *might* influence teamwork over the course of the project, we wonder what future studies might learn about how building community in research teams does or does not impact the work they do together.

Acknowledgments

This research was supported by the National Science Foundation (Grant No. 2010463). Opinions, findings and conclusions or recommendations expressed here are the authors' and do not necessarily reflect the views of the National Science Foundation.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. (2013). *The impact of identity in K-8 mathematics: Rethinking equity-based practices*. National Council of Teachers of Mathematics.

Bartell, T. G., & Johnson, K. R. (2013). Making unseen privilege visible in mathematics education research. *Journal of Urban Mathematics Education*, *6*(1), 35–44.

Battey, D., & Franke, M. L. (2008). Transforming identities: Understanding teachers across professional development and classroom practice. *Teacher Education Quarterly*, *35*(3), 127–149.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101.

Foote, M. Q., & Bartell, T. G. (2011). Pathways to equity in mathematics education: How life experiences impact researcher positionality. *Educational Studies in Mathematics*, *78*, 45–68.

Glesne, C. (2011). *Becoming qualitative researchers: An introduction* (4th ed.). Pearson Education.

Jacobson, D., & Mustafa, N. (2019). Social identity map: A reflexivity tool for practicing explicit positionality in critical qualitative research. *International Journal of Qualitative Methods*, *18*, 1–12.

Koestler, C., Ward, J., Zavala, M. del R., & Bartell, T. G. (in press). *Early childhood lessons to explore, understand, and respond to social injustice*. Corwin Press.

Park, H. (2015, May 19). Learning identity: A sociocultural perspective. *2015 Adult Education Research Conference Proceedings*. https://newprairiepress.org/aerc/2015/papers/41

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, *34*(4), 14–22. https://doi.org/10.3102/0013189X034004014

Social Identity Wheel. (2021). LSA Inclusive Teaching Initiative, University of Michigan. https://sites.lsa.umich.edu/inclusive-teaching/social-identity-wheel/

Vygotsky, L. S. (1979). Consciousness as a problem in the psychology of behavior. *Soviet Psychology*, *17*(4), 3–35. https://doi.org/10.2753/RPO1061-040517043

Wager, A. A., & Foote, M. Q. (2013). Locating praxis for equity in mathematics: Lessons from and for professional development. *Journal of Teacher Education*, *64*(1), 22–34. https://doi.org/10.1177/0022487112442549

EXPLORING WOMEN OF COLOR'S VARIED EXPRESSIONS OF MATH IDENTITY

Ciera Street
Colorado State University
ciera.street@colostate.edu

Jessica Ellis Hagman
Colorado State University
jess.ellis@colostate.edu

One area in the growing call to create more equitable and just mathematics classrooms is supporting and strengthening students' mathematical identities. Limited scholarship in this area considers an intersectional lens when exploring undergraduate mathematical identity (Leyva, 2021; McGee 2016). Informed by Data Feminism, this research uses a transformative, mixed methods approach to consider how women of color in introductory undergraduate mathematics group based on attributes of mathematical identity. We first analyzed quantitative data using cluster analysis followed by the discussion of qualitative data through free response survey questions and future follow-up interviews to better conceptualize the experiences of each cluster. Key ideas emerging from this work include the role of peers and friendship, instructor care, class structure, and mathematical affect in women of color's mathematical identities.

Keywords: Equity, Inclusion, and Diversity, Gender, Undergraduate Education, Calculus

A growing call from research and professional organizations within mathematics education aims to create more equitable and just mathematics classrooms for students with various identities (Mathematical Association of Mathematics Instructional Practices Guide, 2018; National Council of Teachers of Mathematics, 2014). This includes supporting and strengthening students' mathematical identities. Mathematical identity encapsulates students' relationship with mathematics and their perceptions of themselves as a doer of mathematics, including their beliefs about ability and participation, sense of belonging, and institutional and interpersonal supports (Voigt, 2020). Limited scholarship in this area considers an intersectional lens when exploring undergraduate mathematical identity (Leyva, 2016, 2021; McGee 2016). However, these studies suggest that gendered and racialized mathematical discourses uniquely influence expressions of mathematical identity for women of color. Mathematical discourses continue to communicate who belongs in the space by promoting values associated with dominant masculinities and whiteness, therefore often excluding women and students of color (Leyva, 2017; Adiredja & Andrews-Larson, 2017; Martin, 2006). Thus, this research engages critical frameworks at the intersections of gender and race to challenge those exclusionary norms and encourage counternarratives to white, masculine mathematical discourses. Using a transformative, sequential mixed methods design, we ask: How do women of color in introductory undergraduate mathematics group based on attributes of mathematical identity? With respect to which attributes do the groups differ? With respect to which attributes are the groups similar?

Theoretical Perspective

We utilize a Data Feminist lens to frame and support the design of this work. Coined by D'Ignazio and Klein (2020), Data Feminism encourages "a way of thinking about data, both their uses and their limits, that is informed by direct experience, by a commitment to action, and by intersectional feminist thought" (What Is Data Feminism?, para. 9). The authors argue how data have maintained inequitable power structures through promoting discrimination, policing, and surveillance. However, data similarly present an opportunity to challenge these dominant narratives and acknowledge the culture and community inseparable from the data collection and analysis. One area that exists at the intersection of these realms is higher education, and especially mathematics education due to the continued marginalization of women and students of color by maintaining white, patriarchal norms (Leyva, 2017). We chose to center women of color in this analysis to provide counternarratives to the white, patriarchal mathematical context without comparisons to the dominant group. Instead, we highlight the often submerged and undervalued variety of experiences of women of color related to mathematical identity.

Methods

We use a transformative, sequential mixed methods design in this work (Creswell et al., 2003; Mertens, 2012). We first analyzed quantitative data using cluster analysis (n=3296) followed by the discussion of qualitative data through free response survey questions and future follow-up interviews to better conceptualize the experiences of each cluster.

Quantitative Data Methods

The data for this study originate from the *S-PIPS-M* survey instrument administered by the Progress through Calculus (PtC) project. This survey collected information about undergraduate students' experiences in precalculus and calculus, including mathematical activities, interactions, and affect (Street et al., 2021). We considered the subset of the *S-PIPS-M* dataset that includes only women of color. We first classified women as any participant who selected at least woman from the following select-all response options related to gender: Man, Transgender, Woman, Not listed (please specify). We classified a person of color to include any participant who selected at least one of the following select-all response options related to race and/or ethnicity: Alaskan Native or Native American, Black or African American, Central Asian, Hispanic or Latinx, Middle Eastern or North African, Native Hawaiian or Pacific Islander, Southeast Asian, South Asian. Thus, women of color represent the intersection of these two participant classifications.

To narrow in on mathematical identity, we selected 22 survey questions that aligned with one or more of the components of mathematical identity, including students' perceptions of ability, participation in class, feelings of inclusion, and resources such as tutoring (Voigt, 2020). Previous research using this dataset (Voigt, 2020; Apkarian, 2019) suggests correlation between certain survey questions. Thus, we performed a factor analysis under a promax rotation with four factors as informed by the scree plot. We removed any loadings less than 0.2 and allowed factor items to load onto more than one factor. This resulted in four factors we label Class Experience, Inclusion/Positionality by Instructor, Mathematical Peer Interaction, and Self-Efficacy. These labels emerged from themes across the factor items and connections to the components of mathematical identity. We appended four new scores to each participant based on this factor analysis and then performed a cluster analysis over 4-dimensions, one for each factor. The graph of the total within-group sum of squares by cluster number implied an optimal four clusters.

Qualitative Data Methods

Informed by Data Feminism, we used participants' responses to four free-response survey questions about their classroom and mathematical experiences to help contextualize the clusters. This process involved filtering the responses by cluster and reading through each response looking for concepts related to attributes of mathematical identity such as peers, instructor, pedagogy, tutoring, and emotions. This was not intended as a full thematic analysis, rather as a means of helping illuminate common mathematical experiences within each cluster. The next phase of this work will deepen these results by collecting qualitative data from a sample of survey participants. We will conduct interviews with participants from each cluster and ask them to write a brief math autobiography. The interview protocol will capture questions around their relationship with undergraduate mathematics and their perceptions of themselves as doers of math, including questions related to the four key ideas presented in the discussion section.

Results

The results of the cluster analysis suggest four distinct groups related to women of color's mathematical identity. The table below outlines the mean scores for each factor of mathematical identity derived from the cluster analysis. Class experience, mathematics peer interaction, and inclusion have been normalized on a scale of 0 to 1 and self-efficacy on a scale of -0.5 to 0.5.

Table 1: Cluster Analysis Results

		Cluster				Overall Mean
		Cluster 1 (n=857)	Cluster 2 (n=842)	Cluster 3 (n=549)	Cluster 4 (n=1045)	
Mathematical Identity Factor	Self-Efficacy	0.04	-0.02	-0.14	0.04	-.004
	Mathematics Peer Interaction	0.76	0.15	0.52	0.43	0.46
	Inclusion	0.51	0.47	0.39	0.52	0.48
	Class Experience	0.78	0.35	0.34	0.61	0.54

The overall mean for Class Experience was 0.54. This factor measures students' perceptions of the frequency of class activities literature often relates with active, student-centered teaching. Clusters 1 and 4 have class experience scores above average, suggesting these students perceive their class to more frequently include active learning components. Clusters 2 and 3 have scores below average, suggesting these students perceive less frequency of these activities and more lecture. The overall mean for Inclusion was 0.48. A score of 0.5 in this factor implies students feel equally included by their instructor in class as their peers. Clusters 1, 2, and 4 have inclusion scores at about the mean, suggesting the women of color in these clusters feel equally included by their instructor in class as their peers. Cluster 3's below average inclusion score suggests the women of color in this cluster feel less included by their instructor in class as their peers. The overall mean for Mathematics Peer Interaction was 0.46. This factor measures the frequency of students' mathematical interactions with their peers both inside and outside the classroom. Cluster 1 had an above average mathematics peer interaction score, implying more frequent peer interactions. Clusters 3 and 4 present about average scores in this factor, suggesting a moderate amount of peer interaction. Cluster 2 has a far below average score in this factor, suggesting infrequent peer interaction. The overall mean for Self-Efficacy was -0.004 representing a practically null change in students' perceptions of their mathematical ability, interest, and confidence during the course. Clusters 1, 2, and 4 present self-efficacy scores essentially at the mean, suggesting little to no change in self-efficacy during the course. Cluster 3 has a below average score in this factor, suggesting a negative change in self-efficacy during the course.

Discussion and Key Ideas

To contextualize the quantitative results, this discussion presents key ideas that emerged when looking at students' free-response questions grouped by cluster appointment. The next phase of this research will use interviews and written reflections to investigate these key ideas and their potential role in supporting women of color's mathematical identity.

Key Idea 1: Peers and friendship

Students in Clusters 1, 3, and 4 frequently labeled peers and friends as notably helpful aspects of their course. In line with their class experience scores, Clusters 1 and 4 perceived more opportunities to engage in mathematics with their peers through group work and class discussions. Cluster 3 presented peer interaction almost solely with outside of class experiences such as study groups and tutoring. Notably more so than other clusters, these students used the term "friends" in their free responses. Coupled with the low class experience and inclusion scores in Cluster 3, these friendships may have acted as what Leath et al. (2022) describe as "communicative resistance" (p. 9) toward exclusion and frustration in class. Interviews will further explore the potential distinction between peer interactions and peer friendships related to these women of color's mathematical identity.

Key Idea 2: Instructor care

While the majority of the inclusion scores suggest equal feelings of inclusion from their instructor compared to their peers, how each cluster described this inclusion differed. Students in Cluster 1 described their instructor as patient, understanding, and encouraging. They also used more personally directed statements using "my" or "I", such as their "teacher's interest in *my* learning process." Other clusters usually addressed positive qualities about their instructor through teaching style and approachability or appeared fairly neutral toward their instructor. Students in Cluster 3, with their uniquely below average inclusion score, mainly pointed out negative qualities in teaching style or unapproachability in communication and office hours. Interviews will explore not only how general feelings of inclusion in classroom activities and opportunities to participate relate to women of color's mathematical identity, but also the role of students' observations of their instructor's care for them (Rainey et al, 2019; Bartell, 2011).

Key Idea 3: Class structure and implementation

Although the classroom experience score suggests different pedagogy types, additional data are necessary to describe these classrooms and how these activities and their implementation relate to women of color's mathematical identity. Generally for Clusters 1 and 4, students described consistent group work as a helpful class component, where the instructor provided explanation of a topic and then presented students with a task to complete in groups. Students in Cluster 2 described a mostly lecture-based course with some group work components, although some students mentioned discomfort working with peers in-class given the infrequent implementation of this activity (Yoon, 2011). Students in Cluster 3 appeared frustrated with in-class lectures and would have preferred more opportunities to work with others in class. Interviews will prompt participants for further details about their undergraduate mathematics courses and the classroom structures they found supportive of their mathematical identity.

Key Idea 4: What affects affect?

Surprisingly, only Cluster 3 showed a non-zero average change in affective components. While Clusters 1, 2, and 4 differed across various other factors, participants in these clusters on average maintained their same level of mathematical confidence, interest, and perception of ability. However, Cluster 3 resulted in negative average change in each of these components. Many students in Cluster 3 express how the negative experiences in their course contributed to confusion and frustration, which negatively affected how they viewed mathematics and their ability and confidence in the subject. Various studies show that supporting positive experiences in mathematics can improve students' perceptions of their mathematical ability (Hannula, 2006; Weber, 2008), but we appear to see the opposite effect here. While we see in this analysis either a negative or null change in self-efficacy, follow-up interviews will explore which components of undergraduate mathematics classrooms can promote feelings of confidence, interest, and competency for women of color.

References

Abell, M. L., Braddy, L., Ensley, D., Ludwig, L., & Soto, H. (2018). *MAA Instructional Practices Guide*. MAA

Press.
Adiredja, A. P., & Andrews-Larson, C. (2017). Taking the sociopolitical turn in postsecondary mathematics education research. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 444–465. https://doi.org/10.1007/s40753-017-0054-5
Apkarian, N., Smith, W. M., Vroom, K., Voigt, M., Gehrtz, J., PtC Project Team, & SEMINAL Project Team. (2019). *X-PIPS-M Survey Suite.* Available: bit.ly/2wwcSok
Bartell, T. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. *Journal of Urban Mathematics Education, 4*(1), 50-74.
Creswell, J.W, Plano Clark, V., Gutmann, M., and Hanson, W (2003). Advanced mixed methods research designs. In A. Tashakkori and C Teddle (Eds.), *Handbook of mixed methods in social and behavioral research* (pp. 209-240). Thousand Oaks, CA : Sage.
D'Ignazio, C., & Klein, L. (2020). *Data Feminism*. The MIT Press. https://data-feminism.mitpress.mit.edu/
Hannula, M. S. (2006). Motivation in mathematics: Goals reflected in emotions. *Educational Studies in Mathematics, 63*(2), 165–178. https://doi.org/10.1007/s10649-005-9019-8
Leath, S., Mims, L., Evans, K. A., Parker, T. A., & Billingsley, J. T. (2022). "I Can be Unapologetically Who I Am": A Study of Friendship Among Black Undergraduate Women at PWIs. *Emerging Adulthood, 0*(0), 1–15.
Leyva, L. (2016). An intersectional analysis of Latin@ college women's counter-stories in mathematics. *Journal of Urban Mathematics Education, 9*(2), 81-121.
Leyva, L. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education, 48*(4), 397–433. https://doi.org/10.5951/jresematheduc.48.4.0397
Leyva, L. (2021). Black women's counter-stories of resilience and within-group tensions in the white, patriarchal space of mathematics education. *Journal for Research in Mathematics Education, 52*, 117–151. https://doi.org/10.5951/jresematheduc-2020-0027
Martin, D. B. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning, 8*(3), 197–229. https://doi.org/10.1207/s15327833mtl0803_2
McGee, E. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture? *American Educational Research Journal, 53*(6), 1626–1662. https://doi.org/10.3102/0002831216676572
McGee, E., & Martin, D. B. (2011). "You would not believe what I have to go through to prove my intellectual value!" Stereotype management among academically successful Black mathematics and engineering students. *American Educational Research Journal*, *48*(6), 1347–1389. https://doi.org/10.3102/0002831211423972
Mertens, D. M. (2012). Transformative mixed methods: Addressing inequities. *American Behavioral Scientist*, *56*(6), 802-813.
National Council of Teachers of Mathematics. (2014). *Access and equity in mathematics education.*
Ong, M., Smith, J. M., & Ko, L. T. (2018). Counterspaces for women of color in STEM higher education: Marginal and central spaces for persistence and success. *Journal of Research in Science Teaching, 55*(2), 206-245.
Rainey, K., Dancy, M., Mickelson, R., Stearns, E., & Moller, S. (2019). A descriptive study of race and gender differences in how instructional style and perceived professor care influence decisions to major in STEM. *International Journal of STEM Education, 6*(1), 1-13.
Street, C., Apkarian, N., Gehrtz, J., Tremaine, R., Barron, V., Voigt, M., & Hagman, J. E. (2021). *X-PIPS-M Data Summary*. arXiv preprint arXiv:2111.01795.
Voigt, M. (2020). *Queer-spectrum student experiences and resources in undergraduate mathematics* (Order No. 27745009) [Doctoral dissertation, University of California, San Diego]. ProQuest Dissertations & Theses Global. http://www.proquest.com/docview/2435858844/abstract/608F986CC12E4095PQ/1
Weber, K. (2008). The role of affect in learning Real Analysis: A case study. *Research in Mathematics Education, 10*(1), 71–85. https://doi.org/10.1080/14794800801916598
Yoon, C., Kensington-Miller, B., Sneddon, J., & Bartholomew, H. (2011). It's not the done thing: social norms governing students' passive behaviour in undergraduate mathematics lectures. *International Journal of Mathematical Education in Science and Technology, 42*(8), 1107-1122.

BUILDING TOWARDS HARMONY BETWEEN IDENTITY AND MATHEMATICS SYNTHESIZING FIGURED WORLDS AND RIGHTFUL PRESENCE

Brinley Poulsen Stringer
San Diego State University
bstringer@sdsu.edu

Mariah Gabriella Moschetti
San Diego State University
mmoschetti@sdsu.edu

Gabriela Maria Hernandez
San Diego State University
gmhernandez@sdsu.edu

As identity becomes more discussed within education, it becomes crucial to understand identity in relation to power and social justice. In this paper, we discuss the identity frameworks of figured worlds and rightful presence to operationalize the critical consideration of identity within mathematics learning environments. We argue that by themselves, figured worlds and rightful presence have shortcomings that make it difficult to attend to power, and contextualize change, respectively. When considered in tandem, however, these two frameworks complement one another and build a stronger attention to identity that mobilizes student agency in the classroom. We call on educators and researchers to combine and utilize these frameworks to address the dissonance of identity that often occurs for marginalized students, thus building towards a greater harmony of identity and mathematics.

Keywords: Identity, Social Justice

Identity-- the notions and processes by which we recognize ourselves, and others recognize us, as a certain "kind of person" (Gee, 2000)--has become an increasingly discussed topic within education over the past several years. Within education, the development of identity has been articulated as the central purpose of education, where its outcome is, in fact, learning (Bracher, 2006; O'Donnell & Tobbell, 2007). Unfortunately, particularly in mathematics, many students are asked to give up parts of their identity in order to assimilate into the field. Thus, in order to ensure that students, especially those who are marginalized, are not leaving parts of themselves behind in order to study mathematics, it becomes critical to consider identity within mathematics education (Gutierrez, 2012).

As educators, we have the authority to position, or influence the ways that students are perceived, into a variety of identities, both academic and nonacademic (e.g., smart, lazy, collaborative, independent, etc.). The way that students are positioned to participate in mathematics learning environments affects how they learn math as well as how they perceive themselves as learners of mathematics (Turner et al., 2013). With this authority and understanding comes great responsibility. Students ought to be viewed as holistic beings with lives that extend beyond where they learn mathematics; thus, understanding identity is vital if we are to more fully understand and position our students in productive ways that help them succeed both in and outside these learning environments (Wilkes & Ball, 2020). Having a positive mathematical disposition impacts an individual's identity far outside of the walls of education. Positive dispositions towards mathematics–where one's identity is in harmony with the field as opposed to clashing with it– can lead to persistence, agency, and confidence, all of which are useful qualities for navigating everyday life (Bishop, 2012; Brown, 2004).

A focus on identity within mathematics education equips researchers with the analytic lens needed to comprehensively examine ethnicity, gender, culture, and power in mathematics (Brown, 2004; Martin et al., 2010). Identity research also helps us to understand the oppressive narratives, such as positioning students of color as "not being good at mathematics," that exists within mathematics classrooms (Reinholz, 2021). The dissonance that can arise from these dynamics can be leveraged to create harmony within mathematics by reauthoring narratives that allow us to more productively position students and help them succeed within mathematics (Gay, 2018; Gholson & Wilkes, 2017). In this paper, we will discuss two identity frameworks: figured worlds (Holland et al., 1998; Urrieta, 2007) and rightful presence (Calabrese Barton & Tan, 2019; Squire & Darling, 2013). We will discuss some of the main dimensions of these frameworks, and then synthesize the frameworks in tandem to highlight how these ideas, when applied together, can help educators and researchers bring identity as a tool into mathematics learning environments to build towards harmony between identity and mathematics, furthering the possibilities of learning for all students.

Background

While the field of mathematics has long been perceived as an objective field and thus a neutral space in which all students have equal access to succeed (Gutierrez, 2013), a large body of research exists that evidences the reality of mathematics functioning as a field which further marginalizes students (Dubbs, 2016; Gutierrez, 2017, 2019; Leyva et al., 2021; Reinholz, 2021; Solomon & Craft, 2015; Solomon et al., 2021). This body of research highlights dissonance (e.g., stereotype threat, mathematics anxiety, microaggressions, etc.) that exists for many students as they encounter mathematics through an identity-blind paradigm (Jett, 2019; Spencer et al., 2016).

Research shows that attending to the various aspects of students' identities can improve their learning opportunities and experiences (Gay, 2018; Bishop, 2012). In other words, attending to identity can serve as a way to create harmony and to mediate some of the challenges that otherwise exist within mathematics education. We present the identity frameworks of figured worlds and rightful presence to make sense of the vast concept of identity. Figured worlds is a commonly referenced identity framework (Hatt, 2007; Holland et al., 1998; Urrieta, 2007), and rightful presence is a more recent framework that contributes additional considerations to expand our understanding of identity and the possibilities that can be achieved by attending to it.

Figured Worlds

Figured worlds are "socially and culturally constructed realms of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al., 1998, p. 52). Common roles in a mathematics classroom might include instructors, students, and teaching assistants. Certain types of actions are typically associated with each of these roles, such as lecturing, working with peers, and answering questions, respectively. Certain values are also promoted within this context, predicated by the norms that are established, such as solving for correct answers or collaboration. These characters, actions, and values inform how individuals can construct their identities in a given context. Thus, figured worlds function as a production of identity in mathematics education and serve as a framework for understanding the context in which certain identities can or cannot emerge (Voigt et al., 2021; Urrieta, 2007).

Figured worlds as a framework often provides a lens through which to identify the dominant narratives surrounding identity within various contexts (Hatt, 2007; Urrieta, 2007). However, values, such as smartness and success, can be associated with differing actions or roles in various figured worlds, creating inconsistent expectations students must negotiate (Hatt, 2007; Zuckerman & Lo, 2021). While there can be dissonance between various figured worlds, the concept of *third spaces* helps to conceptualize how these worlds can be mediated (Calabrese-Barton & Tan, 2019). According to Jackson and Seiler (2018), third spaces are spaces which blend multiple figured worlds. This enables individuals to incorporate resources from various figured worlds in which they participate into a new blended space which is itself a figured world. Participating in these third spaces also allows new resources to be created within them. An example of this might be students forming a study group outside of class. This space creates the opportunity to form different norms (e.g., sharing personal experiences, playful interaction, etc.) compared to those of the class. Students can pull from the figured world of their classroom, as well as personal figured worlds, to mediate a learning environment that better suits their learning needs. The emergence of third spaces allows us the opportunity to consider, and perhaps even disrupt dominant narratives that are shaped by these interpretations of characters, actions, and values.

Rightful Presence

Rightful presence is a social justice oriented identity framework, which implies that it is a framework that elicits action toward change. This framework was first applied to the concept of sanctuary cities serving refugees (Calabrese Barton & Tan, 2019; Squire & Darling, 2013). In these cities, refugees are often viewed as guests and the city as a host. This guest-host dynamic is thought to perpetuate inequity since the city functions as a host which extends rights to its guests-- the refugees. However, this notion of extended rights means that refugees are extended the opportunity to assimilate into the city and culture which they are now living in; it does not empower the refugees to change anything, rather it maintains the status quo, which is often rife with injustices. Rightful presence challenges this notion of extended rights (Calabrese Barton & Tan, 2019; Squire & Darling, 2013). The rightful presence framework seeks for people to reauthor their rights within a space, making all characters fully fledged members of their own accord rather than guests who must assimilate to the dominant notions that currently define the space. There are three tenets to the rightful presence framework: 1) allied political struggle is necessary to reauthor rights, 2) rightful presence is claimed when (in)justices are made visible, and 3) rightful presence is a shared burden between the guests and host.

We now outline what these three tenets might look like in a mathematical learning environment. In a mathematics classroom, reauthoring rights could take the form of challenging what valid participation or meaningful learning looks like. The first tenet of rightful presence highlights that if allies, like teachers, are not engaged in this political struggle, it is difficult for reauthored rights to gain legitimacy in the classroom (Calabrese Barton & Tan, 2019). For instance, if a teacher helps students challenge what are considered valid ways of knowing and doing mathematics (i.e., encouraging partial solutions, valuing and discussing mistakes, etc.), they are engaging in a politically-oriented act of reauthoring rights in mathematics. The second tenet underscores the need to highlight the cultural historical relevance of a context in order to understand its current dynamics, and how these injustices must be both visible and present in teaching and learning in the classroom. Teachers can make injustices visible in their classrooms by highlighting the voices of their students, creating opportunities to find intersections between students' authentic lives and the discipline of math. These greater insights can help instructors find ways to position students with greater agency (Calabrese Barton & Tan, 2019). Lastly, the third tenet states that rights are not reauthored with top down approaches from teachers to students, rather they must be mediated by both groups working together. While allied political struggle from a teacher is necessary, the teacher alone cannot reauthor their students' rights; together, action must be taken by both the students seeking to reauthor their rights and the teachers who have the power to extend rights, thus disrupting the traditional power dynamics in a mathematics classroom (Calabrese Barton & Tan, 2019).

We now discuss our reasons for bringing the two frameworks together, and what we believe can be accomplished through their synthesis

Purpose and Aim

In this paper, we advocate for research that collates the theoretical frameworks of figured worlds and rightful presence. In particular, we see a need for researchers and educators to consider these perspectives within mathematics learning environments. The racist and sexist narratives that women and people of color are forced to navigate in their pursuit of mathematics can inhibit them from viewing themselves as mathematicians (Leyva et al., 2021; Reinholz, 2021). Thus, we see a need to study identity from a critical lens that both recognizes the unjust reality that exists and leverages the discomfort created by this recognition in order to make mathematics classrooms more equitable.

Figured worlds is a commonly used framework in identity-based research (Holland et al. 1998; Hatt, 2007; Urrieta, 2007). While this framework does discuss notions of power and acknowledges the potential for artifacts to be repurposed for liberation (Holland et al., 1998; Urrieta, 2007), we believe that the insights that figured worlds provide cannot be successfully operationalized to disrupt the status quo without a more critical lens. We argue that while the framework of figured worlds allows us to study how classrooms function within the status quo and how students negotiate their identities within them, rightful presence gives us the tools to work toward disrupting the status quo locally. We believe that considering the figured worlds of mathematics learning environments while seeking rightful presence allows researchers to situate their research within the broader sociopolitical issues that affect traditionally marginalized students while providing tools to work toward healing local injustices.

Synthesis

In this section, we discuss the shortcomings of figured worlds and rightful presence and affordances that the two frameworks provide one another. In particular, figured worlds as a framework falls short of explicitly challenging power dynamics while rightful presence can be challenging to contextualize. By considering these shortcomings, we see opportunities for these frameworks to complement and support one another. Specifically, figured worlds help to identify change that can occur via rightful presence, and rightful presence provides a way to navigate the tensions that are merely identified within figured worlds.

Shortcomings

Figured worlds fall short as a framework in the way it models power and its impact within society. The roles and actors in any given figured world are likely to have rank and status associated with them, creating a hierarchy through which power and privilege play out (Holland et al., 1998). Power and privilege are not preexisting in figured worlds, but are constructed relationally and made visible through interactions (Esmonde & Langer-Osuna, 2013; Holland et al., 1998). While the figured worlds framework does account for power, it is often presented in an amoral, objective way, such as by stating who the actors are, what their roles are, and what significance is bestowed upon their acts. While the framework acknowledges that the constraints in a figured world (such as the actors, roles and actions permitted) can paradoxically be repurposed for change and thus disrupt power dynamics, it does not explicitly provide a way to analyze how such repurposing takes place (Holland et al., 1998). While figured worlds has helped many to conceptualize identity, these holes surrounding the concept of power make it challenging to use this framework to model a dynamic that challenges the status quo.

Because rightful presence seeks to disrupt oppressive power dynamics (Calabrese Barton & Tan, 2019; Squire & Darling, 2013), many may find the framework challenging to contextualize in their own learning environments, which could be interpreted as a shortcoming. Knowing *how* to engage in political struggles, to make invisible injustices visible, and sharing burdens of injustice with those who are marginalized is a different task from simply knowing the importance of these tenets. A framework is only as useful as it is applicable, and rightful presence might require some scaffolding for many in order for the action-oriented tenets to be approachable.

Affordances

While figured worlds and rightful presence both have shortcomings of their own, we see these two frameworks working together and complementing one another, allowing new ways for educators and researchers to consider identity within mathematics. We will first discuss how figured worlds questions and makes explicit the context in which rightful presence needs to occur, and then we will discuss how rightful presence navigates tensions created via third spaces within figured worlds. These affordances provide opportunities to create greater harmony amid the dissonance that often surrounds identity in the world of mathematics because these synthesized ideas attend to power dynamics and their potential oppression within figured worlds and provide actionable steps towards rightful presence.

Figured Worlds for Rightful Presence. While rightful presence in and of itself can seem like a major jump from reality for some, figured worlds provides a way to mediate some of these tensions by asking questions about the roles, actions, and values at play within a mathematics learning environment, which can help bring to light the local injustices that exist within them. When considering rightful presence, Yeh et al. (2021) challenges us to ask the questions “what is mathematics?”, “who can do mathematics?”, and “where is mathematics done?” (p.4). We now use these questions to help us consider the roles, actions, and values that might exist within a mathematical figured world, thus enacting the frameworks in tandem.

When we address the question of what mathematics is, we consider the actions that are expected for individuals to be perceived as successful in math. Is mathematics the process of memorizing procedures? Is it a representation of the universe that allows us to abstractly problem solve? These questions, and their subsequent answers for various mathematical figured worlds then give rise to roles that are allowed to exist, answering the questions who can do mathematics and where. What does it mean to be a mathematics student? Does this identity exist only in the classroom or does it extend outside of school? Are students’ funds of knowledge from outside of class valued in class discussion or are they dismissed and seen as irrelevant (Esmonde & Langer-Osuna, 2013; Langer-Osuna, 2015)?

These questions that naturally arise when considering mathematical figured worlds help us to recognize how and where we can start to enact rightful presence. As we critically consider the ways that mathematics is expected to be done in a given context, we can start to see the ways that certain types of students might be positioned as smart and successful or otherwise (Hatt, 2007; Leyva et al., 2021). As we consider these types of roles that we position students to fulfill within the classroom, we can see the values that are being assigned to math. These critical considerations allow us to bring to light some of the injustices that otherwise go unseen within mathematics, bringing to life the second tenet of rightful presence. Our understanding of the figured worlds that exist within mathematics is what allows us to understand what rights need to be reauthored so that students can be positioned for further success within and beyond the classroom.

Rightful Presence for Figured Worlds. Rightful presence and its orientation toward the illumination of political struggles can help educators be more intentional about creating opportunities for third spaces to arise in the classroom. Because each figured world has its own characters, roles, and values, the convergence of figured worlds in these third spaces can create tensions if characters, roles, or values do not align from one figured world to the next (Jackson & Seiler, 2018; Zuckerman & Lo, 2021). Rightful presence helps bring to light the tensions that can emerge in the process of creating these third spaces, and can also appropriate ways to navigate these tensions. By challenging the status quo of given figured worlds, rightful presence carves a pathway of political struggle, making injustices visible, and sharing the burden of reauthoring rights.

When various figured worlds are recognized within a classroom, then rightful presence as a social justice framework can be operationalized simultaneously as a means to productively leverage tensions that might arise with the emergence of third spaces, allowing for new resources to also emerge that students can utilize in the classroom. Rightful presence does this first by highlighting and giving permission for the political struggle that is a part of this process. While change is not a comfortable process, educators and researchers can take comfort in knowing that engaging in political struggles is to be expected on the journey towards liberation, where education fundamentally should act as a tool and process of empowerment and liberation (Calabrese-Barton & Tan, 2019; Gutierrez, 2013). Rightful presence helps to highlight *why* political struggle is necessary by demanding that injustices at play be brought to light. Where educators might recognize tensions within classrooms, rightful presence can bring these issues to the forefront of attention so that the group (both hosts and guests) can collaborate together to navigate and mediate these challenges. This process of open communication and collaboration is in and of itself the beginning of reauthoring rights, allowing the tensions that inevitably exist within learning environments to serve a purpose. When educators open up these spaces to include students in these ways, the burden of change can be shared by all involved, allowing for more resources to be utilized in navigating tensions by way of third spaces. Once these third spaces have been cultivated through enacting rightful presence, students have an ideal environment that affords the benefit and utility of their full identities, rather than just fractions of them, within mathematics.

Discussion

We see important implications that can stem from drawing upon the figured worlds and rightful presence frameworks together. The figured worlds framework allows educators to characterize how the status quo operates within their classroom. A mathematics classroom exists as a figured world but also serves as a third space where additional figured worlds with highly established hierarchies and power structures, such as race and gender, intersect. While students can draw upon more resources when figured worlds come together like this (Calabrese-Barton & Tan, 2019; Jackson & Seiler, 2018), in mathematics classrooms, the dominant narratives in the figured world of mathematics that regard innate ability, intelligence, race, and gender actively position women and people of color as not belonging in mathematics classes, and even as being incapable of mathematics altogether (Leyva et al., 2021; Reinholz, 2021). Thus, characterizing the extant roles, actions, values, and dominant narratives surrounding student identity in a mathematics classroom reveals to educators what currently exists in these educational spaces, bringing awareness to where change can occur. Rightful presence then provides a student-centered lens oriented toward creating this change.

For researchers, applying these theoretical perspectives concurrently provides a powerful lens through which to analyze how students renegotiate their identities and reauthor their rights within educational contexts. This perspective could be powerful in revealing how problematic narratives within mathematics manifest locally within classrooms and how educators can most effectively leverage the insights figured worlds grant them while studying how the tenets of rightful presence give space for students to renegotiate their beliefs and reconfigure their figured worlds within the classroom (Calabrese Barton & Tan, 2020). Thus, lessons and learning environments can be designed with an orientation toward revealing current injustices in the classroom through the active analysis of figured worlds while the tenets of rightful presence can orient the lesson and learning environment toward remedying these injustices. This creates an explicit focus toward leveraging student agency to use what exists within figured worlds to disrupt and ultimately rewrite the roles, actions, and values within students' figured worlds, creating new counter-narratives for what it means to study mathematics and to be a mathematician.

While we do not claim that this critical lens of identity can be operationalized to change mathematics as a whole, we believe they can greatly assist educators and researchers to better understand how students negotiate their identity and leverage their agency within mathematics learning environments. This can help educators and researchers to then enact these ideas, designing and implementing mathematics lessons and contexts that treat students as holistic beings, creating space for students to mobilize their agency, and working toward disrupting systemic issues within mathematics, one classroom at a time.

Conclusion

In this paper we have discussed the importance of identity in mathematics education and the affordances of synthesizing the frameworks of figured worlds and rightful presence as a means to create harmony among the dissonance in mathematics that surrounds mathematical identity. By viewing mathematics learning environments as figured worlds, researchers can identify how the status quo and power dynamics operate within it. Through explicitly identifying the roles, actions, values, and dominant narratives in the classroom, the possibility for change emerges. By contextualizing the dynamics of injustice, rightful presence equips us with the tools needed to navigate tensions within these environments. Thus, through the illumination of injustices, spaces like classrooms can be cultivated in a way that mobilizes student agency in order to disrupt and reauthor the roles, actions, and values within the pre-existing figured worlds. Through this reauthoring, counter-narratives can be created, blazing the trail for a new conception of what it means to study and do mathematics, as well as what it means to be a mathematician. Together, figured worlds and rightful presence can inform us of *what is* within mathematics and empower us to build towards *what could be.* We call on educators and researchers to consider and apply these frameworks, in tandem as outlined, in their future work so that we might work towards creating mathematics education as a space of liberation.

References

Bishop, J. P. (2012). "She's Always Been the Smart One. I've Always Been the Dumb One": Identities in the Mathematics Classroom. *Journal for Research in Mathematics Education, 43*(1), 34–74. https://doi.org/10.5951/jresematheduc.43.1.0034

Bracher, M. (2006). Radical pedagogy: Identity, generativity, and social transformation. Springer.

Brown, B. A. (2004). Discursive identity: Assimilation into the culture of science and its implications for minority students. *Journal of Research in Science Teaching, 41*(8), 810–834. https://doi.org/10.1002/tea.20228

Calabrese Barton, A., & Tan, E. (2019). Designing for Rightful Presence in STEM: The Role of Making Present Practices. *Journal of the Learning Sciences*, *28*(4–5), 616–658. https://doi.org/10.1080/10508406.2019.1591411
Calabrese Barton, A., & Tan, E. (2020). Beyond Equity as Inclusion: A Framework of "Rightful Presence" for Guiding Justice-Oriented Studies in Teaching and Learning. *Educational Researcher*, *49*(6), 433–440. https://doi.org/10.3102/0013189X20927363
Dubbs, C. (2016). A Queer Turn in Mathematics Education Research: Centering the Experience of Marginalized Queer Students. *J. A.*, 8.
Esmonde, I. & Langer-Osuna, J.M. (2013). Power in Numbers: Student Participation in Mathematical Discussions in Heterogeneous Spaces. *Journal for Research in Mathematics Education, 44*(1), 288-315.
Gay, G. (2018). *Culturally Responsive Teaching* (3rd ed.). Teachers College Press.
Gee, J. P. (2000). Identity as an Analytic Lens for Research in Education. *Review of Research in Education*, *25*, 99–125. https://doi.org/10.2307/1167322
Gholson, M. L., & Wilkes, C. E. (2017). (Mis)Taken Identities: Reclaiming Identities of the "Collective Black" in Mathematics Education Research Through an Exercise in Black Specificity.*Review of Research in Education, 41*(1), 228–252. https://doi.org/10.3102/0091732X16686950
Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education?. In Equity in discourse for mathematics education (pp. 17-33). Springer, Dordrecht.
Gutiérrez, R. (2013). The Sociopolitical Turn in Mathematics Education. *Journal for Research in Mathematics Education*, *44*(1), 37–68. https://doi.org/10.5951/jresematheduc.44.1.0037
Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. *Philosophy of Mathematics Education, 32*(1), 1-34.
Hatt, B. (2007). Street smarts vs. book smarts: The figured world of smartness in the lives of marginalized, urban youth. *The Urban Review*, *39*(2), 145–166.
Holland, D. C., Lachicotte Jr., W., Skinner, D., & Cain, C. (Eds.). (1998). *Identity and agency in cultural worlds*. Harvard University Press.
Jackson, P. A., & Seiler, G. (2018). I am smart enough to study postsecondary science: a critical discourse analysis of latecomers' identity construction in an online forum. *Cultural Studies of Science Education, 13*(3), 761-784.
Jett, C. C. (2019). Mathematical Persistence Among Four African American Male Graduate Students: A Critical Race Analysis of Their Experiences. *Journal for Research in Mathematics Education*, *50*(3), 311–340. https://doi.org/10.5951/jresematheduc.50.3.0311
Langer-Osuna, J. M. (2015). From Getting "Fired" to Becoming a Collaborator: A Case of the Coconstruction of Identity and Engagement in a Project-Based Mathematics Classroom. *Journal of the Learning Sciences, 24*(1), 53–92. https://doi.org/10.1080/10508406.2014.944643.
Leyva, L. A., McNeill, R. T., Marshall, B. L., & Guzmán, O. A. (2021). "It seems like they purposefully try to make as many kids drop": An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics instruction. *The Journal of Higher Education, 92*(5), 784-814. https://doi.org/10.1080/00221546.2021.1879586
Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education, 3*(2), 12-24.
O'Donnell, V. L., & Tobbell, J. (2007). The Transition of Adult Students to Higher Education: Legitimate Peripheral Participation in a Community of Practice? *Adult Education Quarterly*, *57*(4), 312–328. https://doi.org/10.1177/0741713607302686
Reinholz, D. L. (2021). Interrogating Innate Intelligence Racial Narratives: Students' Construction of Counter-Stories within the History of Mathematics. *International Journal of Research in Undergraduate Mathematics Education*. https://doi.org/10.1007/s40753-021-00145-w
Solomon, Y., & Croft, T. (2016). Understanding undergraduate disengagement from mathematics: Addressing alienation. *International Journal of Educational Research*, *79*, 267–276. https://doi.org/10.1016/j.ijer.2015.10.006
Solomon, Y., Hough, S., & Gough, S. (2021). The role of appropriation in guided reinvention: Establishing and preserving devolved authority with low-attaining students. *Educational Studies in Mathematics*, *106*(2), 171–188. https://doi.org/10.1007/s10649-020-09998-5
Spencer, S. J., Logel, C., & Davies, P. G. (2016). Stereotype Threat. *Annual Review of Psychology*, *67*(1), 415–437. https://doi.org/10.1146/annurev-psych-073115-103235
Squire, V., & Darling, J. (2013). The "minor" politics of rightful presence: Justice and relationality in city of sanctuary. *International Political Sociology*, *7*(1), 59-74.
Turner, E., Dominguez, H., Maldonado, L., & Empson, S. (2013). English Learners' Participation in Mathematical

Discussion: Shifting Positionings and Dynamic Identities. *Journal for Research in Mathematics Education, 44*(1), 199–234. https://doi.org/10.5951/jresematheduc.44.1.0199
Urrieta, L. (2007). Figured worlds and education: An introduction to the special issue. *Urban Review*, *39*(2), 107–116. https://doi.org/10.1007/S11256-007-0051-0
Voigt, M., Rasmussen, C., & Martinez, A. E. (2021). The refiguring of students' mathematical identities: A mixed methods study of three tailored calculus courses. *International Journal of Mathematical Education in Science and Technology*, 1–21. https://doi.org/10.1080/0020739X.2021.1940331
Wilkes, C. E., & Ball, D. L. (2020). *How Are Black Learners Positioned in Mathematics Classrooms? What Do We Know and What Do We Need to Know?* [Preprint]. Open Science Framework. https://doi.org/10.31219/osf.io/ycv34
Yeh, C., Tan, P., & Reinholz, D.L. (2021). Rightful Presence in Times of Crisis and Uprisings: A Call for Disobedience. Equity & Excellence in Education, 1-14. https://doi.org/10.1080/10665684.2021.1951631
Zuckerman, A. L., & Lo, S. M. (2021). Transfer student experiences and identity navigation in STEM: Overlapping figured worlds of success. *CBE - Life Sciences Education, 20*(3). https://doi.org/10.1187/CBE.20-06-0121

“I WATCHED AS HE PUT THINGS ON THE PAPER”: A FEMINIST VIEW OF MATHEMATICAL DISCOURSE

Allison S. Theobold
Cal Poly – San Luis Obispo
atheobol@calpoly.edu

Derek A. Williams
Montana State University
derek.williams2@montana.edu

In this study we present results of a discourse analysis of the interactions between two partners, Uma and Sean, through a feminist lens. During roughly five hours of small group work in a teaching experiment, how each partner used language to position each other’s thinking as mathematically significant and establish a collaborative environment varied dramatically. Specifically, Uma shouldered the burden of continuously working to maintain collaboration, oftentimes at the expense of having her thinking positioned as mathematically significant. On the other hand, Sean regularly offered little opportunity for Uma to engage openly with his thinking, which ultimately constrained Uma’s opportunities to learn.

Keywords: classroom discourse; gender; attitudes, affect, and beliefs; equity, inclusion, and diversity

Numerous organizations invested in transforming mathematics education from predominately teacher-center towards student-centered advocate strongly for collaborative group work (e.g., Conference Board of the Mathematical Sciences, 2016; National Council of Teachers of Mathematics, 2014). Group work has been identified as an element of ambitious teaching in college calculus (Sonnert, Sadler, Sadler, & Bressoud, 2014) and has been associated with undergraduate students’ positive attitudes towards learning mathematics (Sonnert et al., 2014). Research focusing on how mathematical ideas emerge as collective ways of reasoning during whole class and small group discussions has demonstrated the importance of interacting with peers’ thinking for students’ own learning (Rasmussen et al., 2020), and aligned mathematical development with the nature of collective mathematical argumentation during class discussions (e.g., Rasmussen, Wawro, & Zandieh, 2015). Additionally, language does much more than establish normative ways of reasoning about mathematics. In fact, discourse allows participants to enact and construct their identities (Gee, 2014; Langer-Osuna & Esmonde, 2017). Therefore, discourse occurring within small groups simultaneously offers students opportunities to learn mathematics and establish their identities. This study presents a discourse analysis of interactions between two students, Uma and Sean, to investigate how their use of language influences opportunities for learning.

Review of Literature

Mathematical Identity and Classroom Discourse

Over the last twenty years, mathematics education research has moved beyond “gap gazing,” (Gutierrez, 2008) at static pictures of inequity, and toward research on the role of identity in teaching and learning mathematics. A variety of perspectives on identity formation have arisen, each emphasizing how learning in classrooms goes beyond understanding new concepts. Students also learn who they are and what they can and cannot do “with respect to the norms, practices, and modes of interaction” (Bishop, 2012, p. 36). Students’ identities are “dynamically negotiated,” representing a synthesis of who they are and who they have learned to be through interactions with others (Bishop, 2012, p. 38). Researchers have found the study of how students enact their identity at the microlevel, through classroom discourse, to elucidate explanations for the variability in mathematical identities within the same classroom.

Building on the framework proposed by Gee (2014), alongside Sfard and Prusak (2005), mathematics education researchers have used classroom discourse as a window into the formation and enactment of students' identities. Every instance of discourse in a mathematics classroom affords students the opportunity to negotiate their identities and respective social positions (Davies & Harre, 2001; Gee, 2014; Sfard, 2001; Wetherell, 2001). Early research on discourse focused on differences in students' desires when participating in group collaborations, finding that girls preferred to work cooperatively in groups, whereas boys "disliked working in groups because they felt it slowed them down" (Boaler, 1997, p. 297).

Langer-Osuna (2011) and Bishop (2012) shifted these conversations to focus on how students used language to position each other. Bishop (2012) described discourse between two students, Bonnie and Teri, and showed how Teri enacted the identity of the mathematical expert—speaking with authority and controlling activity and the uptake of ideas. Whereas Bonnie's discursive actions provided a window into the identity she inhabited and, in fact, helped to author—"dependent, mathematically helpless, and, at times, unknowledgeable" (Bishop, 2012, p. 57). Meanwhile, Langer-Osuna (2011) analyzed discourse to map how two students, acting as leaders in the same group, "developed opposite trajectories of identity" (p. 222). Brianna, originally publicly positioned by her teacher as an example of strong leadership, saw these same qualities positioned as inappropriate by her group. Whereas, Kofi, also positioned as knowledgeable but who was initially unengaged in the group's activities, found himself positioned as an authority figure by his group. Thus, discourse enables participants to establish and enact mathematical identities and also works to position members with (or without) power and authority.

Power & Authority in the Mathematics Classroom

Research on student interactions during small group problem solving has further demonstrated how students' mathematical identities, power, and authority can influence students' opportunities to learn (Chohen, Lotan & Catanzarite, 1990; King, 1993). Group collaborations fall prey to issues of status – the distribution of power among peers. Langer-Osuna and colleagues have documented how power and authority is negotiated among groups of students and is directly connected with how responsibilities for shared work are distributed (Engle et al., 2014; Langer-Osuna, 2016). Students as young as fifth grade have been seen to compete for *intellectual* and *directive* authority, which are heavily influential in determining whose solution strategies are used, whose ideas are assumed to be mathematically correct, and how effort is distributed among partners (Langer-Osuna, 2016). Johnson (2002) states that fights over who gets to speak and whose words are recognized are indicative of power and status.

Historically, male students have been positioned with power in mathematics classrooms (Webb & Kenderski, 1985; Wilkinson, Lindow & Chiang, 1985). Forgasz and Leder (1996) observed two boys refusing to work with their female group members when collaborating on a project with a "female-stereotyped context." Instead, the boys "used sexist language and taunted the two girls" (p. 163), insulting them while the girls completed the project entirely on their own. Jill voiced her disdain for the boys' behavior yet voiced how she and Cara didn't have any recourse— "If we didn't do it, it wouldn't get done" (p. 165).

These chiding remarks by group members took on a different flavor with Brianna (Bishop, 2012), who took on an early leadership role in her group. Less than halfway through the project, however, "group members increasingly positioned Brianna as bossy, claiming that Brianna was overstepping her authority" (p. 212). Two boys, repeatedly interrupted Brianna and scolded her for "being bossy," positioning themselves as a unified front against Brianna's bossiness. While the boys assaulted Brianna's leadership with criticism, they actively positioned Kofi as the group's leader, asserting his ideas deserved attention and seeking out help from Kofi. The leader-related utterances from Kofi were taken up by the group over twice as often as Brianna's, and his utterances were ignored or rejected nearly 25% less often than Brianna's.

These studies offer two perspectives for how students' negotiated identities go beyond the social interactions in the day-to-day classroom. Rather, student's identities also depend on the broader context of a student's life experiences (Gee, 2014; Wenger, 1998) and society's expectations for who they are and how they behave. Society grants students who are members of certain social categories with the right to hold positions of power over others. For decades we have seen research comment on how women—expected to be polite, accommodating, and reassuring—are not afforded positions of power in mathematics classrooms. Discourse has allowed us to see how these subservient positions manifest in classrooms, where women and girls do not have access to the conversational floor, are not able to decide what is correct, and are not seen to contribute meritorious ideas.

Theoretical Framework: Feminist Theory

Boaler (1997) admonished mathematics education as a discipline to "stop blaming girls for the underachievement and non-participation caused by the educational system forced upon them" (p. 304). The root of this underachievement and non-participation, however, potentially lies outside the educational system. The goal in mathematics education is for every student to have access to the resources and opportunities to be successful (NCTM, 2000). However, if discourse is viewed as a resource, then Jill, Cara (Forgasz & Leder, 1996), and Brianna (Bishop, 2012) exhibited pervasive differences in their discursive opportunities.

The gendered macroaggressions Brianna (Bishop, 2012) faced caused her to disengage from her group's mathematical activity and disassociate herself from her ambitious mathematical identity (Bishop, 2012). Villain (1998) argues that experiences such as those seen by Jill and Cara (Forgasz & Leder, 1996)—having their contributions ignored—implicitly conveys the message that their ideas have lesser value. Researchers have discussed these inequitable discursive opportunities as issues of power and authority (Langer-Osuna, 2016), but this research has not named these actions for what they are—*oppression*. A feminist standpoint, however, demands a commitment to understanding and challenging systems of oppression (Crasnow, 2014; Harding, 2004; Intemann, 2010; Intemann, 2016; Wylie, 2003). Adopting a feminist standpoint requires "studying from the margins out," (Intemann, 2016, p. 269) to reveal how power structures shape and limit the scientific phenomena under investigation.

This research expands these political discussions to focus on how discursive patterns between two undergraduate students positioned one student's thinking as significant and restricted another student's opportunities to learn.

Methods

The purpose of this study is to investigate the learning opportunities afforded to two students, Uma and Sean, while they solved problems during a small-scaled teaching experiment (Cobb., 2002). In particular, we pay specific attention to how the students used language and actions to, (1) position each other's thinking as significant, and (2) establish a collaborative environment. Each of these "building tasks" (Gee, 2014) are accomplished through discourse and allow us to understand the learning opportunities afforded to Uma, Sean and their group as a collective.

Setting and Participants

Uma and Sean were among six students participating in a small-scaled teaching experiment covering concepts of logarithms. The experiment consisted of five sessions each lasting one hour. Uma and Sean elected to work with each other. The two did not know each other prior to the experiment. Uma, a traditionally aged white women, had recently changed her major to secondary mathematics education. Sean, a non-traditionally aged white man, had returned to school to major in secondary mathematics education after a military career.

Data Collection and Analysis

Data collected for this study include recordings from the teaching sessions and individual semi-structured recall interviews. Interviews took place after the teaching experiment.

To analyze the discourse between Uma and Sean, we first watched the recordings from the sessions and identified *meaningful interactions*, which we define as an exchange of one or many talk-turns around a common on-task topic, idea or subject. Meaningful interactions may contain intervals of silence as long as subsequent talk-turns continue with the same subject and no changes to conversants occur. We considered instances when the teacher-researcher joined or left the conversation to constitute new interactions. Then, we transcribed all meaningful interactions and coded each conversant's use of language with verbs (e.g., revoicing, clarifying, explaining). Then, we analyzed the groups of codes emerging in the meaningful interactions to characterize how students positioned each other's mathematical thinking as significant, and whether their use of language was open or closed to collaboration. For example, a student asking, "Before, you were thinking..., do you think that still applies?" is using language to revoice their partner's previous mathematical thinking, positioning it as significant, and is also inviting collaboration. We also noted whether Uma or Sean shared ideas or strategies in each interaction.

Participants were shown three clips during the recall interviews. The first clip depicted an interaction where the interviewee had expressed either a high- or low-point in their engagement (see Williams et al., 2020). The second clip depicted the interviewee sharing a mathematical idea, and the third clip depicted an interaction where Uma shared a potentially meaningful mathematical idea that was not pursued on Sean's authority. We judged Uma's contribution to be *potentially meaningful* because it could have led to a solution to the problems being solved based on our evaluation. The first two clips were unique to each student. The third clip was the same. We use interview transcripts as evidence to support our claims from the discourse analysis.

Results and Discussion

Uma and Sean participated in 67 meaningful interactions while working together during the teaching experiment. Uma shared her thinking or a solution strategy in 20 interactions, while Sean presented his thinking or strategy in 54 interactions. From these interactions, Sean positioned Uma's thinking as significant 6 times, compared to the 38 instances in which Uma positioned Sean's thinking as significant. The common clip presented to both students during their recall interviews emerged as an interaction marking a shift in Uma's behavior while working with Sean. In what follows, we describe the progression of Uma's use of language for accomplishing the building tasks before, during, and after clip 3, as compared to Sean's.

Prior to clip 3, Uma was much more forthcoming in sharing her own mathematical ideas or strategies for solving problems with Sean, even though she self-described having low confidence stemming from how long it had been since she studied logarithms. Uma would share her thoughts without solicitation from Sean, and would respond to Sean with questions, even when presenting an argument for her own thinking. These uses of language are exemplified in the following interaction, which took place half-way through the first teaching session when students were attempting to create a timeline that accurately positioned a set of events (note, we use dashes in the transcripts to indicate when a student is cutoff or stops speaking abruptly).

Sean: So, I kinda ran out of room here, but I just went 10 to the 10th, 10 to the 9th... [pointing to his paper and turning to Uma] and then will fit those within there.
Uma: Oh ok! And then we'll fit them. Yeah [both work independently] … [Looking at Sean's paper, thinking aloud] 10 to the 6th
Sean: Yep, 10 to the 5th, and then I'm going to start my line. Over here, I'm actually going to extend it on- [continuing to work on his paper]
Uma: [9 seconds, looking at Sean's paper] and then you're adding a little bit more?
Sean: Mmhmm [7 seconds] Well, and that's turning into a scale that I recognize [bobbing, smiling slightly, looking over his glasses at Uma].
Uma: Yeah, so then that's- it goes to 10 to the 4th and then that's going to be 10 to the 3rd, and then to the 2nd (?) [looking at Sean]
Sean: Yep. And then I just went on out to-
Uma: 10 to the 0, yeah to make it easier [4 seconds]. So then, "Now" falls in at the 0 mark, right? Right? Would that be where we have "Now?"---
Sean: That's "Now," yep [working on his own paper]
Uma: [5 seconds] and then 10 to the 2nd is 100, right?
Sean: Yeah, so you want to kinda like- [continuing to work on his paper]
Uma: So then you have- [working on her own paper]
Sean: So if you wanna like- [7 seconds] the difference from here is actually 900 years, right.
Uma: Yeah, but then with each one the years are going to be more [gesturing "expanding" with her hands]
Sean: Yep. Yep.
Uma: Yeah, that's what I was thinking. I was like- because 500 is going to be- this is 1000, so your 500 is kinda going to be like half-way, right?
Sean: Yeah [slowly and hesitant]
Uma: it should be like here, 500, and then this will be 1000

This interaction begins when Sean states how he is creating his timeline using "eras" based on powers of 10. Uma positions this approach as signification by agreeing with the approach, revoicing his thinking (e.g., "10 to the 0 to make it easier..."), asking clarifying questions, and implementing his approach on her own paper. Uma's use of clarifying questions and posing her argument about where 500 should be positioned on the timeline as a question work to establish an open collaboration between the two students. Alternatively, Sean does not position Uma's thinking about where to position 500 as significant (which he disagrees with in a later interaction), nor does he elect to have a conversation about her suggestion to achieve consensus. Instead, Sean's responses are frequently short acknowledgements ("Mmhmm") that are closed to further collaboration. In fact, during his interview, Sean explained, "I don't think we were working together, we agreed on things, but I don't think we were truly working in unison."

Sean's limited attention towards Uma's mathematical thinking and frequently closed use of language persisted throughout the teaching experiment. In some interactions, Sean would elicit Uma's thinking vaguely—e.g., "What are you thinking?"—while in others Sean would invite Uma to read the next prompt, locate events on the problem sheet, or "unvite" Uma as a way to work independently—e.g., “we’re going to do the math to see what it actually looks like, right?”. However, Sean’s use of language was not exclusively closed towards collaboration. Sean would initiate interactions by inviting Uma to share her thinking, or present his own thinking as an argument (i.e., Toulmin, 1963). Presenting an argument, over stating claims without evidence, is considered an open form of discourse because of the benefits to engaging with the reasoning of others (Rasmussen et al., 2020). In total, Sean’s use of language was open to collaboration in 27 interactions and closed in 37, where it is possible for both to occur in the same interaction. Language that functions to open interactions for collaboration versus that which closed interactions was one primary difference in the discourse between Uma and Sean. Uma never used language that closed collaborations, while Sean’s language was more closed than open.

The patterns of open and closed uses of language between the two students did not change much throughout the teaching experiment. However, Uma’s willingness to share her thinking or solution strategies changed dramatically after clip 3, which took place as the last interaction from session 2. Leading up to this interaction, Uma and Sean were discussing a prompt designed to have students compare two large values, represented as events on a timeline constructed with an exponential scale with base 2. Uma and Sean were discussing the prompt, “Consider events, A which would be positioned at 16.005 and B which would be at 18.013. Which event took place longer ago and by how much? Explain.”

Uma: [reading the prompt] Which event took place longer ago and by how much? So, did that just mean that the one point is going to be over here and the other point is going to be over here? Is that what that-

Sean: [looking back and forth between papers] Yep. Yep.

Uma: [quietly] So, B... and this is A. It says, which one of these took place longer ago. I mean, it would just be B, right?

Sean: It would be B, yep.

Uma: And that’s just because it’s farther back on the timeline.

Sean: [nodding, using his calculator] Yep. And then it’s asking by how much.

Uma: So, then you’d have- you would just subtract 18- [gets quite]

Sean: [11 seconds] So that would give us the, the- [using his calculator] I don’t know if there’s a simpler way of doing that... [4 seconds]. So about 199,000 years. Does that seem right?

Uma: I think- Does it- so I’m just trying to remember like with exponents, can you, if it had the same like number, you can- can you take- so, like if we were to subtract because what I had done- because if you had subtracted the one and then the other one, and if you were to put 2 to the power of that (?)

Sean: So by inspection that’s going to be-

Uma: be a little bit bigger, right?

Sean: be very, very small. Because that’s going to be almost 4. Right, because 2 squared. But I think you’re on to something. There’s probably a relationship there. Um- [working on his own paper]

Uma: [40 seconds, quietly thinking aloud, working on her own paper] 2 to the 18[.013] minus 2 to the...

At the end of this interaction, Uma can be heard implementing Sean’s subtraction approach for determining the amount of time between events A and B from the prompt, which takes place after she presents an argument for using a multiplicative approach that relies on properties of exponents. Implementing Sean’s strategy and foregoing her own positions Sean’s thinking as significant. Alternatively, Sean does not position Uma's thinking as significant in this interaction. In fact, there are two instances when Uma presents an argument (i.e., for event B taking place longer ago, and for how to compare the events using exponents), where Sean's responses are closed. Sean indicates agreement with Uma's first argument in a one-word response. Then, after Uma's second argument, Sean counterargues for why her response is different from his, and ends the interaction stating, "... but you're on to something..." while proceeding to work on his own paper. Sean's limited willingness to further consider Uma's approach while simultaneously comparing her solution to his own suggests he is positioning his own thinking as more significant than Uma's. Additionally, that Uma waited for 40 seconds, looking at Sean, before implementing his strategy may suggest she wished for more discussion.

Prior to and including the interaction from clip 3, Uma shared her thinking in 12 out of 27 meaningful interactions. Sean positioned Uma's thinking as significant in 4 of these instances. Following the interaction from clip 3, Uma only shared her thinking in 7 out of 40 interactions. Also, Uma adapts the way she positions Sean's thinking as significant in future interactions. Prior to clip 3, Uma would agree with, revoice, and implement Sean's thinking as a way of positioning his thinking as significant. Following clip 3, Uma also routinely invites Sean to collaborate with her by posing questions that also revoice his thinking. In other words, Uma found that Sean was a more willing collaborator when she initiated interactions with his thinking. For example, Uma would ask questions such as those in this interaction from the third session.

Uma: Well, it said that each segment is equal, right? So you're saying that this part [pointing to Sean's paper] is the same amount of space as this part, right?

Sean: I think from here there is about the same as there.

Uma: Yeah, okay, so now the question is, 'when you take the overlapped time, does this still preserve the same amount?'

Sean: As that 10 to 1? That is the question. I don't think that it is.

Uma: Mmhmm. Do you have any thoughts leaning you more towards why not?

Sean: [15 seconds] if we graphed them longer, we would end up with something like that. Um, we're saying then that is equal with that, or this is 10 times that, but I think-

Uma: that overlap takes away from it being able to be 10 times that, right?

Sean: Yeah, I just don't know how to- I guess here's what I would say. We know that if this were true, then shouldn't I be able to slide it a little bit further and a little bit further, and then I would have them on top of each other, and they are clearly not 10 to 1 then. You know what I mean? If I keep overlapping more and more, then they become the same time period. So, I don't--- any overlapping, I think is-

Uma: it conflicts, I guess. Yeah, that's a good way to-

Sean: does that make sense?

Uma: Yeah, that does make sense. What do you think about the overlapping? Do they still have a relationship similar to what you were looking at before, or how do you even compare their relationships with the overlapping?

Sean: Well, I think we are subtracting exponents here.

Uma: Mmhmm, and then you'd be comparing from there (?)

From Uma's interview, we know that she valued working with Sean because, "if I was struggling on something, we could talk it out, and I could like go somewhere with it." However, when reflecting on clip 3, Uma also explained, "in that moment, I felt like I actually knew what was going on... and it actually start[ed] to make sense to me." In this way, the interaction from clip 3 emerged as pivotal in how Uma positioned Sean's thinking as mathematically significant and maintained a collaborative working environment. Moreover, Uma appears to have made a conscious decision to sacrifice being viewed as smart by Sean in favor of maintaining collaboration, which proved productive when she positioned Sean's thinking as significant.

Unfortunately, Uma's approach worked. Sean did not appear to value Uma's mathematical contributions, and reflected on this in his interview, "I kind of felt like I wasn't getting much from her." Sean also did not place the same value on working collaboratively as Uma. It is possible that Sean's assessment of Uma's mathematical ability influenced his use of language to position her thinking as significant (or not) and to establish a collaborative environment. However, as demonstrated subtly in the previous example, Sean also began using strategies that involved properties of exponents ("Well, I think we are subtracting exponents here."). In fact, by the end of the last session, Sean posed an argument almost identical to the one Uma shared in the clip 3 interaction. Sean also reflected on this when discussing clip 3 in his interview, "... she's probably contributing more than I gave her credit for."

Conclusion

The discourse patterns emerging between Uma and Sean showcase how Uma took on the responsibility of establishing and maintaining a collaborative environment at the expense of having her own mathematical thinking positioned as significant. We believe Uma made this choice consciously, based on her reflection during the discussion of clip 3 in her interview and in the clear change in her use of language in all subsequent interactions. Uma took on significant cognitive load by deciding whether to share her thinking or create a collaborate environment by constantly posing invitations to Sean. We can imagine how tiresome it must have been for Uma to seek collaboration in this way. In fact, Gholson & Martin (2019) suggest these tedious micro-confirmations "might be experienced as pain" (p. 400). Moreover, seeking out these micro-confirmations, required Uma to "fundamentally distrust her own thinking and intuition" (p. 401), which could have contributed to her sense of co-dependence on Sean. In fact, in the interview, Uma stated, "I was looking to Sean to set an example of what his thoughts were and where he was going...I was watching as he put things on the paper."

Critically, Sean positioned Uma's thinking as significant in six of the 20 times Uma shared her thinking. This is not to say that Uma's ideas were inferior to Sean's, rather Uma's suggestion during clip 3 ultimately became the group's concluding "solution," nearly three hours after Uma's posed the argument to Sean. These constant negations of Uma's thinking as significant could be considered a type of microaggression (Sue et al., 2007).

To be clear, Sean was not outwardly harmful to Uma. In fact, their interactions may appear as "typical" or benign when viewed individually or through a non-critical lens. However, this is precisely our concern. The "typical" gendered interactions students such as Uma, Brianna (Bishop, 2012), Jill, and Cara (Forgasz & Leder, 1996) experience are oppressive and limit their opportunities to learn. Strategies for combatting such oppression consist of establishing classroom norms and cultures that value all mathematical thinking from each and every student such as rough draft thinking (Jansen, 2020) and holding all students responsible for actively engaging with their classmates' thinking and how that thinking contributes to problem solving (e.g., Rasmussen et al., 2020). Moving forward, our future research will investigate classroom and small group norms that elevate all students.

References

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education, 43(1)*, 34-74.

Boaler, J. (1997). Reclaiming school mathematics: The girls fight back. *Gender and Education, 9(3)*, p. 285-305.
Cohen, E. G., Lotan, R., & Catansarite, L. (1990). Treating status problems in the cooperative classroom. In S. Sharan (Ed.), *Cooperative learning: Theory and research* (pp. 203-229). Prager Publishers.
Conference Board of the Mathematical Sciences. (2016). Active learning in post-secondary mathematics education.
Crasnow, S. (2014). Feminist perspectives. In N. Cartwright & E. Montuschi (Eds.). *Philosophy of Social Science* (pp. 145-161). Oxford, UK: Oxford University Press.
Davies, B., & Harré, R. (1999). Positioning and personhood. In R. Harré & L. van Langenhove (Eds.), *Positioning theory: Moral contexts of intentional action* (pp. 32–52). Malden, MA: Blackwell.
Dweck, C. S. (2008). *Mindsets and math/science achievement*. New York: Ballantine Books.
Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 time more likely to leave STEM pipeline after calculus comparted to men: Lack of mathematical confidence a potential culprit. *PLoS ONE*, *11*(7).
Engle, R. A., Langer-Osuna, J. M., & McKinney de Royston, M. (2014). Toward a model of influence in persuasive discussions: Negotiating quality, authority, privilege, and access within a student-led argument. *Journal of the Learning Sciences, 23(2)*, 245-268.
Fennema, E., & Sherman, J. A. (1978). Sex-related differencs in mathematics achievement and related factors: A further study. *Journal for Research in Mathematics Education*, *9*(3), 189–203.
Forgasz, H. J., & Leder, G. C. (1996). Mathematics classrooms, gender and affect. *Matheamtics Education Research Journal, 8(1),* 153-173.
Gee, J. P. (2014). *An introduction to discourse analysis theory and method* (4th ed.). New York, NY: Routledge.
Gholson, M. L. & Martin D. B. (2019). *Blackgirl face: Racialized and gendered performativity in mathematical contexts*, 51, 391-404.
Good, C., Rattan, A., & Dweck, C. S. (2012). Why do women opt out? Sense of belonging and women's representation in mathematics. *Journal of Personality and Social Psychology*, *102*(4).
Gutierrez, R. (2008). A "gap gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Eduction, 39(4)*, 357-364.
Harding, S. (2004). A socially relevant philosophy of science? Resources from standpoint theory's controversiality. *Hypatia, 19(1),* 25-47.
Holmegaard, H. T., Madsen, L. M., & Ulriksen, L. (2014). To choose or not to choose science: Constructions of desirable identities among young people considering a STEM higher education programme. *International Journal of Science Education*, *36*(2), 186–215.
Intemann, K. (2010). 25 years of feminist empiricism and standpoint theory: Where are we now?. *Hypatia, 25(4)*, 778-796.
Intemann, K. (2016). Feminist Standpoint. In L. Disch & M. Hawkesworth (Eds.), *The Oxford handbook of feminist theory* (pp. 261-282). Oxford, UK: Oxford University Press.
Jansen A. (2020). *Rough draft math: Revising to learn.* Stenhouse Publishers.
Johnson, B. (2002). *Introducing linguistics: Vol. 3. Discourse analysis.* Malden, MA: Blackwell.
King, L. H. (1993). High and low achievers' perceptions and cooperative learning in two small groups. *The Elementary School Journal, 93(4)*, 399-416.
Langer-Osuna, J. M. (2011). How Brianna became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education, 11(3)*, 207-225.
Langer-Osuna, J. M. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem solving. *Mathematical Thinking and Learning, 18(2)*, 107-124.
Langer-Osuna, J. M., & Esmonde, I. (2017). Identity in research on mathematics education. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 637–648). NCTM.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.
National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: National Council of Teachers of Mathematics.
Rasmussen, C., Apkarian, N., Tabach, M., & Dreyfus, T. (2020). Ways in which engaging in someone else's reasoning is productive. *The Journal of Mathematical Behavior, 58,* 1-23.
Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259–281.
Rowe, K. J. (1988). Single-sex and mixed-sex classes: The effects of class type on student achievement, confidence and participation in mathematics. *Australian Journal of Education*, *32*(2), 180–202.
Sonnert, G., Sadler, P. M., Sadler, S. M., & Bressoud, D. M. (2014). The impact of instructor pedagogy on college

calculus students' attitude toward mathematics. *International Journal of Mathematical Education in Science and Technology*, *46*(3), 370–387.
Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics, 46*, 13-57.
Sfard A. & Prusak A. (2005). Telling identities: In search of an analytical tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34(4)*, 14-22.
Sue, D. W., Capodilupo, C. M., Torino, G. C., Bucceri, J. M., Holder, A, Nadal, K. L., et al. (2007). Racial microaggressions in everyday life: Implications for clinical practice. *American Psychologist, 62(4),* 271.
Toulmin, S. E. (1958). *The uses of argument* (Updated 2003). Cambridge University Press.
Villan, V. (1998). *Why so slow? The advancement of women*. Cambridge, MA: MIT Press.
Webb, N. M. & Kenderski, C. M. (1985). Gender differences in small-group interaction and achievement in high- and low-achieving classes. In L. C. Wilkinson & C. B. Marrett (Eds.), *Gender differences in classroom interaction* (pp. 209-236). New York: Academic Press.
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, England: Cambridge University Press.
Wetherell, M. (2003). Themes in discourse research: The case of Dianna. In M. Wetherell, S. Taylor, & S. J. Yates (Eds.). *Discourse theory and practice: A reader* (pp. 14-28). Thousand Oaks, CA: Sage.
Wilkinson, L. C., Lindow, J., & Chiang, C. P. (1985). Sex differences and sex segregation in students' small-group communication. In L. C. Wilkinson & C. B. Marrett (Eds.), *Gender influences in classroom interaction* (pp. 185-207). New York: Academic Press.
Williams, D. A., López Torres, J., & Barton Odro, E. (2020). Interactions between student engagement and collective mathematical activity. In S. S. Karunakaran, S. Cook, H. Soto, & M. Wawro (Eds.), *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 655-663).
Wylie, A. (2003). Why standpoint matters. In R. Figueroa & S. Harding (Eds.), *Science and other cultures: issues in philosophies of science and technology* (pp. 26-48). New York: Routledge.

OPPORTUNITIES AND LIMITATIONS OF POETIC TRANSCRIPTION AS A CRITICAL QUALITATIVE METHODOLOGY FOR MATHEMATICS EDUCATION

Rachel Tremaine
Colorado State University
rachel.tremaine@colostate.edu

This work presents an exploration of poetic transcription as a promising critical methodology for use in mathematics education research. I make the case for mathematics education as a specific space in which such methodology could be valuable, and provide a use case for poetic transcription through the poetic analysis of three students' definitions of student success within mathematics. Following a reflection on some of the critiques and limitations of re-presenting data in this way, I conclude that poetic transcription offers mathematics education an imaginative and simultaneously grounded method of understanding, while still necessitating critical thought and methodological malleability.

Keywords: Research Methods, Ethnomathematics

As poetic transcription gains popularity and credibility in research spheres, it is prudent to investigate its efficacy in the context of mathematics education. Poetic transcription draws from the literary tradition of a *found poem*, in which excerpts from existing written media are juxtaposed to create poetry with distilled meaning. In poetic transcription, participants' words from interview or focus group transcripts are pulled in such a way as to honor and reflect the participant's meaning and narrative style (Glesne, 1997). Prendergrast (2009) emphasizes that these poetic transcriptions are built from the participants' words but filtered and "re-presented" through the lens of the researcher by referring to these poems as "participant-voiced poems." Via use of member-checking practices and researcher-participant co-construction of such poetry, poetic transcription can provide direction for research that is "genuinely inclusive and democratic" (Levinson, 2020, p. 195). Further, via the artistic embodiment of experiential knowledge, poetic transcription can present nuances of qualitative data in a way that the quoting and analysis of traditional prose may not, particularly when viewed through a lens of feminist theory (Faulkner, 2018) and critical qualitative research (Burford, 2018; Faulkner & Cloud, 2019; Keith Jr. & Endsley, 2020).

Poetic transcription is already utilized in meaningful ways in other educational spheres (see Mercer-Mapstone et al., 2019; Thunig & Jones, 2020; West & Bloomquist, 2015). Mathematics' well-chronicled existence as a racialized and gendered space which privileges White, masculine, Western ideology (Battey & Leyva, 2016; Gutiérrez, 2018; Leyva et al., 2021; Martin, 2006) makes for a particularly potent case in which critical methodology—that which not only uplifts but foundationally gives power to the voices present in its work—is necessary to explore (Skovsmose & Borba, 2004). As research in this field more closely seeks to align itself with critical perspectives, our own research methodologies should not be above the kind of critical questioning that we ask of mathematical content and practices. Poetic transcription offers opportunities to introduce a humanizing lens to research, center participant voices, and disrupt traditional hierarchies (Fernández-Giménez et al., 2019). If these are goals explicitly stated for our research foci in mathematics education, why would they not also be goals we bring into our methodological practices?

A Use Case for Poetic Transcription in Mathematics Education Research

As asserted by Prendergast (2009), poetic transcription is best utilized when the topic in question "lead[s] into into the affective experiential domain" (p. 546). O'Shea and Delahunty (2018) note that student-constructed definitions of student success are "embodied and emotional," and in this way they are ripe for exploration of and expression through poetic transcription. In the spring of 2021, I conducted semi-structured interviews of five first-year undergraduate women, all of whom identified as First-Generation, Pell-grant eligible, and/or women of color, with the intent of gaining insight into how these women define what it means to be "successful within mathematics."

Each transcript contained between ten and twenty minutes of spoken text, in which the women narrated a time in which they experienced success within mathematics and responded to the question "how would you define student success in undergraduate mathematics?" As in line with previous work on students' definitions of *student success* being nuanced and varied (see Weatherton & Schlusser, 2020, for a comprehensive overview), these women each carried and shared with me diverse experiences and definitions of this amorphous term. Quotes from several of these women students have been previously highlighted in advocacy for the necessity of explicit definitions of student success and acknowledgment of their nuanced and expansive nature with mathematics education (Tremaine, 2021). The purpose of the present work is to contemplate poetic transcription as a viable methodology in mathematics education; as such, I am not including the contextualization or discussion of these data and how they are situated in broader societal issues of how success within mathematics is conceptualized and valued, and by whom—that is outside the scope of this brief report.

My poetic transcription process was primarily informed by my readings of Glesne (1997) and Prendergast (2009). Both of these authors acknowledge that there exist a myriad of ways in which to engage with poetic transcription; I summarize here my methods, but explicitly note that because poetic transcription is inherently a creative process, the process which worked for me in the context of this affective domain of mathematics education research may not be appropriate for other explorations in which poetic transcription might be utilized.

The first phase of my process was an initial parsing of the data, in which meaningful phrases were highlighted and excerpted from the entire interview transcript. This parsing was done in careful conjunction with a revisitation of the audio recordings of the interviews, allowing for attention to vocal emphasis, and is reflective of Lahman et al.'s (2010) concept of "poking around poetically" and Glesne's (1997) concept of "sifting through the data". This selection was based upon my own recollections and perceptions of what the participant emphasized, and what phrases might best represent the meaning from longer narratives or passages. For example, Isabel expressed that she found mathematical success in being able to practically apply geometry skills to make buying fabric for her quilting efficient. While this anecdote was discussed at length, I found the phrase "I really enjoyed being able to know how much fabric to get" an effective and important phrase in her longer narrative, and thus it was selected in this parsing of the data. I also attended to phrasing which I found particularly poetic—sequences of words which gave myself as a listener pause for thought, or provoked imagery and emotion, and perhaps contained deeper meaning. Kenzie referred to her achievement of her definition of success as a "breath of fresh air"; this phrase's attribution to the acquisition of success provided vivid imagery and captured some of the "embodied and emotional" nature of student success definitions.

Once these phrases were distilled, the second phase entailed juxtaposing them in a variety of ways that were reflective of poetic structure. This proved to have many iterations, and resulted in two prominent methods of grouping meaningful phrases: grouping for structure, and grouping for theme. Thematically, certain phrases were grouped based on the topic or thread of thought they related to (i.e. a group of phrases which all related to success in math as pragmatic application). Structurally, phrases were grouped based on rhetorical parallelism (for example, grouping Kenzie's words of "that's where I want to be" and "that's what I was reaching for"). Phrases were often used multiple times in various groupings. These poetic juxtapositions then enabled the creative composition of poems, which were edited and re-edited in additional consultation with the full interview transcript to ensure fidelity to the individual's narrative. Poems and interview transcripts were then revisited several months after their initial composition, and edited in small ways as felt needed.

The final phase of this analysis consisted of a member-checking process, engaged in via email with each participant. Participants were provided a brief of the poetic transcription methodology of the project and were invited to edit and/or affirm their poem as they were written via an editable Google Document. Overall, five finished poems were produced for this work, and I showcase three of them below in *Table 1*.

Table 1. Three of the five poems resulting from poetic transcription, capturing undergraduate student definitions and experiences of student success within mathematics.

Isabel*	Kenzie	Taylor*
It's two different intelligences: being able to do well in school and being able to use what you've learned. Some people enjoy doing math for the sake of math; I really enjoyed being able to know how much fabric to get. Being able to practically apply your math skills- this is what math was meant to be used for.	They determined success: your grades matter a lot. That's what I'm aiming for. Having studied, internalizing the materials, I worked at my dues. Feeling proud after all that work- that's where I want to be, that's what I was reaching for: a breath of fresh air. Here I am.	Math can be really hard. Of course, grades are important. More important than that is understanding what you're learning. I met the requirement and passed it too.
*Poem affirmed by participant as reflective of their experiences and definitions.		

The resultant poems endeavor to present transcript data in a way that is unconstrained by any particular coding scheme or field-specific jargon, while still conveying the nuanced definitions of student success in mathematics held by each of these women. Faulkner (2019) notes that poetic transcription provides qualitative researchers with a "more approachable, powerful, emotionally poignant, and accurate form than prose research reports allow" through the incorporation of rhetorical style and dimensions of emotion that poetry can bring to qualitative work (p. xi). Certainly, the presentation of this data could benefit from an additional thematic or perhaps narrative analysis, as, I argue, would any data set benefit from additional lenses of applicable analysis. However, as Prendergast (2009) argues, the poems produced from poetic transcription are best when they are capable of standing alone. In poetic transcription, the poetry is both the analysis and the presentation of the analysis, rolled into a singular creative process.

Discussion of Limitations Inherent in Poetic Transcription

While poetic transcription affords mathematics education researchers an invigorating and holistic way through which to re-present data, it is important to also make explicit the limitations present in such a methodology as its use is contemplated in our field.

I choose here to acknowledge my positionality in the work I described above; I am a White woman who does not identify as First-Generation or Pell-grant eligible, and while I endeavored to maintain fidelity to participants' rhetoric and meanings, my filtering of participants' words through my own lens and poetic style is in itself a form of colonization of these stories and definitions. Glesne (1997) provides candid reflection on how she and her participant's differing native languages may have construed her ability to accurately reflect her participant's voice and experience in her resultant poetry. Similar to Glesne's concern of not capturing the nuance that would be present in her participant's untranslated responses, I recognize that I cannot fully understand the nuanced responses regarding the lived experiences of my participants. While poetic transcription can help "heal wounds of scientific categorization and technological dehumanization" (Glesne, 1997, p. 214), it also maintains opportunity for withholding and privileging certain lenses that are already dominant in research spaces through its inherent subjectivity. Burdick (2011) and Mercer-Mapstone et al. (2019) are two examples in which participants were explicitly involved in the creation of resultant poetry, thereby more effectively enabling the participant's own creative voice to shine within the work from its very genesis.

I also acknowledge the criticism put forth by Tuck and Wayne Yang's (2014) work, in which they note that, by advocating that poetic transcription join the ranks of established research methodologies in our field, I do nothing to disrupt the problematic exaltation of *research* as the dominant and most reputable form of knowing- *research*, of course, being a space which historically and contemporarily marginalizes important and necessary perspectives. Simultaneously, other scholars highlight that poetic transcription pushes back against the dominant culture of academia by "allowing [researchers] to communicate to cultures other than that of academics" (Vannini & Gladue, 2008, p. 146). Vannini and Gladue assert that, in conjunction with claims made by poetic inquiry powerhouse Laurel Richardson, most informants to research converse in non-academic voices that are better reflected by poetry than written academic prose. In approaching research presentation in a way that is more colloquial, embodied, and free from academic jargon and structure, we re-present research that remains accessible to the individuals who contributed to it. Faulkner (2018) also notes that poetic transcriptions' "evocative critique and resistance of the status quo" enables it to function extremely effectively within necessary critical discourse (p. 23).

In conclusion, I hope this work has provided the reader with food for thought regarding the use of poetic transcription within a mathematics education context. While by no means a methodology which is free from continued tensions regarding colonization and dehumanizing practices, it does endeavor to push back against such influence through a more embodied, accesible way of re-presenting data. Poetic transcription as a methodology remains in its infancy when compared to many other forms of qualitative inquiry, and as such has immense opportunity for development as scholars experience and refine it in ways that continue to grapple with the subjectivity of research. I present it here in this brief report with the hope that it (a) prompts others to engage in and research poetic transcription as a viable methodology for use within mathematics education and (b) allows for a continual reflection on how we, as researchers in this field, can be continually critical in our use of methodology and grow in what we conceptualize as valid and important ways of knowing and re-presenting the experiences of our participants.

References

Battey, D. & Leyva, L. A. (2016). A framework for understanding Whiteness in mathematics education. *Journal of Urban Mathematics Education, 9*(2), 49-80.

Burdick, M. (2011). Researcher and teacher-participant found poetry: Collaboration in poetic transcription. *International Journal of Education & the Arts, 12*(SI 1.10), 1-18.

Burford, J. (2018). Sketching possibilities: Poetry and politically-engaged academic practice. *Art/Research International: A Transdisciplinary Journal, 3*(1), 229-246.

Faulkner, S. L. (2018). Crank up the feminism: Poetic inquiry as feminist methodology. *Humanities, 7*(85).

Faulkner, S. L. (2019) Poetic inquiry as a social justice and political response. In S. L. Faulkner & A. Cloud (Eds.), *Poetic Inquiry as a Social Justice and Political Response* (pp. xi-xv). Wilmington, Delaware, USA: Vernon Press.

Faulkner, S. L. & Cloud, A. (2019). Poetic inquiry as a social justice and political response. Wilmington, Delaware, USA: Vernon Press.

Fernández-Giménez, M. E., Jennings, L. B., & Wilmer, H. (2019). Poetic inquiry as a research and engagement method in natural resource science. *Society & Natural Resources, 32*(10), 1080-1091.

Glesne, C. (1997). That rare feeling: Re-presenting research through poetic transcription. *Qualitative Inquiry, 3*(2), pp. 202-221.

Gutiérrez, R. (2018). Political conocimiento for teaching mathematics. In S. E. Kastberg, A. M. Tyminski, A. E. Lischka, & W. B. Sanchez (Eds.), *Building Support for Scholarly Practices in Mathematics Methods*, (pp. 11-37). Charlotte, NC, USA: Information Age Publishing.

Keith, Jr., A. & Endsley, C. L. (2020). Knowledge of self: Possibilities for Spoken Word Poetry, Hip Hop Pedagogy, and "Blackout Poetic Transcription" in critical qualitative research. *The International Journal of Critical Media Literacy, 2,* 56-82.

Lahman, M. K. E., Geist, M. R., Rodriquez, K. L., Graglia, P. E., Richard, V. M., & Schendel, R. K. (2010). Poking around poetically: Research, poetry, and trustworthiness. *Qualitative Inquiry, 16*(1), 39-48.

Levinson, M. P. (2020). The quest for genuinely participatory and inclusive research approaches: Exploring and expressing experience through cultural animation and transcription poetry. *International Review of Qualitative Research, 13*(2), 182-199.

Leyva, L.A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition and Instruction, 39*(1), 1-34.

Martin, D. (2006). Mathematics learning and participation as racialized forms of experience: African American parents speak on the struggle for mathematics literacy. *Mathematical Thinking and Learning, 8*(3), 197-229.

Mercer-Mapstone, L., Guitman, R., & Acai, A. (2019). Reflecting gendered experiences of student-staff partnership: A student standpoint using poetic transcription. *Teaching in Higher Education, 24*(6), 809-818.

O'Shea, S. & Delahunty, J. (2018). Getting through the day and still having a smile on my face! How do student define success in the university learning environment? *Higher Education Research & Development, 37*(5), 1062-1075.

Prendergast, M. (2009). "Poem is What?" Poetic inquiry in qualitative social science research. *International Review of Qualitative Research, 1*(4), 541-568.

Skovsmose, O. & Borba, M. (2007). Research methodology and critical mathematics education. In P. Valero & R. Zevenbergen (Eds.). *Researching the socio-political dimensions of mathematics education: Issues of power in theory and methodology*, (pp. 207-226). Kluwer Academic Publishers.

Thunig, A. & Jones, T. (2020). 'Don't make me play house-n***er': Indigenous academic women treated as 'black performer' within higher education. *The Australian Educational Researcher, 48,* 397-417.

Tremaine, R. (2021). Explicit and expansive: The importance of (re)defining student success in mathematics. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp. 6-10). Philadelphia, PA.

Tuck, E., & Wayne Yang, K. (2014). R-Words: Refusing research. *Humanizing Research: Decolonizing Qualitative Inquiry with Youth and Communities,* 223-247.

Weatherton, M. & Schlusser, E. (2020). Success for all? A call to re-examine how student success is defined in higher education. *CBE- Life Sciences Education, 20*(3).

West, K. & Bloomquist, C. (2015). Poetic Re-presentations on Trust in Higher Education. *The Canadian Journal for the Scholarship of Teaching and Learning, 6*(2), 1-22.

Vannini, A. & Gladue, C. (2008). Decolonised methodologies in cross-cultural research. In P. Liamputtong (Ed.) *Doing Cross-Cultural Research* (pp. 137-159). Springer Science + Business Media B. V.

TENSIONS OF KNOWLEDGE AND AUTHORITY: CHALLENGES OF TEACHING SJM

Trevor T. Warburton
Utah Valley University
trevor.warburton@uvu.edu

This study seeks to demonstrate how knowledge and authority are linked for preservice mathematic teachers and how this link may pose a challenge for teaching mathematics for social justice. A CDA of multiple preservice class discussions is presented.

Keywords: Social Justice, Equity, Inclusion & Diversity, Preservice Teacher Education

There is a great need for social justice mathematics (SJM) in U.S. schools. Students of color frequently feel disconnected in mathematics classrooms (Gutiérrez, 2012b). Despite the efforts of Frankenstein (1983), Gutstein (2006) and others, SJM has not gained wide acceptance (Gutstein, 2006). Bartell (2013) notes that mathematics teachers prepared to teach SJM often fail in the moment. De Freitas (2008) has suggested that mathematics teacher identities are tied to mathematics and Walshaw (2013) notes that teachers are caught by dominant discourses of mathematics. These represent mathematics as apolitical, acultural and neutral (Ernest, 1991; Walkerdine, 1988). These characteristics contrast sharply with the openly political aims of SJM. Further, these characteristics form the basis of White authority (Frye, 1992), thereby creating an overlap of authority between mathematics and Whiteness (Author, 2015). For mathematics teachers to address social justice requires that they come into conflict with this authority.

One means for teachers to connect to authority is by emphasizing mathematical competency. Since many teachers, White in particular, have limited knowledge of social justice issues they can feel a lack of authority in this area. Further, the mathematical competency of U.S. mathematics teachers is under fire (Wolfmeyer, 2012), potentially leading mathematics teachers to be defensive. This may link mathematical knowledge and authority, leaving little room for SJM. This study seeks to understand the relationship between knowledge and authority for secondary mathematics teachers. To this end I aim to answer the following questions:

1) How do we discursively link mathematical knowledge and teacher authority?
2) How can we disrupt this link to create space for social justice in mathematics education?

Theoretical Framing

Wagner and Herbel-Eisenmann (2014) found that authority issues were prevalent in mathematics classes. These issues may arise because of the perceived authority of mathematics (Amit & Fried, 2005). Skovsmose and Valero (2001) describe a kind of "omniscient authority" embedded in mathematical discourse. Authority relations are primarily between teachers, students, and mathematics. Authority is a key aspect of equity (Gutiérrez, 2012a). A potential problem arises when teachers use a knowledge-authority link to justify authoritative mathematics teaching. I focus primarily on this teacher-student authority relationship linked to knowledge. This link is important because the dominant mathematical discourses may dissuade teachers from questioning their authority and from creating equitable relationships. Amit and Fried (2005) note that when teachers are too authoritative it reduces student authority. One means of understanding authority is through the lens of agency (Sengupta-Irving, 2016). Authority can attempt to limit who (and how) can exercise agency. This may create a dynamic where students are mostly passive (Boaler & Greeno, 2000). While this passivity can be an exercise of agency (or resistance), it does not contribute to the mathematical development of the student or classmates.

Authority and Social Justice in Mathematics

Reforms in mathematics promote a redistribution of authority (Amit & Fried, 2005); this is not sufficient to equitable relationships. Because authority is in the discourses of classes (Skovsmose & Valero, 2001), it does not disappear with the adoption of more “student-centered” pedagogy. Gutiérrez (2009) notes that teachers cannot pretend to lack authority; instead teachers need to explicitly discuss authority (Gutiérrez, 2015), so that its use can be negotiated. In terms of equity, Gutiérrez (2012a) outlines four dimensions: access, achievement, identity and power. The power and identity dimensions intersect with authority in terms of which mathematical practices are valued, whose voices are heard, and in using mathematics to critique structures.

Discourse

Discourses can establish authority. I look at the discourses of mathematics, mathematics education, SJM and teacher and student roles. Discourses both enable and constrain (Fairclough, 2001). Appropriate discourses allow me to be recognized as a mathematics teacher through how I speak, interact, dress, and use mathematics materials. Discourses constrain by limiting what speech and behaviors are acceptable for a mathematics teacher. This enabling/constraining illustrate the operation of power in discourses. As a result the use of discourses is a negotiation of power. This negotiation occurs most directly between the teacher and students. These negotiations are implicit and dependent on a shared idea of teacher or student roles.

The discourses that carry the greatest power are dominant. A discourse becomes dominant by excluding other ways of thinking by establishing a static understanding of truth. Current dominant discourses about mathematics portray it as apolitical, neutral, acultural and objective (Ernest, 1991; Walkerdine, 1988). SJM argues, in part, that mathematics is political, cultural, and subjective. Since the dominant discourses of mathematics specifically disallow this perspective it may be difficult for teachers to understand the why and how of SJM. Walshaw (2013) notes that teachers and students are caught in these discourses. Thus, I expect to find that mathematics teachers generally will struggle to understand SJM, because of the of these dominant discourses.

Discourse and Whiteness. The pressure to conform to dominant discourses and their limits help maintain power structures. In the U.S. dominant discourses are discourses of Whiteness. Whiteness Theory helps uncover how discourses maintain white privilege. Whiteness Theory operates on the assumption that the lives of all people in the U.S., in particular, are racially structured (Frankenberg, 1993; Frye, 1992). Since mathematics achievement follows racial lines \ (Stinson, 2004), race is a significant factor in defining traditional measures of mathematical success. Further, Skovsmose and Valero (2001) explain that mathematics is used to sort students. Battey (2013) specifically applies Whiteness Theory to the material benefits of this sorting. Here I will analyze discursive connections between knowledge and teacher authority.

Methodology

Participants

Like my students, I am caught in the dominant discourses of mathematics education. As a young, white teacher I taught all of the mathematics classes for emergent multilingual students at a rural, public high school. I witnessed the roles of race and class in the lives of my students. There were clear divisions along race and class lines. As a teacher I felt pressure to support school policies; I also wanted to support my students who were not served by those policies.

Teacher Candidates. The teacher candidates included four White males (Pseudonyms: Jeff, Karl, Gavin, and myself), two White females (Stella and Lisa), one Latina (Esperanza), and one

Japanese-American female (Jane). They were working towards a Master's degree in mathematics with certification funded by Mathematics for America (MfA). During the Fall semester I supervised their student-teaching, attended monthly meetings, and a retreat. During the Spring semester I taught their action research class as I continued their supervision. Several of the preservice teachers expressed an interest in SJM and four attended a conference on SJM. I considered each of them to be capable and committed, who struggled with SJM.

Critical Discourse Analysis

Drawing from critical theory, critical discourse analysis (CDA) explains how language is linked to society (Chilton, 2011). Fairclough (2001) draws on Foucault to explore how consent is gained through dominant discourses. Fairclough uses discourse to explain the relationship between social norms and power. Here I will analyze discourses we used to explain the links between mathematics and authority. Critical discourse analysis draws on post-structural understandings of discourse and the circulation of power (Rogers et al., 2005). In particular, as speakers use discourses we construct objects and subject positions. But even these objects and subject positions are multiple and changing; they are negotiated and re-created. I have also framed these discourses within Whiteness Theory to analyze how they maintain white privilege and how we begin to re-work some of these discourses in order to understand SJM.

Data Collection

All data for this study were drawn from a teacher research (with a SJM focus) course for preservice teachers. Each class was recorded and transcribed, resulting in over 25 hours of recorded data. The teacher candidates' written work, as well as my reflection journal were additional sources. I chose sections in the data where there was evidence of, or potential for, SJM for further analysis. I performed line-by-line analysis of each of these transcripts using Gee's (2005) building tasks paying particular attention to authority. From these analyses I identified themes and re-analyzed for key ideas I missed, to fill in details, and to identify deviations.

Results

Linking Knowledge to Authority: A Traditional Frame of Knowledge and Authority

The transcript that follows illustrates the link between knowledge and authority, how that link is normalized, and is representative of other discussions. This discussion focused on the discourses that shape mathematics education. Prior to the transcript we discussed teacher and student roles. These discourses structure teacher-student relationships.

Jeff: It's inevitable I mean if I am more knowledgeable than my students you can't change that, that automatically gives me some more power that they don't have, you can't change it so that's not necessarily a bad thing. Sometimes there's this connotation that we want to remove all power differences.

Teacher: Right

Jeff: I think that that's erroneous.

Teacher: And I think to pretend that you don't have that authority is false and the students are going to recognize it right away and they're not going to respond to it.

Jeff: Well and if they don't respect the fact, if they don't think you have the knowledge you can't teach them in any way

Here Jeff draws on a discourse that links authority to knowledge. Within this discourse greater knowledge necessarily creates greater authority. In drawing on this discourse Jeff positions teachers as naturally having more authority than students since teachers have "more knowledge". Presumably he is talking specifically about mathematical knowledge. This knowledge differential is unchangeable ("inevitable", "you can't change it"). These statements naturalize the relative positions of teacher and student and the teacher's authority as authority over students, which is "automatic". Finally, he evaluates these positions as "not necessarily a bad thing". My comment does nothing to disrupt this link and, in fact, accepts it as natural. Jeff's final comment points to the necessity of this authority as necessary for teaching. Among the assumptions of this discourse is that authority is static (i.e., natural to a teacher), teacher authority and student authority are in an inverse relationship, and that the authority-knowledge link applies only to sanctioned knowledge—especially mathematics.

Lack of Knowledge as Justification

Alternatively, the following transcript shows how this link can be used as justification to not engage SJM. Prior to this, the teacher candidates had read an article by Gutstein (2012) on implementation of SJM. In response, we discussed various topics including the importance of addressing racism. During this discussion Esperanza drew on her experiences as an immigrant student of color to argue that teachers need to discuss these issues. In the transcript Jeff shifts the discussion by asking Esperanza more about her experiences.

Jeff: Because one of my fears is I don't see when I'm doing it wrong for them so but if I said look I want to have a discussion about immigrant reform . . . and I know that I'm not an immigrant so I'm not going to see all of the issues the way that you will and so if I do something you know prelude that says I'm going to probably screw some of this up but it's not because I'm trying to would that have made a difference for you

Esperanza: Letting me know yea and I think it would be great to not take on and shut because I know my U.S. history teacher he thought he knew everything he just knew the newspaper like you're not looking at the families that are being deported you're just bringing the issues the media brings out to the table and not other perspectives and so if you really want to do touchy subjects like that I feel like you have to make sure you know both sides of the coin you know what I mean like

Teacher: Yeah and I think like you said acknowledge the limits of your knowledge

Esperanza: Exactly let them know

Jeff: I don't know, at some point then I might just decide not to have it because the problem is that if there is something that the teacher brings to the class and if nothing else it's some sophistication in how debates can go and realizing that kids just like to argue and so what are they going to do they're going to bring the news to class and their parents views and if you aren't knowledgeable you can't manage that discussion very well right I mean if I personally am not knowledgeable about it it can get out of hand very quickly.

As Jeff directs his question to Esperanza he frames it around his own fear which is, "I don't see when I'm doing it wrong". In Esperanza's response she emphasizes that a teacher should "know both sides" of an issue. This would require a teacher to learn about issues before discussing them in class and positions the teacher as taking responsibility. It is at this point that the direction of Jeff's argument becomes clear. In response he says, "I might just decide not to have [the discussion]". Thus, he draws on a dominant discourse that uses ignorance as a reason not to engage rather than a responsibility to learn. He links this to the idea of "sophistication" that a teacher should bring to the class. This sophistication requires more knowledge than what the students have (who "just like to argue") in order to give the teacher the authority to control "how debates can go", since "if you aren't knowledgeable you can't manage that discussion". The concern is that the discussion "can get out of hand", which may undermine teacher authority. In this way, white ignorance (of racism) creates a fear of losing authority. This fear prevents a willingness to learn to have the difficult discussions that could benefit students. It is a fear of not being a “good” teacher, when a good teacher is understood as knowledgeable and in control. While Esperanza provided an opportunity for a different perspective, the dominant discourse of Whiteness, and a lack of white responsibility to understand racism was reasserted. Again these are the dominant discourses that are available to us.

Problematic Relationships in Social Justice Mathematics

In another moment of this same class Karl brought up a question of how to bring up difficult topics, such as HIV/AIDS, and have students engage appropriately. In our responses to this question we describe the concerns we have about teaching this kind of SJM lesson.

Teacher: So let’s talk about this so somebody there brought up the idea that if you're teaching this lesson on the west side [lower income, more students of color] that it could be possibly discouraging to students. How would you address that?

Gavin: Well just that life sucks and that's how it is.

Multiple: ((laughter))

Jeff: Isn't it all in the context of being able to create change if you can do that it doesn't have to be inherently negative being reminded of reality and if that’s all you're doing is saying yeah tell me something I don't know

Teacher: Isn't well ok I want to hear you out is there any benefit to doing that? . . . .

Jeff: Well maybe I mean to let them know that you're aware

Teacher: To remind them mhmmm why might that be beneficial?

Jeff: Because they may feel that you aren't and they can't approach you and talk . . .

From the beginning I position the teacher as the authority, bringing difficult topics to the class (“how would you address”). While this was Karl’s question, I do not challenge this assumption. This leads us to continue in a way that assumes the authority of the teacher. I ask the teacher candidates to recall a SJM lesson that Esperanza, Stella, Lisa, and Jane had written. Their lesson highlighted an unequal distribution of parks based on geography and population density.

After Gavin’s joke, Jeff mentions the importance of “being able to create change”. The passive construction here leaves it unclear who is creating change. However, if it is parallel to the structurally similar “being reminded of reality” where it is clear that the students are “being reminded,” the implied agent is the teacher. This passive role for the student maintains teacher authority, even if that teacher intends to “create change”. Picking up on Jeff’s point I ask if there is benefit to just pointing out inequity. Jeff mentions a benefit of overcoming some of the teacher’s authority to make them more approachable. This point suggests that students are careful in their relationships, positioning them as agents as they negotiate the school system.

This improved relationship could benefit students, as they will be more likely to have someone they can work with and who will listen to them. In this discussion there are implicit references to knowledge of inequality; this will become more explicit as the discussion continues. As Lisa responds (below) to my questions she seems to suggest that bringing up difficult, social justice topics is beneficial because it can increase teacher authority.

Teacher: And why would they care if you as their teacher are aware?

Jeff: Because you have authority that you're trying to hand out to them.

Teacher: K. other thoughts on that?

Lisa: Yeah. Because you're trying to teach them stuff if they think you have no idea what's going on and they don't get you at all they're not going to participate in your classroom

Jeff: So you can't create relationships you mean

Lisa: Yeah. So I think that's something that I never really thought of that much making them aware that you're aware of it

Teacher: Is there benefit in just making students aware of social issues like this . . . .?

Lisa: I think that you have to talk about the actual thing before you can start to talk about why that thing is occurring and I don't think that that is a given that all students know reasons that could potentially explain why things are unfair.

Lisa highlights the importance of what students think of their teacher. If their opinion of the teacher is low (in terms of knowledge) then “they’re not going to . . . participate in your classroom”. This is a very similar statement to one Jeff made in week 4, “if they [students] don't think you have the knowledge you can't teach them in any way.” Also “your classroom” suggests teacher ownership. Both Jeff and Lisa link teacher knowledge, with students’ willingness to listen. A lack of teacher knowledge may result in students exercising agency disagreeably.

If we discuss teaching mathematics in a traditional sense (as Jeff was) then the knowledge/authority link is restricted to mathematics. However, if we discuss SJM (as Lisa is) then that knowledge is expanded to include sociopolitical knowledge. This is a potentially disruptive view suggesting that being a mathematics teacher requires sociopolitical knowledge. However, increased knowledge is still linked to increased authority. There is no discussion of benefit to the students from this kind of teaching. It seems that our primary concern is how bringing up difficult topics in the classroom will affect the teacher. As I rephrase the question I maintain this focus on the teacher as agent who is “making students aware.” This may reflect my own whiteness where I was largely unaware and may also universalize this white perspective. Lisa also notes that we should not assume that “all students know”. We position ourselves as responsible for SJM and focus on the teacher as authority.

Balancing Teacher and Student Authority

In a later class, Karl shared an example of a SJM lesson in his class. A majority of his students were low-income students of color and bussed to the high school.

Karl: We calculated the area based on those grids that you put over the maps and then talked used that as a segue into you found the area it looks like you guys used a rectangle here this one you guys said was a half what's really going on so like that was kind of the math reason for doing it but the discussion was pretty good it started out kind of poorly . . . . they would say let’s just have it how it is now I don’t care what parks are like and that was about the first five minutes of the discussion was about how people don’t really care about parks

Here Karl explains the lesson that illustrates the ways local parks are distributed inequitably. His focus is on what the students did, positioning them as agents. Notice how he includes himself with his students (“we calculated”) and then the frequent use of “you” to refer to his students and their work. He positions students and their work as central, while he remains on the periphery. When the lesson “started out kind of poorly” he attributes this to his own inexperience. He emphasizes the students’ perspective and identifies with it by using first person (“let’s just”, “I don’t”). Their perspective went against what he was trying, but he valued their thinking. Students are portrayed as exercising agency in valid ways, even when they disagree with the teacher.

After listening to his students, Karl still thought the lesson was valuable and that his students would benefit. As a result, he persuaded them to give the lesson a chance.

Karl: And I had like a little bit of a discussion . . . . like this I know we’re talking about parks right now but take this a little more seriously for a little while this isn’t really about parks and I think you guys will figure out what’s going on here . . . . They brought up were things that I never thought about in terms of this park issue and one thing being cost or a couple things being cost of upkeep for a park I didn’t really think of that as a possible

> reason for maybe how they're distributed like maybe one park has more graffiti than another park and more cost is gone into upkeeping it. One student brought up the distinction between a community park and a city park and I didn't know the difference really and she was kind of thinking she was like schools are kind of supported and I don't know much about taxes and she was kind of asking me about taxes and I was like I don't know your idea could be right or it could be wrong. . . . which I thought was an interesting question that I didn't expect from a 17-year-old

In the beginning part of this section Karl portrays himself as using some of his authority to persuade. He did so through "discussion", recognized the students' perspective, and expressed confidence in their ability to see beyond the parks. The students brought up things, "that [he] never thought about". Karl clearly views this as a positive ("really good") even though it positions him as less knowledgeable in this area than some of his students. He believed that his students' ideas were valuable enough to share them with the rest of us. Karl also admitted his lack of knowledge on the subject to his students (frequent use of "I don't know" and variations). In previous discussions we had constructed teacher authority as closely connected to knowing more than students, this admission appears to work against that link. This admission was not related to mathematical knowledge. It is possible that Karl feels safe admitting a lack of knowledge about park funding, but not mathematics.

Discussion

Dominant discourses appear to link knowledge to authority. By doing so they either leave no space for SJM, an unwillingness to engage SJM, or inequitable relationships within SJM. Only occasionally were we able to break out of this link as Karl did.

No Space for SJM

In the discourses that left no space for SJM we struggled to even understand if or how SJM might be possible. This view seems to be consistent with Bartell's (2013) finding that teachers, after planning a SJM lesson, often focused on either the social justice or the mathematics. This discourse operates by restricting our ability to imagine alternative relationships between knowledge and authority and require a reimagining of what knowledge is valued in a school context (Gutiérrez, 2015). A potential alternative to this view of knowledge and authority would be one that recognized that both teachers and students have varying levels of knowledge in different areas. While the teacher may have more knowledge of school mathematics, students also have knowledge relevant to learning mathematics. However, the dominant discourses of school mathematics shape the roles of teacher and students to prioritize school mathematical knowledge. As a result, our relative lack of knowledge of social justice issues shut down discussions of SJM. Discussions that draw on dominant discourses feel "safe" (Yoon, 2012) by not disrupting the dominant view of mathematics with the racio-political views of SJM. Further, they can help participants maintain an image as "good" (white) teachers (Applebaum, 2008) by staying within knowledge domains that are comfortable.

White Ignorance

Discourses of white ignorance of social issues, especially of racism reflect a colorblind discourse that support racist structures in society (Bonilla-Silva, 2015). It is important for teachers, especially white teachers, to recognize the limits of their knowledge and experience. However, this recognition should be accompanied by a sense of responsibility, rather than a justification. This discourse restricts SJM by reducing the teacher's role to mathematics, since that is the area of knowledge expected for mathematics teachers and prioritized in teacher preparation. Alternatively teacher preparation might value multiple forms of knowledge, such as the classical, critical, and community knowledge suggested by Gutstein (2006). Importantly this discourse came about in response to Esperanza's explanations of how and why teachers must address sociopolitical issues. She was able to successfully draw from her experiences as an immigrant woman of color to access discourses that were temporarily disruptive. She may have done so because of an experience of "double consciousness" (Du Bois, 1903). As a woman of color in a white space she could see both the dominant discourse and alternatives.

Problematic Relationships

Importantly, even when we did successfully discuss SJM we often did so from within a dominant discourse linking knowledge and authority. The result, for us, was that we discursively constructed inequitable classroom relationships between teacher and students. These relationships emphasized teacher authority, the teacher as agent, and the beneficiary of SJM. There is little room left to construct an active role for students. Gutiérrez (2012a) in discussing identity and power explains that students should have opportunities to bring their full self, including knowledge and capabilities. Further, students need opportunities to participate in mathematics that is personally meaningful. This is more likely to happen when students have voice, which is part of Gutiérrez's equity dimension of power. In mathematics education we need to construct classroom dynamics that value and include student voices.

Shared Authority

Finally, Karl's description of his experience attempting to address a social inequity in his mathematics class shows potential for constructing more equitable classroom relationships. This occurred as students brought up ideas that Karl had less knowledge of. He was willing to listen to and learn from his students as they raised their voices and their ideas in the classroom. To further develop these relationships, students would also need opportunities to participate in similar ways with both mathematical and social issues and the teacher would need to demonstrate a similar willingness to listen to and learn from students.

Recommendations

Dominant discourses linking knowledge and authority imply that the simple solution to teachers' lack of knowledge of social issues, such as racism, is to gain knowledge in these areas. While increasing teacher knowledge is necessary, it is insufficient and may lead to inequitable relationships. In addition, teachers will need to create opportunities to learn from their students; white teachers in particular, will need to listen to their students of color. Teacher education programs largely focus on what Gutstein (2006) identifies as classical knowledge. This knowledge is developed both in mathematics content courses and pedagogy courses. Courses, such as multicultural education, may help to develop teacher candidates' critical knowledge, but a single course will not be enough. The embedding of critical themes throughout teacher education and beyond will likely be necessary. Community knowledge is challenging and is typically not included in teacher preparation. While community knowledge will vary and needs to be learned in conjunction with students and their communities, this process can be modeled as teacher educators also learn from and with their students.

Conclusion

Within mathematics education dominant discourses frame knowledge and authority in ways that challenge our ability and willingness to teach SJM. These challenges affect even those who desire to teach SJM. The challenges range from the more obvious where we struggle to conceptualize SJM or are forced to face our lack of critical knowledge to the more subtle concerning classroom relationships and the risk of diminishing student agency as we seek to take responsibility for a socially just mathematics classroom. The need for equitable mathematics is growing and it will require better and more thorough preparation. This preparation will likely need to include modeling and increased emphasis on critical and classical knowledge.

References

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education, 44*(1), 129-163.

Battey, D. (2013). Access to mathematics: "A possessive investment in Whiteness". *Curriculum Inquiry, 43*(3), 332-359.

Boaler, J., & Greeno, J.G. (2000). Identity, agency, and knowing in mathematical worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (171-200). Westport, CT: Greenwood Publishing Group.

Bonilla-Silva, E. (2015). The structure of racism in color-blind, "post-racial" America. *American Behavioral Scientist, 59*(11), 1358-1376.

de Freitas, E. (2008). Troubling teacher identity: Preparing mathematics teachers to teach for diversity. *Teaching Education, 19*(1), 43-55.

Du Bois, W. E. B. (1903). *The souls of black folk*. A Public Domain Book.

Ernest, P. (1991). *The philosophy of mathematics education*. London, UK: The Falmer Press.

Frankenberg, R. (1993). *White women, race matters: The social construction of Whiteness*. Minneapolis, MN: University of Minnesota Press.

Frankenstein, M. (1983). Critical mathematics education: An application of Paulo Freire's epistemology. *The Journal of Education, 165*(4), 315-339.

Frye, M. (1992). White woman feminist. In M. Frye (Ed.) *Willful virgin: Essays in feminism*, 1976–1992, (147–169). Freedom, CA: Crossing Press.

Gee, J.P. (2005). *An introduction of discourse analysis: Theory and method* (2nd Ed.). New York, NY: Routledge.

Gutiérrez, R. (2012a). Context matters: How should we conceptualize equity in mathematics education?. In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm, (Eds.), *Equity in discourse for mathematics education: Theories, practices, and policies* (17-33). New York, NY: Springer.

Gutiérrez, R. (2012b). Embracing Nepantla: Rethinking "knowledge" and its use in mathematics teaching. *Journal of Research in Mathematics Teaching, 1*(1), 29-56.

Gutiérrez, R. (2015). Nesting in Nepantla: The importance of maintaining tensions in our work. In Joseph, N. M., Haynes, C. & Cobb, F. (eds.), *Interrogating Whiteness and relinquishing power: White faculty's commitment to racial consciousness in STEM classrooms*, (pp. 253-282). New York: Peter Lang.

Gutstein, E., (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. New York, NY: Routledge.

Gutstein, E. (2012). Mathematics as a weapon in the struggle. In O. Skovsmose & B. Greer (Eds.), *Opening the cage: Critique and politics of mathematics education* (23-48). Rotterdam, Netherlands: Sense Publishers.

Mathematics for America. (2013). Mission & Vision. Retrieved from http://www.mathforamerica.org/about-us/mission-and-vision

Rogers, R., Malancharuvil-Berkes, E., Mosley, M., Hui, D., & Joseph, G. O. G. (2005). Critical discourse analysis in education: A review of the literature. *Review of Educational Research, 75*(3), 365-416.

Sengupta-Irving, T. (2016). Doing things: Organizing for agency in mathematical learning. *Journal of Mathematical Behavior, 41*, 210-218.

Skovsmose, O., & Valero, P. (2001). Breaking political neutrality: The critical engagement of mathematics education with democracy. In B. Atweh, H. Forgasz, & B. Nebres (Eds.) *Socio-cultural aspects of mathematics education: An international research perspective*, (37-56). Mahway, NJ: Lawrence Erlbaum Associates.

Stinson, D. W. (2004). Mathematics as "gate-keeper" (?): Three theoretical perspectives that aim toward empowering all children with a key to the gate. *The Mathematics Educator, 14*(1), 8–18.

Wagner, D., & Herbel-Eisenmann, B. (2014). Mathematics teachers' representations of authority. *Journal of Mathematics Teacher Education,17*(3), 201-225.

Walshaw, M. (2013). Poststructuralism and ethical practical action: Issues of identity and power. *Journal for Research in Mathematics Education, 44*(1), 100-118.

ICEBERG OF CULTURALLY RELEVANT MATHEMATICS AND SCIENCE PEDAGOGY: A PEDAGOGICAL AND ANALYTICAL TOOL FOR TEACHER EDUCATION

Craig Willey
Indiana University-Purdue University Indianapolis
cjwilley@iupui.edu

Paula A. Magee
Indiana University-Purdue University Indianapolis
pamagee@iupui.edu

There is a clear need for an operational framework that captures the difficulty of implementing culturally relevant pedagogy (CRP) and supports teachers to imagine new forms of pedagogy. In this report, we present the Iceberg of Culturally Relevant Mathematics and Science Pedagogy (CRMSP), a tool, grounded in the tenets of CRP, that delineates practices ranging from the most accessible and easy-to-implement, to the most subversive yet arguably the most significant in terms of their potential to re-characterize the acts of doing and being competent at mathematics and science. We provide examples of CRMSP that re-position marginalize learners in relation to mathematics and science. We discuss the levels in which this tool can serve to disrupt dominant, inequitable systems of instruction and preview ways it can support teachers' efforts to provide meaningful mathematics and science learning experiences for youth.

Keywords: Culturally Relevant Pedagogy, Preservice Teacher Education, Teacher Knowledge, Curriculum

Like other social institutions, schooling has served to disenfranchise and subjugate students of color and their families, and to reaffirm a white ideology that supports white students, teachers and families (Emdin, 2016; Leonardo, 2009; Picower, 2009). Scholars and practitioners have explored ways to confront the inherent racism that permeates schools and have developed critical frameworks such as culturally relevant culturally relevant pedagogy (CRP), culturally responsive teaching and culturally sustaining pedagogies (Paris & Alim, 2014; Ladson-Billings, 1995, 2014; Gay, 2002).

In this report, we describe our work with a teaching protocol that both supports the identification of CRP and the analysis of these practices within the context of culturally relevant science and mathematics classrooms. This approach to understanding *Culturally Relevant Mathematics and Science Pedagogy (CRMSP)* asks preservice teachers (PTs) to use readily identifiable practices and materials and then scrutinize them for how compatible or tenuous they are with CRP. For the final analysis of compatibility/tension, we use the idea of a continuum of criticality from the Iceberg of Culture (Hall & Hall, 1976; Weaver, 1986) to assist PTs in analyzing their own materials, dispositions and practices. To analyze, visualize, and develop responses to the challenges and limitations surrounding PTs' implementation of CRMSP, we use two theoretical frameworks: critical whiteness studies and the iceberg of CRMSP.

Theoretical Perspectives: Culture and Whiteness

Equitable teaching requires PTs be able to analyze their own teaching and to be able to speak clearly about where and how they are supporting all learners and recognizing inherently racist practices. Doing this with efficacy requires a deep understanding of whiteness and the many structurally racist practices that are often "invisible" to white educators (Leonardo, 2009). Supporting PTs to becoming culturally relevant practitioners demands that PTs be able to identify emergent, intermediate and advanced culturally relevant practices in STEM.

Inextricably connected to whiteness, we consider an understanding of culture and how that influences what teachers are willing to do pedagogically. If PTs have a shallow understanding of culture, it is going to be difficult for them to teach in culturally relevant ways. If PTs see culture as *only* stereotypical examples of food, holidays and celebrations, it is going to be challenging for them to see how they reinforce dominant oppressive practices. It also will be exceedingly difficult for PTs to fully understand, both theoretically and practically, the role of culture in equitable science and mathematics teaching and learning (Nasir, Hand & Taylor, 2008; Nasir & de Royston, 2013).

Research Design, Context, and Methods

This work is situated in urban teacher education programs explicitly focused on developing anti-racist, culturally relevant teachers. Major program goals are to develop critically conscious PTs who understand how whiteness permeates schools and classrooms. The PTs we reference in this report are both elementary and secondary students from alternative certification programs.

Using the frameworks above and our experiences with PTs we developed a series of activities for helping PTs better understand and enact CRMSP. Described below, the activities first ask teachers to identify concrete elements of their practice (curricular materials, instructional decisions and teacher dispositions). This is followed by mapping the teacher practices back to the tenets of culturally relevant pedagogy. Finally, we follow this with an alignment of the teacher practices on an "iceberg of CRMSP" to assess the criticality of the practices (see Figure 1 below).

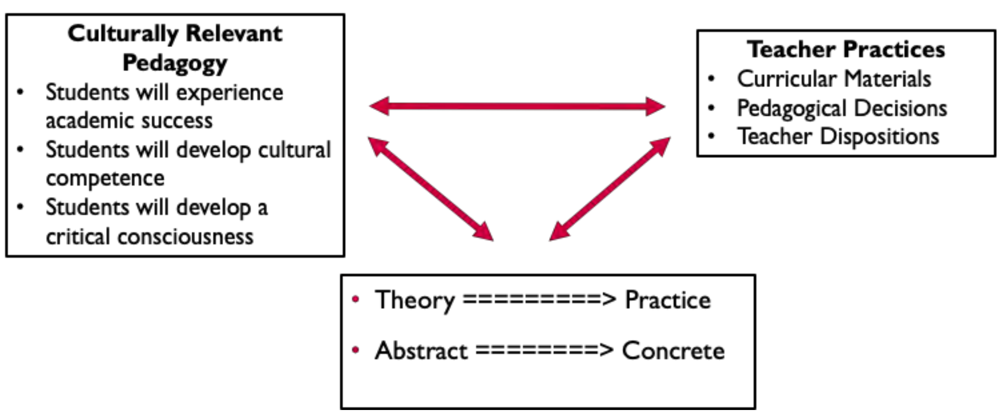


Figure 1: Supporting Teachers to Operationalize CRMSP

Data Sources

The data sources we draw upon include collaborative documents used in class (where PTs document CRMSP), lesson plans that PTs wrote, notes from debriefing sessions with PTs, and PTs' responses to reflection prompts used in class. The data was examined through the lens of culturally relevant pedagogy, specifically how each of the classroom events represents the tenets of CRP and what impact this event would likely have on dramatically shifting the experiences and success of children of color. In this analytical process, we relied heavily on our professional experiences as teachers and teacher educators, and our knowledge of the literature and judgement as to the degree to which PTs were moving beyond superficial notions of culture and critical pedagogy.

Results

As discussed above, PTs overwhelmingly support the *idea* of culturally relevant teaching, and they can, after study, articulate and operationally define the tenets. A disconnect comes when PTs are asked to analyze their own teaching through a CRP lens and identify how they are enacting or plan to enact CRP.

Components of Teaching in Culturally Relevant Ways

The three categories of teacher practice (curricular materials, instructional decisions and teacher dispositions) represent a bridge between the visualization of culturally relevant pedagogy and actual classroom interactions, and potentially offer PTs a way to access a critical analysis of their own science and mathematics teaching. The analysis is embedded within the course that the PTs take. For example, as a class assignment PTs read and analyzed Ladson-Billings' (1995) article, "But That's Just Good Teaching." PTs studied the concept of culture by working on their own understandings of culture and CRP. The analytical tool described here evolved out of this work with PTs and the provocations put forth by Ladson Billings (1995, 2014) and others (e.g., Gay, 2018; Paris, 2012).

In the first step of the analysis PTs are asked to list curricular materials, instructional decisions, and teacher dispositions for a lesson they are developing, have observed, or are going to teach. Some of the listings are easily evident, such as the use of a textbook (curriculum material) or the decision to move to small groups for problem solving (instructional decision). Others, such as how a PT responds to a student's question or "misbehavior" (teacher disposition) are more difficult to identify. We now recognize that the "difficulty" of identifying a practice is linked to the criticality of the practice and that the practices that PTs choose to "not see" are those that often reinforce a white, hegemonic classroom (one example from our practice and research is described below). Despite this recognition, the list offers PTs a series of things to analyze. Once a PT has listed a textbook as a curricular material, the conversation about the textbook begins and opportunities for discussion about what culturally relevant pedagogy looks like continues.

Curricular Materials. *Curricular materials* refer to any physical artifacts or resources that a teacher may use in class. These include, but are not limited to: trade books, textbooks, videos, speakers, handouts, worksheets, and manipulatives. PTs can quite easily identify the concrete curricular materials that they use. Once the PT has identified the curricular material we ask, "How does this choice connect to one of the tenets of CRMSP?" For example, a student teacher, Ned, chose to use an article from *The Atlantic* entitled, "The Myth of 'I'm Bad at Math'" (Kimball & Smith, 2013) after his middle school students took their first math exam. Ned was worried that the students did not see themselves as "good" at math and wanted to challenge the idea that some people are "just good" at math. In our class google doc, where we recorded our examples, Ned explained how the use of the article connected to the tenet of "academic success." Ned was considering how effort makes a much bigger difference in mathematical academic success than we tend to give it credit for, especially in middle and high school. Ned's example offers an opportunity to see how something as simple as a magazine article can be used as evidence of CRMSP. The example also reinforces the idea that the tenets *matter* and that PTs can think about them as a way to dig into what it means to be a culturally relevant teacher. The article also pushed the idea of "culture" since it is not specifically tied to the traits that PTs think of as culture (e.g., holidays, stereotypes, food, etc.). In this way it opened up a place for a conversation about how we can analyze our teaching using a CRMSP lens. It allowed us to talk about how society views "who can do math" as a cultural component. It also begins to set the stage for PTs to push themselves. In this example the connection to academic success is legitimate but it also generates the question, "What about cultural competence and critical consciousness?" Continuing to reinforce the idea that CRMSP *is* all three of the tenets is an important component of this work.

Instructional decisions. Instructional decisions refer to the ways in which teachers engage learners with curriculum. These include the structure of lessons, teacher moves in lesson delivery, managing participation patterns, orchestrating classroom discourse, collecting evidence of student learning, and overseeing interpersonal dynamics. Like curricular materials, instructional practices can be designed for white students (consciously or unconsciously) and guided by school or district-level initiatives, or they can be designed with equity and justice in mind. Our goal is to have PTs identify their own or other teachers' instructional decisions and then to connect them to the tenets of CRP.

Below we describe an observation with Sharon, a student teacher teaching a math lesson to 2nd graders. To prepare for the observation Sharon wrote a lesson plan which included a section specifically addressing culturally relevant pedagogy. She was asked to reflect on how her instructional decisions connected to CRMSP. She wrote:

> I am using a cognitively guided approach. Students will be leading the lesson with their thinking, and I will be able to better scaffold based on what I learn from the process. Rather than open with an algorithm, I want the students to tap into their number sense and sense-making skills to understand the thinking behind the function.

In this example Sharon identifies cognitively guided instruction (e.g., Carpenter et al., 2015) as the foundation for her math lesson. She is intentional about centering student thinking, which reveals her commitment to the value of her students' mathematical ideas, a component of CRP, and articulates how she will engage the students in the content. When asked to move from the concrete to the abstract, from the actual instructional decision to the tenets, Sharon writes the following as a rationale for the instructional decision:

> Soon, students will be introduced to two-digit subtraction with borrowing. Before this happens, I want them to get a better understanding of the math thinking behind the algorithms they will need to learn. I will use a Cognitively Guided approach to present them with a story problem and allow them to work in small groups to find a solution in a way that makes sense to them. The story problem is also based on their literacy text for this week. This way they have background knowledge that contextualizes the math problem and makes it easier for them to connect with. I will not be showing them the algorithm for solving an equation with borrowing. I will be walking around the room, checking in with groups, and asking them questions about their methodology and if their answer makes sense to them. I anticipate that many students will attempt to solve with addition or the improper use of an algorithm, so I will use some talk moves to try to guide them to question their own work and find another way. At the end, we will share what we did to find an answer. We may not arrive at the "correct" answer, but I will get several examples of student thinking that I can use to build upon as we move forward with math instruction.

In Sharon's rationale we can hear how she is building a theoretical base for her teaching. While not explicitly described in her lesson plan, in our debrief after the session we make sure to discuss how the instructional decisions made connect to both academic success and cultural competence. Like the in-class discussions with PTs the debriefs following teaching observations are critical spaces for productive talk about the connections between concrete teaching practices and the tenets of CRP and CRMSP. Often the PTs themselves are not even aware of how they are making these connections in practice. In this case since the use of cognitively guided instruction did not necessarily resonate with Sharon's idea of "culture" she did not recognize using it as evidence for CRMSP. During the debrief we highlight how Sharon's encouraging students to share their sense-making, being open to ideas that students share and facilitating student-to-student talking opportunities, are actual components of culturally relevant math teaching. We want to make sure that Sharon realizes that she is pushing back on the widely used math practices of teacher directed algorithmic memorization for problem solving and is instead demonstrating an actual practice that supports both academic success and cultural competence. Talking with Sharon also creates an opportunity for discussion about the critical consciousness tenet that is not addressed. During our debrief with Sharon, like discussions with Ned, we brainstorm how the lesson can be extended to interrogate something in the students' community or to investigate an injustice.

Teacher dispositions. Teacher dispositions refer to the beliefs about, and behaviors toward children, science, and/or mathematics that shape teachers' decision-making and interactions. With regard to our own science/math teacher education pedagogy (Willey & Magee, 2019), we believe that PTs are more likely to develop an awareness of their, often unconscious, biases if they are consistently asked to reflect on how and why they operate as they do. In discussions with PTs we often ask: How do you/teachers respond when students offer answers that do not match your/their expectations? How did you/teachers respond when students are disengaged or resistant to instruction? How do you/teachers perceive how math and science should be taught in schools and how are these ideas discussed with students? What behaviors do you/teachers display when motivating students to engage in challenging curricula?

Teacher dispositions can be framed as how educators respond to students at various moments. Teacher dispositions, like curricular materials and instructional decisions, are often rooted in long-established, oppressive and racist practices that PTs replay with little scrutiny or awareness. Intentionally including teacher dispositions has the potential to push PTs to look beyond curricular materials used to see CRMSP. Dispositions reveal how underlying beliefs about science and mathematics intersect with views of students, and how these beliefs are externalized in messaging about the importance of science and mathematics and students' success with the content. It is the identification of these dispositions and further analysis of them that offers PTs a way to scrutinize and grow their own teaching practice.

To illustrate teacher dispositions in practice we share Trevor's story. Trevor is a secondary alternative certification student studying to be a science teacher. After learning about CRP and CRMSP he used the categories of curricular materials, pedagogical decisions and teacher dispositions to mine his practice for evidence of CRMSP. He shared the following vignette in a written assignment that was shared with his classmates:

> After being out of the classroom for over a month (and being out of their medication for a longer time) the student was disengaged from what we were doing in class. This student has ADHD and with it has difficulty focusing during a note taking session. During this time, I gave the student the notes but filled in, so they can work ahead or go back in case they missed something and sat them in the back of the room on some comfortable furniture. Since doing this, they have taken all of their notes, actively participated during class time, and has not distracted other students during this time.

Trevor described his actions as supporting the student's academic success through the use of a pedagogical decision - using prepared notes in class and offering a comfortable physical space for the student to work. However, in class discussion with Trevor, it was clear that his teacher dispositions were *also* important here and warranted analysis and discussion. By sharing the prepared notes and finding ways to include the student honorably, and without humiliation, Trevor demonstrated his ability to be supportive and to do whatever needed to be done to re-engage the student. Importantly he did this without bribes, threats or negative talk to the student.

Trevor's story presents an opportunity to talk about the often unanalyzed dispositions that are so difficult for PTs to name. Bringing out and interrogating the dispositions was a key goal for the classroom discussion which occurred during a student teaching seminar. After Trevor shared his story in seminar the following question was posed, "How can the frame of whiteness and cultural norms help us analyze Trevor's vignette?" Given that the story doesn't center race or culture explicitly, the PTs were asked to consider the importance of whiteness and cultural norms in this story. Helping PTs see that race influences and informs all that we do as teachers, especially the often "invisible" dispositions, is a critical component of the preparation program goals. Supporting PTs to see where and how they are negatively impacted by whiteness as well, as then they are resistant to its trappings are difficult skills to develop and take time. We will often ask, "What does this have to do with Whiteness at all?" The PTs know that Trevor is a white male and the student in the vignette is Black.

Iceberg of Culturally Relevant Mathematics and Science Pedagogy

The Iceberg of Culturally Relevant Mathematics and Science Pedagogy (see Figure 2 below) is used to illuminate the complexities of internalizing and enacting CRMSP in school settings, where there is little space readily afforded teachers to innovate and create STEM learning experiences grounded in cultural knowledge and experiences. Moreover, the Iceberg of CRMSP expands the rigid and problematic boundaries of how science and math teaching and learning have historically been characterized.

Like the Iceberg of Culture, the top of the Iceberg of CRMSP includes visible teaching practices that are of low emotional load. These curricular materials, instructional decisions, and teacher dispositions are the most accessible for teachers to employ and the least likely to create friction in school settings, where teachers are often closely surveilled. For example, common practices at the top of the iceberg include, celebrating holidays, using students' names and hobbies in word problems, and using texts and trade books that include diverse characters or historical figures. The use of these types of strategies is not bad, and these are, in fact, the ones that our PTs often use. What we find worrisome, and what we use the iceberg metaphor for, is to help our PTs see that these practices sit at the top of the iceberg in an area of low emotional load. They are generally not controversial, and, while not insignificant, they do not really push into a critical space where the status quo is challenged and reshaped. Pedagogically speaking, they are an acceptable place to start but not where we want our PTs to stay.

The teacher practices in the middle of the iceberg are more emotionally challenging and observed less often in science and math classrooms than those at the top. They represent a level of cultural engagement and awareness more emblematic of CRMSP. These practices draw on more fundamental aspects of students' cultural lives, including invisible cultural practices.

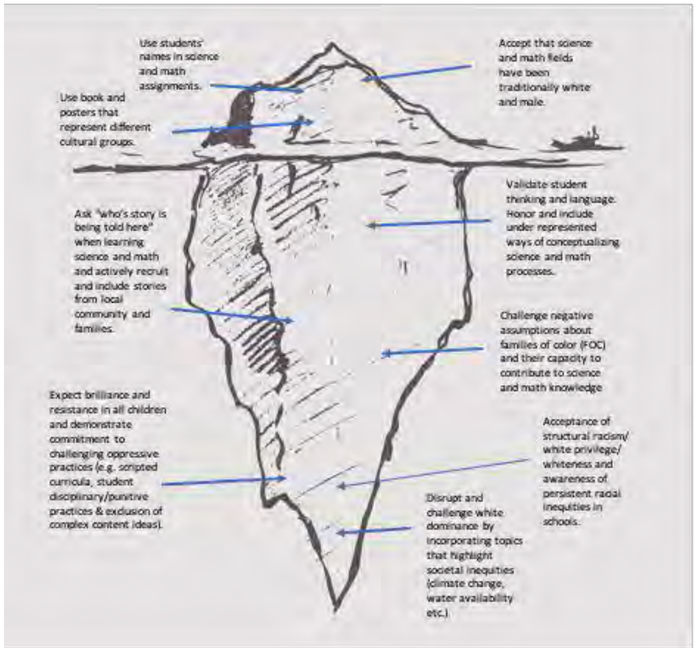


Figure 2: Iceberg of Culturally Relevant Mathematics and Science Pedagogy

Finally, the teacher practices closest to the base of the iceberg are the least likely to be observed in science and math classrooms and are arguably the most tenuous to enact. These moves indicate a strong political and racial consciousness, and a profound understanding of CRP. These practices demonstrate a commitment to all tenets of CRP and to a version of science and math education concerned with the critical science and math identities of the students. For example, one example that we highlight in Figure 2 is "Expect brilliance and resistance in all students and demonstrate commitment to challenging oppressive practices (e.g., scripted curricula, student disciplinary/punitive practices, exclusion of complex content ideas)." In practice this looks like teachers pushing back against skill-based and white-washed curricula and demanding that *all* their students become engaged in learning.

Discussion and Conclusions

Mathematics and science educators have been exploring CRP for at least two decades (e.g., Barton, 2003; Johnson, 2011; Willey & Magee, 2019; Vazquez Dominguez, Allexsaht-Snider, & Buxton, 2018), but it is clear that much work remains to be done. Resistance to change is not surprising given the pervasiveness of structural racism, including how mathematics and science teacher organizations have long appealed to white sensibilities (e.g., Martin, 2015). Preparing new teachers is particularly challenging since most PTs bring to their preparation program unchallenged experiences that support westernized perceptions of math and science, white supremacy, anti-blackness and a superficial understanding of (cultural) difference. As teacher educators immersed in equity work for over 10 years, it is hard to accept the limited progress that PTs make in a year or two of a preparation program. Our response to this dilemma has been to commit to the framework of CRMSP, trusting in the idea that as PTs explore the tenets and CRP holistically, they have the greatest chance of developing long-lasting commitments to racial justice and equitable teaching for all their students. We see that time invested in operationalizing the tenets, interrogating their own cultural identities, and exploring what CRMSP looks like *in practice*, has the potential to support PTs in their long-term development as critical practitioners.

The two-pronged approach that we outline here is grounded in our acceptance that PTs need support as well as opportunities to understand CRMSP. We do not see these activities as a simple protocol, but rather as a way for PTs to understand and internalize the dimensions of CRP, develop a stronger sense of their impact, and to serve as a roadmap for how to live them in real life. We recognize the inherent danger in "breaking down" a framework into manageable pieces as a way to teach it. This is indeed how complex ideas become shallow and superficial (Ladson-Billings, 2014). With this in mind we are mindful of constantly returning to the idea that CRP and CRSMP are not complete without all of the tenets. Working with PTs to see how the concrete and abstract come together - and the theoretical and practical - is the goal of this work.

Supporting PTs to reflect on and analyze their own teaching through the lens of CRP is a critical piece of developing critically conscious teachers. We are hopeful that the approach we describe here can guide research investigating mathematics and science teachers' learning about and implementation of critical pedagogy, and be used within teacher preparation programs and professional development settings to support humanizing, culturally relevant mathematics and science teaching.

Acknowledgements

The preparation of this paper was supported in part by a grant from the National Science Foundation (No. DRL-2101532). The findings and opinions expressed here are those of the authors and do not necessarily reflect the views of the funding agency.

References

Barton, A. C. (2003). *Teaching science for social justice*. New York, NY: Teachers College Press.

Delpit, L. D. (2012). *" Multiplication is for white people": Raising expectations for other people's children*. New York, NY: The New Press.

Emdin, C. (2016). *For White folks who teach in the hood... and the rest of y'all too: Reality pedagogy and urban education*. Beacon Press.

Gay, G. (2002). Preparing for culturally responsive teaching. *Journal of teacher education*, *53*(2), 106-116.

Gay, G. (2018). *Culturally responsive teaching: Theory, research, and practice*. New York, NY: Teachers College Press.

Hall, E. T., & Hall, E. (1976). How cultures collide. *Psychology Today*, *10*(2), 66.

Johnson, C. C. (2011). The road to culturally relevant science: Exploring how teachers navigate change in pedagogy. *Journal of Research in Science Teaching*, *48*(2), 170-198.

Kimball, M., & Smith, N. (2013). The Myth of 'I'm Bad at Math'. *The Atlantic*. Retrieved https://www.theatlantic.com/education/archive/2013/10/the-myth-of-im-bad-at-math/280914/

Kohl, H. (1994). I won't learn from you. *Confronting student resistance in our classrooms. Teaching for Equity and Social Justice*, *134135*.

Ladson-Billings, G. (1995). But that's just good teaching! The case for culturally relevant pedagogy. *Theory into practice*, *34*(3), 159-165.

Ladson-Billings, G. (2006). Yes, but how do we do it? In J. Landsman & C. W. Lewis (Eds.) *White teachers, diverse classrooms: A guide to building inclusive schools, promoting high expectations, and eliminating racism* (pp. 29-42_. Sterling, VA: Stylus Publishing, LLC.

Ladson-Billings, G. (2014). Culturally relevant pedagogy 2.0: aka the remix. *Harvard Educational Review*, *84*(1), 74-84.

Leonardo, Z. (2009). Reading whiteness: Antiracist pedagogy against white racial knowledge. In W. Ayers, T. M. Quinn & D. Stovall (Eds.), *Handbook of social justice in education* (pp. 249-266). New York: Routledge.

Nasir, N. I. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in the mathematics classroom and beyond. *Review of Research in Education*, *32*(1), 187-240.

Nasir, N. I. S. & de Royston, M. M. (2013). Power, identity, and mathematical practices outside and inside school. *Journal for Research in Mathematics Education*, *44*(1), 264-287.
Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational researcher*, *41*(3), 93-97.
Paris, D., & Alim, H. S. (2014). What are we seeking to sustain through culturally sustaining pedagogy? A loving critique forward. *Harvard Educational Review*, *84*(1), 85-100.
Picower, B. (2009) The unexamined Whiteness of teaching: how White teachers maintain and enact dominant racial ideologies. *Race, Ethnicity and Education, 12*(2), 197-215.
Vazquez Dominguez, M., Allexsaht-Snider, M., & Buxton, C. (2018). Connecting soccer to middle school science: Latino students' passion in learning. *Journal of Latinos and Education*, *17*(3), 225-237.
Weaver. G. R. (1986). Understanding and coping with cross-cultural adjustment Stress. In R.M. Paige (Ed). Cross-cultural orientation. New conceptualizations and applications. Lanham MD: University Press of America.
Willey, C. & Magee, P. A. (2019). Whiteness as a barrier to becoming a culturally relevant teacher: Clinical experiences and the role of supervision. *Journal of Educational Supervision, 1*(2), 33-51.
Young, E. (2010). Challenges to conceptualizing and actualizing culturally relevant pedagogy: How viable is the theory in classroom practice?. Journal of Teacher Education, 61(3), 248-260.

SURVEILLANCE AND WORK IN THE MATHEMATICS CLASSROOM

Alexandra Aguilar
University of Texas at Austin
a.aguilar@utexas.edu

Carlos Nicolas Gómez Marchant
University of Texas at Austin
nico.gomez@utexas.edu

Stacy R. Jones
University of Texas at Austin
srjones@utexas.edu

Amy R. Johnson
University of Texas at Austin
arjohnson@utexas.edu

Gerardo Sanchez Gutierrez
University of Texas at Austin
gerardo.sanchez@utexas.edu

Karina Méndez Pérez
University of Texas at Austin
kmendezp@utexas.edu

Chandel F. Burgess
University of Texas at Austin
chandelburgess@utexas.edu

Sara Breidenich
University of Texas at Austin
sarah.breidenich@utexas.edu

There has been growing interest in the development of theoretical knowledge on the design of learning environments supporting nondominant communities (Bell et al., 2017). We explore the ways students are under surveillance (Lugo, 2007) in the mathematics classroom. Particularly, how student responses to being under surveillance inform the development of equitable mathematical learning environments. We identify mathematics as a social practice (D'Ambrosio & D'Ambrosio, 2013), defined by its accompanying context (e.g., school, every day, workplace; Moschkovich, 2002a). School mathematical discourse represents a language of power (Delpit, 1988). Free use of the word 'mathematics 'outside of the classroom represents an important step toward recognizing non-school mathematics as mathematics and challenging the intellectual hierarchy permitting school mathematics dominance over all other mathematics. In our research all practices of mathematics are referred to as *mathematics*, excluding the mathematics typically found in US classrooms, which we refer to as *school mathematics*.

The white construction of school mathematics (D'Ambrosio, 1985; Joseph, 2010) carries with it a history of cultural erasure (Dowling, 1991; Gutiérrez et al., 2017) and predominantly English-only enactment in US classrooms (Gutierrez, 2002; Moschkovich, 2002b). Students are constantly building upon their linguistic and mathematical repertoires of practice (Garcia-Sanchez & Orellana, 2019), and the mathematics classroom is one of many arenas in which their knowledge resists disregard and fights to assert and maintain its value. There is violence when dominant mathematics and languaging practices encroach upon student borders of their own mathematics and language, shifting and reshaping these borders (Anzaldua, 1987). Surveillance in the mathematics classroom is influencing how students speak and what they say. Students are learning, through responses to seen and unseen work, how to adhere to dominant mathematics classroom cultural norms. In order to begin healing from this violence, for students to trust in themselves and their instructors mathematically, we must understand how that violence manifests in the classroom. Work (Star & Strauss, 1999) students do in the mathematics classroom, both traditionally seen and unseen, is one such manifestation.

We discuss how students are responding to their surveillance and what work should be considered in the development of classroom mathematical experiences. In this poster, we analyze how elementary Latiné students discussed the surveillance of their language in the context of the mathematics classroom, how they expressed the role of mathematics in their world, and how they incorporated school mathematical discourse into their communications of mathematics.

References

Anzaldúa, G. (1987). *Borderlands/La frontera: The new mestiza.* San Francisco, CA: Aunt Lute Books.

Bell, P., Van Horne, K., & Haugan Cheng, B. (2017). Special Issue: Designing learning environments for equitable disciplinary identification. *Journal of the Learning Sciences*, 26(3), 367–375. https://doi.org/10.1080/10508406.2017.1336021

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the learning of Mathematics*, 5(1), 44–48.

D'Ambrosio, U., & D'Ambrosio, B. S. (2013). The role of ethnomathematics in curricular leadership in mathematics education. *Journal of Mathematics Education at Teachers College*, 4(1).

Delpit, L. (1988). The silenced dialogue: Power and pedagogy in educating other people's children. *Harvard educational review*, 58(3), 280–299.

Dowling, P. (1991). The contextualizing of mathematics: towards a theoretical map. *Schools, mathematics and work*, 93–120.

García-Sánchez, I. M., & Orellana, M. F. (Eds.). (2019). *Language and cultural practices in communities and schools: Bridging learning for students from non-dominant groups*. Routledge.

Gutierrez, R. (2002). Beyond essentialism: The complexity of language in teaching mathematics to Latina/o students. *American educational research journal*, *39*(4), 1047–1088.

Gutiérrez, K. D., Cortes, K., Cortez, A., DiGiacomo, D., Higgs, J., Johnson, P., Lizárraga, J. R.,

Mendoza, E., Tien, J., & Vakil, S. (2017). Replacing representation with imagination: Finding ingenuity in everyday practices. *Review of Research in Education*, 41, 30–60. https://doi.org/10.3102/0091732x16687523

Joseph, G. G. (2010). *The crest of the peacock: Non-European roots of mathematics*. Princeton University Press.

Lugo, A. (2000). Theorizing border inspections. *Cultural dynamics*, *12*(3), 353–373.

Moschkovich, J. N. (2002a). Chapter 1: An introduction to examining everyday and academic mathematical practices. *Journal for Research in Mathematics Education. Monograph*, 1–11.

Moschkovich, J. (2002b). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical thinking and learning*, 4(2-3), 189–212.

Star, S. L., & Strauss, A. (1999). Layers of silence, arenas of voice: The ecology of visible and invisible work. *Computer supported cooperative work (CSCW)*, 8(1), 9–30.

TEACHERS' USE OF "ROUGH DRAFT MATH" AS A CURE FOR TEACHING DURING THE PANDEMIC

Crystal Collier
University of Delaware
crystalc@udel.edu

Amanda Jansen
University of Delaware
jansen@udel.edu

Keywords: Equity, Inclusion, and Diversity; Instructional Activities and Practices

Rough Draft Math (Jansen, 2020) is a teaching practice in which mathematics teachers strive to welcome and integrate students' in-progress thinking as a valuable part of their learning process, inspired by the concept of "exploratory talk" (Barnes, 2008). When enacting this practice, teachers foster a mathematics learning environment where students feel safe to share their unfinished ideas knowing that they will have the opportunity to revise and refine those ideas as they continue to learn. This practice also has the potential to position students as mathematical authorities in the classroom and foster students' sense of ownership of their learning.

Due to the global pandemic, the landscape of education has changed. K–12 math teachers are being asked to create highly flexible and engaging learning environments in which all students are intellectually challenged. Applying a strengths-based lens to students' thinking has potential to challenge the deficit narrative of "learning loss" that has been prevalent during this pandemic (Chen & Krieger, 2022). As an example of changes in teaching, Roman et al. (2021) found that secondary mathematics teachers increasingly attended to supporting students' affective and social engagement over time in their instruction as the COVID-19 pandemic unfolded.

In the spirit of ambitious mathematics teaching (Lampert et al., 2013), "rough draft math" can support teachers to engage students with mathematical content, specifically by encouraging students to explain their own in-progress thinking and to elaborate on that of their peers. It also provides a way for teachers to create a more equitable classroom by publicly valuing all students' thinking at any stage, as well as validating the notion that learning is as an ongoing process. The guiding question for this research was: *How do K-12 mathematics teachers enact Rough Draft Math in their classrooms within the context of the COVID-19 pandemic?*

Data were gathered, by interview, from 32 teachers who participated in book studies about *Rough Draft Math* (Jansen, 2020) from five states around the U.S.A. During the interviews, teachers were asked to explain how they enacted rough draft math in their classrooms and to share what motivated their use of it. The teachers were also asked to share a digital artifact to illustrate their enactments and to reflect on the perceived benefits and challenges associated with their enactments of "rough draft math." Data analysis allowed for the emergence of themes from teachers' responses (Corbin & Strauss, 1990).

In this poster, we will share unanticipated findings – teachers' voices about how "rough draft math" helped them navigate difficulties of teaching during the pandemic. Teachers expressed that they worked to extend grace to their students as they encountered instability and frustrations. Teachers reported that their responses to changes in the world included becoming more open to welcoming students' draft thinking and embracing continuous revision of ideas; they reported "really needing this" [rough draft math] so that they could meet students as they were, to gradually support students to develop their ideas through revising, and to increase empathy in the classroom. "Rough draft math" helped teachers move away from a deficit perspective toward learners so they could support all students to learn together from one another.

References

Barnes, D. (2008). Exploratory talk for learning. In N. Mercer & S. Hodgkinson (Eds.), *Exploring Talk in School* (pp. 1–15). London: Sage.

Chen, J. J., & Krieger, N. J. (2022). Learning gain rather than learning loss during COVID-19: A proposal for reframing the narrative. *Contemporary Issues in Early Childhood*, 14639491211073144.

Jansen, A. (2020). *Rough Draft Math: Revising to Learn*. Portsmouth, NH: Stenhouse Publishers.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, *64*(3), 226-243.

Roman, T. A., Brantley-Dias, L., Dias, M., & Edwards, B. (2021). Addressing student engagement during COVID-19: Secondary STEM teachers attend to the affective dimension of learner needs. *Journal of Research on Technology in Education*, 1-29.

Corbin, J. M., & Strauss, A. (1990). Grounded theory research: Procedures, canons, and evaluative criteria. *Qualitative Sociology*, *13*(1), 3-21.

LEVERAGING A CASE TO DISRUPT THE OVER-IDENTIFICATION OF BLACK MALES IN MATHEMATICS SPECIAL EDUCATION

Monica Gonzalez
East Carolina University
gonzalezmo18@ecu.edu

Alesia Mickle Moldavan
Georgia Southern University
amoldavan@georgiasouthern.edu

Annie George-Puskar
Fordham University
ageorgepuskar@fordham.edu

Keywords: Preservice Teacher Education; Special Education; Equity, Inclusion, and Diversity

Historically, there has been an over-identification of Black students in special education in early childhood, even more specifically Black males (Cruz & Rodl, 2018). Black males are 1.4 times more likely to be referred to special education than students of other racial backgrounds (Gordon, 2017). Teachers' biases about academic success and the behaviors deemed appropriate and respectful in a classroom can influence instructional decision-making and students' mathematics learning (Moldavan et al., in press). This study reports on a multi-site, interdisciplinary research collaboration where preservice teachers (PTs) from programs of mathematics, special education, and early childhood education participated in case-based instruction (Decker & Pazey, 2017; Gorksi & Pothini, 2018) to explore racial biases during a kindergarten mathematics task. The following research question guided this study: How do PTs recognize and respond to race within a case designed to elicit discussion about racial bias in mathematics special education?

Racial noticing (Shah & Coles, 2020) was used as a lens to attend to the various ways PTs perceive, make sense of, and react to moments where race and racism occur during a mathematics task. We used this framework to guide the case design, which explored a kindergarten student named Tay who exhibited typical 5-year-old behaviors while engaging in a counting collections task. In the case, a White teacher was frustrated with Tay's seemingly disruptive behavior, which overshadowed his counting strategies and knowledge. When faced with the disruptive behavior, the teacher removed Tay from the task, attributing his actions to a learning disability. Following the case, the PTs were asked prompts to engage in discussion related to the teacher's instructional decisions and biases as well as Tay's behavior and mathematical understanding of counting. We analyzed the PTs' responses by first using a priori coding to look for instances of race within the responses (Saldaña, 2016). Once those instances were identified, we then used in vivo and descriptive coding to further analyze how those PTs attended to, interpreted, and responded to race-related phenomena occurring within the case. The codes were cross-checked for inter-rater reliability to corroborate the findings (Grbich, 2013).

Out of the 60 PTs in the study, only 9 PTs (15%) mentioned Tay's race in their responses. Interestingly, the 9 PTs only mentioned Tay's race in one of the seven discussion prompts. That prompt was: How do you think Ms. Caldwell's implicit bias impacted her instruction with Tay? A few PTs (n=4) explicitly stated that Ms. Caldwell had a bias toward Tay due to his skin color, and the others mentioned Tay's race without making a connection to Ms. Caldwell's race (n=3) or an observed racial phenomenon (n=2). In addition, 80% (n=48) of the PTs indicated that Tay's behavior was age appropriate. The other PTs considered Tay's behavior to be "delayed" or "disruptive" from his peers, justifiable for removal from the task. Implications from this study indicate that PTs need explicit opportunities and prompts to recognize racial biases, including how such biases can impact instruction and students' mathematics learning. Cases can be used in teacher education to disrupt the over-identification of Black males in special education to ensure appropriate supports are made to empower rather than dismiss Black males in early childhood.

References

Cruz, R. A., & Rodl, J. E. (2018). An integrative synthesis of literature on disproportionality in special education. *The Journal of Special Education*, *52*(1), 50–63.

Decker, J. R., & Pazey, B. L. (2017). Case-based instruction to teach educators about the legal parameters surrounding the discipline of students with disabilities. *Action in Teacher Education*, *39*(3), 255–273.

Gordon, V. H. A., Jr. (2017). Disciplining students receiving special education. *Online Submission*, 1–9.

Gorski, P. C., & Pothini, S. G. (2018). *Case studies on diversity and social justice education* (2nd ed.). Routledge.

Grbich, C. (2013). *Qualitative data analysis: An introduction* (2nd ed.). SAGE.

Moldavan, A. M., Gonzalez, M. L., & Kaufman, B. A. (in press). Talking about race in mathematics teacher education: An analysis of online community dialogues. *The Teacher Educators' Journal.*

Saldaña, J. (2016). *The coding manual for qualitative researchers* (3rd ed.). SAGE.

Shah, N., & Coles, J. A. (2020). Preparing teachers to notice race in classrooms: Conceptualizing the competencies of preservice teachers within antiracist inclinations. *Journal of Teacher Education*, *71*(5), 584–599.

RECOMMENDATIONS FROM LATINX, INDIGENOUS, LGBTQ+, AND/OR WOMEN STEM-MAJOR STUDENTS TOWARD ONE REHUMANIZING DIMENSION

Kelly MacArthur
Montana State University
kellymacarthur@montana.edu

Keywords: Calculus; Undergraduate Education; Equity, Inclusion and Diversity; Rehumanizing

Many scholars have advocated for more sociopolitically oriented research in mathematics education (e.g. Adiredja & Andrews-Larson, 2017; Aguirre et al., 2017; Gutiérrez, 2013) to address inequities. In that vein, Gutiérrez (2018) created an eight-dimension framework highlighting ways to rehumanize mathematics with a distinct call to center the voices of students from historically marginalized groups. Further, the gatekeeping function of undergraduate calculus courses for STEM majors has been well documented (e.g. Gasiewski et al., 2012), especially for marginalized students (e.g. Leyva et al., 2021). This study considers one particular dimension of the Gutiérrez framework (2018), namely Theirstories/Culture, and investigates how educators can move toward rehumanizing goals for STEM-major students. This dimension aligns with teacher moves that acknowledge and center differing ways for people to do mathematics.

In this mixed methods study, I address two research questions: (1) Is there a difference in how students from dominant vs. focal groups view a hypothetical scenario relating to the Theirstories/Cultures rehumanizing dimension?; and (2) What can instructors do to implement this scenario in ways that positively support students from historically marginalized groups?

To answer these questions, I analyzed survey data from 153 STEM-major Calculus 2 students at two PWIs. From this data, two groups were considered, namely the dominant group (73 white, cis-male, heterosexual students) and the focal group (80 students who identified as being in one or more of the marginalized groups listed above). From the survey responses, 20 students were interviewed, such that all interviewees self-identified as one or more of the following: Native American, Latino/a, Hispanic, Mexican, mixed race, LGBTQ+, or female. In particular, this report focuses on the results of one scenario from the survey regarding the Theirstories/Cultures rehumanizing dimension (Gutiérrez, 2018), which in part states: *Imagine that every week in your calculus course, the instructor spends a few minutes in class to discuss a current-day Indigenous, Black, Latinx, female or LGBTQ+ mathematician and showcase a bit of their work.* Students were asked the following three Likert questions regarding this scenario, with a 7-point scale ranging from Strongly Disagree to Strongly Agree for each question. A. I would likely feel valued and an increased sense of belonging. B. I would likely feel a sense of connection between mathematics and my life. C. I would likely feel supported in my learning.

Using statistical techniques to analyze the Likert questions for this scenario, it was found that the distribution of responses to each question were significantly different with a higher proportion of students from the focal group reporting positive rankings. Focal group interviewees further discussed how this scenario would boost their connections with peers and the instructor as well as their confidence by providing role models of mathematicians who looked like them. They also voiced concern that this scenario could be dehumanizing if the instructor did not implement it thoughtfully and judiciously. Interviewees recommended instructors incorporate this practice in ways that don't interrupt their covering the necessary mathematics curriculum and that take an appropriate amount of class time while also providing links to read about the mathematicians outside of class time.

References

Adiredja, A. P., & Andrews-Larson, C. (2017). Taking the sociopolitical turn in postsecondary mathematics education research. *International Journal of Research in Undergraduate Mathematics Education*, 3(3), 444–465.

Aguirre, J., Herbel-Eisenmann, B., Celedón-Pattichis, S., Civil, M., Wilkerson, T., Stephan, M., Pape, S., & Clements, D. H. (2017). Equity within mathematics education research as a political act: Moving from choice to intentional collective professional responsibility. *Journal for Research in Mathematics Education*, 48(2), 124–147.

Gasiewski, J. A., Eagan, M. K., Garcia, G. A., Hurtado, S., & Chang, M. J. (2012). From gatekeeping to engagement: A multicontextual, mixed method study of student academic engagement in introductory STEM courses. *Research in higher education*, *53*(2), 229-261.

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37–68.

Gutiérrez, R. (2018). Introduction: The need to rehumanize mathematics. In *Rehumanizing mathematics for Black, Indigenous, and Latinx students*. National Council of Teachers of Mathematics.

Leyva, L. A., Quea, R., Weber, K., Battey, D., & López, D. (2021). Detailing racialized and gendered mechanisms of undergraduate precalculus and calculus classroom instruction. *Cognition and Instruction*, 1–33.

BLACK TEACHERS' TRAJECTORIES: STORIES OF EARLY LEARNING, NAVIGATING INEQUITIES, AND TEACHING ELEMENTARY MATHEMATICS

Kayla Myers
Georgia State University
kmyers@gsu.edu

Susan Swars Auslander
Georgia State University
sauslander@gsu.edu

Sterline Caldwell
Georgia State University
scaldwell16@student.gsu.edu

Shani Walker-Jackson
Gwinnett County Public Schools
shani.walker@gcpsk12.org

When I was in kindergarten, which was a long time ago, I was very, very quiet, and because I was so quiet, my teacher felt that I just didn't know the content. So, towards the end of the year, she recommended to my mom that I repeat kindergarten. My mom asked her, of course, "why?" She said, "well, she doesn't know how to read. She doesn't know her numbers." And so, my mom said, "okay well can I come observe tomorrow? Can I come sit in the class?" And so my mom came and every question that the teacher asked, she asked me, and I would answer. So I knew all the content, but just because I didn't raise my hand or present the information, I was labeled as not knowing it. And at the time, as a kindergartener, you don't really think about it [equity or inequity]. But I think my mom, just being a person of color, recognized that this is the only Black child in the class, and she thinks she doesn't know anything, because she's a Black child and because she's so quiet. And the teacher said I didn't know my colors because I colored the sun purple or something. My mom said, "why'd you color the sun purple?" I said, "because all the kids were fighting over the yellow [crayons] and it wasn't that important to me." So just being passive and quiet, I just kind of got labeled in kindergarten as not knowing. And they did move me on to first grade and the first-grade teacher told my mom, like, I definitely should not have been kept back. I was bright. I just kept to myself. And I just didn't want to fight over the crayons.

This presentation highlights a compelling story of dissonance and harmony—Black teachers' early histories as learners of mathematics, and how those early experiences have shaped their identity as learners and teachers of mathematics. There is an aligned dissonance in these stories, one of inequity and barriers, that has harmonized as they find themselves together striving to teach for equity and social justice, to build their students' positive mathematical identities, and to be leaders in their schools.

Teacher identities are deeply connected to early learning, communities, and professional experiences (Aguirre, Mayfield-Ingram, & Martin, 2013). During both individual and focus group interviews with 27 elementary mathematics teachers, we asked about those early learning experiences, the support and community they had as young learners, and how those have shaped them into the teachers they are today. In our analysis of that interview data about histories with mathematics learning, we kept returning to provocative stories about navigating inequities and redirecting obstacles in order to better serve the students in their classrooms. For this poster, and as we prepare for publication, 5 Black elementary mathematics teachers share stories of feeling inequities as young learners of mathematics. These stories spoke to us as mathematics teacher educators, as qualitative researchers, and as social justice advocates. These inequities, though they vary in scope, had a profound impact on the professional trajectory for these 5 Black elementary mathematics teachers, as well as their journey towards teacher leadership. Our poster presentation will highlight these teachers' stories, provoking an important conversation about the significance of racial injustice on mathematics learners and identity.

References

Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K-8 mathematics: Rethinking equity-based practices.* Reston, VA: National Council of Teachers of Mat

SYSTEMIC BARRIERS TO COLLABORATIVELY DESIGNING FOR HIGH QUALITY MATHEMATICS INSTRUCTION

Olanrewaju T. Oriowo
UNC Charlotte
ooriowo@uncc.edu

Christine K. Fisher
UNC Greensboro
cakierna@uncg.edu

P. Holt Wilson
UNC Greensboro
phwilson@uncg.edu

Keywords: Instructional vision; systemic change; equity, sustainability.

Philosophers and critical thinkers have advocated for all students to be given access to high quality education irrespective of ethnicity or social status (e.g., Adler, 1982; Dewey, 1900; Freire, 1970; Ladson-Billings & Tate, 1995). The state of mathematics in the US K-12 education system has not experienced successful mathematical reform that projects a unified vision of high-quality mathematics instruction (VHQMI) (Munter, 2014). The purpose of this poster is to explore some systemic barriers that thwart the best intentions of mathematics educators, administrators, and policy makers at all levels of the US K-12 education system, in developing and designing around a common vision of high-quality mathematics instruction. We articulate three context theories which Edelson (2002) defines as theories that highlight "challenges and opportunities presented by a class of design contexts" (p. 113); these include certain types of education structures, inequitable distributions of funding, and sociopolitical influences.

Education systems in the US are decentralized and hierarchical, exacerbating systemic barriers; curriculum control is an example of such a barrier. When schools and districts with differing fiscal resources make mathematics curriculum decisions but are subject to the same state level accountability measures, inconsistent visions are inevitable (DeBoer, 2012). Inequitable funding across subsystems inhibits the development of shared VHQMI; because funding relies on local revenue, access to curricular resources, high quality teaching, and professional development vary, perceptibly, in districts within a state system. Underfunded districts and schools become scapegoats when they are penalized for failure to meet unfunded mandates and for their inability to provide necessary resources to support instruction (Robinson, 2015). The sociopolitical nature of education is an additional barrier to the growth of a shared VHQMI. Instructional decisions are made by policy makers and stakeholders vying for positions of power without regard to the best interests of students, teachers, and educational leaders (DeBoer, 2012). For example, the adoption of Common Core State Standards for Mathematics (CCSSM) became, according to Larson (2012), a tug-of-war between attention to standards and the need for discussions that impact student learning of mathematics.

We believe that system-wide collaboration and co-designing of implementation resources improves equitable access to material and social resources. Research Practice Partnerships can offer a solution by bringing together policy makers, researchers, and educators (Henrick et al., 2017) to co-design and share resources. Such an initiative will stabilize resources for instruction and learning for transient students and teachers; all schools can have access to the expertise that will be distributed across subsystems.

Acknowledgments

This work was supported by the National Science Foundation (NSF) under grant DRL-2100895 to East Carolina University, DRL-2100903 to UNC Greensboro, DRL-2100947 to UNC Charlotte, and DRL-2100833 to Appalachian State University. Any opinions are solely those of the principal investigators and do not necessarily reflect the views of the NSF.

References

Adler, M. J. (1982). The paideia proposal: An educational manifesto. Macmillan Publishing.
Dewey, J. (1916, 2001). Democracy and education. Pennsylvania State University.
DeBoer, J. (2012). Centralization and Decentralization in American Education Policy. Peabody Journal of Education, 87(4), 510–513. https://doi.org/10.1080/0161956X.2012.705153
Edelson, D. C. (2002). Design research: What we learn when we engage in design. The Journal of the Learning sciences, 11(1), 105-121.
Freire, P. (1970, 2000). Pedagogy of the oppressed. Continuum International Publishing.
Henrick, E.C., Cobb, P., Penuel, W.R., Jackson, K., & Clark, T. (2017). Assessing Research-Practice Partnerships: Five Dimensions of Effectiveness. William T. Grant Foundation.
Munter, C. (2014). Developing visions of high-quality mathematics instruction. Journal for Research in Mathematics Education, 45(5), 584-635.
Robinson, S. (2015). Decentralisation, managerialism and accountability: professional loss in an Australian education bureaucracy. Journal of Education Policy, 30(4), 468–482. https://doi.org/10.1080/02680939.2015.1025241
Ladson-Billings, G., & Tate, W. F. (1995). Toward a theory of Culturally Relevant Pedagogy. American Education Research Journal, 32(3), 465-491. https://doi.org/10.2307/1163320
Larson, M. R. (2012). Will CCSSM matter in ten years? Teaching Children Mathematics, 19(2), 108–115.

MATH AS A GATEKEEPER: NARRATIVE IDENTITY OF ONE COMMUNITY COLLEGE STUDENT IN NON-CREDIT BEARING REMEDIAL MATH

Ranza Veltri Torres
University of Wisconsin-Madison
rveltri@wisc.edu

Keywords: Equity, Inclusion, and Diversity; Systemic Change; Undergraduate Education

Racially minoritized students are disproportionately tracked into non-credit-bearing remedial (NCBR) math classes upon entering college and are often trapped in a cycle of take-fail-repeat. While math is positioned as more important than ever before for a diverse pool of learners, inequitable access to quality math-learning experiences prevails. This poster features a case study of one participant, Marty, a furniture maker and community college (CC) graduate who identifies as Latina, whose story may help us see how certain math requirements and learning experiences may serve as a dehumanizing force in some students' lives and function to shape their mathematical identities. I address the following research questions:

RQ1: How do CC students who have taken NCBR math identify as math learners and doers?
RQ2: What mathematical activities or what kinds of discourses do students articulate for themselves that evidence the kinds of mathematics identities they associate/identify with?

My theories and methods privilege narrative in order to center the voices of mathematics learners and their interpretations of their experiences, rather than analysis about their experiences. I use narrative theory as defined by Langer-Osuna and Esmonde (2017): "[m]athematical identities develop as people make sense of their experiences with mathematics and develop stories of success and failure and belonging or distance." I combine this with the frame of mathematics identity (as defined by Larnell, 2016), who defines math identity as "...a set of personal narrativizations about mathematics-learning or mathematics-teaching experiences that reify, endorse, and signify their subjects as a certain kind of mathematics user or doer."
Two 90-minute interviews are analyzed using Martin's Central Themes of Math Identity (2000), and findings align with four of the six themes:

a) the instrumental value of mathematics, (Marty says she doesn't need math to make furniture, when she engages in mathematical problem-solving techniques in her furniture making.)
d) strategies to learn or participate in formal and informal mathematics contexts, (when Marty sees her strategies as mathless, and in her mind is avoiding formal math)
e) constraints or barriers on participation within mathematics-learning contexts (she describes how math hasn't posed a barrier for her, because she other ways of figuring things out)
f) one's own capacity to perform in mathematics-learning contexts. (Marty refers to herself as "dumb" and unable to engage in formal math)

In relation to RQ1, Marty refers to herself as "dumb" (her words) and unable to engage in formal math. However, Marty often engages in mathematical activity without realizing she is doing so. Marty sees her furniture building as mathless. In relation to RQ2, Marty articulates a discourse of math avoidance, despite the fact that she unknowingly uses math every day in her work. Implications press for deeper thought about how CCs can implement supportive and appropriate math pathways that are best suited to students' academic and career goals.

References

Langer-Osuna, J. M. & Esmonde, I. (2017). Identity in research on mathematics education. *Compendium for research in mathematics education*, 637-648.

Larnell, G. (2016). More than just skill: Examining mathematics identities, racialized narratives, and remediation among Black undergraduates. *Journal for Research in Mathematics Education*, 47(3), 233-269.

Martin, D. B. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency*. Mahwah, NJ: Erlbaum.

SUPPORTS AND CONSTRAINTS FOR LGBT+ STUDENTS SEEKING COLLEGE DEGREES IN STEM

Kellie J. Pop
University of Nevada, Reno
kpop@nevada.unr.edu

Lynda R. Wiest
University of Nevada, Reno
wiest@unr.edu

Keywords: LGBTQIA+; Equity, Inclusion, and Diversity; Higher Education

The purpose of this study was to investigate possible supports and constraints students pursuing STEM degrees experienced during their transition from high school to college. Only recently have scholars begun to recognize the heteronormativity of STEM and the disconnect that can thus occur between the STEM disciplines and students who identify as LGBT+ (e.g., Cech & Waidzunas, 2011; Rubel, 2016). Cultural norms can result in invisibility for LGBT+ students and ultimately academic and social isolation (Bright, 2016; Cech & Waidzunas, 2011; Dzurick, 2018). Gottfried, Estrada, and Sublett (2015) note, "We surprisingly know very little about STEM education as it relates to the fuller spectrum of sexual diversity" (p. 66). Creating a "gender-complex" education requires greater understanding of LGBT+ students' experiences, and this must extend to STEM (Rands, 2013). This research provides insight into this area, which has received little attention to date (Cooper & Brownell, 2016).

Methods

Recruitment of LGBT+ college students took place through social media posts and notifications through LGBT+ college student organizations and community organizations in one region of a Western state. Four students pursuing college STEM degrees participated in one-on-one interviews of 20-40 minutes each with the first author: one lesbian, one gay man, one bisexual man, and one male-to-female transgender student.

Interviews focused on what supports and constraints participants perceived they had in high school and college in relation to pursuing a STEM degree and career. The interview transcripts were analyzed into thematic categories by identifying main ideas in the data, a process that was not linear (Cresswell, 2014; Gay et al., 2012). The hand-coded categories were constructed independently and then jointly by the two authors during multiple readings of the data and adjusted as needed until they were perceived to represent the data accurately. The categories were reviewed for their individual meaning and importance and for larger themes in relation to the research question.

Results & Implications

In terms of encouragement for pursuing STEM, no consistency appeared across the four participants. The following influences were reported once each: self, teacher, family, friends, and nature of the field. In naming deterrents, participants were equally likely to state incongruence of an LGBT+ identity with STEM and the challenge of the subject matter. (Participants often named more than one influence in each of those categories.) Suggestions participants offered for educators for supporting LGBT+ students in STEM were highly varied and included rigor, encouragement, emotional/psychological support (e.g., let them know they are not alone and signal a safe environment), and preparation for possible mistreatment/opposition from others.

References

Bright, A. (2016). The problem with story problems. *Rethinking Schools*, *30*(4), 14-19.

Cech, E. A., & Waidzunas, T. J. (2011). Navigating the heteronormativity of engineering: The experiences of lesbian, gay, and bisexual students. *Engineering Studies*, *3*(1), 1-24.

Cooper, K. M., & Brownell, S. E. (2016). Coming out in class: Challenges and benefits of active learning in a biology classroom for LGBTQIA students. *CBE - Life Sciences Education*, *15*(3).

Cresswell, J. W. (2014). *Research design: Qualitative, quantitative, and mixed methods approaches* (4th ed.). Thousand Oaks, CA: SAGE.

Dzurick, A. (2018). A culture of acceptance: Practicing “science for all” means acknowledging and including lesbian, gay, bisexual, and transgender students in the classroom. *Science Teacher*, *85*(4), 18-21.

Gay, L. R., Mills, G. E., & Airasian, P. (2012). *Educational research: Competencies for analysis and applications* (10th ed.). Boston: Pearson.

Gottfried, M., Estrada, F., & Sublett, C. (2015). STEM education and sexual minority youth: Examining math and science coursetaking patterns among high school students. *The High School Journal*, *99*(1), 66-87.

Rands, K. (2013). Supporting transgender and gender-nonconforming youth through teaching mathematics for social justice. *Journal of LGBT Youth*, *10*(1-2), 106-126.

Rubel, L. H. (2016). Speaking up and speaking out about gender in mathematics. *Mathematics Teacher*, *109*(6), 434-439.

Chapter 5:

Geometry and Measurement

TANGRAM PUZZLES' PRIMING OF SQUARES FOR MATHEMATICS EDUCATORS

Laura Bofferding
Purdue University
lbofferd@purdue.edu

Although solving tangram puzzles is a popular activity for helping students move along the 2D shape composition learning trajectory, there is little evidence of what makes one tangram puzzle with no internal lines more difficult than another. In this study, 46 educators saw either a square in a traditional or non-traditional orientation and then were asked to identify (within 4 seconds) where the square puzzle piece would go in a series of 75 tangram puzzles. Results indicate that adults were more likely to identify the location of the square if it had more sides outlined, but the orientation of the square or number of places it would fit in the puzzle were not associated with their success. These results call for similar studies with children and suggest that educators need to be careful to look beyond their own impressions when choosing suitable puzzles for students.

Keywords: Geometry and Spatial Reasoning; Cognition

As children move through the 2D shape composition learning trajectory, they learn to combine shapes, using turns and flips, and explore what new shapes they can create (Clements & Sarama, 2009, 2014). To encourage students' movement through the levels of the learning trajectory, teachers can use shape puzzles that increase in complexity, moving from shape puzzles—such as tangram puzzles—with each piece separate and outlined, through pieces being connected but still outlined, to puzzles where there are no internal lines (i.e., silhouette puzzles) and pieces need to be turned (Clements & Sarama, 2009, 2014). However, there are a variety of silhouette tangram puzzles, which have varying numbers of sides, piece orientations, and number of solutions that might increase or decrease puzzles' levels of difficulty (Bofferding & Foster, 2021). When choosing a puzzle to support or a challenge a student, educators might consider *their own impressions* of whether particular pieces are easy to place. This study explores whether silhouette tangram puzzles encourage correct or incorrect placement of the square piece based on educators' initial impressions and highlights which factors play a role in their choices.

Framework

Given the 2D shape composition learning trajectory (Clements & Sarama, 2009, 2014) and an analysis of tangram puzzles (Bofferding & Foster, 2021), there are several factors that could contribute to whether the placement of a shape piece is easy or difficult to identify in a silhouette tangram puzzle. One of the initial factors that distinguishes difficulty between puzzles for preschool children is whether each shape piece stands alone or is combined with another piece (Clements & Sarama, 2009, 2014). Therefore, the extent to which each side of the shape pieces are exposed, or part of the puzzle outline, could make it easier to identify. Likewise, silhouette tangram puzzles with more sides would likely have more sides of the individual shapes exposed (Baran et al., 2007) and make it easier to identify where the pieces go (van den Heuvel-Panhuizen & Buys, 2008). Certain puzzles also can be solved in multiple ways (Bofferding & Foster, 2021), which could play a role in how easy placing a particular piece is in the puzzle. Another important factor is the orientation of the pieces. Baran et al. (2007) found that adults had difficulty placing the square piece in tangram puzzles if the square needed to be placed with a corner upward (i.e., a nonstandard orientation for how it is often seen), a difficulty shared by children (Clements & Sarama, 2009, 2014). The adults also placed larger pieces in the puzzle first (Baran et al., 2007), making the case of the square all the more interesting, especially as the square and two small triangles can make up the large triangle.

Given the relative difficulty of the square piece and relative lack of research on tangram puzzle difficulty, we chose to further investigate how adults identify squares in puzzles. The research questions included the following: (1) To what extent are educators' judgements on where the square should go associated with the silhouette tangram puzzles' number of sides, the squares' number of sides outlined in the puzzle, the squares' orientation in the puzzle, number of locations the square could go in the puzzle, and the orientation in which they were shown the square? (2) What characterizes educators' choices on where the square should go?

Method

Participants included 46 educators (e.g., elementary teachers, high school teachers, math coaches, professors) and graduate students in education. Participants accessed a link to a Qualtrics survey through a message posted to mathematics education professional groups on Facebook and the National Council of Teachers of Mathematics message board. The first part of the survey asked participants to indicate their occupation and demographics. Eight participants indicated male pronouns, thirty-six indicated female pronouns, and two did not answer. Further, 41 participants identified as White, one identified as Black or African American, one identified as Hispanic, Latino, or Spanish Origin, one did not disclose, and two people did not answer.

Participants were randomly assigned to one of four versions of the survey, which consisted of ten sections. For sections 1-4 and 6-9 participants were shown a square presented in a standard or nonstandard orientation and asked to indicate where the square would go in a series of silhouette tangram puzzles. Instructions indicated that the square might have to be turned to fit into the puzzle. Participants had four seconds to click on the part of the puzzle where the square would go; therefore, they did not actually have to fit a square in the puzzle. There were ten silhouette puzzles in each section, and if participants saw the square in standard orientation for sections 1-4, they saw the square in nonstandard orientation for sections 6-9 and vice versa. Two more versions of the survey were created by reversing the order of all of the puzzles (except session 10 was always last). In section 5, participants had to identify where a parallelogram would go in 10 tangram puzzles, and in section 10, participants were shown an arrangement of puzzle pieces and had to pick the silhouette tangram puzzle that had the same arrangement (out of four choices). Then, they described how they determined where the square and parallelogram should go in the puzzles (see Figure 1 for an outline of the four versions of the survey).

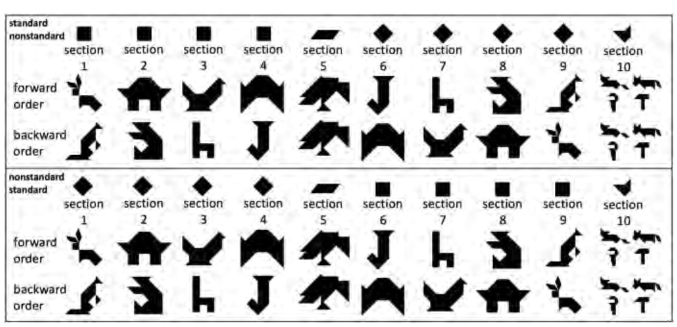


Figure 1: Four Orders of the Test Items (10 Items Per Section)

Of the 80 puzzles where participants identified where the square would go, the square could sometimes fit in multiple places. Five of these puzzles had different numbers of sides of the square visible depending on which of the multiple places it could fit, so they were excluded from further analysis, leaving 75 puzzles in the analysis.

Analysis

To answer the first research question, I grouped participants' responses for each puzzle into two groups: those who saw the shape in a standard orientation and those who saw the shape in a nonstandard orientation. Then, I ran a correlational analysis on the percent of participants successfully identifying where the square would go and the silhouette tangram puzzles' number of sides, the squares' number of sides outlined in the puzzle, the squares' orientation in the puzzle, and number of locations the square could go in the puzzle. Further, I classified participants' descriptions on how they decided where the square should go. To address the second research question, for puzzles where participants' overall success rate was lower than chance (e.g., 1/7=14% if the square can go in one place), I identified common locations where they thought the square should go in relation to their descriptions.

Results

Puzzle Features and Participants' Responses

Based on the correlational analysis for participants who saw the square in a standard orientation, there was a positive and highly significant association between them correctly identifying the location of the square and the number of outlines of the square visible in the silhouette tangram puzzle: r=.550, p<.001, [.366-.689]. They had higher performance when more of the sides were visible. Related to the square outlines, there was also a highly significant correlation for identifying the standard square and the number of sides on the silhouette tangram puzzle: r=.308, p=.007, [.085-.499]. Puzzles with more sides tended to have more sides of the square showing, making the location of the square easier to find. The number of locations that the square could go and the orientation of the square in the puzzle were not associated with participants identifying the correct spot when shown the square in a standard orientation. Participants who saw the square in a nonstandard orientation had similar results. Their performance was highly and significantly correlated with the number of outlines of the square visible in the silhouette tangram puzzle (r=.551, p<.001, [.368-.690]) and significantly correlated with number of sides of the puzzle (r=.275, p=.017, [.049-.471]). Again, number of locations and orientation of the square in the puzzle were not associated with them correctly identifying where the square would go.

Overall, 33 of the participants provided one or more explanations for how they decided where the square should go. The most popular strategy (from 17 participants) was to look for right angles, also described as 90-degree corners. A few elaborated that they started looking for four right corners, then looked for two, and then one. Another anticipated strategy (from 14 participants) was to look for an isolated or obvious square. One person elaborated, "First, I looked for a square shape. Then I looked for somewhere a diamond might fit." Her statement suggests that she intentionally looked for the square in more than one orientation regardless of how it was shown during the survey. Five participants indicated that they paid attention to the sides of the square, looking for parts of the puzzle with the same length sides or part of the square "jutting" out. Seven others tried to make sense of the amount of space the square would take up. One unanticipated type of response came from five educators; these people eliminated spots based on where other shapes would go: "I tried eliminating corners that could be constructed with the large triangle or the parallelogram." Three others considered "if other shapes would hook on to a square to create the figure." Further, three people mentioned symmetry in the puzzles as playing a role. For instance, one person wrote, "Some of them had so much symmetry – with the square being one of a kind…I usually put it in the center."

Incorrect Placements

Out of the 75 puzzles used in the analysis, participants did no better than chance at identifying where the square would go on 16 of the puzzles (in seven of these puzzles the square would work in two or four spots). Two of the puzzles had squares with zero sides outlined, eight puzzles had one side of the square outlined, five puzzles had two sides outlined, and one puzzle had three sides outlined. Although 42% of participants identified that the square piece would go on the inside of the square puzzle; there were four possible inside positions for the square, so they did no better than chance. Their most likely incorrect spot to select was the bottom right corner (35%; see Figure 2a). The shark puzzle was interesting because the square was supposed to go in the head, a common place in several animal puzzles, but only 19% of participants identified it. There was also no strong consensus on where participants thought it should go, with 16% choosing the small triangle poking out on top, 16% selecting the large triangle space below it, and 19% selecting a spot outside of the puzzle (see Figure 2b). Finally, even though the square had three sides outlined in the chair puzzle, only 9% of participants identified it. the chair appeared to have several spots where it might go; however, 45% of participants thought the square should go on the right where a large triangle should go (see Figure 2c).

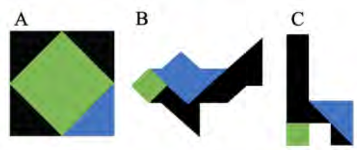


Figure 2: Where the Squares Go (Green) and Popular Locations Participants Chose (Blue)

Discussion

Based on these results, those of Baran et al., (2007), and learning progressions (Clements & Sarama, 2009, 2014), there is additional consensus that how distinguishable a shape is plays a role in how easy it is to place in a silhouette, tangram puzzle. Puzzles where more sides of the square were visible were easier for educators to identify, except in circumstances such as the chair, where there were additional distractor areas that looked like squares. Educators relied on properties of squares (i.e., right angles) to identify places to place the square, as in the square puzzle, and also looked for spots where sides were equal, as in the fin of the shark; this suggests that a strong focus on shape properties might also help children solve these puzzles when the properties align with the shape's locations. Contrary to results suggested by Baran et al. (2007), the participants in this study did not have any difficulty choosing squares based on their orientation. One reason Baran et al. (2007) may have noticed difficulty with this is that they required participants to rotate shapes using a computer app, which is more difficult than selecting a spot where it should go (regardless of its current orientation). Rotating shapes is an area that does cause difficulty for children (Clements & Sarama, 2009, 2014), so if adults, as in this study, do not struggle with rotating shapes, they may not consider this factor when choosing silhouette tangram puzzles for children, especially since there is no classification of puzzles' difficulty based on shape rotation. Likewise, we need more information on whether the rotation and outline of specific shapes (e.g., square versus triangle) are more difficult because multiple shapes could be rotated or outlined in the same puzzle. Investigating these factors in relation to students' success in solving the puzzles will help us better understand the relative difficulty of the puzzles and what factors to attend to when choosing puzzles to support and challenge students.

References

Baran, B., Dogusoy, B., & Cagiltay, K. (2007). How do adults solve digital tangram problems? Analyzing cognitive strategies through eye tracking approach. In J. Jacko (Ed.), *Human-Computer Interaction (HCI) International 2007 - 12th International Conference - Part III* (555–563). Springer-Verlag Berlin Heidelberg.

Bofferding, L., & Foster, A. (2021). A theoretical analysis of tangram puzzles. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 83-97). Philadelphia, PA.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach.* Routledge.

Clements, D. H., & Sarama, J. (2014). *Learning and teaching early math: The learning trajectories approach (2nd Edition).* Routledge.

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* Washington DC: Author. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.

van den Heuvel-Panhuizen, M., & Buys, K. (Eds.) (2008). *Young children learn measurement and geometry: A learning-teaching trajectory with intermediate attainment targets for the lower grades in primary school.* Sense Publishers.

GEOMETRIC ROTATIONS AND ANGLES: HOW ARE THEY CONNECTED?

Navy Dixon
Brigham Young University
navydixon@mathed.byu.edu

Sariah Stevenson
Brigham Young University
sariahblake78@gmail.com

Dawn Teuscher
Brigham Young University
dawn.teuscher@byu.edu

Shannon Dingman
University of Arkansas - Fayetteville
sdingman@uark.edu

With the adoption of the Common Core State Standards for Mathematics 12 years ago, the content of geometric transformations was shifted from high school to grade 8. In our research with middle grades teachers, they often discussed their difficulty in teaching geometric rotations. Therefore, we analyzed 444 middle grade students' responses, across four states, to eight rotation questions from the SMART assessment. The results corroborate teachers' challenges with teaching and student learning of rotations. Results indicate that students have a rigid understanding of angle measure that may impact their understanding of geometric rotations. Although angle measure is introduced in grade 4, we hypothesize that teachers need to provide additional opportunities for students to expand their meaning of angle measure.

Keywords: Geometry and Spatial Reasoning, Curriculum, Assessment, Measurement.

It has been 12 years since many states in the U.S. adopted the *Common Core State Standards for Mathematics* (*CCSSM*, 2010), which shifted mathematics content to different grade levels and connected mathematics content in different ways than was previously done. For example, geometric transformations are now introduced in grade 8 with the purpose to define congruence and similarity, which are concepts that continue to build in high school. The use of geometric transformations to define congruence and similarity was not typically seen in K-12 mathematics curriculum prior to *CCSSM*, yet it is an intuitive way for students to make sense of congruence and similarity (Usiskin, 1972).

For the past five years we researched teachers' curricular reasoning (Dingman et al., 2021), which is teachers' reasoning for the decisions they make related to the teaching and learning of mathematics. We spent time with middle grades mathematics teachers as they prepared and taught lessons in their geometric transformation unit. Many of the middle school teachers reported that rotations were the hardest transformation to teach, often stating: *Rotations are the hardest transformation for students*; *I am not a rotation teacher*; *Rotations are so hard for students*. Although teachers made these statements, we found that teachers seemed to understand and emphasize two important mathematical characteristics of rotations: (1) the center of rotation and (2) the measure of the angle of rotation. We began to consider why teachers thought rotations were such a difficult transformation for students and if students were understanding rotations. To investigate student understanding, we analyzed student responses from an online assessment (https://smartvic.com//smart/index.htm) designed to reveal student mathematical thinking about rotations (Stacey et al., 2013). Our intent was to investigate student understanding of rotations based on the content that was taught in the middle school teachers' classrooms. This report addresses the following research questions:

1. How well do middle school students do on the SMART geometric rotation assessment?
2. What meanings of rotations do middle school students display on the assessment?
3. What unintended meanings may teachers convey to middle school students when teaching rotations?

Theoretical Framework: Mathematical Meanings of Angle

We utilize Thompson's conception of mathematical meanings, which are "... images of the mathematics they teach and intend that students have" (Thompson, 2016, p. 437), to investigate students' understanding of geometric rotations. Teachers plan and enact mathematical lessons with the goal of students understanding specific mathematics content. As teachers enact lessons, students interpret both implicit and explicit actions from the lesson to concur with their current schema (Piaget, 1968). When students become perturbed in their thinking they often adjust their schema, which promotes learning. However, sometimes students interpret implicit actions of the teacher or the task that lead students to adjust their schemas improperly.

Angle measure is a mathematics topic that spans all grade levels. It is introduced in elementary grades and students continue to build on their understanding of angles through college level mathematics courses. Mitchelmore and White (2000) interviewed elementary students to identify how they conceptualized angles in physical settings and how students' conceptualization could progress to understanding abstract angles. Their findings suggest that students' standard angle meanings develop with corner physical angle situations (that show both rays that create the angle). When a turn was used instead, students had difficulty conceptualizing the angle in the situation. Keiser (2004) argues that a static definition "robs students of opportunities to experience angles as rotations that might affect their experiences in trigonometry and other advanced mathematics courses that require some flexibility in this area" (p. 303-304). Moore (2013) argues that teachers and curricula in the United States often define degrees as the standard unit of angle measure and then have students use protractors to measure angles, which may facilitate relating angles measures, "but it fails to address the quantitative structure behind the *process* of determining an angle's measure" (p. 227). Devichi and Munier (2013) found that the word "angle" typically evokes a right angle. The persistent prototypical image of a right angle "opens on the right with arms parallel to the edges of the paper" (p. 3). Browning et al. (2007) adds that "even some adults still struggle with identifying 90º angles that do not have at least one horizontal ray" (p. 286).

When presenting angles beyond 90°, curricula may contribute to a persistent image that assumes specific angle position. A common image students use to identify angles greater than 90° are a 12- hour clock or a compass. With both images one ray of the angle is vertical relative to the paper (i.e.; 0° North, 12 o'clock) and the second ray created by the students is measured clockwise from that initial position. Note that these images maintain the prototypical right angle image. While angles can be measured without one ray being perpendicular to the edge of the paper, are mathematics curricula or teachers providing students with tasks that require them to make sense of angles other than the prototypical right angle image?

Another incomplete meaning of angle is evident when students use measurement tools like protractors. Berry and Wiggins (2001) found that students rely on protractors to measure angles without understanding protractors as measurement tools. The authors' teaching experiment with grade 6 students highlighted how little students understood their standard protractor when asked to generate a nonstandard unit of angle measurement. Hardison and Lee (2020) conducted another teaching experiment with high school and college level students providing them with non-standard protractors and asked them which protractor correctly measured angles in degrees. The students thought that equal changes in angle measurement amounted to equal changes in the linear distance between non-standard protractor units. These students did not demonstrate that equal changes in a circle's circumference signified equal changes in angle measure. It is evident from prior research that K-16 students struggle to conceptually understand angle measurement, which may lead to unproductive meanings that affect their learning of geometric rotations.

Methods

We report on data from 444 middle school students from 13 teachers' classrooms across four states (AR, MI, NV, UT). Students answered an eight-question online formative assessment (http://www.smartvic.com/smart/index.htm) after learning about rotations. The assessment is designed to reveal student mathematical thinking about rotations (Stacey et al., 2013). The questions assess students' understanding of identifying 90° rotations, angle measure based on a given rotation, whether two figures are rotations, and identifying the center of rotation. One question on the assessment had students select a figure and place it so that it represented an image rotated 180°. The assessment contains four attributes of rotations that students should understand after learning about rotations in grade 8: 1) the distance from the center of rotation to any point on the pre-image and image is preserved, 2) identifying the angle measure of rotation from pre-image to image, 3) the orientation of the pre-image and image is preserved, and 4) rotations mean that the pre-image and image are the same size and shape.

Data Analysis

The SMART assessment includes eight questions that reveal student thinking about geometric rotations. First, we analyzed all answer choices for each question to identify common student thinking. We then matched the common student thinking with four conceptions identified by the assessment developers. We created a spreadsheet of all student responses for each question part. Two questions (Q1, Q8) on the assessment had multiple parts, which we scored as separate questions. Q1 has four parts and Q8 has three parts giving a total composite score of 13 points with each question part weighted the same. We then calculated a composite score for each student based on their correct or incorrect answer choices. With these data we then created frequency tables to identify the answer choices that were most prevalent for each question. We also grouped similar questions to identify trends in students' understanding. Finally, we identified different mathematical meanings that students have about rotations based on their answer choices that may not typically be addressed in mathematical tasks students see in their classrooms.

Only one question on the assessment was open-ended. Figure 1 displays the four "E" options students were given for the image. Students selected one image and then placed it on the circle to create an 180° rotation from the pre-image (grey). The variability of responses were first coded for students' selected image and location on the circle. This created 31 distinct student responses, with 30 variations of an incorrect answer. From there, we categorized these incorrect answers into similar meanings. This categorization allowed for trends to be considered across students' varied responses.

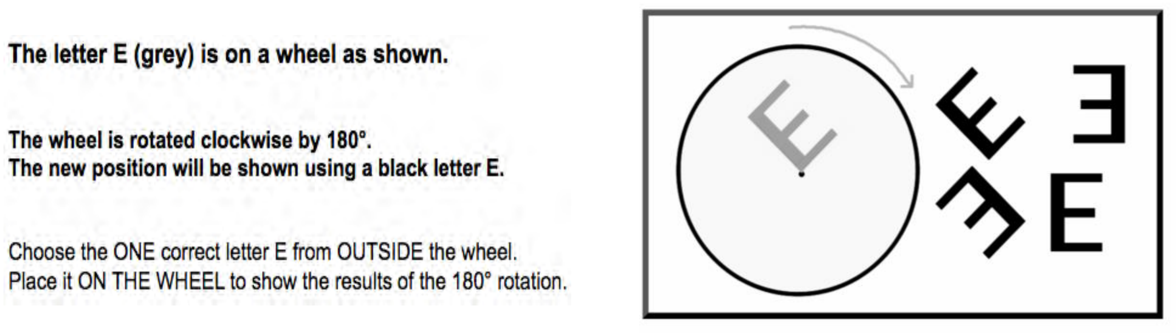


Figure 1: SMART Question 3 (Stacey et al., 2013)

Results

We begin by sharing overall results for the assessment across all students in the study using student composite scores. We then elaborate on the different meanings we found across the students, regardless of their teacher or the state in which they live. Finally, we present findings related to students' mathematical meaning for rotations that teachers should be aware of as they teach topics related to rotations.

Research Question 1

How well do middle school students do on the SMART geometric rotation assessment? The average composite score for all students was 7.73, with a standard deviation of 3.52. This suggests that on average students answered 59% of all questions correctly after being taught about rotations. The percentage of correct answers ranged from 46-74% across the eight questions. Question 1b, which had students identify the amount of rotation between the pre-image and image that was rotated 180° was one of the easiest questions for students as 74% (*n*=321) of all students answered it correctly. Conversely question 8a, which had students determine if the pre-image and image were a rotation or not and if it was a rotation, then students identified the center of rotation was the most difficult for students as only 46% (n=196) of all students answered correctly. From these results on the SMART assessment, we conclude that students held an incomplete meaning of geometric rotations. Therefore, we further analyzed the four attributes of geometric rotations that were challenging for students.

Research Question 2

What meanings of rotations do middle school students display on the assessment? We illustrate middle school students' meanings displayed on the assessment with the results of questions 1, 2, and 3, which are displayed in Figures 1 and 2. The percentage of all students who correctly answered Q1 (parts a-d) completely was 63.6%. The percentage of all students who correctly answered Q2 was 68.7% and for Q3, 62.8%. Student responses to these questions demonstrate their meanings we identified throughout the assessment.

One of the most surprising findings was the difference between students who answered Q1(parts a-d) correctly and those who missed any part of Q1. Students were given the pre-image and had to identify the amount of rotation for the given image (either 45°, 90°, 180°, or 270°). The simplicity of the object (i.e., a circle with a radius) allowed this question to assess students' understanding of angle measure. The results indicate that when students answered all parts of Q1 correctly, they scored significantly higher on the remaining questions ($F = 99.488$, $p \leq 0.001$). Students who answered all parts of Q1 correctly also had a higher composite score on Q2-8 ($\bar{x} = 5.5548$, $\sigma = 2.28444$) than those that answered at least one part of Q1 incorrectly ($\bar{x} = 3.4627$, $\sigma = 2.13336$). This indicates that students who have a productive meaning of angle measure are more likely to understand the concept of rotation.

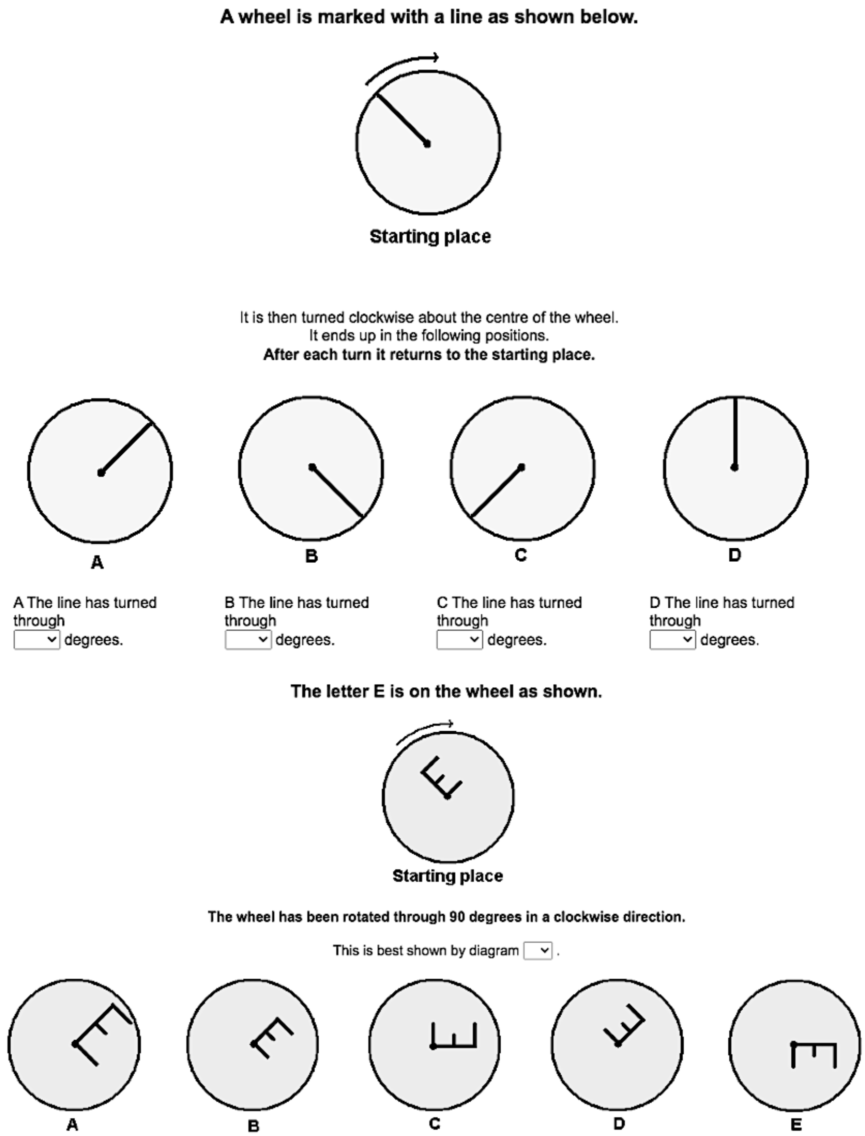


Figure 2: SMART Questions 1 and 2 (Stacey et al., 2013)

In Q2 (see Figure 2, bottom question) students were given a pre-image (the letter "E") and asked to select the image that is rotated 90°. The answer choices targeted different meanings that students form about rotations. Approximately 5.2% (*n*=23) of all students selected answer A that assesses students' meaning that the size of the image stays the same when it is rotated. Approximately 5.4% (*n*=24) of all students responded with answer D that assesses students' meaning of the orientation of an image when rotated. Answers C and E assess students' meaning of a rotation of 90° from the pre-image. Approximately 4.5% (*n*=20) of all students chose answer C; however 16.1% (*n*=71) of all students chose answer E. Among the students that answered Q2 incorrectly, more than half of them chose answers that suggest their meaning of angle measure is not productive. While answer choice E has the image with the proper size and orientation, it is rotated to the 3 o'clock position. From here, we conclude that students were more challenged by distractor answers that showed an incorrect angle measure than answers that showed incorrect orientation or size. We then analyzed student results based upon whether students answered all parts of Q1 correctly. We found that 81% of students who correctly answered all parts of Q1 also got Q2 correct. However, of the students that did not get all parts of Q1 correct, there were only 45% of them that answered Q2 correct. This seems to indicate that students who have a productive meaning of angle measure are more likely to have an understanding of rotations when asked to rotate an object 90° than the students who do not have a productive meaning of angle measure.

Q3 is the open-ended question (see Figure 1). Students were able to place their selected image (of four options with varied slant and orientation) anywhere on the circle to represent a 180° rotation for the pre-image in gray. There was great variety in student responses, which we categorized based on their incorrect answers that were associated with different incomplete meanings of rotations: (1) an orientation meaning, (2) a distance meaning, and (3) an angle measure meaning. A student response was categorized as an incomplete orientation meaning if their chosen image did not preserve orientation of the pre-image. Student responses were categorized as an incomplete distance meaning if their placed image was located away from the given center of rotation (since the pre-image was connected to the center). For incorrect responses in which the students' image is connected to the center of rotation, there may still be an overlooked distance meaning based on their final image location. For a simple categorization, we categorized a student's response as "an incomplete distance meaning" if the image did not touch the center of rotation. Responses were categorized as an incomplete angle measure meaning if their image was not 180° from the pre-image. These were not exclusive groups, so a single student response could be categorized in two or three of the meanings.

Of the students that answered Q3, 7.5% chose an image that suggests an incomplete meaning of orientation. By our categorization of the distance meaning, 11.2% of students responded without attending to distance preservation, indicating an incomplete meaning of distance preservation for rotations. By contrast, 31.9% of students responded with an image placement that suggests an incomplete meaning of angle measure. It seems that students have the most difficulty, of those three attributes of geometric rotations, attending to the angle of rotation.

Research Question 3

What unintended meanings may teachers convey to middle school students' when teaching rotations? Student responses to SMART test Q1, Q2, and Q3 demonstrate implicit student meanings of angle measure and the salience of the prototypical angle position. In these questions, students were asked about a 45°, 90°, and 180° rotation. Students associated an image halfway between the 12 o'clock and 3 o'clock position as a 45° rotation, an image at the 3 o'clock position as a 90° rotation, and an image at the 6 o'clock position as a 180° rotation, regardless of where the pre-image was located.

In Q1, students were given the pre-image and identified the angle of rotation (either 45°, 90°, 180°, or 270°) based on the given image (see Figure 2). Each of the images are shown within a circle with the center of rotation as the center of the circle. If the circle were a clock, the pre-image is located around the 10 o'clock position. When students were shown an image located around the 2 o'clock position, many students indicated that the object had rotated 45° instead of 90°. However, if the pre-image was located around the 12 o'clock or 0° North position, the 45° angle would be accurate. Table 2 summarizes the percent of students who selected the different angle measures for Q1a. For this question, 25.8% of all students indicated that the angle of rotation was 45°. Regardless of the location of the pre-image, these students seem to associate a segment located around the 2 o'clock position with a 45° angle.

Table 1: Question 1 Part A Student Frequencies

Student Response	Percent
45°	25.8
90° (Correct)	71.4
180°	1.4
270°	1.4

Next, in Q2 (see Figure 2), students were given the pre-image of a rotation with the center of rotation and were asked to identify which of five images was a 90° rotation. Table 3 displays the percentage of students who selected the different answer choices. Answer E is particularly popular, which has the image with the correct size and orientation, but incorrect angle measure. However, the distractor showed the image at the 3 o'clock position, 90° from the 12 o'clock position but more than 90° from the pre-image's position. Answer E was chosen by 16.1% of all the students. Of the students who did not answer Q2 correctly , 51.45% (n=139) chose answer E. Answer E also has the image parallel to the bottom of the screen, instead of tilted like the pre-image.

Table 2: Question 2 Student Frequencies

Student Response	Percent
A	5.2
B (Correct)	68.7
C	4.5
D	5.4
E	16.1

Q3 asked students to choose an image from four options and place the image on the given circle to create a 180° rotation from the pre-image (see Figure 1). The image is the same as in Q2, which is the letter "E". About 6.8% (n=30) of all students chose an image that was perpendicular with the bottom of the screen, instead of slanted like the pre-image. Figure 3 shows a few typical student responses that indicate a rigid meaning of angle measure.

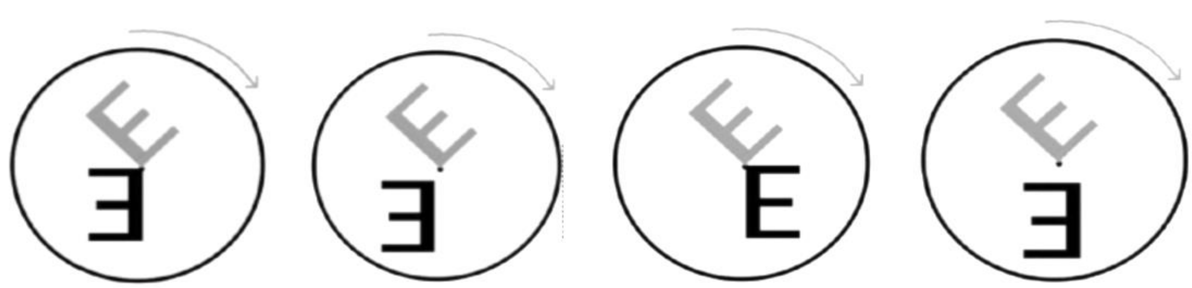


Figure 3: SMART Assessment Q3 sample student responses with 6 o'clock placement

These students did not attend to a 180° rotation from the pre-image, but seem to interpret 180° as the location at the base of a circle. The choice of the upright letter E instead of the slanted letter E shows the comfort students have with objects/images that have a parallel side to the base of the paper or screen.

Students appear to have developed an implicit meaning for angle measurement relating object position to a clock or compass. Perhaps students infer this meaning through their experiences measuring angles. Alternatively, student experiences with geometric rotation may be limited to objects aligned with the x- or y-axes on a Cartesian grid. It appears that teachers must provide students with opportunities to disrupt this implicit meaning.

Implications and Conclusion

Given the results presented on middle school students' understanding, incomplete meanings, and implicit meanings of geometric rotations we recommend teachers explicitly address the meaning of angle measure with students in multiple grades starting at grade 4. The incomplete orientation and distance meanings of rotation may suggest that students are reasoning colloquially with rotations as simply a turn or a flip. Instead, however, the most common incomplete meaning was related to the student's mathematical meanings of angles. It is important that teachers spend more time on having students understand what an angle is measuring and how that measuring unit is different from a linear unit of measure. As can be seen on students' composite scores on the SMART assessment, they are not understanding geometric rotations based on one lesson in grade 8; however, more problematic seems to be students' meanings related to angle measure. These grade 8 students rely on the visual placement of an image to determine the angle measure, rather than the amount of openness of the angle. While this specific meaning of angle measure is introduced in grade 4 of *CCSSM*, we conjecture that teachers and curricula should provide students with opportunities across multiple grades to think about angle measure in this way.

Many of the students in our study displayed a prototypical right angle meaning of angle measure. Students need to be given opportunities to work with angles that are not aligned with the horizontal and vertical axes as well as angles that are not 90° to create dissonance in the students about their own meaning of angle measure. Without students having the opportunity to work with different angle measures they seem to implicitly assume that all angles are the prototypical right angle, which may be reinforcing an unproductive meaning of angle. Without a deeper understanding of angles, students will continue to struggle with the concept of rotations, which may also impact future learning of content for which knowledge of geometric transformations is prerequisite.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant Nos. 1561542, 1561554, 1561569, 1561617. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Berry, R., & Wiggins, J. (2001). Spotlight on the Principles/Standards: Measurement in the Middle Grades. *Mathematics Teaching in the Middle School, 7*(3), 154-156.

Browning, C. A., Garza-Kling, G., & Sundling, E. H. (2007). What's your angle on angles? *Teaching Children Mathematics, 14*(5), 283-287.

Devichi, C., & Munier, V. (2013). About the concept of angle in elementary school: Misconceptions and teaching sequences. *The Journal of Mathematical Behavior, 32*(1), 1-19.

Dingman, S. W., Teuscher, D., Olson, T. A., & Kasmer, L. (2021). Conceptualizing curricular reasoning: A framework for examining mathematics teachers' curricular decisions. *Investigations in Mathematics Learning, 13*(4), 267-286.
Hardison, H. L. & Lee H. Y. (2020). Funky protractors for exploring angle measure. *Mathematics Teacher: Learning & Teaching PK-12, 113*(3), 229-232.
Keiser, J. M. (2004). Struggles with developing the concept of angle: Comparing sixth-grade students' discourse to the history of the angle concept. *Mathematical Thinking and Learning, 6*(3), 285-306.
Mitchelmore, M. C., & White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41*(3), 209-238.
Moore, K.C. Making sense by measuring arcs: a teaching experiment in angle measure. *Educ Stud Math* 83, 225–245 (2013). https://doi.org/10.1007/s10649-012-9450-6
Piaget, J. (1968). Six psychological studies, D. Elkind, ed. New York: Vintage Books.
Stacey, Kaye, Beth Price, Eugene Gvozdenko, and Vicki Steinle. 2013. Specific Mathematics Assessments that Reveal Thinking. Accessed Feb 15, 2022. www.smartvic.com/teacher.
Thompson, P. W. (2016). Researching mathematical meanings for teaching. In English, L., & Kirshner, D. (Eds.), *Handbook of International Research in Mathematics Education* (pp. 435-461). London: Taylor and Francis.
Usiskin Z. P. & Coxford, A. F. (1972). A transformation approach to tenth-grade geometry. *The Mathematics Teacher, 65*(1), 21-30.

HOW MANY ANGLES DO YOU SEE? PROSPECTIVE TEACHERS' ASSIMILATORY DOMAINS FOR ANGULARITY

Hamilton L. Hardison
Texas State University
HHardison@txstate.edu

Hwa Young Lee
Texas State University
hylee@txstate.edu

Lino Guajardo
Texas State University
lrg74@txstate.edu

Mai T. Bui
Texas State University
mtb104@txstate.edu

Given the centrality of angle in mathematics curricula and scarcity of research in this area, we investigated 64 PTs' assimilatory domains of angularity by analyzing the angles they indicated when presented with four segments mutually sharing an endpoint. In both interview and written settings, we found PTs were more likely to recognize convex angles than reflex angles. Additionally, they were more likely to assimilate disjoint angles than angles formed via additive angular compositions. In particular, we found that PTs were unlikely to recognize full angles or additive compositions involving reflex angles. We consider future directions and implications.

Keywords: Geometry and Spatial Reasoning, Cognition, Preservice Teacher Education

Teachers play a significant role in students' mathematical learning. Therefore, prospective teachers (PTs) enrolled in undergraduate teacher education programs need opportunities to develop strong mathematical content knowledge to support their future teaching (AMTE, 2017). Geometry and Measurement are critical domains of mathematics, and, within these domains, angle and angle measure are pervasive topics throughout mathematics curricula from elementary school through higher education (Barabash, 2017).

Although many researchers have studied and worked to support prospective and practicing teachers' content knowledge related to other quantities (e.g., length, area, and volume), research on teachers' conceptions of angle and angular measure is scarce (Smith & Barrett, 2017). However, we know from the few extant studies in this area that teachers tend to experience challenges when it comes to angle concepts (Smith & Barrett, 2017). Prior to the study of precalculus mathematics, conceiving of angle measures in degrees is particularly important given current conventions and curricular standards. For example, in the Common Core State Standards for Mathematics (CCSSM), angle measure is explicitly introduced in Grade 4:

> An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. (NGA Center and CCSSO, 2010, p. 31)

In subsequent grade levels, robust understandings for angles and their measures are essential for topics including the analysis of geometric figures and relationships, constructions, transformations, proof, and trigonometry. Given this, more research is needed to understand teachers' conceptions of angles and their measures. Working toward this goal, we focus this report on a fundamental question: What angles do prospective teachers recognize when presented with multiple line segments mutually sharing an endpoint? We view this question as fundamental because PTs' conceptions of angles as objects will necessarily impact their understanding of angular measure.

Theoretical Components

The study reported here was informed by principles of quantitative reasoning (Thompson, 1994; 2011). A quantity is an individuals' conception of a measurable attribute of an object or phenomenon. Quantities are conceptual entities existing in the minds of individuals. Quantities vary both within an individual over time and across individuals. Characterizing a quantity involves attending to an individuals' conception of three interrelated components—an object of interest, a particular attribute of the object, and a conceived measurement process (i.e., a quantification). Although all three components are important, we focus primarily on the object of interest in this report. Specifically, we are interested in PTs' conceptions of angles as mathematical objects and variation in these conceptions.

To investigate these conceptions, we leverage a radical constructivist (von Glasersfeld, 1995) perspective. In this perspective, knowledge is not seen in relation to a "true" understanding of the real world (or a mathematical one); instead, knowledge consists of conceptual structures an individual has produced to organize experiential constraints (von Glasersfeld, 1997). Within an individual, conceptual structures are constructed and modified over time, and the structures that persist do so because they remain viable. In other words, the existence of an individual's knowledge implies that the knowledge is useful (for the individual).

Radical constructivists often offer nuanced characterizations of individuals' conceptual structures via scheme theory, which has roots in Piaget's (1970) genetic epistemology. According to von Glasersfeld (1995), schemes consist of three parts (a) an individual's recognition of a prior situation in a present experience, (b) an executable activity associated with the recognized situation, and (c) an expected or beneficial result. In this report, we focus in particular on the first component, which is often referred to as assimilation. Regarding assimilation, von Glasersfeld remarked, "The mind primarily assimilates, that is it perceives and categorizes experience in terms that are already known" (1997, p. 301). Specifically, we are interested in specifying the perceptual material PTs assimilate as angle models. In other words, we are interested in what PTs recognize as angles and the assimilatory operations of recognition, which are crucial steps toward better understanding how individuals quantify angularity.

Methods

The findings reported here come from a larger, design-based research (DBRC, 2003; Cobb et al., 2003) project. In this project, we designed and iteratively refined task-based lessons for elementary and middle-grades PTs enrolled in geometry content courses in order to foster critical ways of reasoning about angularity. For more information on the project and lessons, see Hardison and Lee (2019). In this report, we focus on a single task we used in two different contexts: (a) follow-up interviews conducted approximately one year after eight PTs' enrollment in the course (Method 1) and (b) written responses from 58 different PTs prior to a subsequent implementation of the lessons (Method 2).

Method 1: Follow-up Interviews (Spring & Summer 2020)

In Fall 2018, we implemented task-based lessons with three sections of elementary and middle-grades PTs enrolled in a geometry course. We have argued elsewhere that these lessons were successful in engendering productive quantifications of angularity during the course (Hardison & Lee, 2019). However, we wanted to investigate whether participants sustained these ways of reasoning about angle measure beyond the course. Therefore, we contacted the PTs previously enrolled during Fall 2018 to solicit participants for follow-up interviews. Eight PTs volunteered, and we conducted interviews with these eight participants during Spring and Summer 2020, a little more than one year after the PTs' original enrollment in the course.

Interviews, task, and protocols. The one-on-one, task-based clinical interviews (Clement, 2000; Goldin, 2000) with PTs were semi-structured and conducted by their previous course instructor, which was one of the first two authors. Each interview was approximately 60 minutes in length. Due to restrictions associated with Covid-19, each interview occurred remotely via Zoom. Microsoft OneNote was used to pose tasks and for PTs to record written work. Each interview was video-recorded. Data sources for subsequent analyses included interview recordings, PTs' digitally written work, interview transcripts, and researcher notes. Each interview consisted of 6–8 tasks on angle or angle measure, which were either tasks previously presented during the geometry course, variations on these tasks, or entirely original tasks.

Here, we focus on a single original task, *How Many Angles*, which is shown in Figure 1. We designed this task to investigate what perceptual material PTs would assimilate as an angle as well as the assimilatory domains for angularity (e.g., whether PTs considered reflex angles to be angles in this context). In this task, PTs were asked how many angles they could identify in the provided configuration of line segments. If a PT did not spontaneously draw to indicate the angles they counted, the interviewer asked the PT to indicate the angles they counted using the drawing function. Once a PT committed to a final number of angles, the interviewer pressed by asking whether it would be possible to find any more angles. This aspect of the interview protocol was repeated until the PT indicated it was not possible to find any more angles.

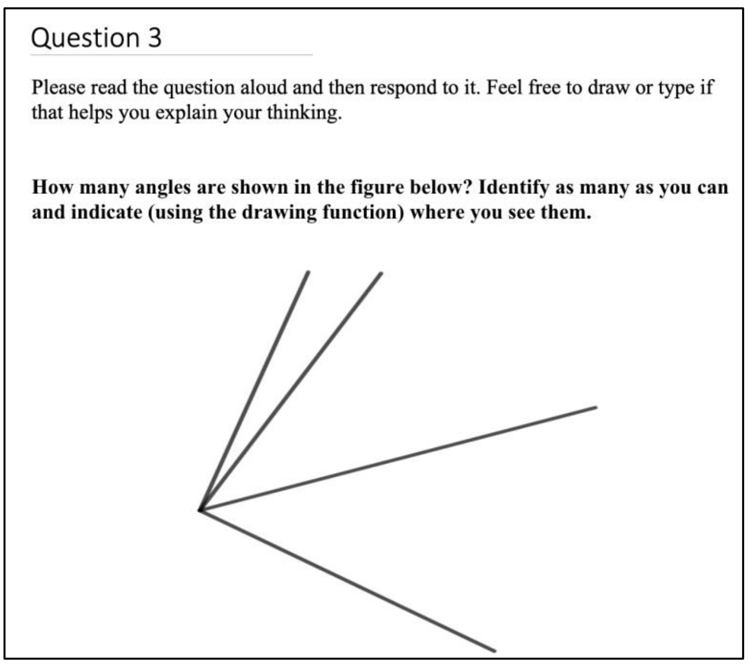


Figure 1: How Many Angles Task

Coding and analysis. Procedures for analyzing the data were established after the interviews were conducted. In order to systematically track the angles PTs identified, we created a reference and notation for the angles participants considered in the interviews (Figure 2). In Figure 2, convex angles are designated C1–C6 and reflex angles R1–R6 with matching enumeration for conjugate angles, (e.g., C6 and R6 are conjugate angles in that they share the same sides but have different interiors). Designations for full, closed, and straight angles were F, O, and S, respectively. We tracked the angles PTs indicated regardless of the indication method. Indication methods included usage of arcs, shading, and labels. As shown in Figure 3, we tracked the angles

PTs identified, the order in which angles were indicated, and the number of angles indicated. If PTs identified more angles after an interviewer press (e.g., "Would it be possible to identify any more?"), we continued additional rounds of coding. Each interview was coded by three of the authors, codes were compared, and discrepancies were discussed until consensus was achieved.

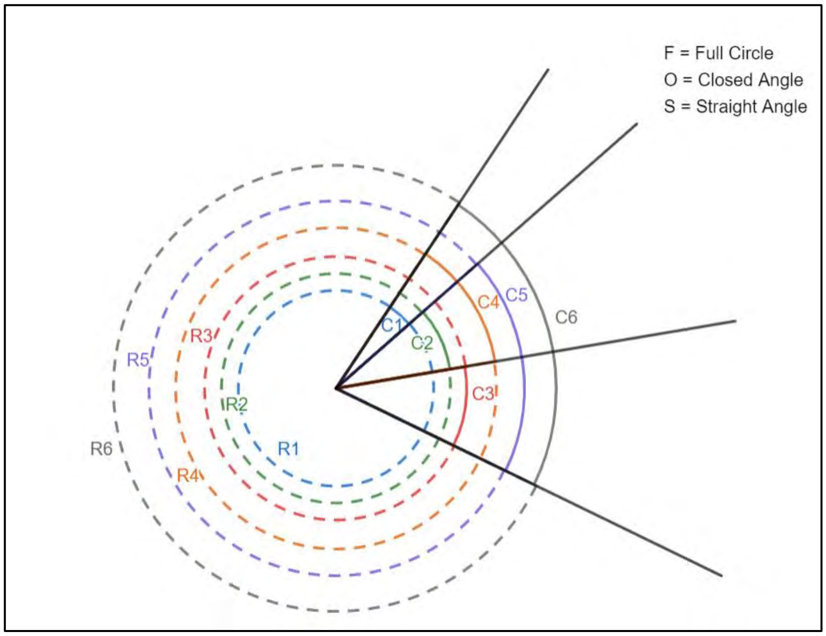


Figure 2: Reference for Angle Names

				How Many Angles			
Student	Round 1	Number 1	Round 2	Number 2	Round 3	Number 3	Round 4	Number 4
Example	C1, C2, C3	3	C4, C5, C6	6	R6	7		

Figure 3: Sample Coding Spreadsheet

Method 2: Written Responses (Fall 2021)

In Fall 2021, the first author taught two sections of the same geometry content course for elementary and middle-grades PTs. Written data was collected throughout the course from 58 PTs, none of whom were participants in the previous study. The How Many Angles task, with minor modification in phrasing (see Figure 4 below), was posed in a written journal prompt, which PTs submitted electronically, during the second week of class. This occurred prior to any lessons on angle or angle measure. Given the written format of the task, we were limited to the responses PTs submitted. So, in contrast to the interview task, we were unable to request clarification, track the order in which PTs identified angles, or press for finding additional angles. As a consequence, we tracked only the angles PTs indicated and the number of angles reported. These responses were coded by two of the authors, and discrepancies in coding were discussed until resolution was achieved. We noted the angles indicated using the coding shown in Figure 2, with one significant modification described in the paragraph below.

In their written responses, some PTs indicated a pair of sides without clearly indicating an interior (e.g., Figure 4). In these cases, since we were unable to determine whether the PT intended to indicate a convex angle or a reflex angle, we adopted the notation P1–P6 to indicate the pair of sides that was indicated. For example, we coded the response shown in Figure 4 below as "P1, P2, P3, P4, P5, P6" since all six pairs of sides were designated in different colors.

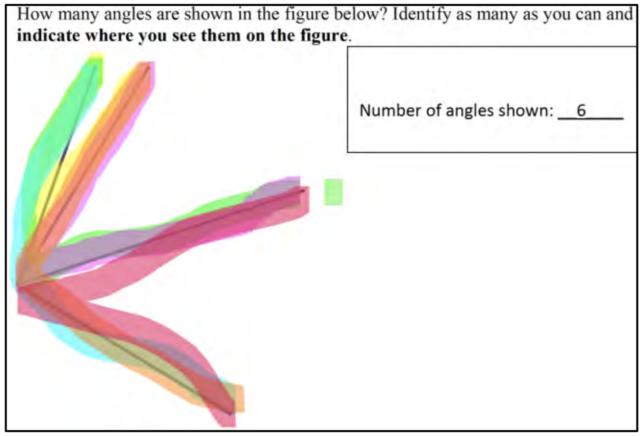


Figure 4: Indicating Pairs of Sides Without Interiors

Findings

We organize the findings in two sections according to the methods described above. We first present findings from follow-up interviews and then present findings from written responses.

Findings from Method 1: Follow-up Interviews (Spring & Summer 2020)

We begin by discussing the angles PTs identified prior to any interviewer press for additional angles (i.e., Round 1). In Round 1, PTs' numbers of stated angles ranged from four to seven, with 50% of the eight PTs indicating six angles were shown in the task (see Table 1).

Table 1: Distribution of Number of Angles Stated in Interviews in Round 1

Number Stated	4	6	7	**Total**
Number of PTs (% of 8 PTs)	2 (25)	4 (50)	2 (25)	58 (100)

The distribution of the angles PTs indicated is shown in Table 2. PTs were more likely to indicate convex angles than reflex angles in Round 1 (100% vs. 38%). Thus, PTs were more likely to assimilate convex angles than reflex angles. Additionally, we noted that disjoint angles (i.e., angles with no perceptual segments penetrating their interiors; C1–C3) were more frequently indicated than composite convex angles (i.e., angles that could be formed through additive compositions of C1–C3, namely C4–C6). Furthermore, the only reflex angle indicated in Round 1 was R6, which was indicated by three PTs. We hypothesize that other reflex angles were not indicated since R6 is the only disjoint reflex angle that might be assimilated without additive composition. Thus, PTs' assimilatory domains for angularity appeared not to spontaneously include additive compositions involving reflex angles (i.e., R1–R5).

Table 2: Angles Indicated by PTs in Round 1 of Interviews

Angle Indicated	C1	C2	C3	C4	C5	C6	R6
Number of PTs (% of 8 PTs)	8 (100)	8 (100)	8 (100)	6 (75)	6 (75)	7 (88)	3 (38)

At the end of Round 1, each PT was pressed by the interviewer for whether it would be possible to find any more angles in the figure. Five of the eight PTs indicated that it was not possible; one subsequently entertained R6 as a possibility and discarded it; leaving only two PTs who ultimately changed their final number of angles after Round 1. We discuss the responses of these two PTs (Lori and Taneem) in further detail in the subsequent sections.

Lori's subsequent rounds. At the conclusion of Round 1, Lori had identified 4 angles (C1–C3 and R6). That is to say, Lori initially assimilated all possible disjoint angles (both reflex and convex). In Round 2, she considered whether additive compositions of angles were permitted:

Lori: If you can use, like any lines, like in any way that you wanted, you know, like I'm not sure if that's allowed, but if you could there's tons of possibilities of angles.

Int: Oh yeah? So, what would, what other possibilities would that introduce if that was allowed?

…

Lori: Okay, does it go from here to here, here to here, this one to here, this one *all the way back around* that'd be 1, 2, 3, 4. And this one would have four – oh, would it be 16?

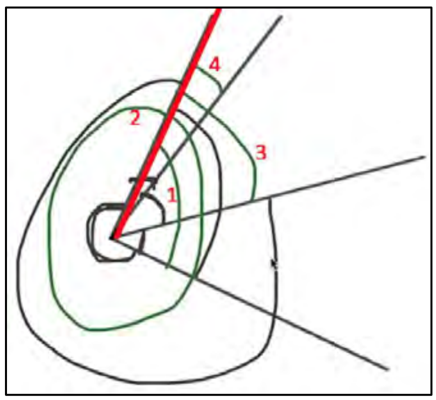


Figure 5: Lori's indication of 4 green angles initiating clockwise from the red segment.

Lori's consideration of using lines "like in any way" indicated that she began to consider all possible pairings of the line segments as a way to determine the number of angles shown. Lori's last line of transcript indicates her picking a single initial segment and considering pairing it with each of the four segments (including the initial segment she selected) and concluding that four possible pairings would be possible. Lori's conclusion that it might be 16 angles was indicative of her awareness that the enumeration of four pairings could be repeated taking any of the four segments as the initial segment. Because Lori's gestures were always enacted in a clockwise fashion from the initial segment, she did not double count any of the 12 convex or reflex angles (i.e., C1–C6, R1–R6). The additional four angles in Lori's final count of 16 were four full angles, each of which went "all the way back around" and initiated at a different segment. Her illustration of the four possible clockwise pairings (in green) with a single initial segment (in red) is shown in Figure 5.

Taneem's subsequent rounds. At the conclusion of Round 1, Taneem had identified 7 angles (C1–C6 and R6). That is to say, Taneem initially assimilated all possible convex angles (disjoint and additive compositions) and the only disjoint reflex angle. In Rounds 2 and 3, she continued considering additive compositions until she had identified all remaining possible reflex angles (i.e., R1–R5) and concluded she saw 12 angles in the task. When pressed again (Round 4), Taneem considered whether full angles should be counted and how many.

Taneem: So, like, if you were just using like this the line [*highlighting a line segment in pink in Figure 6*], I'm thinking about and then, if you just counted that as like, if that was, like the only line here then this would be like a, hold on. [*Drawing the circle in green*] Around there, like a 360-degree angle. I don't know.

Int: Okay. If you, if you counted the one that you just drew, how many would there be then?

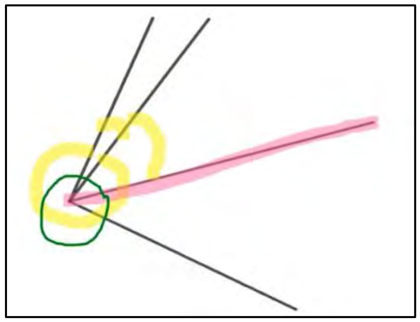


Figure 6: Taneem indicates a full angle initiating and terminating at the pink segment.

Subsequently, Taneem questioned whether each of the four segments could be used in the generation of a distinct full angle remarking, "Well, actually I think that will all be the same for all these lines because they all meet up with the same point. So just be like one more angle possibly…so it'd be like the same 360-degree angle for all of them." Taneem ultimately determined that if she counted full angles, then only one should be counted, bringing her total angle count to 13. Taneem's decision to count only one full angle, rather than four, was rooted in that the four angular candidates shared the same vertex and interior. Ultimately, she decided her final count for angles in the figure was 12 as she did not wish to count any full angles remarking, "So, I'm not sure if that counts an angle because it's like 360-degrees. So, it's like a circle."

Findings from Method 2: Written Responses (Fall 2021)

We now shift our focus to an analysis of 58 PTs written responses to the same task in Fall 2021. In these responses, the number of angles PTs stated ranged from 3 to 12, with a plurality of PTs (47%) stating six angles were shown (Table 3). Only three PTs stated that more than six angles were shown, with two PTs stating 7 shown angles and one PT stating 12.

Table 3: Distribution of Number of Angles Stated for How Many Angles Task (Written)

Number Written	3	4	5	6	7	12	Total
Number of PTs (% of 58 PTs)	14 (24)	11 (19)	3 (5)	27 (47)	2 (3)	1 (2)	58 (100)

Regarding the angles PTs indicated, we begin first by noting two findings different from the interview setting. First, six PTs (10%) stated the number of angles shown but did not indicate any angles on the figure. Second, another nine PTs (16%) did not indicate an interior for any angle. We attribute these findings to a constraint in our second research method, namely that we could not press for clarification in the written setting. An alternative interpretation is that, for some of the PTs, assimilating an angle did not necessarily mean assimilating an angular interior.

We restrict our remaining findings to analyses of the 43 PTs who indicated the interior of at least one angle, either reflex or convex. In other words, we are excluding from subsequent analyses the six PTs who did not indicate any angles on the figure and the nine PTs who indicated only pairs of sides without indicating an interior for any angle. All of the remaining 43 PTs indicated at least one convex angle. In other words, convex angles were in the assimilatory domain of angularity for each of the 43 PTs. As shown in Table 4, all 43 PTs indicated C1, C2, and C3, while only about half of PTs indicated C4, C5, and C6. We attribute these differences in percentages to the disjoint nature of C1–C3 and the composite nature of C4, C5, and C6 in the figure. This indicates additive compositions of convex angles are outside of some PTs' assimilatory domain of angularity.

Table 4: Convex Angles Indicated by PTs (Written)

Convex Angle Indicated	C1	C2	C3	C4	C5	C6
Number of PTs (% of 43 PTs)	43 (100)	43 (100)	43 (100)	21 (49)	22 (51)	23 (53)

Reflex angles were indicated less often than their convex counterparts with only 10 of 43 PTs (23%) indicating a reflex angle. This suggests that reflex angles are outside of the assimilatory domains of the majority of PTs in our study. Moreover, R6 was the only reflex angle indicted by these 10 PTs. In other words, none of the 43 PTs indicated R1, R2, R3, R4, or R5 on the provided figure. We suggest that the disjoint nature of R6 accounts for PTs assimilation of this angle and not R1–R5 since these later angles can be viewed as additive compositions of reflex and convex angles. This suggests that additive compositions involving reflex angles are rare in the assimilatory domains of the PTs we studied. We do note that, as shown in Table 3, one PT stated that 12 angles were shown in the figure, which suggests all 12 possible convex and reflex angles noted in Figure 2; however, this PT was one of the six PTs who did not indicate any angles on their figures. Finally, we note that none of the 58 PTs indicated consideration of closed, straight, or full angles in their responses to this task.

Brief Conclusions and Considerations

In this report, we considered PTs' assimilatory domains of angularity through analyzing the angles they indicated when considering four segments with a mutually shared endpoint. In both interview and written settings, we found PTs were more likely to assimilate convex angles than reflex angles. Additionally, we found PTs were more likely to assimilate disjoint angles than angles formed via additive composition of disjoint angles. In particular, we found that PTs were unlikely to assimilate additive compositions involving reflex angles. Finally, we note that across the 64 PTs from both methods, only two PTs (Lori and Taneem) indicated consideration of a full angle and this consideration was occasioned through interviewer prompts. We also acknowledge that our results only reflect PTs assimilatory domains of angularity for static angle contexts and that dynamic angle models might indicate different assimilatory domains of angularity. Still, these results suggest purposeful interventions targeting angular operations may be necessary to support PTs conceptions of angularity.

As noted at the beginning of this report, angle measurement in degrees is typically introduced in Grade 4 by taking a full angle as a 360-unit composite. Considering the PTs we studied have had 12+ years of experience in school mathematics classrooms, we hypothesize that their assimilatory domains of angularity would tend to exceed those of the students they are to teach. Of course, more research is needed to verify this hypothesis. Nevertheless, given this hypothesis and our finding that PTs tended to assimilate angles in excess of 180° less frequently than the convex counterparts, future research is needed to investigate whether approaches that introduce degrees using other composite units (e.g., a right angle as a 90-unit composite) might better support students in productively quantifying angularity in the context of static angle models.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics*.

Barabash, M. (2017). Angle concept: A high school and tertiary longitudinal perspective to minimize obstacles. *Teaching Mathematics and Its Applications, 36*, 31–55.

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. Educational Researcher, 32(1), 5–8.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547–589). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational researcher, 32(1), 9–13.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Hardison, H. L., & Lee, H. Y. (2019). Supporting prospective elementary teachers' non-circular quantifications of angularity. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1340–1344). St. Louis, MO: University of Missouri.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Author.

Piaget, J. (1970). *Genetic Epistemology.* Colombia University Press.

Smith, J. P., III., & Barrett, J. E. (2017). Learning and teaching measurement: Coordinating quantity and number. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 355–385). Reston, VA: NCTM.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181–234). Albany, NY: SUNY Press.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 35–57). Laramie, WY: University of Wyoming.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Washington, D. C.: Falmer.

von Glasersfeld, E. (1997). Homage to Jean Piaget (1896–1982). *The Irish Journal of Psychology, 18*(3), 293–306.

K-2 STUDENT'S COMPOSING OF LEGO STRUCTURES

Sezai Kocabas
Purdue University
Skocabas@purdue.edu

Yi Zhu
Purdue University
Zhu966@purdue.edu

Yiheng Liang
Purdue University
Liang273@purdue.edu

Laura Bofferding
Purdue University
Lbofferd@purdue.edu

Block building activities help develop students' spatial reasoning, but few studies focus on the development of block building skills beyond preschool. We worked with four kindergarten, four first grade, and four second grade students to learn more about their Lego block building. We compared students' accuracy, building strategies, and spatial language as they used manuals versus pictures of final Lego structures (presented in color versus grayscale) to build two Lego structures. On the first structure, students using color manuals or pictures had an easier time choosing correct bricks but had difficulty correctly placing them; students using grayscale manuals or pictures had difficulty picking the correct bricks but placed them more accurately. By the second design, students did better with the manuals, regardless of color. Students need more support to use specific spatial language and building with depth versus height.

Keywords: Elementary School Education; Geometry and Spatial Reasoning; Instructional Activities and Practices.

The K-2 mathematics standards in the United States emphasize geometric reasoning, which includes the ability to identify and describe shapes and to create structures by analyzing and predicting the outcome of composing and decomposing shapes (National Governors Association Center for Best Practices & the Council of Chief State School Officers, 2010). Children use geometric reasoning to make sense of the world through multiple practices (Goldenberg & Clements, 2014). The first practice in making sense of the world is classification, which involves children identifying the elements in the environment and establishing relationships among them. Then, children might use spatial relationships, which help identify an object's location relative to reference points by using spatial words (e.g., right, under, top) or numbers (e.g., 3 units away from an object). Another practice to understand the environment is noticing the transformations of objects at various orientations or distances (e.g., symmetry, rotation). Geometric reasoning also entails measuring or counting to identify the relationships between objects in the environment. To identify certain properties of the objects, direct measurements (e.g., length, area, volume), indirect measurements (e.g., comparing an object to another object being measured), or using various units (e.g., one-unit length, block should be placed in the middle) might be utilized (Goldenberg & Clements, 2014).

Play with blocks is a popular early childhood activity that helps children develop a variety of concepts, including geometric and spatial reasoning (Casey et al., 2008; Phelps & Hanline, 1999; Ramani et al., 2014), part-whole relationships (Gura, 1992), and other early mathematics concepts (e.g., aspect of numbers, lines, area surfaces, and volume; Cross et al., 2009; Gura, 1992; see also Kamii et al., 2004). There is also a 3D shape composition learning trajectory (Clements & Sarama, 2009) focused primarily on preschoolers' block-building; however, research on block building beyond the preschool has been limited. Since the development of block-building skills may not be fully established until school age, more research on block-building behaviors after preschool is needed (Tian et al., 2020).

Unstructured (free-play) and structured (guided) block building activities are two pedagogical approaches to building blocks. Unstructured tasks are more open-ended, allowing children to create their own structures without being given specific goals, i.e., "Build the best thing you can with these blocks" (Caldera et al., 1999, p. 860). In structured activities, children, on the other hand, copy and reproduce a specific structure from a design (Caldera et al., 1999; Stiles & Stern, 2009). Structured activities focused on improving skills in sorting and classifying blocks and sometimes focused on "...estimation, measurement, patterning, part-whole relationships, visualization, symmetry, transformation, and balance" (Casey & Bobb, 2003, p. 2). At the preschool level, Verdine et al. (2014) used the Test of Spatial Assembly (TOSA) to measure students' block building accuracy. For these tasks, students must copy a block design (e.g., a three-piece Lego structure). However, Lego sets targeted at students ages 5-8 typically range from having about 50 pieces to multiple hundred pieces, and the features (i.e., use of color, types of pieces, orientation of pieces, placement of pieces) increase in complexity. Therefore, to better understand K-2 students' block building behaviors, we need more investigations into how they coordinate these features and how pictures and manuals can support their efforts.

Framework

Worked examples are an instructional aid for helping students understand challenging concepts. Worked examples show step-by-step solutions to problems that help students understand the problem-solving process (Atkinson et al., 2000; Sweller & Cooper, 1985) while reducing cognitive load (Paas et al., 2003; Sweller et al., 1998). Analyzing worked examples promotes learning in different fields, such as in mathematics (Catrambone, 1998; Congdon et al., 2018; Durkin & Rittle-Johnson, 2012) and programming (Bofferding et al., 2022; Joentausta & Hellas, 2018; Margulieux & Catrambone, 2016). Congdon et al. (2018) investigated first-graders' ability to measure using rulers starting at zero or a whole number. The students' measurement conceptions improved when they analyzed worked examples of taking measurements that were not aligned with the zero point.

Worked examples organized by subgoals, on the other hand, may help students learn since subgoals make problem steps explicit by explaining the purpose of each step and providing clues on how to achieve them (Atkinson et al., 2003; Atkinson & Derry, 2000; Catrambone, 1998). Additionally, students can concentrate on the important components in the worked examples (Margulieux et al., 2016) and engage in more self-explanations (Catrambone, 1998; Renkl & Atkinson, 2002).

Studies of young children's block building provide some insight into factors that children pay attention to (e.g., spatial language, see Bower et al., 2020; Cohen & Emmons, 2017; Pruden & Levine, 2017) or struggle with when recreating block structures (e.g., placement, see Stiles & Stern, 2009; Verdine et al., 2017). For example, Verdine et al. (2014) evaluated 102 children's (38 to 48 months) spatial assembly skills beyond basic building accuracy as they attempted to construct seven models using 2 to 4 Mega Blocks of various sizes and colors. When determining if the children's constructions matched the models, the researchers created a dimensions score. They scored the accuracy of blocks relative to the base block, taking into account the *vertical* location of the blocks, *rotation* of the blocks, and the *translation* or horizontal location of the blocks based on the child placing the blocks on the right studs. Based on children's decreasing dimension scores, they had difficulty coordinating rotation and translation as the number of pieces (e.g., 2 versus 3 pieces) or the spatial complexity (e.g., two blocks sharing the two-studs width of the base) increased.

Other researchers have investigated the language children use as they work with blocks for insight into the factors they find important. Pruden and Levine (2017) investigated boys versus girls' (14-46 months) spatial language in terms of dimensions (e.g., big, little, tall, short), shape terms (e.g., circle, square), and spatial features (e.g., curvy, bent). Compared with girls, boys produced more spatial words in preschool years. On the other hand, Cohen and Emmons (2017) investigated school aged children's (4-12 years) production of spatial language during structured block building activities. Like Pruden and Levine (2017), Cohen and Emmons (2017) identified children's spatial language regarding dimensions (e.g., big, wide, length), shapes (e.g., square), and spatial features (e.g., vertical, flat, curvy, side, corner). Additionally, they identified children's spatial language regarding location/direction referring to relative position of blocks (e.g., high, under, up), orientation/transformation referring to relative orientation or transformation (e.g., rotate, upright, right side up), continuous amount (e.g., a lot, same, half, inch), deictic (e.g., here, there, anywhere), and pattern (e.g., order, next, first, increase)(Cohen & Emmons, 2017). Children were more likely to produce words in the location/direction category than they were in the shape and orientation categories (Cohen & Emmons, 2017).

Current Study

Block features (e.g., length, color, shape, size) contribute to spatial complexity, especially as the number of blocks increases. The preschool studies involved a few bricks (e.g., up to 4, Verdine et al., 2014; up to 8, Stiles & Stern, 2009) and typically focused on children's final structures. Instead, we explored how school-aged children deal with spatial complexity by concentrating on the process, particularly what is easy or difficult for them and what they pay attention to in relation to language while building 30-40 piece Lego structures.

Structured (guided) block building can be interpreted as a form of using worked examples, where the final structure is shown but also includes all information needed to build it (e.g., someone can trace the steps from bottom to top and see the needed pieces). Manuals, such as those included in Lego sets, can be interpreted as a worked example with subgoals, where the final structure is broken down into smaller chunks to help explain how the pieces fit together to make the final structure. Subgoal labels may be of increasing importance as structures increase in size and complexity because they make each step explicit. Likewise, colors might reduce cognitive load when structural complexity increases by helping children distinguish among pieces.

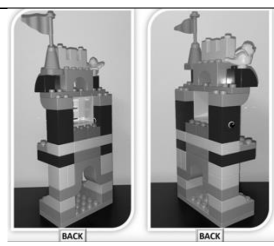
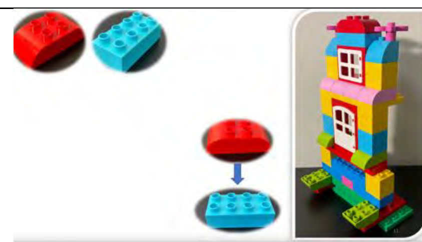
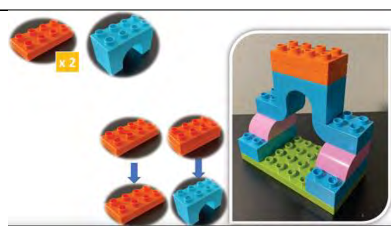
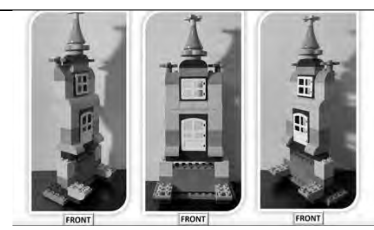
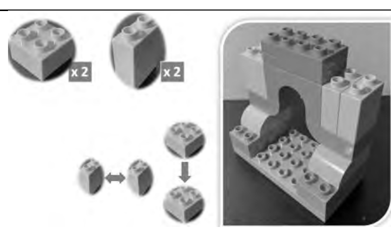
In this study, we explored: How do students' composing of Lego structures compare when they build with manuals of steps versus a picture of the final structures? (a) What spatial language do they use? (b) Which Lego brick features (i.e., location, size, orientation, shape, color) are most difficult to coordinate in their building?

Method

For this study, we analyzed data from four kindergarteners, four first graders, and four second graders in an afterschool program at a Montessori school in the midwestern United States. We tried to strike a balance in regard to students' gender and background. During three sessions of a Lego building project, the students composed, decomposed, and fixed Lego structures. We met with students individually and video recorded them using two cameras to capture the front and back of their Lego structure. Interviewers also asked students questions about where they were looking in the picture or manual as they built, how they knew which brick to use and where to place it. The data for this study comes from the composing portions of the tasks from the project's first two sessions.

In the first session, the students either used a picture (as a worked example) or a step-by-step manual (as a worked example with subgoals) to build a Lego structure (see Table 1). We showed the picture or manual in color to half of the students and in grayscale to the other half. The students built a different Lego structure in the second session but started with half of the structure already built. Students who used a manual in the first session used a picture in the second session (and vice versa). Likewise, students used a color picture or manual in the first session used a grayscale one in the second session (and vice versa).

Table 1: Examples of Pictures and Manuals Used for Each Design

	Design 1	**Design 2**
Order 1	 Example color pictures	 Example grayscale manual steps
Order 2	 Example grayscale pictures	 Example color manual steps
Order 3	 Example color manual steps	 Example grayscale pictures
Order 4	 Example grayscale manual steps	 Example color pictures

Analysis

In order to analyze students' building process, we first recorded the number of differences between students' structures and the given picture or manual. Differences were divided into several categories based on Verdine et al.'s (2014) coding scheme and included students using the wrong brick, including an extra brick, leaving out a brick, placing a brick with an incorrect orientation, and placing a piece with an incorrect left-to-right, forward-to-backward, or vertical placement. We took notes of the specific bricks students had difficulty with in order to identify patterns. Next, we coded the students' building process based on how they used the bricks, pictures, and manuals (see Table 2 for description of codes). During this process, we also took notes of changes students made as they built as well as any help they received from the interviewers.

Table 2: Codes for How Students Composed the Lego Structure

Composing strategy	Description
Resource	Where students referred when building and explaining:
Picture of structure	The composed picture of the structure (solo picture/in manual)
Manual of steps	Steps in manual for how to compose the bricks
Lego structure	Parts of the Lego structure
Turning pieces	Turned the direction of bricks horizontally
Flipping pieces	Turned the direction of bricks vertically
Turning structure	Turned the direction of the Lego structure
Direction	Built the structure from *bottom to top* or *top to bottom*
Symmetric	Built one side and a similar brick on the other side (1 piece, 2 pieces, or 3 pieces at a time)
One side	Built up more than 3 bricks of one side, then did the other side
Lines	Used the lines between bricks on the picture
Counting studs	Counted raised dots on bricks to decide the location or brick

Finally, we used Cohen and Emmons' (2017) classification to code students' spatial language as they built and answered questions about the building of the Lego structures. We did not use their pattern category, but we also included a separate color category given our design focus on the role of color in building (see Table 3). We calculated the percent of students' language for each of the categories to look for trends.

Table 3: Codes for How Students Composed the Lego Structure

Language (Cohen & Emmons, 2017)	**Description**
Dimension	Length and Width: Students reference the brick size using specific numbers (4x2) or in generic terms (e.g., long, big, wide) Height: Students reference the brick's thickness (e.g., thick, thin) or how tall it is.
Shapes	Specific: Students use shape words or names of the bricks (e.g., square, window, flower) Generic: Students refer to the brick using "this" "that" or other generic referents

Location/Direction	Specific: Students use the number of studs on Lego bricks or number of bricks to justify placement Generic: Students refer to locations without using numbers (e.g., on the top, the other side, on the right)
Orientation/ Transformation	Students talk about turning or flipping bricks
Continuous amount	Students use number words to describe number of pieces or the count pieces using numbers
Deictic	Students use generic language (e.g., here, there) to describe brick placement
Spatial features	Students describe the appearance or features of the brick (e.g., sharp, slanted, curved, has a shadow, has an eye) Generic: Students describe the appearance using vague language and actions (e.g., “like this,” “the same,” “looks good”)
Color (we separated this out from spatial features)	Students refer to the color or shading of a brick (e.g., red, darker)

Results

Design One

When building design 1, all students built from the bottom to the top and used the lines between bricks to help guide their building. Further, one student composed several bricks into a substructure on one side before building the same sub-structure on the other side; whereas, the other 11 students built symmetrically (see Figure 1), placing bricks back and forth between sides. Students used more spatial language as their grade level increased, but overall their language was pretty generic. They referred to the bricks in general shape terms (27% of language terms) and often paired it with a continuous amount (e.g., this one, those two; 24% of terms), leading to those categories having the highest percentage. Their language involved specific location terms 11% of the time and general location terms (e.g., “there”) 13% of the time. Overall, students had a median of six differences with the target structure (ranging from 2 to 17) when composing the first Lego structure. Students who used the grayscale picture had a total of 38 differences, those who used the color picture had 21 differences, those who used the color manual had 16 differences, and those who used the grayscale manual had 14 differences. Participants who had the highest numbers of differences left out chunks of the structure (e.g., the window and surrounding pieces or arch and surrounding pieces) and referred to the picture of the final structure.

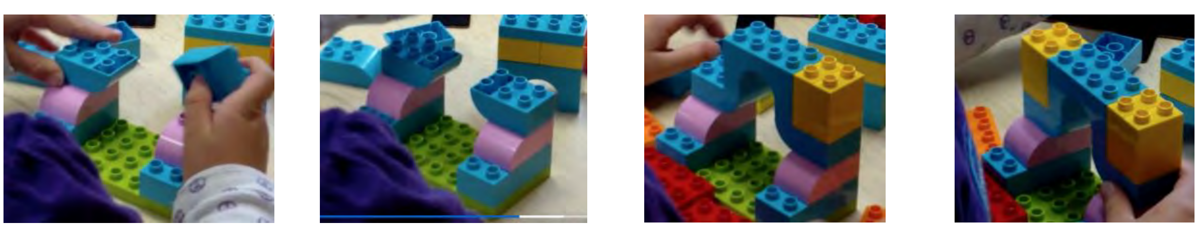


Figure 1: Second Grader Building Symmetrically

Aside from general differences between students who built using the manual versus the picture, there were interesting differences between students who used color pictures or manuals and those who only used grayscale pictures or manuals. Students in the color conditions had more difficulty placing Lego bricks in the correct spots; their typical difficulties in placing the bricks aligned with students not making proper use of the forward/backward dimension (see Figure 2). Students in the grayscale conditions had more difficulty using the correct Lego bricks; their typical difficulties with pieces involved using the incorrect thickness of pieces, regular 2x2 bricks instead of taller 2x1 bricks, or incorrect dimensions (see Figure 3).

Incorrect: red brick does not overhang	Correct overhang for red bricks	Incorrect: window and blue 4x2s set back	Correct placement of window and blue 4x2s

Figure 2: Difficulty with Placing Lego Bricks

Incorrect: 4x2 curved bricks instead of 3x2	Correct use of 3x2 curved bricks	Incorrect: 2 stacked 2x2 bricks on right	Correct use of tall 2x1 bricks on right

Figure 3: Difficulty with Choosing Lego Bricks

Design Two

In general, most students had fewer differences when they completed design 2, which was not surprising since they only had to build half of the structure. Students also did not have much difficulty figuring out where to start from the manual or general picture. They continued to build symmetrically from the bottom up but only eight continued to use the lines between bricks to guide them. As with the first design, they continued to use generic language, referring to bricks in generic ways (21% of language terms) and paired this with continuous amounts (20% of terms). They also continued to refer to locations in generic ways (13% of terms) and increased their focus on the bricks' colors or shading (e.g., darker; 11% of terms). Overall, students had a median of two differences in their final designs (ranging from zero to six difficulties, excluding one kindergartener who left off 10 bricks). Interestingly, there were fewer differences between the color and grayscale conditions with this design. Rather, the biggest difference occurred between students who used a manual (who had a total of six differences in their final structures) versus those who used the picture (who had a total of 29 differences in their final structures). Students' difficulties were similar to those from the first design.

Discussion

Overall, students who built the Lego structures using the manuals had fewer differences than those who built the Lego structures using the pictures. Manuals (a kind of worked example with subgoals) may reduce students' cognitive load by engaging them in more self-explanation (Atkinson et al., 2003; Catrambone, 1998; Renkl & Atkinson, 2002) and helping them focus on transformations (Goldenberg & Clements, 2014). There were benefits and drawbacks to students' building their first design using the *color* picture or manual. These students were largely able to pick out the correct pieces, perhaps aided by the color (see also classification, Goldenberg & Clements, 2014) and a reduction to cognitive load (Atkinson et al., 2000; Paas et al., 2003; Sweller et al., 1998); however, they had difficulty placing them, especially in relation to the forward-backward dimension (see also transformation, Goldenberg & Clements, 2014). One potential reason may be that their focus was more on the correct use of pieces vertically in relation to each other. Students who saw the pictures and manuals in grayscale may have looked more closely at the placement to help them figure out both the pieces involved as well as how to place them because they did not have the color cues. The Lego structures were designed to emphasize the vertical element, so future work could explore if students have similar color-placement difficulty with Lego structures that involve a stronger depth element and little vertical change. Another possible avenue to explore would be to give students manuals where the pieces for the sub-goals are in color but the composed structure is in grayscale. This change might help students find the correct pieces but then encourage them to examine the picture more closely to interpret how to place the pieces. In fact, by design 2, the advantage of having color appeared to wane, and the benefits of the manual, with its sub-goals, took on more importance. Interestingly, students were more likely to use general words than specific words to describe the bricks and their locations, similar to findings from Cohen and Emmons (2017). Therefore, another fruitful avenue may be to help students use specific language when building to determine if that helps them have a better sense of the spatial dimensions in the pictures or manuals.

Acknowledgments

This research was supported by a Ross-Lynn grant through Purdue University.

References

Atkinson, R. K., Catrambone, R., & Merrill, M. M. (2003). Aiding transfer in statistics: Examining the use of conceptually oriented equations and elaborations during subgoal learning. Journal of Educational Psychology, 95(4), 762. https://doi.org/10.1037/0022-0663.95.4.762

Atkinson, R., Derry, S., Renkl, A., & Wortham, D. (2000). Learning from examples: Instructional principles from the worked examples research. Review of Educational Research, 70(2), 181-214. https://doi.org/10.3102/00346543070002181

Atkinson, R.K., & Derry, S.J. (2000). Computer-based examples designed to encourage optimal example processing: A study examining the impact of sequentially presented, subgoal- oriented worked examples. In B. Fishman & S.F. O'Connor-Divelbiss (Eds.), Proceedings of the Fourth International Conference of Learning Sciences (pp. 132-133). Erlbaum. https://doi.org/10.4324/9780203763865-31

Bofferding, L., Kocabas, S., Aqazade, M., Haiduc, A., & Chen, L. (2022). The effect of play and worked examples on first and third graders' creating and debugging of programming algorithms. In A. Ottenbreit-Leftwich & A. Yadav (Eds.), Computational thinking in PreK-5: Empirical evidence for integration and future directions. A Special Research Publication (pp. 19-29). Association for Computing Machinery, Inc. and the Robin Hood Learning + Technology Fund.

Bower, C. A., Foster, L., Zimmermann, L., Verdine, B. N., Marzouk, M., Islam, S., ... & Hirsh-Pasek, K. (2020). Three-year-olds' spatial language comprehension and links with mathematics and spatial performance. Developmental Psychology, 56(10), 1894. https://doi.org/10.1037/dev0001098.supp

Caldera, Y. M., Culp, A. M., O'Brien, M., Truglio, R. T., Alvarez, M., & Huston, A. C. (1999). Children's play preferences, construction play with blocks, and visual-spatial skills: Are they related? International Journal of Behavioral Development, 23, 855–872. https://doi.org/10.1080/016502599383577

Casey, B. M., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. Cognition and Instruction, 26(3), 269-309. https://doi.org/10.1080/07370000802177177

Casey, B., & Bobb, B. (2003). Early childhood corner: The power of block building. Teaching Children Mathematics,10(2), 98-102. https://doi.org/10.5951/tcm.10.2.0098

Catrambone, R. (1998). The subgoal learning model: Creating better examples so that students can solve novel problems. Journal of Experimental Psychology: General, 127(4), 355-376. https://doi.org/10.1037/0096-3445.127.4.355

Clements, D. H., & Sarama, J. (2009). Learning and teaching early math: The learning trajectories approach. Routledge. https://doi.org/10.4324/9780203883389

Cohen, L. E., & Emmons, J. (2017). Block play: spatial language with preschool and school-aged children. Early Child Development and Care, 187(5-6), 967-977. https://doi.org/10.1080/03004430.2016.1223064

Congdon, E. L., Kwon, M. K., & Levine, S. C. (2018). Learning to measure through action and gesture: Children's prior knowledge matters. Cognition, 180, 182–190. https://doi.org/10.1016/j.cognition.2018.07.002

Cross, C. T., Woods, T. A., & Schweingruber, H. (2009). Mathematics learning in early childhood: Paths toward excellence and equity. National Academies Press. https://doi.org/10.17226/12519

Durkin, K., & Rittle-Johnson, B. (2012). The effectiveness of using incorrect examples to support learning about decimal magnitude. Learning and Instruction, 22(3), 206-214. https://doi.org/10.1016/j.learninstruc.2011.11.001

Goldenberg, E. P., & Clements, D. H. (2014). Developing essential understanding of geometry and measurement in grades prek-grade 2. National Council of Teachers of Mathematics.

Gura, P. (1992). Developmental aspects of block play. In P. Gura (Ed.), Exploring learning: Young children and block play (pp. 48–74). Paul Chapman Publishing Ltd.

Joentausta, J., & Hellas, A. (2018). Subgoal labeled worked examples in K-3 education. In Proceedings of the 49th ACM Technical Symposium on Computer Science Education (pp. 616-621). https://doi.org/10.1145/3159450.3159494

Kamii, C., Miyakawa, Y., & Kato, Y. (2004). The development of logico-mathematical knowledge in a block-building activity at ages 1–4. Journal of Research in Childhood Education, 19(1), 44-57. https://doi.org/10.1080/02568540409595053

Margulieux, L. E., & Catrambone, R. (2016). Improving problem solving with subgoal labels in expository text and worked examples. Learning and Instruction, 42, 58-71. https://doi.org/10.1016/j.learninstruc.2015.12.002

National Governors Association Center for Best Practices [NGA] & the Council of Chief State School Officers [CCSSO]. (2010). Common core state standards for mathematics. http://corestandards.org/

Paas, F., Renkl, A., & Sweller, J. (2003). Cognitive load theory and instructional design: Recent developments. Educational Psychologist, 38(1), 1-4. https://doi.org/10.1207/S15326985EP3801_1

Phelps, P., & Hanline, M. F. (1999). Let's play blocks! Creating effective learning experiences for young children. Teaching Exceptional Children, 32(2), 62-67.

Pruden, S. M., & Levine, S. C. (2017). Parents' spatial language mediates a sex difference in preschoolers' spatial-language use. Psychological Science, 28(11), 1583-1596. DOI: 10.1177/0956797617711968

Ramani, G. B., Zippert, E., Schweitzer, S., & Pan, S. (2014). Preschool children's joint block building during a guided play activity. Journal of Applied Developmental Psychology, 35(4), 326-336. https://doi.org/10.1016/j.appdev.2014.05.005

Renkl, A., & Atkinson, R. K. (2002). Learning from examples: Fostering self-explanations in computer-based learning environments. Interactive Learning Environments, 10(2), 105-119. https://doi.org/10.1076/ilee.10.2.105.7441

Stiles, J., & Stern, C. (2001). Developmental change in spatial cognitive processing: Complexity effects and block construction performance in preschool children. Journal of Cognition and Development, 2(2), 157-187. https://doi.org/10.1207/S15327647JCD0202_3

Sweller, J., & Cooper, G. A. (1985). The use of worked examples as a substitute for problem solving in learning algebra. Cognition and Instruction, 2(1), 59-89. https://doi.org/10.1207/s1532690xci0201_3

Sweller, J., Van Merrienboer, J. J., & Paas, F. G. (1998). Cognitive architecture and instructional design. Educational Psychology Review, 10(3), 251-296.

Tian, M., Luo, T., & Cheung, H. (2020). The development and measurement of block construction in early childhood: A review. Journal of Psychoeducational Assessment, 38(6), 767-782. https://doi.org/10.1177/0734282919865846

Verdine, B. N., Golinkoff, R. M., Hirsh-Pasek, K., Newcombe, N. S., Filipowicz, A. T., & Chang, A. (2014). Deconstructing building blocks: Preschoolers' spatial assembly performance relates to early mathematical skills. Child Development, 85(3), 1062-1076. DOI: 10.1111/cdev.12165

INTERTWINING SPATIAL AND LOGICAL REASONING IN QWIRKLE

P. Janelle McFeetors
University of Alberta
janelle.mcfeetors@ualberta.ca

Krista Francis
University of Calgary
kfrancis@ucalgary.ca

Stefan Rothschuh
University of Calgary
stefan.rothschuh@ucalgary.ca

Nimrah Ahmed
University of Alberta
nimrah@ualberta.ca

Munesah Rahman
University of Calgary
munesah.rahman@ucalgary.ca

Dominic Lodge
University of Alberta
dominic@ualberta.ca

We argue that playing board games provides crucial experiences for developing logical and spatial reasoning. Drawing on Dewey, growth in spatial and logical reasoning requires the accumulation of sufficient experiences in action. Our design-based research involving Grade 4-6 students playing Qwirkle weekly revealed their significant improvement using three key tile arrangements that contributed to strategic play. These strategies occurred as a result of enacting intertwining spatial and logical reasoning. Both are necessary for flourishing in mathematics.

Keywords: Geometry & Spatial Reasoning; Reasoning & Proof; Elementary School Education; Instructional Activities & Practices.

To disrupt the hegemonic discourse of mathematics class as creating arithmetic fluency, using commercial games holds promise for mathematics learning through playfulness. Spatial reasoning and logical reasoning in abstract strategy games are processes critical for all mathematics learning. Games have long been recommended as a way for students to develop a meaningful understanding of mathematical ideas before they move toward abstractions (Diénès, 1971; Skemp, 1993). Only limited systematic research is published on the use of commercial games for mathematics learning (e.g., Marshall, 2004; McFeetors & Palfy, 2018; Reid, 2002). We argue that playing board games provides crucial experiences for enacting logical and spatial reasoning. Play as learning is devalued in compulsory schooling and contexts to develop reasoning are needed. Mathematical play is a universal cultural activity (Bishop, 1988) and is about the opportunity to flourish (Su, 2020), engage in mathematical thought (Taylor, 2014), actively do mathematics (Vogel, 2013) while finding joy as motivation to continue playing (Gerofsky, 1999). Contexts which promote the development of reasoning are familiar and stimulating in reaching all students – like playing games. Higher achievement in mathematics is correlated to spatial reasoning (Young et al., 2018) and logical reasoning (Nunes et al., 2007).

In this paper, we illustrate the connections of spatial and logical reasoning in the growth of game play strategies in Qwirkle. In particular, we ask: **How does students' play with Qwirkle occasion moments of spatial and logical reasoning that contribute to strategic engagement?** Our aim was to understand ways in which students increase strategies for game play as their spatial and logical understandings of the game evolved.

Theoretical Framework

We ground students' growth of spatial and logical reasoning in Dewey's theory of experience. Dewey (1938/1997) saw experience as the essence of education, where the aim of education is growing. Students learn by (inter)acting in and on the world around them, including both physical and intellectual engagement so that the event is meaningful and leads to growth. Doing is not sufficient for learning to occur, but students transform activity into experience through reflection where they interpretively ascribe meaning to their activity. Learning requires the accumulation of sufficient experiences in action (Kahn et al., 2015; Varela et al., 1991) through curated opportunities to ensure each and every learner can engage in a sufficient set of educative experiences over time (Francis & Kahn, 2021).

We view logical reasoning as a systematic pattern of behaviour (Reid, 2002a) to "develop lines of thinking or argument" (Brodie, 2010, p. 7) that encapsulates both process and product (Jeannotte & Kieran, 2017). As a process, logical reasoning includes specializing, conjecturing, representing, generalizing, investigating, explaining, justifying, refuting, modifying, and convincing (Lannin et al., 2011; Mason et al., 2010; NCTM, 2000). As a product, logical structures include deductive and plausible/inductive (Polya, 1954), analogy and metaphor (English, 1997), abductive (Conner et al., 2014) and transformational (Simon, 1996), to name a few. Students' development of logical reasoning follows a trajectory, moving from analysis of specific cases to systematic generalizations and proof (Stein & Burchartz, 2006; Tall, 2014).

Spatial reasoning involves the capacity to relate to and navigate the world, and the ability to create and mentally manipulate actual and imagined shapes (Cohen & Hegarty, 2012), recognize relations amongst objects (Bruce et al., 2017), manipulate visual patterns (Carroll, 1993), and locate shapes as they move along a path (Newcombe & Frick, 2010). It is about being able to visualize or picture things in the mind's eye and to mentally move objects and shapes in space. Davis et al. (2015) attempted to collect the many competencies and habits associated with spatial reasoning into a model that represents the emergent complexity of spatial reasoning skills as co-evolved and the complementary nature of the mental and physical actions. Spatial reasoning elements include: altering, moving, situating, sensating, interpreting, [de]constructing.

Methods

This paper reports on successive cycles of Qwirkle game play as part of a larger design-based research project (Cobb et al., 2003; McKenney & Reeves, 2012). The research occurred in one elementary school in Calgary (2 teachers and 48 Grade 6 students) and one in Edmonton (4 teachers and 44 Grades 4-6 students). Qwirkle was played weekly in each class for 60 minutes over 4-5 weeks. Weekly sessions included: a class discussion about students' reasoning from the previous week, game play, and students recording their thinking through words and drawings on reflection sheets. Data collected included videos, photos, students' reflection sheets and score sheets, teaching materials, and field notes.

Qwirkle is a tile-laying game of matching six different colors or six different shapes to score points (see ultrabaordgames.com/qwirkle/game-rules.php). Qwirkle has been established as containing mathematical features, namely combinatorics (Britton et al., 2018), visuo-spatial processing (Nyamsuren & Taatgen, 2013), computational thinking (Scirea & Valente, 2020), and executive functioning (Allee-Herndon & Roberts, 2018).

For this report, we limited our analysis to photos, reflection sheets, and field notes generated in three grade 6 classes where Qwirkle was played. We conducted two phases of data analysis. First, our emergent analysis of reflection sheets revealed three types of tile arrangements for game play: 1-line/crane, 2-line/corner, and grid/add-on (see Fig. 1). We coded responses on reflection sheets with the three arrangements and collected examples in images from class game play. Second, we used Davis et al.'s (2015) elements of spatial reasoning and McFeetors and Palfy's (2017) elements of logical reasoning to analyze the students' enactment of spatial and logical reasoning during their game play, as recorded in photos and field notes. For each action or utterance, we coded each manifestation of a spatial and/or logical reasoning element with a corresponding verb (e.g., investigate, conjecture, analyze, convince, integrate, generalize for logical reasoning; e.g., alter, move, situate, sensate, interpret, (de)construct for spatial reasoning).

Figure 1: Strategic Tile Arrangements in Qwirkle

Results

Students' analysis of game boards for tile placements indicated that they were enacting both spatial and logical reasoning (elements indicated in italics, as coded in the data analysis process) in their Qwirkle play. Logical reasoning occurred as students *investigated* possible tile arrangements, *specialized* while considering their hand against the current board, and *analyzed* current and future moves by arranging their hand for maximum points. Spatial elements of *modelling* by *deconstructing* various *paths* and *imagining* how the *paths* could be *reconstructed* by *locating* and *arranging* new tiles informed students' moves.

On the weekly reflection sheet, students were asked to *analyze* a given game board and six-tile hand by *diagramming* their preferred move and *justifying* it. Table 1 shows how quickly students eliminated illegal moves and increased awareness of tile arrangements. In the first week 15% of the students applied the grid on the reflection sheet. By the third week 57% of the students chose to make a grid with the tiles. Awareness of a range of tile arrangements increases the opportunity to make strategic moves. A repeated measures ANOVA (Huck, 2012) reveals that Mauchly's assumption of sphericity was violated $X^2(5) = 14.89$, $p < .05$, therefore the degrees of freedom were corrected using Huynh-Feldt estimates of sphericity ($\varepsilon = .83$). The results show a significant change in means $F(2.69, 89.93) = 23.68$, $p < .05$. A post-hoc linear contrast analysis reveals significant improvement each week $F(1,36) = 41.89$ $p < .05$.

Table 1: Cycle 1 Reflection Sheet Results

	Week 1	Week 2	Week 3	Week 4
Illegal moves	8	3	0	0
1-line / crane	12	23	3	1
2-line / corner	20	4	17	17
Grid / Add-on	7	12	27	26
Total (out of 48)	47	42	47	44

While student analysis on reflections sheets was important for demonstrating reasoning, it was decontextualized from their immediate experience. However, we observed strategic use of the three tile arrangements within the flow of game play. Figure 2 depicts a game in the second class of Cycle 2 with analysis of each move. It is apparent that the two players in this game were able to employ all three tile arrangements throughout the game.

We provide commentary on a few key moves. After Move 8, the two students discussed that no red tiles had been played yet as they *analyzed* the board in hopes of drawing a red clover to complete the first qwirkle of the game (*projecting* and *pathfinding*). In Move 10, Cora's *conjecture* of playing a red clover expressed as "Jackpot!" when she drew the tile resulted in *justification* of her play. The 15-point creation of the red line of circle-rhombus-clover required *intersecting*, *locating*, *composing*, and *modelling* to complete the maximum score compared to any other move possible with the tiles in hand as a corner tile placement. In Move 11, Amy's play of adding the purple line of star-rhombus provided the maximum points of any move possible and she was fortunate to have these tiles in her hand as an add-on tile arrangement. Between Moves 15 and 16, Amy demonstrates an approach to *investigating* (*modelling* with *arranging, locating,*) a tile arrangement to *explain* (*composing* and *pathfinding*) the possible point value with her opponent without revealing the specific tiles before completing her turn. The students *generalized* a tile arrangement when they named the red clover placement in Move 18 as a “corner,” as it became an object of class conversation because a similar play had yielded a double-qwirkle for Amy the previous week as a great payoff of 24 points for one tile played. Amy *convinced* her classmates and it became a coveted board arrangement to set up in advance. While Move 22 also involves the coordination of two lines similar to Move 18, the students were adamant that this was a distinct tile arrangement as they *refuted* a claim of similarity (*composing* and *arranging*). In Move 24 where Amy placed the green line of square-circle at the bottom of the picture to score a qwirkle and two more points. The qwirkle at 12 points is such a large bonus that strategically it may supersede other tile placements that make a board “crowded” as Amy *explained* (*modelling*). We observed a common strategy across students, where they first looked for a qwirkle and then added extra tiles to the line to increase their score on a single turn.

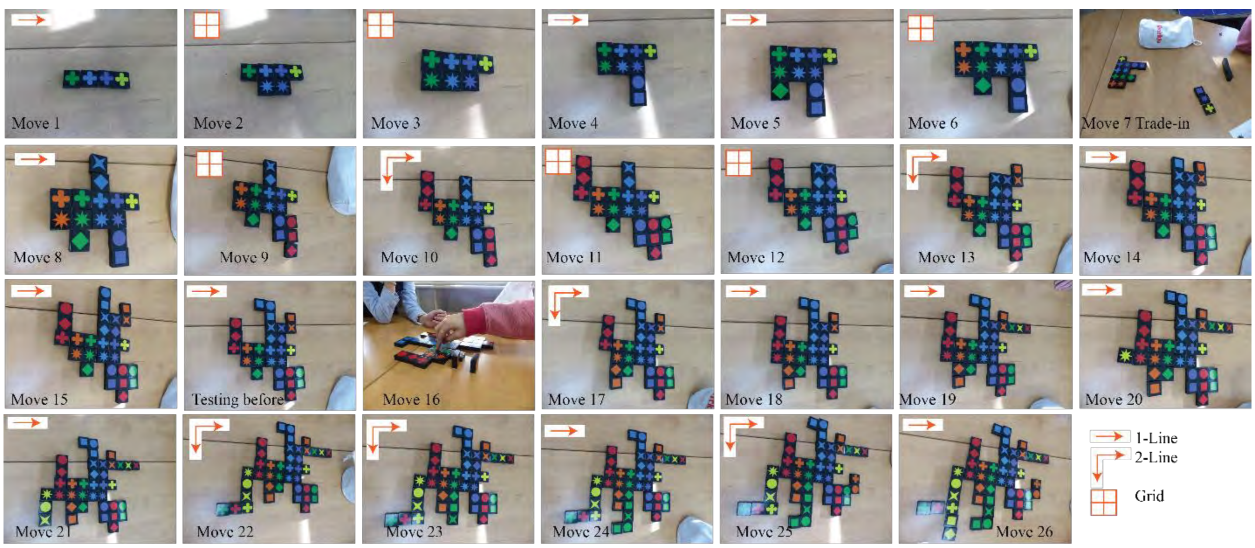


Figure 2: Observing Placements in the Flow of a Game

Discussion

What came as a surprise to us as researchers is how just a few experiences with Qwirkle led to significant gains in strategic play through developing and using a variety of tile arrangements. Over the weeks of Qwirkle play, students were able to analyze the board and make strategic plays to maximize scoring. Knowing the three tile arrangements and having sufficient experiences in playing the game provided more possibilities for strategic game play.

Our study reveals that students were continuously drawing on key components of spatial and logical reasoning. We found it incredibly challenging to tease apart spatial and logical reasoning. In fact, it is not meaningful in understanding students’ actions and mathematical thinking to separate moments of spatial reasoning and logical reasoning. Rather, we see these two specialized types of reasoning as intertwined. We conjecture that these two types of reasoning evolve together. That is, students became more proficient in spatial reasoning as they used processes of logical reasoning, and the converse is the case. We suggest that not only further research needs to be conducted to explore this conjecture, but that teachers are invited to use intertwined spatial and logical reasoning tasks in classrooms. Such tasks invite the complexity of mathematics and simultaneously invite each and every learner into flourishing in mathematics.

References

Allee-Herndon, K., & Roberts, S. K. (2018). Neuroeducation and early elementary teaching: Retrospective innovation for promoting growth with students living in poverty. International Journal of the Whole Child, 3(2), 4-18.

Bishop, A. J. (1988). Mathematics education in its cultural context. Educational Studies in Mathematics, 19(2), 179-191. https://doi.org/10.1007/bf00751231

Brodie, K. (2010). Teaching mathematical reasoning in secondary school classrooms. Springer. https://doi.org/10.1007/978-0-387-09742-8_1

Britton, S., DiMuro, J., Lundstrom, D., & Sansonetti, J. (2018). Probabilities of Qwirkle hand values. The College Mathematics Journal, 49(4), 270-276. https://www.doi.org/10.1080/07468342.2018.1490592

Bruce, C. D., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, M., Francis, K., Hawes, Z., Moss, J., Mulligan, J., Okamoto, Y., Whiteley, W., & Woolcott, G. (2017). Understanding gaps in research networks: Using spatial reasoning as a window into the importance of networked educational research. Educational Studies in Mathematics, 95(2), 143-161. https://doi.org/10.1007/s10649-016-9743-2

Carroll, J. B. (1993). Human cognitive abilities: A survey of factor-analytic studies. Cambridge University Press.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9-13. https://doi.org/10.3102/0013189X032001009

Cohen, C., & Hegarty, M. (2012). Inferring cross sections of 3D objects: A new spatial thinking test. Learning and Individual Differences, 22(6), 868-874. https://doi.org/10.1016/j.lindif.2012.05.007

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Identifying kinds of reasoning in collective argumentation. Mathematical Thinking and Learning, 16(3), 181–200. https://doi.org/10.1080/10986065.2014.921131.

Davis, B., Okamoto, Y., & Whiteley, W. (2015). Spatializing school mathematics. In B. Davis & The Spatial Reasoning Group (Eds.), Spatial reasoning in the early years: Principles, assertions, and speculations (pp. 139-150). Routledge.

Dewey, J. (1938/1997). Experience and education. Touchstone.

Diénès, Z. P. (1971). An example of the passage from the concrete to the manipulation of a formal system. Educational Studies in Mathematics, 3(3-4), 337-352. https://doi.org/10.1007/BF00302302

English, L. D. (Ed.). (1997). Mathematical reasoning: Analogies, metaphors, and images. Lawrence Erlbaum Associates.

Francis, K. & Khan, S. (2021). Syntonic appropriation for growth in mathematical understanding: An argument for curated robotics experiences. In S. A. Costa, M. Danesi, & D. Martinovic (Eds.), Math (education) in the information age (pp. 143-169). Springer.

Gerofsky, S. (1999). Genre analysis as a way of understanding pedagogy in mathematics education. For the Learning of Mathematics, 19(3), 36-46. https://doi.org/10.2307/40248309

Huck, S. W. (2012). Reading statistics and research (6th ed.). Pearson.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. Educational Studies in Mathematics, 96, 1-16. https://doi.org/10.1007/s10649-017-9761-8

Khan, S., Francis, K., & Davis, B. (2015). Accumulation of experience in a vast number of cases: Enactivism as a fit framework for the study of spatial reasoning in mathematics education. ZDM Mathematics Education, 47, 269-279. https://doi.org/10.1007/s11858-014-0623-x

Lannin, J. K., Ellis, A. B., & Elliott, R. (2011). Developing essential understanding of mathematical reasoning for teaching mathematics in prekindergarten-grade 8. National Council of Teachers of Mathematics.

Marshall, J. A. (2004). Construction of meaning: Urban elementary students' interpretation of geometric puzzles. Journal of Mathematical Behavior, 23(2), 169–182. https://doi.org/10.1016/j.jmathb.2004.03.002

Mason, J., Burton, L., & Stacey, K. (2010). Thinking mathematically (2nd ed.). Prentice Hall/Pearson.

McFeetors, P. J., & Palfy, K. (2017). We're in math class playing games. Mathematics Teaching in the Middle School, 22(9), 534-544.

McFeetors, P. J., & Palfy, K. (2018). Educative experiences in a games context: Supporting emerging reasoning in elementary school mathematics. Journal of Mathematical Behavior, 50, 103-125. https://doi.org/10.1016/j.jmathb.2018.02.003

McKenney, S., & Reeves, T. C. (2012). Conducting educational design research. Taylor & Francis.
National Council of Teachers of Mathematics. [NCTM]. (2000). Principles and standards for school mathematics. NCTM.
Newcombe, N. S., &. Frick, A. (2010). Early education for spatial intelligence: Why, what, and how. Mind, Brain, and Education, 4(3), 102-111. https://doi.org/10.1111/j.1751-228X.2010.01089.x
Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. British Journal of Developmental Psychology, 25(1), 147-166. https://doi.org/10.1348/026151006X153127
Nyamsuren, E. & Taatgen, N.A. (2013). Qwirkle: From fluid reasoning to visual search. Unpublished manuscript. https://bcogs.info/publications/pdf/nyamsuren_qwirkle_preliminary.pdf
Pólya, G. (1954). Mathematics and plausible reasoning: Induction and analogy in mathematics. Princeton University Press.
Reid, D. A. (2002). Conjectures and refutations in grade 5 mathematics. Journal for Research in Mathematics Education, 33(1), 5-29. https://doi.org/10.2307/749867
Scirea, M., & Valente, A. (2020). Boardgames and computational thinking: How to identify games with potential to support CT in the classroom. In G. N. Yannakakis, A. Liapis, P. Kyburz, V. Volz, F. Khosmood, & P. Lopes (Eds.), Proceedings of FDG '20: International Conference on the Foundations of Digital Games (art. 114). Association for Computing Machinery. https://doi.org/10.1145/3402942.3409616
Simon, M. A. (1996). Beyond inductive and deductive reasoning: The search for a sense of knowing. Educational Studies in Mathematics, 30(2), 197–210. https://doi.org/10.1007/BF00302630
Skemp, R. (1993). SAIL through mathematics: Structured activities for intelligent learning, an elementary school resource book. EEC.
Stein, M., & Burchartz, B. (2006). The invisible wall project: Reasoning and problem solving processes of primary and lower secondary students. Mathematical Thinking and Learning, 8(1), 65–90. https://doi.org/10.1207/s15327833mtl0801_4
Su, F. (2020). Mathematics for human flourishing. Yale University Press.
Tall, D. (2014). Making sense of mathematical reasoning and proof. In M. N. Fried, & T. Dreyfus (Eds.), Mathematics and mathematics education: Searching for common ground (pp. 223–235). Springer.
Taylor, P. (2014). My students deserve better. For the Learning of Mathematics, 34(1), 34-35. https://www.jstor.org/stable/43894879
Varela, F. J., Thompson, E., & Rosch, E. (1991). The embodied mind: Cognitive science and human experience. MIT Press.
Vogel, R. (2013). Mathematical situations of play and exploration. Educational Studies in Mathematics, 84(2), 209-225. https://doi.org/10.1007/s10649-013-9504-4
Young, C. J., Levine, S. C., & Mix, K. S. (2018). The connection between spatial and mathematical ability across development. Frontiers in Psychology, 9(JUN), 1-7. https://doi.org/10.3389/fpsyg.2018.00755

AREA AND PERIMETER: DESIGN OF AUTHENTIC TASKS ASSISTED BY 3D SOFTWARE

Romario Montaño-Ramos	Honorina Ruiz-Estrada	Josip Slisko Ignjatov
Benemérita Universidad Autónoma de Puebla	Benemérita Universidad Autónoma de Puebla	Benemérita Universidad Autónoma de Puebla
romario.montanor@gamil.com	hruizestrada@gmail.com	jslisko@fcfm.buap.mx

This document presents the central aspects for the design of authentic tasks and some results obtained during their implementation, which were applied to a group of six students belonging to the Consejo Nacional de Fomento Educativo (CONAFE) of the state of Puebla. The analysis considered the previous knowledge of the students and their interaction with each of the tasks. The results obtained show that students present misconceptions regarding the relationships between the concepts of area and perimeter and the use of extracurricular knowledge is evidenced when working with authentic tasks assisted by a 3D software.

Keywords: Authentic tasks, extracurricular knowledge, Area and perimeter

En este documento se presentan los aspectos centrales para el diseño de tareas auténticas y algunos resultados obtenidos durante su implementación, las cuales fueron aplicadas a un grupo de seis estudiantes pertenecientes al Consejo Nacional de Fomento Educativo (CONAFE) del estado de Puebla. En el análisis se tuvieron en cuenta los conocimientos previos de los estudiantes y su interacción con cada una de las tareas. Los resultados obtenidos muestran que los estudiantes presentan concepciones erróneas con respecto a las relaciones existentes entre los conceptos de área y perímetro y se evidencia el uso de conocimientos extraescolares cuando se trabajan con tareas auténticas asistidas por un software de realidad virtual.

Palabras clave: Tareas auténticas, Conocimientos extraescolares, Área y Perímetro.

Different researchers, teachers, and educators in mathematics throughout history have shown interest in the teaching and learning processes of mathematics in the classroom. This interest has exposed that some of the contend in geometry are presented as finished products, promoting the memorization of formulas (Araya & Alfaro, 2010).

The area and perimeter are geometric concepts that are presented explicitly and implicitly in many situations of daily life of the students, however, Amadeo and Yáñez (2006) affirm that the students have a confusion between these concepts, due to use of meaningless formulas. Gómez and Reyes (2015) highlight the fact that many of the students perform the proposed exercises in class, ignoring the origin of the algorithms used. In parallel, D'Amore and Fandiño (2007) emphasize that the concepts of area and perimeter of flat figures present from the scientific field some elements in common, but many others are only assumptions created by students and even by the teachers themselves.

However, it should be noted that extracurricular knowledge plays a role of great relevance within the training process of students, because better results are obtained when extracurricular knowledge is integrated with that acquired in the classroom (Schliemann et al., 2002). Similarly, Palm (2009) mentions that great contributions to the field of mathematics education could be generated if authentic word problems can be incorporated into the classroom.

Considering what has been mentioned so far and the fact that one of the places where technology has influenced considerably is in the school and in turn, the work of the teacher (Parra, 2012), this word poses the following research question: What designs of authentic tasks assisted by the Sweet Home 3D software that simulate aspects of students' daily lives under the theoretical approach proposed by Palm and Nyström (2009), promote the learning of the relationships between the concepts of area and perimeter of flat figures?

To answer this question, this study set as a general objective:

- Design authentic tasks assisted by sweet home 3D software that simulate aspects of students' daily lives under the theoretical approach proposed by Palm and Nyström, which promote the learning of the relationships between the concepts of area and perimeter of flat figures.
- To achieve the general objective set, the following specific objectives are proposed:
- Characterize some aspects of the theory of authentic task situations and some mathematical elements, which strengthen the work with the concepts of area and perimeter in extracurricular contexts.
- Articulate the aspects of the theory of authentic task situations and the mathematical elements in the design of authentic situations assisted by the Sweet Home 3D software, which comply with the aspects raised by Palm and Nyström.
- Analyze the data obtained in the implementation of authentic tasks and determine if the students managed to establish the existing relationships between the concepts of area and perimeter.

Theoretical and Referential Framework

Theory of authentic task situations

This is a local theory put forward by Dr. Torulf Palm in 2009. In it, word problems are considered as situations that are real for a group of people who are immersed within the described context and that allow to establish a connection between abstract mathematics and mathematics that are immersed in real life, which can be manipulated to eventually be proposed in the classroom as authentic tasks.

This theory seeks to link extracurricular mathematics with mathematics learned in the classroom through simulations of real-world situations. However, it is to be recognized that not all aspects surrounding a problem situation can be simulated in a word problem. Therefore, Palm and Nyström (2009) present three aspects and two sub-aspects that allow us to identify an authentic task, which are described below:

Event: The statement proposed in the task must be a phenomenon, event or incident that has existed in real life or failing that may occur, in such a way that the student recognizes the situation as an extracurricular environment and allows him to locate himself mentally in that situation.

Question: The questions that arise within the school task must have a close relationship with real life, since within the simulation it is essential to establish a concordance between the questions posed in the school task and the context of the phenomenon or event proposed in the event of the task.

Information/data: When word problems are raised, they not only describe the event, but also provide relevant information that will allow the solution of this, they also have explicitly and / or implicitly the conditions of the problem, the possible models to be built and even the purpose of the situation.

Specificity of the data (sub): In the event proposed in the task, it is not spoken in general terms, but rather, it seeks to give all the specific characteristics of the simulated situation, describing with proper names the people, objects and places that are linked to the situation.

Purpose in the figurative context (sub): The purpose of the task must be clear to the student within the simulated situation as it would be within the real-life situation, since the considerations that the student considers for the solution, will be framed with the purpose of the task.

Mathematical Referent

Fandiño and D'Amore (2009) present in detail the relationships between the concepts of area and perimeter of flat figures, which can be summarized as follows:

1. At parity of sides and perimeter, the one with the largest area is the regular polygon.
2. Among several isoperimetric polygons, the one with the largest area is the polygon with the most sides.
3. Among all isoperimetric figures (figures in general), the circle is the figure that has the largest area.

However, it is necessary to clarify what is meant by area and perimeter in this research and understanding that it is not possible to take for granted the definition of a mathematical object if the preceding elements that allow us to understand its beginning are unknown (Fandiño & D'Amore, 2009).

Perimeter conception: It is called perimeter to the measurement of the contour of a figure, this is measured in linear units which represents a length (for example: m, cm), understanding that the perimeter is different from what is understood by contour or border of a flat figure, since the contour is a closed line, and the perimeter is the measure the same.

Area Conception: The area is a two-dimensional measure accompanied by a conventional measure (for example: m2, cm2), understanding that the area is different from the surface, since the surface is a part of the plane and the area is the measure of said surface, understanding that parallel to the perimeter the area is usually defined as a synonym of surface.

Research Method

This research is qualitative, more specifically it is a descriptive case study. Six students belong to the Consejo Nacional de Fomento Educativo (CONAFE) of the municipality of Tzicatlacoyan in the state of Puebla. The students belong to an agricultural community, which is in the process of expansion, for this reason, in the community there are many houses in the process of construction, an aspect that turns out to be of relevance for the design of authentic tasks, since the students have been involved in the building process.

The study subjects do not have computing devices in their homes and have only made use of two computers installed in the Educational Institution, for this reason their knowledge is basic regarding the use of these devices. Another aspect to highlight is the fact that students only have access to the internet in the Institution since there is no coverage of this service within the community.

For the data collection, the following were used: a test of previous knowledge, which allowed to know if there are erroneous conceptions regarding the relationships between the concepts of area and perimeter; three designs of authentic tasks, which were constructed using their everyday language and situations that occur or came to occur within the Community; housing designs in Sweet Home 3D software, in this software students work on a 2D plane and visualize their effects in a 3D plane, students here can build or rebuild homes; video recordings, which allow the in-depth study of students at the time of the implementation of the tasks; and a semi-structured interview, with this it is possible to know how authentic the task was for the students.

Results and Conclusions

Initially, a Knowledge Test was presented to the study population, which allowed to identify that the students consider that there is a direct relationship between the concepts of area and perimeter, since, the students consider as true that if you have two iso-perimeter plane figures then these figures are necessarily equi- extensive, in turn, if you have a figure A with a greater perimeter than a figure B, then figure A has a greater area than figure B, it is also possible to identify that within a set of iso-perimeter figures they cannot identify that the figure that has the largest area is the figure with the largest number of sides. All these initial data agree with the assumptions that students can generate, which were raised in D'Amore and Fandiño (2007).

About working with authentic tasks, the students when facing them in one way were able to identify that these contexts emulated in the verbal problem are related to their environment, this aspect allowed them to think about the situation and forget the academic part, seeking to propose solutions that fit their reality. From this application you can rescue the fact that the software allowed them to quickly visualize the present magnitudes of their constructions. The following figure shows some of the students' work:

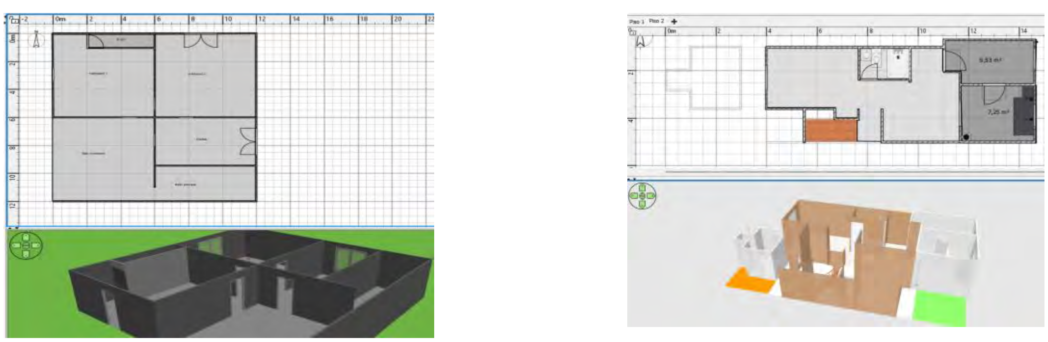


Figure 1. Student design in Sweet Home 3D Software.

This research is still in the process of analyzing the data collected, however, the information analyzed so far shows that working with authentic tasks allows the student to use their extracurricular mathematical knowledge and even their knowledge acquired in the classroom. In addition, these authentic tasks when linked to the use of the software allow the student to think differently than they would in a verbal problem that is worked on pencil and paper, since this software not only allowed them to have a realistic image of the environment that is described in the task, but also, it allows them a quick view of the considerations they come to have as possible solutions.

References

Amadeo, G. G., y Yáñez, J. C. (2006, septiembre 6-10). Relación entre perímetro y área: el caso de Patricia y las interacciones. Investigación en educación matemática. X Simposio de la Sociedad Española de Investigación en Educación Matemática, Huesca. Instituto de estudios Altoaragoneses.

Araya, R. G., y Alfaro, E. B. (2010). La enseñanza y aprendizaje de la geometría en secundaria, la perspectiva de los estudiantes. Revista Electrónica Educare 14(2), 125-142.

D'Amore, B. y Fandiño, M. (2007). Relaciones entre área y perímetro: convicciones de maestros y de estudiantes. Revista Latinoamericana de investigación en Matemática Educativa 10(1), 39-68.

Fandiño, M. y D'Amore, B. (2009). Área y perímetro. Aspectos conceptuales y didácticos. Bogotá: Magisterio.

Gómez, S. T. G., y Reyes, K. J. V. (2015, mayo 3-7). Área y perímetro de cuadriláteros en estudiantes colombianos de grado 5° de educación formal. XIV Conferencia Interamericana de Educación Matemática, Chiapas, México.

Palm, T. (2009). Theory of authentic task situations. In B. Greer, L. Verschaffel, W. Van Dooren, y S. Mukhopadhyay (Eds.), Word and worlds: Modelling verbal descriptions of situations (pags.3-19). Sense Publishers.

Palm, T. y Nyström, P. (2009). Gender aspects of sense making in word problem solving. Journal of Mathematical Modelling and Applications, 1(1), 59-76.

Parra, C. (2012). TIC, conocimiento, educación y competencias tecnológicas en la formación de maestros. Nómadas, 36, 145-159.

Schliemann, A. L., Carraher, T., Carraher, D., y Schliemann, A. (2002). La comprensión del análisis combinatorio: desarrollo, aprendizaje escolar y experiencia diaria. In En la vida diez, en la escuela cero (pp. 90-105). México: Siglo veintiuno editores.

DEVELOPING A FRAMEWORK FOR CHARACTERIZING STUDENT-CREATED DIAGRAMS IN DGES

Xiangquan Yao
The Pennsylvania State University
xzy73@psu.edu

Dynamic geometry environment (DGE) offers a wide range of tools for its users to create and interact with geometry diagrams. In this paper, I propose a framework to characterize the geometry diagrams learners create in DGEs. This framework considers a learner's approach to creating a geometry diagram (i.e., perceptual-based, measurement-based, construction-based, and transformation-based), the driving force that guides the learner's specific actions in a DGE (i.e., tool-driven, and property-driven), and constraints in the resulting diagram (i.e., drawing, underconstrained, overconstrained, and appropriately constrained). Examples of student work on one geometry construction problem are used to illustrate the use of the framework.

Keywords: Geometry Diagram; Dynamic Geometry Environments; Geometric Construction

Geometry diagrams are a distinct mode of mathematical representation and communication that uses the visual features of specific drawn objects to convey meaning about abstract geometric concepts. Geometry diagrams are both signs that represent abstract geometric concepts and visual images that offer graphical-spatial properties, which can mobilize the perceptual activities of an individual. The duality of geometry diagrams has been discussed by researchers in mathematics education (Fischbein,1993; Laborde,2005). A geometry diagram can be used as a visual representation of a geometric figure, which displays the theoretical properties that define a geometric figure. Meanwhile, a diagram also has other properties that are not necessitated by the defining properties of the geometric figure but rather depend on the spatiographical features of the diagram. Geometry diagrams are thus ambiguous representations of sets of geometric concepts and their relations. The ambiguity of geometry diagrams as a semiotic mode of representation has urged researchers to examine the use of diagrams in teaching and learning geometry (Duval, 1995; Herbst, 2004; Alshwaikh, 2010; Jones, 2013; Dimmel & Herbst, 2015). However, learners' activity of constructing geometry diagrams is less examined. This is probably due to the limited number of tools to create a geometry diagram and the static nature of a geometry diagram in a paper-and-pencil environment. Traditionally, a geometry diagram is created through either freehand drawing or a straightedge-and-compass construction. In both cases, the diagram is static and cannot be manipulated directly.

The appearance of dynamic geometry environments (DGEs) not only allows users to create and then manipulate geometry diagrams with a wide range of tools but also adds a dynamic feature to them. DGEs like the Geometer's Sketchpad (GSP) and GeoGebra contain multiple sets of tools, such as free drawing tools, construction commands, measuring commands, and transformation commands. As a result, a user can create a diagram by using different DGE tools. Moreover, once a DGE diagram is created by a sequence of primitive commands chosen by the user, the diagram can be modified by dragging an element of it while still preserving all the geometric relations used in its construction. Therefore, when a user drags an element of a DGE diagram, it changes according to the geometry of its construction rather than the wishes of the user. In other words, once created, DGE diagrams become quasi-independent from the user and react to manipulations of the users by following the laws of geometry. This separates DGE diagrams from paper-and-pencil diagrams, which are statics and can be slightly distorted by students in order to meet their expectations. The variety of tools and the dynamism in a DGE enrich learners' construction activities. When asked to create a DGE diagram that represents a geometrical object, learners can be guided by different knowledge, take distinctive approaches, and use various tools, which can result in different diagrams. Therefore, it is important to develop a theoretical tool to analyze the different geometry diagrams in DGE that are intended to represent the same geometrical figure. In this paper, I will first present a framework for characterizing geometry diagrams in a DGE and then use the framework to describe some diagrams that students created with GSP. The affordances of the framework will also be discussed.

A Framework for Characterizing Student-Created Diagrams in DGE

The framework was developed based on a review of existing research on learning geometry in DGEs and my teaching and research work in DGEs (Figure 1). In developing the framework, I considered the possible approaches students could take to create a diagram, the driving force that guides students' specific actions in a DGE, and the constraints in the resulting diagram.

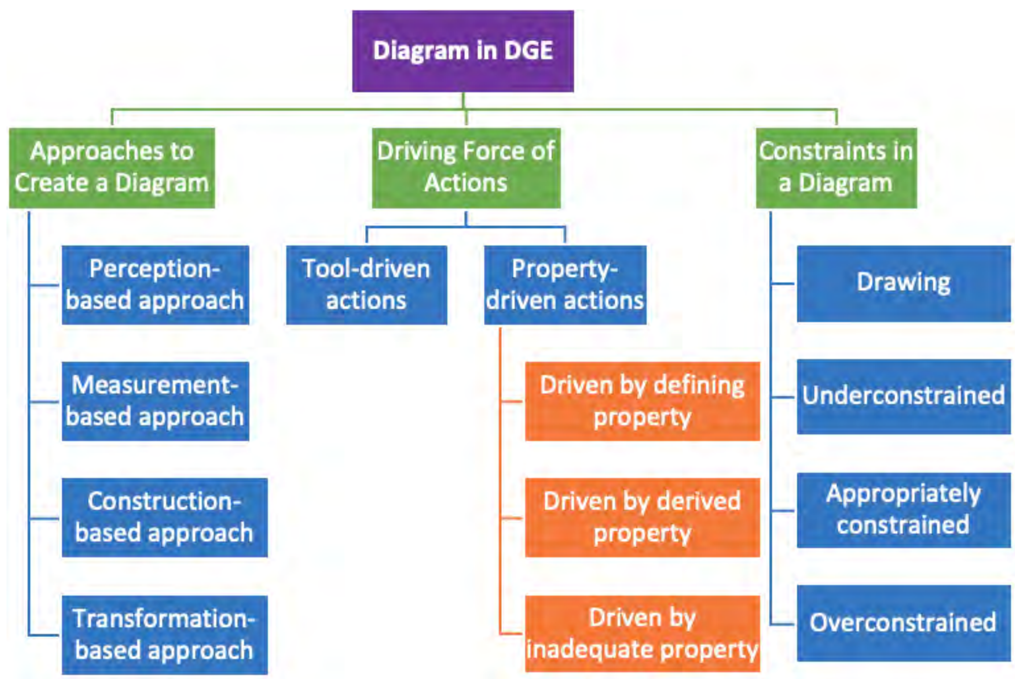


Figure 1: A framework for characterizing student-created diagrams in DGEs

Approaches to Create a Geometry diagram

Dynamic geometry software includes both free drawing tools and construction commands that intend to create a dynamic version of a compass and a straightedge. The accessibility to different sets of tools in a DGE allows learners to take different approaches to create a geometric figure. The preference for drawing over construction for beginning users of dynamic geometry software suggests that visual perception can sometimes dominate diagram creation in a DGE. Moreover, in the process of moving from drawing to construction, learners are likely to develop different construction techniques. Laborde (2005) differentiated between *soft* and *robust* constructions. A robust construction is a construction that passes the "dragging test" and is obtained by using geometrical objects and relationships characterizing the construction. In contrast, a soft construction is a partial construction in which variation is part of the construction itself, and a property becomes evident only at the point at which another property is satisfied. In discussing the validity criterion of constructions in DGEs, Stylianides and Stylianides (2005) identified *measurement constructions* as constructions that pass the “dragging test” but violate the Euclidean construction restrictions because of their use of measurement. Measurement constructions are different from the constructions that are compatible with straightedge-and-compass constructions. The two types of robust constructions suggest two distinct construction approaches in DGEs. Meanwhile, geometric transformations and the invariant properties under them offer a new approach to study geometry opposite to the classical synthetic geometry approach. The transformation commands in DGEs make it easier for learners to construct geometric objects through translation, reflection, rotation, and dilation. For instance, Villiers (2016) used transformation tools in GSP to construct geometric objects when solving the classic treasure-hunt problem and its generalization. Yao (2020) described how one preservice mathematics teacher used the rotation tool in GSP to construct an equilateral triangle with its vertices on three parallel lines.

Based on the above literature, I propose that learners could take at least four approaches to creating a diagram in a DGE. In a *perception-based* approach, a learner relies on visual perception to create the intended diagram. The use of free drawing tools and dragging is prominent in this approach. A learner who takes a *measurement-based* approach relies heavily on numerical measures to create the intended geometric figure. In this approach, the use of visual perception and primitive commands is heavily guided by numerical measures. In a *construction-based* approach, a learner mainly relies on geometric properties and primitive construction commands that are compatible with straightedge and compass to construct the intended geometric figure. A learner who takes a *transformation-based* approach largely relies on properties of transformations and transformation commands in a DGE to construct the intended geometric figure.

Driving Force of Actions

Studying the behaviors students exhibit when working with DGE tools can lead to insights about the appropriateness of their use of those tools and their understanding of geometric ideas. Researchers have observed that some students use technological tools judiciously and some students use technological tools inappropriately (Ball & Stacey, 2005). This indicates that students might use digital tools differently even though they take the same approach to create diagrams in a DGE. Guin and Trouche (1999) used *work method* to characterize students’ behaviors when using technological tools and identified five work methods for students’ use of symbolic calculators: random work method, a mechanical work method, a resourceful work method, a rational work method, and a theoretical work method. These different work methods reveal different roles technological tools and mathematical knowledge can play in students’ mathematical activities. Based on whether students’ actions with technology are guided by mathematics knowledge and their ability to interpret and predict results produced by technology, Hollebrands (2007) described *reactive* and *proactive* strategies as two distinct ways that students interact with tools in a DGE. Students who use a tool reactively perform an action based on what is produced on the computer screen rather than their knowledge of the geometric figure to be constructed. As a result, they might not be able to fully anticipate what may result from their technology-mediated action prior to performing it. In contrast, students who use the tool proactively are guided by their knowledge of the geometric figure to be constructed. They are likely to anticipate specific results of their technology-mediated action.

Based on the above literature, I argue that the dominant force that drives student actions in DGEs can be either the technological tools or knowledge of geometry. In *tool-driven actions*, students are guided by instantaneous feedback from the use of DGE tools rather than their knowledge of the geometric figure to determine their subsequent actions. It is parallel with Guin and Trouche's mechanical work method and Hollebrands' reactive strategy. In *property-driven actions*, students are driven by what they know about the properties of the geometric figure to be constructed and then choose DGE tools to produce these properties. These properties can be visually salient properties that define the geometric figure (e.g., a parallelogram has opposite sides congruent and parallel), less visually salient properties that still define the geometric figure (e.g., diagonals of a parallelogram bisect each other), properties that are derived from the defining properties but not sufficient to define the geometric figure (e.g., two pairs of adjacent angles of a parallelogram are supplementary), or properties that are inadequate or irrelevant to the geometric figure. In other words, students' actions can be driven by defining properties, derived properties, and inadequate or irrelevant properties. Property-driven actions are parallel with Hollebrands' proactive strategy.

Constraints in a Diagram

The dynamism of DGEs has provoked researchers to consider the robustness of a DGE diagram under dragging. Finzer and Benett (1995) established four categories (i.e., drawing, underconstraint, overconstraint, and appropriate constraints) for diagrams built in a DGE and used the case of drawing a parallelogram to exemplify these categories. Scher (2005) applied these four categories to assess squares created in a DGE. In this paper, I define the four categories according to the relationship between a diagram and the theoretical properties that it intends to represent. A *drawing* is a geometric object that perceptually appears to be the intended geometric figure but possesses none of its property under dragging test. It can easily be messed up by dragging. An *underconstrained* diagram is a geometric object that is constructed by using one or more properties that are necessary but not sufficient to define the intended geometric figure. Like a drawing, an under-constrained diagram can appear to be the intended geometric figure but collapse when particular elements are dragged. Laborde (2005) used soft construction to describe this type of geometric figure in a DGE. An *overconstrained* diagram is a geometric object that is constructed by using one or more properties that are sufficient but not necessary to define the intended geometric figure. Since excessive constraints are imposed on it, when dragged, an overconstrained diagram can only generate a subset of all the possible configurations of the intended geometric figure. An *appropriately constrained* diagram is a geometric object that is constructed by using properties that are necessary and sufficient to define the intended geometric figure. Laborde (2005) used robust construction to describe this type of geometric figure in a DGE. When dragged one or more elements of it, a diagram created by a robust construction maintains all of its significant properties and is possible to generate all the possible configurations of the intended geometric figure.

Example of Student-Created Diagrams

To illustrate the affordance of this framework, I will use some geometry diagrams that learners created with GSP when solving the following construction problem:

The center of a circle is accidentally erased, can you find it?

In addition to the above statement, a circle with its center hidden was shown in GSP. These geometry diagrams were collected through a larger research project that used a sequence of task-based interviews to examine the dynamic relationship between instrumented activities and the development of mathematical knowledge in solving geometric construction problems. The participants of this research project were a group of preservice secondary mathematics teachers from a large public university in the United States. All of the participants reported that they had not taken any geometry courses after high school. Before participating in this research project, these preservice teachers (PSTs) had learned how to use the GSP through a course that focused on teaching and learning mathematics with mathematics action technologies. Figure 2 presents a few of the methods that the preservice teachers used to find the center of a circle. The geometry diagrams in these methods are used to illustrate how the framework can be used to characterize student-created diagrams.

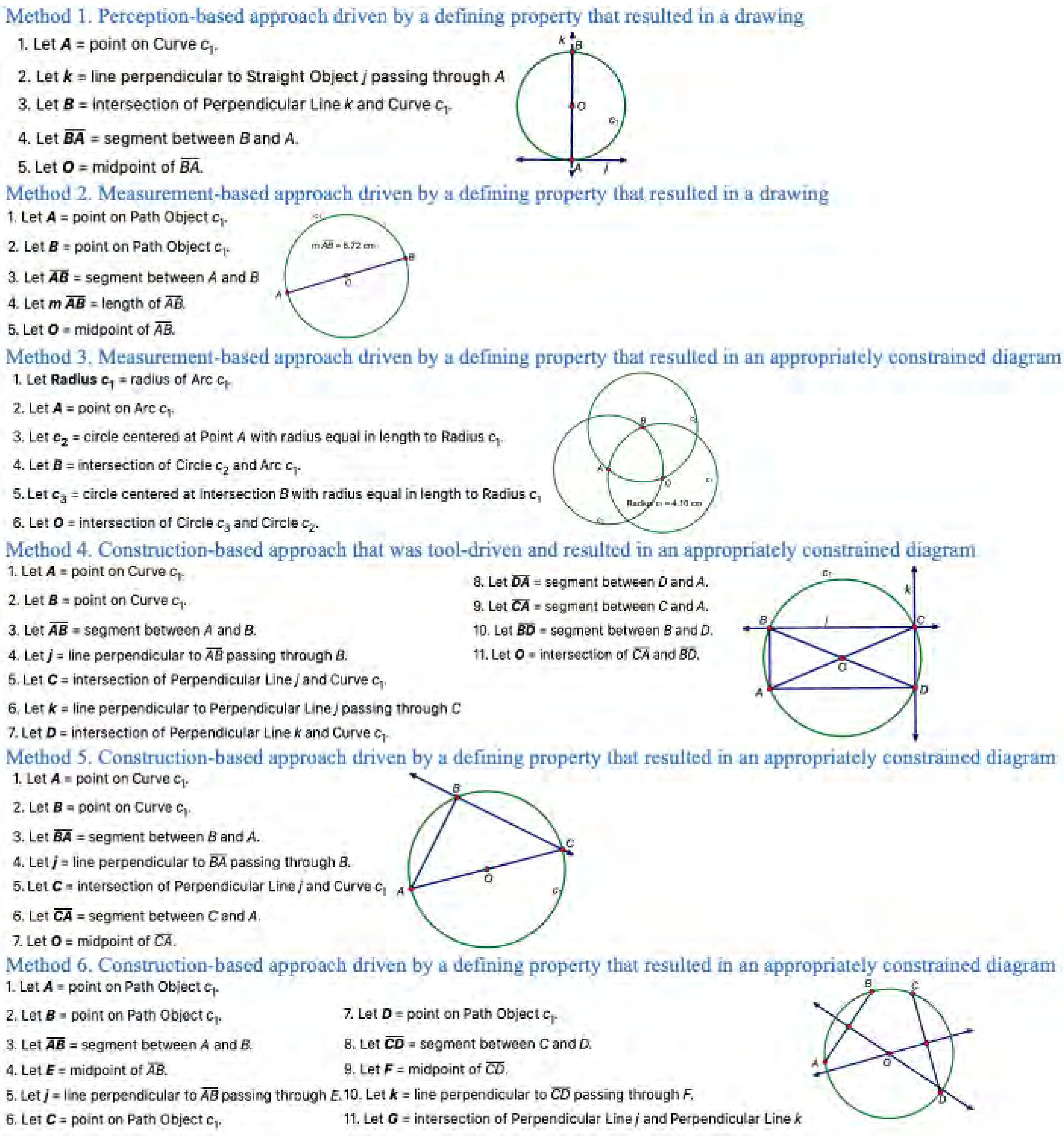


Figure 2: Methods for Finding the Center of a Circle

In Method 1, the PST was guided by the knowledge that the radius of a circle is perpendicular to the tangent line through its endpoint on the circle's circumference. He drew a line j and made it appear to be tangent to the circle through dragging. After marking the “tangent” point A, he constructed a line k perpendicular to line j at point A, which intersects the circle at point B. After creating segment $\overline{AB}$, he found the midpoint of $\overline{AB}$ and claimed that the midpoint was the center of the circle because $\overline{AB}$ was the diameter of the circle. In this method, the PST was guided by a property that could define the center of the circle. However, he relied on perception when creating a tangent line of the circle. Therefore, the diagram in Method 1 can be characterized as a perception-based approach driven by a defining property that resulted in a drawing. In Method 2, knowing that the diameter is the longest chord of a circle, the PST drew a chord, measured its length, and then gradually dragged an endpoint of the chord to make it the longest based on the numerical measure on the screen. After that, he found the midpoint of the chord and claimed that it was the center of the circle. Although guided by a defining property of the center of a circle, he relied on measurement to create the longest chord (i.e., diameter) of the circle. Therefore, the diagram in Method 2 can be characterized as a measurement-based approach driven by a defining property that resulted in a drawing.

In Method 3, the PST was driven by the idea that all points on the circle are equidistant to its center. After measuring the radius of the given circle by using the “Radius” command under the “Measure” menu, she chose a point A on the circle and the length of the radius of the circle to construct a new circle by using the “Circle by Center+Radius” command under the “Construct” menu, intersecting the given circle at point B. Using point B as the center and the length of the radius, she then constructed another circle by using the “Circle by Center+Radius” command again. She claimed that the point of intersection of the two new circles would be the center of the given circle. In this method, the PST was guided by a property that defines the center of the given circle. To use this property, she turned to measurement since the length of the radius of the circle was not given. The method did result in a diagram that passes the dragging test. Therefore, the diagram in Method 3 can be characterized as a measurement-based approach driven by a defining property that resulted in an appropriately constrained diagram.

In Method 4, the PST was initially guided by the idea that a square and its circumscribed circle share the same center. However, she did not know how to inscribe a square in the given circle. As a result, she planned to first make a rectangle and then adjust the side lengths to make it a square. After drawing a chord $\overline{AB}$ of the given circle, she constructed a perpendicular line j of $\overline{AB}$ from point B intersecting the circle at point C, and then a perpendicular line k of j from point C interesting the circle at point D. This resulted in a rectangle $ABCD$. After constructing the diagonals $\overline{AC}$ and $\overline{BD}$, she noticed that the point of intersection of the diagonals of the rectangle seemed to be the center of the circle. While dragging the vertices of the rectangle, she observed that it remained to be the center of the circle. As a result, she claimed that the point intersection of the diagonals of an inscribed rectangle is the center of the circle. In this method, the PST only used construction commands and dragging. Her actions were initially informed by a mathematical idea but then heavily driven by the instantaneous feedback from dragging. The feedback from dragging made her realize that she had already constructed the center of the circle. Therefore, the diagram in Method 4 can be characterized as a construction-based approach that was tool-driven and resulted in an appropriately constrained diagram.

In Method 5, guided by the idea that the hypotenuse of a right triangle inscribed in a circle is the diameter of the circle, the PST decided to inscribe a right triangle in the given circle. He drew a chord $\overline{AB}$ of the circle and then constructed its perpendicular line from an endpoint B of the chord. He then connected the endpoint of the chord and the point of intersection between the perpendicular line and the circle to create a right triangle ABC. He then found the midpoint of the hypotenuse $\overline{AC}$ of the right triangle ABC and claimed that it was the center of the circle. In this method, the PST relied on a property that defines the center of the circle and only used construction commands to locate the center. Therefore, the diagram in Method 5 can be characterized as a construction-based approach driven by a defining property that results in an appropriately constrained diagram. In Method 6, guided by the idea that the perpendicular bisector of a chord of a circle passes through the center of the circle, the PST drew a chord of the circle and constructed its perpendicular bisector by using the “Midpoint” and “Perpendicular Line” commands. Then, she drew another chord and constructed its perpendicular bisector in the same fashion. The point of intersection of the two perpendicular bisectors was marked as the center of the circle. This process of finding the center of the circle shows that the diagram in Method 6 can be characterized as a construction-based approach driven by a defining property that results in an appropriately constrained diagram.

Discussion and Conclusion

In this new framework, I consider learners' approaches to creating a geometry diagram, the mathematical knowledge that guides specific actions, and constraints in the resulting diagram. As a result, it provides a tool to describe geometry diagrams in a DGE that goes beyond any one of the three dimensions in the framework.

In DGEs a solution to a construction problem is often considered valid if the diagram is not possible to be messed up by dragging. In other words, a DGE geometry diagram is appropriately constrained when it passes the “dragging test”. However, the framework in this paper shows that an appropriately constrained diagram can be constructed in different ways. First of all, appropriately constrained diagrams can be created by using a measurement-based approach, a construction-based approach, or a transformation-based approach. For example, in the problem of finding the center of rotation, although both Method 3 and Method 5 produced the exact center of the circle, they took different approaches. While Method 3 relied heavily on measurement tools, Method 5 used construction tools. The existence of different approaches to constructing an appropriately constrained diagram has raised the issue of validity for diagrams in a DGE because the "dragging test" does not distinguish between diagrams involving measurement and diagrams involving only those software tools that are compatible with straightedge and compass. Stylianides and Stylianides (2005) introduced the compatibility criterion, which states that a DGE diagram is valid if and only if it maintains its geometrical properties under dragging and its construction process does not violate the DGE-construction constraints equivalent to those imposed on straightedge-and-compass constructions. The proposed framework can capture the subtle differences among appropriately constrained diagrams that are built upon different approaches.

Moreover, appropriately constrained diagrams can be created based on different geometric knowledge. For example, when finding the center of a circle, both Method 5 and Method 6 took a construction-based approach and yielded a construction for the center of the circle, but the two methods were built on different mathematical ideas. While Method 5 was guided by the idea that the hypotenuse of a right triangle inscribed in a circle is a diameter of the circle, Method 6 was built on the property that the perpendicular bisector of a chord of a circle passes through the center of the circle. The knowledge that is mobilized to create a geometry diagram reveals students’ understanding of the geometric object to be constructed and how it connects with other geometric ideas. The proposed framework affords us to account for the mathematical knowledge mobilized in the diagram construction process.

Appropriately constrained diagrams can also result from actions driven by the instantaneous feedback from the use of construction commands and dragging in DGE. For instance, when finding the center of a circle, although the preservice teacher in Method 4 did not anticipate that the intersection of diagonals of the inscribed rectangle would give the center of the circle, the instantaneous feedback from dragging made her realize that she just constructed the center of the circle. This suggests that an appropriately constrained diagram might not build on a learner's appropriate knowledge of geometry. Rather, it might be created unintentionally by the indirect invariants in DGE (Mariotti, 2014). This is particularly true when a learner is not aware of the invariant relationship resulting from the use of one and more construction commands. The proposed framework can capture this nuance among appropriately constrained diagrams. Meanwhile, teachers may leverage such realizations as opportunities to support the making and proving of conjectures.

The proposed framework also provides a tool to identify methods for creating geometry diagrams that are only possible in DGEs. As shown in Method 2 for finding the center of a circle, an appropriately constrained diagram can be created through a measure-based approach that is driven by a defining property of the intended geometry figure. However, this method is only possible in a DGE because it is impossible to measure the radius of a circle without knowing its center in a paper-and-pencil environment. Indeed, any method that is characterized as a measure-based approach driven by a defining property that results in an appropriately constrained diagram is unique to DGEs. Another unique method in DGEs is dragging an underconstrained diagram to create the intended geometry figure. Barabash (2019) suggested adding dragging as a geometric construction tool in DGE because it is an application of the continuity reasoning and intermediate-value theorem to geometry. In essence, dragging is the continuous variation of one of the parameters of a problem, which is only made possible by dynamic mathematics software.

Creating, interpreting, manipulating, and communicating with geometry diagrams is an important part of geometry learning. Dynamic geometry environment offers a wide range of tools for its users to create and interact with geometry diagrams. This paper proposes a framework to characterize a geometry diagram in a DGE that considers a learner's approach to create the geometry diagram (i.e., perceptual-based, measurement-based, construction-based, or transformation-based), the driving force that guides the learner's specific actions in a DGE (i.e., tool-driven, or property-driven), and constraints in the resulting diagram (i.e., drawing, underconstrained, overconstrained, and appropriately constrained). Examples of student work on a construction problem are presented to illustrate the use of the framework. These examples show that the proposed framework can capture the subtle but important differences among geometry diagrams that learners create in a DGE. The examples also show that the framework can be used to identify methods for creating geometry diagrams that are only possible in DGEs and moments of instrumental synergy.

References

Alshwaikh, J. (2010). Geometrical diagrams as representation and communication: A functional analytic framework. Research in Mathematics Education, 12(1), 69-70. https://doi.org/10.1080/14794800903569881

Ball, L., & Stacey, K. (2005). Teaching strategies for developing judicious technology use. In W. J. Masalski & P. C. Elliott (Eds.), Technology-supported mathematics learning environments, 2005 Yearbook of the National Council of Teachers of Mathematics (pp. 3-15). Reston, VA: National Council of Teachers of Mathematics.

Barabash, M. (2019). Dragging as a geometric construction tool: Continuity considerations inspired by students' attempts. Digital Experiences in Mathematics Education, 5(2), 124-144. https://doi.org/10.1007/s40751-019-

0050-2
de Villiers, M. (2017). An explanatory, transformation geometry proof of a classic treasure-hunt problem and its generalization. International Journal of Mathematical Education in Science and Technology, 48(2), 260-267. https://doi.org/10.1080/0020739X.2016.1210245
Dimmel, J. K., & Herbst, P. G. (2015). The semiotic structure of geometry diagrams: How textbook diagrams convey meaning. Journal for Research in Mathematics Education, 46(2), 147-195. https://doi.org/10.5951/jresematheduc.46.2.0147
Duval, R. (1995). Geometrical pictures: Kinds of representation and specific processings. In R. Sutherland & J. Mason (Eds.), Exploiting mental imagery with computers in mathematics education (pp.142-157). Berlin, Germany: Springer. https://doi.org/10.1007/978-3-642-57771-0_10
Finzer, W. F., & Bennett, D. S. (1995). From drawing to construction with The Geometer's Sketchpad. The Mathematics Teacher, 88(5), 428-431. https://doi.org/10.5951/MT.88.5.0428
Fischbein, E. (1993). The theory of figural concepts. Educational Studies in Mathematics, 24(2), 139-162. https://doi.org/10.1007/BF01273689
Guin, D., & Trouche, L. (1999). The complex process of converting tools into mathematical instruments: The case of calculators. International Journal of Computers for Mathematical Learning, 3, 195-227. https://doi.org/10.1023/A:1009892720043
Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. ZDM, 36(5), 129-139. https://doi.org/10.1007/BF02655665
Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. Journal for Research in Mathematics Education, 164-192. https://doi.org/10.2307/30034955
Jones, K. (2013), Diagrams in the teaching and learning of geometry: some results and ideas for future research, Proceedings of the British Society for Research into Learning Mathematics, 33(2), 37-42.
Laborde, C. (2005). Robust and soft constructions: Two sides of the use of dynamics geometry environments. In Proceedings of the Tenth Asian Technology Conference in Mathematics (pp. 22–35). Korea National University of Education.
Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, O. Skovsmose, & P. Valero (Eds.), Meaning in mathematics education (pp. 159-179). New York, NY: Springer. https://doi.org/10.1007/0-387-24040-3_11
Mariotti, M. A. (2014). Transforming images in a DGS: The semiotic potential of the dragging tool for introducing the notion of conditional statement. In S. Rezat, M. Hattermann, & A. Peter-Koop (Eds.), Transformation—A fundamental idea of mathematics education (pp. 155–172). New York: Springer. https://doi.org/10.1007/978-1-4614-3489-4_8
Scher, D. (2005). Square or not? Assessing constructions in an interactive geometry software environment. In W. J. Masalski & P. C. Elliott (Eds.), Technology-supported mathematics learning environments, 2005 Yearbook of the National Council of Teachers of Mathematics (pp. 113-124). Reston, VA: National Council of Teachers of Mathematics.
Stylianides, G. J., & Stylianides, A. J. (2005). Validation of solutions of construction problems in dynamic geometry environments. International Journal of Computers for Mathematical Learning, 10(1), 31-47. https://doi.org/10.1007/s10758-004-6999-x
Yao, X. (2020). Unpacking learner's growth in geometric understanding when solving problems in a dynamic geometry environment: Coordinating two frames. The Journal of Mathematical Behavior, 60, 100803. https://doi.org/10.1016/j.jmathb.2020.100803

GEOMETRIC REASONING OF K-5 IN-SERVICE TEACHERS

Teruni Lamberg
University of Nevada, Reno
Terunil@unr.edu

Lynda Wiest
University of Nevada, Reno
wiest@unr.edu

Keywords: Geometry and Spatial Reasoning; Teacher Knowledge

The purpose of this study was to investigate K-5 in-service teachers' geometric reasoning. Twenty teachers completed two types of problem sets. The first consisted of three problems in which participants were to examine three geometric figures, mark which did not belong, and tell why. In the second, participants were to answer five questions on a Venn diagram with three intersecting circles (classify figures in each circle and then draw a figure that would fit into the intersection of all three and one that wouldn't fit into any (with explanation). Developing geometric thinking involves understanding polygons as a class (Battista, 2007; Clements & Battista, 1992). Understanding polygons and relationships among polygons draws on students' ability to recognize and modify relevant attributes of polygons. This process also involves the ability to transform a non-example of a polygon into a polygon (Bernabeu et al., 2021).

Methods

Participants were 20 in-service teachers in a small city in the West. All were engaged in a mathematics professional development project.

We analyzed written work by marking the eight responses for each participant as correct, partially correct, or incorrect and writing notes about possible student thinking. We discussed these until we agreed on our assessments of participant work.

Results & Implications

Influencing factors in participants' work were content knowledge (e.g., knowing specific polygon and angle types), reasoning (ability to identify similarities and differences, to name the most constrictive class describing a set of figures, such as "triangles" versus "polygons," and to interpret a Venn diagram), and visual perception (e.g., noticing key mathematical features without relying on visual impressions, such as having "diagonal sides" or a "short base").

Participants operated at different levels of geometric reasoning (e.g., Pujawan et al., 2020) ranging from use of visual perception to properties of figures to classes of figures (with most falling into the second two categories). Participants who based decisions on properties of figures relied almost entirely on the sides (number and relationship to each other) with little consideration of angles. This points to the importance of that which can be directly observed. This can limit an ability to perceive and examine geometry problems more flexibly and could interfere with reasoning about shapes. Indeed, creativity in perceiving likenesses and differences among figures where several answers were possible was very limited (two examples where this did occur involved two isosceles and one scalene triangle that was noted by almost all participants, albeit worded differently, but one participant identified the scalene as different due to not having line symmetry and another chose the figure that had no obtuse angle). Finally, a key issue noted was lack of language precision (e.g., "straight three-sided shapes" instead of "triangles" and "all have right angles" instead of all have "at least one right angle").

We conclude that teachers need more work with content knowledge, reasoning (visual, oral, textual), and language precision in relation to teaching elementary geometry (e.g., Shockey & Pindiprolu, 2015).

References

Bernabeu, M. Moreno, M., & Llinares, S. (2021). Primary school students' understanding of polygons and the relationships between polygons. *Educational Studies in Mathematics*, *106*, 251-270.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), Second handbook of research on mathematics teaching and learning (pp. 843–908). Reston, VA: NCTM-IAP.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 420–464). New York, NY: MacMillan.

Pujawan, I. J. N., Suryawan, I. P. P., & Prabawati, D. A. A. (2020). The effect of Van Hiele learning model on students' spatial abilities. *International Journal of Instruction*, *13*(3), 461-474.

Shockey, T., & Pindiprolu, S. (2015). Uniquely precise: Importance of conceptual knowledge and mathematical language. *Journal on School Educational Technology*, *11*(1), 28-33.

SPATIAL ANXIETY MODERATES THE EFFECT OF SPATIAL ABILITY IN GEOMETRIC REASONING

Kelsey E. Schenck
University of Wisconsin - Madison
keschenck@wisc.edu

Mitchell J. Nathan
University of Wisconsin - Madison
mnathan@wisc.edu

Keywords: Spatial Anxiety, Spatial Ability, Geometry Reasoning

Geometry is a highly spatial domain of mathematics with studies investigating the relationship between spatial skills and geometry from both psychometric accounts (Newcombe, 2013) and domain-centric analyses (Battista et al., 2018). While spatial ability has been linked to general mathematics performance (Newcombe, 2013) and STEM achievement (Shea et al., 2001), little is known about the impact of spatial anxiety on specific domains of mathematics such as geometry. *Spatial anxiety*, a type of domain-specific anxiety, is a relatively new construct linked to spatial ability (Malanchini et al., 2017) and varies considerably among individuals (Lyons et al., 2018). Recently, studies have begun to investigate its links to mathematics (Schenck & Nathan, 2020), but these studies have focused on standardized assessments that encompass various mathematics domains. To the best of our knowledge, no studies have investigated the links between spatial anxiety and geometric reasoning, which drives the research question: What is the relationship between spatial anxiety and geometric reasoning as seen in verbal insight, and transformational proof?

Methods

The dataset includes undergraduate students (N = 94) recruited from a large Midwestern university in the United States. First, participants completed a series of demographic and covariate measures including a truncated spatial ability assessment based on the *Spatial Reasoning Instrument (*Ramful et al., 2017) and the *Novel Spatial Anxiety Scale* (Lyons et al., 2018). Next, participants played an embodied videogame, *The Hidden Village*, in which participants gave a verbal response to the veracity of eight geometry conjectures (Nathan & Swart, 2021). Videos were transcribed verbatim and time stamps were added to split the transcripts into eight conjectures for coding. This segmentation resulted in 752 videos clips to be coded. Verbal insight and transformation proof were given a 0/1 binary code based on the coding scheme developed by Authors (Nathan et al., 2021).

Results and Discussion

For verbal insight, the interaction between spatial ability and spatial anxiety was significantly associated with a decrease in the chance of producing correct verbal insight. Gender and spatial ability scores remained significantly associated with correct verbal insight. The results for the model predicting transformational proof showed that the interaction and spatial anxiety scores were significantly associated with a decrease in the chance producing a transformational proof. Gender and spatial ability scores also remained significantly associated. These results suggest that spatial anxiety may not have a direct effect on geometric reasoning but appears to moderate effect of spatial ability on geometry reasoning. This effect may need to be considered by researchers and educators who are looking to improve students' geometric reasoning.

Acknowledgments

This project was supported by the Institute of Education Sciences, U.S. Department of

Education, through Grant R305A160020 to University of Wisconsin - Madison. The opinions expressed are those of the authors and do not represent views of the IES or the U.S. Dept. of Ed.

References

Battista, M.T., Frazee, L. M., & Winer, M. L. (2018). Analyzing the relation between spatial and geometric reasoning for elementary and middle school students. In K. S. Mix & M. T. Battista (Eds.), *Visualizing Mathematics. Research in Mathematics Education* (pp. 195 –228). Springer, Cham.

Lyons, I. M., Ramirez, G., Maloney, E. A., Rendina, D. N., Levine, S. C., & Beilock, S. L. (2018) Spatial anxiety: A novel questionnaire with subscales for measuring three aspects of spatial anxiety. *Journal of Numerical Cognition, 4*(3), 526-553.

Malanchini, M., Rimfeld, K., Shakeshaft, N. G., Rodic, M., Schofield, K., Selzam, S., Dale, P. S., Petrill, S. A., & Kovas, Y. (2017). The genetic and environmental aetiology of spatial, mathematics, and general anxiety. *Scientific Reports, 7*, 42218.

Nathan, M. J., Schenck, K. E., Vinsonhaler, R., Michaelis, J. E., Swart, M. I., & Walkington, C. (2021). Embodied geometric reasoning: Dynamic gesture during intuition, insight, and proof. *Journal of Educational Psychology, 113*(5), 929 - 948.

Nathan, M. J., & Swart, M. I. (2021). Materialistic epistemology lends design wings: educational design as an embodied process. *Educational Technology Research and Development*, *69*, 1925 – 1954.

Newcombe, N. S. (2013). Seeing Relationships: Using Spatial Thinking to Teach Science, Mathematics, and Social Studies. *American Educator, 37*, 26–40.

Ramful, A., Lowrie, T., & Logan, T. (2017). Measurement of spatial ability: Construction and validation of the Spatial Reasoning Instrument for Middle School Students. *Journal of Psychoeducational Assessment, 35*(7), 709 -727.

Schenck, K.E. & Nathan, M.J. (April, 2020). *Connecting mathematics, spatial ability and spatial anxiety.* [Paper presentation]. American Educational Research Association Conference, San Francisco, CA, United States.

Shea, D. L., Lubinski, D., & Benbow, C. P. (2001). Importance of assessing spatial ability in intellectually talented young adolescents: A 20-year longitudinal study. *Journal of Educational Psychology, 93*(3), 604-614.

WORLD OF PATTERN: GEOMETRY, BODY, AND BEAUTY

Mina Sedaghatjou
Alfred University
sedaghatjou@alfred.edu

Colleen Wahl
Alfred University
Wahl@alfred.edu

Chen Lin
University Western Ontario
clin393@uwo.ca

Keywords: Geometry and Spatial Reasoning, Integrated STEM / STEAM, Cognition, Body Movement.

Following the introduction of embodied cognition by Lakoff and Núñez (2000), several studies have expanded body-mind unity in mathematics education (Arzarello & Robutti, 2008; Radford, 2009; Nemirovsky & Ferrara, 2009; Roth, 2010). We desired to investigate mathematical embodied cognition in student's body movements, including choreographic choices based on mathematical ideas, where the students did not share a Science, Technology, Engineering, and Mathematics (STEM) background at their undergraduate level. This study, inspired by the aesthetic geometrical nature of the human movement categorized and perceived in the Laban Movement Analysis tradition (Dörr, 2003; Wahl, 2019). To address this purpose, we suggested the following research question: How do non-STEM background students perceive, demonstrate, leverage, and perform geometrical concepts and patterns in their moving bodies?

Methods

In this exploratory mixed-method study, we offered an elective interdisciplinary undergraduate course called "World of Pattern: Geometry, Body, and Beauty", where eight students are enrolled and the course is co-taught by faculty in STEM Education and Dance, both with unique specialties. For this study, some course experimental activities were developed in the classroom, which consisted of recognizing, planning, drawing, and performing movements. The course is still in process, and we video record and analyze students' body movements, their performances, gestures, and communications. The theoretical lens of perceptuomotor integration (Nemirovsky, Kelton & Rhodehamel, 2013) is used to articulate mathematical embodied communications. The figure below illustrates a portion of the student's planned performance while maintaining an awareness of spatial and geometric patterns and directions, followed by dancing in an octahedron.

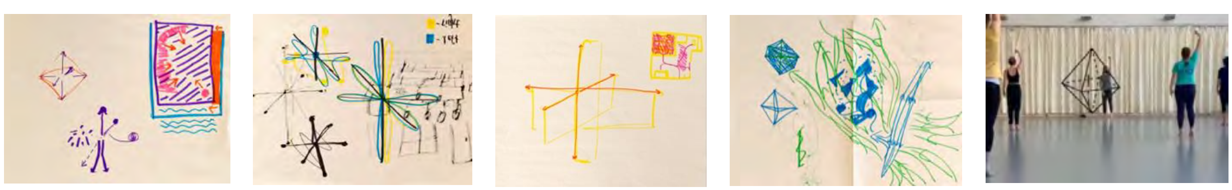


Figure 1 : Geometry is used to communicate, plan, and perform body movements.

Conclusion

The initial analysis of the students' actions and feedback shows the students search for patterns, directions, and dimensions in the space even beyond the obvious primary ones; they also create algorithms, plan, and perform sophisticated 3D geometrical shapes—individually or jointly—and propel themselves through the space in specifically patterned ways. We also found the interdisciplinary approach promoted spatial reasoning skills and awareness, which are vital in both geometry and dance. Furthermore, we saw the emergence of challenging creative ideas based on the concept of interconnectedness between mathematics and art, demonstrating mathematical embodied cognition with implications for both mathematics education and dance.

References

Arzarello, F., & Robutti, O. (2008). Framing the embodied mind approach within a multimodal paradigm. *Handbook of International Research in Mathematics Education*, 720–749.

Dörr, E. (2003). *Rudolf von Laban: The "Founding Father" of Expressionist Dance*. https://doi.org/10.1081/DNC-120018851

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being* (1st ed). Basic Books.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*(2), 159–174 https://doi.org/10.1007/s10649-008-9150-4

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing Mathematical Instruments: Emerging Perceptuomotor Integration With an Interactive Mathematics Exhibit, *Journal for Research in Mathematics Education JRME*, *44*(2), 372-415. Retrieved Feb 15, 2022, from https://pubs.nctm.org/view/journals/jrme/44/2/article-p372.xml

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*(2), 111–126. https://doi.org/10.1007/s10649-008-9127-3

Roth, W. M. (2010). Incarnation: Radicalizing the Embodiment of Mathematics. *For the Learning of Mathematics*, *30*(2), 8–17. http://www.jstor.org/stable/20749443

Wahl, C. (2019). *Laban/Bartenieff Movement Studies: Contemporary Applications*. Human Kinetics.

EXPLORING MEASUREMENT STRATEGIES AMONG NOVICE AND ADVANCED LEARNERS

Hannah Smith
Worcester Polytechnic Inst.
Hsmith2@wpi.edu

Luisa Perez Lacera
Worcester Polytechnic Inst.
lmperezlacera@wpi.edu

Avery Harrison Closser
Worcester Polytechnic Inst.
ahclosser@wpi.edu

Keywords: Geometry and Spatial Reasoning, Instructional Activities and Practices, Measurement.

Accurately measuring and estimating spatial quantities is important for success in and out of school and is contingent on grasping core concepts of measurement (Lehrer, 2003). In particular, *tiling*, or successive measurement without gaps, is critical for measurement (Leher, 2003). The concept of tiling is that measurement must be done from an acute start point such as the edge of an object. The current study explores whether and how elementary and college students use a start-point marker strategy for measurement, as an indicator of their understanding of tiling.

As part of a larger study, we compared the measurement strategies of 15 novice (elementary) and 25 advanced (college) learners (Harrison, 2019). Students were given a basketball, which had a diameter of 9.5 inches, and asked, “Can you estimate the diameter of this sphere?”

Students were provided with a 6- or 12-inch stick to help. Students were video recorded, and researchers coded measurement strategies during problem solving. Here, we analyze whether or not students used a start-point marker strategy, defined as: “the edge of the object lining up with the end of the reference tool” (Harrison, 2019). We compare the use of start-point markers across students’ accuracy on the task by percent error, calculated by subtracting the correct answer from the given answer, dividing it by the correct answer, and retrieving the absolute value. We ask: 1) How often do college and elementary students use a start-point marker measurement strategy? 2) How does using a start-point marker strategy affect measurement accuracy? 3) Does the accuracy differ between college and elementary students?

First, we found that 72% of college and 67% of elementary students used a start-point marker. Second, a 2 (age: college vs. elementary) × 2 (start-point marker: yes or no) ANOVA revealed a significant main effect of age, $F(1,36) = 18.00$, $p < .001$. College students' average percent error ($M = 8.39$, $SD = 5.88$) was significantly lower than elementary students' ($M = 20.71$, $SD = 19.94$). Next, we found a significant main effect of the start-point marker, $F(1,36) = 9.04$, $p = .005$. Those who used a start-point marker ($M = 10.04$, $SD = 9.64$) had a lower percent error than those who did not ($M = 19.9$, $SD = 20.08$). Third, we found a significant Age × Strategy interaction, $F(1,36) = 10.30$, $p = .003$. When participants used a start-point marker, there were no significant differences in accuracy between age ($p = .79$). However, when participants did not use a start-point marker, elementary students had a significantly higher percent error than college students ($p < .001$). There were no differences in percent error between college students based on strategy. However, elementary students who used a start-point marker had a significantly lower percent error than those who did not ($p = .002$).

The results from this study have implications for instruction and practice in classrooms. We found that when novice learners use a start-point marker strategy, they have a much more accurate measurement estimation, adding to the literature on concepts of measurement as a critical component of instruction (Lehrer, 2003). Researchers should expand this work with a larger sample to see if this strategy is successful in other contexts and to identify other successful measurement strategies that might inform classroom practice.

References

Harrison, A. (2019). Exploring measurement estimation through learners' actions, language, and gestures [Unpublished master's thesis]. Worcester Polytechnic Institute.

Lehrer, R. (2003). Developing understanding of measurement. In Kilpatrick, J. W., Martin, G., & Schifter, D. (Eds.), A research companion to principles and standards for school mathematics (pp. 179-192). Reston, VA: National Council of Teachers of Mathematics

SCALE IT UP? I DO NOT KNOW WHAT THAT MEANS

Lili Zhou
Purdue University
Zhou756@purdue.edu

The concept of *scale/scaling* is important in mathematics as well as in other disciplines and in real life. In mathematics, the concept development of *scale/scaling* derives measurements from qualities and links to spatial reasoning and multiple mathematical topics (e.g., proportion). I observed a group of 8th-grade girls (n=10) completing a scaling task in an afterschool STEM club; the girls were asked to scale the original design by a factor of two. Eighty percent of the girls scaled the original design in one direction or two directions. See Figure 1, the sample of student work showing that she only scaled the original design in one direction.

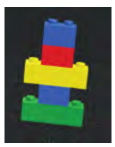
Initial Design	Student Work	Correct Model
		

Figure 1. Scaling Task

I interpret this student's work as her having understood the concept of scaling by a factor of two to mean using double the original number of blocks to make the design bigger. She chose to make the design wider, showing her incomplete understanding of the concept of scale/scaling. My observation prompted me to investigate how scale/scaling is described and used in the mathematics curricula, including the official curriculum and the intended curriculum (Remillard & Heck, 2014). Here, I only focus on scale/scaling in the official curriculum (i.e., CCSSM).

Exploring the Concept of Scale in CCSSM

I chose CCSSM because, as an official curriculum, it is widely adopted and greatly influences the intended curriculum (Reys, 2014). Throughout CCSSM, the word *scale* appears nineteen times, *scaled* appears three times, and *scaling* appears twice. These appearances are distributed throughout multiple topics: (a) measurement, in the area of Measurement and Data, CCSSM expects students to use scaled pictures and scaled bars to represent and interpret data; (b) multiplication, in Grade 5, CCSSM expects students to "interpret multiplication as *scaling* (resizing)" (G5, p. 36); (c) drawing, CCSSM expects students to "solve problems involving *scale* drawings of geometric figures" (G7, p. 49); and (d) transformation, in high school geometry, scale is used in similarity, e.g., "…formalizing the similarity ideas of 'same shape' and 'scale factor'…" (HS-G, p. 74).

Conclusion and Discussion

Scale/scaling covers multiple mathematics topics in CCSSM and is used to describe and explain other concepts. However, there is no clear description of *scale/scaling,* and no learning expectations for the development of students' understanding of *scale/scaling*. CCSSM expects students to interpret multiplication as *scaling*, which interprets *scale/scaling* as a linear quantitative concept. Later, in the topic of similarity, *scale/scaling* is presented as both qualitative and quantitative transformation of figures. Given the importance of scale in mathematics and other disciplines (e.g., engineering), I suggest that mathematics curricula present *scale/scaling* as a formal concept, providing a rigorous definition and explicating learning expectations to support the development of understanding.

References

Common Core State Standards Initiative (CCSSI). (2010). *Common Core State Standards for Mathematics*. Washington, DC: National Governors Association Center for Best Practices and Council of Chief State School Officers.

Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM Mathematics Education*, *46*(5), 705-718.

Reys, B. J. (2014). Mathematics curriculum policies and practices in the U.S.: The Common Core State Standards Initiative. In Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 35- 48). Springer.

Chapter 6:
Mathematical Knowledge for Teaching

DEVELOPMENT AND MEASUREMENT OF STATISTICAL KNOWLEDGE FOR TEACHING

Stephanie Casey
Eastern Michigan University
scasey1@emich.edu

Andrew Ross
Eastern Michigan University
aross15@emich.edu

Jeremy Strayer
Middle Tennessee State University
jeremy.strayer@mtsu.edu

We describe novel teacher education curriculum materials designed to develop secondary PSTs' Statistical Knowledge for Teaching (SKT) along with a new test for measuring teachers' SKT. We report PSTs' changes in SKT from learning with the materials in a preliminary study.

Keywords: Teacher Knowledge; Data Analysis and Statistics

Teachers create meaningful student learning opportunities by drawing on a variety of knowledge domains—encompassing not only knowledge of the content to be learned but also additional knowledge connected to the work of teaching (Hill et al., 2008; Shulman, 1986; Thwaites et al., 2011). Assisting preservice teachers (PSTs) in developing this sort of mathematical and statistical knowledge (termed mathematical knowledge for teaching (MKT) and statistical knowledge for teaching (SKT)) is of particular interest to those who teach mathematics and statistics content courses in secondary mathematics teacher preparation programs because we know that there is an association between teachers' MKT/SKT and high quality instruction (Hill et al., 2007). However, there is dissonance between the preparation PSTs are receiving in these courses and the preparation they need to develop MKT/SKT (Goulding et al., 2003). Additionally, understanding how PSTs develop MKT/SKT and how researchers observe and/or measure MKT/SKT is still an open question (Groth, 2017; Silverman & Thompson, 2008; Thwaites et al., 2011). The MODULE(S^2) project works in this space and has created curricular materials that provide opportunities for secondary PSTs to develop MKT and SKT in university algebra, geometry, modeling, and statistics courses. With regard to measures, some tests exist for measuring MKT (e.g., Hill et al., 2007; Mohr-Schroeder et al., 2017), but none exist for measuring secondary (grades 6-12) teachers' SKT. In this brief report, we describe the development of a test we created to assess secondary PSTs' SKT prior to and after learning with MODULE(S^2) materials, and report PSTs' changes in SKT from learning with the statistics MODULE(S^2) materials in a preliminary study.

Just as mathematics and statistics are related yet distinct fields (Cobb & Moore, 1997), MKT and SKT are related yet distinct domains of knowledge (Groth, 2007, 2013). Both MKT and SKT include subject matter knowledge (SMK: understanding the logical necessity of aspects of the content to be taught) and pedagogical content knowledge (PCK: understanding how to make content understandable to students) (Groth, 2013; Hill et al., 2008; Simon, 2006). However, SKT differs from MKT because it contains nonmathematical elements, such as assessing the suitability of statistical questions for the context of a statistical investigation (Groth, 2007). The *MODULE(S^2): Statistics for Secondary Mathematics Teaching* curriculum (Casey et al., 2019) was created to develop PSTs' SKT regarding study design, exploratory data analysis, statistical inference, and association, through enactment of the statistical investigation cycle. The materials align with the content of a modern college introductory statistics course while also preparing PSTs to address the CCSS-M (CCSSI, 2010) statistics standards in their teaching and meeting the recommendations from professional societies (AMTE, 2017; Bargagliotti et al., 2020; Franklin, 2015). Attention is given to developing PSTs' critical statistical literacy (Weiland,

2017) and abilities to implement equitable teaching practices. This is accomplished through a sustained focus on data sets addressing inequities in educational and social contexts, asking PSTs to wrestle with the dissonances that analyses of these data sets illuminate, and prompting PSTs to reflect on how they might teach lessons with their future students to leverage these contexts for learning. Opportunities for PSTs to develop their PCK are integrated throughout the materials, including learning common conceptions of students when learning statistics, responding to student work, constructing lessons, analyzing curriculum standards, and learning how to use statistics to empower action for social change. The materials are organized into three modules that could collectively be taught in a semester course.

SKT Test

We developed a 7-item SKT test along with an accompanying scoring rubric for the MODULE(S^2) project. It is designed to assess secondary teachers' knowledge for teaching the statistics standards in the CCSS-M (CCSSI, 2010). It is aligned with Groth's (2013) conceptualization of SKT, comprehensively assessing SMK and PCK. Its design drew upon previous work to develop SKT assessment items (e.g., Casey et al., 2019; Groth, 2014). Items assessing PCK utilized released student work from LOCUS (n.d.) to document student conceptions of statistics. We piloted and revised the instrument and its accompanying rubric for two years. Revisions focused on improving the clarity of the language as well as the reliability of the scoring. Thus, we used multiple methods to work towards test content validity (AERA et al., 2014) for the use of this SKT test for this study.

The test contains 12 selected response prompts (e.g., multiple choice) and 20 open-ended prompts across the 7 items. The maximum number of points possible on the test is 45. Three of the items, worth a total of 8 points, address SMK. An additional 3 items, totaling 29 points, address PCK. Finally, a single item addresses both SMK (parts i and iii, worth 3 points each) and PCK (part ii, worth 2 points). This item is shown in Figure 1 as a sample.

Imagine you were teaching a high school (non-AP) class and presented the following task to your students:

A simple random sample of 100 high school seniors was selected from a large school district. The gender of each student was recorded, and each student was asked the following questions: 1. Have you ever had a part-time job? 2. If you answered yes to the previous question, was your part-time job in the summer only? Their responses are shown in the table to the right.

Construct a graphical display that represents the association between gender and job experience for the students in the sample.

Job Experience	Gender		
	Male	Female	Total
Never had a part-time job	21	31	52
Had a part-time job during summer only	15	13	28
Had a part-time job but not only during summer	12	8	20
Total	48	52	100

As your students work on the task, you notice a student named Juan has created this graph (shown at the right).

i. Comment on the strengths and weaknesses of this graph.
ii. How would you respond to Juan in order to advance his thinking concerning graphical displays to show association between categorical variables?
iii. Complete the task yourself, creating the best graphical representation for representing the association between gender and job experience.

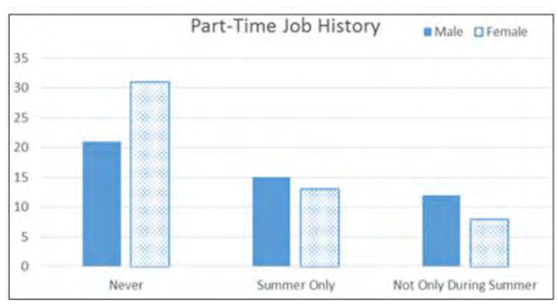


Figure 1: Sample item; parts (i) and (iii) focus on SMK while (ii) focuses on PCK

The scoring rubric for the item in Figure 1 delineates how the points should be awarded for this item. To earn 3 points for part (i), the response needs to include at least one strength and the following two notable weaknesses: (1) there is no label on the vertical axis; and (2) it displays frequencies instead of conditional relative frequencies. Relatedly, when creating the best graphical representation in part (iii), a representation received all three points if it was a segmented bar graph showing percentages conditioned by either categorical variable and there were no flaws in the execution of this representation. The rubric tiered down the point values when some elements were missing. For part (ii), a response received 2 points if it included questions or proposed activities for Juan that addressed either of the notable weaknesses from part (i). A response to part (ii) received 1 point if it accurately gave an explanation about what is incorrect or correct about Juan's graph but did not prompt Juan to think about it himself.

Data Collection and Results

The SKT test was given to PSTs in two statistics-for-teachers classes at separate universities that used the MODULE(S^2) materials in the 2020-21 academic year. It was given at the start of the semester as a pre-test and at the end of the semester as a post-test. It was administered as a take-home test without a time limit. The tests were independently scored by 2 graduate assistants using the scoring rubric, and they reconciled any scoring discrepancies for each item. The sample sizes were 11 on the pre-test and 11 on the post-test; we present results here for the 10 PSTs who submitted both tests.

Figure 2a shows the pre- and post-test scores for PSTs' overall SKT, with line segments linking each PST's pre- and post-test scores.

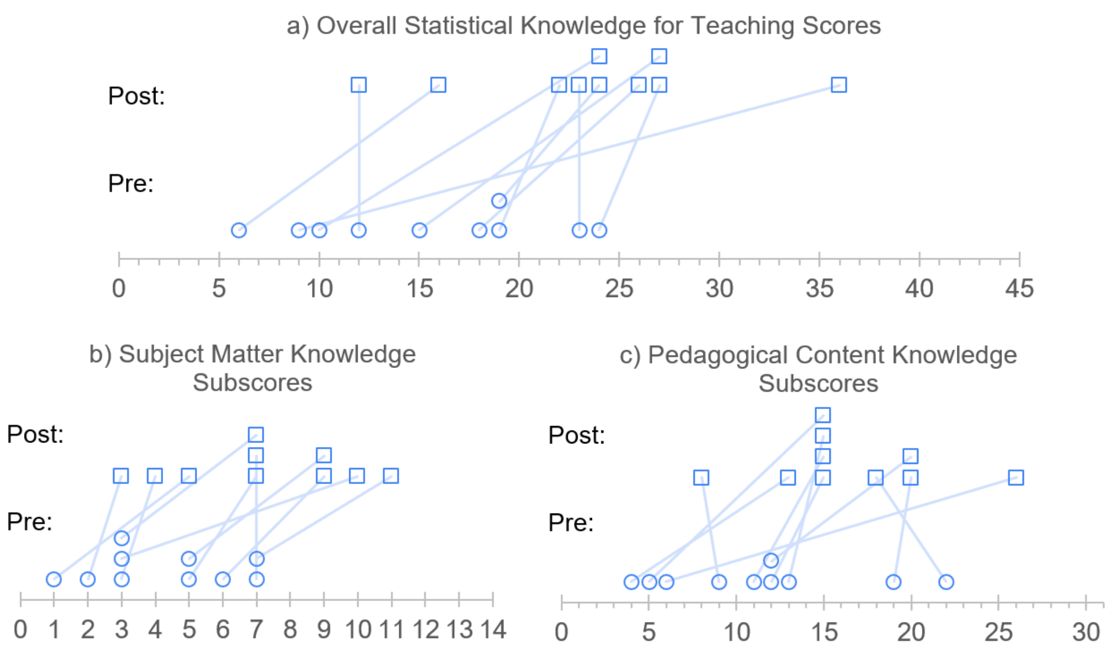


Figure 2: Paired dotplots of pre-test (circles) and post-test (squares) scores. (a) Overall scores, (b) SMK subscores, (c) PCK subscores

We observe a general pattern of increases from pre to post; the mean pre-test and post-test scores were 15.5/45 (34.4%) and 23.7/45 (52.7%), for an average increase of 18.22 percentage points (standard deviation 18.20). This results in a strong effect size, measured as 0.92 by Cohen's d with Hedge's correction. Similar graphs are provided in Figure 2 for SMK (2b) and PCK (2c). These also show that PSTs generally improved from pre- to post-test in both of these areas, some quite substantially. The mean SMK pre-test and post-test subscores were 4.2/14 (30.0%) and 7.2/14 (51.4%), a mean increase of 21.4. The mean PCK pre-test and post-test subscores were 11.3/31 (36.5%) and 16.5/31 (53.2%), a mean increase of 16.8. We concluded that PSTs made similar gains on the SMK and PCK portions of the test. An item-by-item analysis revealed that, on average, the PSTs considerably improved their performance on all items from pre- to post-test and that their mean change was roughly the same for SMK- and PCK-related items overall.

Discussion

The MODULE(S^2) project works to resolve the conflict between the preparation PSTs have historically received in their university content courses and the preparation they need to develop their MKT/SKT. The statistics teacher education curriculum materials made by the MODULE(S^2) project were designed to address this by attending to the development of PSTs' SKT, including SMK and PCK development simultaneously. This preliminary study has provided initial evidence that harmony in developing these aspects of SKT concurrently is possible and productive. Use of the MODULE(S^2) project's materials considerably enhanced PSTs' SKT, including both their SMK and PCK. Additionally, we believe that strong SKT supports teachers to effectively implement data-driven lessons focused on equity. The design of the MODULE(S^2) project's materials to improve PSTs' critical statistical literacy along with evidence from this study that PSTs' SKT improved through their use of the MODULE(S^2) project's materials is promising for the development of teachers who can teach equity-focused statistical lessons. Further research on a larger scale to study the effects of the MODULE(S^2) project's materials on SKT and critical statistical literacy is needed to move to more substantial findings. In addition, further research that explores how PSTs' SKT evolves through use of the MODULE(S^2) project's materials—including features of the materials' implementation that support or inhibit PSTs' SKT development—is recommended.

An additional contribution of this preliminary study is the development of a SKT test for secondary teachers, addressing a longstanding lack of teacher instruments in statistics education (Groth & Meletious-Mavrotheris, 2018). Future projects can conduct large-scale validation studies of the test to work towards establishing more validity evidence for its use in measuring teachers' SKT.

Acknowledgements

The work reported here was supported by the National Science Foundation (Awards 1726252 and 1726723). The content of this manuscript does not necessarily reflect the views of the National Science Foundation.

References

AERA, APA, & NCME. (2014). *Standards for Educational and Psychological Testing*. American Educational Research Association. https://www.testingstandards.net/open-access-files.html

Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for Preparing Teachers of Mathematics.* Retrieved from: amte.net/standards.

Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. (2020). *Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education (GAISE) report II*. American Statistical Association and National Council of Teachers of Mathematics.

Casey, S.A., Albert, J., & Ross, A. (2018). Developing knowledge for teaching graphing of bivariate categorical data, *Journal of Statistics Education, 26*(3), 197-213. https://doi.org/10.1080/10691898.2018.1540915

Casey, S., Ross, A., Maddox, S., & Wilson, M. (2019). *MODULE(S2): Statistics for Secondary Mathematics Teaching.* Mathematics of doing, understanding, learning, and educating for secondary schools. APLU. http://modules2.com
Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *The American Mathematical Monthly, 104*(9), 801–823. https://doi.org/10.2307/2975286
Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/the-standards/mathematics
Franklin, C.A., Bargagliotti, A.E., Case, C.A., Kader, G.D., Schaeffer, R.L., & Spangler, D. A. (2015). *Statistical Education of Teachers.* American Statistical Association. https://www.amstat.org/asa/files/pdfs/EDU-SET.pdf
Goulding, M., Hatch, G., & Rodd, M. (2003). Undergraduate mathematics experience: its significance in secondary mathematics teacher preparation. *Journal of Mathematics Teacher Education*, *6*(4), 361-393.
Groth, R. E. (2017). Developing statistical knowledge for teaching during design-based research. *Statistics Education Research Journal*, *16*(2), 376.
Groth, R. E. (2014). Using work samples from the National Assessment of Educational Progress (NAEP) to design tasks that assess statistical knowledge for teaching. *Journal of Statistics Education, 22*(3), Available: http://www.amstat.org/publications/jse/v22n3/groth.pdf
Groth, R. E. (2013). Characterizing key developmental understandings and pedagogically powerful ideas within a statistical knowledge for teaching framework. *Mathematical Thinking and Learning*, *15*(2), 121–145. https://doi.org/10.1080/10986065.2013.770718
Groth, R. E. (2007). Toward a conceptualization of statistical knowledge for teaching. *Journal for Research in Mathematics Education*, *38*(5), 427–437. https://doi.org/10.2307/30034960
Groth, R. E. & Meletious-Mavrotheris, M. (2018). Research on statistics teachers' cognitive and affective characteristics. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 327-355). Springer.
Hill, H. C., Ball, D. L., Blunk, M., Goffney, I. M., & Rowan, B. (2007). Validating the ecological assumption: The relationship of measure scores to classroom teaching and student learning. *Measurement: Interdisciplinary Research and Perspectives*, *5*(2–3), 107–118. https://doi.org/10.1080/15366360701487138
Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*, 430–511.
LOCUS (n.d.) *Levels of Conceptual Understanding in Statistics.* https://locus.statisticseducation.org/
Mohr-Schroeder, M., Ronau, R. N., Peters, S., Lee, C. W., & Bush, W. S. (2017). Predicting student achievement using measures of teachers' knowledge for teaching geometry. *Journal for Research in Mathematics Education*, *48*(5), 520–566.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15,* 4–14.
Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, *11*, 499–511. https://doi.org/10.1007/s10857-008-9089-5
Simon, M. A. (2006). Key developmental understandings in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, *8*(4), 359–371. https://doi.org/10.1207/s15327833mtl0804_1
Thwaites, A., Jared, L., & Rowland, T. (2011). Analysing secondary mathematics teaching with the knowledge quartet. *Research in Mathematics Education, 13*(2), 227–228. https://doi.org/10.1080/14794802.2011.585834
Weiland, T. (2017). Problematizing statistical literacy: An intersection of critical and statistical literacies. *Educational Studies in Mathematics, 96*(1), 33-47.

HOW TEACHERS INTERPRET AND RESPOND TO MISUNDERSTANDINGS: PATHWAYS TOWARDS INSTRUCTIONAL DECISIONS

John Ezaki
University of Southern California
jezaki@usc.edu

Michael Lawson
Kansas State University
mlawson1@ksu.edu

Yasemin Copur-Gencturk
University of Southern California
copurgen@usc.edu

Knowledge of student misunderstanding is a critical aspect of teacher knowledge, with implications for teacher practice and student achievement. However, there is no mechanism describing the progression from how teachers identify student misunderstandings to the types of strategies used when addressing misunderstandings. In this study, we analyze 342 cases of student misunderstandings, underlying rationales, and instructional strategies reported by 153 elementary mathematics teachers. Using fsQCA, we identified pathways showing the conditions that lead to teachers providing instructional responses that target the misunderstanding and/or build students' conceptual understanding. Overall, we found that if teachers provide mathematical rationales to explain why students struggle with a concept/procedure, then their instructional responses help students develop conceptual understanding.

Keywords: teacher knowledge

Many conceptualizations and studies of teacher knowledge place an emphasis on teacher knowledge of students, due to its broad implications for how teachers plan instruction, deliver content, and interact with students (e.g., Hill, Ball, & Schilling, 2008; Shulman, 1986; Carlsen, 1999). A subdomain of teachers' knowledge of students is their knowledge of student misunderstandings (KOSM), a theoretically relevant subdomain which can be traced back to Shulman's (1986) conceptualization of pedagogical content knowledge. Research on teachers' KOSM has shown evidence that there is a positive relationship between teachers' KOSM and student achievement (Sadler et al., 2013; Chen et al., 2020). However, while studies have linked teacher KOSM to student achievement, it is unclear *how* this knowledge plays out in teacher decision-making about instructional strategies. To capture *how* this knowledge plays out, we use a novel research method, *fuzzy set qualitative comparative analysis* (fsQCA; Ragin, 2000, 2008), to begin modeling pathways that link how teachers describe student misunderstandings to the types of instructional decisions they report using to address the misunderstandings.

Sadler et al. (2013), Chen et al. (2020), and Hill and Chin (2018) studied science and mathematics in-service teachers' ability to anticipate the most common distractor for students on multiple-choice items. Sadler et al. (2013) found that science teachers who could identify the misunderstanding through the multiple-choice distractors had larger classroom gains than those teachers who could only identify the correct answer. Chen et al (2020) replicated the previous findings but with biology content rather than physical science. Relatedly, Hill and Chin (2018) noted that although knowledge of student misunderstandings can be difficult to measure, they were able to identify a link between this aspect of mathematics teachers' knowledge and student outcomes. These studies indicate a link between teacher knowledge of misunderstandings, instruction, and student outcomes by way of how teachers identify and address student misunderstandings in conceptual and procedural ways. However, research has yet to identify the mechanism that describes the progression from how teachers are identifying student misunderstandings (procedural, conceptual, or a combination) to the types of strategies used when addressing the student misunderstanding. Thus, the present study aims to understand the association between how teachers think about student misunderstandings and their planned instructional strategies to address the misunderstandings. The research questions were:

What pathways describe how teachers' descriptions and rationales for a misunderstanding are associated with:

1. The type of instructional response (i.e., targeted to build students' conceptual or procedural understanding) they use to address student misunderstandings?
2. Whether the instructional response is specific or not specific to the misunderstanding teachers are aiming to address?
3. Whether the instructional response is or is not both specific and building students conceptual understanding when addressing student misunderstandings?

Conceptual Framework

We use the constructs of procedural and conceptual understanding (or knowledge) to describe the ways teachers identify student misunderstandings and the instructional strategies used to address the misunderstandings. Procedural understanding is commonly defined as an understanding of procedures and carrying them out flexibly and appropriately (Eisenhart et al., 1993; National Research Council, 2001; Rittle-Johnson et al., 2015). For instance, knowing the steps involved adding two fractions with unlike denominators is an indicator of procedural knowledge (i.e., find a common denominator, find equivalent fractions, add across the numerator, keep the denominator the same). Instructional strategies focusing on students' procedural understanding of adding fractions would highlight step-by-step overviews of how to carry out algorithms or ways to remember the steps (i.e., a song or mnemonic device).

Conceptual understanding is commonly defined as an understanding of concepts and the interconnectedness of topics that provide the structure for understanding procedures (Eisenhart et al., 1993; NRC, 2001; Rittle-Johnson et al., 2015). Returning to the fraction addition algorithm, a student with conceptual understanding of adding fractions would show an understanding of ideas such as part-whole relationships and equivalent fractions, as well as the meaning of addition. Instructional strategies supporting students' conceptual understanding of adding fractions would highlight strategies linking the ideas of part-whole relationships, equivalent fractions, and adding fractions.

Methods

This study used data from a sample of 153 elementary mathematics teachers located in Indiana and North Carolina. The teachers completed an online, open-ended survey with three items asking teachers to (1) identify common student misunderstandings related to fraction content, (2) provide a rationale for why students have a given misunderstanding, and (3) report the instructional steps they would take to address the misunderstandings. Since many teachers answered the three survey questions referring to more than one misunderstanding, the dataset was parsed out such that each row contained one misunderstanding and subsequent rational and reported instructional strategies. This means the 153 teachers provided 342 complete cases of a student misunderstanding and the subsequent rationale and instructional decision.

To investigate the links between how teachers describe student misunderstandings and the types of instructional decisions they report using to address the misunderstanding, this study employed fsQCA (Ragin, 2000, 2008). fsQCA views a collection of individual cases through a set-theoretic lens, where Boolean logic is used to look across the cases and identify solution pathways that represent the logical paths linking conditions and outcomes. Using fsQCA to find these solution pathways allows us to describe the ways in which how teachers identify and understand the reasons for a student's misunderstanding leads to the types of instructional strategies they use to address misunderstandings.

For the analysis, each case was coded for three conditions related to how teachers described a misunderstanding. First, whether the student misunderstanding was described as procedural or conceptual in nature (student misunderstanding type). Second, whether the source of the misunderstanding was attributed to students' understanding of procedures or students' underlying thinking about mathematical concepts. Third, whether there was an external factor provided as the source of the misunderstanding (e.g., issues concerning curriculum or home support).

We also coded each case for three outcomes related to the instructional strategies teachers reported using to address the misunderstanding. First, whether the instructional strategy aimed at building students' procedural or conceptual understanding Second, whether the instructional strategy was specific to the misunderstanding or generic. Third, whether the instructional strategy was specific to the misunderstanding and built students' conceptual understanding or not. For an overview of the conditions and outcomes, see Table 1. All coding was completed by two raters who initially coded a small number of cases (10-20) and checked their coding until interrater reliability was >90%. Then, a larger number of cases were coded independently and all disagreements were reconciled.

After coding, we used fsQCA software (Ragin & Davey, 2016) to carry out the quantitative analysis, referred to as a truth table analysis (Ragin, 2008). The truth table analysis identified the combinations of the presence and absence of conditions (pathways) that led to the presence or absence of a given outcome.

Results

We report the eight pathways leading to the four different types of instructional responses (Table 2). Table 2 indicates which combinations of factors were present in how teachers identified student misunderstandings (e.g., procedural versus conceptual) and the underlying rationales for these misunderstandings (e.g., based on students' mathematical or non-mathematical thinking) and the subsequent instructional response.

Due to the limited space, we specifically focus on one pathway that leads to teachers reporting instructional strategies aimed at building students' conceptual understanding. Teachers in Pathway 1 describe the student misunderstandings conceptually *and* provide a mathematical rationale for why the students have the misunderstanding which led to a conceptual instructional response. An example of this pathway is below. The teacher described how students have a misunderstanding about the result of multiplying and dividing fractions because they do not make connections between the meaning of a fraction and the operation.

When you multiply, the number gets bigger and when you divide it gets smaller [referring to whole number operations]. When students move into operations with fractions, their understanding of what a fraction means is lost, and they don't understand why their product is smaller than the factors or the quotient is bigger than the dividend/divisor.

The teacher identified the misunderstanding as a conceptual issue with students' understanding of fractions as different from whole numbers. This teacher later explains the rationale for this confusion could be that students, "miss the understanding that a fraction means that you divide a whole into parts… a number less than one." To address this misunderstanding this teacher:

I have used explicit examples of multiplying by 0 (where the product is 0 - less than the factors), by 1 (where the product is the same as a factor), and by 2 (where the product is greater than the factors) to show students that there is a pattern to the value of the product compared to the factors. When they see that x0 is less than the factors and x1 is equal to a factor, it becomes easier for them to accept that multiplying by a proper fraction will result in a product less than the factors. I continue with multiplying by 1/2 because it's an easy fraction and students typically know that it's the same as dividing by 2, which they know will result in a smaller product. Through targeted examples and a slow approach to looking for patterns, students "get it" and are able to correctly predict whether the product will be smaller or larger than the factors.

By identifying a misunderstanding through conceptual issues, along with a mathematical rationale for the misunderstanding, this teacher is able to provide an instructional approach that uses targeted examples and pattern-making to help students overcome the misunderstanding conceptually.

Overall, across all pathways, it is interesting to note the role of teachers' rationale for student misunderstandings. When teachers are describing strategies that support students' conceptual understanding (Pathway 1, 3), it is important that teachers base the rationale for the misunderstanding on students' mathematical thinking. In contrast, when teachers are not providing specific or conceptually grounded instructional strategies, then they are not providing a mathematical rationale for the misunderstanding (Pathway 4-7).

How teachers interpret the student misunderstandings also seems relevant to their instructional decision. Two of the three pathways leading to a conceptual instructional response include the teacher interpreting the misunderstanding conceptually, whereas two pathways leading to a procedural response include teachers interpreting the misunderstanding procedurally.

Significance of the Study

This study adds depth to the field's understanding of teacher knowledge of student misunderstandings by capturing an initial snapshot of how this knowledge is operationalized in relation to instructional strategies. While student misconceptions are well studied in the literature on teaching and learning, there remains a need to identify how teachers progress from *identifying* misunderstandings to *addressing* them. We found multiple descriptive pathways that highlight the complex ways in which teachers make sense of student misunderstandings in order to address them. Each path is important, but the repeated occurrence of providing a mathematical rationale about the misunderstanding appears to be an important condition to better understand why teachers provide the instructional strategies they use. It is plausible that teachers who have considered what causes common misunderstanding are more able to craft an instructional response that target the conceptual issues rather than simply addressing the procedural components. This suggests that time spent analyzing student responses and discerning why the mistake was made could be a valuable addition to teacher preparation and planning.

Of similar note, is how teachers conceptualize and explain student errors. If teachers explained the misunderstanding procedurally, their response to it was procedural. This is not unexpected, but reestablishes the importance of teachers having a deep conceptual understanding of the content and of their students (Shulman, 1986). Without that conceptual understanding, it is unreasonable to expect a conceptual instructional response, which is what we see in these pathways.

Overall, we find value in the fsQCA methodology to assist in answering questions about the mechanisms describing how and why teachers' make particular instructional decisions. How teachers interpret a misunderstanding (either procedurally or conceptually) and the rationale for the misunderstanding (mathematical or non-mathematical) seem to be key pieces in understanding how teachers choose to respond to student issues.

Table 1: Description of Conditions and Outcomes

Conditions and Outcomes	Description
Conditions	
Students' misunderstanding	Captures the extent to which teachers described student misunderstandings as solely an issue of procedural or conceptual understanding.
Mathematical rationale	Captures the extent to which teachers described the cause of student misunderstandings through students' understanding of procedures or students' underlying thinking about mathematical concepts.
Non-mathematical rationale	Captures instances where teacher provided non-mathematical reasons for student misunderstandings (e.g., issues around curriculum or home-support).
Outcomes	
Instructional strategy aimed at building students' conceptual understanding	Captures the extent to which the strategies teachers reported were supporting students' conceptual understanding of the mathematical concepts underlying the misunderstanding.
Instructional Strategy Specific to Students' Misunderstanding	Captures whether or not the strategy teachers described using specifically addressed the student misunderstanding.
Instructional strategy that is specific to students' misunderstanding and builds students' conceptual understanding	Captures whether or not the strategy teachers described using both specifically addressed the student misunderstanding and supported students in making sense of math concepts.

Table 2: Pathways leading to instructional responses to student misunderstandings

Solution Pathways by Outcome	Conditions: Student Misunderstanding	Conditions: Mathematical Source of Misunderstanding	Conditions: Non-mathematical Source of Misunderstanding	Pathway Consistency and Coverage: Consistency	Pathway Consistency and Coverage: Raw Coverage	Pathway Consistency and Coverage: Unique Coverage
Occurrence of instructional strategies aimed at building students' conceptual understanding						
Pathway 1	•	•		0.773	0.411	0.355
Pathway 2	•		•	0.765	0.149	0.092
Pathway 3		•	•	0.824	0.075	0.019
					Solution Consistency: 0.755 Solution Coverage: 0.523	
Nonoccurrence of instructional strategies aimed at building students' conceptual understanding						
Pathway 4		⊗	⊗	0.703	0.495	0.495
					Solution Consistency: 0.703 Solution Coverage: 0.495	
Nonoccurrence of instructional strategies that are specific to the student misunderstanding						
Pathway 5		⊗	⊗	0.853	0.437	0.097
Pathway 6	⊗	⊗		0.814	0.490	0.151
					Solution Consistency: 0.829 Solution Coverage: 0.588	
Nonoccurrence of instructional strategies that are specific to the student misunderstanding and build students' conceptual understanding						
Pathway 7		⊗		0.923	0.633	0.146
Pathway 8	⊗			0.897	0.650	0.163
					Solution Consistency: 0.895 Solution Coverage: 0.796	

Note: Presence of a condition is indicated with a closed circle (•); absence of a condition is indicated with an open circle with an x (⊗); empty cells indicate that a condition is not part of the recipe and can be present/absent and not impact the outcome.

References

Ashlock, R. B. (2006). Error patterns in computation: *Error Patterns in Computation: Using Error Patterns to improve instruction* (9th ed.). Pearson Prentice Hall.

Ausubel, D., Novak, J., & Hanesian, H. (1978). *Educational psychology: A cognitive view* (2nd ed.). Holt, Rinehart, and Winston.

Brown, G., & Quinn, R. J. (2006). Algebra students' difficulty with fractions: An error analysis. *Australian Mathematics Teacher, The, 62*(4), 28.

Carlsen, W. (1999). Domains of teacher knowledge. In Gess-Newsome J., Lederman N.G. (Eds.) *Examining pedagogical content knowledge* (6th ed., pp. 133-144): Springer. https://doi.org/10.1007/0-306-47217-1_5

Hill, H. C., & Chin, M. (2018). Connections between teachers' knowledge of students, instruction, and achievement outcomes. *American educational research journal, 55*(5), 1076-1112.

Kilpatrick, J., Swafford, J., Findell, B., & council, N. r. (2001). *Adding it up: Helping children learn mathematics* (Vol. 2101): Citeseer.

Loughran, J., Berry, A., Mulhall, P. J., & Woolnough, J. (2006). Understanding and valuing the development of pedagogical content knowledge in science teacher education.

Sadler, P. M., Sonnert, G., Coyle, H. P., Cook-Smith, N., & Miller, J. L. (2013). The influence of teachers' knowledge on student learning in middle school physical science classrooms. *American educational research journal, 50*(5), 1020-1049.

Ragin, C. and Davey, S. (2016). *Fuzzy-Set/Qualitative Comparative Analysis 3.0*. Irvine, California: Department of Sociology, University of California.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., . . . Wray, J. (2010). Developing Effective Fractions Instruction for Kindergarten through 8th Grade. IES Practice Guide. NCEE 2010-4039. *What Works Clearinghouse*.

Silver, E. A. (1986). Using conceptual and procedural knowledge: A focus on relationships.

Son, J.-W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar

rectangles. *Educational Studies in Mathematics, 84*(1), 49-70.
Spillane, J. P., & Miele, D. B. (2007). chapter 3Evidence in Practice: A Framing of the Terrain. *Yearbook of the National Society for the Study of Education, 106*(1), 46. doi:10.1111/j.1744-7984.2007.00097.x
Spillane, J. P., Reiser, B. J., & Reimer, T. (2002). Policy implementation and cognition: Reframing and refocusing implementation research. *Review of educational research, 72*(3), 387-431.
Strauss, S. (2001). Folk psychology, folk pedagogy, and their relations to subject-matter knowledge. *Understanding and teaching the intuitive mind: Student and teacher learning*, 217-242.
Weick, K. E. (1995). Sensemaking in organizations.

DEVELOPMENT OF A MATHEMATICS DISCIPLINE-SPECIFIC LANGUAGE SCALE

Alanna Gibbons
Teachers College, Columbia University
ag3729@tc.columbia.edu

Irina Lyublinskaya
Teachers College, Columbia University
il2369@tc.columbia.edu

Existing studies have defined and assessed disciplinary literacy, mathematical literacy, and general academic language. However, there is a need to define and assess mathematics discipline-specific language (MDL), particularly for elementary school teachers. Therefore, the purpose of this study was to develop a research instrument to assess the MDL of elementary school teachers. The final instrument developed through iterative analysis included 20 items on a 4-point Likert-like scale distributed between three distinct MDL categories: technical, symbolic, and visual. Instrument validity was confirmed using Confirmatory Factor Analysis with the set of 211 video recordings and corresponding lesson plans of mathematics lessons taught by pre-service elementary school teachers enrolled in a graduate special education program.

Keywords: Mathematical Knowledge for Teaching, Preservice Teacher Education, Teacher Knowledge

Purpose of the Study

Mathematical literacy is essential for solving problems encountered in today's rapidly evolving world. It is defined as "an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen" (OECD, 2009, p. 14). Mathematically literate students should know how to do mathematics and how to speak the language of mathematics. This is not learned by simply memorizing definitions; but rather, by using the language of mathematics in their learning experiences (Hill, et al, 2008). The Common Core State Standards for Mathematical Practice emphasize the importance of developing mathematical literacy and discipline-specific mathematics language (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Discipline-specific language is defined as "the decontextualized oral and written language used within a specific discipline or profession with specialized vocabulary, syntax, and discourse patterns" (Zhang, 2014, p. 40). Mathematics discipline-specific language (MDL) is different from everyday language (Machaba, 2017). MDL is used to encode information and it allows members of the discipline to communicate in discipline-specific ways (Bernstein, 1996). Examples of MDL include vocabularies, symbols, and notations. A lack of appropriate MDL can hinder student learning and become a source of misconceptions carried over to later mathematics education (Di Domenico, 2014; Köse, 2008). It is vital that attention be given to MDL beginning at the elementary level (Siffrinn & Lew, 2018) since high-quality early mathematics instruction serves as a sound foundation for later learning in mathematics (Darling-Hammond & Bransford, 2005). In order to develop students' MDL, elementary school teachers must be proficient in their own MDL.

Studies that assess teachers' knowledge of MDL have been mostly conducted at the secondary level (Colwell & Gregory, 2016; Di Domenico, 2014; Spires et al., 2018) and in other content areas (Cisco, 2016; Feez & Quinn, 2017; Ruzycki, 2015), but it remains unclear what MDL looks like in the lower grades. Therefore, the purpose of this study was to develop and validate an instrument, titled the Mathematics Discipline-specific Language Scale (MDLS), for assessing the MDL of elementary school teachers. This study was guided by the following research question: What are constructs of MDL for elementary school teachers?

This study addresses the conference themes by resolving the dissonance between what is expected of elementary school teachers' MDL and the lack of the assessment instrument through the process of developing and refining the MDLS.

Literature Review and Theoretical Frameworks

MDL is a major component of disciplinary literacy (McConachie & Petrosky, 2009; Fang, 2012). Analysis of literature on disciplinary literacy theory provided background information on what is known about MDL of elementary school teachers. Disciplinary literacy is defined as "the use of reading, reasoning, investigating, speaking, and writing required to learn and form complex content knowledge appropriate to a particular discipline" (McConachie & Petrosky, 2009, p. 70). Shanahan and Shanahan (2014) proposed that disciplinary literacy should be introduced as early as elementary school. Therefore, elementary school teachers must develop MDL to make their instruction more accessible (Siffrinn & Lew, 2018).

McConachie and Petrosky's (2009) disciplinary literacy framework defined criteria necessary for students to develop discipline-specific literacy in the core subject areas. This framework suggests the following components of literacy: reading, reasoning, investigating, speaking, and writing. Gee (2012) suggested that language incorporates behaving, interacting, thinking, reading, speaking, and writing. Therefore, McConachie and Petrosky's (2009) speaking and writing components of disciplinary literacy can be categorized as language. In this study, McConachie and Petrosky's (2009) framework was adapted to focus solely on the language component of disciplinary literacy for the discipline of mathematics. The discipline of mathematics has a language of its own that is functional for constructing knowledge and reasoning in the subject. Further, Fang (2012) suggested that different language patterns of mathematics can be categorized as technical, symbolic, and visual. Fang (2012) defines technical language as discipline-specific grammatical features, structure, and vocabulary. Symbolic language is represented by mathematical symbols that are used to describe relationships between mathematical objects. Based on topics covered in elementary school mathematics curriculum, the following symbols are learned in grades K-5: basic operations signs, the plus sign ($+$), the minus sign ($-$), the multiplication signs ($\times$, $\cdot$, or $*$), the division signs ($\div$ or $/$), the relation signs ($=, >, <, \geq$, and $\leq$), the fraction notation ($\frac{\square}{\square}$ or $\square/\square$), the place value signs (decimal point . and coma ,), units of measurement signs (feet $'$ and inches $''$), grouping symbols (parentheses, brackets, braces), and money signs (\$ and ¢). Visual language at this level is represented by number lines, number paths, array and area models, strip diagrams, schematic diagrams, drawings, tables, and graphs. These three categories of MDL were used to classify the patterns of MDL in this study.

Theoretical frameworks need to be tested in practice, and in research that is accomplished through development of research instruments that are consistent with the theory. There are various research instruments that assess MDL of teachers (Stanford: Center for Assessment, Learning, & Equity, 2016; Hill, 2010). However, most of these instruments focus on a single aspect of the language. For example, the Teacher Performance Assessment (edTPA) evaluates pre-service teachers' precision in language, where MDL precision is defined as being accurate with definitions and symbols in labeling, measurement, and numerical answers (Stanford: Center for Assessment, Learning, & Equity, 2016). The Mathematical Quality of Instruction (MQI) Coding Tool assesses pre- and in-service teachers' explicitness of mathematical terminology and technical language fluency as part of overall instruction quality rather than focusing specifically on quality of MDL (Hill, 2010). In the MQI, fluency is defined as the density of MDL during periods of teacher talk. The MQI also defines explicitness as teachers' accurate use of technical terms. Further, most research instruments focus on the teacher's ability to support language development of students, rather than assessing the teacher's own MDL. In order to assess the quality of teachers' MDL, this study adapted assessment criteria from edTPA and MQI, e.g., precision, fluency, and explicitness for each language category specifically for the MDL of elementary school teachers.

This study's theoretical framework for the development of the MDLS aligned the three assessment criteria of MDL quality for each of the three language categories (Figure 1).

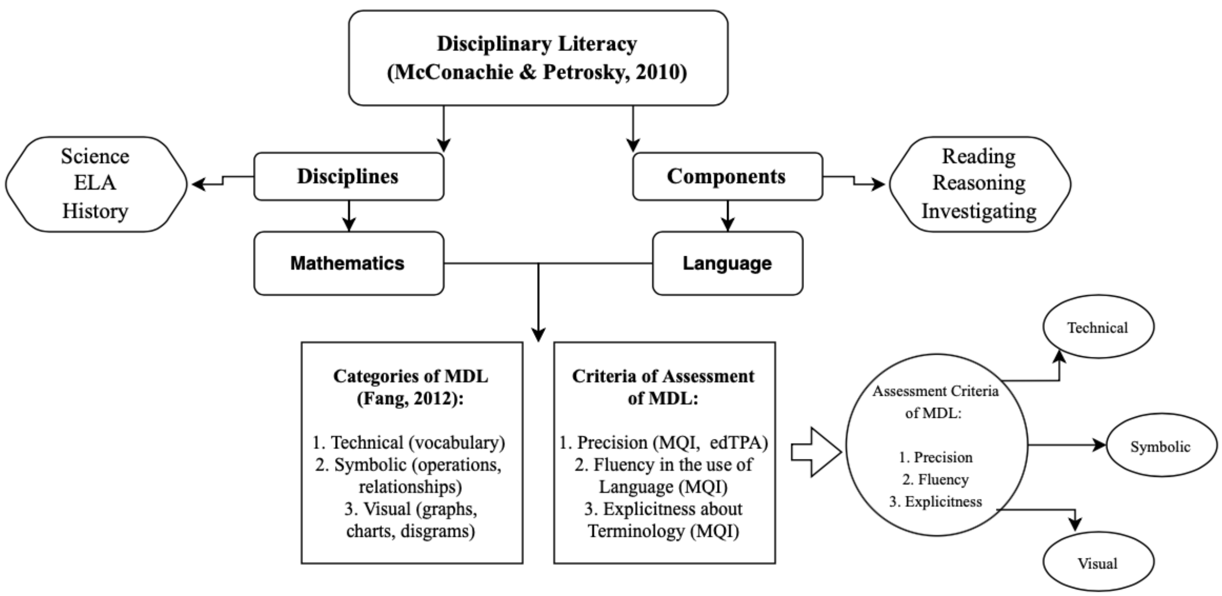


Figure 1: Framework for Development of the MDLS

Methods

Development of MDLS items was guided by the criteria defined by Hathcoat, Sanders, and Gregg (2016). First, category-specific items had to describe directly observed characteristics of MDL in video recordings and corresponding lesson plans. Second, the following practices were used: 1) generating twice as many items as needed, 2) making items simple and specific, and 3) ensuring that items are unidimensional and easy to read.

Explicit phrases that were found in the literature in relation to MDL characteristics were collected and analyzed to generate the preliminary statements for the MDLS items. Further, these statements were revised into performance-based statements. Then, each item was classified into one of the three categories of MDL. A 4-point Likert-like scale (never, rarely, often, always) was used in the MDLS to measure frequency of occurrence for each MDLS item. Iterative process of item revisions at this stage aimed to ensure that each item was observable and independent of others.

The initial version of the MDLS was sent to two mathematics literacy experts to evaluate the content validity. Based on their feedback, wording of several items was revised for clarity. The reliability and construct validity of the MDLS were tested on video recordings and corresponding lesson plans developed by pre-service elementary teachers enrolled in a special education graduate program and collected over the course of six years in grades K-5 mathematics classrooms in urban public schools in Northeast USA.
In order to test inter-rater reliability, two raters were trained to use the MDLS. An initial round of scoring included independent assessment of a set of thirty video recordings and corresponding lesson plans. Pearson correlation analysis was used on this set of scores. Debriefing with the raters was conducted and based on debriefing, additional revisions to wording of items and the structure of Likert-like scale were made. This version of the MDLS was then independently scored by the same two raters on a new set of thirty video recordings and corresponding lesson plans. Inter-rater reliability was tested again using Pearson correlations on this set of scores. Construct validity was tested using a Confirmatory Factor Analysis (CFA). The CFA was conducted in SPSS 28 using varimax rotation with three factors according to the number of categories of MDL defined by the theoretical framework. The goodness-of-fit of this model was tested using SPSS AMOS 28. This procedure led to removal of several items. Internal consistency of the final MDLS was tested using the split-half reliability method in SPSS 28.

Results

Initial MDLS consisted of 27 items, with nine items in each category (Table 1). Based on the feedback from the experts, the wording of three items in technical language (TL4, TL6, and TL8), two items in symbolic language (SL14 and SL17), and all items in visual language was revised for clarity (see Revision 1 in Table 1).

Table 1: Iterative Analysis of MDLS

Label	Initial Version	Revision 1	Revision 2	Final Version
TL1	The teacher states/writes numerical answers that are relevant to the problem's context.	The teacher states/writes numerical answers that are relevant to the problem's context.	The teacher states/writes numerical answers that are relevant to the problem's context.	
TL2	The teacher uses discipline-specific words to provide clear definitions.	The teacher uses discipline-specific words to provide clear definitions.	The teacher uses discipline-specific words to provide clear definitions.	
TL3	The teacher correctly states/specifies units of measurements (when applicable).	The teacher correctly states/specifies units of measurements (when applicable).	The teacher states/specifies units of measurements, or puts meaning to the numerical value.	The teacher states/specifies units of measurements, or puts meaning to the numerical value.
TL4	The teacher correctly uses at least two different terms to describe the same mathematical idea (when applicable).	The teacher correctly uses at least two different terms to describe the same mathematical concept (when applicable).	The teacher makes connections between mathematical concepts.	The teacher makes connections between mathematical concepts.

TL5	The teacher discusses the meaning of used discipline-specific words.	The teacher discusses the meaning of used discipline-specific words.	The teacher discusses the meaning of used discipline-specific words.	The teacher discusses the meaning of used discipline-specific words.
TL6	The teacher compares and contrasts everyday language with technical language.	The teacher compares and contrasts everyday language with mathematical concepts.	The teacher compares and contrasts everyday language with at least two mathematical concepts.	The teacher compares and contrasts everyday language with at least two mathematical concepts.
TL7	The teacher provides correct explanations.	The teacher provides correct explanations.	The teacher provides correct explanations (when applicable).	The teacher provides correct explanations (when applicable).
TL8	The teacher correctly uses terms to describe mathematical ideas.	The teacher correctly uses mathematical terms to describe concepts.	The teacher uses mathematical terms to describe concepts.	The teacher uses mathematical terms to describe concepts.
TL9	The teacher uses mathematical language as a vehicle for conveying content.	The teacher uses mathematical language as a vehicle for conveying content.	The teacher uses mathematical language as a vehicle for conveying content.	The teacher uses mathematical language as a vehicle for conveying content.
SL10	The teacher uses symbols correctly.	The teacher uses symbols correctly.	The teacher uses symbols correctly (when applicable).	The teacher uses symbols correctly (when applicable).
SL11	The teacher correctly uses the equals sign.	The teacher correctly uses the equals sign.	The teacher correctly uses the equals sign.	
SL12	The teacher uses mathematical notation to provide clear definitions.	The teacher uses mathematical notation to provide clear definitions.	The teacher uses mathematical notation to provide clear definitions.	The teacher uses mathematical notation to provide clear definitions.
SL13	The teacher discusses the meaning of the symbols used.	The teacher discusses the meaning of the symbols used.	The teacher discusses the meaning of mathematical symbols.	The teacher discusses the meaning of mathematical symbols.
SL14	The teacher makes connections between symbols and mathematical ideas.	The teacher makes connections between symbols and mathematical concepts.	The teacher makes connections between symbols and mathematical concepts.	The teacher makes connections between symbols and mathematical concepts.
SL15	The teacher supports the meaning of the equal sign as relational rather than operational.	The teacher supports the meaning of the equal sign as relational rather than operational.	The teacher supports the meaning of the equal sign as relational rather than operational (when applicable).	The teacher supports the meaning of the equal sign as relational rather than operational (when applicable).

SL16	The teacher uses mathematical symbols when providing explanations.	The teacher uses mathematical symbols when providing explanations.	The teacher uses mathematical symbols when providing explanations.	The teacher correctly uses mathematical symbols when providing explanations.
SL17	The teacher correctly uses symbols to describe mathematical ideas.	The teacher correctly uses symbols to describe mathematical concepts.	The teacher correctly uses symbols to describe mathematical concepts (when applicable).	
SL18	The teacher uses simple, concise language to connect symbols to their meaning.	The teacher uses simple, concise language to connect symbols to their meaning.	The teacher uses simple, concise language to connect symbols to their meaning.	The teacher uses simple, concise language to connect symbols to their meaning.
VL19	The teacher selects visual representations that are appropriate for the structure of the problem.	The teacher selects mathematical models that are appropriate for the structure of the problem.	The teacher selects mathematical models that are appropriate for the structure of the problem.	
VL20	The teacher correctly labels elements of a visual representation.	The teacher correctly labels elements of a mathematical model.	The teacher labels elements of a mathematical model.	
VL21	The teacher correctly converts visually represented information into mathematical notation.	The teacher correctly converts mathematical models, or visually represented information, into mathematical notation.	The teacher converts mathematical models, or visually represented information, into mathematical notation.	The teacher converts mathematical models, or visually represented information, into mathematical notation.
VL22	The teacher states what the visuals represent.	The teacher states what the mathematical models represent.	The teacher states what the mathematical models represent.	The teacher states what the mathematical models represent.
VL23	The teacher links visual representations with quantities in the problem.	The teacher links mathematical models with quantities in the problem.	The teacher links mathematical models with quantities in the problem.	The teacher links mathematical models with quantities in the problem.
VL24	The teacher includes only necessary details in visual representations.	The teacher includes only necessary details in mathematical models.	The teacher includes only necessary information in mathematical models.	
VL25	The teacher correctly uses visual representations to	The teacher correctly uses mathematical models to describe concepts.	The teacher uses mathematical models to describe concepts.	The teacher uses mathematical models to describe concepts.

	describe mathematical ideas.			
VL26	The teacher uses discipline-specific words when describing visual representations.	The teacher uses discipline-specific words when describing mathematical models.	The teacher uses discipline-specific words when describing mathematical models.	The teacher uses discipline-specific words when describing mathematical models.
VL27	The teacher uses different visual representations for different types of problems.	The teacher uses different mathematical models for different types of problems.	The teacher uses different mathematical models for the same problem.	The teacher uses different mathematical models for the same problem.

The Pearson *r*-values for Revision 1 of MDLS ranged from .63 to .86. All items with *r*-values below .7 were discussed with the raters during debriefing and further revisions were made (see Revision 2 in Table 2). The raters also suggested changing the original 4-point Likert-like scale that was based on frequency of occurrence to a 4-point scale based on quality of MDL (not evident, incorrect, somewhat correct, correct). For the scoring with Revision 2 of MDLS, the Pearson *r*-values ranged from .71 to .89 confirming high inter-rater reliability (Asuero, Sayago, & Gonzalez, 2006). The remaining video recordings and corresponding lesson plans were divided between the raters to complete the scoring using this version of MDLS.

In order to confirm the construct validity of MDLS, CFA on the set of 211 video recordings and corresponding lesson plans was completed in SPSS 28 using principal component analysis extraction method with varimax rotation. Loadings less than .42 were suppressed which is consistent with assumption about significant loadings for this sample size (Guadagnoli & Velicer, 1988). Five items (TL1, TL2, VL19, VL20, and VL24) that did not load to their specific categories were removed.

The goodness-of-fit of this three-factor model was examined using maximum likelihood estimation performed in SPSS AMOS 28. The analysis resulted in a significant chi-square, $\chi^2(206) = 658.0$, $p < .001$. The comparative fit index (CFI = .834) and the Tucker-Lewis index (TLI = .813) were both below the accepted values. In addition, the standardized root mean square residual (SRMR = .082) and the root mean square error for approximation (RMSEA = .102) were higher than acceptable. In order to improve the fit of the model, two items (SL11 and SL17) that had very high modification indices were removed. The final model (Figure 2) resulted in a significant chi-square, 2(167) =368.7, $p < .001$, although this can be sensitive to the sample size.

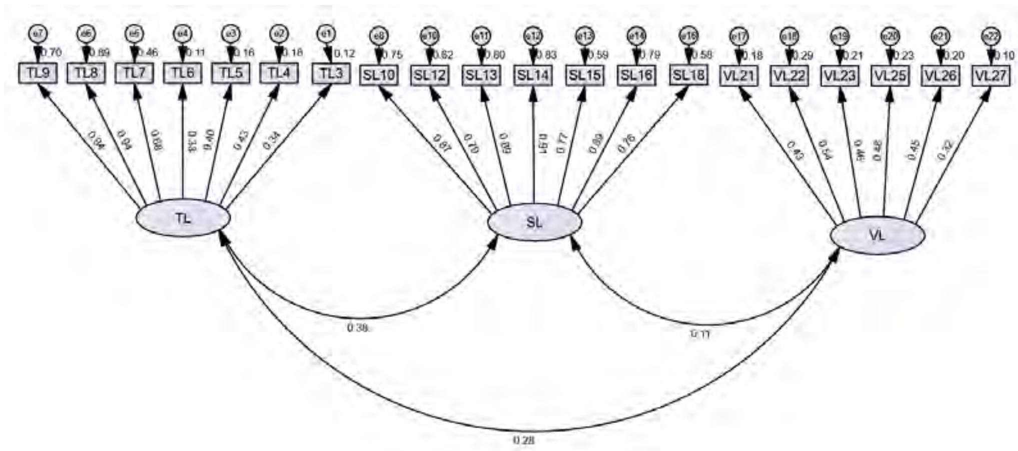


Figure 2: AMOS Path Diagram for a Final 3-factor Model of MDLS

The ratio $\chi^2 / df = 2.208 < 3.0$ indicates a good fit (Kline, 2005). The comparative fit index (CFI = .90) shows relatively good fit and the Tucker-Lewis Index (TLI = .886) is just a little below the acceptable value of .9, which suggests a reasonable fit (Bentler, 1990). Moreover, both SRMR = .047 < .08, and RMSEA = .076 < .08 are acceptable. Based on these indices, the three-factor model has a reasonable fit and CFA procedure confirmed the three factors matching the three theoretical categories of MDL.

The final version of the MDLS consists of 20 items with seven items in technical language and symbolic language categories each, and six items in visual language category (Table 1). Cronbach's alpha was used to test internal consistency of the final scale. The value of alpha for symbolic language subscale (α = .954) is excellent, for the technical language subscale (α = .795) is good, and just a little below acceptable for visual language subscale (α = .594) with very good overall value of 0.897 indicating acceptable internal consistency of the MDLS.

Conclusion

This study developed and validated a quantitative instrument for external assessment of the MDL of elementary school teachers. To the best of our knowledge, this is the first instrument that defines constructs of MDL for elementary school teachers. Another significance of this study is that through validation of MDLS it confirmed three categories of MDL suggested in the theoretical framework by Fang (2012). Thus, this study contributes to the field by bridging the gap between the theoretical definitions of MDL and the practical measurements of MDL in the field. The process of iterative development and analysis of the MDLS items led to the higher clarity of MDL categories and clear distinction between them.

The results of this study have practical implications for teacher education and professional development programs. The MDLS could be used to assess gaps and deficiencies in the MDL of preservice and in-service elementary school teachers and therefore to support their MDL development through focused programs. Further studies are needed to develop better understanding of the visual language category and to analyze how the teachers' MDL influences student learning of mathematics. Although this study made progress in operationalizing different categories of MDL, more work is also needed to operationalize qualities and correctness of MDL. Therefore, future studies will 1) identity levels of the MDLS using the criteria of precision, fluency, and explicitness, and 2) connect these levels to the three categories of MDL.

References

Asuero, A. G., Sayago, A., & Gonzalez, A. G. (2006). The correlation coefficient: An overview. *Critical reviews in analytical chemistry, 36*(1), 41-59.

Bentler, P. M. (1990) Comparative fit indexes in structural models. *Psychological Bulletin, 107*(2), 238–246.

Bernstein, B. (1996). *Pedagogy, symbolic control and identity: Theory, research, critique.* London, UK: Routledge & Kegan Paul.

Cisco, J. (2016). A case study of university honors students in humanities through a disciplinary literacy lens. *Literacy Research and Instruction, 55*(1), 1–23.

Colwell, J. & Gregory, K. (2016). Exploring how secondary pre-service teachers' use online social bookmarking to envision literacy in the disciplines. *Reading Horizons, 55*(3), 62-97.

Darling-Hammond, L. & Bransford, J. (2005). *Preparing teachers for a changing world: What teachers should learn and be able to do*. San Francisco, CA: Jossey-Bass.

Di Domenico, P. (2014). *High school teachers' disciplinary literacy knowledge: A mixed-method study* (Doctoral dissertation) Available from ProQuest Dissertations and Theses database. (UMI No. 3681928).

Fang, Z. (2012). Language correlates of disciplinary literacy. *Topics in language disorders, 32*(1), 19-34.

Feez, S. & Quinn, F. (2017). Teaching the distinctive language of science: An integrated and scaffolded approach for pre-service teachers. *Teaching and Teacher Education, 65*, 192 204.

Gee, J. (2012). *Social linguistics and literacies: Ideology in discourses.* Abingdon, Oxon: Routledge.

Guadagnoli, E. & Velicer, W. (1988). Relation of sample size to the stability of component patterns. *Psychological Bulletin, 103*, 265-275.

Hathcoat, J., Sanders, C., & Gregg, N. (2016). Selecting and designing instruments: Item development, reliability, and validity. Retrieved from: https://www.jmu.edu/assessment/_files/Hathcoat_et_2016_Selecting%20and%20Designing%20Instruments.pdf

Hayes, R., Kyer, B., & Weber, E. (2015). *The case study cookbook.* Worcester, MA: Worcester Polytechnic Institute.

Hill, H., (2010). *Mathematical Quality of Instruction (MQI). Coding Tool.* Lansing MI: University of Michigan.

Hill, H., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and instruction*, *26*(4), 430-511.

Kline, R.B. (2005) *Principles and practices of structural equation modeling* (2nd Ed.) New York, NY: The Guilford Press.

Köse, S. (2008). Diagnosing student misconceptions: Using drawings as a research method. *World Applied Sciences Journal, 3*(2), 283-293.

Machaba, F. (2017). Pedagogical demands in mathematics and mathematical literacy: A case of mathematics and mathematical literacy teachers and facilitators. *Eurasia Journal of Mathematics, Science and Technology Education, 14*(1), 95-108.

McConachie, S. & Petrosky, A. (2009). *Content matters: A disciplinary literacy approach to improving student learning.* San Francisco, CA: John Wiley & Sons.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards.* Washington, DC: Authors.

OECD., (2009). *PISA 2009 assessment framework. Key competencies in reading, mathematics, and science.* Paris: OECD.

Ruzycki, N. (2015), *Writing, Speaking, and Communicating – Building Disciplinary Literacy in Materials Science Undergraduate Students.* Paper presented at 2015 ASEE Annual Conference & Exposition, Seattle, Washington. 10.18260/p.25115.

Shanahan, C. & Shanahan, T. (2014). Does disciplinary literacy have a place in elementary school? *The Reading Teacher, 67*(8), 636-639.

Siffrinn, N. & Lew, S. (2018). Building disciplinary language and literacy in elementary teacher training. *The Reading Teacher, 72*(3), 325-341.

Spires, H., Kerkhoff, S., Graham, A., Thompson, I., & Lee, J. (2018). Operationalizing and validating disciplinary literacy in secondary education. *Reading and writing, 31*(6), 1401 1434.

Stanford Center for Assessment, Learning, and Equity. (2016). *Understanding rubric progressions*. Stanford, CA: Author.

STRUCTURAL VALIDITY OF 2021 KNOWLEDGE FOR TEACHING EARLY ELEMENTARY MATHEMATICS (K-TEEM): A PSYCHOMETRIC STUDY

Ahmet Guven	Robert C. Schoen	Gizem Solmaz
Florida State University	Florida State University	Florida State University
aguven@fsu.edu	rschoen@lsi.fsu.edu	gsolmaz@fsu.edu

In this paper, we describe the method and results of our investigation of structural validity of the 2021 Knowledge for Teaching Early Elementary Mathematics test, which measures teachers' mathematical knowledge for teaching at the early elementary level. The 2021 Knowledge for Teaching Early Elementary Mathematics test was used in a long-term, randomized controlled trial of the effect of a Cognitively Guided Instruction professional-development program on teachers, teaching, and students. Our sample includes data collected in spring 2021 from 651 grades K–2 elementary educators in Florida. Exploratory factor analysis and analyses based on classical test theory and item response theory were conducted to examine the psychometric properties of the test. Results of factor analysis suggest a one-factor solution. The marginal reliability for response pattern scores was 0.84, and the test difficulty appears to align well with the levels of mathematical knowledge for teaching of the target population.

Keywords: Assessment, Mathematical Knowledge for Teaching, Measurement, Teacher Knowledge

Introduction

The quality and usefulness of quantitative research rests on high-quality measurement practices (APA, AERA, & NCME, 2014). Hill and Shih (2009) presented a compelling argument for improvements in quantitative research methods in mathematics education. In this paper, we present a method for examining the structural validity (Flake et al., 2017) of the 2021 Knowledge for Teaching Early Elementary Mathematics (K-TEEM), a web-based assessment of mathematical knowledge for teaching (MKT) at the early elementary level (Ball et al., 2008; Schoen et al., 2017; 2019; 2021; 2022a). The intended use of the K-TEEM is to estimate impact of three consecutive years of a professional development program based on Cognitively Guided Instruction (CGI) and to examine the role of MKT as a mediator of the effect of the CGI program on teaching and students. Through this paper, we will describe the process and results of a psychometric analysis of the data from the 2021 Knowledge for Teaching Early Elementary Mathematics (K-TEEM) test.

Methodology

The 2021 K-TEEM test serves as a measure of teacher knowledge in a randomized controlled trial of the CGI program. A previous version of the K-TEEM test, which included 31 items consisting of a combination of open-ended and multiple-choice questions, was used in 2019 and 2020. Based on some concern about potential ceiling effects in 2020, the 2021 version of the K-TEEM included 34 items after adding three items in attempt to improve measurement of teacher ability for those with high levels of MKT. Teachers (n = 651) in the K–2 track of the CGI program (Schoen et al., 2022) completed the web-based assessment (Qualtrics, 2020) in spring 2021. Our sample included teachers, math coaches, and other instructional support personnel, such as interventionists. K-TEEM contains items designed to measure early elementary teachers' knowledge in five subcategories: common content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum.

To examine the structural validity of the 2021 KTEEM test, the exploratory factor analysis (EFA) and analyses based on classical test theory (CTT) and item-response theory (IRT) were conducted. We followed these steps in the investigation: 1) initial item analysis, 2) dimensionality, 3) item- and test-level statistics based on the CTT and EFA analyses, 4) reliability, and 5) IRT item calibration and person scoring.

Analysis & Results

In this section, we present the results from dimensionality analysis, initial item analysis, and analyses based on CTT and IRT. For the sake of transparency and scientific rigor, we also share the data, syntax, and output files through Open Science Framework (OSF; Schoen et al., 2022b).

Initial Item Analysis

The EFA was run via Mplus 8.0 (Muthén & Muthén, 1998-2017) using the weighted least square estimation with mean and variance adjusted method with tetrachoric correlations. The structure of the 34-item test was examined by requesting solutions with 1–6 factors. The tetrachoric correlation matrix was also checked to investigate the interrelations among the items. One of the new items was removed before subsequent analyses due to non-significant factor loadings identified through the EFA and negative correlations with the other items. The remaining 33 dichotomously scored items contributed to the final scale for the 2021 K-TEEM.

Dimensionality Analysis & Results

To investigate the dimensionality of the 2021 KTEEM test, we repeated the EFA with the remaining 33 items. Table 1 shows the first six eigenvalues provided in the Mplus EFA output with the corresponding percentages of explained variation for the one-factor model. The largest variation explained by the first factor was 26.51 percent, whereas second largest variation was found to be 5.49 percent. The difference of the Root Mean Square Error of Approximation (RMSEA) values between the 1-factor EFA (Model 1) and the 2-factor EFA (Model 2) can be a supplementary approach used to identify the optimal number of factors to retain. Specifically, if the difference is smaller than a prescribed cut-off value (we used 0.015 for this case), the more parsimonious model is retained. Finch's (2020) study recommended this approach and showed that the RMSEA_0.015 approach performed better than Parallel Analysis when the indicators are categorical with small factor loadings. The difference was calculated to be 0.007 (RMSEA_Model1 = 0.030; RMSEA_Model2 = 0.023), again suggesting the more parsimonious 1-factor model. This result implies that unidimensionality is a reasonable assumption.

Table 1: Eigenvalues Estimated from Mplus and Their Corresponding Percentages of Explained Variation

	Eigenvalue	% Variation explained
1	8.75	26.51
2	1.81	5.49
3	1.59	4.81
4	1.51	4.56
5	1.28	3.87
6	1.21	3.67

Classical Test Theory (CTT) Analysis & Results

After concluding the dimensionality analysis (and removing the item with poor fit), we conducted the CTT analyses using SPSS 26.0 (IBM corp., 2019). Figure 1 shows the distribution of the raw sum scores. The scores ranged from 2 to 33 with a mean of 18.35 and a standard deviation of 6.20. Less than one percent (0.61 percent) of examinees in this sample received a perfect score of 33. Item means (i.e., item difficulties based on the percentages of correctly endorsing an item) ranged from 0.19 to 0.79 with a mean of 0.56 and a standard deviation of 0.17 and were approximately normally distributed. The coefficient alpha (Cronbach, 1951) and standard error of measurement were calculated to be 0.84 and 2.50, respectively.

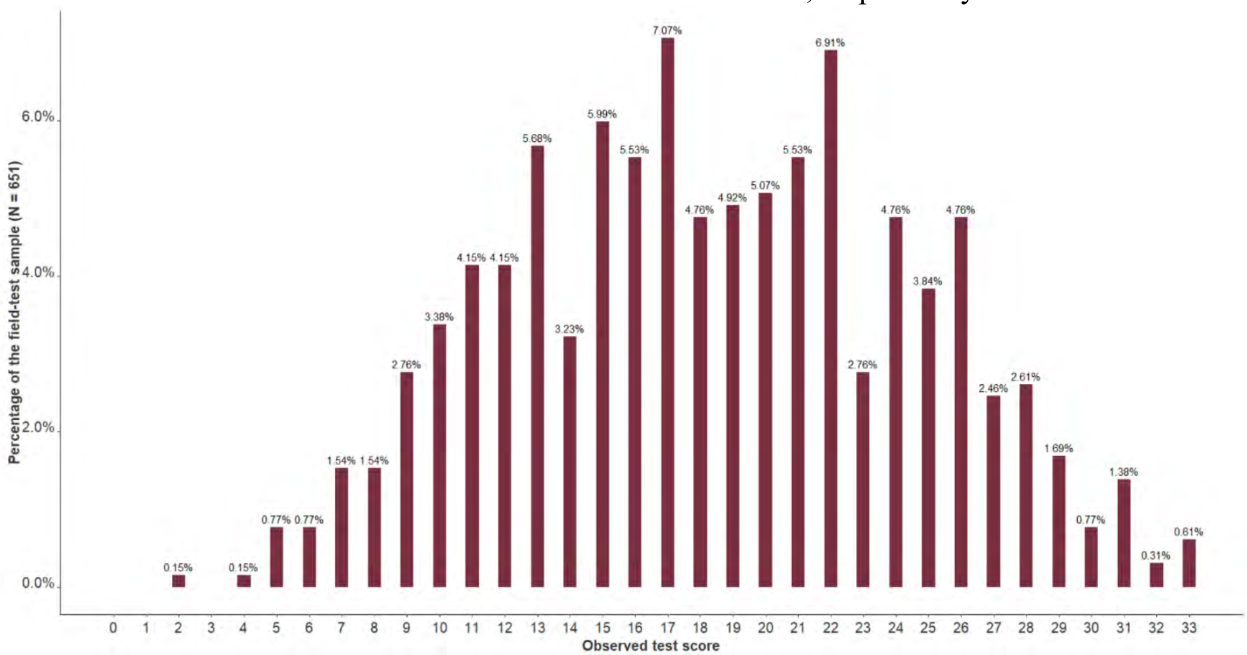


Figure 1. Observed Test Score Distribution for the 33-item 2021 K-TEEM

Item Response Theory (IRT) Analysis & Results

We used flexMIRT 3.6 (Cai, 2020) to perform the IRT analyses. Because the final test consists of 33 dichotomously scored open-ended and multiple-choice items, and the dimensionality analysis suggested one dominant construct underlying the relationships of the items, we fit a unidimensional IRT model to the data. Many cognitive assessments in educational practices involve data with binary scores (i.e., assigning “0” to an incorrect response and “1” to a correct response). Measurement practitioners may consider fitting a three-parameter (3PL) model for multiple-choice items as a way to take a guessing effect into account. However, the optimum number of observations required by a unidimensional three-parameter model is recommended to be at least 1,000 (de Ayala, 2009).

Following this suggestion and based on the fact that our test includes open-ended items as well as multiple-choice items, a unidimensional two-parameter logistic (2PL) model was used to calibrate two items added to the 2019 version of our test. For the other test items, the parameter estimates from the 2019 sample were used as anchor items to vertically link the metrics (Schoen et al., 2021; 2022). This is known as fixed item parameter calibration (FIPC) approach under the common item nonequivalent group test design, which could be used to place scores from the tests of different levels on a common scale (Kolen & Brennan, 2004). The population mean and standard deviation were freely estimated. Results of flexMIRT showed that the model converged successfully. The formula of the 2PL model is shown in Equation 1 according to the parameterization of de Ayala (2009).

$$P_j(\theta) = \frac{\exp[a_j(\theta - b_j)]}{1+\exp[a_j(\theta - b_j)]}, \tag{1}$$

where a_j is the discrimination index of item j ($j = 1, 2, \ldots, J$), b_j is the difficulty index of item j, $P_j(\theta)$ is the probability of correct answer, θ is the person ability.

Figures 2 and 3 display the item-difficulty estimates and item-discrimination estimates of each item. The item difficulty estimates ranged from –2.48 to 1.99 with a mean of –0.20 and a standard deviation of 1.12. The item discrimination estimates ranged from 0.41 to 1.35 with a mean of 0.80 and a standard deviation of 0.39. As seen in the figures, the discrimination estimates of those two items (see the last two items in the graphs below) added to the 2019 version were 0.75 and 1.12, and their difficulty estimates were relatively larger as compared to many items in the test. We note that the difficulties of the test items are heterogeneous, and the difficulty of the test aligns reasonably well with the ability range. The marginal reliability for response pattern scores calculated in flexMIRT was 0.84.

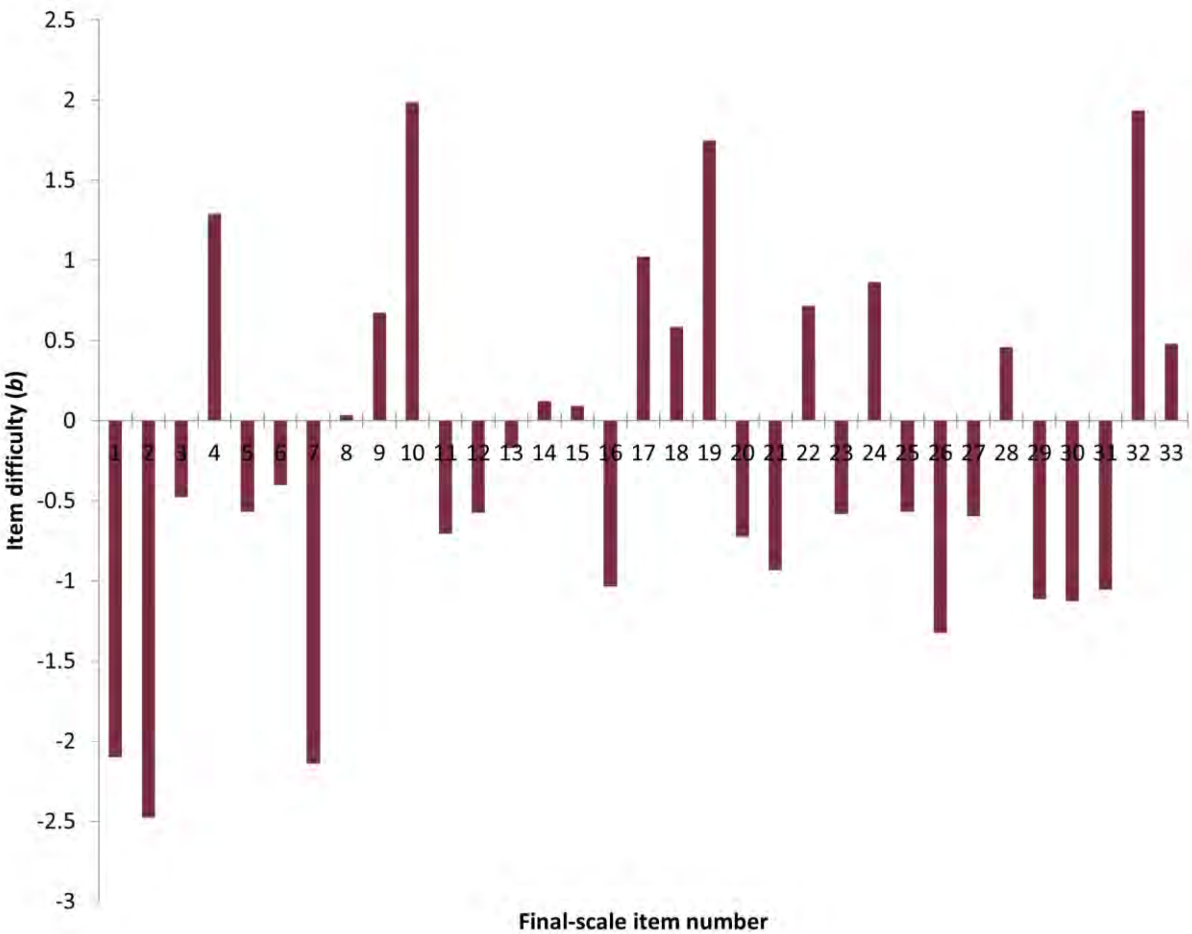


Figure 2. Item-difficulty estimate (b) of each final-test item, presented in sequential order.

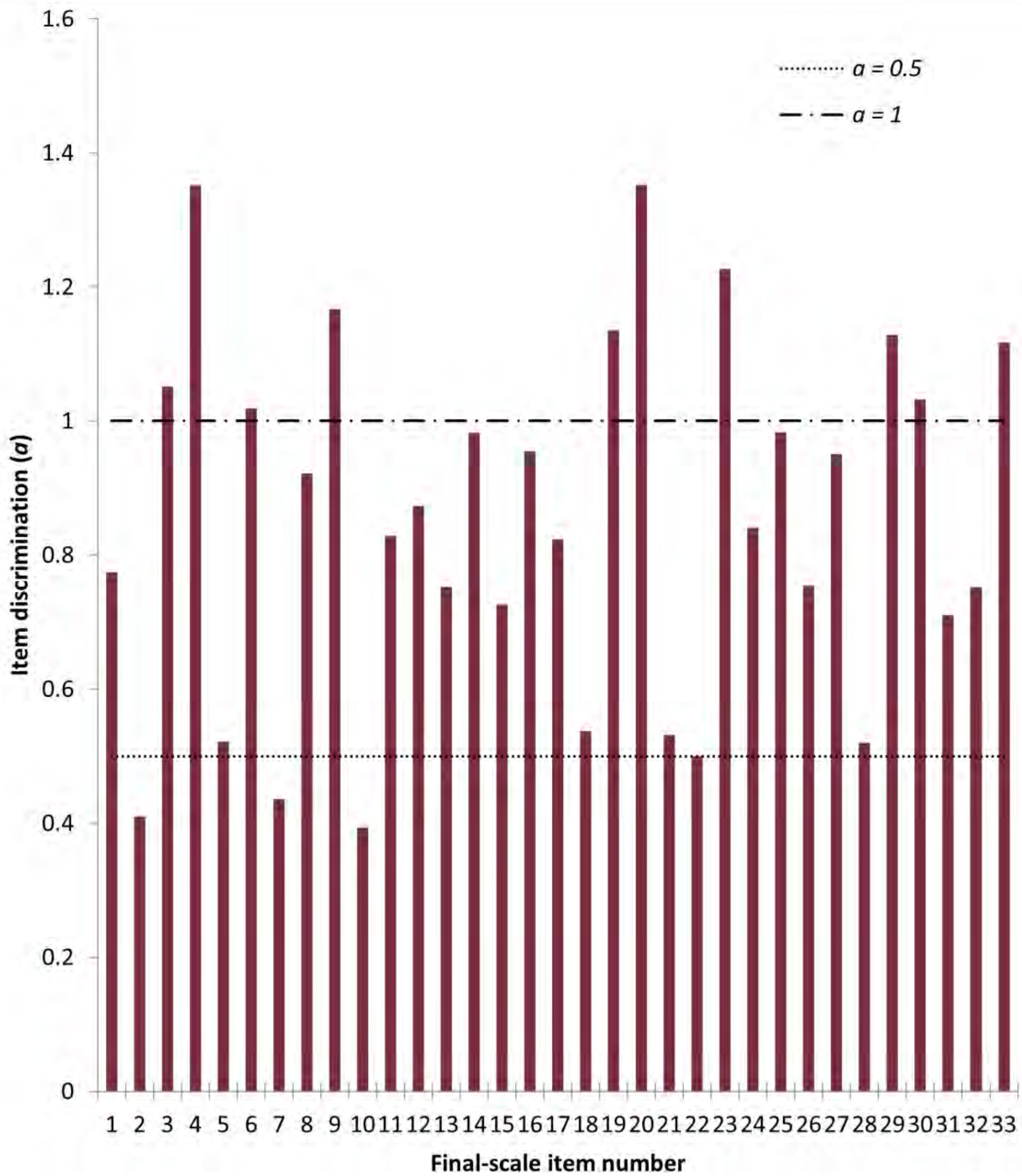


Figure 3. Item-discrimination estimate (b) of each final-test item, presented in sequential order.

Discussion

The 2021 K-TEEM is one of more than a half-dozen test forms in the K-TEEM set of assessments. It is being used in a three-year randomized controlled trial involving teachers in more than 200 schools. It is being used as a baseline measure of teacher MKT and to estimate the effects of a CGI professional-development program on teacher MKT, and as a measure of MKT as a potential mediator of the effect of the CGI professional-development program on teaching and student learning.

Item-level analysis suggested that 33 out of 34 items on the 2021 K-TEEM test form were adequate. Dimensionality analysis demonstrated that one dominant construct (i.e., MKT) described the interrelatedness among the items. The difficulty of the test was reasonably well calibrated to align with the levels of MKT of the target population. Reliability estimates for the observed sum scores and response pattern scores indicated that the test scores may be useful for the planned, subsequent, group-level analyses.

The construct validation process is an ongoing effort of collecting evidence to justify the interpretation and use of test scores. Structural validity is one of the phases of the construct validation process, which investigates the psychometric/measurement properties of an assessment tool. According to Flake, Pek & Hehman (2017), "if the measurement properties of a scale do not replicate, then the replicability of the results from analyses using those measures is suspect (p. 3)". Notably, the best practices for conducting construct validation are to continually collect evidence to support measures underlying the latent construct using rigorous measurement practices. Therefore, the process used for analysis and scoring of the K-TEEM can present a model for researchers in mathematics education to use as they increase the methodological rigor of their measurement practices.

Acknowledgments

The research and development reported here was supported by the Institute of Education Sciences and United States Department of Education through Award Numbers R305A180429 and U423A180115 to Florida State University. The opinions expressed are those of the authors and do not represent the views of U.S. Department of Education.

References

American Educational Research Association, American Psychological Association, & National Council on Measurement in Education (Eds.). (2014). *Standards for educational and psychological testing*. American Educational Research Association.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Cai, L. (2020). flexMIRT version 3.6: Flexible multilevel multidimensional item analysis and test scoring [computer software]. Vector Psychometric Group.

Cronbach, L. J. (1951). Coefficient alpha and the internal structure of tests. *Psychometrika*, *16*(3), 297–334.

de Ayala, R. J. (2009). The theory and practice of item response theory. The Guilford Press.

Finch, W. H. (2020). Using fit statistic differences to determine the optimal number of factors to retain in an exploratory factor analysis. *Educational and Psychological Measurement*,80(2), 217–241. https://doi.org/10.1177/0013164419865769

Hill, H. C. & Shih, J. C. (2009). Examining the quality of statistical mathematics education research. *Journal for Research in Mathematics Education*, *40*, 241–250.

IBM Corp. (2019). IBM SPSS Statistics for Windows, Version 26.0. IBM Corp.

Kolen, M. J., Brennan, R. L. (2014). *Test equating, scaling, and linking: Methods and practices* (3rd ed.). New York: Springer.

Li, C. H. (2016). Confirmatory factor analysis with ordinal data: Comparing robust maximum likelihood and diagonally weighted least squares. *Behavior research methods*, *48*(3), 936-949. https://doi.org/10.3758/s13428-015-0619-7

Meyer, M. N. (2018). Practical tips for ethical data sharing. *Advances in Methods and Practices in Psychological Science*, *1*(1), 131–144. https://doi.org/10.1177/2515245917747656

Muthén, L. K. and Muthén, B. O. (2017). *Mplus User's Guide* (8th ed.). Muthén & Muthén

Qualtrics. (2005). *Qualtrics software* (Version 2020). [Computer software]. Qualtrics.

Flake, J. K., Pek, J., & Hehman, E. (2017). Construct validation in social and personality research: Current practice and recommendations. *Social Psychological and Personality Science*, *8*(4), 1–9.

Schoen, R. C., Bray, W., Wolfe, C., Nielsen, L., & Tazaz, A. M. (2017). Developing an assessment instrument to measure early elementary teachers' mathematical knowledge for teaching. *The Elementary School Journal*, *118*(1), 55–81. https://doi.org/10.1086/692912

Schoen, R. C., Yang, X., Tazaz, A. M., Bray, W., & Farina, K. (2019). *Development and Initial Field Test of the*

2016 K-TEEM (Knowledge for Teaching Early Elementary Mathematics) Test (Research Report No. 2019-01). Tallahassee, FL: Florida State University. DOI: 10.33009/fsu.1581610055

Schoen, R.C., Yang, X., & Solmaz, G. (2021). *Psychometric Report for the 2019 Knowledge for Teaching Early Elementary Mathematics (K-TEEM) Test* (Research Report No. 2021-04). Florida State University. https://doi.org/10.33009/lsi.1620243057

Schoen, R.C., Yang, X., & Solmaz, G. (2021), 'Psychometric Analysis Of 2019 Knowledge For Teaching Early Elementary Mathematics (K-TEEM) ', *The forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA),* Philadelphia, USA, 14–17 October, pp. 496–497. https://www.pmena.org/pmenaproceedings/PMENA%2043%202021%20Proceedings.pdf

Schoen, R. C., Bray, W. S., Tazaz, A. M., & Buntin, C. K. (2022a). A Description of the Cognitively Guided Instruction Professional Development Program in Florida: 2013–2020. Retrieved from https://doi.org/10.33009/fsu.1643828800

Schoen, R. C., Guven, A., Solmaz, G., Tazaz, A., & Iuhasz-Velez, N. R. (2022b). *Psychometric Report for the 2021 Knowledge for Teaching Early Elementary Mathematics (K-TEEM) Test*. https://doi.org/10.17605/OSF.IO/AWSZC

Yang, Y., & Green, S. B. (2011). Coefficient alpha: A reliability coefficient for the 21st century?. *Journal of psychoeducational assessment*, *29*(4), 377-392. https://doi.org/10.1177/0734282911406668

IMPACTO DE LA REFLEXIÓN EN LOS CONOCIMIENTOS DE PROFESORES MIEMBROS DE UNA COMUNIDAD DE APRENDIZAJE

IMPACT OF REFLECTION ON THE KNOWLEDGE OF TEACHERS WHO ARE MEMBERS OF A LEARNING COMMUNITY

Miguel Ángel Huerta-Vázquez
Cinvestav (México)
mhuerta@cinvestav.mx

Olimpia Figueras
Cinvestav (México)
figuerao@cinvestav.mx

Sandra Areli Martínez Pérez
UNAM (México)
sandraareli.martinez@cch.unam.mx

Un grupo de profesores de matemáticas de nivel medio superior buscan mejorar su labor docente, para lo cual crean una comunidad profesional de aprendizaje, en dónde los docentes producen de manera colectiva actividades de enseñanza mediante un ciclo reflexivo de varias etapas que llaman 'Enseñanza-Reflexión'. En este reporte se describe cómo se amplía el conocimiento para la enseñanza de las matemáticas durante un ciclo reflexivo sobre la función cuadrática, así como el impacto de la reflexión en el conocimiento de los docentes. Un resultado importante de este trabajo es que el marco de Rowland et al. permite analizar el conocimiento de profesores, que emerge cuando ellos hacen actividades reflexivas en una comunidad profesional de aprendizaje autogestiva.

Palabras clave: Conocimiento matemático para la enseñanza, Función cuadrática, Álgebra y pensamiento algebraico, Desarrollo profesional

Un grupo de profesores de una escuela de nivel medio superior mexicana, se organizaron para diseñar, aplicar y discutir reflexivamente actividades de enseñanza, con el fin de mejorar su labor docente ya que consideran que tienen deficiencias para la enseñanza de las matemáticas, debido a que ninguno de estos docentes tuvo formación inicial para la enseñanza, Sosa y Ribeiro (2014, p. 1) señalan que es una situación muy común en los profesores de ese nivel en México hasta 2014.

Para atenuar sus deficiencias los profesores deciden crear una comunidad profesional de aprendizaje, Sowder (2007, p. 185) la define como una agrupación en la cual se comparten mismos propósitos, generalmente con el entendimiento de que los docentes son responsables unos con otros por lograr sus metas; coordinar sus esfuerzos para asegurar el aprendizaje de los estudiantes, aprender juntos para mejorar su práctica y compartir responsabilidades para la toma de decisiones sobre asuntos pertenecientes a la mencionada comunidad.

Los investigadores acuerdan junto con los profesores implementar una metodología de trabajo dentro de la comunidad en la que se planean, aplican en el aula y se discuten reflexivamente actividades de enseñanza, las cuales están definidas como actividades que facilitan el aprendizaje significativo, de temas específicos de conocimiento conceptual o procedimental, que pueden estimular la investigación aplicada a la enseñanza diaria de las clases (Moreira, 2011).

También determinan que el tema en el que se van a centrar es en la enseñanza de la función cuadrática ya que tiene un lugar relevante en el curriculum, así que a partir de ese momento las actividades de enseñanza que van elaborando se centran en ese tema.

El propósito de este trabajo es mostrar evidencias de que el conocimiento de los profesores para la enseñanza de la función cuadrática puede ampliarse cuando los docentes participan en una comunidad de aprendizaje, y hacen actividades reflexivas en las que producen actividades de enseñanza. Hay una versión previa de esta investigación en Huerta-Vazquez & Figueras (2020).

Se analizan esas evidencias aplicando el trabajo de Kula Ünver (2018) como herramienta de análisis, a partir del marco teórico del Cuarteto del Conocimiento de Rowland, Huckstep y Thwaites (2005).

Con la herramienta teórica antes descrita se busca, además, observar cómo el trabajo reflexivo dentro de la comunidad impacta en el conocimiento de los profesores participantes del colectivo.

El conocimiento del profesor

Shulman (1986, 1987) es de los primeros en centrar la investigación en el conocimiento del profesor en general, proponiendo tres dimensiones: conocimiento del currículo, del contenido de la materia (o del objeto matemático) (subject matter knowledge SMK), y conocimiento pedagógico del contenido (pedagogical content knowledge PCK).

Ball et al. (2008) profundizan el trabajo de (Shulman (1986) y lo adecuan para el profesor de matemáticas el cuál denominan cómo el Conocimiento Matemático del Profesor para su Enseñanza (MKT), que corresponde al conocimiento matemático necesario para la enseñanza de las matemáticas a los estudiantes, y se centran en dos dominios del trabajo de Shulman: el conocimiento del contenido o de la materia (SMK) y el conocimiento pedagógico del contenido (PCK).

Rowland et al. (2005), proponen un marco que también retoma de Shulman (1986), y lo denominan Cuarteto del Conocimiento (KQ del inglés), lo desarrollan inicialmente para describir y analizar las observaciones hechas en el aula. Este marco está divido en cuatro dimensiones:

Fundamentación: La primera tiene sus raíces en la base de los antecedentes y creencias teóricas del profesor. Se refiere a su conocimiento, la comprensión y capacidad a recurrir a lo que se aprendió en etapa de instrucción (escuela, universidad, etc.), incluyendo la educación inicial del profesorado, en preparación (intencionadamente o no) para su papel en el aula.

Transformación: La segunda tiene que ver con el conocimiento en acción demostrado tanto en la planeación de lo que se va a enseñar, como en el mismo acto de enseñar. Al respecto Kula Ünver (2018) señala que este componente se puede visualizar como la transformación del conocimiento del contenido (SMK) en el conocimiento pedagógico del contenido (PCK) de ahí que la transformación requiere del uso de modelos, analogías, ejemplos, ilustraciones, representaciones y demostraciones que puedan construir un puente entre la comprensión de los profesores sobre la materia y la comprensión que se espera que alcancen los alumnos.

Conexión: La tercera combina ciertas elecciones y decisiones que se hacen en partes concretas del contenido matemático. En gran medida, estas reflejan las deliberaciones y decisiones que implican no sólo conocimiento de conexiones estructurales dentro de las matemáticas en sí mismas, sino también el conocimiento de las demandas cognitivas relativas a diferentes temas y tareas.

Contingencia: La cuarta se refiere a la respuesta del profesor a todos los eventos no previstos en el aula durante la planificación. A pesar de tener todo planeado, libro de texto, programa de estudios, secuencia de actividades, etc. las respuestas de los estudiantes son impredecibles, y está probado que la respuesta a estas situaciones es de las tareas más difíciles para los profesores novatos

Kula Ünver (2018) revisa de manera exhaustiva las dimensiones del Cuarteto del Conocimiento (KQ) de Rowland y colaboradores, y concluye que es un marco detallado y exhaustivo para examinar cómo se manifiesta el conocimiento del contenido (SMK) y el conocimiento pedagógico del contenido (PCK) de los profesores de matemáticas en su proceso de enseñanza. Además, encuentra conexiones teóricas en cada una de las dimensiones con el marco teórico del MKT de Ball et al. (2008).

Comunidad Profesional de Aprendizaje en acción

La inspiración para establecer la forma de trabajo en la comunidad fue a partir de la metodología de estudio de lección (lesson study) que en consiste que un grupo de profesores hagan investigación acerca de lo que planifican, enseñan, observan y comparten con el fin de observar lo que pasa en el aula, además tiene también como objetivo la mejora de la práctica educativa. Dicha metodología se aplica en muchos países de diferente forma dependiendo del objetivo (Soto Gómez & Pérez Gómez, 2015); en Lewis et al. (2019, pp. 17–33) se sintetizan varios de estos estudios de lección y se puntualiza que en general están compuestos en cuatro etapas: de estudio de lección, de planificación, de enseñanza y de reflexión.

Basados en lo anterior los investigadores sistematizaron la metodología de trabajo de la Comunidad en un ciclo que llamaron Ciclo de Enseñanza-Reflexión, está dividido en tres etapas.

Primera etapa, planeación /reflexión-colectiva, basado en las ideas de Dewey (1989) se hace una reflexión colectiva de la secuencia para ser ajustada antes de su aplicación por algún profesor, en este momento se hace una re-revisión de los conocimientos matemáticos para la enseñanza, necesarios para la aplicación de la secuencia.

Segunda etapa, puesta-a-prueba/reflexión-individual, en esta fase se desarrolla lo que Schön (1983) llama la reflexión durante la acción, la secuencia se pone a prueba por el docente el cual debe tomar decisiones durante el proceso de enseñanza. Esta etapa es videograbada para que pueda ser analizada

Tercera etapa, análisis-síntesis /reflexión-colectiva es lo que Schön llama la reflexión después de la acción, en este momento - se analizan los videos de las puestas a prueba, los conocimientos que se esperaba que los estudiantes construyeran, de los modos de enseñanza de los profesores, de las dificultades de los estudiantes, de cómo se intentó que las superaran y como resultado del proceso del diseño de la secuencia enriquecida con la experiencia de todos los miembros de la comunidad. Surge una reflexión individual de lo que cada uno de los miembros 'aprendió' durante el ciclo. (Figura 1)

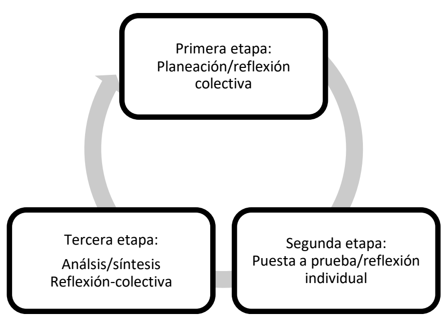


Figura 1: Ciclo de Enseñanza-Reflexión de la comunidad de aprendizaje

El Ciclo de Enseñanza-Reflexión del problema del área mínima de un cuadrilátero

El Ciclo de Enseñanza-Reflexión de la Comunidad para la secuencia didáctica planteada del área mínima de un cuadrilátero se desarrolló de la siguiente manera:

Dado el rectángulo $ABCD$ con base de 6 cm y altura de 4 cm Sea P un punto en el lado $\overline{AB}$, M un punto en el lado BC, N un punto en CD y S un punto en DA, tal que las distancias de BM, CN, DS y AP son iguales. Obtener el área mínima del cuadrilátero que se forma con los puntos P, M, N y S.

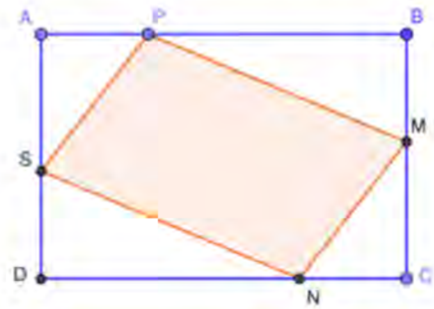


Figura 2: El problema del área mínima del cuadrilátero

1. Etapa 1 *planeación*: la profesora Lola propuso el problema a los docentes de la comunidad y se discutieron los pormenores para usarse en la enseñanza y ella dijo que esta serie de actividades podría ayudar a la enseñanza de la función cuadrática, pues en estas actividades los estudiantes pueden observar tres registros de la función cuadrática: tabular, geométrico y algebraico, además de observar la importancia de obtener los máximos y mínimos de dicha función. Se acordó que la primera clase la aplicaría el profesor Mario y luego el profesor Tadeo.
2. Etapa 2 *puesta-a-prueba*: el profesor Mario aplicó la secuencia en el grupo de Tadeo, esta clase se desarrolló de acuerdo con la planificación de la secuencia didáctica en cuya enseñanza se busca que el alumno haga el tránsito a los distintos registros de la función cuadrática asociada al problema.
3. Etapa 3 *análisis-síntesis*: en el interior de la comunidad se hace una reflexión colectiva usando los videos que se grabaron en la clase de Mario; Lola señaló lo siguiente: Mario no dejó que los estudiantes exploraran la solución al problema por sí mismos, sino que los apresuró a dibujar el rectángulo ABCD, haciendo que la enseñanza fuera muy dirigida. También sugiere que los estudiantes tengan que hacer la actividad de dibujo en parejas de tal manera que uno dibuja y el otro calcula el área para varios valores AP. Tadeo comentó que Mario debió dejar que los alumnos dibujaran el problema para resolverlo y no apresurarlos entre otros comentarios.
4. Etapa 4 *puesta-a-prueba*: el profesor Tadeo aplica la secuencia propuesta en el grupo de Mario, siguiendo el desarrollo de acuerdo con la planificación de la secuencia y toma en cuenta los comentarios de la etapa anterior y de la clase de Mario.
5. Etapa 5 *análisis-síntesis*: los profesores llevan a cabo una nueva reflexión colectiva acerca de la secuencia didáctica en la que resaltan varios aspectos.
   En particular para pasar del *registro verbal al geométrico*: el docente debe escribir el problema en el pizarrón o llevarlo escrito en un papel para que no se pierda tiempo en dictado a los estudiantes. También señalan que los estudiantes deben traducir el texto para hacer el dibujo del rectángulo y del cuadrilátero en su cuaderno y eventualmente algún estudiante pasar al pizarrón a hacerlo. Así también observaron que muchos estudiantes probablemente se van a equivocar al colocar los puntos P, M, N, S como puntos medios del rectángulo $ABCD$ en vez de preservar la igualdad $AP = BM = CN = DS$, así que el profesor que aplique la secuencia debe estar preparado.

Para el *registro tabular* sugieren que los propios alumnos deben proponer diferentes valores de AP para calcular el área del cuadrilátero para que así en el registro gráfico vayan trazando la parábola con los valores que hayan propuesto para AP y así observen cómo el área varía en cada caso.
Para el *registro algebraico* los estudiantes deben generalizar la función cuadrática con base a lo hecho en las actividades anteriores, mientras el docente acompaña a los estudiantes en el proceso para lograr la expresión algebraica esperada. Y para encontrar el área mínima: sería conveniente que usaran los registros tabulares y gráficas, así los estudiantes podrían identificar el área mínima del área $PMNS$ del cuadrilátero, por último, sería conveniente que llevaran la función cuadrática a la forma $y = a(x-h)^2 + k$ donde (h, k) (ver Figura 3 registro algebraico) es el vértice para así interpretar k del vértice, cómo el área mínima del cuadrilátero $PMNS$ pero saben que es probable que no les alcance el tiempo.

La enseñanza de esta secuencia didáctica requiere que haya tránsito entre los registros de representación de la función cuadrática (Figura 3), además estos registros sirven para señalar segmentos útiles para el análisis de cada etapa del Ciclo de Enseñanza-Reflexión.

Registro	Acciones para tener el registro
Verbal Geométrico	Traducir del texto al dibujo geométrico, el problema demanda hacer la conversión del texto al cuadrilátero $PMNS$ y el rectángulo $ABCD$ (Figura 1)
Tabular	Encontrar el área del cuadrilátero cuyos vértices son los puntos $PMNS$ en función de diferentes valores de $\overline{AP}$, para hacer lo siguiente es necesario calcular el área para cada valor propuesto de $\overline{AP}$
Gráfico	Representar la parábola asociada a los valores de la representación tabular, haciendo la conversión de la tabla a la gráfica
Algebraico	Encontrar la expresión algebraica que relaciona el área del cuadrilátero $PMNS$ $A(x)$ en función de valores de $x = AP$ $A(x) = 2x^2 - 10x + 24$ Forma general de la función cuadrática $A(x) = 2(x-2.5)^2 + 11.5$ Ecuación estándar de la parábola vertical

Figura 3: Registros del problema del área mínima del cuadrilátero

Metodología para análisis del Ciclo de Enseñanza-Reflexión del problema

Previamente se describieron las cinco etapas que se desarrollan en el ciclo de Enseñanza-Reflexión, las cinco etapas de dicho ciclo se grabaron en video usando un celular colocado en un tripié fijo, grabando a una resolución *full HD*.

Para las dos etapas de puesta a prueba la cámara fue colocada al fondo del aula de tal manera que el pizarrón fue grabado de manera frontal con el objetivo de captar al profesor y los dichos de los estudiantes. Los nombres de los profesores han sido cambiados por confidencialidad.

Para el caso de las demás etapas la cámara fue colocada al fondo del salón donde se llevaron las actividades de discusión de tal manera que los profesores podían ser grabados al menos en su audio, además se grabaron dos entrevistas individuales tanto de Tadeo como de Mario obteniéndose en total sietes videos para este análisis.

Para el análisis de este Ciclo de Enseñanza, la participación de tres profesores es analizada, Lola la que propone la actividad en la etapa 1, Mario y Tadeo etapa 2 y 4 los que aplican la secuencia en aula, y los tres participan en las etapas 3 y 5.

El software que se usa para el análisis de los datos es el MaxQDA 2020, para codificar y transcribir los videos, y el análisis de los datos obtenidos en la investigación descrita en este artículo, a saber: el Cuarteto del Conocimiento (KQ) de Rowland mediante la interpretación de Kula Ünver (2018) y como complemento de dichas observaciones se utiliza el marco de Ball y colaboradores y las relaciones entre los marcos de Rowland y Ball resumidos en Turner, (2012, p. 257)

Análisis de los datos obtenidos

En todas las etapas podemos encontrar evidencias del conocimiento para enseñanza de la función cuadrática, se muestran algunos ejemplos de momentos donde aparecen indicadores de cada una de las dimensiones del Cuarteto del Conocimiento de Rowland, bajo el lente de Kula Ünver. Las entrevistas ayudan a entender las actuaciones de los profesores en el aula.

En la etapa 1, planeación, Lola propone la secuencia didáctica, ya que la ha aplicado durante varios semestres previos, así que muestra conocimiento de cómo reaccionan los estudiantes ante la secuencia y cómo puede ser implementada, algunos ejemplos se muestran en la Figura 4.

Etapa 01 Planeación Docente: Lola		
Dimensiones (KQ)	Ejemplo	Comentario
Fundamentación	"Esta secuencia se puede usar de 2 formas, se puede usar para iniciar, se propone un problema y los alumnos empiezan a hacer los dibujitos"	Lola ha aplicado varias veces las secuencia así que conoce como van a actuar los estudiantes cuando se les enseña a ellos, se muestra un conocimiento del pedagógico del contenido (PCK) respecto a esta secuencia y los alumnos.
Transformación	"es que los alumnos deben dominar los 3 registros"	Lola sabe que en esta secuencia se hace el tránsito en los tres registros: tabular, gráfico y algebraico. (Aunque le falta especificar los registros verbales y geométricos relativos al problema)
Conexión	"Se puede introducir incluso lo que es dominio y el contradominio. Eso es más ideal como para mate 4".	Lola conoce como en que otras lecciones se puede usar esta secuencia en este caso para la asignatura de matemática IV

Figura 4: Evidencias de las dimensiones del KQ de Rowland en la etapa 1

En la etapa 2 puesta-a-prueba el profesor Mario aplica la secuencia didáctica, al ser una lección nueva para presenta algunas situaciones que no previó, que lo llevan a contingencias, pero logra sortear las dificultades, además muestra conocimiento acerca de la función cuadrática y conceptos asociados como la ecuación cuadrática Figura 5.

Etapa 02 Puesta a prueba Docente: Mario		
Dimensiones (KQ)	Ejemplo	Comentario
Fundamentación Registro Algebraico	Mario explica la diferencia entre ecuación y función cuadrática, igualando el área a la expresión.	Mario explica esa diferencia para hacer notar la variación en la función y lo estático en la ecuación
Transformación Registro Tabular	Pregunta a los estudiantes si se puede tabular con un valor de AP = 4.5 y llegan a la conclusión que no se puede porque un de los lados del rectángulo es de 4 cm. siendo este el valor máximo para tabular.	Mario interroga a los estudiantes con el objetivo de que se den cuenta de los límites de los valores permitidos para el problema,
Conexión Registro Gráfico	Le recuerda a los estudiantes lo visto en el curso de matemáticas I, los conceptos de variable independiente y dependiente para empezar la gráfica de la tabla antes hecha.	Mario presenta conocimiento de la materia (SMK) acerca de estos conceptos relacionados a la definción de función
Contingencia Registro Verbal/Geométrico	Los estudiantes tardan en hacer el dibujo del cuadrilatero PMNS dentro del rectángulo ABCD ya que los alumnos no toman en cuenta que $AP = BM = CN = DS$	En la entrevista el profesor Mario este relata que en esta parte de la secuencia didáctica tuvo muchas dificultades pero considera que logra sortear esta contingencia de tal manera que logra terminar de dar la lección a los alumnos
Etapa 04 Puesta a prueba Docente: Tadeo		
Fundamentación Registro Algebraico	Tadeo espera que los estudiantes coloquen los puntos AP, BM, CN, DS en los puntos medio del rectángulo, así que les pregunta acerca de la igualdad de AP, BM, CN, DS con el fin de que se den cuenta y logren dibujar el cuadralatero esperado, y una alumna se da cuenta y lo dibuja.	Tadeo observó que esta situación fue de contingencia para Mario y lo tomó en cuenta en la planeación de la aplicación de la secuencia
Transformación Registro Gráfico	Los estudiantes afirman que el valor del área mínima es de 12 cm y mediante la interrogación Tadeo los hace ver que el área mínima es de 11.5 cm^2	Tadeo tiene bien claro el rumbo de la secuencia didáctica por lo que sabe hacer las preguntas correctas, de hecho en la entrevista comenta que su estilo de enseñanza busca ser más constructivista que conductista, por eso considera necesario hacer preguntas para que los alumnos entiendan conceptos matemáticos

Figura 5: Evidencias de las dimensiones del KQ de Rowland en la etapa 2 y 4

Dimensiones (KQ), etapa y docentes	Ejemplo	Comentario
Fundamentación Etapa 03 Lola y tadeo	Para Tadeo encontrar el modelo del área para este problema es más que suficiente ya que se cumple con el aprendizaje indicado en el currículum del Colegio donde laboran, y para Lola introducir comentarios no planificados dentro de la secuencia didáctica no es pertinente ya que se pierde tiempo, refiriéndose al comentario que hace Mario en la etapa de anterior acerca de la función y la ecuación cudrática.	En la entrevista Tadeo considera que el introducir el comentario acerca de la función y la ecuación cuadrática es algo valioso para enriquecer esta secuencia
Transformación Etapa 03 Lola	"tú debes de cuestionar a los alumnos de cómo van a resolver el problema, al final algunos de ellos te voy a hacer un dibujo"	Lola se refiere a qué hay que preguntarle a los estudiantes como deben proceder para empezar a resolver este problema, en la medida de lo posible hay que evitar guiarlos de manera 'conductista'
Fundamentación Etapa 05 Lola	Lola considera que sería pertinente agregar a la secuencia la posibilidad de que el alumno encuentre el modelo de área a partir de 3 puntos, ya que en una parábola vertical sólo se necesitan 3 puntos para encontrar la ecuación.	Lola demuestra conocimientos importantes acerca de geometría analítica y la función cuadrática lo que es un claro ejemplo del conocimiento de la materia (SMK)
Conexión Mario	Mario considera que esta secuencia didáctica puede ser utilizada al empezar la unidad relacionada con la función cuadrática o al terminar ya que considera que puede adaptarse a las necesidades del profesor	Mario muestra que conoce el currículum del colegio donde labora ya que sabe al menos para la unidad relacionada a la función cuadrática como esta secuencia puede ser adaptada en cualquiera de los momentos de la enseñanza de dicha unidad.

Figura 6: Evidencias de las dimensiones del KQ de Rowland en la etapa 3 y 5

Etapa 3 análisis-síntesis, en esta etapa se revisa lo hecho por Mario en la etapa anterior, los profesores en particular hacen una crítica a Mario en el sentido de que no les dio tiempo a los alumnos para que pensaran en el problema lo que, a juicio del Lola, no lograron entender el texto del problema de manera correcta Figura 6.

Etapa 4 Puesta a prueba, es en esta etapa que el profesor Tadeo aplica la secuencia en el aula, él tuvo la oportunidad de observar la aplicación del profesor Mario y de estar presente en la etapa 3 donde se hizo una reflexión acerca de lo hecho por el profesor Mario, la actuación del profesor toma en cuenta las sugerencias hechas en la etapa anterior ya que incorpora en su planeación los errores que pudieran tener los estudiantes al resolver el problema.

Etapa 5 análisis-síntesis, en esta etapa se hace una reflexión final acerca de lo hecho por los 2 profesores se llega a varias conclusiones, la primera es que los profesores deben llevar bien planeada la secuencia para evitar problemas en el momento de su aplicación, también los profesores consideran que es necesario que los estudiantes tenga la oportunidad de fallar y que se les dé tiempo para que rectifiquen sus errores y consideran que se podría mejorar la secuencia con el uso de tecnología como GeoGebra o agregar nuevos retos dentro de la secuencia, como el cálculo de los coeficientes usando diferencias finitas.

Discusión de los resultados

El análisis de las evidencias encontradas en este ciclo de Enseñanza-Reflexión muestran que la reflexión de las actuaciones de los profesores al aplicar la secuencia didáctica ayudó a qué situaciones de contingencia como la que le ocurrió a Mario durante la transición del registro verbal al geométrico, se convierta en una situación de fundamentación de Tadeo ya que sabe que los estudiantes fallan al no considerar la anterior igualdad (Figura 6).

También existen situación donde las creencias, se imponen sobre conocimientos para la enseñanza, es el caso de la explicación de la diferencia de la ecuación y función hecha por Mario en la etapa 2 (ver Figura 2) puede ser una pérdida de tiempo ya que no está en la planificación original, pero puede ser considerada valiosa para otro profesor, ver el caso de Lola y Tadeo (Figura 6, Etapa 03, Fundamentación).

Por otro lado, también se muestra que un docente con experiencia, como fue el caso de Lola puede ayudar a mejorar la enseñanza de otros profesores, ya que ella ha enseñado con el problema y ha visto cómo actúan los estudiantes cuando hacen las actividades de la secuencia, pero los demás profesores pueden enriquecer de manera colectiva el conocimiento para la enseñanza a pesar de no conocer la secuencia.

Otro aspecto que debe ser incorporado a las discusiones reflexivas de la comunidad es el hecho de que los profesores podrían involucrarse en el análisis de cada etapa, sobre todo aquellas relacionadas con la aplicación de las actividades didácticas, con el objetivo de que conozcan el marco del Cuarteto del Conocimiento de Rowland, como una manera de hacer mejores reflexiones acerca de su actividad docente.

De manera más global hay que señalar que esta forma de trabajo colectiva y autogestiva puede contribuir a la ampliación del conocimiento de los profesores que participan en una comunidad profesional de aprendizaje, un ejemplo que podemos señalar es el enriquecimiento al conocimiento de los profesores durante las etapas de 3 y 5, donde se dan discusiones en las que se intercambia la experiencia de parte de Lola, las reflexiones en las aplicaciones en aula de parte de Mario y de Tadeo sobre todo cuando Mario experimenta momentos de contingencia que Tadeo aprovecha para incorporar a su fundamentación. Es claro también que evidencias sólidas del crecimiento del conocimiento de los docentes del colectivo solo se podrá observar en periodos largos.

Referencias

Ball, D., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Dewey, J. (1989). *Cómo pensamos: nueva exposición de la relación entre pensamiento y proceso educativo*. Paidós.

Duval, R. (2004). *Semiosis y pensamiento humano: registros semióticos y aprendizajes intelectuales*. Universidad del Valle.

Huerta-Vazquez, M. A., & Figueras, O. (2020). Towards reflection on the practice in a learning community of higher-level mathematics teachers. *Proceedings of the Forty-Second Meeting* of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico*, 1873–1877. https://doi.org/10.51272/pmena.42.2020-306

Kula Ünver, S. (2018). The Knowledge Quartet in The Light of The Literature on Subject Matter and Pedagogical Content Knowledge. *Acta Didactica Napocensia*, *11*(2), 27–42. https://doi.org/10.24193/adn.11.2.3

Lewis, C., Friedkin, S., Emerson, K., Henn, L., & Goldsmith, L. (2019). How Does Lesson Study Work? Toward a Theory of Lesson Study Process and Impact. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and Practice of Lesson Study in Mathematics* (pp. 13–37). Springer International Publishing. https://doi.org/10.1007/978-3-030-04031-4_2

Moreira, M. A. (2011). Unidades de enseñanza potencialmente significativas - ueps. *Aprendizagem Significativa Em Revista/Meaningful Learning Review*, *2*, 43–63.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary Teachers' Mathematics Subject Knowledge: The Knowledge Quartet and the Case of Naomi. *Journal of Mathematics Teacher Education*, *8*(3), 255–281. https://doi.org/10.1007/s10857-005-0853-5

Schön, D. A. (1983). *The reflective practitioner: how profesionals think in action*. Basic Books.

Shulman, L. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Shulman, L. (1987). Knowledge and Teaching: Foundations of the New Reform. *Harvard Educational Review*, *57*(1), 1–23. https://doi.org/10.17763/haer.57.1.j463w79r56455411

Sosa, L., & Ribeiro, C. M. (2014). La formación del profesorado de matemáticas de nivel medio superior en México: una necesidad para la profesionalización docente. *Revista Iberoamericana de Producción Académica y Gestión Educativa*, *1*(1).

Soto Gómez, E., & Pérez Gómez, Á. I. (2015). Lessons Studies: un viaje de ida y vuelta recreando el aprendizaje comprensivo. *Revista Interuniversitaria de Formación Del Profesorado*, *29*(3). https://www.redalyc.org/jatsRepo/274/27443871002/html/index.html

Sowder, J. T. (2007). The Mathematical Education and Development of Teachers. In *Second handbook of research on mathematics teaching and learning : a project of the national council of teachers of mathematics* (Vol. 1, pp. 157–223). Information Age Publishing.

Turner, F. (2012). Using the Knowledge Quartet to develop mathematics content knowledge: the role of reflection on professional development. *Research in Mathematics Education*, *14*(3), 253–271. https://doi.org/10.1080/14794802.2012.734972

IMPACT OF REFLECTION ON THE KNOWLEDGE OF TEACHERS WHO ARE MEMBERS OF A LEARNING COMMUNITY

A group of high school mathematics teachers seek to improve their teaching work, for which they create a professional learning community, where teachers collectively produce teaching activities through a reflective cycle of several stages that they call 'Teaching-Reflection '. This report describes how knowledge is expanded for teaching mathematics during a reflective cycle on the quadratic function, as well as the impact of reflection on teachers' knowledge. An important result of this work is that the framework of Rowland et al. It allows to analyze the

knowledge of teachers, which emerges when they do reflective activities in a self-managed professional learning community.

WHAT'S IN A NAME? SEEKING GEOMETRY IN GEOMETRIC SEQUENCES

Andrew Kercher
Simon Fraser University
andrew_kercher@sfu.ca

Anna Marie Bergman
Simon Fraser University
anna_marie_bergman@sfu.ca

Rina Zazkis
Simon Fraser University
zazkis@sfu.ca

Mathematical terminology is sometimes created according to conventions that are not obvious to students who will use the term. When this is the case, investigating the choice of a name can reveal interesting and unforeseen connections among mathematical topics. In this study, we tasked prospective and practicing teachers to consider: What is geometric about geometric sequences? Participants embedded their explanations within a scripted dialogue between teacher- and student-characters in a mathematics classroom, provided commentary on this dialogue, and expanded on its mathematical content. Participants most often leverage the concept of a geometric mean to explain why geometric sequences are named as such. To capture this informal arguments, we built on the work of Toulmin (1958/2003) to conceptualize and develop the Toulmin-Reversed (Toulmin-R) model.

Keywords: Teacher Knowledge, Communication, Professional Development, Preservice Teacher Education

Introduction

Mathematical communication relies on the use of terms that have a particular meaning within the mathematical register. However, confusion may arise when a learner interprets a mathematical term by referencing the use of that word outside of mathematics. For example, the phrase "only a fraction of students participated in the Olympiad" implies that a "fraction" is a small amount and may interfere with a learner's ability to recognize fractions that are greater than one (Hackenberg, 2007). To indicate such fractions within a mathematical conversation, a particular adjective is chosen: *improper* fractions are fractions greater than one.

Such adjectives often resonate with the intuitive interpretation of the words. Consider for example increasing functions, whose graphical appearance is often exactly in accord with students' preconceived notions of what increasing entails. There are also instances of particular adjectives which may at first seem unrelated to the mathematical object, but which are immediately explained by the associated definition. For example, *rational* numbers are those expressed as ratios (under some constraints).

However, there are examples of adjectives where the choice is somehow puzzling even after learning the corresponding definition. Consider for example a *perfect* number, defined as an integer that is equal to the sum of its factors (excluding itself). It is not immediately obvious why "perfect" is an appropriate adjective to describe this particular property. It is only in light of the fact that perfect numbers are usually considered together with *deficient* (smaller than the sum of its factors) and *abundant* (larger than the sum of its factors) numbers that the choice of adjective is given some context; and even so, the adjectives "sufficient" or "balanced" could be seen as more appropriate than associating the property with perfection.

We acknowledge that mathematical convention, including terminology, is sometimes *arbitrary* in the sense that it cannot be deduced using logical principles (Hewitt, 1999). For example, a student simply cannot deduce the arbitrary fact that circles are agreed to have 360 degrees—they must be told as much by an authority. But as Hewitt illustrates, it is sometimes possible to embed a seemingly arbitrary convention within a historical or mathematical context to justify its use. When introducing degree measure, a teacher may appeal to the Babylonian number system in which the degree originated in order to rationalize our modern conventions. Our study is concerned with similar explanations surrounding the arbitrary use of the adjective “geometric” in connection with a geometric sequence. It is stimulated by a student question, “What is geometric about geometric sequences?”, to which we collected responses from prospective and practicing teachers.

Our investigation addresses the following research question: How do teachers justify a convention of mathematical terminology? In particular, what is the most common argument invoked by teachers to explain the adjective “geometric” in a reference to a geometric sequence?

Background: Toulmin’s Model

Toulmin’s model is comprised of six elements which, when taken in relation to each other, outline the structure of an informal argument. The *data* is an assertion from which follows the truth of the *conclusion* (sometimes called the claim: cf. Inglis et al., 2007; Conner et al., 2014). The degree of confidence with which the arguer believes the conclusion follows from the data is inferred from the use of a *modal qualifier* (e.g., “therefore, it is necessary that…” or “so, it is probably the case that…”). Throughout this report, we refer to the combination of data, claim, and modal qualifier as the core argument.

A lower degree of confidence in the conclusion is often caused by *rebuttals*, the existence of which may be known or only anticipated at the time the argument is made. Rebuttals are statements that present contrary evidence to the conclusion by describing how it may not follow from the data. Conversely, a higher degree of confidence could be the result of a convincing *warrant* and any associated *backing*. A warrant is an attempt to support the relationship between the data and conclusion with reasoning or evidence; a backing statement is further evidence in support of the warrant (Toulmin, 1958/2003). These elements are not considered part of the core argument because an argument need not contain rebuttals, warrants, or backings—on the other hand, it may contain multiple.

In mathematics, informal argumentation often serves as a foundation for, or supplement to, logical proof; as such, Toulmin models can be used to visualize the structure of mathematical activity surrounding more rigorous mathematics. For example, Inglis et al. (2007) examined the work of mathematics graduate students who had been tasked with deciding the truth value of certain mathematical statements; the authors used Toulmin models to coordinate instances of intuitive and inductive reasoning as their participants informally developed the ideas of a rigorous proof. Weber et al. (2008) and Conner et al. (2014) used Toulmin models to analyze whole-class discussions of middle and secondary school mathematics classrooms, respectively. These data were used to illustrate the role of explicit warrants in facilitating learning opportunities (Weber et al., 2008) and to generate a framework that categorizes the ways in which teachers can support collective argumentation (Conner et al., 2014).

In this paper, we extend the use of Toulmin models to a situation in which the conclusion is known, but the data used to reach that conclusion is not. It is certainly true that geometric sequences are named as such—but what set of data and accompanying warrants might have led to that conclusion?

Methods

Participants and Setting

A total of 24 participants (referred to here as T-1 through T-24) took part in the study. Of these, 9 were prospective teachers in the last term of their teacher certification program. At the time of data collection they were enrolled in a course that used mathematical problem solving as a lens through which secondary mathematics could be connected to advanced mathematics. The remaining 15 participants were practicing teachers enrolled in a professional development course that provided them with an opportunity to investigate and extend their own mathematical thinking by focusing on the learning and teaching of mathematics. The practicing teachers held bachelor's degrees in mathematics or science; in the latter case, the degree also included a number of mathematics courses sufficient for teaching certification.

Both populations completed several scripting tasks (see next section) as part of their regular coursework. Their responses to one such task, along with their responses to accompanying discussion prompts, serve as the dataset for this report.

The Task

The data analyzed in this report is based on participants' responses to a *scripting task*. Scripting tasks were initially introduced in mathematics teacher education as *lesson plays*, which were envisioned as a more robust form of lesson planning. In a lesson play, the scriptwriter constructs dialogue that captures key interactions between a teacher and student characters (Zazkis et al., 2009; Zazkis, et al., 2013). More recently, the idea of a lesson play has been extended to the activity of writing an imaginary dialogue between interlocuters in any mathematical context. In this expanded scope, scripting tasks can provide multiple affordances not only for teachers but also teacher educators and researchers.

Scripting tasks often begin with a prompt, which serves as the beginning of a dialogue between the scripted characters. In prior research, prompts have introduced a student error (e.g., Zazkis et al., 2013), a student question (e.g., Bergman et al., in press; Marmur & Zazkis, 2018; Zazkis & Kontorovich, 2016), or a disagreement among students (e.g., Marmur et al., 2020; Zazkis & Zazkis, 2014). The scriptwriter responds to a prompt by continuing the dialogue. These dialogues reveal mathematical understanding and, for teachers, demonstrate "awareness-in-action" (Mason, 1998, p. 255). That is, they show an envisioned response in practice, rather than in theory, to students' errors, queries, or unusual ideas.

Part 1 of the geometric sequence task consists of a scripting task, the prompt for which is seen in Figure 2. We refer to the scripted characters as "teacher-characters" and "student-characters" throughout this report. We refer to participants also as respondents and scriptwriters, interchangeably. In Part 2, participants were asked to explain the actions taken by the characters in the script. This included justifying both the explanation(s) chosen by the teacher-character as well as the responses given by the student-characters. In Part 3, participants were asked to elaborate on how their personal understanding of the mathematics might have differed from what was presented in the script; that is, they were given the opportunity to clarify the mathematics using more formal or advanced language appropriate for a colleague rather than a student.

> You are starting a unit on geometric sequences. After you have provided several examples, you are faced with a student's question.
>
> **Student: What is GEOMETRIC about geometric sequences?**
> **Teacher: What do you mean?**
> **Student: You called these sequences "geometric", but these are just sequences of numbers…**
> **Teacher: …**
>
> YOUR TASK is to develop an imaginary dialogue in which the student question is discussed and to justify your course of action.

Figure 2: The prompt for the geometric sequences task

Data Analysis

Analysis of participants' responses began with reflexive thematic analysis (Braun & Clarke, 2006, 2019; Nowell et al., 2017). First, the research team thoroughly familiarized themselves with the data by reading and rereading both the scripts and the accompanying commentary. The coding process began by identifying and classifying within Part 1 how the teacher-character chose to answer the student-character's initial question from the prompt: "What is geometric about geometric sequences?" Explanations that only appeared in Part 2 or 3 were also identified and classified in the same way. During this process, supplementary codes emerged from the data that captured other common aspects of the participants' submissions. These included: ways in which the teacher-characters' explanations were unsatisfactory, either to a student-character or the scriptwriter themselves; how characters chose to define the adjective "geometric" and what mathematical objects should be described as such; and how comparisons to other types of sequences bolstered or diminished the explanatory power of the teacher-characters' justification. In total, the initial codes and supplementary codes combined to form a preliminary codebook.

Upon review of the submissions, the research team recognized that Toulmin's model of informal argument could capture the scripted dialogue. When a student-character opposed the teacher-character's explanation, we perceived this as voicing a rebuttal. When the dialogue attended to what is or is not geometric, the characters were seen to be establishing a warrant or its associated backing. However, the existing Toulmin model needed modification given that the scripted characters were attempting to find data that supported a forgone conclusion. That is, it is certainly the case that geometric sequences are named as such—but one cannot establish that they "should" be named this way through formal, deductive logic, as one might establish that the square root of 2 is an irrational number. To capture this novel dynamic, we developed the Toulmin-Reversed (Toulmin-R) model pictured in Figure 3. The Toulmin-R model contains the same elements as a Toulmin model but reverses the direction of the arrows within the core argument and draws attention to the fact that the conclusion is already known; instead, the data is the subject of the argument.

The codebook was then reorganized and refined in light of the Toulmin-R model. Finally, the research team created Toulmin-R models to visualize the informal arguments as they were used by participants to answer the student-character's question from the prompt.

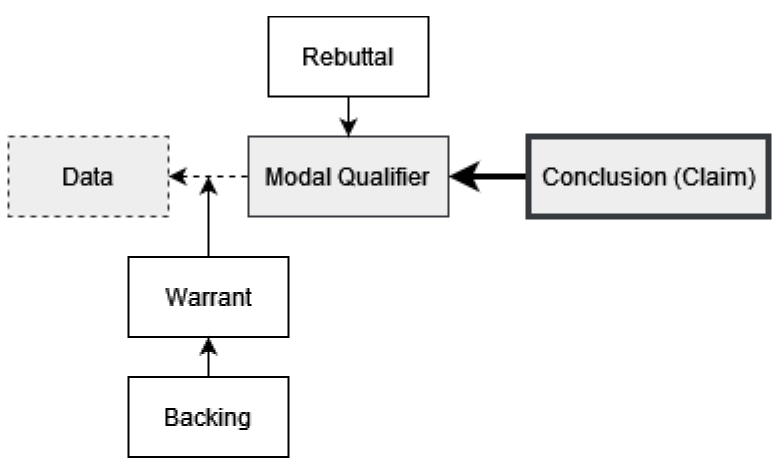


Figure 3: A Toulmin-R diagram template, with the reversed core argument emphasized

Findings

In 13 of the 24 submissions, participants used a connection to the geometric mean as data that could rationalize the naming convention of geometric sequences. This made it the most prevalent source of data employed in arguments.

Scripts that invoked the geometric mean in Part 1 sometimes introduced that concept as an analogue to the arithmetic mean. When this was a purely computational comparison, the student-characters typically responded with skepticism. For example, in T-17's script, an unsatisfied student-character observed that the two means are just "more things called 'arithmetic' and 'geometric' where one's about addition and the other is about multiplication." In these cases the geometric mean was, like the sequence, being introduced and handled entirely with arithmetic—and so its power to explain the choice of "geometric" as an appropriate adjective was limited. A student-character in T-20's script voices this concern explicitly: "If a geometric sequence is geometric because it involves geometric means, then why do we call the geometric mean 'geometric'?"

In response to these student-characters, the teacher-character typically provided one of two geometric explanation that warranted the geometric nature of the mean, and thus, the associated sequence. Both warrants were sometimes accompanied by diagrams, as exemplified in Figure 4.

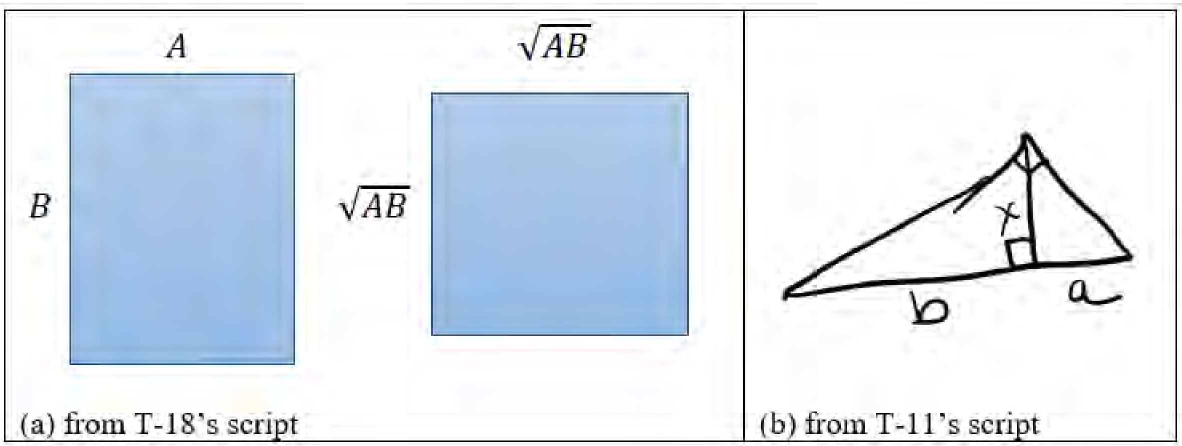


Figure 4: Diagrams used by teacher-characters as warrants

The more common warrant was that if two given numbers were used as the sides of a rectangle, then the geometric mean was the side of a square with the same area as that rectangle. T-18's teacher-character provided the accompanying visualization of this relationship pictured in Figure 4(a). More infrequently, it was explained that the geometric mean could be visualized as the altitude of a right triangle that had been cut into two similar right triangles. The teacher-character in T-11's script drew the diagram seen in Figure 4(b) to illustrate this warrant.

For each of these warrants, an associated backing (sometimes explicit, but usually only implied) was that the polygons in these diagrams were clearly geometric, thereby justifying the use of that adjective for the associated mean and sequence. Figure 5 provides a *composite* Toulmin-R model illustrating these arguments. This model is a composition in the sense that it includes all the rebuttals, warrants, and backings of any submission that argued for the geometric mean as a source of data. We also note that the model in Figure 5 express sequences with notation familiar to a research audience but, as exhibited in the visualizations provided in Figure 4, not typically used by participants in their scripted dialogues.

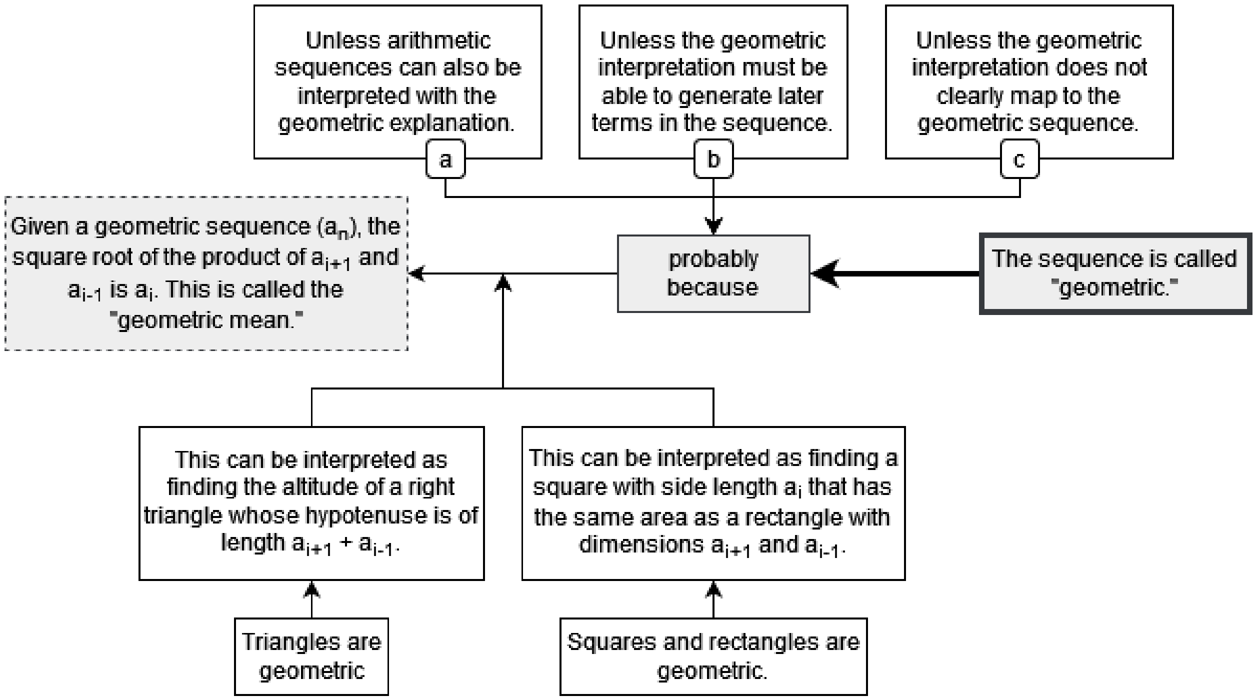


Figure 5: Composite Toulmin-R model featuring geometric mean as a source of data

The selection of the geometric mean as the likely source of data sometimes led to rebuttals, even with the accompanying warrants described above. For example, the teacher-character in T-18's script extended his explanation warranting the geometric nature of the geometric mean to also include a geometric explanation of the arithmetic mean. After drawing the diagram in Figure 4(a), the following dialogue occurs:

Teacher: […] Suppose you have A and B are the lengths of two sides of a rectangle. So the geometric mean $\sqrt{AB}$ is the length of a side of a square having the same area of the rectangle.

Student 1: Okay, then?

Teacher: And the arithmetic mean $\frac{(A+B)}{2}$ is the length of the side of a square having the same perimeter of the rectangle.

Student 1: I still not see your point. Why they are both expressed geometrically?

From Student 1's perspective, accepting the geometric mean explanation as a likely source of data meant that the arithmetic sequence was also similarly geometric. This perspective corresponds to rebuttal-a in Figure 5. Even though the geometric mean does justify that the geometric sequence is in some way geometric, it does not clearly establish why the adjective "geometric" was chosen to apply to one sequence and not the other—thereby calling into question whether it is the correct source of data. Some participants avoided this rebuttal by providing additional historical context. In one such case, T-23's teacher-character explained that "in ancient times" measures of length, such as perimeter, were not considered geometric.

Other rebuttals hinged on the fact that the geometric mean did not immediately lend itself to generating subsequent terms of a geometric sequence, as seen in rebuttal-b in Figure 5. For example, T-17 recognized that the geometric mean could interpolate additional points between two known terms of a geometric sequence, but called into question whether this was a useful property: "Why would we ever be in a position where the nth element is unknown but its 2 neighbours are known?" Similarly, T-14 noted in Part 2 that she was unsatisfied with the geometric mean explanation because it "doesn't nicely describe the progression of numbers […] and it isn't exactly logical what number would come next in the sequence." The characters in T-6's script handled this concern by deriving the common ratio in terms of the geometric mean:

Allen: Uh… A, $\sqrt{AB}$, and B.
Teacher: Exactly! Short rectangle side, then square side, then long rectangle side, which makes A, $\sqrt{AB}$, and B. Now, what's the pattern here? […] To go from A to $\sqrt{AB}$, what do you have to multiply?
Allen: Multiply? Uh… $\sqrt{AB}/A$?
Teacher: Exactly! […] This "pattern" you found is the sequence: the geometric sequence.

Finally, T-11's script contained an example of rebuttal-c in Figure 5. This rebuttal was based on the fact that warranting the geometric mean explanation with a geometric representation actually obscured the underlying sequence. This is illustrated in the following conversation between the characters:

Teacher: […] This is why this progression is called a geometric progression since it's exactly like growing similar triangles.
Student: ………………………….. what is a similar triangle? I forget………………..
Teacher: I'll draw similar triangles for you. Here we have 3 similar triangles: There's the large triangle and the two small triangles.
Student: But where's the geometric sequence?

The visualization drawn by the teacher-character in this excerpt is the right triangle in Figure 4(b). The teacher-character does not answer the student-character's final question in Part 1, but the scriptwriter later explains in Part 3 that she might clarify the construction of this triangle for a mathematically mature colleague.

Discussion

Mathematics teachers must be careful that they do not take language for granted. This is especially true of mathematical names, which are (perhaps naively) expected to encapsulate the essence of the object they denote. An expert in mathematics might take familiar terminology for granted. But before a novice connects mathematical properties to a name, they might associate with it their personal connotations and experiences—which may sometimes cause conflict. It is an awareness of this fact that we hope to explore by assigning the geometric sequences task to teachers.

In analyzing participants' submissions, we considered the following research questions: *How do teachers justify a convention of mathematical terminology? In particular, what is the most common argument invoked by teachers to explain the adjective "geometric" in a reference to a geometric sequence?* Our findings indicate that the most common argument leveraged by teachers to explain why geometric sequences are in fact geometric relies on the relationship between a geometric sequence and the geometric mean. This approach necessitated a warrant justifying that the geometric mean is in fact geometric. Some participants handled this with a geometric diagram: either of a square that preserves the area of a given rectangle, or of a right triangle decomposed into two similar right triangles. In both cases, the fact that these diagrams were made of familiar polygons implied that they were genuinely "geometric."

In addressing our research questions, we considered the ways in which each explanation was considered insufficient to either student-characters or scriptwriters. These observations were captured in the findings as rebuttals. Here, we frame these rebuttals as the inability on the part of the core argument to meet an intellectual need (as in Harel, 2013). Rebuttal-a in Figure 5 does not satisfy the need for *structure*. For student-characters who voiced this rebuttal, the argument from geometric mean did not logically reorganize their understanding of mathematical terminology by meaningfully differentiating between geometric and arithmetic sequences. Rebuttal-b in Figure 5 does not satisfy the need for *computation*. Participants already knew how to compute the next term of a geometric sequence arithmetically by multiplying by the common ratio; the inability of the geometric mean argument to replicate that capability in a geometric context was therefore seen as a shortcoming. Finally, rebuttal-c in Figure 5 captures a need for *communication*. The student-character who voiced this rebuttal could not productively translate between their symbolic understanding of geometric sequences and the visual embedding of that concept in the triangle diagram.

Recasting other elements of an informal argument in light of intellectual need is one direction for future research; for example, how do certain warrants successfully attend to intellectual need? Another direction for additional research is to consider the origin of other mathematical terms. What is linear about linear algebra? What is natural about the natural logarithm? Whether posed to prospective teachers or to other students of mathematics, we anticipate that such questions will provoke meaningful introspection on the nature, structure, and history of the subject. This will, in turn, lead to further exemplification of the role of informal argumentation in mathematical discourse.

Our study contributes to research in mathematics education by developing the Toulmin-R model as a tool for representing informal arguments in which the goal is to establish a likely source of data rather than a meaningful conclusion. We have used this model in the context of explaining mathematical terminology; we suggest it could also be of use when a mathematical effect is observed but its cause is unclear. For example, a novice will eventually prove that the product of any two odd numbers is itself odd. Before their thinking is rigorously expressed by deductive proof, however, the Toulmin-R model might be leveraged to capture their informal arguments as they seek data leading to this conclusion.

This report also contributes to the body of knowledge on methods for mathematics teacher education. In their *Standards for Preparing Teachers of Mathematics*, the Association of Mathematics Teacher Educators recommend that to be “well-prepared,” teachers must “understand that mathematics is a human endeavor” (2017, p. 9). It is empowering for teachers to understand mathematics as the cumulative result of centuries of human effort—even though, as a result, it may sometimes seem disorganized or arbitrary. A well-prepared teacher should recognize that this is not always to the subject’s detriment. It may not be set out clearly why a geometric sequence is geometric, but by attempting to answer the question, teachers are apt to discover unexpected connections between areas of mathematics. Paradoxically, it is often those things that seem inexplicable that ultimately reveal a more cohesive and interrelated subject.

Acknowledgements

On behalf of all authors, the corresponding author states that there is no conflict of interest. This research was supported by a grant from the Social Sciences and Humanities Research Council of Canada. The datasets generated and analyzed during the current study are not publicly available due the fact that they constitute an excerpt of research in progress but are available from the corresponding author on reasonable request.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at amte.net/standards.

Bergman, A. M., Gallagher, K., & Zazkis, R. (in press). Prospective Teachers' Responses to Students' Dialogue on Fractions: Attribute Substitution and Heuristic Approaches. *Research in Mathematics Education*.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, *3*(2), 77-101.

Braun, V., & Clarke, V. (2019). Reflecting on reflexive thematic analysis. *Qualitative Research in Sport, Exercise and Health*, *11*(4), 589-597.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, *86*(3), 401-429.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, 26(1), 27-47.

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum. *For the learning of Mathematics*, *19*(3), 2-9.

Inglis, M., Mejia-Ramos, J. P., & Simpson, A. (2007). Modelling mathematical argumentation: The importance of qualification. *Educational Studies in Mathematics*, *66*(1), 3-21.

Kontorovich, I., & Zazkis, R. (2016). Turn vs. shape: Teachers cope with incompatible perspectives on angle. *Educational Studies in Mathematics*, 93(2), 223-243.

Marmur, O., Moutinho, I., & Zazkis, R. (2020). On the density of $\mathbb{Q}$ in $\mathbb{R}$: Imaginary dialogues scripted by undergraduate students. *International Journal of Mathematical Education in Science and Technology*, 1-29.

Marmur, O., & Zazkis, R. (2018). Space of fuzziness: Avoidance of deterministic decisions in the case of the inverse function. *Educational Studies in Mathematics*, 99(3), 261-275.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. Journal of Mathematics Teacher Education, 1(3), 243-267.

Nowell, L. S., Norris, J. M., White, D. E., & Moules, N. J. (2017). Thematic analysis: Striving to meet the trustworthiness criteria. *International journal of qualitative methods*, *16*(1).

Toulmin, S. E. (1958/2003). *The uses of argument*. Cambridge: Cambridge university press.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*(3), 247-261.

Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching versus teaching planning. *For the learning of mathematics*, *29*(1), 40-47.

Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). Lesson play in mathematics education: A tool for research and professional development. New York: Springer.

Zazkis, R., & Zazkis, D. (2014). Script writing in the mathematics classroom: Imaginary conversations on the structure of numbers. *Research in Mathematics Education*, 16(1), 54-70.

INVENTED STRATEGIES CHANGING TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

Jana Lunt
Southern Utah University
janalunt@suu.edu

This study investigates how utilizing student-invented strategies in the classroom can inform teachers' pedagogical content knowledge. Two elementary school teachers participated in professional development discussing the benefits of invented strategies. Data was then gathered as the participants implemented this practice in their classrooms. Data was analyzed qualitatively to show the ways in which invented strategies can be useful in a teacher's development of their pedagogical content knowledge, including their Knowledge of Content and Students, Knowledge of Content and Teaching, as well as Knowledge of Content and Curriculum.

Keywords: Teacher Knowledge, Instructional Activities and Practices, Elementary School Education, Curriculum

As one elementary school teacher in this study told me, "In 20 years of teaching I have never come across a math curriculum that encourages teachers to ask students to solve problems with their own invented strategies." In her experience, teaching in three different states, she had not been exposed to the importance of giving students time to develop their own algorithms. After the first day of implementing this practice, I talked with the teacher, expecting to hear validation of the research about students' capacity for thought and their flexibility in their understanding (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998; Chambers, 1996). I was surprised that the conversation centered on what she, as the teacher, had learned. Her understanding of the students, of her own teaching, and of the curriculum changed after watching students invent their own strategies to solve problems.

The teachers in this study reveal how their pedagogical content knowledge (PCK) increased. Some of this growth was pleasing to them. It amplified their current beliefs and ways of thinking. It resonated with past experiences and harmoniously pushed them in ways that excited them. On the other hand, some of this growth was jarring for the teachers to experience. They found this practice confusing, and claimed that it created dissonance between their past teaching practices and what they wanted to accomplish. This study investigates how teachers can increase their PCK directly from observing their students use invented strategies.

Objectives

We know the value of invented strategies for students (Carroll, 1999). In this study I begin to uncover the value of using student-invented strategies for the teachers themselves. My study is exploratory in nature and seeks out possible ways in which teachers can use this practice to change how they see and understand their students, what they teach and how they teach, and how they use the curriculum in their teaching.

Framework

My research uses Deborah Ball's construct of the mathematical knowledge needed for teaching (Ball, Phelps, & Thames, 2008). As Ball illustrates, there are several components of PCK. Knowledge of Content and Students (KCS) is knowledge that involves the dynamic between understanding specific mathematical concepts, and familiarity with students and their thinking about those concepts. This includes knowing student conceptions as well student misconceptions. Teachers need to be able to predict how students will perform on a task, and what students will find challenging, what students will find easy, and what students will find confusing. As students are developing their own concepts, even while those concepts are still incomplete, teachers need to be able to understand the students' language and ideas.

Knowledge of content and teaching (KCT), is knowledge that combines knowing about teaching and knowing about mathematics. Teachers need to understand the architecture of a task and what types of responses will be elicited by the design of the task. Teachers need to know how to sequence particular content for instruction. They need to discern the advantages and disadvantages of different tasks, manipulative, and technologies. When developing content with students, they need to make decisions about which student ideas to pursue, when to pause for clarification, and when to pursue a new avenue.

Knowledge of Content and Curriculum (KCC), the intersection between knowledge of mathematical content and knowledge of the curriculum from which it is taught, is not developed in the literature, but Ball theorizes that KCC is a factor in a teacher's PCK. As Hill suggests, currently KCC is an "interim placement…still in need of revision and refinement…it may run across several categories or be a category of its own" (Hill, Ball, & Schilling, 2008)

My study also relies on the importance of student-invented strategies in the classroom. Carpenter's longitudinal study of invention and understanding in multi-digit addition and subtraction with children shows that students benefit from inventing their own algorithms and strategies in a variety of ways (Carpenter, Franke, Jacobs, Fennema, & Empson, 1998). The authors explain that, in contrast to standard algorithms, invented strategies generally are derived directly from the underlying concepts. Therefore, students who are asked to devise and use their own strategies demonstrate knowledge of underlying concepts before students who relied primarily on standard algorithms. Invented strategies demonstrate a characteristic hallmark of understanding, and children who use them are able to use them flexibly to transfer their use to new situations.

In this study, I helped practicing elementary school teachers understand the benefits of invented strategies. They implemented this practice in their classrooms and, in doing so, they learned more about their students, they discovered benefits of and problems with their teaching methods, and changed their relationship to the curriculum.

Methods

After teaching a series of professional development courses at an elementary school, one second-grade teacher, Julie (all names used are pseudonyms), approached me and asked me to look over the text they were using to "see what was missing." I noticed that student-invented strategies were not addressed in the text, nor in her curriculum. She was very interested in exploring the use of this practice. For the second half of that school year, Julie intermittently gave time to students to try their own strategies before teaching them the standard algorithm. She was impressed with what the students gained from this experience, and I was surprised by and interested in what *she* was learning from the experience. This informal interaction developed into a pilot study. I began observing Julie teach each time she used invented strategies in her class and interviewed her afterwards. There was so much she was learning that we decided to continue the study the following school year and to include other teachers at the school.

Julie approached all of the other second and third-grade teachers at her school to see if they were interested in participating. Over the summer I met with four teachers to discuss the research regarding invented strategies and the benefits for students. Unfortunately, by the end of the summer two of the teachers had left the school. This was the summer of 2020, and it was unclear if we were going to be able to continue the study at all because of the COVID-19 pandemic. However, this particular school district did teach face-to-face for the 2020-21 academic year. So, although the number of participating teachers was down to just two, these two teachers were still willing to take on this new challenge, in the midst of all of the other pandemic-related difficulties, and learn more about student-invented strategies.

I met with Julie, a second-grade teacher, and Kamila, a third-grade teacher, several times during the summer and at the beginning of the school year to discuss general benefits of invented algorithms. We focused on things teachers should know about the importance of and reasons for employing invented algorithms (Trafton, P., & Thiessen, D., 2004). We also discussed content-specific invented strategies such as children's conceptual structures for addition and subtraction (Fuson, Wearne, Hiebert, Murray, Human, Olivier, & Fennema, 1997; Huinker, Freckman, & Steinmeyer, 2003). The teachers decided, completely on their own, how they would use this knowledge to change and enhance their teaching over the following year. Both teachers decided to use invented strategies before each section that involved a new algorithm.

I observed and took notes during any class time during which teachers were using invented strategies. I made copies of materials used by the teacher, took photographs of board work, and recorded other relevant data including occasional student work. After each observed class, I interviewed the teachers. I audio-recorded those interviews as well as took field notes. The field notes, curricular materials, board work, student work, and teacher interviews were used to analyze how the teachers' PCK was being affected by utilizing student-invented strategies.

The analysis was conducted qualitatively. Learning what the teachers noticed, what surprised them, what they expected, and what changes they made to future lessons due to their findings all provided me with an important avenue for investigating what they were learning. I reviewed the materials looking for patterns, and ultimately found evidence of several developments in the teachers. It was not until after I looked for patterns that I began thinking of the teachers' growth as improving their PCK. This became the avenue for how I grounded by research and began analyzing the data

Results

I discovered ways in which students using invented algorithms informed each aspect of the teachers' PCK. The following paragraphs will highlight some of the ways that teachers experienced change in their Knowledge of Content and Students (KCS), Knowledge of Content and Teaching, (KCT) as well as possibly Knowledge of Content and Curriculum (KCC).

Examples of Change in Teacher's Knowledge of Content and Students

Students as imparters and constructors of knowledge. Because students were given more of a voice in the classroom, teachers were able to get to know their students' capabilities on a deeper level. The teacher, as well as the students, began to see the children as 'imparters of information," according to Julie. Everyone in the classroom saw the students as "capable of knowing and sharing things" and not just the teacher. Julie and Kamila frequently described how they watched the students build confidence as they displayed their thinking to others or made connections to previous strategies. As Julie said, "I'd never asked what they already know. I know much more about that now." The teachers were able to see even their struggling students in a different way, focusing not just on what these students did not understand, but also seeing what they *did* understand. After a day of student-invented strategies Julie reported, "One of my struggling students today was able to solve a subtraction problem on her own today, without having been taught how. I think the flexibility in solving problems in different ways has enabled her to get better at using the strategy that makes most sense to her. Even when posed with the traditional algorithm she uses drawings to help her solve problems and is making great gains. This is so exciting!"

Students spent a lot of time comparing and critiquing other students' strategies. The teachers realized this forced students to explain themselves more clearly and visibly. It encouraged everyone to participate. As Kamila said, "Everyone is working on *something*. No one is just sitting there. They are all willing to come up and share and talk about strategies. They feel successful and proud." Both teachers saw the benefits of students working with partners. They acknowledged that they often used partner-work with reading, but rarely with math. Now they could see how working in partnerships allowed students to help one another, and built confidence and ownership of their work.

Julie used invented strategies in the previous school year, as part of our pilot study. One day after school, one of the students from the previous year approached Julie and asked, "Did you teach them my strategy yet?" Julie did not immediately remember which strategy the student claimed as hers, but the student quickly reminded her. Julie told me, "It made me realize how powerful invented strategies are because kids take ownership of them. They are the constructors of their own knowledge. For her to remember that a year later made quite an impression on me!" The teachers began to look at their students differently: they were now capable of constructing their own knowledge and sharing it with other students.

Student learning styles. The teachers frequently talked about understanding individual students' ways of thinking more clearly. They did not only view their class as a whole, but the teachers could describe each student, what types of strategies each preferred and which strategies posed a struggle. This understanding helped the teachers group students more effectively. Julie described this process:

> After the lesson I sorted the student work into three categories: 1) students who have at least one strategy and were able to solve all of the problems, 2) students who had a start (part of a strategy or a whole strategy, but may have had a misconception or mistakes, and 3) those who don't seem to have any strategies at their disposal. In thinking about these groups of students, it confirmed for me which students have solid strategies and can work independently and most likely transfer these skills. It also showed me which students would benefit from scaffolding…which students need to work with concrete base ten blocks before working with pictures or abstract problems.

Teachers could describe individual students' learning styles after watching them repeatedly invent strategies because they could see which strategies their students relied on, and which strategies the students were using when they made mistakes. As one teacher reported, "When kids get to the standard algorithm they won't use it if they don't understand it. They will fall back on other strategies they have used." The teachers were more aware of how their students thought, how the students could best express themselves, and what tools helped them learn.

Student misconceptions. Julie and Kamila are seasoned teachers, each having taught their grade level for over 20 years. They were both surprised that letting students use invented strategies revealed student misconceptions of which they were previously unaware. For example, many of the strategies in these grades relied on an understanding of place value. But, as some students used their invented strategies, it became clear to the teachers that the students had unanticipated misconceptions about place value. Frequently, these misconceptions surfaced when students were discussing their strategies and they would use words incorrectly, demonstrating to the teachers that the students did not fully understand the math concept behind the words.

The teachers were more informed when words such as “equal” or “whole,” for example, were concepts that needed more instruction. It was also apparent to teachers when students did not understand *why* they were doing certain algorithms. Teachers described identifying a lack of connection between tasks within a unit that was building to a cumulative goal. There were many times when teachers thought the students understood a topic, but recognized that the understanding was not there when the teachers invited the students to build on that topic for the next topic, using student invented-strategies.

Student transfer. The teachers were very concerned about and interested in transfer as they discussed their students with me. Sometimes the teachers saw the students using previous knowledge to solve new problems more than they had expected. They saw invented strategies as a tool that students used to help make connections between different topics. There were also many times when the teachers expressed discouragement that the students were *not* able to build on what they had discussed previously. In some ways, invented strategies left as many questions as answers when it came to students’ ability to transfer knowledge. After a year of using this teaching strategy, the teachers are still interested in invented strategies to investigate transfer.

Examples of Change in Teacher’s Knowledge of Content and Teaching

Assessment. The teachers immediately saw invented strategies as a form of assessment. Julie described using invented strategies, and asking the students to articulate their thinking in writing, as one of the best things she could use to see what her students were understanding. “That was more helpful than seeing the problem and how they solved it,” she explained. Julie now wants to use invented strategies mid-chapter as a formative assessment in her teaching.

Julie and Kamila also found themselves thinking more about assessment in general after using invented strategies. They examined the type of assessments they were using and realized a need for changes. In a conversation with the two teachers they said, “There needs to be some changes on a school level about how students are assessed. There should be more one-on-one assessment with math. We do it with reading; why don’t we do that in math?”

In addition to assessment at the end of, or even during a chapter, the teachers realized invented strategies were an excellent tool to use for pre-assessment. After the very first day that Julie taught with invented strategies she said, “Truly today worked as a pre-assessment and helped me understand what skills my students already have in this arena, as well as helped me identify some action steps for how to address their needs.” The teachers were looking to see how students were able to build on concepts they had learned previously and what new boundaries they could investigate all on their own.

Sometimes the pre-assessment went well and sometimes it did not. When the pre-assessment did not go well, the teachers learned they might need to go back and address concepts from previous lessons. As Kamila described, “If their foundation isn’t there, they can’t see it. They can’t move forward.” Based on how the students performed when trying strategies of their own, the teachers were able to make decisions about the upcoming lessons. The teachers were able to look beyond whether the students performed well or not on the pre-assessment. They were able to decide which concepts they needed to review more, the tasks that would most benefit their students, which students were best suited to work together, and which students needed more instruction.

Manipulatives. At the end of the study, when asking the teachers how they had changed, Kamila expressed that her relationship to manipulatives possibly had changed more than anything else. She said, “I am the type of person that I like to have everything in its place, but now my intention is to have manipulatives accessible so that students can take any tool they feel will help. I already have this math station envisioned in my mind and am so excited about it!” She was looking forward to a time (past the pandemic) when it would be possible to set up a table of manipulatives for students to use at their discretion.

Because of COVID-19, each of Julie’s students had access to their own individual bag of manipulatives that they could use anytime they chose. Even if individualized bags are no longer necessary after the pandemic, Julie plans to continue this practice so that her students have better access to a larger number and a larger variety of manipulatives.

Along with deciding that students need to have more freedom and more choices with manipulatives, teachers were able to tell when, as a classroom, they needed to revert to using a tangible tool instead of proceeding more abstractly. For example, when transitioning from single digit subtraction to double digit subtraction in Julie’s classroom, she noticed that many of the students who tried their version of the traditional algorithm were subtracting the top number from the bottom number. She said this helped her understand that “I need to go back to using blocks and have them see that they cannot subtract in any order.” Using invented strategies helped her see when it would be helpful to revisit using manipulatives.

The teachers also saw how more use of manipulatives changed the students. Julie noticed that her students seemed less intimidated by new problems that had not been explained to them. The students knew they could find a tool that could help them think through the problems. “Everyone has a starting point now,” explained Kamila. The teachers found value in helping students increase their confidence and independence by giving students autonomous access to manipulatives when using invented strategies.

Student-led discussions. According to the teachers, using invented strategies has guided them to a more inquiry-based approach for teaching. Julie learned that getting her students to think deeply about a topic was more easily done as a discussion that was student-led and centered on their own approaches. She realized that when she had been relying on the textbook, the discussions were always teacher-led. She could see that even when the class was not discussing invented strategies, if she wanted deeper thinking, students needed to become more involved. She resolved to plan for times when students would lead the dialogue in all subjects.

Kamila discovered that, when she asked her students a question about their strategy, the students initially would assume that they were wrong. This helped her understand that she was not routinely asking enough questions about how the students were thinking. Both teachers suggested that using invented strategies has taught them to use questioning in a different way. They are more likely to ask questions such as, “What do you already know?” or “What strategy have you used before that could help you?” They describe using these questions as a way to maintain a high cognitive demand in the tasks, allowing students to think more deeply over a longer period of time. In a reflection at the end of the school year Julie said she wants to include “at least three thinking questions in every math class, no matter if it is a textbook lesson or an invented strategy lesson.” Using invented strategies changed her teaching style to include more questioning, which elicited student-led conversations.

Examples of Change in Teacher’s Knowledge of Content and Curriculum

Previous research is unclear about the details of PCK with respect to the teacher’s knowledge of content and curriculum. As I was listening to the teachers talk about their relationship to the textbook and how it was changing, I saw possibilities of how using student-invented strategies could change a teacher’s KCC.

Freedom from the textbook. After using invented strategies, teachers decided that they were relying on the textbook too heavily for their curriculum. According to Kamila, “the Common Core was, at least in part, designed to encourage students to think more critically. But we are still teaching systematically. We give students more freedom in reading. We need to do that more with math.” Both teachers realized this could not be done if they taught straight from the textbook, as they had previously been doing. In a conversation with Kamila, Julie said, “Teachers need to feel enough freedom, and trust themselves enough, to not rely on the text. There needs to be willingness from teachers and administration to supplement the text so kids are getting opportunities to develop conceptual understanding.” She said she felt an expectation to teach with fidelity to the text but that “if we do it with fidelity, we will never create deep thinkers.”

Students need to have an opportunity to select and invent strategies and, in their experience, the text always tells them which strategy to use. The teachers decided that metacognition is not present when students work from their books because students are merely following procedures, compared to using invented strategies which gives students opportunities to think about their thinking. Kamila decided, “teaching the Common core is what we need to do with fidelity, not the textbook.”

Thinking critically while using textbook. Even when using the textbook, teachers realized they need to be thinking more about their pedagogical choices. They realized that they could still facilitate student-led discussions and group work, and start a lesson with invented strategies, while still approaching the lesson from the text. In their opinion, most textbook lessons were not complex enough, so they could incorporate activities such as making an argument and critiquing others’ work using textbook lessons. One of their conclusions at the end of the study, one they had not previously acknowledged, was that they could omit some parts of the units in the curriculum and add in other lessons. Invented strategies helped them feel freer to make pedagogical decisions. They could decide which units needed to be longer or shorter, and use their assessments to guide their practice. They could use modeling, even if it was not an explicit part of the textbook lesson. They learned that they needed to decide how to balance things like invented strategies versus the textbook’s specific algorithms.

The teachers also realized that there were parts of the text they now understood better. By the end of the year, they discovered sections of the text that were intended as inquiry-based class activities, but they were not able to recognize them as such before they started using invented strategies. Kamila described a section in the textbook where the students were asked to write about their thinking. She realized she had not fully utilized that section in previous years, but was now able to see the purpose of the activity. Kamila and Julie both believed that teachers would benefit from professional development regarding textbook use, discussing how to encourage deeper levels of thinking among students and assessing the curriculum as a group.

Discussion

Learning about new ideas and changing teaching practice is both exciting and interesting, as well as difficult and discouraging. Kamila and Julie felt all of these emotions as they were asked to think more critically about their PCK.

Critical Dissonance

It was jarring for the teachers to watch their students fail to see patterns, make connections, or transfer knowledge in unanticipated ways when the teachers asked students to use invented strategies. This practice illuminated problems in ways that were occasionally surprising and disheartening. Teachers sometimes finished a lesson and realized students did not understand the topic to the extent the teachers had supposed. Teachers sometimes realized that they had been using a word or an expression for some time that the students did not actually understand, but the teachers did not discover this until the students tried to employ their own strategies. These realizations were sometimes painful for the teachers, causing teachers to second-guess past lessons, and wondering what they had missed before. The teachers frequently wished that they had more time.

Resonant Harmony

Of course, much of what the teachers learned resonated with them perfectly. Imparting a stronger voice to their students helped the teachers accomplish their goals. Because of COVID-19 each student was given their own set of manipulatives to use. This was a providential turn of events that led to teachers learning that students need better access to manipulatives. It was not a challenging new idea; the teachers already knew the value of manipulatives. However, they learned how to improve the implementation of this idea.

In addition, learning that their students need to work more with others fit perfectly with the teachers' PCK. They were already implementing this practice in other subjects, but they realized that they were not doing it enough in math. These realizations amplified and improved their teaching, helping them push through the pandemic's obstacles to find a better way.

One of the most remarkable things about teachers such as these is their ability to change. These teachers took in new information about PCK and used it to change the experience of their students. Regardless of whether the change in PCK was difficult or easy for them, these teachers knew that it was necessary to improve their teaching. As Julie said, "Invented strategies are a way to open up the dialogue to include more student-directed discussion, to enable teacher reflection and analysis of student work, and to allow students to build on their conceptual understanding. I will include invented strategies in future lessons to ensure more student voice and encourage a more authentic mathematics community."

Limitations

There were many limitations to the study, specifically because of COVID-19. I had four teachers who were excited about participating and were planning on recruiting others in their grade bands before the pandemic hit. Then two of the teachers decided it was a good time to retire from teaching and others realized that, with all the difficulties that were facing them, it was not a good time to be in a study. So only being able to work with two teachers was very limiting.

Additionally, rules for being in classrooms at this time became a lot stricter. Schools were not interested in having more people interact with students and teachers. In fact, the only way I was able to attend the classrooms was to become a teacher aide. I was not able to conduct interviews with or film the students, but I did interact quite a bit with them and even served in other capacities, such as a running reading groups and going on field trips. This was a major reason for shifting the data collecting to only interviews with the teachers. This seemed permissible because the teachers were truly the participants of the study, but videos of the classroom would clearly have been helpful for data analysis.

References

Ball, D. L., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? J*ournal for Research in Mathematics Education*, 59(5), 389-407.

Carpenter, T. P., Franke, M. L., Jacobs, V. R., Fennema, E., & Empson, S. B. (1998). A longitudinal study of invention and understanding in children's multidigit addition and subtraction. *Journal for research in mathematics education*, 29(1), 3-20.

Carroll, W. M. (1999). Invented computational procedures of students in a standards-based curriculum. *The Journal of Mathematical Behavior*, *18*(2), 111-121.
Chambers, D. L. (1996). Direct modeling and invented procedures: Building on students' informal strategies. *Teaching Children Mathematics*, *3*(2), 92-96.
Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education,* 28(2), 130-162.
Hill, H.C., Ball, D.L., & Schilling, S.G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.
Huinker, D., Freckman, J. L., & Steinmeyer, M. B. (2003). Subtraction strategies from children's thinking: Moving toward fluency with greater numbers. *Teaching Children Mathematics*, *9*(6), 347-353.
Trafton, P., & Thiessen, D. (2004). Linking mental and written computation via extended work with invented strategies. *Beyond written computation*, 113-125.

TEACHER-RESPONSES: HIGHLIGHTING CHARACTERISTICS OF LOW RESPONSE PROCESS VALIDITY FOR ITEM(S) MEASURING TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

Martha L. Epstein
UMass Dartmouth
mepstein1@umassd.edu

Hamza Malik
UMass Dartmouth
hmalik1@umassd.edu

Kun Wang
UMass Dartmouth
kwang1@umassd.edu

Chandra Hawley Orrill
UMass Dartmouth
corrill@umassd.edu

Response Process Validity (RPV) reflects the degree to which items are interpreted as intended by item developers. In this study, teacher responses to constructed response (CR) items to assess pedagogical content knowledge (PCK) of middle school mathematics teachers were evaluated to determine what types of teacher responses signaled weak RPV. We analyzed 38 CR pilot items on proportional reasoning across up to 13 middle school mathematics teachers per item. By coding teacher responses and using think-alouds, we found teachers' responses deemed indicative of low item RPV often had one of the following characteristics: vague answers, unanticipated assumptions, a focus on unintended topics, and paraphrasing. To develop a diverse pool of items with strong RPV, we suggest it is helpful to be aware of these symptoms, use them to consider how to improve items, and then revise and retest items accordingly.

Keywords: Item development; constructed response; pedagogical content knowledge; response process validity

Purpose of the Study

The mathematics education community has shown a growing interest in mathematics teacher knowledge assessment (e.g., Izsák et al., 2016 and Mosvold & Hoover, 2016). There is still much to do to understand the domain of mathematics teacher knowledge as well as to understand how to best assess this knowledge through a variety of types of scenarios and both CR (constructed response) and SR (selected response) types of items. When creating items for any assessment, one important task in which developers should engage is the determination of whether items exhibit response process validity (RPV). RPV is a measure of whether the person reading the items understands them in a way that is intended by the item developers. Our goal in this study was to determine the reasons PCK items may fail to demonstrate RPV. By being aware of symptoms of low-RPV, item developers can work to improve the assessment items, accordingly. Engaging in the RPV process can allow test developers to move beyond a binary decision to keep an item, or not, and may facilitate the development of assessment items with strong RPV that can also tap more complex, hard to communicate concepts that may take several rounds of revision to build strong RPV. We were guided by the following research question: What characteristics of middle school mathematics teachers' responses to CR PCK assessment items suggest low RPV?

Perspective and Review of Relevant Literature

Effective assessment development requires that relevant content domain (e.g., elements of proportional reasoning) first be successfully identified (Orrill & Cohen, 2016). For example, Hill and colleagues' (2008) efforts to measure mathematical knowledge for teaching, started by defining the knowledge domain they hoped to measure. Defining the knowledge domain is challenging. Rowan et al. (2001) noted, “One difficulty we faced was developing items (and scenarios) that adequately tapped the full range of underlying “abilities” or “levels” of teachers’ content and pedagogical knowledge…” (p. 16). Once the domain has been adequately conceptualized, the work of trying to measure that domain begins with the development of assessment items. Item development is complicated, in part, because the items must invoke the intended knowledge from test takers. This means that a participant needs to understand the question being asked in ways that align to the developer’s intent. Therefore, it is important to determine the RPV of items. RPV has garnered recent attention in STEM education (e.g., Deng et al., 2021; Padilla & Benítez, 2014), and think-alouds is one method used to investigate RPV. Using methods such as think-alouds (Bostic, 2021), researchers sought to determine the match between test developer intent and test taker interpretation and to garner insights into what may be causing any mismatches (e.g., Mo et al., 2021). An area of opportunity is to continue to investigate sources and symptoms of low RPV and to work to hone our RPV research methods, so they lead to diverse items with strong RPV.

Methods

Context and Participants

Participants included a convenience sample of 13 middle school mathematics teachers from across the United States (nine female, four male). All the teachers were given pseudonyms. Assessment items used in this research study came from an assessment development effort seeking to measure mathematics teachers’ content knowledge and PCK about proportional reasoning. The findings reported here are based on an analysis of only PCK items, all of which were constructed response. The items all involved asking teachers about realistic classroom scenarios (i.e., student work and classroom situations) and were based on Kersting and her colleagues’ (Kersting, 2008; Kersting et al., 2012) successful work with similar items. Many of the items included short video clips that the teachers were asked to comment on.

Data Collection and Analysis

The 38 CR assessment items evaluated were spread across five surveys completed online. Depending on their schedule, teachers completed between one and five surveys and typed their responses to items. A follow-up Zoom think-aloud interview was recorded and transcribed. A coding template was developed to facilitate the analysis process that involved three researchers independently reviewing each item to determine, (1) if the item communicated as intended, (2) if the item did not communicate as intended, how had the teacher likely interpreted the item, and (3) if the item did not communicate as intended, characteristics of teachers’ responses that led to this conclusion. Divergent views triggered a review of a teacher’s data until the research team reached a consensus regarding how the assessment item “worked” for that teacher (e.g., did the teacher understand it as intended and did it invoke the kinds of reasoning intended).

Results

Teacher responses that created RPV issues tended to be: vague/overly general, predicated on assumptions, focused on unintended topics, and/or paraphrased information already provided. Below, we expand on our findings for each of these characteristics.

Vague, Overly General Answers

The intent of all CR PCK items was to elicit answers from teachers that provided rich insights into their PCK. Vague, overly general teacher answers to questions were, therefore the antithesis of the types of answers item developers had hoped to solicit, thus, such answered were considered indicative of low item RPV. As an example of one vague response, participants were asked to watch a short video of a 7th-grade classroom discussing proportions. They were then asked, "Would you have led the class discussion in this video clip differently to support the student's understanding of proportional relationships?" If "yes," "explain how you would have led the discussion differently to support student's understanding of proportional relationships." Seven out of 10 teachers' responses to this item were coded as vague/overly general. For example, Christie offered, "…getting input for more than one student and perhaps leading them in a direction to the correct answer using keywords from students." We considered this and other answers like it vague or general because Christie did not reference specific mathematical concepts, and while she mentioned she would look for "keywords," she did not provide detail about which key math words she was looking for or how she could help students make relevant connections among them to support an understanding of proportional relationships. We found vague/overly general responses were often associated with items that: were not specific enough, provided insufficient information, contained distracting elements, and/or did not adequately take the test-takers vantage point into account.

Answering Based on Assumptions

We considered another indicator of low item RPV when teacher responses suggested that teachers made assumptions to answer an item. Assumptions were deemed problematic because if test takers are making different assumptions, they are, in essence, answering different assessment questions. For example, for an item in which teachers were asked to watch a video and comment on a student's understanding, Lydia, noted, "I think it was difficult to understand what he [the student] truly understood…" and Lydia went on to tell us that she "made an assumption about what he was sort of thinking …" In other instances, teacher responses suggested an assumption may have been inadvertently made which altered the question being asked. One such item presented an inverse proportion task (i.e., $y = k/x$), and the student's use of cross multiplication to solve it—a strategy appropriate for solving proportional problems; not inverse proportions—was therefore problematic. Yet, seven out of 10 teachers' answers were predicated on the assumption or assessment that the student's work was correct, when item RPV depended on teachers recognizing the student's work was not correct. In summary, when teacher responses explicitly or implicitly suggested an answer was based on an unanticipated assumption, we considered it a sign of poor RPV. We found teacher responses that were based on assumptions tend to be associated with items that provided insufficient information and/or did not ask specific enough questions.

Focusing on Unintended Topics

We deemed an item to have low RPV if teachers' responses focused on an unintended topic, as this suggested they had interpreted the item quite differently than anticipated by item developers. An item in our research study that resulted in several responses that did not address the intended topic included a graph and table that featured a proportional relationship between mango weight and cost. The item stated, "how could you help students understand how the key characteristics of proportional relationships are demonstrated in both representations?". This item's goal was to solicit PCK regarding helping students build conceptual understanding of proportions across different types of representations. The item's goal was not to tap PCK regarding optimal graph labeling. However, three of nine teachers' answers focused solely on the suboptimal qualities of the graph. For example, Christie noted "you don't see any clear ordered pairs. If it was in units of one on the y axis, I think it would be easier for the students to visualize the proportionality." Emma noted, "we can create ordered pairs: (1,6) (2,12), etc. Then I would place these points on the graph, so they can see that the points all fall on the line on the graph." In our study, we noticed teachers tended to have off-topic responses when an item contained a distracting element, was not specific enough, or did not adequately take the test-takers vantage point into account.

Paraphrasing the Information

Some teachers responded to items by simply paraphrasing information provided to them in the item scenario. No items were written asking participants to summarize the information provided. Instead, all items were designed to solicit PCK specific to a scenario. Hence, we considered paraphrasing a sign of RPV issues. In one example, teachers viewed a video clip in which a student (Evan) explained why he thought two ratios (6/14 and 15/35) were equivalent. Teachers were asked to "Comment on Evan's understanding based on his method, 'If you divide down, you'll get the same answer'." The intent was for teachers to use information about Evan and their own PCK to project what else Evan likely understood. Seven out of 10 teachers' responses did not go beyond paraphrasing Evan's response. For example, Christie noted that Evan "… understands that you can check for equivalent fractions by simplifying them." Similarly, Emma offered, "Evan understands that equivalent ratios will always simplify to the same numbers." In our study, paraphrasing was linked with overly general questions as well as items that did not adequately reflect the test-takers' perspective.

Discussion

Soliciting teacher feedback on assessment items via think-aloud or other relevant methodology is critical for developing items with high RPV. We posit that it is not only important to perform such RPV research to refine a given assessment, but also it is important to share "lessons learned" so that other research teams can benefit from better understanding potential pitfalls. Our findings are situated in one study, and we do not suggest the four response characteristics we found in our study are exhaustive of responses that signal RPV issues. We suggest identifying responses that signal low RPV is most beneficial if it is used to drive subsequent refinement of assessment items. We posit that using RPV as the basis for refinement may allow developers to create assessments that better measure challenging and harder to communicate ideas. We hope by sharing our analysis of "what teachers told us" in our assessment pilot items we will enhance existing knowledge for item development as well as trigger discussion regarding how one's RPV methodology may impact the diversity of assessment items that ultimately result.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1813760. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. We would also like to acknowledge Dr. Yasemin Copur-Gencturk for her support in providing teachers' dataset and other useful information for this research study.

References

Orrill, C. H., & Cohen, A. S. (2016). Why defining the construct matters: An examination of teacher knowledge using different lenses on one assessment. *The Mathematics Enthusiast*, *13*(1), 93-110.

Bonner, S., Chen, P., Jones, K., & Milonovich, B. (2021). Formative assessment of computational thinking: Cognitive and metacognitive processes. *Applied Measurement in Education, 34*(1), 27-45.

Bostic, J. D. (2021). Think alouds: Informing scholarship and broadening partnerships through assessment. *Applied Measurement in Education, 34*(1), 1-9.
de Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*(3), 311-334.
Deng, J. M., Streja, N., & Flynn, A. B. (2021). Response process validity evidence in chemistry education research. *Journal of Chemical Education, 98*(12), 3656-3666.
Dolan, R. P., Burling, K., Harms, M., Strain-Seymour, E., Way, W. D., & Rose, D. H. (2013). *A universal design for learning-based framework for designing accessible technology-enhanced assessments.* Pearson Assessment Research Report. http://images.pearsonclinical.com/images/tmrs/DolanUDL-TEAFramework_final3.pdf
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.
Hogan, T. P., & Murphy, G. (2007). Recommendations for preparing and scoring constructed-response items: What the experts say. *Applied Measurement in Education, 20*(4), 427-441.
Izsák, A., Remillard, J. T., & Templin, J. (2016). Psychometric methods in mathematics education [Monograph]. *Journal for Research in Mathematics Education*.
Kersting, N. (2008). Using video clips of mathematics classroom instruction as item prompts to measure teachers' knowledge of teaching mathematics. *Educational and Psychological Measurement, 68*(5), 845-861.
Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. *American Educational Research Journal, 49*(3), 568-589.
Mo, Y., Carney, M., Cavey, L., & Totorica, T. (2021). Using think-alouds for response process evidence of teacher attentiveness. *Applied Measurement in Education, 34*(1), 10-26. https://doi.org/10.1080/08957347.2020.1835910
Mosvold, R., & Hoover, M. (2016). *The Mathematics Enthusiast, 13*(1-2).
Padilla, J. L., & Benítez, I. (2014). Validity evidence based on response processes. *Psicothema, 26*(1), 136-144.
Rowan, B., Schilling, S. G., Ball, D. L., Miller, R., Atkins-Burnett, S., & Camburn, E. (2001). Measuring teachers' pedagogical content knowledge in surveys: An exploratory study. *Ann Arbor: Consortium for Policy Research in Education, University of Pennsylvania, 1*, 1-20.
Salkind, N. J. (2017). Tests & measurement for people who (think they) hate tests & measurement. Sage Publications.

TYPES OF MATHEMATICAL REASONING EVIDENCED BY A MIDDLE SCHOOL TEACHER IN PATTERN GENERALIZATION

Karina Nuñez-Gutierrez
Universidad Autónoma de Guerrero
kgutierrez@uagro.mx

Guadalupe Cabañas-Sánchez
Universidad Autónoma de Guerrero
gcabanas@uagro.mx

The objective of this article is to describe types of mathematical reasoning evidenced by a middle school mathematics teacher, when answering two generalization questions in a figural pattern generalization task, related to quadratic sequences. Reasoning is delimited from teacher's arguments, reconstructed from a theoretical-methodological proposal that combines Peirce's definitions with some elements of Toulmin's argumentative model. The results show that the teacher evidenced abductive, inductive, and deductive reasoning, based on cognitive actions such as the decomposition of the figure, strategic counts, recognition of the behavior of the figural pattern, formulation, verification, and validation of conjectures.

Keywords: reasoning, argumentation, figural pattern, teacher.

Introduction

Reasoning is one of the fundamental cognitive processes in teaching and learning mathematics. Organizations such as the National Council of Teachers of Mathematics ([NCTM], 2000) conceive of reasoning as a skill or proficiency that must be developed at all levels. On this regard, teacher assumes an important role to promote it in students through different mathematical experiences in teaching conditions (Brodie, 2010). Research has reported that teacher have limited knowledge both the nature of reasoning and pedagogical knowledge to support development in their students (Clarke et al., 2012). On the other hand, the investigations on mathematical reasoning that have taken as a case study to the preservice teacher and in service have been of an exploratory nature (e.g., Bozkus & Ayvaz, 2018; Mata-Pereira & Ponte, 2017) and teaching proposals to develop it (e.g., Bragg & Herbert, 2018).

The study of mathematical reasoning has focused mainly on the central and classic forms of reasoning: abductive, inductive, and deductive (Arce & Conejo, 2019; Conner et al., 2014; Soler-Álvarez & Manrique, 2014), which, in their most have showed cognitive processes developed by middle school students and future mathematics teachers when solving tasks in arithmetic and geometry context. They connected the structures of Peirce and Toulmin in their analysis to show through the arguments, the reasoning they evidenced when solving them. However, the same has not happened with the study of these forms of reasoning about the in-service mathematics teacher. Patterns are one of the contexts that favor the study of reasoning because it contributes to the formulation and justification of generalizations (e.g., Barbosa & Vale, 2015; Rivera, 2013; Mata-Pereira & Ponte, 2017). Kirwan (2015) states that if students are expected to generalize and justify, then it is important to reflect on teachers' thinking about these concepts. The research is focused in describing the mathematical reasoning that is evidenced a middle school mathematics teacher in the framework of the generalization of quadratic patterns. Patterns are one of the contexts that favor the study of reasoning because it contributes to the formulation and justification of generalizations (e.g., Barbosa & Vale, 2015; Rivera, 2013; Mata-Pereira & Ponte, 2017). The research question of the study is, what types of mathematical reasoning does a mathematics teacher evidence when solving a quadratic pattern generalization task?

Theoretical Framework

Mathematical reasoning

In mathematics education there is no consensus on the definition of mathematical reasoning, because it is a polysemic term that encompasses a wide range of mathematical practices (Conner et al., 2014; Yackel & Hanna,2003). In this research, it is understood as any action or procedure that allows obtaining new information from: a) previous or known information, which corresponds to that provided by a statement as initial hypotheses; b) resolution of a problem or c) that derived from previous knowledge (Saorin et al., 2019; Torregrosa et al., 2010). In addition, it is related to other processes, such as inference, justification, and generalization (McCluskey et al., 2016).

Argumentation and argument

Argumentation is a sequential process that allows conclusions to be inferred from premises, through interactive communication between people (Toulmin, 1958/2003). Mathematical reasoning groups a set of arguments based on a series of propositions that implies a conclusion inferred from the data (Toulmin et al.,1984). The argument is a complex data structure that involves a movement from data (D) to a conclusion (C). The movement of the evidence to the conclusion is the certainty that the argumentative line has been carried out successfully, a movement or connection that is allowed by the warrant (W), which in turn has a backing (B), a modal qualifier (Q) that indicates the degree of strength or probability of the assertion and occasionally, objections or refutations may be presented (R). Toulmin´s model is considered to analyze the content of the arguments (Inglis et al., 2007) (see Figure 1).

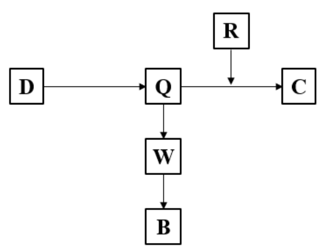


Figure 1: Toulmin's argumentative model (Inglis et al., 2007).

The study of arguments in this research is based on the core of the argumentative Toulmin´s Model (See figure 2). This structure allows to identify the typology or mathematical reasoning.

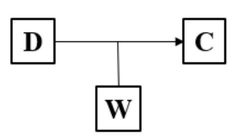


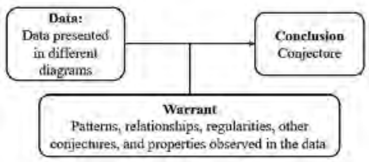
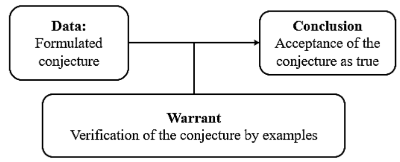
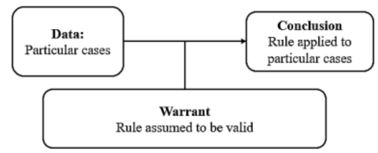
Figure 2: Toulmin's basic argumentative model.

Types of Reasoning

Mathematical reasoning is classified into three types: abductive, inductive, and deductive. *Abduction* and *induction* represent plausible and experimental reasoning, supported by the formulation and verification of conjectures, while *deduction* consists of validating the conjecture to warrant its veracity (Polya, 1966). We are based on the theoretical-methodological proposal of

Soler-Álvarez and Manrique (2014) (see Table 4) for the identification and description of the types of mathematical reasoning.

Table 1: General schemes of the types of mathematical reasoning (Soler-Álvarez & Manríque, 2014).

Abductive	Inductive	Deductive
		

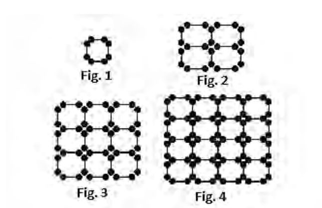
Generalization

When a conjecture expresses a plausible mathematical rule related to all the particular cases and it represents the mathematical pattern, we speak of generalization, which consists of the process of identifying a regular behavior in some particular cases of a sequence, in order to extend that identified regularity and construct an expression or general rule that represents and relates all the terms of the sequence (Radford, 2008).

Method

The research is a qualitative case study (Merriam & Tisdell, 2015). It was developed through a course-workshop (CW) in a virtual setting. Participants were 16 middle school mathematics teachers (MMT) from three Latin American countries. One of these teachers was selected for the analysis of the case study unit with the following criteria: a) to solve seven quadratic pattern generalization tasks and b) to participate in a semi-structured interview. For the purposes of this article, the analysis focuses on one of the tasks (see Table 2).

Tabla 2: Quadratic pattern generalization task.

Task of the points and sides	Characteristic
Considering the figures, answer the following questions and give a detailed argument for your answer. a. Find a general rule for the total number of points in any figure. b. Find a general rule for the total number of sides in any figure. 	Adapted from Rivera (2010). The context of the task is a figural pattern. The explanation of the behavior of the pattern in any of its stages involved the construction of two general rules, one associated with the variable sides and the other with points. Promotes the use of visual strategies.

Data analysis

The data analysis followed three stages: 1) reconstruction of teacher's argument from the task resolution process and the semi-structured interview. Toulmin's argumentative model (1958/2003) was useful in this process, 2) identification and description of the reasoning based on the typology of Soler-Álvarez and Manrique (2014) and 3) triangulation of data from previous stages, with expert researchers in argumentation, generalization and mathematical reasoning.

Results

The results are described by type of reasoning that the MMT evidenced when solving the task, which challenged him to build two general rules, one associated with the points variable and the other with the sides variable. In the construction of the general rule linked to the first type of variable, the teacher evidenced three forms of mathematical reasoning and two for the second.

Abductive reasoning

a) Reasoning in the points variable

In the construction of the general rule associated with the variable points, this way of reasoning appears when the MMT recognizes how it can explain the behavior of the pattern in each figure (see Figure 3). He identifies that the total number of points in the corners is a constant number, eight, that, on the sides, the number of points equals four times the figure number multiplied by three and in the center, it is four times the figure number minus one, squared. This fact constitutes a conjecture, which establishes three actions: a) to decompose the points of each figure in three parts: corners, sides and center, b) to count strategically the points in each of the parts, c) to establish a correspondence relation between the number of points of this decomposition, in relation to each figure.

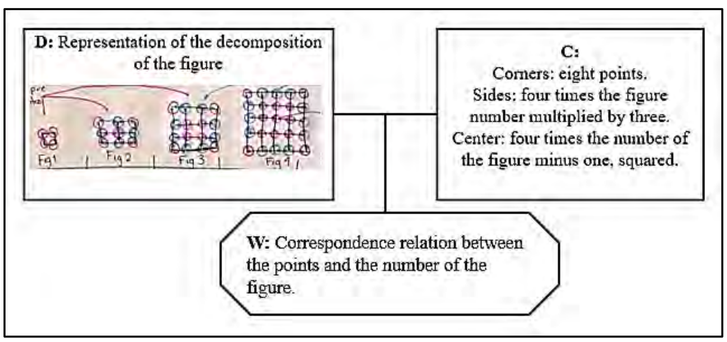


Figure 3: Abductive argument associated with the variable points.

The behavior identified by the teacher regarding the variable points when decomposed it into three parts, represents it through algebraic expressions where the figure number is written down in terms of n (see Table 3). The formulated conjecture is: $8 + 3[4(n - 1) + 4(n - 1)^2]$.

Table 3. Algebraic expressions about the points and the figure number.

Points	*Algebraic expressions*
Corners	8
Sides	$3[4(n-1)]$
Center	$4(n-1)^2$
Total (conjecture)	$8 + 3[4(n-1)] + 4(n-1)^2$

b) Reasoning in the variable sides

In the construction of the general rule associated with the variable sides, the abductive reasoning appears when the MMT recognizes that the increase of this variable from one figure to another has a quadratic type of behavior, that is, the number of the figure squared multiplied by two, plus twice the number of the figure. This fact constitutes a conjecture, which establishes four actions (see Figure 4): a) to count the total number of sides of each figure, b) to establish the first and second differences between the total number of sides of one figure and the next, c) to recognize that the first differences are not constant, and d) that the second differences are constant.

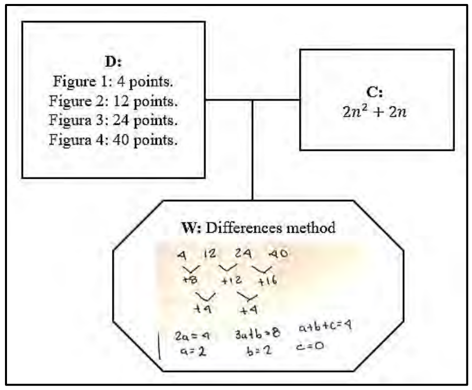


Figure 4: Abductive argument associated with the variable sides.

The behavior identified by the teacher with respect to the variable sides is based on the method of differences, which consists of determining the coefficients of the quadratic sequence. He represents them in terms of n. The formulated conjecture is: $2n^2 + 2n$.

Inductive reasoning

This type of reasoning was identified when the MMT evaluated the conjectures formulated by abduction within the framework of the variables points and sides of the figural pattern. Therefore, he relied on a double-entry table in which he recorded the total points and sides per figure. This evaluation consisted of substituting the total number of points and sides of four figures, from 1 to 4. These data are compared with those obtained by substituting the number of each figure (n) in the algebraic expression that he formulated by abduction (see Figure 5).

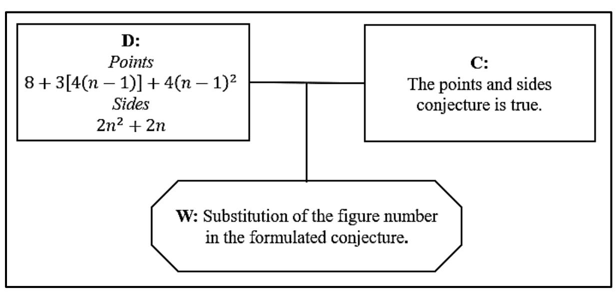


Figure 5: Inductive argument of the MMT in the context of the variables points and sides.

Deductive reasoning

This type of reasoning appears in the MMT when validating the conjecture associated with the points variable, obtaining the general expression. In addition, he used the method of differences to verify the general expression obtained (see Figure 6). The general expression obtained through the difference method is $4n^2 + 4n$, equivalent to $8 + 3[4(n-1)] + 4(n-1)^2$(see Table 3). He realized this by comparing them.

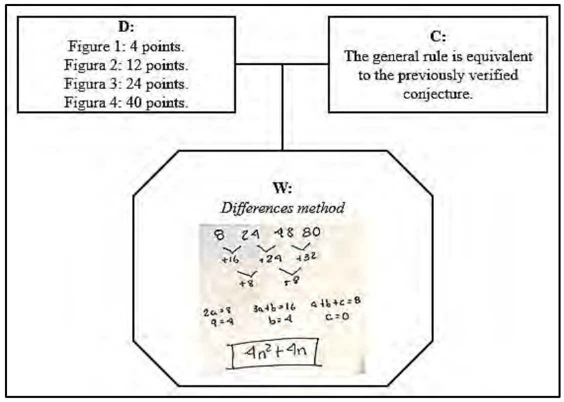


Figure 6: Deductive argument of the MMT on the variable of the points.

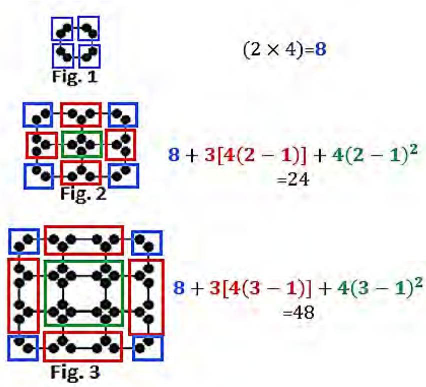
Discussion and conclusion

In this paper we described types of mathematical reasoning of a MMT when solving a task of generalization of figural patterns in the context of quadratic sequences. The mathematical reasoning evidenced by the MMT allows characterizing the thought process regulated by mental actions and relating it to the typology of reasoning (abduction, induction, and deduction). The way of reasoning abductively in the MMT begins with the observation of particular cases to formulate a conjecture. Then, inductively, he verifies the conjecture through the particular cases known, in order to verify its veracity. Finally, deductively, he validates the conjecture with known particular cases.

According to the characteristics and demands of the figural task, in the construction of the general rule associated with the points variable, the MMT evidences the three types of reasoning and in the sides variable, it reasoned abductively and inductively. In this sense, it is recognized that the types of mathematical reasoning can be presented in an ideal way and not necessarily in a linear way.

The context of quadratic pattern generalization contributed to the MMT evidencing different actions when solving the task in the figural context (see Table 4).

Table 4: Evidence of the reasoning by the MMT.

Type of reasoning	Cognitive Actions	
	Points	
Abductive	Decomposition of the figure, grouping and strategic counting of the points, identification of the regularity and formulation of the conjecture from the visual. 	Formulated conjecture $8 + 3[4(n-1)] + 4(n-1)^2$

<table>
<tr><td></td><td colspan="2">Sides</td></tr>
<tr><td></td><td>Total count of the sides by figure number, differences between terms, recognition of regularity and formulation of the conjecture from the numerical through the method of differences.
4 12 24 40
8 12 16 First difference
4 4 Second difference
Coefficients of the sequence in general terms $an^2 + bn + c$
$2a = 4$, $a = 2$ $3a + b = 8$, $b = 2$ $a + b + c = 4$, $c = 0$
$l_n = 2n^2 + 2n$</td><td>Formulated conjecture

$2n^2 + 2n$</td></tr>
<tr><td></td><td colspan="2">Points and sides</td></tr>
<tr><td>Inductive</td><td colspan="2">Total count of the points per figure and comparison with the results of the substitution in the points and sides conjecture.
<table>
<tr><td>Figure</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>Points</td><td>8</td><td>24</td><td>48</td><td>80</td></tr>
<tr><td>Sides</td><td>4</td><td>12</td><td>24</td><td>40</td></tr>
</table></td></tr>
<tr><td></td><td colspan="2">Points</td></tr>
<tr><td>Deductive</td><td>Difference method
4 24 48 80
16 24 32 First difference
8 8 Second difference
Coefficients of the sequence in general terms $an^2 + bn + c$
$2a = 8$, $a = 4$ $3a + b = 16$, $b = 4$ $a + b + c = 8$, $c = 0$
$p_n = 4n^2 + 4n$</td><td>The formulated conjecture has been justified by the method of differences, which are equivalent.
$8 + 3[4(n-1)] + 4(n-1)^2 = 4n^2 + n$</td></tr>
</table>

In the construction of the two rules of thumb in relation to the points and sides variables, the MMT involved his visual skills to recognize the behavior of the pattern. In the analysis of the points variable, the arrangements allowed him to establish correspondence relationships through decompositions of the figure, while for the sides variable, the MMT decided to work with the differences between the total number of points in each figure. According to Rivera (2010), working with figural patterns allows different interpretations of their behavior and the organization or configurations of the objects of some figural patterns are complex to interpret, even when their construction is well defined (Nuñez-Gutierrez & Cabañas-Sánchez, 2020).

In general, it is recognized that teaching experience and professional knowledge about mathematics, supported by the teacher's facts, conceptual images, and beliefs (Lithner, 2006), influence the choice of his strategies to formulate, verify and validate conjectures. Furthermore, it was evidenced that the mathematics teacher does not make his random choice of his actions but is supported by his knowledge of the subject and uses it to respond to the demands of the task, which, for his criteria, is the most accurate and quickest method.

Referencias bibliográficas

Arce, M., & Conejo, L. (2019). Razonamientos y esquemas de prueba evidenciados por estudiantes para maestro: relaciones con el conocimiento matemático. *Investigación en Educación Matemática XXIII* (pp. 163-172). Valladolid: SEIEM.

Barbosa, A., & Vale, I. (2015). Visualization in pattern generalization: Potential and Challenges. *Journal of the European Teacher Education Network*, *10*, 57-70.

Bozkuş, F., & Ayvaz, Ü. (2018). Middle school mathematics teachers' knowledge of mathematical reasoning. *European Journal of Education Studies*. *4*(9), 16-34.

Bragg, L., & Herbert, S. (2018). What Can Be Learned from Teachers Assessing Mathematical Reasoning: A Case Study. *Proceedings of the 41st annual conference of the Mathematics Education Research Group of Australasia* (pp. 178-185). Auckland: Merga.

Brodie, K. (2010). *Teaching Mathematical Reasoning in Secondary School Classrooms*. Johannesburg, Sudáfrica: Springer.

Clarke, D. M., Clarke, D. J., & Sullivan, P. (2012). Reasoning in the Australian Curriculum: Understanding its meaning and using the relevant language. *Australian Primary Mathematics Classroom, 17*(3), 28–32.

Conner, A., Singletary, L., Smith, R., Wagner, P. A., & Francisco, R. T. (2014). Identifying Kinds of Reasoning in Collective Argumentation. *Mathematical Thinking and Learning, 16*(3), 181-200.

Inglis, M., Mejia-Ramos, J. P. & Simpson, A. (2007) Modelling mathematical argumentation: the importance of qualification. *Educational Studies in Mathematics,* 66, 3–21. https://doi.org/10.1007/s10649-006-9059-8

Kirwan, J. (2015). Preservice Secondary Mathematics Teachers Knowledge of Generalization and Justification on Geometric-Numerical Patterning Tasks. Springfield, Illinois: Theses and Dissertations. Paper 392.

Lithner, J. (2006). *A framework for analysing creative and imitative mathematical reasoning*. Department of Mathematics and Mathematical Statistics, Umeå universitet.

Mata-Pereira, J., & da Ponte, J. P. (2017). Enhancing students' mathematical reasoning in the classroom: teacher actions facilitating generalization and justification. *Educational Studies in Mathematics*, 96(2), 169-186.

McCluskey, C., Mulligan, J., & Mitchelmore, M. (2016). The Role of Reasoning in the Australian Curriculum: Mathematics. *Mathematics Education Research Group of Australasia*.

Merriam, S. B., & Tisdell, E. J. (2015). *Qualitative research: A guide to design and implementation*. John Wiley & Sons.

National Council of Teachers of Mathematics [NCTM] (2000). *Principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

Nuñez-Gutierrez, K., & Cabañas-Sánchez, G. (2020). Inductive reasoning in mathematics teachers when resolving generalization tasks. *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 790-802). Mazatlán, México: Cinvestav / AMIUTEM.

Polya, G. (1966). *Matemática y razonamiento plausible.* Madrid: Technos-Madrid.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM Mathematics Education*, *40*(1), 83–96.

Rivera, F. (2010). Visual templates in pattern generalization activity. *Educational Studies Mathematical*, *73*(3), 297–328.

Rivera, F. (2013). *Teaching and Learning Pattern in School Mathematics*. Springer.

Saorín Villa, A., Torregrosa Gironés, G., & Quesada Vilella, H. (2019). Razonamiento configural y organización discursiva en procesos de prueba en contexto geométrico. *Revista latinoamericana de investigación en matemática educativa*, *22*(2), 213-244.

Soler-Álvarez, M., & Manrique, V. (2014). El proceso de descubrimiento en la clase de matemáticas: los razonamientos abductivo, inductivo y deductivo. *Enseñanza de las ciencias*, *2*(32), 191-219.

Torregrosa, G., Quesada, H., & Penalva, M. C. (2010). Razonamiento configural como coordinación de procesos de visualización. *Enseñanza de las Ciencias*, 28(3), 327-340.

Toulmin, S. E. (1958/2003). The uses of argument. In *Cambridge University Press*.

Toulmin, S. E., Rieke, R. D., & Janik, A. (1984). *An introduction to reasoning*. New York London: Macmillan; Collier Macmillan Publishers

Yackel, E., & Hanna, G. (2003). Reasoning and proof. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 227–236). Reston: National Council of Teachers of Mathematics.

ONE TEACHER'S KNOWLEDGE OF PROPORTIONS IN PRACTICE

Chandra Hawley Orrill
University of Massachusetts Dartmouth
corrill@umassd.edu

Rachael Eriksen Brown
Pennsylvania State University Abington
reb37@psu.edu

Rohini Thapa
University of Massachusetts Dartmouth
rthapa@umassd.edu

Emmanuel Nti-Asante
University of Massachusetts Dartmouth
entiasante@umassd.edu

This exploratory study offers a unique qualitative approach to exploring a teacher's knowledge of proportional reasoning and how that knowledge appears in the classroom. Using two theoretical frameworks, Knowledge in Pieces and the Knowledge Quartet, the study analyzes one middle grades teacher's proportional reasoning knowledge in two content interviews as well as in classroom data. Results suggest the teacher has both procedural and conceptual knowledge. Interestingly, the teacher tended to rely on procedural approaches with a preference for efficiency. This was evident across data types. A challenge was coordinating the teacher's understanding with classroom practice and raises questions about needed modifications to the Knowledge Quartet to include approaches to teaching that are less didactic.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge

Purpose

While it is a commonly held belief that a teacher's knowledge of mathematics shapes the instructional opportunities available to students, proving this conjecture is very complicated. Most efforts to date have focused on large scale assessments of teacher knowledge and students' outcomes (e.g., Hill et al., 2005) or more modest efforts using premade tools to assess teachers' practices from a particular lens of practice (e.g., Boston, 2012; Hill et al, 2008). Sometimes these studies also align teacher performance on the assessment of practice to the assessment of knowledge (e.g., Hill et al., 2008). Further, some studies have attempted to determine the relationship of teachers' knowledge to their practice through assessment items designed to engage teachers in life-like scenarios (e.g., Copur-Gencturk, 2015; Kersting et al., 2012)

In contrast to these approaches, we have undertaken to understand how one teacher reasons about proportional situations and how his content knowledge was observable in his classroom teaching using a qualitative approach. That is, we relied on Epistemic Network Analysis (ENA; described below) to build an understanding of the ways in which the teacher could reason about proportions. Then we used the Knowledge Quartet (KQ; described below) to understand how his knowledge was implemented in the classroom. This is a first step for understanding, in fine-grained detail, how teachers' knowledge aligns to their practices. For this exploratory study, we asked: How do we see one teacher's knowledge of proportions in his practice?

Theoretical Framework

We worked from two theoretical frameworks. We used Knowledge in Pieces (KiP; diSessa, 2016; diSessa, Sherin, & Levin, 2006) to understand teacher knowledge of mathematics. KiP posits that knowledge is comprised of fine-grained pieces of understanding connected in a variety of ways. As we learn, we refine those understandings, build on them, and make connections between them to create a rich understanding. Consistent with the research on expertise (Bédard & Chi, 1992; Bransford et al., 1999), KiP considers greater understanding to be characterized by the connections made between the knowledge resources and not just on the number of knowledge resources. Therefore, focusing on the connections between these knowledge resources provides a view into a teacher's knowledge that goes beyond simply understanding the amount of knowledge they have.

For this paper, we specifically relied on coding of one teacher's knowledge that was done as part of a larger project. This coding was done using a framework for the robust understanding of proportions that was created by the larger research team (Weiland et al., 2021). Each utterance of a teacher's response was coded as they thought aloud about mathematics tasks using a binary coding where 1 means they are invoking the knowledge resource in that utterance and 0 means they are not invoking the knowledge resource.

For the teacher interview data, we use ENA (Shaffer et al., 2009) to attend to the connections between knowledge resources that were invoked. ENA presents a mapping of knowledge resources when used in connection with other resources (Figure 1). Each node represents a knowledge resource. The size of the node indicates the relative number of co-occurrences. Thus, a larger node indicates a knowledge resource that co-occurred more frequently in this teacher's interview than a smaller node. The visual map also highlights the strength of the connection with the thickness of the connecting segment showing the relative frequency of pairs of codes co-occurring in the interview. Thus, a darker, thicker line indicates knowledge resources that were used together more than a thinner, paler line. For the purposes of this analysis, we assume that the co-occurrence of codes in an utterance indicates a connection between those codes for the participant. Using KiP and ENA allowed us to create a visual image of the teacher's content knowledge as expressed in two interviews.

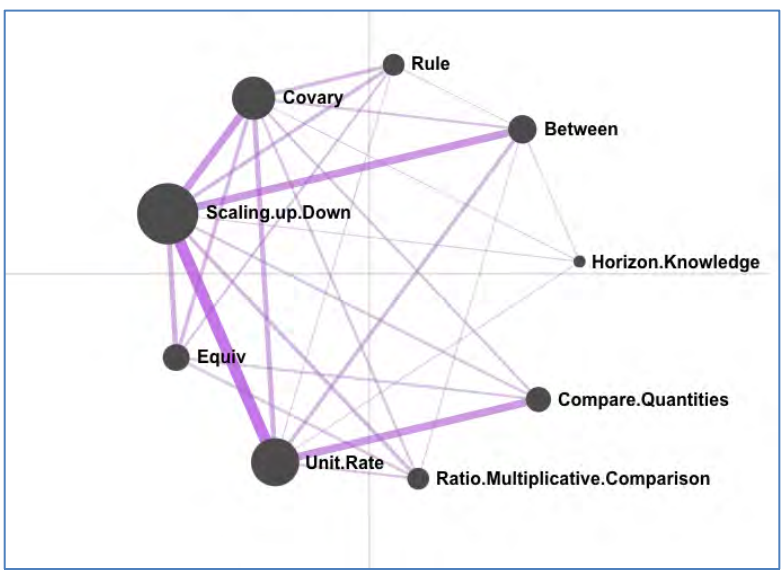


Figure 1: ENA plot of Matt's interview data.

The second theoretical lens upon which we draw is the KQ (e.g., Rowland, 2011; Rowland et al., 2005; Rowland & Turner, 2007; Turner & Rowland, 2011). The KQ is an operationalization of Shulman's (1986) construct of Pedagogical Content Knowledge (PCK). One of the elements that makes the KQ particularly useful and intriguing for our work is that it is a theory of mathematics knowledge *in* teaching, meaning that it is a framework that aims to make sense of that ways that PCK manifests itself in teachers' mathematics teaching. The KQ is divided into four key categories of knowledge: Foundation, Transformation, Connection, and Contingency. By using this framework, we were able to think about the ways in which one teacher's knowledge of mathematics was drawn on throughout his teaching. We were unable to use the same framework across data types. Originally, we intended to use KiP and ENA to look at teacher knowledge in the classroom; however, there was little direct evidence of the teacher's content knowledge during facilitation, in part because this teacher relied on high press questioning (Kazemi & Stipek, 2001) and mathematical argument for instruction more than on direct instruction. Thus, KQ allowed us to consider teacher knowledge (as well as other important categories of teacher knowledge) for the classroom data.

Methods

Participant and Data Collected

In this study, we examine one 7th grade teacher, Matt (pseudonym), as he solved a series of proportional reasoning tasks in two interviews and as he taught his own class proportional reasoning across four lessons. Matt was a white, male teacher who had been teaching mathematics for seven years at the time of data collection. He worked in an urban district where he had been able to "wrap" with his class – thus the 7th grade students we observed him teach were with him for the second year in a row.

Matt completed a paper-based think-aloud protocol using a LiveScribe pen, which allowed us to capture his speech as he wrote on the paper. He also completed one face-to-face clinical interview. Both interviews engaged teachers with a range of proportional reasoning tasks including many novel tasks such as tasks that made use of dynamic representations.

In addition to the two content knowledge interviews, Matt also allowed us to video record his mathematics class and conduct brief interviews with him about those sessions. For this paper, we analyzed four mathematics lessons and three brief teacher interviews focused on those lessons.

Data Analysis

As mentioned above, we used a framework for robust understandings of proportional reasoning to analyze the content interviews. For this paper, we particularly focused on knowledge resources that were focused on key understandings about the structures of proportional situations (e.g., between measure space, covariation, ratio as a multiplicative comparison, and compare quantities) and those that were key solution approaches (e.g., unit rate, equivalent fractions, scaling up or down). We also considered adherence to a given rule and horizon knowledge (Ball et al., 2008), which refers to knowledge that relies on proportional reasoning, but extends beyond what 7th graders typically learn (e.g., slope). See Weiland et al., (2021) for definitions of each knowledge resource code represented in Figure 1.

For classroom video, we trimmed the videos to focus exclusively on the main task of the day, thus cutting the "do now" or any test preparation, etc. Once the video was trimmed, we cut it into three-minute segments. Each segment was coded using the codes set forth by Rowland and Turner (2007), however, we also allowed for emergent codes (Charmaz, 2014). Emergent codes were included when we noticed a trend in Matt's teaching that seemed mathematically important, but not included in the original coding set. We classified the new codes into the original categories based on the original definitions for each category. Each video segment was viewed by at least two researchers. Discussion about codes was brought to the whole team and any points of disagreement were discussed until there was 100% agreement.

We then compared our observations about Matt's classroom teaching to his ENA graph. Noting some important similarities and differences. We then used his classroom interview transcripts to inform our understanding of particular decisions he made about his instruction.

Results

Our results suggest that Matt had both procedural and conceptual understandings of proportional reasoning. In both his own mathematical practice and in his teaching, however, he relied heavily on procedural approaches, with a strong preference for efficient procedures. This was evident in the tasks he used as well as in his interactions with his students. Further, Matt demonstrated a commitment to deep procedural knowledge (Star, 2005) through his use of high press questions and fostering of mathematical argumentation.

In Matt's ENA plot (Figure 1), we can see he relied considerably on scaling and unit rate, which we consider as two approaches for finding solutions to proportional reasoning tasks. We can also see that he used covariation and between measure space reasoning with scaling as well as comparing quantities with unit rate. These codes are all about understanding and applying the structures of proportional situations. That is, covariation, between measure space reasoning, and comparing quantities are all codes that consider the invariant relationship between the two quantities that are being compared in the ratio. Thus, Matt demonstrated both procedural and conceptual knowledge.

In his classroom, Matt gave the students tasks that could be solved with unit rate or scaling up and down. For example, one task asked students to determine how far they would need to run to burn off the calories in a sports drink. He allowed students to solve these any way they wanted, but in whole-class discussion, he highlighted "efficient" approaches. For example, he often asked the students to think about the ways different approaches were the same. And, sometimes, he introduced the idea of efficiency in the discussion. In our interviews with Matt, we learned this was part of his intentional practice. He noted, "The kids all use ratio tables right, that's what they all had coming in, which is fine. They just need to be able to use these other strategies…they need to be able to understand them… I want them to choose one that's efficient." The focus was primarily on making sure students had key solution approaches.

Discussion

This analysis is the first step in our ongoing effort to understand how a teacher's knowledge of proportional reasoning shapes his practice. In this case, we were able to see the teacher has more knowledge resources available to him than we saw him emphasize in his instruction. His decisions about what to emphasize in instruction were deliberate and had a focus on students learning to use more than one method, to focus on efficiency, and to have confidence in their ability to use at least one strategy.

The analysis is interesting because we learned one of the hardest parts of this research is learning how to coordinate the ENA analysis of the teacher's understanding with classroom practices. In this classroom, the struggle was amplified by Matt's lack of use of direct instruction. Thus, we needed understand how he leveraged his mathematics knowledge to ask students questions, to assess their understandings, and to pose rich tasks. Based on our work in this case, we conjecture that the KQ needs to be extended to include less didactic teaching approaches. That said, we also note that the four overarching categories of the KQ do hold up to Matt's teaching style, it is only the code set that needed to be extended.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1054170. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Boston, M. (2012). Assessing instructional quality in mathematics. The Elementary School Journal, 113(1), 76-104. https://doi.org/10.1086/666387

Bédard, J., & Chi, M. T. H. (1992). Expertise. Current Directions in Psychological Science, 1(4), 135-139.

Bransford, J. D., Brown, A. L., & Cocking, R. R. (1999). How people learn. Washington D.C.: National Academy Press.

Charmaz, K. (2014). Constructing grounded theory (2nd ed.). Sage Publications Ltd.

Copur-Gencturk, Y. (2015). The effects of changes in mathematical knowledge on teaching: a longitudinal study of teachers' knowledge and instruction. Journal for Research in Mathematics Education, 46(3), 280-330.

diSessa, A. A. (2006). A history of conceptual change research: Threads and fault lines. In R. K. Sawyer (Ed.). The Cambridge handbook of the learning sciences (pp. 265-282). New York: Cambridge University Press.

diSessa, A. A., Sherin, B. L., & Levin, M. (2016). Knowledge analysis: An introduction. In A. A. diSessa, M. Levin, & J. S. Brown (Eds.), Knowledge and interaction: A synthetic agenda for the learning sciences, pp. 30-71. New York: Routledge.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. Cognition and Instruction, 26, 430-511. https://doi.org/10.1080/07370000802177235

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal, 42(2), 371–406. doi:10.3102/00028312042002371

Kazemi, E., & Stipek, D. (2001). Promoting conceptual thinking in four upper-elementary mathematics classrooms. The Elementary School Journal, 102(1), 59-80.

Kersting, N. B., Givvin, K. B., Thompson, B. J., Santagata, R., & Stigler, J. W. (2012). Measuring usable knowledge: Teachers' analyses of mathematics classroom videos predict teaching quality and student learning. American Educational Research Journal, 49(3), 568–589. doi:10.3102/0002831212437853

Rowland, T. (2011). The knowledge quatret: The genesis and application of a framework for analysing mathematics teaching and deepening teachers' mathematics knowledge. Sisyphus, 1(3), 15-43.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. Journal of Mathematics Teacher Education, 8, 255-281. http://doi.org/10.1007/s10857-005-0853-5

Rowland, T., & Turner, F. (2007). Developing and using the 'Knowledge Quarter': A framework for the observation of mathematics teaching. The Mathematics Educator, 10(1), 107-123.

Turner, F., & Rowland, T. (2011). The knowledge quartet as an organising framework for developing and deepening teachers' mathematics knowledge. In T. Rowland & K. Ruthven (Eds.,), Mathematical knowledge in teaching (pp. 195-212). Springer. http://doi.org/10.1007/978-90-481-9766-8_12

Shaffer, D. W., Hatfield, D., Svarovsky, G. N., Nash, P., Nulty, A., Bagley, E., … Mislevy, R. (2009). Epistemic Network Analysis: A Prototype for 21st-Century Assessment of Learning. International Journal of Learning and Media, 1(2), 33–53. doi:10.1162/ijlm.2009.0013

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

Star, J R. (2005). Reconceptualizing procedural knowledge. Journal of Research in Mathematics Education, 36(5), 404-411.

Weiland, T., Orrill, C. H., Nagar, G. G., Brown, R. E., & Burke, J. (2021). Framing a robust understanding of proportional reasoning for teachers. Journal of Mathematics Teacher Education, 24, 179-202. https://doi.org/10.1007/s10857-019-09453-0

EXAMINING CHANGES IN PRESERVICE SECONDARY MATHEMATICS TEACHERS' DEFINITIONS OF FUNCTION (FROM PRETEST TO POST-INTERVENTION)

Radhika Pathak Kharel
Illinois State University
rpathak@ilstu.edu

Matthew S. Winsor
Illinois State University
mwinsor@ilstu.edu

This study examined preservice secondary mathematics teachers' (PSTs') definitions of a function and how those definitions changed because of a teaching intervention. Fourteen PSTs in a senior-level content capstone course participated in the study. PSTs' definition of a function before and after the intervention were analyzed using the comparison method. The findings showed that the PSTs struggled to revise their rote-learned definitions of a function to incorporate both definition univalence and arbitrariness in their descriptions of functions. In addition, the findings indicated that PSTs were able to identify whether the given relations or graphs represented a function.

Keywords: Preservice Teachers, Functions, Knowledge

Introduction

The concept of a function is fundamentally important in mathematics, relating to topics including algebraic representations and expressions, dependency, ordered pairs and graphs (Bannister, 2014; Biber & Incikabi, 2016). Indeed, a function has sometimes been described as the most powerful and useful concept in all of mathematics (Bannister, 2014). It is regarded as basic to undergraduate mathematics, (e. g., to calculus) and essential to applying mathematics in many sciences (Oehrtman et al., 2008). Yet, despite the acknowledged importance of functions, teaching functions in a conceptual manner has been found to be challenging for many teachers (Biber & Incikabi, 2016).

Part of the challenge has been to enable learners to arrive at a mathematically acceptable definition of a function. Providing accurate definitions of important mathematical concepts is crucial in teaching and learning mathematics. A student's construction of appropriate definitions is "a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that its [importance] has been neglected in most mathematics teaching" (De Villiers, 1998, p. 249).

Researchers noted that preservice teachers should understand univalence and arbitrariness to successfully teach the concept of functions to their future students (Even, 1990, 1993; Haciomeroglu, 2005; Nyikahadzoyi, 2013; Steele et al., 2013; Hatisaru & Erbas, 2017; Viirman, 2014). According to the Common Core State Standards Initiative [CCSSI] (2010), secondary school students should know that a function is a relationship between elements in an "input" set (the "domain") and an "output" set (the "co-domain") which assigns each input element to exactly one output element. This relationship between input and output elements is what determines the *univalent* nature of a function (Even, 1993). Thus, for example, if x and y represent real numbers then the equation $y = x^2$ exhibits univalence because each element x gets assigned to a unique element y. Similarly, functions are arbitrary because, "the elements of the domain and range need not be numbers and that the rule relating them does not have to be described by a regular expression" (Steele et al., 2013, p. 455).

It is well documented that school students' mathematics learning depends on their teachers' mathematical knowledge (Ball & Bass, 2002; Tasdan & Koyunkaya, 2017). As such, it is necessary to develop prospective mathematics teachers' mathematical knowledge. With respect to functions, researchers (Even, 1993; Even and Tirosh, 1995; Hatisaru & Erbas, 2017; Tasdan & Koyunkaya, 2017) found, however, that some preservice teachers have only a partial understanding of functions, which limits their ability to teach functional relationships thoroughly. Helping PSTs understand the concept of functions should be a priority for mathematics teacher educators. Mathematics education researchers are required to examine PSTs' prior knowledge of function concepts, a foundation for constructing a complete understanding of univalence and arbitrariness function. This study, therefore, assesses how PSTs knowledge of function changed, as measured by their definition of function, because of a teaching intervention in a content course for secondary preservice mathematics teachers.

Theoretical Perspective

Shulman (1986) defined *content knowledge* as "…the amount and organization of knowledge per se in the mind of the teacher." When examining different theories of teacher knowledge (e.g., Shulman, 1986, 1987; Hill et al., 2008), content knowledge serves as the anchor for other types of knowledge (e.g., pedagogical knowledge, knowledge of learners). For example, Shulman coined the term pedagogical content knowledge (PCK) (Shulman, 1986). PCK combine the knowledge of content with the knowledge of pedagogy and the knowledge of learners to teach content in the classroom. Without the anchor of content, PCK would not exist. Similarly, with Mathematical Knowledge for Teaching (MKT) (Hill et al., 2008) several subsets of MKT rely on mathematics (e.g., Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT), and the Knowledge of Content and Curriculum (KCC).) A strong anchor of mathematics content knowledge is a starting point to help PSTs build their teaching knowledge. This study examines PSTs' definition of function, which can serve as the foundation for successfully teaching the concept of function to students (Chesler, 2012, Edwards & Ward, 2008; Hill et al., 2008).

Related Literature

Researchers have investigated pre- and in-service teachers' knowledge of mathematical definitions of functions. Chesler (2012) found that most of the PSTs used the definition of functions from several resources such as dictionaries, textbooks, and websites. Chelser noted that some PSTs provided insufficient details about the definition of functions. Nineteen out of 21 students defined functions as relations, rules, mappings, correspondences, or relationships between variables, sets, or inputs/ outputs (Chesler, 2012). Although participants presented correct memorized definitions, intuitive understandings of the concept of functions were usually lacking (Chesler, 2012). Haciomeroglu (2005) observed that PSTs' weak subject matter hindered PSTs from acknowledging misconceptions of their students' mathematical work which caused students to make incorrect mathematical assertions.

Vinner and Dreyfus (1989) investigated junior high teachers and college students' concepts of mathematical functions. Only 82 of the 307 participants in the study noted the univalent nature of a function in their definition. Furthermore, while working on function-related tasks, those 82 candidates exhibited discrepancies in their performance—46 of the 82 did not incorporate any well-defined conception of a function while answering function-related tasks. Linchevsky et al., (1992) reported that only 21of the 82 junior high school mathematics PSTs in the study articulated an understanding of the arbitrary nature of functions.

Purpose

The purpose of this paper is to examine the effect of a teacher intervention aimed at helping PSTs refine their understanding of functions. Chesler (2012) explained that creating appropriate definitions promotes both teachers' and students' understanding of mathematics and helps to deepen teachers' abilities to teach mathematics conceptually. As such, it is imperative for PSTs to develop a clear understanding of definition of function, which will strengthen PSTs' ability to effectively teach the concept of functions.

Research Question

How did PSTs' definition of function change because of a teaching intervention?

Method

Participants

This research was conducted in a large public university in the Midwest region of the United States. Fourteen PSTs (nine female and five male) enrolled in a senior level content capstone course participated in this study. The content capstone course used college-level mathematics to develop a deep and connected understanding of secondary-school concepts such as functions, complex numbers, and polynomials. The instructor's pedagogy emphasized group work, problem solving, connections, justification, and mathematical communication. The aim of the course was to help PSTs conceptually understand both the mathematics of the secondary school and the more nuanced mathematical underpinnings that can aid in the implementation of meaningful instruction. This course was a part of PSTs' degree plan to become secondary mathematics teachers.

Procedures

The PSTs took a pretest on the first day of their course. Post-tests were taken at the end of the unit on functions. For two weeks, PSTs examined the content on functions from Usiskin, Peressini, Marchisotto, & Stanley (2003). Each class lasted 95 minutes and was structured to start with a problem to solve. The problem was designed to lead into the topic of the day as well as to generate discussion. The lesson would then proceed into a variety of planned activities. The end of the study session consisted of completing several problems out of the text, worked as a group. PSTs' solutions were offered and discussed. Finally, a journal prompt was put on the board for PSTs response and homework was assigned.

As the first classroom activity, PSTs were asked to write down their "best" definition of a function. PSTs then shared their definitions of functions with peer groups and were given the opportunity to revise their definitions if needed. The instructor recorded the groups' definition of functions on the board and conducted a discussion comparing the different groups' definitions.

With the groups' definitions of function on the board, the instructor provided circle diagrams (see figure 1) that represented examples and non-examples of functions. The instructor asked PSTs to identify which representations were functions and which were not, based on the groups' definitions of functions. Based on the activity, PSTs were asked to determine the robustness of their definitions of function and make edits if necessary. Finally, the entire class worked to establish a common definition of functions. The instructor helped PSTs recognize the characteristics of univalence and arbitrariness in the circle diagrams presented.

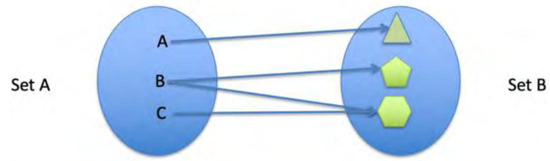


Figure 1: Circle Diagram Representation of Potential Function

In the remaining classes, PSTs examined different characteristics of functions (e.g., one-to-one functions, inverse functions) that were central to high school mathematics. The instructor made every effort to connect the topic of the day to the concepts of univalence and arbitrariness. A post-intervention test was administered to detect change in PSTs' definition of functions and mathematical work about functional relationships.

Research Design

This study was qualitative in nature. The pre- and post-intervention assessments were taken from a larger assessment on functions used by Winsor (2003). For this study, only three tasks were chosen for analysis. The first assessment item asked PSTs to provide a definition of function. The second item asked the PSTs to determine whether a given example was a function or not and to provide justification. The third assessment item showed a representation of function and PSTs were asked whether the item was an example or non-example of a function, and to provide a justification for their answer (see figure 2). All 14 PSTs took both assessments.

Pretest Items:

1. Give a definition of a function.
2. Is this a function? Why or why not?

$$g(x) = \begin{cases} x, \text{if } x \text{ is a rational number} \\ 0, \text{if } x \text{ is an irrational number} \end{cases}$$

3. Do the following ordered pairs describe a function? Why or why not?
   {(1,4), (2,5), (3,9), (4,4)}

Post-Intervention Items:

1. Give a definition of a function.
2. Is this a function? Why or why not?

$$g(x) = \begin{cases} x, \text{if } x \text{ is a rational number} \\ 0, \text{if } x \text{ is an irrational number} \end{cases}$$

3. Does the following table describe a function? Why or why not?

X	12	5	7	9	4	5
Y	5	12	8	4	9	7

Figure 2: Pre and Post Questionnaire for the Tests

A comparative method (Steele et al., 2013) was applied to compare the level of correctness of student responses to all the three tasks on both the pre-and the post-intervention test. Before analyzing the data, a rubric was created, with five different categories being defined for the purpose of analyzing appropriateness of the definitions of functions which would be given. Commonalities of definitions of function found in previous research (Steele et al., 2013) were incorporated in the rubric. PSTs' responses to assessment items two and three were based on the extent to which valid justifications were given. All data were analyzed by the first author and another doctoral student in mathematics education.

The following categories are how PSTs definitions were classified. Special attention was paid to how PSTs addressed arbitrariness and univalence in their definition of functions.

Category 4 Assessment. The definition of function should explicitly refer to both univalence and arbitrariness. For definition of univalence function a statement like– "*y* is a function of *x* when there is a relationship between *x* and *y* such that for every value of *x* there is one and only one corresponding value of *y*" was expected (Steele et al., 2013, p. 461). For arbitrariness, "elements of the domain and range need not be numbers and that the rule relating them does not have to be described by a regular expression" (Steele et al., 2013, p. 455). Definitions that had the same meaning as the definitions found in Steele, but not the same wording also were classified as a "Category 4" response. For example, the letters *x* and *y* could be replaced by identical mathematical terminologies, for instance, input and output, independent variable and dependent variable, domain, and range, etc.

Category 3 Assessment. The definition of a function would include a formal definition of arbitrariness or univalence and informally or intuitively refer to the other concept. For example, a definition might say that each element in the set of baseball stadiums is assigned to the city where it was built. This definition addresses univalence and implicitly hints at arbitrariness.

Category 2 Assessment. The definition of function only focuses on univalence or arbitrariness, but not both. As a sample response for this category is, "a function is any relation such that for every number input into a formula result in exactly one number as an output.

Category 1 Assessment. The definition of function is incomplete. As an example, "Functions map inputs to outputs."

Category 0 Assessment. The PSTs did not give a response or gave a definition which did not include any idea of univalence or arbitrariness. For example, "functions need to be linear relationships."

Findings

Assessment Item 1

The PSTs' responses were categorized based on their descriptions of definitions of functions that they provided for the pre- and post-intervention (see Table 1).

Table 1: Results of Assessment item 1

Categories from the rubric	Number of PSTs responses on the pre-test	Number of PSTs responses on the post- test
0	1	0
1	9	2
2	1	1
3	1	6
4	2	5

The data showed that in both their pre- and post-intervention test responses, some PSTs had difficulty in providing accurate definitions of a function. At the pre-test stage, only two of the 14 participants offered definitions which included detailed information of both the univalent and arbitrary nature of functions explicitly (a rating of 4). Five PSTs received a rating of 4 on the post-test. An example of a definition of function given at the pre-test stage which was classified as Category 4 was "A function is a rule which maps each element of the domain to a unique element in the co-domain." The same participant's definition of function at the post-intervention stage was “A function is a rule which maps every element of a set A (domain) to a unique element in a set B (co-domain)” which was also classified as a *Category 4* response.

Similarly, nine PSTs’ pre-test definitions of functions were categorized as *Category 1, and* two PSTs’ post-intervention definitions of functions were categorized as *Category 1*. At the pre-test stage, one of the participants explained definition of function as “a *relationship where no 2 same x’s are paired with different y’s”* and was assessed as *Category 1*. The same participant posed, *“a relationship where every input has exactly one output*” at the post-intervention test. That definition was classified as a *category 1* response as well.

Some PSTs who struggled to provide adequate definitions at the pre-test stage improved to provide better definitions of functions at the post-intervention stage. In fact, five PSTs’ definitions were deemed to have improved from Category 1 to Category 3 and two from Category 1 to Category 4. Additionally, one participant improved from Category 0 to Category 3, another from Category 3 to Category 4, and another from Category 2 to Category 4. However, one participant’s responses were assessed as decreasing from Category 4 to Category 3. A participant whose responses improved from Category 0 (pre-test) to Category 2 (post-intervention test) wrote, “a function represents an equation/ graph” (at the pre-test stage) and “a function is a domain and range of elements that maps in X (1 2 3) to a unique element in Y (w y z) (at the post-intervention stage).

Assessment for Item 2

This task required PSTs to determine if the given mathematical statement described functions, and to provide justifications for their decisions. The correct response for this question was “yes,” and a justification such as, “there is an assignment of a single value to each number" (Even, 1990, p. 105). Table 2 has a summary of the results. At the pre-test stage, 50 percent of the PSTs gave correct answers and acceptable descriptions. Similarly, at the pre-test stage, two of the 14 participants provided a correct response but did not give an acceptable justification, and five participants responded with an incorrect answer or did not provide a justification. But, at the post-intervention stage, most of them—indeed 13 of the 14, provided correct answers with acceptable justifications.

Table 2: Findings of Assessment for Task 2

Category of Response	Number at Pre-test	Number at Post-test
Participants with correct responses and acceptable justifications	7	13
Participants who gave correct responses without providing appropriate justifications	2	–
Participants with incorrect responses and/or no responses	5	1

Most of the justifications for correct responses of the PSTs were in the form: “Yes, this is a piece-wise function”, or “Yes because ‘it passes the vertical line test.” Justifications that were deemed not correct included “No, several *x* values were mapped to zero” and “No, because these are boundaries.”

Assessment for Item 3

This task included to check whether participants could identify if a set of ordered pairs was a function. Participants were required to explain why they gave their explanations. The correct answer at the pre-test stage was “yes” and at the post-intervention stage, it was “no” (questions of assessment item 3 in the pre-test and post-intervention were little bit different). Table 3 has a summary of the results. At the pre-test stage, eight PSTs gave correct responses and offered valid justifications, and at the post-intervention stage, 13 participants responded with correct answer and valid reasoning. All 14 participants attempted to explain their justifications in both tests, however six of their responses were incorrect at the pre-test stage and one was incorrect at the post-intervention stage.

Table 3: Findings for Assessment for Task 3

Category of Response	**Number at the Pre-Test**	**Number at the post-test**
Correct response and an acceptable justification	8	13
Correct response but no justification	0	0
Incorrect and/or no response	6	1

Some of the PSTs’ responses at the pre-test stage were, “Each x-value in the (x, y) pair maps to one y-value”, “no two same x-values have different y-values". Similarly, some correct responses in the post-intervention test—for which the correct answer was “no,” were, “no, because each x element does not have a unique y element”. Most PSTs appealed to the univalent nature of functions to justify why a set of ordered pairs was or was not a function. Some correct responses at the post-intervention stage were: “No, the following does not pass the vertical line test,” and “No, because there is not a common relation between the x and y coordinates.”

Discussion

This study assessed the quality of PSTs’ definitions of a function at pre-test and post-intervention stages. Analysis of pre-test responses has shown that some of the PSTs did not understand the concept of a function. However, it was also found that although most of the PSTs improved their understanding of the function concept by the post-intervention stage, some of their explanations were still inadequate. This study revealed that at the pretest stage the PSTs focused on the univalent nature of a function. Similarly, studies found that their participants were aware of the univalent nature of functions but did not know about the arbitrary nature of function (Chesler, 2012; Haciomeroglu, 2005). Steele et al., (2013) reported that the *arbitrariness* of a function was not well known by participants in her study, and the same was true for this present study.

Similar trends have been observed in previous research—for example, Even (1993) found that many preservice teachers were not aware of both univalence and arbitrariness. This finding is not surprising given that most of the concept of functions that preservice teachers experience during mathematics classes at school and in undergraduate mathematics programs were presented via graphs and expressions which clearly demonstrate univalence. Unfortunately, arbitrariness is often overlooked in the high school and undergraduate curriculum. Even (1993) further reasoned that the application of the concept of the arbitrariness of a function is found within advanced calculus, which is a part of PSTs' education. Unfortunately, that is not enough for students to reinforce that arbitrariness should be included as a part of the definition of function.

Additionally, an examination of PSTs' data generated by assessment items two and three, revealed that there was significant pre-test to post-intervention improvement so far as clarifying whether the given expressions were or were not functions. After completion of all the three tasks by the PSTs, some of them showed inconsistency in the justifications which they provided with *assessment item 2* and *assessment item 3* in comparison with the definitions of function they provided with *assessment item 1*. Some of their explanations in the second and third assessment items were valid but often their definitions of functions seemed implicit or incomplete. Tabach and Nachlieli (2016) found that although students explicitly expressed the definition of function, some of them were not able to use the definition to identify the mathematical model of a function.

Even (1990) claimed that the mathematical knowledge of the PSTs in her study was "fragile and weak." PSTs did not hold comprehensive and well-articulated definitions of the mathematics which they were going to teach. Even (1993) further elaborated that when the teacher participants in her study were asked to define a function, many of them chose to offer rote definition and did not refer to univalence and arbitrariness because their minds were influenced by the functions with which they had regularly worked on in classes, rather than by the definitions of a function which they had memorized. Wilson (1997) found that although the PSTs' understanding of the concept of function had grown significantly, because of their participation in different types of classroom activities and had been influenced by their involvement in classroom teaching practices, interviews, written work, and observations, they still had limited understandings of functions and, more generally, of mathematics.

Conclusion

Most PSTs referred only to the univalent nature of a function, omitting the arbitrary nature of functions. None of the PSTs mentioned univalence or arbitrariness by their names nor provided any distinct examples of those two aspects. Based on previous research (e.g., by Tabach & Nachlieli, 2016) and the analysis provided in this paper, it appears to be unlikely that the PSTs' concepts of a function will be changed significantly because of participating in a single mathematics course. Shulman's (1987) concept of pedagogical content knowledge which has content as an anchor for a teacher's pedagogical knowledge and knowledge of learners, suggests that if PSTs do not acquire a strong mathematical knowledge of the concept of functions, their future teaching on the concept of functions will be limited. PSTs should have multiple opportunities to study about functions in multiple classes with the aim of helping them develop a clear understanding of the concept of functions. Moreover, it seems as if it is challenging to restructure a PST's definition of functions, which suggests that the concept of functions should be taught more carefully in the high school mathematics courses (Vinner & Dreyfus, 1989). High school students should be given the opportunity to grapple with the concept of function starting in algebra 1.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Retrieved from amte.net/standards.

Ball, D. L., & Bass, H. (2002, May). Toward a practice-based theory of mathematical knowledge for teaching. In *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3–14).

Bannister, V. R. P. (2014). Flexible conceptions of perspectives and representations: An examination of pre-service mathematics teachers' knowledge. *International Journal of Education in Mathematics, Science and Technology, 2*(3), 223–233.

Biber, A. C., & Incikabi, L. (2016). Problems posed by prospective elementary mathematics teachers in the concept of functions: An analysis based on SOLO taxonomy. *Mersin University Journal of the Faculty of Education*, *12*(3), 796–809. DOI: http://dx.doi.org/10.17860/mersinefd.282381

Chesler, J. (2012). Pre-service secondary mathematics teachers making sense of definitions of functions. *Mathematics Teacher Education and Development, 14*(1), 27–40.

Common Core State Standards Initiative (CCSSI) (2010). *The standards: Mathematics*. Retrieved May 15, 2011, from http://www.corestandards.org/the-standards

Conference Board of the Mathematical Sciences (2012). *The mathematical education of teachers II.* Providence, RI and Washington, DC: American Mathematical Society and Mathematical Association of America.

De Villiers, M. (1998). To teach definitions in geometry or teach to define? In A. Olivier & K. Newstead (Eds.), Proceedings of the 22nd conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 248–255). Stellenbosch, South Africa: University of Stellenbosch.

Edwards, B. & Ward, M. B. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson, & C. Rasmussen (Eds.). *Making the connection: Research and teaching in undergraduate mathematics education (p. 223*–232*).* Mathematical Association of America. doi:10.5948/UPO9780883859759.004

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics, 21*(6), 521–544.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal of Research in Mathematics Education, 24*(2), 94–116.

Even, R., & Tirosh, D. (1995). Subject-matter knowledge and the knowledge about students as sources of teacher presentations of the subject-matter. *Educational Studies in Mathematics, 29*, 1–20.

Haciomeroglu, G. (2005). *Prospective secondary teachers' subject matter knowledge and pedagogical content knowledge of the concept of function.* (Publication No. 3232384) [Doctoral dissertation, The Florida State University]. ProQuest Dissertations & Theses Global.

Hatisaru, V., & Erbas, A. K. (2017). Mathematical knowledge for teaching the function concept and student learning outcomes. *International Journal of Science and Mathematics Education, 15*(4), 703–722.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372–400.

Linchevsky, L., Vinner, S., & Karsenty, R. (1992). To be or not to be minimal? Student teachers' views about definitions in geometry. In W. Geeslin, & K. Graham (Eds.), *Proceedings of the 16th Conference of the International Group for the Mathematics Education* (vol. 2, pp. 48–55). Durham, New Hampshire.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Authors.

Nyikahadzoyi, M. R. (2015). Teachers' knowledge of the concept of a function: A theoretical framework. *International Journal of Science and Mathematics Education, 13*(2), 261–283.

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. Carlson, & C. Rasmussen (Eds.). *Making the connection: Research and teaching in undergraduate mathematics education (p. 27–42).* Mathematical Association of America. doi:10.5948/UPO9780883859759.004

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. (1987). Knowledge and teaching: foundations of the new reform. Harvard Educational Review, 57(1), 1–23.

Steele, M. D., Hillen, A. F., & Smith, S. S. (2013). Developing mathematical knowledge for teaching in a methods course: The case of function. *Journal of Mathematics Teacher Education, 16*, 451–482.Stylianides, A.J. and Stylianides, G.J. (2014). Viewing "Mathematics for Teaching" as a Form of Applied Mathematics: Implications

for the Mathematical Preparation of Teachers. Notices of the AMS, 61, 266–276
Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*(4), 639–663.
Tabach, M., & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: Prologue. *Educational Studies in Mathematics*, *91*(3), 299–306.
Tasdan, B. T., & Koyunkaya, M. Y. (2017). Examination of pre-service mathematics teachers' knowledge of teaching function concept. *Acta Didactica Napocensia, 4*(3), 1–18.
Usiskin, Z., A. Peressini, E. A. Marchisotto, and D. Stanley. (2003). Mathematics for high school teachers: An advanced perspective. Upper Saddle River, NJ: Prentice-Hall. Viirman, O. (2014). The function concept and university mathematics teaching (Doctoral dissertation. Karlstad University, Karlstad, Sweden). Retrieved from https://www.diva-portal.org/smash/get/diva2:693890/FULLTEXT01.pdfFeva
Viirman, O. (2014). The function concept and university mathematics teaching (Doctoral dissertation. Karlstad University, Karlstad, Sweden). Retrieved from https://www.diva-portal.org/smash/get/diva2:693890/FULLTEXT01.pdfFeva
Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356–366.
Wilson P. S. (1990). Inconsistent ideas related to definitions and examples. *Focus on Learning Problems in Mathematics*, *12*(3–4), 31–47.
Winsor, M. S. (2003). *Preservice mathematics teachers' knowledge of functions and its effect on lesson planning at the secondary level* (Publication No. 3114407) [Doctoral dissertation, The University of Iowa]. ProQuest Dissertations & Theses Global.

ELEMENTARY SCHOOL TEACHERS' UNDERSTANDING OF UNIT FRACTIONS

Arthur B. Powell
Rutgers University-Newark
powellab@newark.rutgers.edu

Muteb M. Alqahtani
State University of New York at Cortland
muteb.alqahtani@cortland.edu

Özlem Doğan Temur
Dumlupinar Üniversitesi
ozlem.dtemur@dpu.edu.tr

Daniela Tirnovan
Rutgers University-New Brunswick
dt594@scarletmail.rutgers.edu

Teachers' mathematical knowledge is critical for their teaching of mathematics. We investigated whether elementary teachers believe that a unit fraction, $1/n$*, results only from a whole equipartitioned into* n *parts. We adapted Ciosek and Samborska's (2016) Frame Task, presenting a frame consisting of three unequal segmented squared rings, with one squared ring shaded. In semi-structured interviews, 19 teachers engaged the task and reasoned whether the shaded portion equals 1/3 of the frame. Our findings indicate that about three-quarters of the participants believe that either (1) to have one-third of a quantity, a section needs to be one of three parts, or (2) a section cannot be 1/3 of an object if the object is partitioned into three unequal sections. Finally, we hypothesize how an iterative perspective of unit fractions may mitigate against the false beliefs that Ciosek and Samborska (2016) and our study document.*

Keywords: Number Concepts and Operations, Rational Numbers, Teacher Beliefs, Teacher Knowledge

Introduction

Foundational interpretations of what is a fraction are numerous and interwoven. They are pedagogical, historical, mathematical, and neurocognitive (Cuisenaire & Gattegno, 1954; Davydov & Tsvetkovich, 1991; Kieren, 1976, 1980; Matthews & Chesney, 2015). Those interpretations shape how learners perceive a whole or unit and an associated unit fraction, that is, a fraction with 1 as its numerator. Studies illustrate how assigning a unit of measure to a given quantity, or unitizing, is critical for working adeptly with fractions (Lamon, 1996, 2007; Mack, 2001; Olive, 1999; Steffe & Olive, 2010; Van Ness & Alston, 2017a, 2017b, 2017c). To inaugurate students' study of fractions, textbooks and disciplinary policy (Common Core State Standards Initiative, 2010) present the part/whole interpretation. However, research suggests that a fraction's part/whole understanding leads learners to conceive and represent with difficulty fractions greater than a unit or whole (Gabriel et al., 2012, 2013; Mack, 1990; Tzur, 1999; Zhang et al., 2017). To understand further the difficulty of learners to conceive and represent improper fractions, a potentially fruitful line of inquiry concerns investigating how a part/whole interpretation of fractions shapes learners' comprehension of a unit fraction. Exploring an aspect of unit fraction comprehension, Ciosek and Samborska (2016) present a hitherto undocumented belief among elementary to university students, science graduates, and mathematics teachers that a unit fraction, $1/n$, results only from a whole equipartitioned into n parts. Using Ciosek and Samborska's (2016) study instrument, our work extends theirs by examining the related ideas of practicing elementary school teachers. Teachers of those grades are responsible for supporting elementary school students' development of ideas about fractions and their operations. Therefore, understanding teachers' ideas may provide valuable insights into the origins of students' beliefs and indicate percipient ways to challenge and enhance students' fundamental awareness of unit fractions.

Related Literature and Conceptual Framework

Teachers' Mathematical Knowledge

Teachers' mathematical knowledge is critical for teaching mathematics and their students' achievement. Several models have theorized the knowledge that teachers need to teach mathematics effectively. Our study draws from the Mathematics Teacher's Specialised Knowledge (MTSK) model by Carrillo-Yañez et al. (2018). Extending the work of Shulman (1986) and Ball et al. (2008), Carrillo-Yañez et al. (2018) state that teachers' knowledge consists of three major domains: beliefs, pedagogical content knowledge, and mathematical knowledge. Since teachers' beliefs influence teaching practice, MTSK centers the two other domains around teachers' beliefs about mathematics and mathematics teaching and learning. Each of the remaining two domains includes three subdomains. The pedagogical content knowledge domain involves knowledge of features of learning mathematics (KFLM), knowledge of mathematics teaching (KMT), and the knowledge of mathematics learning standards (KMLS). The KFLM subdomain encompasses the features inherent to learning certain mathematical content. The second subdomain, KMT, concerns awareness of mathematical teaching theories, including knowing about activities, strategies, and techniques for teaching specific mathematical content. The last subdomain, KMLS, is the knowledge about mathematical standards and curricula for the content at different levels.

The mathematical knowledge domain includes knowledge of topics (KoT), knowledge of the structure of mathematics (KSM), and knowledge of practices in mathematics (KPM). The KSM subdomain describes teachers' knowledge about the relations among different mathematical ideas. This knowledge influences teachers' decisions when connecting a mathematical topic to previous or future topics. Carrillo-Yañez et al. (2018) define the KPM subdomain as "any mathematical activity carried out systematically, which represents a pillar of mathematical creation and which conforms to a logical basis from which rules can be extracted" (p. 9). This knowledge does not focus on teaching mathematics but on the workings of mathematics such as mathematical demonstrations, justifications, definitions, and making deductions and inductions. Finally, the KoT subdomain describes "*what* and *in what way* the mathematics teacher knows [or may know] the topics they teach" (Carrillo-Yañez et al., 2018, p. 7, original emphasis). An example of this type of knowledge includes concepts, procedures, and theorems.

This study investigates the mathematical knowledge domain and focuses on teachers' KoT. The specific topic of our research concerns teachers' knowledge of the unit fraction concept, its relation to a whole and other parts of the whole. Moreover, regarding elementary school teachers who teach mathematics, this study also seeks to explore Ciosek and Samborska's (2016) hypothesis: "An iterative procedure of dissecting something into n equal parts to constitute a fraction of $1/n$ (as it is defined) may lead to the false belief of the learner of mathematics that if a whole is divided into n unequal parts, none of them can be $1/n$ of its size" (p. 22). Investigating how teachers conceptualize and identify unit fractions is essential since, as Chapman (2014) states, "it is not only important what mathematics teachers know but also how they know it and what they are able to mobilize for teaching" (p. 295).

Unit Fractions

Fractions, one of three representations of rational numbers, have several interpretations. Kieren (1976, 1980) calls them sub-constructs and identifies five: part-whole, quotient, measures, ratios, and operators. These interpretations, Kieren further notes, are united by the act of equipartitioning a whole. "Partitioning is seen here as any general strategy for dividing a given quantity into a given number of 'equal' parts. Thus, it can be seen as important in developing all of the five sub-constructs" (1976, p. 138). In that quote and elsewhere in our text, the term *quantity* means a measurable quality of an object such as its length, area, or volume. Positing partitioning a quantity as the cognitive basis for fraction knowledge has in practice implied that instructionally the part/whole interpretation is the foundational and initial fraction concept. This stance perseveres as curricular policy (Common Core State Standards Initiative, 2010).

This partitioning approach Vergnaud (1983) identifies as the first of two categories of ratios or fractions as comparisons or proportions. The first category he calls inclusive fractions, represented by a whole and a part of it (p out of q) such as this set model statement: Aaliyah ate two-thirds of the cookies. For an area model, one compares pizza slices to the whole pizza. The part/whole interpretation is the ubiquitous basis for initial learning about fractions among students in the United States and elsewhere. The inclusive category is how fractional parts of a whole are understood, including unit fractions. For example, the unit fraction, $1/n$, quantifies the part-whole relationship. It represents one part of the whole's equal parts. Its quantification or magnitude results from equipartitioning a whole and considering the ratio between one part and the whole.

In contrast, Vergnaud (1983) denotes the second category of fraction representations as exclusive. In it, fractions multiplicatively compare two distinct quantities with no inclusion relationship (p to q)—for example, the volume of Karma's luggage is three-fifths of Samir's luggage. The second quantity is the unit to which the first is compared. The quantities are of the same kind (volume) and compare proportionally. As a further instance, Davydov and Tsvetkovich (1991) present this situation: A student compares two distinct objects, sharing length as a common attribute, a ruler to a table's side. To compare the objects, the student assigns a quantity to equal one and measures, actions corresponding to the fraction concept's origin. For example, if n iterations of the student's ruler equal the table's side, and she considers the table the unit, the ruler is one-nth or $1/n$ of the table.

Building on the preceding paragraph's conceptual ideas, we subscribe to a generalized formulation of a unit fraction. A unit fraction, $1/n$, quantifies a particular multiplicative relation between two quantities of the same kind (e.g., lengths, areas, or collections). Specifically, the multiplicative relation is where n iterations of one quantity measure the other quantity considered the unit of measurement. In other words, definitionally, a quantity is $1/n$ of a unit if and only if n iterations of the quantity equal the unit. Though this definition integrates inclusive and exclusive categories of fractions and their corresponding nonsymbolic models, the crucial difference for the two categories is whether the n iterations are internal or external to the unit quantity. In either case, a unit fraction is a unit of a unit or a subunit.

Research evidences learners' movement from inclusive to exclusive fraction representation. This movement is essential for, as Vergnaud (1983) notes, "comparisons and ratios between any two quantities of the same kind are a more powerful model than inclusive fractions, providing a more general foundation for scalar operators or ratios" (p. 164). Hunt et al. (2016) show how tasks for learners with learning disabilities can conceptually prime them to use unit fractions to construct non-unit fractions in and out of equal sharing contexts. Tzur's (1999) study evinces how children nontrivially reorganize their numerical operations with a unit fraction to construct fractions less than or equal to a whole to represent fractional magnitudes greater than a whole.

Other than initiating fraction learning with the part/whole interpretation and its inclusive representations, challenging the settled partitioning perspective of fraction learning, there are instructional research and pedagogical materials that develop fractional knowledge with exclusive models (Cuisenaire & Gattegno, 1954; Davydov & Tsvetkovich, 1991; Dougherty & Simon, 2014; Dougherty & Venenciano, 2007).

Assigning a quantity to equal one or a unit is a mental and bodily act. Lamon (1996) calls the "assignment of a unit of measurement to a given quantity" (p. 170) unitizing. For a given whole or unit, a fraction whose numerator is 1, a unit fraction, is a subunit. From the part/whole perspective, if a whole is partitioned into n equal parts, one part is $1/n$ of the whole. As Ciosek and Samborska's (2016) study suggests, from that statement, learners may incorrectly conclude the statement's converse: If one part of a whole is $1/n$ of it, then the whole is divided into n equal parts. Besides, learners may falsely believe versions of the following two statements:

- If a whole is partitioned into n parts, one part is $1/n$ of the whole.
- If a whole is partitioned into n unequal parts, one cannot equal $1/n$ of the whole.

The part/whole interpretation of a unit fraction means that if both conditions—n parts and all parts equal—are true for how a unit is partitioned, each part equals $1/n$ of the unit. However, the statement does not imply that a part cannot equal $1/n$ of the unit when the unit is not partitioned into n parts, or the parts are unequal. That is, a part of a unit can equal $1/n$ of it even (a) when the unit is partitioned into n unequal parts or (b) without the unit being partitioned into n parts.

To engage learners' awareness of the meaning and conditions for the existence of unit fractions, educators can develop instructional tasks that challenge potential false beliefs (see, for example, Problems 1 to 6 in Ciosek and Samborska (2016, pp. 30-31)). However, to support the appropriate implementation of those tasks, it is essential to know how elementary school teachers understand the consequences of a fraction's part/whole interpretation for ideas about unit fractions. Then, if needed, teacher educators can support teachers to be aware of what the interpretation means and does not mean about the existence of unit fractions.

Methods

Study Instrument

To explore how elementary school teachers understand unit fractions and, if necessary, support them in educating their awareness of false ideas, we implemented the Frame Task (see Figure 1 below), adopted from the study instrument in Ciosek and Samborska (2016, p. 22). This task reveals participants' unit fraction understanding as one of n equal parts of a whole. It also allows us to examine whether they have the false belief that if a whole is divided into n unequal parts, none of them will equal $1/n$ of the whole.

In Figure 1, the frame consists of three squared annuli or rings—an outer, a middle or shaded, and an inner—each segmented into squares of the same size. Not part of the frame is the unsegmented center square.

Participants and Setting

From a pool of 47 teachers of grades 1 to 5 participating in a larger professional learning project about rational numbers, this study involved 19 of the teachers. They are from a public charter school in the Bronx, New York City, with two locations, which we call B1 and B2. They volunteered to participate in individual interviews, responding to the Frame Task, justifying and illustrating their responses, reacting to challenges to their answers or justifications, examining alternative explanations, and discussing their take-aways from the experience. For the interview, the teachers meet with one of the two first authors for an average of 18.2 minutes during the school day through videoconferencing. All interviews occurred on the same day. So that participating teachers did not receive information in advance about the task and the nature of the interview, we asked each participant not to reveal their experience to their colleagues until the next school day. All participating teachers signed an online informed-consent form, previously approved by the Institutional Review Board of Rutgers University.

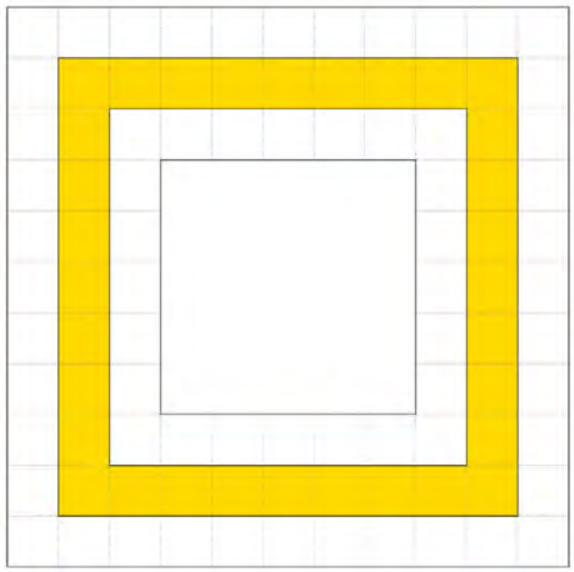


Figure 1: The Frame Task

Data Production

Guided by an interview procedure from Ciosek and Samborska (2016), the current study's two interviewers performed semi-structured interviews (Lune & Berg, 2017). We randomly assigned each participant to one of the two researchers for a specific one-half-hour time slot. Each researcher interviewed teachers from both B1 and B2. At the assigned time, teachers logged into the videoconferencing application, and their interviewer informed them that the session would be videorecorded. In addition, the interviewers confirmed that teachers knew how to use the application's annotation tool to illustrate their work and then shared their screens to display the Frame Task.

After displaying the task, we asked participants to read the question—Is the shaded part $1/3$ of the "frame"—and, for the context, explain their understanding of the word "frame." Their explanation was essential since the word as a noun or verb has several meanings. Furthermore, even when considering a picture frame, the consideration is variable as to which portion of the figure constitutes the frame in question. Once the participant and interviewer agreed that, for the task, the "frame" was the area of the figure surrounding the central large white square, the interviewer asked the participant to respond to the task. A complete response entailed two components: an answer to the task's question and a justification of the answer. Therefore, aside from their answers, we listened to how participants justified them If they did not, we then invited them to do so. If the participants' answers and justifications were correct, we encouraged them to provide an alternative rationale for their answers. If they did not have one, we offered an alternative explanation based on counting or mentally decomposing and recomposing the task's graphical figure.

If participants' answers or justifications were incorrect, we prompted their awareness by offering a monetary scenario that challenged their erroneous reasonings such as whether a unit fraction can only be defined for an equally partitioned object. After participants responded to the challenge and revised their answer or justification, we invited them to provide an alternative rationale for their answer. If they had none, we offered an alternative justification.

Finally, the interview ended with participant and researcher reflections. First, we encouraged participants to reflect on their experience in the interview from the perspectives of mathematics, learning, and teaching. Second, to complement or extend what the participants said, we offered mathematical perspectives to contribute further to their KoT of fractions.

Our mathematical perspectives comprise two considerations. The first KoT idea concerns this statement: If a whole is partitioned into n equal parts, one part is $1/n$ of the whole. This partition or part/whole interpretation of a unit fraction means that if both conditions—n parts and all parts equal—are true for how a unit whole is partitioned, each part equals $1/n$ of the unit. However, the statement does not imply that a part cannot equal $1/n$ of the unit when either the unit is not partitioned into n parts or when the unit's parts are unequal. That is, a part of a unit can equal $1/n$ of it (a) without the unit being partitioned into n parts or (b) even when the unit is partitioned into n unequal parts.

The second KoT consideration we offered pertains to the following generalized interpretation of unit fraction: A quantity is $1/n$ of a unit if and only if n iterations of the quantity equal the unit. We hypothesize that this understanding of a unit fraction, including that it is considered a subunit, emerges naturally from a measuring perspective of fraction knowledge (Powell, 2019b).

Data Analysis

The data consisted of the video recordings of the online interviews, including participants' annotations on a digital version of the Frame Task and transcripts of the videos' audio tracks. To probe the interview data, the authors developed an analysis spreadsheet containing columns for components of the interview and a row for each participant. The columns pertinent to this report captured participants' initial answers, whether their justification was unprompted or prompted and its content, whether the interviewer challenged their answer or explanation and the content of the challenge, the conclusion of how a participant addressed the challenge. In addition, two of the authors independently analyzed the interview data. For their analyses, the inter-rater reliability measure was 93.42% agreement, and the Cohen's Kappa coefficient is 0.858, indicating strong agreement between the two raters (McHugh, 2012).

Results

We report results corresponding to segments of our interview that parallel those Ciosek and Samborska (2016) detailed. Consequently, we do not discuss participants' reflections about alternative explanations for why the frame's yellow-shaded portion is one-third of it that they offered or that we supplied and comments about what they learned from the interview.

A complete response to the Frame Task entails two parts, an answer to the task's question and its justification. However, our data analysis suggests that a small proportion of the participants initially responded correctly to the Frame Task, and most teachers struggled at first to reason correctly about the conditions necessary for a portion of an object to represent a unit fraction.

Participants' initial responses comprised an answer to the question—"Is the shaded part 1/3 of the 'frame?'"—and the reason for their answers. An answer essentially was either "yes" or "no," coupled with a justification. Nevertheless, two participants' initial answers were a version of "Let me check" and ultimately responded "yes." Table 1 contains a tally of participants who provided each of the three initial responses and provides justification examples from teachers, T19, T5, and T3. In Table 1, of the 11 who responded, "yes," about 32 percent or six provided false justifications, noted with the letter "I" next to a paraphrasing of their explanation, and the five others or roughly 26 percent proffered correct reasonings, indicated with the letter "C."

Table 1: Initial responses and examples

Initial Answer	Justification	Count	Example
Yes	(I) The shaded part is one of three parts.	6 (31.6%)	T19: "the shaded area represents one of three parts."
	(C) The number of small shaded squares equals one-third of the total number of squares.	3 (15.8%)	T5: "So, the yellow part of this frame is 32... So, it is one-third of 96."
No	The three parts are not the same size.	8 (42.1%)	T3: "I see three divisions of the frame, but they're not all equal."
Let me check-Yes	The number of shaded squares to the total number of squares is $1/3$.	2 (10.5%)	
	Totals	19 (100%)	

As participants' initial responses, the incorrect to correct responses were 26 percent to 74 percent. Among participants whose initial answer was "yes," we observed false and valid justifications. Six teachers argued incorrectly that the frame's shaded part is $1/3$ of the frame since it is one of three parts. The other three participants reasoned correctly by attending to the number of the shaded squares (32) to the total number of squares (96). The eight teachers who initially answered "no" justified their answers, stating that the three squared rings of the frame are not the same size. The two teachers who first answered some version of "Let me check" compared the number of the shaded squares to the total number of squares in the frame before answering. They inspected the frame, compared the three squared rings, and stated that the shaded part is one-third of the frame. Then, they mentally shifted squares from the frame's outer ring to its inner ring and verified that the shifting resulted in two squared rings that are the same size as the shaded squared ring. For example, one of the "Let-me-check" teachers, T12, reasoned by mentally decomposing and recomposing the outer and inner squared rings to visualize that the average number of small squares is the same for the frame's three squared rings.

For participants who responded incorrectly, we prompted their awareness by offering one of two scenarios that challenged their erroneous reasoning: (a) Suppose you divide $90 into these three parts: $45, $30, and $15. Is any of these 1/3 of $90? or (b) Suppose you divide $90 into these three parts: $40, $35, $15. Is any of the parts 1/3 of $90? We gave the first scenario to participants whose initial answer was "no." We offered the second scenario to participants who answered "yes" but incorrectly justified. In all cases, after we challenged their thinking with a scenario, participants revised either their answer or justification or both and, ultimately, responded correctly based on the portion of the shaded square ring to the total of the three squared rings.

Discussion

A goal of our study was to investigate how elementary teachers who teach mathematics understand how unit fractions are constituted. Specifically, we were interested in knowing whether they believe that a unit fraction, $1/n$, results only from a whole equipartitioned into n parts, as Ciosek and Samborska (2016) documented among a range of elementary to university students, science graduates, and mathematics teachers. Understanding elementary teachers' ideas may provide valuable insights into the origins of students' beliefs. Also, teachers' understanding is associated with the learning opportunities they create for their students (Borko et al., 1992; Fisher, 1988). Furthermore, discerning whether teachers have robust conceptual insights into the unit fraction concept might indicate whether they need support to engage students with counterexamples to challenge mistaken notions.

Our study and Ciosek and Samborska's (2016) results indicate that, respectively, about 75% and 65% of the participants initially harbor incorrect interpretations of how the part/whole perspective defines unit fractions. Specifically, in our study, elementary-grade teachers who teach mathematics believe either (1) for a section of an object to equal one-third of it, the section needs to be one of three parts of the object, or (2) a section cannot be equal to one-third of an object if the object is partitioned into three unequal sections. Notably, when we presented scenarios to challenge their conceptual beliefs, the teachers revised their thinking and responded correctly to the task.

Our investigation suggests that, like teachers, without working through counterexamples, students may develop incorrect conclusions about the constitution of unit fractions such as this idea: The only way to obtain $1/n$ of a given whole is to divide it into n equal parts. Educators will want to challenge and enhance students' fundamental awareness of how unit fractions are constituted.

Earlier in this report, we mentioned the following generalized interpretation of a unit fraction: A quantity is $1/n$ of a unit if and only if n iterations of the quantity equal the unit. From a part/whole perspective, Tzur (1999) analyzes a constructivist teaching experiment to show how children, engaging in an iterative fraction scheme (Olive, 1999), nontrivially reorganize their numerical operations with a unit fraction to construct fractions less than or equal to a whole then later to represent fractional magnitudes greater than a whole. This cognitive feat is noteworthy since the fundamental basis for conceiving of the unit fraction is the equipartitioning of a whole. That criterion causes learners to hesitate iterating a unit fraction beyond the magnitude of the whole (Gabriel et al., 2012, 2013; Mack, 1990; Tzur, 1999; Zhang et al., 2017) and, as the present study and Ciosek and Samborska (2016) show, perceiving a unit fraction among parts of a whole partitioned unequally. To attenuate cognitive obstacles associated with founding fractions on equipartitioning a quantity, we hypothesize that learners' iterative fraction scheme, handling comeasurement units (Olive, 1999), and construction of proper, improper, and mixed numbers may emerge naturally from a measuring perspective of fraction knowledge (Alqahtani & Powell, 2018; Powell, 2019a, 2019b). This perspective defines fractions as ratios that express the multiplicative comparison of two quantities of the same kind. We surmise that working from a measuring perspective, learners spontaneously encounter the meaning of a general fraction, m/n, as an outcome of m iterations of the unit fraction $1/n$. Our research team intends to explore this hypothesis in a future study, challenging the settled partitioning perspective of fraction learning.

Acknowledgment

This study is part of a larger research project funded by the Bronx Charter Schools for Better Learning. The results and opinions reported are those of the authors and do not necessarily reflect the views of the funding agency.

References

Alqahtani, M. M., & Powell, A. B. (2018). Investigating how a measurement perspective changes pre-service teachers' interpretations of fractions. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 484 - 487). University of South Carolina & Clemson University.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content Knowledge for Teaching: What Makes It Special? *Journal of Teacher Education, 59*(5), 389-407.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to Teach Hard Mathematics: Do Novice Teachers and Their Instructors Give up Too Easily? *Journal for Research in Mathematics Education, 23*(3), 194-222. https://doi.org/10.2307/749118

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D., Rojas, N., Flores, P., Aguilar-González, Á., Ribeiro, M., & Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education, 20*(3), 236-253. https://doi.org/10.1080/14794802.2018.1479981

Chapman, O. (2014). Overall Commentary: Understanding and Changing Mathematics Teachers. In J.-J. Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research Trends in Mathematics Teacher Education* (pp. 295-309). Springer. https://doi.org/10.1007/978-3-319-02562-9_16

Ciosek, M., & Samborska, M. (2016). A false belief about fractions – What is its source? *The Journal of Mathematical Behavior, 42*, 20-32. https://doi.org/https://doi.org/10.1016/j.jmathb.2016.02.001

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics* National Governors Association Center for Best Practices, Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Cuisenaire, G., & Gattegno, C. (1954). *Numbers in colour: A new method of teaching the process of arithmetic to all level of the Primary School*. Hienemann.

Davydov, V. V., & Tsvetkovich, Z. H. (1991). The object sources of the concept of fractions. In V. V. Davydov & L. P. Steffe (Eds.), *Soviet studies in mathematics education. Volume 6. Psychological abilities of primary school children in learning mathematics* (pp. 86-147). National Council of Teachers of Mathematics.

Dougherty, B., & Simon, M. (2014). Elkonin and Davydov Curriculum in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 204-207). Springer Netherlands. https://doi.org/10.1007/978-94-007-4978-8_56

Dougherty, B. J., & Venenciano, L. C. H. (2007). Measure up for understanding: Reflect and discuss. *Teaching Children Mathematics, 13*(9), 452-456. https://doi.org/10.5951/TCM.13.9.0452

Fisher, L. C. (1988). Strategies Used by Secondary Mathematics Teachers to Solve Proportion Problems. *Journal for Research in Mathematics Education, 19*(2), 157-168. https://doi.org/10.2307/749409

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2012). Developing children's understanding of fractions: An intervention study. *Mind, Brain, and Education, 6*(3), 137-146. https://doi.org/10.1111/j.1751-228X.2012.01149.x

Gabriel, F., Coché, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology, 4*, 1-12. https://doi.org/10.3389/fpsyg.2013.00715

Hunt, J. H., Tzur, R., & Westenskow, A. (2016). Evolution of Unit Fraction Conceptions in Two Fifth-Graders with a Learning Disability: An Exploratory Study. *Mathematical Thinking and Learning, 18*(3), 182-208. https://doi.org/10.1080/10986065.2016.1183089

Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational number. In R. A. Lesh (Ed.), *Number and measurement* (pp. 101-144). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kieren, T. E. (1980). The rational number construct-Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Lamon, S. J. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170-193. https://doi.org/10.5951/jresematheduc.27.2.0170

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In J. F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning: A Project of the National Council of Teachers of Mathematics* (pp. 629-667). Information Age.

Lune, H., & Berg, B. L. (2017). *Qualitative research methods for the social sciences* (9th ed.). Pearson.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*(1), 16-32. https://doi.org/10.5951/jresematheduc.21.1.0016

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex content domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267-295. https://doi.org/10.2307/749828

Matthews, P. G., & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology, 78*, 28-56. https://doi.org/10.1016/j.cogpsych.2015.01.006

McHugh, M. L. (2012). Interrater reliability: the kappa statistic. *Biochemia medica: Biochemia medica, 22*(3), 276-282.

Olive, J. (1999, 1999/12/01). From Fractions to Rational Numbers of Arithmetic: A Reorganization Hypothesis. *Mathematical Thinking and Learning, 1*(4), 279-314. https://doi.org/10.1207/s15327833mtl0104_2

Powell, A. B. (2019a). How does a fraction get its name? *Revista Brasileira de Educação em Ciências e Educação Matemática, 3*(3), 700-713.

Powell, A. B. (2019b). Measuring perspective of fraction knowledge: Integrating historical and neurocognitive findings. *Revista Sergipana de Matemática e Educação Matemática, 4*(1), 1-19.

Shulman, L. S. (1986). Those who understand knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Steffe, L. P., & Olive, J. (Eds.). (2010). *Children's fractional knowledge*. Springer.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

Van Ness, C. K., & Alston, A. S. (2017a). Establishing the importance of the unit. In C. A. Maher & D. Yankelewitz (Eds.), *Children's reasoning while building fraction ideas* (pp. 49-64).

Van Ness, C. K., & Alston, A. S. (2017b). Justifying the choice of the unit. In C. A. Maher & D. Yankelewitz (Eds.), *Children's reasoning while building fraction ideas* (pp. 83-94).

Van Ness, C. K., & Alston, A. S. (2017c). Switching the unit. In C. A. Maher & D. Yankelewitz (Eds.), *Children's reasoning while building fraction ideas* (pp. 65-81).

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127-174). Academic.

Zhang, D., Stecker, P., & Beqiri, K. (2017). Strategies students with and without mathematics disabilities use when estimating fractions on number lines. *Learning Disability Quarterly, 40*(4), 225-236. https://doi.org/10.1177/0731948717704966

PRESERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: FOCUS ON LESSON PLANNING AND REFLECTION

Bima Sapkota
Purdue University
bsapkota@purdue.edu

Amanda Huffman
Purdue University
huffma33@purdue.edu

Mathematics teacher educators have suggested that preservice mathematics teachers' (PMTs') practices provide evidence of their Mathematical Knowledge for Teaching (MKT). In an effort to explore connections between MKT and PMTs' practices, we developed a framework that operationalizes Ball et al.'s (2008) six MKT domains in terms of approximations of practice. We then used our framework to investigate which domains were evidenced in eleven PMTs' lesson plans and how PMTs described MKT in their lesson plan reflections. We found Knowledge of Content and Teaching most evidenced and Horizon Content Knowledge least evidenced. Also, PMTs made few instances of Knowledge of Content and Students as they struggled to address students' mathematical thinking in their plans. We propose alternative forms of approximations of practice to optimize PMTs' opportunities to demonstrate and conceptualize MKT.

Keywords: Mathematical knowledge for teaching, Teacher knowledge, Preservice teacher education; Instructional activities and practices

Educators have emphasized that teacher knowledge should be discipline-specific, and evidenced through teacher practices (e.g., Ball et al., 2008; Shulman, 1986; Tchoshanov, 2010). In mathematics education, discipline-specific knowledge is commonly conceptualized as Mathematical Knowledge for Teaching (MKT, Ball et al., 2008), which provides a way to contextualize mathematical knowledge during mathematics teaching (Thomas et al., 2017). Preservice mathematics teachers (PMTs) often engage in approximations of practices (referred to as "approximations") such as peer teaching (Grossman et al., 2009) during their teacher education programs, thus, their MKT could be evidenced in approximations. However, the forms of approximations of practice that are appropriate for PMTs to demonstrate specific aspects of MKT (e.g., anticipating students' mathematical reasoning; formulating meaningful questions to students) are underexplored. In addition, mathematics teacher educators would benefit from opportunities to consider the operationalization of Ball and colleagues' (2008) six MKT domains in terms of associated practices (see the description of the domains in the Perspectives section). In this study, we operationalized MKT domains in terms of peer lesson planning and reflections. First, we synthesized the literature to develop descriptors for the MKT domains. Then, we used this framework to investigate the following questions: (a) How did PMTs demonstrate MKT in their lesson planning? (b) How did PMTs describe the use of MKT domains in their lesson plan reflections? We conceptualized lesson planning (e.g., formulating teacher and student actions) as approximations of practice. As such, PMTs engaged in approximations through peer teaching; thus, the actual practices were altered by changing the context of the practice (i.e., PMTs planned a lesson to implement in a university classroom) and by offering scaffolding such as peer and instructor feedback and a guided lesson plan template (Tyminski et al., 2014).

Perspectives

We framed our study around the concept of discipline-specific knowledge, or MKT, with foundations in Shulman's (1986) interpretation of Pedagogical Content Knowledge (PCK)—

teachers' skills to identify and unpack critical mathematical components that are fundamental for the teaching of mathematics and use those components in a way that are comprehensible to students. PCK was further elaborated by Ball and Bass (2000) and other mathematics educators (e.g., Hill et al., 2005; Wasserman & Stockton, 2013). These educators proposed that teacher knowledge is discipline-specific; suggesting that focusing solely on generic pedagogy, such as classroom management, poses a risk of simplifying the complexities of teaching because generic pedagogy does not include the unique characteristics associated with content or discipline. Moreover, Shulman and recent educators (e.g., Styers et al., 2021) proposed teacher knowledge should be evidenced through their practice as opposed to paper and pencil tests.

Ball et al. (2008) formalized MKT through a framework, which consists of two domains—Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK), each consisting of three subdomains (referred to as "MKT domains"). SMK consists of the following three subdomains: Common Content Knowledge (CCK), Horizon Content Knowledge (HCK), and Specialized Content Knowledge (SCK). CCK comprises the general mathematical knowledge required to solve mathematics problems. HCK is the knowledge of core disciplinary values and major structures of the discipline. SCK refers to teachers' conceptualization of mathematics in nuanced ways that include mathematical reasoning and multiple mathematical representations. PCK consists of Knowledge of Content and Student (KCS), Knowledge of Content and Teaching (KCT), and Knowledge of Content and Curriculum (KCC). KCS includes the understanding of students' mathematical conceptions and reasoning. KCT includes knowledge of effective mathematics teaching strategies. KCC comprises knowledge of learning goals and horizontal and vertical organizations of mathematics across grade levels. Even though the MKT framework provides a way to unpack MKT in terms of six domains, how these domains are operationalized in PMTs' teaching is still underexplored. In addition, teacher education programs have yet to ensure practices for PMTs to conceptualize and demonstrate MKT (Wasserman et al., 2019); which forms of approximations of practice (our second conceptual framework) are appropriate for PMTs to engage with and demonstrate MKT is still underexplored.

As we worked to operationalize Ball et al.'s (2008) six MTK domains in terms of approximations of practice, we looked at how approximations have been defined in research. Approximations of practice are defined as "opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (Grossman et al., 2009, p. 2058). Approximations are different from actual practices because they often (a) mimic components of teaching practice, (b) include scaffolding, and (c) do not always replicate the complexities of teaching (Campbell et al., 2020; Janssen et al., 2015; Tyminski et al., 2014). Through approximations, teacher educators focus explicitly on the enactment of teaching and aim to develop skilled teachers (Forzani, 2014; Janssen et al., 2015). By using the concept of approximations, we designed instructional activities that provided PMTs with opportunities to rehearse several practices in a setting different than the actual classroom; PMTs rehearsed peer teaching and received feedback from their instructors and peers to improve their teaching. We explored the explicit connections between MKT and approximations of practice— how MKT domains are operationalized in approximations involving lesson planning and reflections.

Methods

Using a single case-study design (Yin, 2017), we identified the MKT domains associated with teacher and student actions described in the PMTs' lesson plans and lesson plan reflections. We considered all PMTs' lesson plans as one case and investigated all MKT domains demonstrated across all PMTs' lesson plans. This merging of all data into one case was appropriate for our study because we were interested, not in individual PMT's demonstration of the MKT domains, but rather in the total instances of MKT domains in all PMTs' lesson plans and the way they describe MKT domains in their reflections.

Context, Participants, and Instructional Activities

Using convenience sampling (Nielsen, et al., 2017), we recruited 11 secondary PMTs enrolled in a secondary mathematics methods course at a large Midwestern University. The first author was a course instructor and developed and implemented instructional activities based on the fundamental concepts of approximations of practice (Grossman et al., 2009). The PMTs had completed content and general education courses prior to taking this course, but this was their first methods course. During this course, the PMTs planned, reflected on, and implemented lessons with their peers. In the first phase, the PMTs worked with a peer to plan a lesson ("Lesson Plan I") for middle school students using the Connected Mathematics curriculum (Connected Mathematics Project, n.d.). Afterward, they individually reflected on their lesson plans with prompts (e.g., Describe how the examples, strategies, and representations that you have listed in the lesson plan help to build on students' understanding of mathematics). We used González et al. (2020) and Özgün-Koca (2020) to develop prompts. In the second phase, the instructor facilitated a discussion of Ball and colleagues' (2008) MKT framework. In the third phase, PMTs individually planned a lesson ("Lesson Plan II") by adapting activities from the College Preparatory Mathematics curriculum (CPM Educational Program, n.d.). For both lessons, PMTs were provided a lesson plan template, which partitioned the lesson into three sections: Launch (helping students understand the problem setting, the mathematical context, and the challenge), Explore (inviting students to explore mathematical ideas), and Summary (inviting students to present and discuss problem-solving strategies).

Data Collection and Analysis

Our primary data was the PMTs' lesson plans. We used the PMTs' reflections on the lesson plans as an additional data source to understand how and why the PMTs selected certain teacher and student actions. We used content analysis methods (Chi, 1997; Schreier, 2012) to analyze the data. We first synthesized the literature to develop descriptors for the MKT domains (Table 1) and then used these descriptors to code lesson plans and reflections.

Table 1: Descriptors of MKT Domains

MKT Domains and Associated Descriptors/Codes
Common Content Knowledge -CCK (Ball et al., 2008; Baumert et al., 2010)
Define mathematical terms
Unpack mathematics concepts and/or symbols
Solve mathematical problems
Specialized Content Knowledge- SCK (Ding & Heffernan, 2018; Hill et al., 2005; Morris & Hiebert, 2017)
Anticipate alternative representations, provide explanations, evaluate unconventional student solution approaches
Identify multiple representations for mathematics problems or concepts
Offer multiple representations of mathematical solutions
Unpack mathematical concepts and concrete models for justifying standard procedures utilized in the process
Horizon Content Knowledge- HCK (Ribeiro et al., 2013; Wasserman & Stockton, 2013)
Address how a topic is connected with the broader disciplinary territory
Include big mathematical ideas that contribute to the teaching of mathematics topics
Knowledge of Content and Curriculum-KCC (Ball et al., 2008)
Address curricular trajectory

Knowledge of Content and Students-KCS (Hill et al., 2005; Özgün-Koca, 2020; Schilling & Hill, 2007)	
Anticipate the contextual factors that would support (or impede) the development of students' understanding	

MKT Domains and Associated Descriptors/Codes
Anticipate what students are likely to do with a mathematical task and/or concepts
Evaluate the diagnostic potential of tasks or recognize typical student errors
Foresee students' alternative conceptions and plan to address those conceptions
Predict what students will find interesting or motivating or useful
Knowledge of Content and Teaching-KCT (Ding & Heffernan, 2018)
Anticipate an item's difficulty level and plan for mathematical concepts that address rigor demanded by the item
Describe mathematical tasks or procedures that students would be engaged in
Formulate, sequence, pose questions to students
Identify strategies to assess students' mathematical understanding
Identify supplemental resources associated with mathematics topics
Identify what different methods and procedures might afford during instruction
Select and sequence examples or performance tasks that would allow students to understand the topic
Generic-No MKT-domains
Include generic teacher and student actions without referencing mathematics

We also modified the descriptors based on our coding. For example, we identified the code "anticipate the contextual factors that support (or impede) the development of students' understanding" under KCS from PMTs' descriptions of contextual factors associated with students' mathematical understanding. The first author used the data from the pilot study (i.e., four PMTs' lesson plans and their reflections collected in an earlier semester) to test the initial codes and further develop the coding scheme. The first and second author (a mathematics education graduate student and an instructor of the methods course in subsequent semesters) identified new descriptors in the data which we added to the framework. For example, we found that the PMTs planned strategies to assess students' thinking. It was not in the original set of codes, and we added it as a descriptor of KCT because it is related to teachers' knowledge associated with selecting instructional activities to understand students' mathematical thinking.

We coded teacher and student actions described and the assessment strategies noted in the lesson plans; the standards, instructional materials, students' prior knowledge, and vocabulary were articulated while describing the teacher and student actions. For the second data set, we coded all the reflections. We identified each instance in which the MKT domains were evidenced in PMTs' lesson plans as one coding unit. For example, one PMT mentioned that "students also need prior knowledge of probability notation, including that P(x) means that we are looking for the probability of x." We identified this as one coding unit. In addition, we identified instances wherein PMTs described generic teacher and student actions without referencing mathematical content. We added those activities under the new category "Generic-No MKT-domains." In addition, we refined our coding scheme by revising codes. For example, we aimed to distinguish between types of specific questions PMTs planned to pose versus types of tasks PMTs planned to implement while selecting mathematical tasks. Thus, we added the code: "formulate, sequence, and pose questions." We each used the coding scheme to code all lesson plans independently, then discussed our coding processes and codes to resolve any discrepancies. We collaborated to decide which codes or descriptors needed to be added or eliminated.

Findings

We first report the frequency of the MKT domains that appeared in the PMTs' Lesson Plan I and Lesson Plan II. We then elaborate on the context of the frequencies— how we identified these frequencies from PMTs' lesson plans and what these frequencies suggest in terms of

PMTs' MKT; we first report MKT domains that were evidenced the most and the shifts in the frequency from Lesson Plan I to Lesson Plan II. Afterward, we include examples of less evidenced domains in PMTs' lesson plans. We used PMTs' lesson plan reflections to provide the context and description of potential reasons that PMTs included certain activities in their lesson plans. In Table 2, we present the MKT domains that we identified in the PMTs' lesson plans.

Table 2: MKT Domains in PMTs' Lesson Plans

MKT Domains	KCC		KCS		CCK		KCT		HCK		SCK	
Lesson Plan	I	II	I	II	I	II	I	II	I	II	I	II
Total Instances	16	22	14	45	8	7	87	102	4	1	11	21

As indicated in Table 2, KCT was the most highlighted domain in both lesson plans, including 87 instances in Lesson Plan I and 102 instances in Lesson Plan II. KCT was primarily identified when PMTs described the following instructional plans related to teacher actions: (a) selecting and describing mathematical tasks, (b) formulating, sequencing, and posing questions for students, and (c) identifying what different methods or procedures would afford during instruction. PMTs had more instances (102) in their Lesson Plan II associated with teacher actions compared to Lesson Plan I (87 instances), suggesting that PMTs may have begun to think more about mathematics-specific teacher actions. PMTs presented examples of mathematical tasks from textbooks while other times they elaborated on the mathematical tasks from the textbook to describe which mathematical concepts students would be engaged in and which mathematical concepts or procedures that given mathematical tasks would afford. PMTs also discussed how tasks would allow students to engage with certain mathematical concepts. For example, Adam (pseudonym) discussed the following task and procedure in Lesson Plan I:

> This [Factor Game] provides students the chance to improve their understanding of factors. While playing the game, students become familiar with the factors of the numbers from 2 to 30. They also review the multiplication and division of small whole numbers while playing.

In Lesson Plan I, John anticipated that students would be confused by the problems that he planned: "I anticipate that students will have trouble initially figuring out what to do and will tell the students that the solution lies in using the tools they were given, e.g., a ruler and protractor."

In Lesson Plan II, the PMTs began to anticipate students' mathematical thinking; however, their skill of identifying and incorporating students' mathematical thinking was still evolving and they even stated in their reflections they needed to engage with authentic student work. In Lesson Plan I, the PMTs' included generic teacher and student actions (48% of the instances were generic). In Lesson Plan II, PMTs began to include more specific teacher and student actions.

Thus, we identified more instances that were related to both KCT and KCS. For example, Sarah included more specific questions in Lesson Plan II than in Lesson Plan I. In Lesson Plan I, she included the following generic statement: "Pose further questions to keep them thinking and understanding their answers to the questions." In Lesson Plan II, Sarah included specific students' mathematical thinking: "How did you know where to put each variable? If you had a word problem that required three equations, do you think it would be possible to solve that?"

In addition, the PMTs began planning for more mathematics-specific student actions in Lesson Plan II with 45 instances of specific student actions compared to only 14 instances in

Lesson Plan I. Specifically, in Lesson Plan II, the PMTs began describing how their students would be engaged in the mathematical tasks, including how their students might solve mathematical problems or what questions students might ask. For example, Sarah anticipated that her students could have the following response to a task she planned to pose: "[Students] could have an incorrect single equation that doesn't incorporate all the information from the problem." However, many PMTs still included generic teacher and student actions in Lesson Plan II. Some of their generic teacher actions included teacher questioning, division of groups in their classrooms, how students would respond to teacher actions. For example, George and Kate included the following generic student action: "[Students will] discuss the process with peers to gain a better understanding and will respond to teacher prompting to dig deeper and prepare responses for class discussion." PMTs had to use a Lesson Plan template asking them to write teacher and student actions separately. Not surprisingly, KCS was evidenced in the PMTs' descriptions of student actions. However, there were only a few instances wherein PMTs described which mathematical topics students would find interesting, what would be motivating to students, and what kinds of mathematical problems their students might find difficult to understand. For example, Jacob in his Lesson Plan II included the following: "I anticipate there will be questions and confusion. Students in the past have only worked with equations, so they have the tools they just are unaware of how to implement them into solving the problems."

We identified more MKT domains from the Launch and Explore sections of the lesson plans than in their Summaries. In these sections, PMTs described mathematical tasks building on their students' prior knowledge, what the teacher would present to the students, and what actions the teacher and students would perform. We identified only a few instances of the MKT domains in the Summary sections. We noticed that the PMTs struggled to plan for those specific actions because they had to anticipate what their students would explore in the Explore section. As a result, most PMTs planned for generic actions, specifically inviting students to share their ideas. For example, Griffin and Sarah wrote: "Walk around to each group and observe what they are discussing. Pose further questions to keep them thinking. Decide which groups should share."

KCC and CCK were also evidenced in the PMTs' lesson plans. KCC was evidenced when PMTs discussed students' prior mathematical knowledge and how they would build on it in their lessons. For example, in Lesson Plan II, Kate included: "Prior knowledge about calculating area and probability knowledge is needed for this lesson. Students also need prior knowledge of probability notation, including that P(x) means that we are looking for the probability of x." CCK was evidenced when PMTs presented the specific definitions or solution methods. For example, in her Lesson Plan I, Kiara included her plan for defining a *factor of a number*: "Teacher will review what a factor of a number is by reminding them that it can be thought of as both the numbers multiplied to get a product and the divisor of a number." This quote indicates Kate's knowledge of MKT in the CCK domain as she defined factors.

In their reflections, PMTs mostly described KCT, including which mathematical tasks their students were engaged in. For example, in Lesson Plan II reflection, Kate noted:

> I think she wanted the students to think of solutions on their own because there were several ways for the students to solve the problems. Through the student presentations, other students were able to discover different strategies so that they could figure out which method works best for them or what they understand the most.

In this reflection, the notion of students sharing multiple strategies and different thinking allows the teacher to assess their students' understanding, one descriptor of KCT. In Tania's

Lesson Plan II reflection, another code of the KCT domain, “Identify what different methods and procedures might afford during instruction,” was referenced:

> I encourage each activity to involve some sort of discussion with their peer, and the activities themselves are asking the students to present the information in different ways, which is allowing the students to enhance their conceptual understanding of the material.

While this reflection could be seen as more generic, the mentioning of “enhance their conceptual understanding of the material” points to understanding how students' conceptual knowledge of simplifying exponential expressions was important.

From Table 2, we also notice that there were only a few instances of HCK (four instances in Lesson Plan I and one instance in Lesson Plan II). Recall that HCK includes logical connections of mathematics topics within broader mathematical disciplines when presenting big mathematical ideas. Thus, this domain could have been evidenced when the PMTs presented instructional strategies in a way that made connections with advanced mathematical topics. For example, in the following excerpt, Sarah used her knowledge of “transitivity” to plan her lesson: “If we know that t is equal to two different things, does that mean those two things are equal to each other? (transitivity).”

In addition, SCK, KCC, and CCK were less evidenced in the PMTs’ lesson plans. SCK was evidenced in instances where the PMTs were able to offer alternative mathematics solution strategies, multiple mathematical representations, and concepts needed to understand mathematics. For example, Jacob, in Lesson Plan II, mentioned:

> You are looking for something a little bit bigger but will fit in your TV cabinet and you only know the width and height of the TV cabinet. When you get to the store, they do not list the Width and Height of the TVs, instead, they only have the diagonal. What are some possible ways that you could find out the size of the TV you need?

Here, Jacob presented his mathematics task, and it included a real-world problem. Tayra planned to address students' conceptions in Lesson Plan II: “If the students aren’t understanding conceptually, then I’ll spend more time individually with the students and track specifically what things they’re finding difficult to understand.” Later, while reflecting on the lesson plan, Tayra mentioned that she anticipated the task to be difficult for students; however, she did not explain how: “we anticipated this task being difficult for many. It involves students to get creative and think outside the classroom.” This finding suggested that while PMTs had generic explanations of what their students might find confusing, there were fewer instances wherein PMTs mentioned which mathematical concepts students might find interesting and their anticipation of students’ alternative problem-solving strategies (e.g., patterns of student errors).

In their reflections, PMTs noted that they did not anticipate students’ mathematical thinking because they wanted to have flexibility about how to modify students’ task-solving strategies during lesson plan implementation. For example, Kate reflected that she did not try to anticipate students’ thinking because she planned to be open to students’ ideas: “I do not feel like we tried to anticipate a lot because I did not want to get caught up in what I think the students will answer so that I am more flexible for when they do not answer the way someone typically would.” This finding suggests that PMTs need more opportunities to explore students’ authentic work. KCC was evidenced when PMTs described how they built their plans on students’ prior knowledge. For example, Jacob planned for addressing students’ prior knowledge: “While this is a new chapter, this lesson relates back to measurements, triangles and shapes, and heights and areas.”

Discussion and Implications

Our findings indicated that KCS and KCT were the primary MKT domains identified from PMTs' explanations of teacher and student actions. We identified two possible reasons associated with this finding. First, due to the lesson plan template, the PMTs demonstrated some domains more than others. For example, the template did not include an obvious place for PMTs to demonstrate HCK. Each PMT discussed CCK (i.e., addressing students' prior knowledge). Given that the lesson plan template explicitly included the prompts about students' prior knowledge, each PMT included at least one instructional activity that described how they would address students' prior knowledge. Also, the lesson plan template had prompts for the PMTs to describe teacher and student actions and not many options to demonstrate how their advanced mathematical knowledge connected with the teacher and student actions, which could be a reason we identified less evidence of HCK.

Second, when the PMTs had to explain student actions, they demonstrated fewer instances related to mathematics-specific student actions. PMTs noted they could not plan for student actions because they had not had a chance to work with specific students in the field. We identified that our approximations of practice (i.e., peer teaching, reflections) did not contain students' authentic work. To address this limitation, incorporating authentic student work prior to lesson planning could evoke PMTs' conceptions about mathematics-specific student work. Moreover, engaging PMTs with the research about students' possible mathematical thinking and rehearsing how to respond to students' mathematical thinking could be a way to cultivate PMTs' skills to plan for activities to address students' mathematical reasoning. Analysis of student work could assist PMTs in identifying activities to understand and develop students' mathematical reasoning (Álvarez et al., 2020). We propose authentic forms of approximations wherein PMTs first read and discuss students' potential mathematical reasoning, then engage with students' authentic work in some form. Afterward, they will have opportunities to plan instructional activities to develop students' mathematical understanding of a certain topic, including possible ways to address students' existing conceptions. Finally, PMTs could implement those strategies with their peers first and with real students later during their field experiences.

We found minimal instances associated with SCK, indicating alternative representations of mathematics topics and contextual factors associated with the given mathematics topics were less discussed. Prior research suggested textbooks played a significant role to enhance PMTs' MKT; for example, textbooks predominantly promoted CCK (Atanga, 2021). Given that the textbooks that PMTs used in our study focused on conceptual understanding, our findings do not align with Atanga's findings. Thus, exploring the explicit connections between how MKT domains in PMTs' lesson plans were influenced by the textbook should be a question for further research. In addition to textbook content, a section in the lesson plan template about the contextual factors and alternative representations of mathematics could help PMTs explore, demonstrate, and conceptualize how mathematics topics relate to other concepts and real-life.

Our study has implications for mathematics teacher educators and is connected with the conference theme of Critical Dissonance and Resonant Harmony. As opposed to assessing teacher knowledge through paper and pencil tests to identify what teachers know versus do not know, we invite educators to critically reflect on approximations of practice so PMTs have the opportunity to explore how they can contextualize their advanced mathematical knowledge during mathematics teaching. Our MKT domain descriptors provide a way to design instructional activities that engage PMTs in both teacher knowledge and practices.

References

Álvarez, J. A., Arnold, E. G., Burroughs, E. A., Fulton, E. W., & Kercher, A. (2020). The design of tasks that address applications to teaching secondary mathematics for use in undergraduate mathematics courses. *The Journal of Mathematical Behavior*, *60*(2020), 1-18. https://doi.org/10.1016/j.jmathb.2020.100814

Atanga, N. A. (2021). Assessing the quality of mathematics in Cameroon primary school textbooks and its implications to learning, *International Journal on Integrated Education,* 4(3), 215-222. https://doi.org/10.31149/ijie.v4i3.1398

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Ablex.

Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407. https://doi.org/10.1177%2F0022487108324554

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., & Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180. https://doi.org/10.3102%2F0002831209345157

Campbell, M. P., & Elliott, R. (2015). Designing approximations of practice and conceptualising responsive and practice-focused secondary mathematics teacher education. *Mathematics Teacher Education and Development, 17*(2), 146-164. https://files.eric.ed.gov/fulltext/EJ1085903.pdf

Campbell, M. P., Baldinger, E. E., & Graif, F. (2020). Representing student voice in an approximation of practice: Using planted errors in coached rehearsals to support teacher candidate learning. *Mathematics Teacher Educator, 9*(1), 23-49. https://doi.org/10.5951/MTE.2020.0005

Chi, M. T. H. (1997). Quantifying qualitative analyses of verbal data: A practical guide. *The Journal of Learning Science, 6* (3), 271-315. https://doi.org/10.1207/s15327809jls0603_1

Connected Mathematics Project. (n.d). Retrieved February 2021 from https://connectedmath.msu.edu/

CPM Educational Program: More math for more people. (n.d.). Retrieved March 2021 from https://cpm.org/

Ding, M., & Heffernan, K. (2018). Transferring specialized content knowledge to elementary classrooms: Preservice teachers' learning to teach the associative property. *International Journal of Mathematical Education in Science and Technology*, *49*(6), 899-921. https://doi.org/10.1080/0020739X.2018.1426793

Forzani, F. M. (2014). Understanding "core practices" and "practice-based" teacher education: Learning from the past. *Journal of Teacher Education*, *65*(4), 357-368. https://doi.org/10.1177%2F0022487114533800

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9), 2055-2100.

Gutierez, S. B. (2015). Teachers' reflective practice in lesson study: A tool for improving instructional practice. *Alberta Journal of Educational Research*, *61*(3), 314-328.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*(2), 371-406. https://doi.org/10.3102%2F00028312042002371

Humboldt County College of Education (n.d.). College Preparatory Mathematics. https://my.hcoe.net/project/cpm/

Janssen, F., Grossman, P., & Westbroek, H. (2015). Facilitating decomposition and recomposition in practice-based teacher education: The power of modularity. *Teaching and Teacher Education*, *51*, 137-146. https://doi.org/10.1016/j.tate.2015.06.009

Morris, A. K., & Hiebert, J. (2017). Effects of teacher preparation courses: Do graduates use what they learned to plan mathematics lessons? *American Educational Research Journal*, *54*(3), 524-567. https://doi.org/10.3102%2F0002831217695217

Nielsen, M., Haun, D., Kärtner, J., & Legare, C. H. (2017). The persistent sampling bias in developmental psychology: A call to action. *Journal of Experimental Child Psychology*, *162*, 31-38. https://doi.org/10.1016/j.jecp.2017.04.017

Özgün-Koca, S. A., Lewis, J. M., & Edwards, T. (2020). Fostering middle school teachers' mathematical knowledge for teaching via analysis of tasks and student work. *Mathematics Teacher Educator*, *9*(1), 50-62. https://doi.org/10.5951/MTE.2020.0004

Ribeiro, C. M., Mellone, M., & Jakobsen, A. (2013). Characterizing prospective teachers' knowledge in/for interpreting students' solutions. *In Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 89-96).*

Schreier, M. (2012). *Qualitative content analysis in practice.* SAGE.

Schilling, S. G., & Hill, H. C. (2007). Assessing measures of mathematical knowledge for teaching: A validity

argument approach. *Measurement*, *5*(2-3), 70-80. https://doi.org/10.1080/15366360701486965
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.
Styers, J. L., Nagle, C. R., & Moore-Russo, D. (2020). Teachers' noticing of students' slope statements: attending and interpreting. *Canadian Journal of Science, Mathematics and Technology Education*, *20*(3), 504-520. https://doi.org/10.1007/s42330-020-00107-5
Tchoshanov, M. A. (2011). Relationship between teacher knowledge of concepts and connections, teaching practice, and student achievement in middle grades mathematics. *Educational Studies in Mathematics*, *76*(2), 141-164. https://doi.org/10.1007/s10649-010-9269-y
Thomas, J., Jong, C., Fisher, M. H., & Schack, E. O. (2017). Noticing and knowledge: Exploring theoretical connections between professional noticing and mathematical knowledge for teaching. *The Mathematics Educator*, *26*(2), 3-25. http://tme.journals.libs.uga.edu/index.php/tme/article/view/418/305
Tyminski, A. M., Zambak, V. S., Drake, C., & Land, T. J. (2014). Using representations, decomposition, and approximations of practices to support prospective elementary mathematics teachers' practice of organizing discussions. *Journal of Mathematics Teacher Education, 17*(5), 463-487. https://link.springer.com/article/10.1007%2Fs10857-013-9261-4
Wasserman, N., & Stockton, J. (2013). Horizon content knowledge in the work of teaching: A focus on planning. *For the Learning of Mathematics, 33*(3), 20–22.
Wasserman, N., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: fostering mathematics teachers' "attention to scope". *Journal of Mathematics Teacher Education*, *22*(4), 379-406. https://doi.org/10.1007/s10857-019-09431-6
Yin, R. K. (2017). *Case study research and applications: Design and methods* (6th ed.). SAGE.

CONCEPTUALIZING MATHEMATICS TEACHER EDUCATOR KNOWLEDGE: COMPARING AND CONTRASTING EXISTING FRAMEWORKS

Rachael M. Welder
University of Nevada, Reno
rwelder@unr.edu

Alison Castro Superfine
University of Illinois, Chicago
amcastro@uic.edu

Priya V. Prasad
University of Texas at San Antonio
priya.prasad@utsa.edu

Dana Olanoff
Widener University
dolanoff@widener.edu

Over the past two decades, the landscape of research on mathematics teacher educators (MTE) has grown considerably. One particular area of interest has focused on the knowledge needed by MTEs for their work with preservice K-8 teachers (PTs). In an effort to understand this varied landscape, we conducted an extensive review of research on frameworks of MTE knowledge. This report explores the theoretical underpinnings of MTE knowledge and highlights similarities and differences among theoretical frameworks. By mapping the terrain of research on MTEs' knowledge, our goal is to identify aspects of MTE knowledge to inform the types of research that may be needed for its further development.

Keywords: teacher educators, teacher knowledge, mathematical knowledge for teaching

The issues of what mathematics teachers need to know (Ball et al., 2008; AMTE, 2017; CBMS, 2012) and how to prepare prospective teachers (PTs) as part of teacher preparation programs (e.g., Adler, 2010; Grossman et al., 2009; Dinham, 2013) have become a central focus in both research and policy arenas in many countries. The term Mathematics Teacher Educators (MTEs) refers to individuals who work with PTs in a variety of contexts to develop and improve the teaching of mathematics (Jaworski, 2008a), including mathematicians, graduate students, mathematics education researchers, and classroom teachers. As such, these individuals play a critical role in the mathematical preparation of PTs.

Over the past two decades, the landscape of research on MTEs has grown considerably. In an effort to understand this varied landscape, the authors conducted an extensive review of research on the nature of MTE knowledge. This paper reports the theoretical underpinnings of the existing frameworks for MTE knowledge identified in this review. We focus on MTEs who work with elementary teachers because of the currently limited number of studies focused exclusively on MTEs who teach secondary coursework. Much of the extant research on MTEs draws on the construct of mathematical knowledge for teaching (MKT; Ball et al., 2008), a model of teacher knowledge that is grounded in elementary school teaching. As such, research on MTEs is largely focused on MTEs who teach elementary PTs. We sought to answer the question: What frameworks for teacher knowledge are leveraged in conceptualizing MTE knowledge, and in what ways?

Conceptual Framework

Conceptualizing MTE Knowledge

Broadly defined, *knowledge* is the information and skills that teachers develop through experience and education. Researchers have long recognized the tension that exists for teachers between their general content-specific knowledge, developed in teacher education programs, and the *craft* knowledge that is developed through teaching practice (Shulman, 1986; Leinhardt et al., 1991). *Practice* refers to the things that teachers do consistently in their work with students (Lampert, 2010). Craft knowledge, for example, includes "know-how for teaching based on past experiences, empirical data, and well-reasoned arguments and predictions" (Hiebert & Morris, 2009, p. 476), the knowledge that is developed through experience working with students. Like others (Cochran-Smith & Lytle, 1999; Ball & Bass, 2000), we argue that teacher knowledge is intimately related to teaching practice – teachers' knowledge is further developed and enhanced over time as they teach.

Mathematical Knowledge for Teaching Teachers

Shulman (1986) introduced the term *pedagogical content knowledge* (PCK), which linked knowledge of teaching pedagogy with knowledge of the specific content being taught. Since then, a great deal of research has focused on identifying the knowledge needed for teaching mathematics (e.g., Ball et al., 2008; Davis & Simmt, 2006; Leinhardt et al., 1991; Ma, 1999; Rowand et al., 2005). Many of these studies argue that teachers need to understand the mathematics in the curriculum they teach in deep and connected ways that are specific to the needs of teachers. When combined with PCK, this knowledge base is commonly referred to as *mathematical knowledge for teaching* (MKT) (e.g., Ball et al., 2008).

Building on MKT research (Shulman, 1986; Davis & Simmt, 2006; Leinhardt et al., 1991; Ma, 1999), central to this review is an assumption that the knowledge needed by MTEs for their work with teachers differs from the knowledge needed by teachers for working with students. Researchers generally agree that the work of MTEs involves working with PTs and/or practicing teachers to develop their MKT (Jaworski, 2008a). As there is considerable diversity in the nature of MTEs' work, the range of expertise shared by MTEs is similarly diverse, involving varying levels of mathematical expertise, pedagogical expertise, and/or expertise derived from their experiences as teachers (Bergsten & Grëvholm, 2008). However, in addition to developing new content knowledge, MTEs need to help PTs' understand the ways in which the concepts they are learning connect to their future teaching of students. This requires MTEs to also understand what is involved in teaching mathematics to students. To simultaneously enhance different levels of PTs' awareness (as learners and future teachers), work that is different from what teachers do with students, MTEs need *mathematical knowledge for teaching teachers* (MKTT).

Method

Literature Search

We conducted a literature search using ERIC and Dissertation Abstracts using the following search terms: (1) *mathematics teacher educator(s)* AND *knowledge,* (2) *mathematics* AND *teacher educator* AND *knowledge*, (3) *teacher trainer* AND *knowledge*, and (4) *mathematics* AND *teacher trainer*. We also searched edited books and book series that were cited in many of the articles we reviewed, recent special issues of peer-reviewed journals, and conference proceedings of the Research in Undergraduate Mathematics Education (RUME) annual meetings, as these were not captured in the ERIC search results. The search covered articles written in English and published in peer-reviewed scientific journals or conference proceedings, as well as dissertations. Hereafter we use the term "articles" to refer to the resulting set of scientific studies, books, book chapters, dissertations, and conference proceedings. Because Shulman's seminal article about teacher knowledge was published in 1986 (Shulman, 1986) and the majority of the research on MTEs emerged in the early 2000s, we confined the search to 1986-2021. The literature search and initial review of abstracts for studies of MTE knowledge or its development (beyond just implications for MTE knowledge) resulted in 87 articles (including 6 dissertations). We then divided up the articles and individually reviewed each one in its entirety, summarizing each in terms of research goals, method(s), research questions, researcher's role, and findings, resulting in a final tally of 15 articles that proposed different frameworks for conceptualizing and characterizing MTE knowledge. Of these, we categorized 10 as proposing Complete Frameworks for MTE Knowledge and five as proposing Components of a Framework for MTE Knowledge. Additionally, we found one paper (Goos, 2009) that proposed a framework for *analyzing* MTE knowledge, which we will not discuss in this review.

Results: Frameworks for MTE Knowledge

Complete Frameworks for MTE Knowledge

We begin with Mason (1998) who presented one of the earliest frameworks for MTE knowledge. In particular, Mason conceptualized MTE knowledge not as different types of knowledge, but rather as different levels of awareness. Mason identified the key notions underlying teaching practice as the nature of awareness and the structure of attention (which encompasses the locus, focus, and form of attention moment by moment). Teaching teachers involves the refinement and development of a complex of awareness on all three levels, and this is manifested in alterations to the structure of attention. He proposed three levels of awareness: (a) *awareness-in-action* involves an awareness of one's actions and own learning, i.e., the development of one's knowledge of mathematical content; (b) *awareness-in-discipline* includes an awareness of one's awareness-in-action, i.e., the development of one's mathematical knowledge needed for teaching so that they can develop others' knowledge of mathematical content; and (c) *awareness-in-counsel* encompasses an awareness of how to develop awareness-in-discipline in others, i.e., the development of one's knowledge for teaching teachers so that they can develop other's mathematical knowledge for teaching. In this way, each level of awareness encompasses all prior levels. For Mason, MTEs develop awareness-in-discipline in PTs, as opposed to just awareness-in-action. Mason points to the unique nature of MTE knowledge and the ways in which it builds on, but is different from, teacher knowledge. His work provided a foundation for much of the research on MTE knowledge that followed.

Zaslavsky and Leikin (2004) expanded Jaworski's (1992) model for the practice of teaching mathematics to better understand the practice of teaching mathematics teachers. Jaworski's (1992) model, known as the "teaching triad of mathematics teachers," highlights the interactions among three important components of teaching: *challenging content for students* (mathematics), *the management of students' learning*, and *sensitivity to students*. Zaslavsky and Leikin created analogous terms for knowledge important to the teaching of teachers: *challenging content for mathematics teachers*, *the management of mathematics teachers' learning*, and *sensitivity to mathematics teachers*. In developing their "teaching triad of MTEs," Zaslavsky and Leikin considered Jaworski's "teaching triad for mathematics teachers" to be a subdomain of MTE knowledge, contained within the component of *challenging content for mathematics teachers*.

Similarly, Perks and Prestage (2008), reflecting on their work with PTs and their experiences as teachers themselves, proposed the "teacher-educator knowledge tetrahedron," positioning their entire "teacher knowledge tetrahedron" framework (Prestage & Perks, 1999) as a subdomain of MTE knowledge. Both frameworks highlight the interactions among four aspects of classroom practice: *teacher knowledge*, *professional traditions*, *practical wisdom, and learner knowledge*. However, in the teacher-educator knowledge tetrahedron, the learners in question are PTs and the teachers in question are MTEs. For Perks and Prestage, teacher-educator knowledge refers to the knowledge MTEs develop over time as they teach PTs; professional traditions refer to knowledge of teacher preparation coursework and research on mathematics teacher education; practical wisdom refers to the tasks and activities used with PTs; and, learner knowledge refers to the content that PTs need to understand. Notably, the teacher knowledge tetrahedron (Prestage & Perks, 1999) is entirely contained within the learner knowledge portion of their teacher-educator knowledge tetrahedron.

Reflecting on her work as an MTE, Chauvot (2009) conceptualized MKTT as consisting of different subdomains that encompass the varied responsibilities of a teacher educator who works with PTs. Specifically, she constructed a knowledge map consisting of subject matter content knowledge, PCK, and curricular knowledge that parallel knowledge domains for teachers described by Shulman (1986), as well as subsequent research on teacher knowledge (e.g., Ball et al., 2008). Chauvot's model expanded upon Shulman's work by placing all three of his domains of knowledge for teaching within the *subject matter content knowledge* of MTEs. She also extended several of the domains identified by Ball and colleagues (2008) and described how they are relevant to her work as a MTE-researcher (MTE-R), including knowledge of how to develop PTs' specialized and PCK and knowledge of how to engage PTs with content in ways that are connected to teaching. Chauvot also included the notion of *knowledge of context*, describing this knowledge subdomain for MTEs as including an understanding of contextual factors affecting the teachers with whom they work, e.g., standards and policies for teacher preparation programs, state teacher certification, and national accreditation. Like others, Chauvot's framework includes a teacher knowledge framework as one of the subdomains of MTE knowledge.

Shaughnessy and colleagues (2016) expanded upon the *instructional triangle* suggested by Cohen and colleagues (2003) by positioning the instructional triangle for teacher knowledge (i.e., interactions among teachers, students, and content) as the content knowledge needed by MTEs. Furthermore, they proposed analogous subdomains of MTE knowledge that parallel those of the MKT framework offered by Ball and colleagues (2008), arguing that MKT is a subdomain of MTE knowledge. These subdomains include *MTE common content knowledge* (e.g., knowledge of how to explain multi-digit subtraction that allows PTs to engage in this instructional practice); *MTE knowledge of content and students* (e.g., knowledge of common errors that PTs tend to make when engaging in instructional practices like regrouping using base-10 blocks); and *MTE knowledge of content and teaching* (e.g., knowledge of the types of tasks, representations, etc., that are useful in helping PTs learn an aspect of mathematics teaching).

Hauk and colleagues (2017) also expanded upon the MKT framework offered by Ball and colleagues (2008) in their conceptualization of MTE Knowledge, which they call Mathematical Knowledge for Teaching Future Teachers (MKT-FT). They argued that akin to the ways in which teachers require specialized mathematical knowledge for teaching students, MTEs require specialized knowledge specific to helping PTs develop MKT. The "subject matter" for MKT-FT is a combination of mathematics and mathematics education, as they conceptualize MKT becoming MTEs' common content knowledge. They then propose a framework for MKT-FT PCK, which builds upon a framework for teacher PCK proposed by Hauk and colleagues (2014). By adding a fourth domain, *knowledge of discourses*, connected to each of the three components of PCK in the framework offered by Ball and colleagues (2008), this model for teacher PCK becomes a tetrahedron with vertices that represent anticipatory, curricular, and implementation thinking. Hauk and colleagues (2017) expanded this tetrahedron model for teacher PCK to create an analogous tetrahedron to model MTE PCK, where the subdomain of knowledge of content and students for MTEs contains the entire framework for teacher PCK. The authors describe this subdomain as including knowledge of how to support PTs' development of MKT, as well as knowledge of how to engage PTs in learning to unpack mathematical ideas in ways needed for teaching students. Hauk and colleagues concluded by illustrating ways in which MTEs have used the MKT-FT framework in designing and implementing teaching-related tasks with PTs.

The conceptualization of MKTT that we have proposed (Castro Superfine et al., 2020; Olanoff et al., 2018; Welder et al., 2017) similarly extends from and is connected to the domains of MKT (Ball et al., 2008). We posit that akin to how MKT is composed of subject matter knowledge and PCK, MKTT is composed of subject matter knowledge for MTEs (including MKT) and PCK for MTEs (for facilitating PTs' learning of MKT). We conceptualize MTE-subject matter knowledge as being composed of three subdomains analogous to those comprising subject matter for teachers: *MTE common content knowledge* (which includes the entire framework for MKT), *MTE specialized content knowledge* (mathematical content knowledge that is specific to developing PTs' MKT), and *knowledge at the mathematical horizon for PTs*. We conceptualize MTE-PCK as being composed of three subdomains analogous to those comprising PCK for teachers: *knowledge of content and PTs* (i.e., MTE-knowledge of content and students), *knowledge of content and teaching PTs* (i.e., MTE-knowledge of content and teaching), and *knowledge of curriculum for PTs* (i.e., MTE-knowledge of curriculum). Our conceptualization of MKTT consists of not only the mathematical knowledge needed by teachers but also specialized knowledge of content that is unique to teaching PTs and knowledge of how to facilitate PT learning (i.e., relearning (Castro Superfine et al., 2020)). Like Beswick and Chapman (2012), we consider MKTT to be an elaborated extension of teacher knowledge that also includes domains of MTE knowledge that are characteristically different from teacher knowledge.

The most recently offered framework for MTE knowledge was published in a chapter of a book on the learning and development of MTEs edited by Goos and Beswick in 2021. Based on a review of research with and about mathematics teachers, Escudero-Avila and colleagues (2021) constructed a framework for MTE knowledge composed of seven subdomains, the last three of which the authors consider to be aspects of MTE PCK: 1) *mathematical knowledge*, which includes both knowledge of mathematics and MKT, 2) *knowledge about teachers' PCK*, which includes theories of teaching, key features in learning mathematics, and learning standards, 3) *knowledge about mathematics teaching practices and skills*, 4) *knowledge about professional identity*, 5) *knowledge of the features of the professional development of mathematics teachers*, 6) *knowledge of teaching the content of initial mathematics teacher education programmes*, and 7) *knowledge of the standards of mathematics teacher education programmes*.

In contrast to all of the frameworks discussed above, in the concluding chapter of a volume on MTE knowledge and practice that she edited, Jaworski (2008b) suggested that there are aspects of teacher knowledge that are unique to teachers, just as there are aspects of MTE knowledge that are unique to MTEs. In doing so, she conceptualized the relationship between MTE knowledge and teacher knowledge using a Venn diagram. Similar to Chavout (2009), Jaworski posits that unique to MTE knowledge is the professional and research literature related to mathematics teaching and learning, including knowledge of theories of learning and teaching and knowledge of methodologies of research focused on learning and teaching in educational systems. MTEs utilize their knowledge of methodologies used to study teaching and learning in schools as they work with PTs both in and out of school settings. According to Jaworski, knowledge unique to the needs of teachers, and not necessarily needed by MTEs, includes knowledge of school contexts and elementary mathematics curricula.

Components of a Framework for MTE Knowledge

Five articles do not include complete frameworks for MTE knowledge, but rather, they present components of a framework. Building on Shulman's (1986) subdomain curricular knowledge, in particular, Chauvot (2008) characterized what such knowledge entails for MTEs. Relevant to this review, she identified four components of curriculum knowledge for MTEs based on her experiences as an MTE and as a mathematics education researcher. These components include (a) *knowledge of programs and materials* (e.g., different models for teacher preparation, textbooks, and materials for use in courses for PTs), (b) *knowledge of indications and contraindications of curricula* (e.g., use of curricula or program materials in particular circumstances and effectiveness of curriculum programs), (c) *lateral curriculum knowledge* (e.g., knowledge of other courses PTs are enrolled in), and (d) *vertical curriculum knowledge* (e.g., knowledge of coursework that precedes and follows current courses in which PTs' are enrolled). Chauvot concluded by highlighting the use of MTE curricular knowledge in current studies of MTEs' professional learning, arguing for the centrality of curricular knowledge in MTEs' work.

In their studies of an MTE's practice, Chick and Beswick (2013; 2017) proposed a framework for the subdomain of MTE PCK, building their descriptions of PCK for MTEs from descriptions of the PCK teachers require. Analyzing the teaching practice of the first author, Chick and Beswick identified several components of MTE PCK, including knowledge of examples, curriculum, student thinking, and common misconceptions, among others. Using vignettes of practice from the first author's practice, they provided evidence of the existence of the various components, illustrating ways in which such knowledge is leveraged as an MTE teaches PTs. For each component, they identified ways in which such components of MTE knowledge are similar to and different from teachers' knowledge.

Olanoff (2011) used observations of and interviews with three MTEs teaching fraction multiplication and division concepts, to identify several components of MKTT. Unlike other authors, Olanoff examines the components of the domain of MKTT, with a particular focus on the knowledge components leveraged while teaching a particular concept to PTs. These components include knowledge of (a) multiple representations of the topics, how the representations relate to other topics, and which representations best support PTs in making connections, (b) how to set specific goals for student learning, and (c) how to design and use assessments effectively. Like Chick and Beswick (2013; 2017), Olanoff provided evidence of the existence of MTE knowledge components that are unique to MTEs.

Felton-Koestler (2020) proposed a framework for *knowledge for sociopolitical mathematics teaching* (KSMT), knowledge that teachers at all levels, including teacher preparation, need for addressing issues of equity and social justice by what he calls *mathematizing sociopolitical issues*. This framework extends MKT, which he uses as a blanket term to address the specialized knowledge for teachers and MTEs, to include knowledge of sociopolitical issues and knowledge of sociopolitical curriculum. To have and develop KSMT, teachers and MTEs must be aware of current sociopolitical issues and be able to turn a critical lens on how they are presented in the general discourse. This work goes beyond just understanding the effects of individual biases on current events to include institutional and structural forms of oppression. Although this framework is generalized to both teachers and MTEs, Felton-Koestler does identify a need for future work to address differences in KSMT for teachers and MTEs. Regardless of these potential differences, it is noteworthy to consider KSMT as a component of MTE knowledge.

Discussion

There is general consensus within the teacher education community that the knowledge MTEs require in their work with PTs includes not only the knowledge teachers need to know but also unique elaborations of that knowledge. In fact, many of the articles we reviewed conceptualize MTE knowledge as an extension of teacher knowledge. That is, these frameworks not only position teacher knowledge as a subdomain of MTE knowledge but also partition MTE knowledge into subdomains similar to those found in frameworks for teacher knowledge. In our previous work (Castro Superfine et al., 2020; Olanoff et al., 2018; Welder et al., 2017), similar to Hauk and colleagues (2017), we propose a fractalization metaphor to describe the ways in which various articles conceptualize MTE knowledge as the visualizations of many of these frameworks resemble part of a fractal. We use the term *fractalization* to refer to the process by which one component or subdomain is entirely contained within a larger subdomain, where the larger subdomain is analogous in structure to the smaller one. In many instances, a teacher knowledge framework becomes the content knowledge subdomain of a framework for the knowledge needed by MTEs. We refer to this fractalization metaphor throughout the discussion.

Three main themes emerged from our review of frameworks for MTE knowledge. First, many of these frameworks build on existing frameworks for teacher knowledge, and in many instances, represent fractalizations of teacher knowledge frameworks (e.g., teaching triad (Cohen et al., 2003), instructional triangle (Jaworski, 1992), teacher knowledge tetrahedron (Prestage & Perks, 1999)). Therefore, many frameworks for MTE knowledge included a teacher knowledge framework in its entirety as one of its subdomains. Only Jaworkski (2008b) proposed a Venn Diagram to suggest that teachers require additional knowledge (e.g., school context, elementary curriculum) not needed by MTEs in their work with PTs. Furthermore, researchers conceptualized certain subdomains (and related components) of MTE knowledge as being "meta" forms of analogous teacher knowledge subdomains (e.g., Chauvot, 2009; Hauk et al., 2017; Perks & Prestage, 2008; Zaslavsky & Leikin, 2004). For example, similar to Shaughnessy and colleagues (2016), we (Castro Superfine et al., 2020; Olanoff et al., 2018; Welder et al., 2017) have proposed MTE knowledge subdomains analogous to the MKT framework from Ball and colleagues (2008). These include MTE common content knowledge (which contains MKT), MTE knowledge of content and students, and MTE knowledge of content and teaching. In other words, as teachers of teachers, MTEs require similar types of knowledge needed by teachers, but MTEs need knowledge of these subdomains in ways that are specific to teaching PTs. Thus, we posit that fractalization can be a useful metaphor for conceptualizing MTE knowledge.

Second, despite some overall similarities, there are important differences in the subdomains of the frameworks we reviewed. Broadly speaking, all of the knowledge frameworks in our review include some or all of four main subdomains representing extensions of teacher knowledge domains (e.g., Ball et al., 2008; Shulman, 1986): knowledge of content, knowledge of curriculum and context, knowledge of PTs, and knowledge of ways of supporting PT learning. However, some frameworks included subdomains unique to MTEs, such as Zaslavsky and Leikin's (2004) sensitivity to mathematics teachers (PTs). Considering the uniqueness of PTs as a population of learners, this knowledge includes understanding that PTs often enter teacher preparation programs with limited conceptual understandings of mathematics. As such, the work for MTEs is to support PTs' relearning of mathematics, which involves PTs ultimately reconstructing their previously developed knowledge of mathematics (Author 2020; Zazkis, 2011). Further, Chauvot (2009) included MTEs' knowledge of mathematics education research as part of the knowledge needed to effectively prepare PTs. In fact, Chauvot posited that knowledge of research in mathematics education underlies all other knowledge subdomains MTEs require in their work with PTs (e.g., knowing the research on how children learn can inform MTEs' content course design). This subdomain echoes other researchers who describe conducting research in mathematics teacher education as a form of professional learning (e.g., Rowland et al., 2014). Notably, these different subdomains are unique to MTEs and arguably do not have analogous subdomains in a teacher knowledge framework. In addition, there are important differences in the grain size at which researchers conceptualized MTE knowledge. While the majority only identified subdomain levels using the four main subdomains described above, a few deconstructed their subdomains into components. For example, Chauvot (2008) described components of MTE curricular knowledge (e.g., lateral and vertical curricular knowledge); whereas we (e.g., Castro Superfine et al., 2020) specified components of MTEs' content-specific knowledge (e.g., specialized content knowledge, knowledge of content and teaching). Such variation is indicative of the fragmented research landscape on MTE knowledge.

A third theme that emerged is the process by which MTE knowledge has been conceptualized, which has largely been from a knowledge-in-practice perspective. Through an analysis of various artifacts of practice and reflections on MTEs' work with PTs, researchers applying a knowledge-in-practice perspective to the work of MTEs highlight the types of knowledge leveraged as they teach PTs, reinforcing the dynamic relationship between knowledge and practice. A majority of articles on MTE knowledge describe a self-study process wherein one or more authors reflect on and describe the types of knowledge they leveraged in their work with PTs (e.g., Masingila et al., 2018; Muir et al., 2017; Zazkis & Mamolo, 2018). Such a process is productive for understanding the types of resources (e.g., experiences, beliefs, knowledge) that impact MTEs' practice to support drawing on different types of knowledge in light of their expertise. However, more work needs to be done to explicate the analytic processes taken in research on MTE knowledge so that others can employ similar methods and contribute to the growing knowledge base on MTEs. Moreover, while self-studies (i.e., research *by* MTEs) provide unique insights into the nature of MTE knowledge, the research base would be strengthened by research *on* MTEs to corroborate and further specify the various MTE knowledge subdomains and components.

The goal of this review was to explore the theoretical underpinnings of the existing frameworks for MTE knowledge. We find that MTE knowledge frameworks that are fractalized versions of teacher knowledge frameworks tend to miss the same aspects of MTE knowledge that are missing from teacher knowledge frameworks. For example, Felton-Koestler's (2020) KSMT framework addresses MTE's knowledge of equity, a domain that does not explicitly appear in any of the fractalized frameworks. Additionally, none of the frameworks discussed the need for MTEs to be knowledgeable of classroom technologies. The COVID-19 pandemic and subsequent shifts to online courses demonstrated the need for MTEs to have knowledge of relevant technological tools, not only for their practices as teacher educators but also for preparing PTs to use such tools in their future classrooms. There exists a corpus of literature on *technological pedagogical and content knowledge* (TPACK) for mathematics teachers (e.g., Kohler & Mishra, 2009), but this work has not yet been integrated into frameworks for MTE knowledge. To address some of these missing aspects and further build a mutually agreed-upon knowledge base for MTEs, researchers might analyze MTEs' reflections of their teaching practices to understand their knowledge-in-practice and the types of knowledge MTEs draw on in their work with PTs. Such a knowledge base could inform the design and implementation of opportunities to improve the preparation and professional development of MTEs.

References

Adler, J. (2010). *Professional knowledge matters in mathematics teaching*. In the Proceedings of the International Congress of Mathematicians. Hyderabad, India.

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at http://amte.net/standards

Bai, H., & Ertmer, P. (2008). Teacher educators' beliefs and technology uses as predictors of preservice teachers' beliefs and technology attitudes. *Journal of Technology and Teacher Education*, *16*(1), 93–112.

Ball, D., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.) *Multiple perspectives on mathematics of teaching and learning*. (pp. 83–104). Westport, CT: Ablex Publishing.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.

Bergsten, C., & Grëvholm, B. (2008). Knowledgeable teacher educators and linking practices. In T. Wood & B. Jaworski (Eds.), *The mathematics teacher educator as a developing professional*, (pp. 223–246). Rotterdam: Sense Publishers.

Beswick, K., & Chapman, O. (2012). Mathematics teacher educators' knowledge for teaching. In S. J. Cho (Ed.), *Proceedings of the 12th International Congress on Mathematical Education* (pp. 629–634). Seoul, Korea.

Castro Superfine, A., Prasad, P. V., Welder, R. M., Olanoff, D., & Eubanks-Turner, C. (2020). Exploring mathematical knowledge for teaching teachers: Supporting prospective teachers' relearning of mathematics. In A. Appova, R. M. Welder, and Z. Feldman, (Eds.), Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. *Special Issue: The Mathematics Enthusiast*, ISSN 1551-3440, vol. 17, nos. 2 & 3, pp. 367–402. ScholarWorks: University of Montana.

Chauvot, J. (2008). Curricular knowledge and the work of mathematics teacher educators. *Issues in Teacher Education*, *17*(2), 83.

Chavout, J. (2009). Grounding practice in scholarship, grounding scholarship in practice: Knowledge of a mathematics teacher educator-researcher. *Teaching and Teacher Education*, *25*, 357–370.

Chick, H., & Beswick, K. (2013). Educating Boris: An examination of PCK for mathematics teacher educators. In V. Steinle, L. Ball, & C. Bardini (Eds.). Mathematics education: Yesterday, today and tomorrow. *Proceedings of the 36th Annual Conference of the Math Education Research Group of Australasia* (pp. 170–177). Melbourne: MERGA.

Chick, H., & Beswick, K. (2017). Teaching teachers to teach Boris: A framework for mathematics teacher educator PCK. *Journal of Mathematics Teacher Education*, *21*(5), 475–499.

Cochran-Smith, M. & Lytle, S. (1999). Relationships of knowledge and practice: Teacher learning in communities. *Review of Research in Education, 24*, 249–305.

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119–142.

Conference Board of the Mathematical Sciences (2012). *The Mathematical Education of Teachers II*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Council for the Accreditation of Educator Preparation. (2013). *CAEP accreditation standards and evidence: Aspirations for educator preparation*. Author: Washington DC.

Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the math that teachers (need to) know. *Educational Studies in Mathematics*, *61*, 293–319.

Dinham, S. (2013). The quality teaching movement in Australia encounters difficult terrain: A personal perspective. *Australian Journal of Education, 57*, 91-106.

Escudero-Avila, D., Montes, M., & Contreras, L.C. (2021). What do mathematics teacher educators need to know? Reflections emerging from the content of mathematics teacher education. In M. Goos & K. Beswick (Eds.) *The learning and development of mathematics teacher educators: International perspectives and challenges* (pp. 23–40). Springer.

Felton-Koestler, M. (2020). Teaching sociopolitical issues in math teacher preparation: What do mathematics teacher educators need to know? In A. Appova, R. Welder, & Z. Feldman, (Eds.), Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. *Special Issue: The Mathematics Enthusiast*, ISSN 1551-3440, Vol. 17, nos. 2 & 3, pp. 435–468. ScholarWorks: University of Montana.

Garofalo, J., Drier, H., Harper, S., Timmerman, M.A., & Shockey, T. (2000). Promoting appropriate uses of technology in math teacher preparation. *Contemporary Issues in Technology and Teacher Education*, *1*(1), 66–88.

Goos, M. (2009). Investigating the professional learning and development of math teacher educators: A theoretical discussion and research agenda. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 209–216). Palmerston, NZ: MERGA.

Goos, M., & Beswick, K. (Eds.). (2021). *The learning and development of mathematics teacher educators: International perspectives and challenges*. Switzerland: Springer Nature Switzerland AG.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching*, *15*(2), 273–289.

Gutierrez, R. (2013). Why (urban) math teachers need political knowledge. *Journal of Urban Mathematics Education*, *6*(2), 7–19.

Hiebert, J., & Morris, A. (2009). Building a knowledge base for teacher education: An experience in K-8 mathematics teacher preparation. *The Elementary School Journal, 109*(5), 475–490.

Hauk, S., Jackson, B., & Tsay, J. (2017). Those who teach the teachers: Knowledge growth in teaching for math teacher educators. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education* (428-439). San Diego, CA.

Hauk, S., Toney, A., Jackson, B., Nair, R., & Tsay, J.-J. (2014). Developing a model of pedagogical content knowledge for secondary and post-secondary mathematics instruction. *Dialogic Pedagogy: An International Online Journal*, *2*, 16-40.

Huang, D. (2017). *Unpacking math teacher educators' decision making when designing online professional development programs* [Unpublished doctoral dissertation]. The Ohio State University.

Jaworski, B. (1992). Mathematics teaching: What is it? *For the Learning of Mathematics*, *12*(1), 8–14.

Jaworski, B. (2008a). Mathematics teacher educator learning and development: An introduction. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (pp. 1–17). Rotterdam, Netherlands: Sense Publishers.

Jaworski, B. (2008b). Development of the mathematics teacher educator and its relation to teaching development. In B. Jaworski & T. Wood (Eds.), *The International Handbook of Mathematics Teacher Education: The Mathematics Teacher Educator as a Developing Professional* (Vol. 4, pp. 335–361). Rotterdam, Netherlands: Sense Publishers.

Kalonde, G., & Mousa, R. (2016). Technology familiarization to preservice teachers: Factors that influence teacher educators' technology decisions. *Journal of Educational Technology Systems*, *45*(2), 236–255.

Kersaint, G. (2003). Technology beliefs and practices of mathematics education faculty. *Journal of Technology and Teacher Education*, *11*(4), 549–577. Norfolk, VA: Society for Information Technology & Teacher Education.

Koehler, M., & Mishra, P. (2009). What is technological pedagogical content knowledge (TPACK)?. *Contemporary Issues in Technology and Teacher Education, 9*(1), 60-70.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal*, *109*(*5*), 491–509.

Leinhardt, G., Putnam, R., Stein, M. & Baxter, J. (1991). Where subject matter knowledge matters. In J. Brophy (Ed.), *Advances in Research on Teaching* (Vol. 2, pp. 87–113). JAI Press: Greenwich, CT.

Ma, L. (1999). *Knowing and teaching elementary math: Teachers' understanding of fundamental math in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Masingila, J., Olanoff, D., & Kimani, P. (2018). Mathematical knowledge for teaching teachers: Knowledge used and developed by math teacher educators in learning to teach via problem solving. *Journal of Mathematics Teacher Education*, *21*(5), 429–450.

Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, *1*(3), 243–267.

McLeman, L., & Vomvoridi-Ivanovic, E. (2012). Understanding the knowledge and practices of math teacher educators who focus on developing teachers' equitable math pedagogy. *REDIMAT - Journal of Research in Mathematics Education*, *1*(3), 278–300.

Muir, T., Wells, J., & Chick, H. (2017). Developing an understanding of what constitutes mathematics teacher educator PCK: A case study of a collaboration between two teacher educators. *Australian Journal of Teacher Education*, *42*(12), 60–79.

Niess, M. (2006). Guest Editorial: Preparing teachers to teach math with technology. *Contemporary Issues in Technology and Teacher Education*, *6*(2), 195–203.

Olanoff, D. (2011). *Mathematical knowledge for teaching teachers: The case of multiplication and division of fractions* [unpublished doctoral dissertation]. Syracuse University.

Olanoff, D., Welder, R. M., Prasad, P., & Castro Superfine, A. (2018). Fractalization as a metaphor for mathematical knowledge for teaching teachers: Synthesizing the frameworks and exploring the consequences. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 500-503). Greenville, SC: University of South Carolina & Clemson University.

Perks, P. & Prestage, S. (2008). Tools for learning about teaching and learning. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 31–56). Rotterdam, Netherlands: Sense Publishers.

Prestage, S., & Perks, P. (1999). Towards a pedagogy of initial teacher education. In L. Bills (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, *19*(3), 91-96.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' math subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8*, 255–281.

Rowland, T., Turner, F., & Thwaites, A. (2014). Research into teacher knowledge: A stimulus for development in MTE practice. *ZDM Mathematics Education*, *46*(2).

Shaughnessy, M., Garcia, N., Selling, S., & Ball, D. (2016). What knowledge and skill do math teacher educators need and (how) can we support its development? In M. Wood, E. Turner, M. Civil, & J. Eli (Eds.), *Proceedings of the 38th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 813–820). Tucson, AZ: University of Arizona.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14.

Welder, R. M., Prasad, P. V., Castro Superfine, A., & Olanoff, D. (2017). Developing a framework for mathematical knowledge for improving the content preparation of elementary teachers. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (p. 637). Indianapolis, IN: Purdue University.

Zaslavsky, O., & Leikin, R. (2004). Professional development of math teacher educators: Growth through practice. *Journal of Mathematics Teacher Education*, *7*, 5–32.

Zazkis, R. (2011). *Relearning mathematics: A challenge for prospective elementary school teachers.* Information Age Publishing: Charlotte: NC.

Zazkis, R., & Mamolo, A. (2018). From disturbance to task design, or a story of a rectangular lake. *Journal of Mathematics Teacher Education, 21*, 501–516.

EXAMINING THE NATURE OF PEDAGOGICAL CONTENT KNOWLEDGE (PCK) WITH A VALIDATION ARGUMENT FOR THE PCK-FRACTIONS MEASURE

Maryam Zolfaghari	Karl W. Kosko	Christine K. Austin
Kent State University	Kent State University	Kent State University
mzolfagh@kent.edu	**kkosko1@kent.edu**	**causti17@kent.edu**

This study presents an extension of the validity argument for the PCK-Fractions measure. PCK-Fractions is designed to assess the effectiveness of professional experiences in facilitating teachers' pedagogical content knowledge (PCK) for children's fraction reasoning in grades 3-5. We examined data across 101 participants from two Midwest universities including non-education majors, education majors, and education majors with grade 3 to 5 math field experiences. Results provide additional validity evidence for the PCK-Fractions. Namely, evidence indicates differences in scores between groups of participants—preservice teachers grade 3 to 5 field experience had higher scores than those without, and all preservice teachers had higher scores than non-education majors.

Keywords: Mathematical Knowledge for Teaching; Rational Number.

Introduction

Mathematical knowledge for teaching (MKT) is a practice-based theory that investigates the nature of professional knowledge of mathematics that teachers use to make effective instructional decisions to help students' learning (Ball & Bass, 2002, Ball, Thames, & Phelp, 2008). MKT as a theory and set of constructs has been largely developed and disseminated with quantitative assessments across mathematics content. For instance, many scholars designed different kinds of MKT assessments in geometry, rational numbers, and number sense from grades K to 12 (Herbst & Kosko, 2014; Hill et al., 2008; Kazemi & Rafiepour,2018; Khakasa & Berger, 2016). MKT includes two primary domains of content knowledge (CK) and pedagogical content knowledge (PCK). While both domains are considered vital to effective teaching, most MKT assessments either target teachers' CK explicitly (Lo & Luo; 2012), or include both CK and PCK in the same assessment (Depaepe et al., 2015). There has been little attention to designing MKT assessments that focus on PCK as a domain worthy of investigation on its own (Copur-Gencturk et al., 2019; Hill et al., 2008; Zolfaghari et al., 2021). Rather, by incorporating PCK assessments within efforts to develop CK measures, many items designated as 'PCK items' have later been found to assess CK instead (Copur-Gencturk et al., 2019; Hill et al., 2008). This led to Zolfaghari et al. (2021) focusing on a PCK exclusive measure for fractions. Piloting items with a focus on the 'task of teaching' of assessing children's fraction reasoning resulted in a more accurate representation of the domain.

The current study is our continued effort to validate our framework for how PCK for Fractions develops, and our associated construct map for the PCK-Fractions measure (Zolfaghari et al., 2020, 2021). In this validation process, we hypothesize that PCK develops in a particular way that is useful in designing items of varying difficulty – described as *construct maps* within validity argument literature. We also found that exposure to coursework and field experience had a positive association with PCK scores. However, there was a need to better examine this phenomenon with a larger sample. Given prior results (Zolfaghari et al., 2020; 2021), we conjecture individuals' experiences inform the development of their PCK. In particular, we sought to understand the degree to which majoring in teacher education and having certain forms of field experience affected PCK for fractions.

Theoretical Framework

MKT is "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill et al., 2005, p.373). It involves abilities such as analyzing and interpreting students' reasonings and determining the related materials and information based on that. Mastering these skills promotes good instruction and, as a result, effective mathematics learning for students. The emphasis on knowledge of teaching a subject was primarily introduced by Shulman (1986) as PCK. Later, Ball and colleagues applied Shulman's theory to mathematics, thus framing MKT (Ball et al., 2008). MKT contains two primary domains of CK and PCK, with each of these domains consisting of several subconstructs (Hill et al., 2008b). To explore teachers' MKT, various assessments were designed within several mathematical topics, including geometry (Herbst & Kosko, 2014), rational numbers (Kazemi & Rafiepour, 2018), multiple mathematical topics at the secondary level (Khakasa & Berger, 2016) and elementary level (Hill et al., 2008), statistic subject (Siswono et al., 2018), and so forth. These various scholars explored several components of MKT, with many finding that MKT scores aligned with teachers' professional experience (Herbst & Kosko, 2014) and effective instruction (Hill et al., 2008).

MKT is professional knowledge; thus, various forms of professionalized experience have been found to affect and/or facilitate MKT. For instance, in examining the effect of types of experiences on teachers' MKT, Hill (2010) found characteristics such as grade taught, math content course, years of experiences, and math self-concept associated with teachers' MKT. Although these associations vary in terms of their strengths. Jakobsen et al. (2011) noted the grade level at which the teachers taught was related to their MKT scores. Similarly, in the study of mathematic teachers grade 3-7, Copur-Gencturk (2020) found that teachers with experiences teaching higher grades had stronger MKT scores. However, the number of years of mathematics teaching had a weak associate with teachers' MKT scores.

Research on MKT for fractions typically includes both domains of CK and PCK (Depaepe et al., 2015; Tirosh, 2000; Trobst et al., 2018). For instance, in studying PSTs' knowledge of teaching fractions, Tirosh (2000) found that most PSTs know how to solve fraction division (CK) but are unable to explain children's strategy or misconception (PCK). Similarly, examining secondary and elementary preservice teachers, Depaepe et al. (2015) noticed that PSTs with significant differences in demonstrated CK did not demonstrate differences in their PCK. A common premise across such studies is that CK is a prerequisite for higher PCK. Indeed, in describing PCK applied to assessing a student's error, Hill et al. (2008) noted that "teachers must be able to examine and interpret the mathematics behind student errors prior to invoking knowledge of how students went astray" (p. 390). Yet, Trobst et al. (2018) found that PSTs were able to increase their PCK despite limited CK. Coupled with Depaepe et al.'s (2015) findings of no statistical relationship between the constructs, there appears to be inconsistent evidence towards the common stance that CK is a prerequisite for PCK.

To be clear, there is a large body of evidence that supports a relationship between CK and PCK (Agathangelou & Charalambous, 2020; Depaepe et al., 2015; Hill et al., 2008). However, various findings in reports such as those described in the prior paragraph should not be disregarded. We believe the inconsistent findings in the literature point to an issue noticed by Hill et al. (2008) and expanded upon by Copur-Gencturk et al. (2019): development of PCK measures has focused more on CK than PCK and the resulting assessments have led to conflicting reports and an incomplete understanding of PCK as a theoretical construct. For this reason, PCK-Fractions was designed exclusively to focus on PCK (Zolfaghari et al., 2020; 2021), with an initial focus on the PCK sub-domain of knowledge of content and students (KCS). This targeted focus allowed us to decrease the risk to validity in designing unintended items that might measure CK instead of PCK.

Measuring PCK-Fractions

The intended purpose, or use, of the PCK-Fractions measure is to assess the effectiveness of professional experiences in promoting or facilitating teachers' PCK for upper elementary fractions concepts. The present version of PCK-Fractions focuses on the KCS domain, and represents one of several studies that collectively construct a validity argument for PCK-Fractions (Zolfaghari et al., 2020; 2021). Hill et al. (2008) define KCS as "used in tasks of teaching that involve attending to both specific content and something particular about learners" (p. 375). *Tasks of teaching* are the fundamental means Ball and colleagues have designed items for MKT, focusing on what a teacher must do in a specific aspect of the profession. For KCS, we focused on assessing children's reasoning as the task of teaching for fractions. Zolfaghari et al. (2021) initially adapted the sequence children learn certain fraction concepts but found that "student actions being assessed [were] a better explainer of why certain items have different difficulties" (p. 241). Table 1 presents a construct map developed from the initial pilot data (Zolfaghari et al., 2020; 2021). At Level 1, teachers are able to assess how children partition, fragment, or fair share fractional parts. At Level 2, teachers assess whether and how children coordinate parts to whole. Specifically, children may or may not have developed part-whole reasoning, but a teacher is able to distinguish between children's actions demonstrating such reasoning or not. Level 3 demonstrates an ability to assess how children compare and use different fractions. Level 4 focuses on a teachers' ability to assess how children coordinate non-unit fractions with the whole – actions often demonstrated in fraction multiplication and division.

Table 1: Construct map for PCK-Fractions.

Level	Description
Level 1	Assess children's creation and/or use of fractional parts.
Level 2	Assess children's coordination of parts and of the whole.
Level 3	Assess children's creation and use of non-unit fractions & comparison of fractions.
Level 4	Assess children's coordination of non-unit fractions with the whole & comparison of fractions and wholes.

Key in understanding the role of the construct map presented in Table 1 is that it is child-focused, not task-focused. Rather, a teachers' ability to assess children's reasoning begins with a focus on the child. As an example, consider item F44 in Figure 1. The mathematical task at-hand is fraction addition that requires the child to convert to a common denominator to obtain a correct answer. However, that's not what this child does. Rather, Tim counts the total shaded and unshaded and draws a representation illustrating that total. Thus, Tim is coordinating parts of the fraction, but without maintaining the whole. A teacher correctly assessing Tim's reasoning is, therefore, at Level 2 or higher. This focus on the student's reasoning within a mathematical task was key in aligning items with the construct map (Table 1). Additionally, all PCK-Fraction items were designed as multiple choice. Most items followed a common template in wording, as evidenced in responses for F44 (Figure 1). Specifically, response options were adapted from fraction diagnostic literature written for teachers (i.e., Battista, 2012; Hackenberg et al., 2016) and were listed in order of lower to higher level reasoning evidenced by the student depicted in the prompt. This design was purposeful, as prior data collection suggested randomization of such levels and changes in language decreased consistency in responses across items (Zolfaghari et al., 2020; 2021).

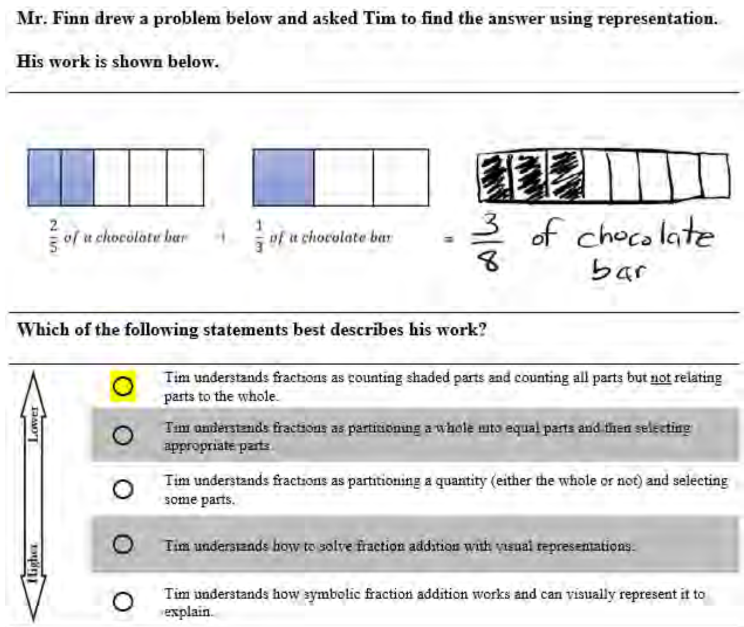


Figure 1: Example item (F44) from PCK-Fractions with correct response highlighted.

As noted previously, the present study is one of several papers that present evidence towards a validity argument for PCK-Fractions (Zolfaghari et al., 2020; 2021). Validity arguments consist of various claims regarding the validity of a measure, and evidence that warrants such claims (Kane, 2012). Further, such arguments are conveyed through “the accumulation of evidence from various sources for claims” (Krupa et al., 2019, p. 11) and “should occur over several studies in order to provide adequate warrants for claim of inference for a measure” (Kosko, 2019, p. 19). Prior validity evidence for PCK-Fractions suggests a unidimensional construct that aligns with the construct map presented in Table 1 (Zolfaghari et al., 2021). There is evidence that exposure to mathematics methods coursework facilitates PCK-Fractions, but prior samples also represented skew relative to higher scorers (Zolfaghari et al., 2020; 2021). Rather, Zolfaghari et al. (2021) identified a need for sampling participants with a much wider range of ability to better support the validity argument for PCK-Fractions. The present study focused efforts at recruiting a sample we believed would have a wider range in ability, but in doing so we also sought to align this range in ability with the construct map itself. Rather, we sought to better understand how participants with little to no exposure to teaching (outside of once being an elementary student) and those with more experience in such contexts would respond to our items. In doing so, we also sought to understand how responses aligned with our construct map. These efforts fulfill the purpose of the study which is to understand the effect of different types of professional experience on the PCK-Fractions measure by investigating the validity argument process. Particularly, we addressed the following research question:

How does the construct map for revised PCK-Fractions measure align with validity evidence for individuals with different professional teaching experiences?

Method

Participants included 101 undergraduate students enrolled in two Midwest universities. To ensure a sample with a wider range of PCK-Fraction scores, both high and low, we recruited undergraduate students enrolled in a Marketing 101 course (*n*=39) who were not, had not, and did not plan to major in education. Non-education majors primarily included financing, marketing, fashion design, and business. We also recruited participants from 36 early childhood (certification preK to grade 5) and 26 middle childhood majors (certification grades 4-8) across both universities (*n*=62). Because grades 3-5 (the focus grade band for PCK-Fractions) was included in the teacher licensure for both majors, these participants were asked to report whether, and in which grade levels, they had field experience. This resulted in 21 education majors reporting having grades 3-5 field experiences that included teaching mathematics (20.8% of total sample). Across the entire sample, almost all participants self-identified as white (97%). Also, 79.2 % of the participants self-identified as female, 18.8 % male and 2% nonbinary.

Measure

As previously described, PCK-Fractions items were designed to measure teachers' PCK for grades 3-5 students' fraction reasoning. Prior versions of the PCK-Fractions measure included 15 questions, which included several 'multiple-response' items that allowed participants to select 'all that apply' (Zolfaghari et al., 2020). However, following the pilot of the measure, such items were either revised into multiple-choice items or removed to reduce variance in response data. Previous Rasch modeling of the PCK-Fractions measure indicated sufficient item reliability of .90, but less than ideal person reliability of .41 (Zolfaghari et al., 2021). A primary reason identified for the poor person reliability was a negative skew in participants' scores with "75.3% of participants having a score above 0.00, or average ability" (p. 238). The current version of the measure includes 19 multiple-choice items (see Figure 1 for an example item). As noted earlier, these items were revised to further improve item reliability, reduce variance in responses, and better align theoretically to the improved construct map. Responses were all coded dichotomously (0 = incorrect, 1 = correct) for Rasch modeling.

Analysis and Findings

This paper examined validity evidence for revised PCK-Fractions and its construct by adopting the *Standards for Educational and Psychological Testing* (AERA et al., 2014) and building upon the prior validity evidence for PCK-Fractions measure (Zolfaghari et al., 2020; 2021). For the present study, we focused particular on validity evidence related to response processes andinternal structure. Evidence towards *response processes* focuses on how participant responses correspond with the intended theoretical design of the items. Evidence towards *internal structure* was used to provide study how response processes correspond to the conceptual framework for the revised PCK-Fractions' measure. The Rasch analysis and the data from one-way ANOVA allowed us to seek the validity evidence for revised PCK-Fractions as well as how participants' different level of experiences related their scores in PCK construct map.

Rasch Modeling

Rasch modeling was conducted to examine internal structure validities for PCK-Fraction assessment. Namely, we used item and person reliability aligned with unidimensionality and fit statistics to examine the validity of survey (AERA et al., 2014). The initial item analysis indicated the acceptable item reliability of .95, which exceeded the acceptable threshold of .90 (Linacre, 2021). Likewise, the item separation index showed a sufficient value of 4.23 (above 2.0 implies a good differentiation of item difficulty). In particular, this item separation value (4.23), indicates the items can be differentiated into four different tiers, which coincidentally aligns with the four levels of construct map (see Table 1). This result suggests that the hierarchy of difficulty of the items for PCK-Fractions would remain constant for different samples in similar contexts. Additional psychometric results from the average mean square for item infit (MNSQ= 1.00, Z=.00) and outfit (MNSQ= 1.01, Z= .10) provide another indicator that the data fit the model. These findings suggested that our revised PCK-Fractions measure represents a significantly wider range of item difficulties than the initial PCK-Fractions—improving from an item reliability of .90 and item separation of .30 (Zolfaghari et al., 2021) to item reliability of .95 and item separation of 4.23. Collectively, these results provide validity evidence that the measure is aligned to the construct map in Table 1.

Despite otherwise ideal psychometric data, the person reliability estimated for participants was .42 which is lower than the acceptable threshold of near or above .80. Further analysis showed that, although the person reliability was low, the average mean square for person infit (MNSQ= .99, Z=.00) and outfit (MNSQ= .99, Z= .00) aligned with Rasch Model expectation. This indicated that all participants' responses to the items behaved as expected (Bond & Fox, 2015). Two potential reasons for a low person reliability are: 1) the spread of item difficulty is too narrow or 2) the ability range of participants assessed is too narrow. In order to examine these issues, the Wright Map was investigated (see Figure 2). As indicated in the Wright Map, items are almost evenly distributed between -2.00 and 2.00 logits. Recalling that item reliability and separation index values were both ideal, this data suggests that the spread of item difficulty was not too narrow. Thus, we further examined the second most common reason for low person reliability.

Mean scores for participants were near the model average score of 0.00 (M = -.14, SD = .65), which did not clearly indicate an issue with skew in responses. To better understand whether, or how, the range of participants may have been too narrow, we visually examined the Wright Map, but distinguished participants into three groups: non-education majors, education majors without grades 3-5 experience, and education majors with grades 3-5 experience. We also juxtaposed a Box-and-Whisker plot to help explore such patterns. Notably, the third quartile is relatively smaller than the fourth quartile, suggesting some skew in the upper half of scores. In particular, this data suggests a need for more participants with higher scores to improve person reliability. The Wright Map in Figure 2 suggests that a targeted sampling of participants with grades 3-5 experience may be warranted in this regard. To test this conjecture, we compared the person reliability of the Rasch model ran with and without participants having grades 3-5 experience. Notably, excluding such participants reduced person reliability from .42 to .37. Note this reduction is not due to sample size but is due to a lower variance in scores. Thus, despite a sample of 101 participants with what visually appears to be a normal distribution (see Figure 2), results suggest a need for more teachers with grades 3-5 experience for our sample to be representative enough to reliably measure PCK across samples.

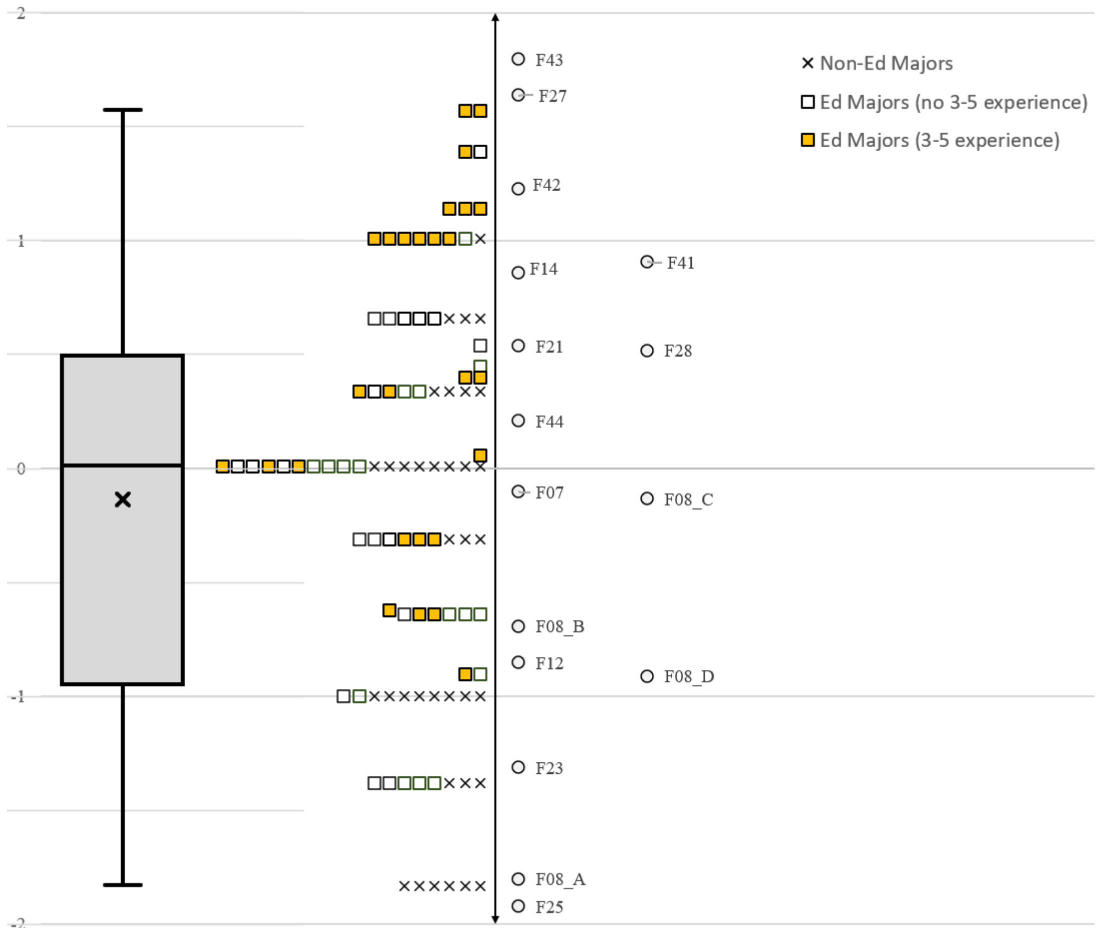


Figure 2: Wright Map for PCK-Fraction measure

ANOVA

The use of a one-way between subjects' analysis of variance (ANOVA) provided evidence to our claim in which differences in PCK scores are associated with participants' professional experiences. We compared Rasch PCK scores of non-education majors, education majors without grades 3-5 field experience, and education majors with such experience. Results from ANOVA found that there was a statistically significant difference between the three groups [F (2, 94) = 11.320, $p < 0.001$]. A Tukey HSD post hoc analysis indicates a statistically significant difference ($p < 0.001$) between the education majors with grade 3-5 experience ($M = 0.16$, $SD = 0.62$) and non-education majors ($M = -0.65$, $SD = 0.74$), as well as statistically significant difference ($p = 0.034$) between education majors with experience and education majors without grades 3-5 field experience ($M = -0.27$, $SD = 0.63$). Comparatively, the Tukey HSD post hoc analysis did not detect a statistically significant difference between non-education majors and education majors without 3-5 experience ($p = .057$). However, non-education majors did have lower PCK-Fraction scores than education majors without grades 3-5 experience, and the p-value was marginally non-significant at the .05 level. A slightly larger sample may have yielded a statistically significant result. Results indicate that differences in PCK-Fractions scores is associated with participants' experience with grades 3-5 students. However, the content and quality of such field experiences were not examined, and further investigation is warranted.

Discussion

This study explored differences in participants' PCK-Fractions scores considering varying professional experiences. Results suggest participants with field experience (particularly with grades 3-5) demonstrated higher PCK scores for fractions than participants without. Additionally, while marginally not statistically significant at the .05 level, education majors tended to have higher PCK scores than non-education majors. Prior studies indicated the importance of teachers having different types of professional experiences including coursework, higher CK, and experience teaching specific grade levels (Agathangelou & Charalambous, 2020; Copur-Gencturk, 2021; Hill, 2010). Our results support some of this prior literature (Copur-Gencturk, 2021; Hill, 2010) as we observed that PSTs' PCK may benefit from having upper elementary field experience. To our knowledge, this paper is the first to compare PCK scores of education and non-education majors. Although it is difficult to distinguish how much upper-elementary field experience may have interacted with some of our participants' coursework, there does appear to be some benefit to pedagogy courses prior to such field experience with education majors lacking grades 3-5 field having much higher PCK scores (M = -.27), than non-education majors (M = -.65).

Results from our psychometric analysis provided strong validity evidence that the PCK-Fractions measure is aligned well with our construct map (see Table 1). Recall that this version of our construct map was a result from analysis in our pilot study (Zolfaghari et al., 2021). Thus, in finding that items designed at specific levels appeared to ordinally fall in the sequence they were designed (see Figure 2), and that our item separation statistic allows for distinguishing four levels of items, we believe there is strong validity evidence for our construct map. This set of findings is non-trivial, as PCK in general is undertheorized. By successfully designing items as child-centered and not task-centered, we believe this study provides guidance to those constructing PCK measures in similar or other domains. Further, the construct map itself provides a framework for examining PSTs' PCK for fractions with or without our PCK-Fractions measure. For example, a PST's explanation of a child's mathematics when viewing a video of their own teaching could be examined in relation to the construct map in Table 1. Such research is needed to further theorize PCK for fractions, as well as other mathematical domains.

The validity evidence from the present study supports and expands that of the initial validation process for PCK-Fractions (Zolfaghari et al., 2020; 2021). Specifically, validity evidence suggests the items in the PCK-Fractions measure are strong and do not appear to need any further revision currently. However, there is a need to sample a wider range of teachers with varying levels of upper-elementary experience. This may include inservice teachers, as well as PSTs' with said experience. Future research in this area may also consider teachers' professional beliefs and other facets.

Acknowledgments

Research reported here received support from the National Science Foundation (NSF) through DRK-12 Grant #1908159. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author(s) and do not necessarily reflect the views of NSF.

References

American Educational Research Association [AERA], American Psychological Association [APA], & National Council on Measurement in Education [NCME] (2014). *Standards for educational and psychological testing.* Washington D.C: AERA.

Battista, M. T. (2012). *Cognition-based assessment & teaching of geometric shapes: Building on students' reasoning*. Heinemann.

Ball, D. L., & Bass, H. (2002, May). Toward a practice-based theory of mathematical knowledge for teaching. In *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group* (pp. 3-14).
Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?
Copur-Gencturk, Y. (2021). Teachers' knowledge of fraction magnitude. *International Journal of Science and Mathematics Education*, 1-16.
Copur-Gencturk, Y., Tolar, T., Jacobson, E., & Fan, W. (2019). An empirical study of the dimensionality of the mathematical knowledge for teaching construct *Journal of Teacher Education*, *70*(5), 485–497. https://doi.org/10.1177/0022487118761860
Depaepe, F., Torbeyns, J., Vermeersch, N., Janssens, D., Janssen, R., Kelchtermans, G., & Van Dooren, W. (2015). Teachers' content and pedagogical content knowledge on rational numbers: A comparison of prospective elementary and lower secondary school teachers. *Teaching and Teacher Education*, *47*, 82–92. https://doi.org/10.1016/j.tate.2014.12.009
Hackenberg, A. J., Norton, A., & Wright, R. J (2016) *Developing fractions knowledge* Sage.
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, *39*(4),372–400.
Herbst, P., & Kosko, K. (2014). Mathematical knowledge for teaching and its specificity to high school geometry instruction. In *Research trends in mathematics teacher education* (pp. 23-45). Springer, Cham.
Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *41*(5), 513-545.
Jakobsen, A., Fauskanger, J., Mosvold, R., & Bjuland, R. (2011). What can be Learned from Comparing Performance of Mathematical Knowledge for Teaching Items found in Norway and in the US?
Lo, J. J., & Luo, F. (2012). Prospective elementary teachers' knowledge of fraction division. *Journal of Mathematics Teacher Education*, *15*(6), 481-500.
Kane, M. (2012). Validating score interpretations and uses. *Language Testing*, *29*(1), 3-17. https://doi.org/10.1177%2F0265532211417210
Kazemi, F., & Rafiepour, A. (2018). Developing a scale to measure content knowledge and pedagogy content knowledge of in-service elementary teachers on fractions. *International Journal of Science and Mathematics Education*, *16*(4), 737–757. https://doi.org/10.1007/s10763-016-9792-0
Kosko, K. W. (2019). The form of mathematics in assessment items: How items convey and measure multiplicative reasoning differently. In J. D. Bostic, E. E. Krupa, & J. C. Shih (Eds.), *Quantitative measures of mathematical knowledge: Research instruments and perspectives* (pp. 14-40). Routledge.
Krupa, E. E., Bostic, J. D., & Shih, J. C. (2019). Validation in mathematics education: An introduction to *Quantitative measures of mathematical knowledge: Researching instruments and perspectives*. In J. D. Bostic, E. E. Krupa, & J. C. Shih (Eds.), *Quantitative measures of mathematical knowledge: Research instruments and perspectives* (pp. 1-13). Routledge.
Siswono, T. Y. E., Kohar, A. W., & Hartono, S. (2018). Designing Tasks to Examine Mathematical Knowledge for Teaching Statistics for Primary Teachers. In *Journal of Physics: Conference Series* (Vol. 947, No. 1, p. 012008). IOP Publishing.
Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for research in Mathematics Education*, *31*(1), 5-25. https://www.jstor.org/stable/749817
Trobst, S., Kleickmann, T., Heinze, A., Bernholt, A., Rink, R., & Kunter, M. (2018). Teacher knowledge experiment: Testing mechanisms underlying the formation of preservice elementary school teachers' pedagogical content knowledge concerning fractions and fractional arithmetic. *Journal of Educational Psychology*, *110*(8), 1049–1065. https://doi.org/10.1037/edu0000260
Zolfaghari, M., Kosko, K. W., & Heisler, J. (2020). Exploring preservice teachers' pedagogical content knowledge of teaching fractions. In Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1703-1706). Mazatlán, Sinaloa, Mexico: Cinvestav & the Mexican Association for Research on the Use of Technology in Mathematics Education (AMIUTEM). https://doi.org/10.51272/pmena.42.2020-270
Zolfaghari, M., Austin, C. K., & Kosko, K. W. (2021). Exploring teachers' pedagogical content knowledge of teaching fractions. *Investigations in Mathematics Learning, 13*(3), 230-248. https://doi.org/10.1080/19477503.2021.1963145

WHAT KNOWLEDGE IS NEEDED FOR TEACHING MATHEMATICS: USING TOPIC MODELING

Eunhye Flavin
Stonehill College
eflavin@stonehill.edu

Sunghwan Hwang
Seoul Gaju Elementary School
ihwang@gmail.com

Keywords: Mathematical knowledge for Teaching, Teacher Knowledge

Purposes of the Study

The question of what knowledge is needed for teaching mathematics has been studied over the last three decades in the community of mathematics researchers and educators. Extensive theoretical and empirical studies have been conducted in this area. However, there is limited systemic review identifying the trends of topics. This study proposes to use a natural language processing method to identify the "unseen" patterns of the topics emerged in all relevant articles published over the last three decades. This study aims to gain insights on what topics get more attention and which other topics are relatively neglected. This study also aims to suggest what areas need to be studied more in mathematics teachers' knowledge to ensure promoting access and attainment in learning mathematics.

Theoretical Framework

This study was built on the theory of mathematics teachers' knowledge (Ball, Thames, & Phelps, 2008; Davis & Simmit, 2006; Peng, 2007; Rowland, Huckstep & Thwaites, 2005; Carrillo-Yañez et al, 2018) that identified the components of it. The previous studies had proposed a slightly different, but similar framework of mathematics teachers' knowledge including subject matter knowledge and pedagogical content knowledge. These studies provided information on the nature of mathematics teachers' knowledge, which helped us to interpret the topics detected by automatic computation.

Methods

This study analyzed the abstracts of 3,485 scholarly articles published in peer-reviewed journals. The quantity of data is the largest in the literature review of mathematics teachers' knowledge within our search scope. Topic modeling was used as an analytical method for this study. Particularly, Latent Dirichlet Allocation (LDA) has been used as a tool for topic modeling.

Results and Implications

Using the topic modeling techniques, eleven underlying topics were found in the articles. The most popular topic was the professional development of mathematics teachers followed by pedagogical content knowledge and students' mathematical understanding. Second, sociocultural context knowledge in teaching mathematics emerged as one topic, which has not been proposed as the component of mathematics teachers' knowledge. Third, the topics were consistent in their popularity over time, which indicates interest in one topic can be quite perpetuating. This study suggests that more diverse topics need to be studied in mathematics teachers' knowledge.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?.

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L. C., Flores-Medrano, E., Escudero-Ávila, D., ... & Muñoz-Catalán, M. C. (2018). The mathematics teacher's specialised knowledge (MTSK) model. Research in

Mathematics Education, 20(3), 236-253.
Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. Educational studies in mathematics, 61(3), 293-319.
Peng, A. (2007). Knowledge growth of mathematics teachers during professional activity based on the task of lesson explaining. Journal of Mathematics Teacher Education, 10(4), 289-299.
Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. Journal of mathematics teacher education, 8(3), 255-281.

USING VARIATION THEORY TO DEVELOP SECONDARY TEACHER MATHEMATICAL KNOWLEDGE CONNECTED TO DIOPHANTINE EQUATIONS

Veronica Hoyos
National Pedagogical University, Mexico
vhoyosa@upn.mx

Christina Eubanks-Turner
Loyola Marymount University, LA (USA)
Christina.Eubanks-Turner@lmu.edu

Eileen Murray
mathforamerica.org
EMurray@mathforamerica.org

David Silva
Claustro Universitario de Oriente, Mexico
david@cudeoriente.edu.mx

Keywords: Mathematical Knowledge for Teaching, Professional Development, Teacher Knowledge

Here it is presented part of the results of an exploratory study on the application of advanced mathematics topics in the professional development of in-service secondary school teachers. The germ of this work was born during the international conference "Connections between Advanced and Secondary Mathematics: Exploring the Impact of Abstract Algebra on the Teaching and Learning of Secondary Mathematics" (CASM Conference), which took place on May 20-22, 2019, funded by the NSCF-IUSE of the USA. Part of the information shared there was the MET II report. This report stated that courses for teachers should contain content that examines high school mathematics from an advanced standpoint throughout their courses and not just as a capstone. Secondary teacher's educators, therefore, following the MET II recommendations, should be interested in generating explicit connections between advanced and secondary mathematics to support or enhance teacher's teaching.

Following this line of thought, we have then applied variation theory (Koichu et al, 2016; Leung, 2012; Marton, 2015) for setting up a four-month online course for professional development of secondary teachers. The content of learning we have chosen was how to find the general solution to Diophantine equations (see Cárdenas et al, 1988; Guelfond, 1981), a topic of superior (or abstract) algebra that immediately could be connected to solving linear (and quadratic) equations in integers, a topic of secondary mathematics. Diophantine equations is a topic mathematically interesting because to find their general solution all basic properties of integers enter in play This topic could be considered as an introduction to Numbers' Theory.

In this work we were interested in exploring how we could use variation theory to approach finding the general solution of Diophantine equations in a teacher professional development course, and how this approach could be reflected in secondary teachers' practice. In other words, what would be possible connections, established by participant teachers, between the general solution of Diophantine equations and the mathematics to be taught in high school?

Some Results

We observed three different connections between the general solution of Diophantine equations and the math participants taught in high school, namely: 1) Transforming fractions from improper to continuum; 2) Getting a general formula to solve linear Diophantine equations using the Euclid's Algorithm; 3) Getting a general formula to solve Diophantine equations of second degree. That evidenced participants' pedagogical content knowledge in teaching (see Petrou & Goldin, 2011), in this case associated with the topic studied.

References

Campbell, S., and Zaskis, R. (2002). Learning and teaching number Theory: Research in cognition, learning and instruction (Mathematics, learning and instruction, V 2). Greenwood Publishing Group.

Cárdenas, H., Lluis, E., Raggi, F., y Tomás, F. (1988). *Algebra Superior*, pp.174-176 y 193-197. México: Trillas

CBMS (2012). *The Mathematical Education of Teachers II*. American Mathematical Society. Retrieved on January 16th, 2022, at: https://cbmsweb.org/archive/MET2/met2.pdf

Gelfond, A. O. (1981). *Solving Equations in Integers*. MIR Publishers

Koichu, B., Zaslavsky, O., and Dolev, L. (2016). Effects of variation in task design on mathematics teachers' learning experiences: A case of sorting task. *J. Math Teacher Educ*, *19*: 349-370.

Leung, A. (2012). Variation and mathematics pedagogy. In J. Dindyal, L.P. Cheng, & S.F. Ng (Eds.), *Proceedings of the 35th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 433–440). Singapore: MERGA. Available from http://www.merga.net.au/documents/Leung_2012_MERGA_35.pdf

Marton, F. (2015). *Necessary conditions of learning*. New York, NY: Routledge.

Petrou, M., and Goulding, M. (2011). Conceptualising teachers' mathematical knowledge in teaching. In T. Rowland & K. Ruthven (eds.), *Mathematical Knowledge in Teaching*, pp. 9-25. Springer

ADAPTING THE KNOWLEDGE QUARTET FOR NON-DIDACTIC CLASSROOMS

Chandra Hawley Orrill
University of Massachusetts Dartmouth
corrill@umassd.edu

Rachael Eriksen Brown
Pennsylvania State University Abington
reb37@psu.edu

Rohini Thapa
University of Massachusetts Dartmouth
rthapa@umassd.edu

Emmanuel Nti-Asante
University of Massachusetts Dartmouth
entiasante@umassd.edu

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge, Research Methods

One of the challenges for researchers interested in studying teachers' mathematical knowledge is characterizing that knowledge in a way that allows it to be visible in practice. Some high-profile projects have attempted to create instruments for this (e.g., IQA (Boston, 2012)) and researchers in the United States have considered a variety of ways to apply the "egg" proposed by Ball and colleagues (Ball et al., 2008; Hill et al., 2008). Another approach has been proposed by Rowland and his colleagues (Rowland, 2013; Rowland et al., 2005; Rowland & Turner, 2007; Turner & Rowland, 2011) who have developed a framework called the Knowledge Quartet (KQ) for characterizing teacher knowledge by organizing observable activities into four discrete categories: Foundation, Transformation, Connection, and Contingency.

The developers of the KQ have provided numerous codes that emerged through their work with preservice teachers in the United Kingdom. The codes are all observable behaviors that fit into one of the four categories. For example, "Identifying Errors" is one way a teacher may demonstrate Foundation knowledge or "Making Connections between Procedures" can demonstrate Connection knowledge.

In our work, we were interested in understanding how one seventh grade teacher, Matt (pseudonym), applied his content knowledge of proportional reasoning in four video recorded lessons. We adopted the KQ for this analysis. However, as we began to apply the KQ, we began to understand that it was developed for a different teaching and learning context than we saw in Matt's classroom. To this end, we created additional codes, situating them within the four knowledge categories by using the definitions provided by Rowland and his colleagues (Rowland & Turner, 2007).

In this poster, we will present the original KQ code set in addition to the added codes necessary to make sense of Matt's knowledge in his classroom teaching. We will also include discussion of the original codes that were cut from our analysis and those that were merged. Examples from the data will be included to illustrate the needs for these changes. We discuss the implications which consider the extent to which the mathematical knowledge teachers need for teaching is culturally grounded. A teacher who does not demonstrate proficiency with the original KQ codes may have other proficiencies valued by the math education community that may not be captured by the KQ.

Acknowledgments

The work reported here was funded by the National Science Foundation under grant DRL-1054170. The opinions expressed here are those of the researchers and do not necessarily reflect those of the NSF.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking*

Equity-based Practices. Reston, VA: National Council of Teachers of Mathematics.
Baker, B., Cooley, L., & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education*, 557–578.
Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389-407.
Bos, B. (2011). Professional development for elementary teachers using TPACK. *Contemporary Issues in Technology and Teacher Education, 11*(2). Retrieved from http://www.citejournal.org/vol11/iss2/mathematics/article1.cfm
Boston, M. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal, 113*(1), 76-104. https://doi.org/10.1086/666387
Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics*, *52*(3), 243–270.
Herbst, P., & Chazan, D. (2006). Producing a viable story of geometry instruction: What kind of representation calls forth teachers' practical rationality? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez. (Eds.), *Proceedings of the 28th North American Group for the Psychology of Mathematics Education Conference* (Vol. 2, pp. 213–220). Mérida, México: UPN.
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*(4), 372-400.
Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & Ball, D. L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*, 430-511. https://doi.org/10.1080/07370000802177235
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Rasmussen, C., & Ellis, J. (2015). Calculus coordination at PhD-granting universities: More than just using the same syllabus, textbook, and final exam. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.). *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 107–115). Washington, DC: The Mathematical Association of America.
Rowland, T. (2011). The knowledge quatet: The genesis and application of a framework for analysing mathematics teaching and deepening teachers' mathematics knowledge. *Sisyphus, 1*(3), 15-43.
Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education, 8,* 255-281. http://doi.org/10.1007/s10857-005-0853-5
Rowland, T., & Turner, F. (2007). Developing and using the 'Knowledge Quarter': A framework for the observation of mathematics teaching. *The Mathematics Educator, 10*(1), 107-123.
Turner, F., & Rowland, T. (2011). The knowledge quartet as an organising framework for developing and deepening teachers' mathematics knowledge. In T. Rowland & K. Ruthven (Eds.,), *Mathematical knowledge in teaching* (pp. 195-212). Springer. http://doi.org/10.1007/978-90-481-9766-8_12

Chapter 7:

Mathematical Processes and Practices

TRANSITION FROM PROBLEM SOLVING TO MODELING IN STUDENTS IN HONDURAS

Noé Fernando Aguilar Perdomo
Cinvestav-IPN
noe.aguilar@cinvestav.mx

Claudia Margarita Acuña Soto
Cinvestav-IPN
claudiamargarita_as@hotmail.com

In the present work, we propose to investigate which elements of problem solving could allow the transition to mathematical modeling. The study involved two students from Honduras with standard instruction in problem solving, where one of them has participated in mathematical Olympiads. We consider that the detection and organization of information, the explanation of the results, and the validation processes could be points of contact between these positions and that their adequate management would allow the transition between them. We found that the students were good problem solvers, but only with the support of the researcher did they develop activities of analysis, validation, and determination of the conditions of the problem to achieve school modeling.

Keywords: Problem solving, modeling, metacognition.

Introduction

In the Republic of Honduras, since 2003, mathematics education has been implemented under the problem-solving approach. At the same time, current trends point to the importance of students giving meaning to contextual problems that could enrich their perception of the usefulness of mathematics and that can be used in everyday life, positions that have been part of what has been called school modeling.

Thus, we face Honduran students with a background in problem solving but will soon have to move towards the modeling perspective, which we must prevent.

The essential background for this research considers the formulations of Polya (1989) and Schoenfeld (1985) on problem solving, as well as the positions of educational organizations such as PISA (2015), which establish the objectives and conditions of this position. We also identified that currently, institutions realize that the use and detection of the appropriate strategy are relevant for teaching mathematics. Still, it is necessary to contextualize the problems. We find this phenomenon more clearly in the PISA (2015) position. Over time, they have developed an interest in including skills that traditionally are not considered part of problem solving, which is the basis of their approach. Now they tend to get the benefits that can be obtained from them, for example, when students realize that a problem is associated with specific strategies, where the need to interpret them may arise, in addition to giving a solution which would be bringing them closer to the interests of modeling.

To establish the character and depth of the contents treated in problem solving, we analyzed the official textbooks of Honduras to identify 1. The perspective present in them, and 2. The skills developed based on these materials, so we can say that students in Honduras, from the use of these textbooks, have had some experiences that would allow them to address the problems making use of: 1. The detection of the aspects that are relevant to the task, 2. The approach and identification of the adequate strategies for solving the problems, 3. The organization of the information, and 4. To a lesser extent, the acquisition of tools related to the processes of validation and generalization, aspects that we consider particularly useful for the creation of models.

Referential Framework

In Schoenfeld's (1985) proposal, we identify the dimension called *control*, which refers to the student's ability to realize whether his strategy is going in the right direction and explain his results. Both points could place the student in the metacognition field, which is also related to "monitoring and control" (Schoenfeld, 2016, p.2), where, if the student takes control of his solution process, he can develop an overview of the activities and conditions of the problem, which allows him to increase his chances to solve it correctly.

In our work, by including the development of metacognitive processes, we intend to promote a possible and slight change in the paradigm of problem solving. The object of study is no longer focused solely on recognizing and using strategies to solve problems. Still, it is required to identify the conditions under which it is possible to solve them.

To achieve this paradigm shift, aspects related to decision-making and the interpretation of the problem can be incorporated, including identifying the specific conditions of the problem. If we add to this the realistic contextual component, we would be approaching the field of school mathematical modeling since: "Mathematical models appear when there is a need to answer specific questions in real situations, when decisions need to be made or when it is imperative to make predictions related to natural and social phenomena" (Trigueros, 2009, p. 76).

In this paper, we will consider modeling from a cognitive perspective. In particular, "they foreground how students' thinking leads to modeling while serving as a basis for pedagogical or curricular goals" (Czocher, 2018, p.138).

Students' thinking can be abstracted and illustrated through the structures known as modeling cycles, which establish the nodal points of the process. We will use the proposal of Blum and Borromeo Ferri (2009), which is susceptible, in our understanding, to be used both in problem solving and modeling and which is proposed from four activities: 1. understanding task, 2. Establishing model, 3. using mathematics, and 4. explaining result.

This proposal does not state validation activities; however, these are presented through the cycle iteration, which indicates that it requires validation to be restarted. In addition, when the student can explain his results, he is also developing validation activities.

To achieve this paradigm shift in problem solving, we consider the need to focus on two essential factors 1. The development of metacognitive aspects promotes a reflection on the conditions under which the problem can be solved, and 2. The teacher's role enables a greater understanding of the relationships and implications of the problem, which has as a starting point a recognition of the contextual conditions, which could result in a transition from one position to another.

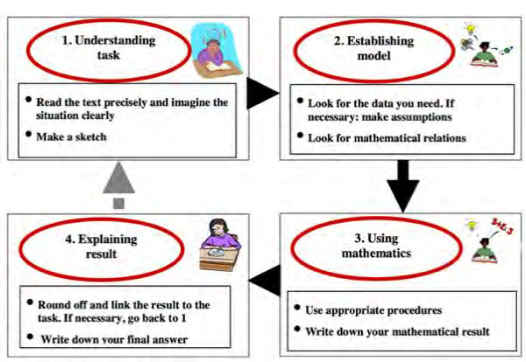


Figure 1: Blum and Borromeo Ferri's Modeling Cycle (2009, p.54)

So, if we consider the definition of modeling competence as a starting point to introduce its teaching, in this case, the one proposed by Niss et al. (2007) that refers to:

> the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation, as well as the ability to analyse or compare given models (p.12)

We are interested in analyzing how metacognitive aspects could promote some of the minimum requirements to develop modeling activities.

Methodology

We developed a qualitative interpretative study to find out how the competencies and skills in problem solving taught, both school and extra-school in Honduras, could serve as a basis for introducing school mathematical modeling and how the transition process from one didactic proposal to the other. Our research hypothesis is that both students who have had regular instruction and those who have acquired additional instruction on problem solving can transition to a didactic proposal based on modeling through relevant problems and collaborative work with the teacher that support the use of metacognitive processes.

The study was conducted in the context of the COVID-19 pandemic in 2021. Two Honduran students of academic excellence participated in the study, both were in the ninth grade at the time of the investigation, and both were 15 years old. Student 1 (E1) has experience in mathematics Olympiads, while student 2 (E2) does not have this experience.

The study was based on 1. The students solved an online questionnaire individually on the GeoGebra platform, which included open-ended questions and the filling in of tables, and 2. From the questionnaire, we detected the level of development of problem-solving skills through four problems, and both students showed: detection of relevant aspects in the problem, organization of information, use of appropriate strategies, as well as good management of the mathematics to solve them.

In the interview-intervention, we asked the students about their procedures and results to deepen the answers to the questionnaire. Also, we took advantage of the incomplete or incorrect questions to understand how the students faced the challenges posed by the tasks, to provoke a metacognitive type of work to observe the possibility of expanding their perception of the requirements of the problem and thus be able to support them in the development of the proposed activities.

Metacognitive reflection in these activities has two functions: The first goes toward explaining the procedure, which can provoke students a need for certainty about the arguments they offer when communicating their solution. The second they can be developed by the need for validation.

In the following section, we will present the analysis and discussion of the data obtained from 1. the answers presented in the questionnaire, 2. the video of the interview-interventions, and 3. the audio of their transcripts.

Results

In the following, we present the results of our study on the points that could enable the transition from problem solving to modeling. This discussion is organized into two moments. In the first, which is before the interview-intervention, we will identify the problem-solving skills possessed by the students and the type of production they carry out individually. In the second moment, we will establish the important elements identified by the interview-intervention to know how the collaborative production with the researcher brought the students closer to modeling.

We will present the results related to activity 2 of the questionnaire since we consider that it recovers the work done by the students, which is shown below:

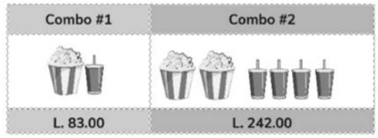


Figure 2: Activity 2

Before the interview-intervention

At this point, we were interested in the problem-solving skills used by the students to solve the activity individually. The following table shows the students' answers to question 1 of activity 2.

Table 1: Answers to question 1 of Activity 2

E1	Tarea 23: Pregunta 1 Explore la imagen y comente sus observaciones: Combo #1: Se observa una caja de palomitas y refresco Combo #2: Se observa dos cajas de palomitas y cuatro refrescos. (El doble de la caja de palomitas del combo #1 y el cuádruple del refresco del combo #1)
E2	Tarea 23: Pregunta 1 Explore la imagen y comente sus observaciones: En el combo 1 nos venden 1 caja de palomitas y 1 refresco por L. 83. En el combo 2 nos venden 2 cajas de palomitas y 4 refrescos por L. 242.

We can observe that E1 recognizes proportionality relationships between the objects that appear in the combos but do not find the total value of each one because it is probably not important to do so for her, at that moment of the solution. While E2 focuses on the composition of the combos, and she gets all the numerical information. We can say that both students detect the relevant information of the situation, but with different approaches, which seem to be directly linked to the possible strategies they will use. This will be reflected in the way they establish the data, which we show below:

Table 2: Answers to question 2 of Activity 2

E1	Tarea 24: Pregunta 2 ¿Qué datos se ofrecen en la figura? Por una caja de palomitas y por un refresco el precio es de L. 83 y por dos cajas de palomitas y por cuatro refrescos el precio es de L. 242
E2	Tarea 24: Pregunta 2 ¿Qué datos se ofrecen en la figura? X= Cantidad de refrescos Y= Cantidad de palomitas x+y=83 2xr+4y=242

The purpose of this question is to know how the students organize the information provided by the problem. In this task, E1 describes the composition of the combos incorporating the numerical relations that appear in the problem. Still, she does not yet show the elements that allow us to know which strategy she will use. However, in answer to question 1, she identified compensation relations between the elements of the combos. On the other hand, E2 expresses his strategy by organizing the information using a system of linear equations.

After retrieving the relevant information from the problem, in questions 1 and 2, students could be able to find the value of a box of popcorn and soft drink in questions 3 and 4. This is shown in Figure 3.

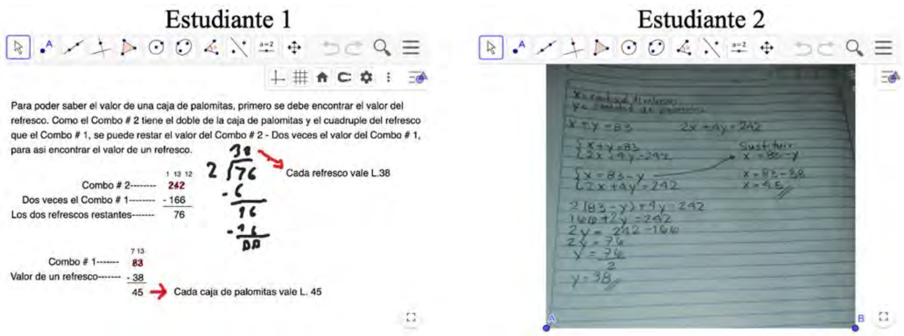


Figure 3: Solution to Activity 2 (questions 3 and 4)

The students' backgrounds explain the difference in the approaches used. When E1 identifies the comparison relationship by compensation, she recognizes a heuristic that consists of duplicating combo #1 to obtain combo #2, realizing that in doing this, there will be a surplus; in this case, it is two sodas. He refers to: "the value of Combo #2 - Twice the value of Combo #1, to find the value of one soda". This recognition allows him to solve the activity using a strategy that is not emphasized in the curriculum and is more typical of a mathematical Olympiad.

On the other hand, in work done by E2, we recognize the use of a strategy emphasized in the curriculum, which is to construct a system of linear equations. Still , she does not adequately label the variables when she applies, which does not allow her to find the correct values. This aspect does not have repercussions on the choice of the strategy, but it does have implications in the results, a situation that is taken up and discussed in the interview.

With question 5, we aimed to identify the metacognition activities that students might be performing, and that relate to the suggestions they would make to another student to solve this activity:

Table 3: Answers to question 5 of Activity 2

E1	Tarea 29: Pregunta 5 Si tuviera que explicar este problema a otro estudiante, ¿Qué sugerencias le daría para que pueda resolverlo? Que primero debe encontrar el valor del refresco y luego el valor de la caja de palomitas.
E2	Tarea 29: Pregunta 5 Si tuviera que explicar este problema a otro estudiante, ¿Qué sugerencias le daría para que pueda resolverlo? Le diría que usara un sistema de ecuaciones de 2 variables para resolver el problema, ya que en esta situación es la mejor forma de hacerlo y que lo resuelva por cualquiera de los métodos, ya sea sustitución o eliminación.

At this stage and before the interview, the suggestion offered by E1 does not give the key elements for another person to solve the problem using her strategy, which seems to show a tendency to make use of a self-explanation since for her, these elements are already codified and therefore should be understood by others. For her part, E2 proposes a method to solve the problem by formulating a system of linear equations but does not explain how to construct the equations from which the solution is obtained. Both explanations lack the important points that could help another person replicate their strategies, which shows us that they could not do metacognitive work on their solutions.

The questionnaire results show that the student's previous experiences condition the type of approach to the proposed problems. E1, having a background in mathematical Olympiads, uses ways of thinking such as compensation strategies that allow her to understand and solve the problem. E2, on the other hand, has only as a resource the elements that have been offered to her through regular instruction. This aspect limits her strategies, but she manages to operate them correctly.

In addition, we identified that the students possess skills related to detecting the relevant information of the problem and its organization. Still, we observed that individually they are not showing the use of metacognitive processes associated with explaining procedures and results.

Analysis of the interview-intervention

Next, to promote reflection, we will address the results from the intervention-interviews we conducted with the students, based on the incomplete or incorrect answers to activity 2 of the questionnaire. In particular, our focus will be on identifying the activities that could be recognized as an approach to modeling and showing how the researcher promoted the need for the development of metacognitive processes that would allow them to reflect on their productions.

In the following excerpt, which corresponds to the discussion of activity 2, E1 is asked to explain *what it is like to complete?* to which she responds:

(161) **E1:** That is to say, the value, well, ... of a box of popcorn, that by doubling it I have two boxes of popcorn and the soft drinks, ... I would subtract two soft drinks from the combo, the combo is multiplied twice, to go like, I do not know how to explain it, sometimes I do not understand my logic, but sometimes I find myself thinking like this
(162) **I:** I understand it; we are doing well.
(163) **E1:** So, to more or less complete, what it was, more or less, to achieve an equality between combo 1 and combo 2.
(164) **I:** Ok, you say you duplicated combo 1 and compared it to combo 2.
(165) **E1:** Yes, and when I subtracted it, I was left with the two soft drinks.
(166) **I:** ou had the soft drinks left, so by subtracting it, you could get what...?
(167) **E1:** The value of one soda, because I only had two sodas left and the answer I had would be to divide it by 2, because they were the two sodas I had leftover.
(168) **I:** Ok, Excellent, divide by two because they were the two sodas you had leftover. So, ... now knowing that, ... how would you tell a classmate of yours? Can you think of this, how would you do it? What things would you ask him to think about so he could solve the activity?
(169) **E1:** Let him look, ... try to compare combo 1 with combo 2, so that he can see a relationship between them and make a shorter procedure, try to duplicate combo 1 in order to find the quantity.
(170) **I:** What quantity?
(171) **E1:** The one for soda ... for the case I used that, first in order to find the popcorn value, I first had to find the soda value.

From the above, we note that E1 realizes that: explaining his procedure can be a complex task *(line 163 - E1)* that involves ordering his ideas so that they can be conveyed clearly. From the questions posed by the researcher: (could you get what ... What things would you ask him to think about so that he can solve the activity? ... What amount?) he pushes E1 to clarify his ideas and be able to explain his approach. Through a continuous dialogue with the teacher, this type of approach allows students to develop a self-demand on their production, where they could establish a mental schema that serves them to order their ideas and explain their procedures and approach.

When we discussed this activity with E2, the following question was posed to her: *what elements of the problem allowed her to think of a system of equations?* she responded:

(106) **E2:** I realized (that) we could make a system of 2-variable equations and solve it fast…
(107) **E2:** I noticed that, ... to begin with ... that there were two situations ... two different situations, but involving the same variables. So, if we organize by variables, we have x on one side and y on the other, then x could represent one of the things they sell us, in this case, the popcorn, as I put it there, and the other would be the amount of soft drinks ... because they are selling us a certain amount of each thing for a specific price. So, I came ... so, I saw that you could make two ... two different equations and make them into a system.

In the dialogue we show, we observe that E2 already recognizes a solution strategy, which is curricular emphasized; this gives him a lot of confidence about the approach he uses to solve the problem since he has used it before and knows that it works.

When the researcher asks E2 to explain the strategy she used to solve the activity, she has a moment of doubt. In the course of her explanation, she realizes that she had not constructed the equations according to the labels she had chosen for the variables *(line 150 - E2)*, as we see in the following dialogue:

(150) **E2:** But... now I have a doubt; I don't know if I solved it right because it says that 38 would be the price of popcorn because... I think that the... the equations that I did were correct, which would be $x + y = 83$ and we have that x is the amount of soft drinks, one soft drink and one popcorn, it is the first one. The second one is $2x + 4y = 242$. It would be the one in the second combo, 2... x... so, there's the mistake.
(151) **I:** But how could we correct that mistake?
(152) **E2:** This would be $4x + 2y$

The uncertainty in E2's explanation produced the need to verify the procedures used, allowing him to have an approach consistent with the initial interpretation *(line 152 - E2).*

From this, we can say that: placing students in a situation where they must explain their procedures could provoke the need to perform validation tasks to obtain answers that are consistent with their strategies and activity conducive to working with models that seek to justify the structure of the problem.

In addition, we can say that the teacher, from the development of metacognitive processes and the questions that the teacher asks, can propitiate in the students a dialogue that allows them to overcome their particular zone of proximal development, encouraging the creation of adequate mental schemes to use the necessary resources and order their ideas through the suggestion of explaining their procedures and approaches, realizing that, in some cases, they can perform validation tasks that go beyond verifying their immediate results, which could help them to improve the solution of problems and provide a broader view of the variety of problems that can be solved with the resources present, which can then be applied to the model idea from the point of view of its general structure.

Conclusions

As shown in the study, we identified that metacognitive reflection was fulfilling two functions, one as a means of explanation of the solution and the other as a validation resource, which allowed the students to identify the conditions under which the proposed problem was solved.

In some cases, certainty about a method may cause the student to resist change or to make his strategy explicit, as happened in the case of E2, who developed an internal dialogue to establish a finer validation process than the one initially developed because the error he presented was exclusively in the process of constructing the equations and in the consistency of the labels he used for the variables and not in the approach, since it was correct.

Finally, in this work, we can say that the skills related to the detection of relevant information and the organization of this, as points of contact between these proposals, can be used as a necessary condition to bring students closer to modeling, which allows the researcher to take advantage of these resources of the students to promote metacognitive processes, which allowed them to be able to analyze and explain their solution and strategy in an orderly manner, as well as to identify the conditions under which the problem was solved, which are activities that go beyond the objectives of problem solving, and that in particular would be approaching the interests of modeling, in an elementary approximation.

Acknowledgments

To Conacyt and Cinvestav-IPN for funding the research project.

References

Blum, W. & Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt?. *Journal of mathematical modelling and application*, 1(1), 45-58.

Czocher, J. A. (2018). How does validating activity contribute to the modeling process?. *Educational Studies in Mathematics*, 99(2), 137-159.

Niss, M., Blum, W., & Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 3–32). New York: Springer.

PISA, O. (2015). Released Field Trial: Cognitive Items.

Polya, G. (1989). Cómo plantear y resolver problemas (decimoquinta reimpresión). México: Editorial Trillas.

Schoenfeld, A. H. (1985). Mathematical Problem Solving. Academic Press. INC.

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education*, 196(2), 1-38.

Trigueros, M. (2009). El uso de la modelación en la enseñanza de las matemáticas. *Innovación educativa*, 9(46), 75-87

TRÁNSITO DESDE LA RESOLUCIÓN DE PROBLEMAS HASTA LA MODELIZACIÓN EN ESTUDIANTES DE HONDURAS

TRANSITION FROM PROBLEM SOLVING TO MODELING IN STUDENTS IN HONDURAS

Noé Fernando Aguilar Perdomo
Cinvestav-IPN
noe.aguilar@cinvestav.mx

Claudia Margarita Acuña Soto
Cinvestav-IPN
claudiamargarita_as@hotmail.com

En el presente trabajo se plantea investigar cuáles elementos de la resolución de problemas podrían permitir el tránsito hacia la modelización matemática. En el estudio participaron dos estudiantes de Honduras con instrucción estándar en resolución de problemas, donde una de ellas ha participado en olimpiadas matemáticas. Consideramos que la detección y organización de la información, la explicación de los resultados y los procesos de validación podrían ser puntos de contacto entre estas posturas y que su adecuado manejo permitiría el tránsito entre ellas. Encontramos que las estudiantes se manejan bien en resolución de problemas, pero que sólo con el apoyo del investigador desarrollaron actividades de análisis, validación y la determinación de las condiciones del problema para lograr la modelización.

Palabras clave: Resolución de problemas, modelado, metacognición.

Introducción

En la República de Honduras, desde el año 2003 se ha venido implementado la educación matemática bajo el enfoque de resolución de problemas. Mientras que las tendencias actuales, señalan la importancia de que los estudiantes den sentido a los problemas llamados *contextuales* que podrían enriquecer su percepción sobre la utilidad de la matemática y que pueden usarse en la vida cotidiana, posturas que han sido parte de lo que se ha dado en llamar *modelización*.

De esta manera, nos encontramos frente a estudiantes hondureños que poseen los antecedentes de la resolución de problemas, pero que pronto deberán transitar hacia la perspectiva que se apoya en la modelización, situación que debemos prevenir.

Los antecedentes básicos para esta investigación consideran las formulaciones de Polya (1989) y Schoenfeld (1985) sobre la resolución de problemas, así como las posturas de organizaciones educativas como PISA (2015), que establecen los objetivos y las condiciones de esta postura. También identificamos que actualmente las instituciones se percatan de que: no sólo el uso y detección de la estrategia adecuada es relevante para la enseñanza de la matemática, sino que es necesario contextualizar los problemas. Este fenómeno lo encontramos más claramente en la postura de PISA (2015), dónde a través del tiempo han desarrollado un interés por incluir habilidades que tradicionalmente no se consideran parte de la resolución de problemas, que es la base de su aproximación y ahora se inclinan por los beneficios que se pueden obtener de esta ampliación, por ejemplo, cuando los estudiantes se dan cuenta de que una problemática está asociada a ciertas estrategias, donde puede surgir la necesidad para interpretarlas, además de darle solución, lo que estaría acercándolos a los intereses de la modelización.

Para establecer el carácter y profundidad de los contenidos tratados en la resolución de problemas, analizamos los libros de texto oficiales de Honduras, para identificar 1. La perspectiva presente en ellos y 2. Las habilidades desarrolladas con base en esos materiales, por lo que podemos decir que los estudiantes de Honduras, a partir del uso de estos libros de texto han tenido algunas experiencias que les permitirían abordar los problemas haciendo uso de: 1. La detección de los aspectos que son relevantes para la tarea, 2. El planteamiento e identificación de las estrategias idóneas para la solución de los problemas, 3. La organización de la información y 4. En menor medida la adquisición de herramientas relacionadas con los procesos de validación y generalización, aspectos que consideramos particularmente provechosos para la creación de modelos.

Marco Referencial

En la propuesta de Schoenfeld (1985) identificamos la dimensión llamada de *control*, la cual hace referencia a las habilidades del estudiante para: darse cuenta de que su estrategia va en la dirección correcta o no y poder explicar sus resultados. Ambos puntos podrían colocar al estudiante en el terreno de la metacognición, que también se relaciona con el "seguimiento y control" (Schoenfeld, 2016, p.2), en dónde, si el estudiante toma el control de su proceso de solución, puede desarrollar una visión general de las actividades y las condiciones del problema, lo que le permite aumentar sus posibilidades para resolverlo adecuadamente.

En nuestro trabajo, al incluir el desarrollo de procesos metacognitivos, pretendemos propiciar un posible y ligero cambio en el paradigma de la resolución de problemas, en dónde el objeto de estudio ya no se centra únicamente en el reconocimiento y uso de las estrategias para resolver los problemas, sino que se requiere de identificar las condiciones bajo las cuales es posible resolverlos.

Para lograr este cambio de paradigma, se pueden incorporar aspectos que se relacionen con la toma de decisiones y la interpretación de la problemática, que incluye identificar las condiciones específicas de ésta y si a esto le sumamos el componente contextual realista, estaríamos acercándonos al terreno de la modelización matemática escolar, debido a que: “Los modelos matemáticos aparecen cuando se tiene la necesidad de responder preguntas específicas en situaciones reales, cuando se requiere tomar decisiones o cuando es imperativo hacer predicciones relacionadas con fenómenos naturales y sociales” (Trigueros, 2009, p. 76).

En este trabajo vamos a considerar la modelización desde una perspectiva cognitiva, porque en particular: “ponen en primer plano la forma en que el pensamiento de los estudiantes conduce al modelo, al tiempo que sirven de base para los objetivos pedagógicos o curriculares” (Czocher, 2018, p.138).

El pensamiento de los estudiantes puede abstraerse e ilustrarse a través de las estructuras conocidas como ciclos de modelización, que establecen los puntos nodales del proceso. En particular, vamos a utilizar la propuesta de Blum y Borromeo Ferri (2009), la que es susceptible, a nuestro entender, de ser usada tanto en la resolución de problemas como en la modelización y que se plantea a partir de cuatro actividades: 1. Entendiendo la tarea, 2. Estableciendo el modelo, 3. Usando matemáticas y 4. Explicando los resultados.

Esta propuesta no declara actividades de validación, sin embargo, éstas se presentan por medio de la iteración del ciclo, lo cual indica que requiere de la validación para ser reiniciadas. Además, cuándo el estudiante está en condiciones de poder explicar sus resultados, también está desarrollando actividades de validación.

Para poder lograr este cambio de paradigma en la resolución de problemas, consideramos la necesidad de centrarse en dos factores esenciales: 1. El desarrollo de los aspectos metacognitivos, que promuevan una reflexión sobre las condiciones bajo las cuales el problema puede ser resuelto y 2. El papel del profesor, que posibilita un mayor entendimiento de las relaciones e implicaciones del problema, lo que tiene como punto de inicio un reconocimiento de las condiciones contextuales, lo que podría dar como resultado una transición de una postura a otra.

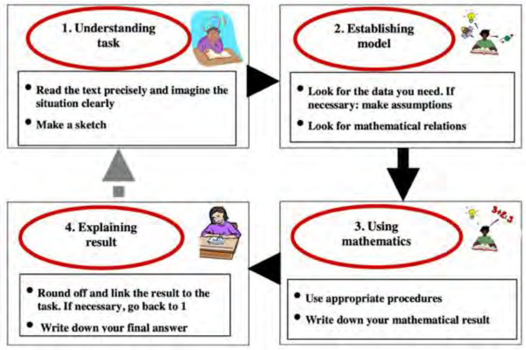


Figura 1: Ciclo de Modelización de Blum y Borromeo Ferri (2009, p.54)

De manera que, si consideramos a la definición de la competencia de la modelización como punto de partida para introducir su enseñanza, en este caso la propuesta por Niss et al. (2007) que se refiera a:

> La habilidad para identificar las cuestiones, variables, relaciones o supuestos relevantes en una situación del mundo real, traducirlos a las matemáticas e interpretar y validar la solución del problema matemático resultante en relación con la situación dada, así como la capacidad de analizar o comparar modelos. (p.12)

Por ello, estamos interesados en analizar cómo los aspectos metacognitivos podrían estar promoviendo algunos de los requisitos mínimos para desarrollar actividades de modelización.

Metodología

Desarrollamos un estudio cualitativo interpretativo para conocer la forma cómo las competencias y habilidades sobre resolución de problemas impartidas, tanto curricularmente como extra curricularmente en Honduras, podrían servir como base para una introducción de la modelización matemática escolar y cómo es el proceso de tránsito desde una propuesta didáctica a la otra. Nuestra hipótesis de investigación es que tanto los estudiantes que han tenido instrucción regular, así como los que han adquirido una adicional sobre resolución de problemas, están en condiciones de transitar a una propuesta didáctica basada en la modelización, a través de problemas pertinentes y un trabajo colaborativo con el profesor que favorezca el uso de los procesos metacognitivos.

El estudio se realizó en el marco de la pandemia por COVID-19 en el 2021. Participaron dos estudiantes hondureñas de excelencia académica, que al momento de la investigación cursaban el noveno grado y ambas tenían 15 años. La estudiante 1 (E1), cuenta con experiencia en olimpiadas de matemáticas, mientras que la estudiante 2 (E2) no tiene esta experiencia.

El estudio se apoyó en: 1. Un cuestionario en línea que las estudiantes resolvieron de manera individual en la plataforma de GeoGebra y que consideró preguntas abiertas y llenado de tablas, y 2. En una entrevista-intervención en línea, luego de haber resuelto el cuestionario.

A partir del cuestionario, detectamos el nivel de desarrollo de las habilidades sobre resolución de problemas a través de cuatro problemas y ambas estudiantes mostraron: detección de los aspectos relevantes en el problema, organización de la información, uso de estrategias adecuadas, así como del buen manejo de la matemática necesaria para resolverlos.

En la entrevista-intervención, preguntamos a las estudiantes sobre sus procedimientos y resultados, para profundizar en las respuestas del cuestionario. También, se aprovecharon las preguntas incompletas o incorrectas, para conocer cómo las estudiantes enfrentaron los retos que les plantearon las tareas, de manera que se provocara un trabajo de tipo metacognitivo para observar la posibilidad de ampliar su percepción sobre los requisitos del problema y así poder apoyarlas en el desarrollo de las actividades propuestas.

La reflexión metacognitiva en estas actividades tiene dos funciones: La primera va en la dirección de lograr una explicación del procedimiento, lo puede provocar en los estudiantes una necesidad de certeza sobre los argumentos que ofrecen cuando comunican su solución y la segunda puede ser desarrollada por la necesidad de la validación.

En el siguiente apartado, presentaremos el análisis y la discusión de los datos que se obtuvieron a partir de: 1. Las respuestas presentadas en el cuestionario, 2. El video de las entrevistas-intervención y 3. La transcripción del audio de estas.

Resultados

A continuación, presentamos los resultados de nuestro estudio sobre los elementos que podrían posibilitar el tránsito desde la resolución de problemas hacia la modelización. Esta discusión se organiza en dos momentos. En el primero, que es previo a la entrevista-intervención, identificaremos: las habilidades sobre resolución de problemas que poseen las estudiantes y el tipo de producción que realizan de manera individual. En el segundo momento, estableceremos los elementos importantes identificados en la entrevista-intervención, para conocer de qué manera la producción conjunta con el investigador acercó a las estudiantes a la modelización.

Presentaremos los resultados relativos a la actividad 2 del cuestionario, ya que consideramos que, recupera de maneral, el trabajo realizado por las estudiantes, la que mostramos enseguida:

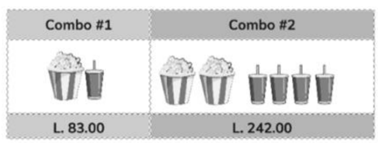


Figura 2: Actividad 2

Previo a la entrevista-intervención

En este momento nos interesa conocer las habilidades sobre resolución de problemas que utilizan las estudiantes para resolver la actividad de manera individual. En la siguiente tabla presentamos las respuestas de las estudiantes a la pregunta 1 de la actividad 2.

Tabla 1: Respuestas de la pregunta 1 de la Actividad 2

E1	Tarea 23: Pregunta 1 Explore la imagen y comente sus observaciones: Combo #1: Se observa una caja de palomitas y refresco Combo #2: Se observa dos cajas de palomitas y cuatro refrescos. (El doble de la caja de palomitas del combo #1 y el cuádruple del refresco del combo #1)
E2	Tarea 23: Pregunta 1 Explore la imagen y comente sus observaciones: En el combo 1 nos venden 1 caja de palomitas y 1 refresco por L. 83. En el combo 2 nos venden 2 cajas de palomitas y 4 refrescos por L. 242.

Podemos observar que E1 reconoce relaciones de proporcionalidad entre los objetos que aparecen en los combos, pero no rescata el valor total de cada uno, porque para ella en ese momento de la solución probablemente no sea importante hacerlo. Mientras que E2 se enfoca en la composición de los combos y extrae toda la información numérica. Podemos decir que ambas estudiantes detectan la información relevante de la situación, pero con enfoques distintos, lo que parece estar ligado directamente con las posibles estrategias que utilizarán. Esto se verá reflejado en la manera cómo establecen los datos, lo que mostramos enseguida:

Tabla 2: Respuestas de la pregunta 2 de la Actividad 2

E1	Tarea 24: Pregunta 2 ¿Qué datos se ofrecen en la figura? Por una caja de palomitas y por un refresco el precio es de L. 83 y por dos cajas de palomitas y por cuatro refrescos el precio es de L. 242
E2	Tarea 24: Pregunta 2 ¿Qué datos se ofrecen en la figura? X= Cantidad de refrescos Y= Cantidad de palomitas x+y=83 2xr+4y=242

Con esta pregunta se pretende conocer cómo las estudiantes organizan la información que ofrece el problema. En esta tarea, E1 describe la composición de los combos incorporando las relaciones numéricas que aparecen en el problema, pero aún no muestra los elementos que nos permitan conocer cuál es la estrategia que utilizará, aunque en la respuesta a la pregunta 1 identificó relaciones de compensación entre los objetos de los combos. Por su parte, E2 deja expresada su estrategia al organizar la información utilizando un sistema de ecuaciones lineales.

Después de recuperar la información relevante del problema, en las preguntas 1 y 2, las estudiantes podrían estar en condiciones de encontrar el valor de una caja de palomitas y de un refresco, preguntas 3 y 4. Lo que mostramos en la figura 3.

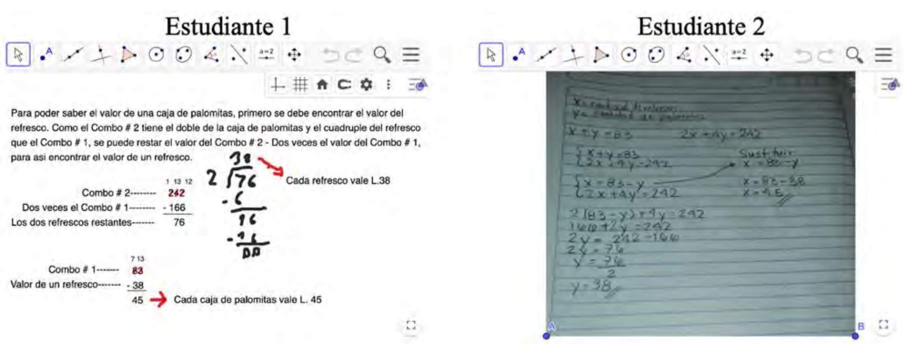


Figura 3: Solución de la Actividad 2 (preguntas 3 y 4)

Los antecedentes de las estudiantes explican la diferencia de los enfoques utilizados. Cuando E1 identifica la relación de comparación por compensación, reconoce una heurística que consiste en duplicar el combo #1 para obtener el combo #2, dándose cuenta de que al hacer esto habrá un excedente, en este caso son dos refrescos. En sus palabras se refiere a: “el valor del Combo #2 – Dos veces el valor del Combo #1, para así encontrar el valor de un refresco”. Este reconocimiento le permite resolver la actividad utilizando una estrategia que no es enfatizada curricularmente y es más propio de una olimpiada matemática.

Por otra parte, en el trabajo que realiza E2 reconocemos el uso de una estrategia que es enfatizada curricularmente, esta es construir un sistema de ecuaciones lineales, pero cuando la aplica no etiqueta adecuadamente las variables y esto no le permitió encontrar los valores correctos. Este aspecto no tiene repercusión en la elección de la estrategia, pero si en los resultados, situación que se retoma y discute en la entrevista.

Con la pregunta 5, pretendíamos identificar las actividades de metacognición que podrían estar realizando las estudiantes y que se relaciona con las sugerencias que le harían a otro estudiante para resolver esta actividad:

Tabla 3: Respuestas de la pregunta 5 de la Actividad 2

E1	Tarea 29: Pregunta 5 Si tuviera que explicar este problema a otro estudiante, ¿Qué sugerencias le daría para que pueda resolverlo? Que primero debe encontrar el valor del refresco y luego el valor de la caja de palomitas.
E2	Tarea 29: Pregunta 5 Si tuviera que explicar este problema a otro estudiante, ¿Qué sugerencias le daría para que pueda resolverlo? Le diría que usara un sistema de ecuaciones de 2 variables para resolver el problema, ya que en esta situación es la mejor forma de hacerlo y que lo resuelva por cualquiera de los métodos, ya sea sustitución o eliminación.

En esta etapa y previo a la entrevista, la sugerencia que ofrece E1 no proporciona los elementos claves para que otra persona pueda resolver el problema utilizando su estrategia, lo que pareciera mostrar una tendencia a hacer uso de una auto explicación, ya que para ella estos elementos ya están codificados y por tanto deberían ser entendidos por los demás. Por su parte, E2 propone un método para resolver el problema formulando un sistema de ecuaciones lineales, pero no explica cómo construir las ecuaciones de las que se obtiene la solución. Ambas explicaciones carecen de los puntos importantes que podrían ayudar a otra persona a replicar sus estrategias, lo que nos muestra que no estaban en condiciones de hacer un trabajo metacognitivo sobre sus soluciones.

Los resultados del cuestionario muestran que las experiencias previas de las estudiantes condicionan el tipo de acercamiento a los problemas propuestos. E1, al tener antecedentes en olimpiadas matemáticas usa formas de pensamiento como las estrategias de compensación que le permiten entender y resolver el problema. Por su parte E2, al poseer como único recurso los elementos que se le han ofrecido a través de una instrucción regular, sus estrategias están limitadas por este aspecto, pero logra ponerlos en funcionamiento correctamente.

Además, identificamos que las estudiantes poseen habilidades relacionadas con la detección de la información relevante del problema y con la organización de esta, pero observamos que de manera individual no están mostrando un uso de procesos metacognitivos, en lo relacionado con la explicación de los procedimientos y los resultados.

Análisis de la entrevista-intervención

A continuación, abordaremos los resultados obtenidos de las entrevistas-intervención que realizamos con las estudiantes, las que se apoyaron en las respuestas incompletas o incorrectas de la actividad 2 del cuestionario, para promover una reflexión. En particular, nuestro foco de atención estará en identificar las actividades que podrían ser reconocidas como un acercamiento a la modelización, además de mostrar cómo el investigador promovió la necesidad del desarrollo de procesos metacognitivos que les permitieran reflexionar sobre sus producciones.

En el siguiente fragmento, que corresponde a la discusión de la actividad 2, se le pide a E1 que explique *¿cómo es eso de completar?,* a lo que ella responde:

(172) **E1:** Es decir, el valor, pues, … de una caja de palomitas, que al duplicarlo ya me quedan dos cajas de palomitas y los refrescos, … me restarían dos refrescos del combo, se multiplica dos veces el combo, para ir como, no sé cómo explicarlo, a veces no entiendo a mi lógica, pero a veces me resulta pensar así

(173) **I:** Yo la entiendo, vamos bien.
(174) **E1:** Entonces, para completar más o menos, lo que era, más o menos, para lograr una igualdad entre el combo 1 y el combo 2.
(175) **I:** Ok, Usted dice que duplicó el combo 1 y lo comparó con el combo 2.
(176) **E1:** Sí y al restarlo, me quedaron los dos refrescos.
(177) **I:** Le quedaban los refrescos, entonces al restarlo, ¿usted podría obtener qué…?
(178) **E1:** El valor de un refresco, porque a lo que queda como al restarle, ya solo me quedaron dos refrescos y la respuesta que tenía sería dividirla entre 2, porque eran los dos refrescos que me sobraban.
(179) **I:** Ok, Excelente, divide entre 2 porque eran los dos refrescos que le sobraban. Entonces, … ahora ya sabiendo eso, … ¿cómo le diría a un compañero suyo? Puedes pensar en esto, ¿cómo lo haría? ¿En qué cosas le pediría que piense él para que pueda resolver la actividad?
(180) **E1:** Que mire, … que intente comparar el combo 1 con el combo 2, para que pueda ver una relación entre el y hacer un procedimiento un poco ya más corto, intentar ya duplicar el combo 1 para poder encontrar ya la cantidad.
(181) **I:** ¿Cual cantidad?
(182) **E1:** La de los refrescos … para el caso que yo utilicé que, primero para poder encontrar el valor de las palomitas, primero tuve que encontrar el valor de refresco.

De lo anterior, observamos que E1 se da cuenta de que: explicar su procedimiento puede ser una tarea compleja *(línea 163 - E1)* que implica ordenar sus ideas de manera que puedan ser transmitidas con claridad. A partir de las preguntas planteadas por el investigador: (¿podría obtener qué … ¿En qué cosas le pediría que piense él para que pueda resolver la actividad? … ¿Cual cantidad?) empuja a E1 para que pueda aclarar sus ideas y sea capaz de poder explicar su enfoque. Este tipo de acercamientos, a través de un diálogo continuo con el profesor permite que los estudiantes desarrollen una autoexigencia sobre su producción, dónde podrían establecer un esquema mental que les sirva para ordenar sus ideas y explicar sus procedimientos y enfoque.

Cuando discutimos esta actividad con E2, se le planteó la siguiente pregunta: *¿qué elementos del problema le permitieron pensar en un sistema de ecuaciones?* ella respondió:

(108) **E2:** Me di cuenta (de) que podíamos hacer un sistema de ecuaciones de 2 variables y resolverlo rápido…
(109) **E2:** Me fijé que, … para empezar… que había dos situaciones… dos situaciones diferentes, pero que implicaban las mismas variables. Entonces, si nos organizamos por variables, tenemos a x por un lado y a y por el otro, entonces x podría representar una de las cosas que nos venden, en este caso, las palomitas, así como lo puse ahí y el otro sería la cantidad de refrescos … porque nos están vendiendo una cantidad determinada de cada cosa por un precio específico. Entonces, llegué … entonces, vi que se podían hacer dos… dos ecuaciones distintas y hacerlas en un sistema.

En el diálogo que mostramos, observamos que E2 ya reconoce una estrategia de solución, la que es enfatizada curricularmente, esto le proporciona mucha seguridad sobre el enfoque que utiliza para resolver el problema, ya que anteriormente lo ha utilizado y sabe que funciona.

Cuando el investigador le pide a E2 que explique la estrategia que utilizó para resolver la actividad, ella tiene un momento de duda y en el transcurso de su explicación, se da cuenta de que no había construido las ecuaciones en función de las etiquetas que había elegido para las variables *(línea 150 – E2)*, como vemos en el siguiente diálogo:

(153) **E2:** Pero… ahora sí tengo una duda, no sé si lo resolví bien porque, es que dice que 38 vendría siendo el precio de las palomitas porque… creo que las… las ecuaciones que hice si estaban bien, que serían $x + y = 83$ y tenemos que x son la cantidad de refrescos, un refresco y una palomita, es la primera. La segunda es $2x + 4y = 242$. Estaría siendo la del segundo combo, 2… x… así, ahí está el error
(154) **I:** Pero ¿cómo podríamos corregir ese error?
(155) **E2:** Sería $4x + 2y$

La incertidumbre en la explicación de E2 produjo la necesidad de verificar los procedimientos usados, permitiéndole tener un planteamiento consistente con la interpretación inicial *(línea 152 – E2)*.

De esto, podemos decir qué: colocar a los estudiantes en la situación donde deben de explicar sus procedimientos, podría provocar en ellos la necesidad de realizar tareas de validación para obtener respuestas que sean consistentes con sus estrategias, actividad propicia para el trabajo con modelos que pretenden justificar la estructura del problema.

Además, podemos decir que el profesor, a partir del desarrollo de procesos metacognitivos y de las preguntas que realiza, puede propiciar en los estudiantes un diálogo que les permita superar su zona de desarrollo próximo particular, propiciando la creación de esquemas mentales adecuados para usar los recursos necesarios y ordenar sus ideas a través de la sugerencia de explicar sus procedimientos y enfoques, dándose cuenta de que, en algunos casos pueden realizar tareas de validación que van más allá de verificar sus resultados inmediatos, las que podrían ayudarles a mejorar la solución de los problemas y proporcionar una mirada más amplia sobre la variedad de problemas que se resuelven con los recursos presentes, lo que luego puede ser aplicado a la idea de modelo desde el punto de vista de su estructura general.

Conclusiones

Cómo se muestra en el estudio, identificamos que la reflexión metacognitiva estuvo cumpliendo dos funciones, una como medio de explicación de la solución y la otra como recurso de validación, lo que permitió que las estudiantes pudieran identificar las condiciones bajo las cuales fue resuelto el problema propuesto.

En algunos casos, la certeza sobre un método puede provocar que el estudiante se resista al cambio o a dar una explicitación de su estrategia, como sucedió en el caso de E2, quien desarrolló un diálogo interno para establecer un proceso de validación más fino que el desarrollado inicialmente, debido a que el error que presentó se encontraba exclusivamente en el proceso de construcción de las ecuaciones y en la consistencia de las etiquetas que utilizó para las variables y no en el enfoque, ya que era correcto.

Finalmente, en este trabajo podemos decir que las habilidades relacionadas con la detección de la información relevante y la organización de esta, como puntos de contacto entre estas propuestas, puede usarse como una condición necesaria para acercar a los estudiantes a la modelización, lo que permitió al investigador aprovechar estos recursos de las estudiantes para promover procesos metacognitivos, lo que les permitió poder analizar y explicar ordenadamente su solución y estrategia, así como identificar las condiciones bajo las cuales se resolvía el problema, que son actividades que van más allá de los objetivos de la resolución de problemas, y que en particular estarían acercándose a los intereses de la modelización, en una aproximación elemental.

Agradecimientos

A Conacyt y al Cinvestav-IPN por el financiamiento del proyecto de investigación.

Referencias

Blum, W. y Borromeo Ferri, R. (2009). Mathematical modelling: Can it be taught and learnt?. *Journal of mathematical modelling and application*, 1(1), 45-58.

Czocher, J. A. (2018). How does validating activity contribute to the modeling process?. *Educational Studies in Mathematics*, 99(2), 137-159.

Niss, M., Blum, W., y Galbraith, P. (2007). Introduction. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education* (pp. 3–32). New York: Springer.

PISA, O. (2015). Released Field Trial: Cognitive Items.

Polya, G. (1989). Cómo plantear y resolver problemas (decimoquinta reimpresión). México: Editorial Trillas.

Schoenfeld, A. H. (1985). Mathematical Problem Solving. Academic Press. INC.

Schoenfeld, A. H. (2016). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics (Reprint). *Journal of Education*, 196(2), 1-38.

Trigueros, M. (2009). El uso de la modelación en la enseñanza de las matemáticas. *Innovación educativa*, 9(46), 75-87

USING STRUCTURE TO FIND THE DISTANCE ON AN EMPTY NUMBER LINE

Laura Bofferding
Purdue University
LBofferd@Purdue.edu

Mahtob Aqazade
Rice University
ma119@rice.edu

Looking for and making use of structure along with using the mathematical tools (i.e., number line) are important mathematical practices. In this paper, we explore 32 first-grade and 36 third-grade students' noticing and making use of the structure of two identical number lines when determining the distance of a bunny's jumps. The number lines represented a distance from 11 to 27, and students could choose to jump to 20 (nearest multiple of ten), 21 (adding ten), or 15 (nearest multiple of five). Choosing 21 (distance of 10) was the most frequent choice, and overall, 71% of third graders' and 17% of first graders' attempts showed a correct distance. However, less than one-third of students made connections between their two number lines and noticed their structure. This study informs the importance of number combination knowledge and highlighting structure in order to leverage efficient addition strategies.

Keywords: Number Concepts and Operations, Elementary School Education, Problem Solving

An important feature of number lines is that they are continuous representations of numbers that can represent distances. Although object-based interpretations of quantity are prevalent when students first work with addition, connecting addition on number lines to distance is also important (National Research Council [NRC], 2001). Counting jumps can help children bridge between discrete situations and using distance, but they need to pay attention to the length of the jumps. In other words, students need to attend to structure, particularly that distance units can be combined or partitioned into larger or smaller units (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010; NRC, 2001). When adding on number lines, some students might count tick marks instead of spaces, resulting in answers that are off by one (Ernest, 1985). Students might also count the number of jumps they take (or distance units they use), regardless of the distance of their jumps (or units; e.g., Saxe et al., 2010).

Number lines can serve as important tools for helping students visualize and advance their thinking (i.e., moving from a model of thinking to a model for thinking; Gravemeijer, 1999; Klein et al., 1998) around addition (and other) problems, and can also reveal important connections between addition and measurement (NRC, 2001). Number lines align well with students' counting strategies but also allow for more sophisticated jump strategies (Bobis, 2007; Klein et al., 1998). Further, multiple number lines can help students notice different combinations of jumps to get to the same number (e.g., see work on contrasting cases; Schwartz et al., 2016), highlighting the structure of numbers. Looking for and making use of structure are characteristics of mathematically proficient students (NGA & CCSOO, 2010); "when young students use the relationships in and among mathematical content and processes, they advance their knowledge of mathematics and extend their ability to apply concepts and skill more effectively" (National Council of Teachers of Mathematics, 2000, p. 132). For this study, we explore to what extent first and third graders notice and use the structure of two number lines (representing the same distance from 11 to 27) to solve a missing addend problem.

Theoretical Framework

In terms of finding the difference (or distance) between two numbers, missing addend problems (e.g., 2 + __ = 5) can support a counting up strategy, which is often easier for students than counting down for corresponding subtraction problems (Fuson & Fuson, 1992). When finding the distance between two points on a number line, the distance can be broken up in ways that support students' number sense (Cain et al., 2020). Therefore, when trying to find a missing addend, students might use different addition strategies. Two important addition strategies for students to learn are making 10's and adding on 10's. For example, if a student starts at 11 and is trying to figure out the distance to 27, they might use an *add tens* strategy (e.g., Fuson et al., 1997) to get to 21 and then know that they need six more to get to 27, resulting in a distance (or missing addend) of 16. On the other hand, they might use a *make ten* strategy (e.g., Fuson et al., 1997) by adding 9 to get 20 and then know they need seven more to get to 27, resulting in the same distance of 16. Less efficient strategies involve counting on from 11 by ones or drawing out jumps to represent the addends and counting the total (Fuson et al., 1997; Murata, 2004). Overall, students will use more efficient strategies as they become more comfortable with place value and as students receive support for using tens-based strategies (Fuson et al., 1997; Murata, 2004).

Exploring how distances can be broken up in multiple ways can help students notice patterns and structure in numbers, such as with the compensation strategy or different combinations of numbers (see Bobis & Bobis, 2005). Take for instance a ten frame filled with seven yellow circles and three red circles to show 7 + 3 = 10. If one yellow circle is flipped over to become a red circle (showing 6 + 4 = 10), students may notice that having one fewer yellow circle means one more red. Likewise, when jumping to ten on a number line, a jump of six is one less than seven, so the second jump of four is one more than a jump of three. During a number string math routine, teachers illustrate students' strategies on a series of empty number lines, which could highlight how different sized jumps can lead to the same endpoints (Lambert et al., 2017). However, students may not pay attention to relations among jumps if they are not the ones they might make themselves. Having students try different jumps themselves might help them notice the structure of the jumps or use the structure of one to solve the other. In this study, we presented students with two number lines showing the same distance but asked them to break up the distance by making jumps in two different ways (i.e., creating two addition problems with the same answer). Through this task we hoped to gain insight into the following research questions: How do first and third graders make sense of the distance between numbers on number lines? (a) Are students more likely to jump to the nearest ten, make a jump of ten, or make a five? (b) How do students determine the distances of their jumps? (c) To what extent do they notice and use structure between the number lines in determining distances?

Methods

Participants and Design

Participants included 32 first and 36 third graders from a public suburban elementary school in the midwestern US who completed the target item. The data from this paper came from a larger set of items exploring the relation between students' explanations and debugging in programming and mathematics. This paper focuses on a mathematical explanation item that specifically targeted students' noticing of the common structure between distances on two number lines. Although the first graders had likely been exposed to number lines, they were not a regular part of their instruction. We only presented the target numbers on the number lines, making them similar to students' use of empty number lines. Empty number lines only show the numbers being used (as opposed to labeling all whole numbers, for example), which reduces their passive use (Klein et al., 1998) and can also make it easier for students to focus on the length of the jumps rather than just the number of jumps. Although we did not ask students to identify differences, we structured the two parts of the item to implicitly highlight a contrast in jumps on a number line as students solved each part (see Figure 1). During individual, video-recorded interviews, we read the instructions to the students and helped record their jumps and answers.

2a. The bunny starts at 11, jumps to one apple, and then jumps to get to the carrot at 27. Which apple do you want to jump to? How far is it to the apple? How far is it from the apple to the carrot? How far did the bunny go from 11 to 27?

2b. Now which apple do you want the bunny to jump to? How far is it to the apple? How far is it from the apple to the carrot? How far did the bunny go from 11 to 27?

2c. Which way is better? Explain.

Figure 1. Mathematical Explanation Item - Bunny Hops (11 + __+ __ = 27)

Analysis

First, we recorded which apple they jumped to for their first (Figure 1, 2a) and second choices (Figure 1, 2b) to determine if they were more likely to *choose 20* (the nearest ten), *choose 21* (a jump of ten), or *choose 15* (the nearest multiple of five). Then, we analyzed students' distance strategies and noted their use of structure.

Finding the distances. For each sub-item (2a and 2b in Figure 1), we coded their strategies for how they determined the length of each distance: (1) first jump from 11 to the apple, (2) second jump from the apple to 27, and (3) total. We coded students' strategies as *known fact* if they answered quickly and were correct; we used *guessing* if they answered quickly but were incorrect. The guessing strategies included answers that were close to the correct distances, answers that were wildly off, and answers that resulted from students concatenating the numerals from the jumps (e.g., saying the jump from 11 to 20 is 1120). *Counting* strategies included counting in their head, counting all, or counting on, which often involved finger motions, drawing marks, or mouthed counting words. A couple students subtracted by counting or added in chunks to find the distances, which we also classified under counting. After students found the distances on both number lines, we asked them which way they thought was better.

Noticing and using structure. Across the two sub-items, we coded their strategies as using structure if they used knowledge of the other sub-item to determine the length of a jump (i.e., through compensation) or the total distance (i.e., indicated the answer would be the same because the rabbit starts and ends at the same points). If students' answers to the two sub-items did not align, we took this as further evidence that they did not notice the common structure between the two number lines.

Results

Both first and third graders were least likely to choose 15 for their combined choices (the higher percentage on first graders' first choice appeared to be for 15 because it was the first option for where to jump). First graders were most likely to choose 21 (a jump of ten) first, followed by 20 (a jump to the nearest ten); whereas, third graders were slightly more likely to choose 20 first then 21 (see Table 1).

Table 1: Number of Students' First and Second Choice for Where to Jump

	Choose 15	**Choose 20**	**Choose 21**
First choice			
First grade (*n*=32)	34% (*n*=11)	22% (*n*=7)	44% (*n*=14)
Third grade (*n*=36)	19% (*n*=7)	33% (*n*=12)	47% (*n*=17)
Second choice			
First grade (*n*=32)	13% (*n*=4)	44% (*n*=14)	44% (*n*=14)
Third grade (*n*=36)	17% (*n*=6)	53% (*n*=19)	31% (*n*=11)
Choices Combined			
First grade (*n*=32)	47% (*n*=15)	66% (*n*=21)	88% (*n*=28)
Third grade (*n*=36)	36% (*n*=13)	86% (*n*=31)	78% (*n*=28)

Twenty-eight students (nine first and 19 third graders) indicated that choosing 21 first was better than their other choices because it was faster, or they indicated that the addition problem 10 + 6 was easier to solve than 9 + 7 or 4 + 12. A few students chose 21 as their favorite choice based on the jumping-bunny context. For example, one student said, "Because you use a lot of energy and then just slow down," hinting that the bunny would be tired after the first jump and would want a smaller jump. Choosing 20 first (i.e., making a ten) was a favorite of 17 students (eight first and nine third graders) who thought the numbers were easier or indicated that they liked jumping closer to the middle of the number line. Two of these students (one first and one third grader) also referred to the context and explained why choosing 21 was not good, "Because I think of this [pointing to the jump from 10 to 21], I don't think anyone can jump ten meters (third grader)" and "You don't have a lot of energy, so you go a little [each time] (first grader)."

Finding the Distances

Regardless of their first jump, third graders correctly found the distance from 11 to 27 on 71% of their attempts across both sub-items, and first graders correctly found the distance on 17% of their attempts across both sub-items (see Table 2).

Table 2: Percent of Students Showing the Correct Distances on Both Sub-items

Bunny's first jump	**First jump** (from 11 to chosen apple)	**Second jump (**from chosen apple to 27)	**Total** (overall distance from 11 to 27)
Chooses 15			
First grade (*n*=15)	33% (*n*=5)	27% (*n*=4)	13% (*n*=2)
Third grade (*n*=11)	82% (*n*=9)	64% (*n*=7)	73% (*n*=8)
Chooses 20			
First grade (*n*=21)	19% (*n*=4)	24% (*n*=5)	24% (*n*=5)

Third grade (*n*=31)	71% (*n*=22)	81% (*n*=25)	68% (*n*=21)
Chooses 21			
First grade (*n*=28)	21% (*n*=6)	32% (*n*=9)	14% (*n*=4)
Third grade (*n*=28)	82% (*n*=23)	89% (*n*=25)	75% (*n*=21)

For first graders, finding the distance of smaller jumps was easier (i.e., jump from 11 to 15 and from 21 to 27, compared to 11 to 27; see Table 2). Interestingly, when first graders chose 20 first, they were slightly more likely to find the total distance correctly than when choosing 15 or 21. Although third graders also did well on the small jumps, they had high performance when jumping ten (i.e., jumping from 11 to 21) and jumping from a ten (i.e., jumping from 20 to 27), suggesting they were better able to make use of place value (see Table 3).

Table 3: Percent of Students' Strategies When Finding Distances for Jumps and Total

	Strategy			
	Guess	**Known Fact**	**Counting**	**Using Structure**
Choosing 15 first				
First grade (*n=15)*	33% (15/45 times [a])	11% (5/45 times)	40% (18/45 times)	2% (1/45 times)
Third grade (*n*=11)	26% (10/39 times)	36% (14/39 times)	36% (14/39 times)	3% (1/39 times)
Choosing 20 first				
First grade (*n*=21)	57% (36/63 times)	10% (6/63 times)	22% (14/63 times)	0% (0/63 times)
Third grade (*n*=31)	25% (23/93 times)	49% (46/93 times)	23% (21/93 times)	5% (5/93 times)
Choosing 21 first				
First grade (*n*=28)	55% (49/84 times)	11% (9/84 times)	17% (14/84 times)	2% (2/84 times)
Third grade (*n*=28)	13% (11/84 times)	57% (48/84 times)	23% (19/84 times)	7% (6/84 times)

[a] For each sub-item, we coded three strategies per person. A total of 15 first graders chose to jump to 15 first, so there were 15 * 3 = 45 possible times for them to use a strategy.

Students often provided their answers quickly. In the case of the first graders, over 75% of their quick answers (i.e., known facts and guessing) aligned with them incorrectly guessing what the answer would be.

Guessing. Third graders were more accurate but still incorrectly guessed between 19%-42% of the time out of their quick answers. Students' guesses sometimes indicated a focus on the endpoints of the jumps. For example, one third grader, when asked how long the jump from 11 to

21 is, said it was 21, even though he determined the following jump from 21 to 27 was six. One first grader concatenated the labeled points on the number; for example, when determining the distance from 11 to 20, he indicated the distance was "eleven twenty" or 1120. Another first grader added up the labeled numbers, getting 11 + 15 = 26 for the distance from 11 to 20 and then he took that 26 + 20 + 21 to get the distance from 20 to 27.

Aside from endpoints, some students focused on the number of jumps rather than the distance of the jumps. In fact, several students at both grade levels interpreted each jump as a unit of movement rather than a distance. When asked what the distances from 11 to 21 and 21 to 27 were, a third grader replied, "One and one." Likewise, he indicated that the total distance jumped was "Two." Using the apples as his unit of distance, another third grader (like many first graders) said the distance from 11 to 21 was "two" and from 21 to 27 was "one." When asked what the total distance was, he said, "Three" and then counted the three apples to double check.

Known facts. First graders did not show much difference in their use of known facts across their attempts, suggesting the number choices did not affect them much. When identifying the length of the jump from 20 to 27, a first grader indicated that it was seven because it was "twenty and seven more" to 27. Third graders, on the other hand, were noticeably more likely to use known facts when jumping ten or making a multiple of ten than when jumping to and from 15.

Counting. Students were mostly likely to count if they chose 15 first compared to choosing 20 or 21; however, of the 18 times first graders counted, only five times (28%) were correct (five additional attempts were off by one). Third graders were more successful; of the 14 times they counted, nine times (64%) were correct (and three were off by one). Several students miscounted when trying to find the distance between two numbers or adding two jumps. For example, one third grader counted the distance from 15 to 27 as 13. Another third grader raised fingers to count from 15 to 27. At the end of her count she had two fingers up (after counting fingers on both hands), forgot how many hands she had counted before that, and reported the distance as 17 instead of 12. Interestingly, when jumping to 20, another third grader used subtraction (i.e., solved 20 - 11), and one added up in chunks (11 + 4 = 15 and 15 + 5 = 20).

Using structure. Only three first graders (9%) and eleven third graders (31%) explicitly referred (with or without prompting) to using the structure of one number line to help them solve one or more of the distances in the other number line. On the first sub-item, a third grader identified the jump from 11 to 21 as a jump of 10 and from 21 to 27 as a jump of 6. When determining the length of the jump from 11 to 20 on the next sub-item, he said, "That's obvious, nine, since you're just removing one from [that first jump]." Completing the same items in reverse order, a first grader found the distance from 11 to 20 as 9 on the first sub-item. On the next sub-item, when determining the distance from 11 to 21 as 10, she said, "Last time there was nine, and there was just one more." When another third grader went to find the total distance on the second sub-item, he said that he started to add the two jumps and then realized he could just use the ones from the first sub-item. Six additional first graders and five additional third graders noticed the totals were the same after answering or had reactions, such as laughing and quickly answering, that implied they noticed that the total distances were the same.

Discussion

Students' more frequent choice to the jump of ten or to the next multiple of ten and their greater accuracy when doing so, reinforces the importance of helping students make use of 10-based reasoning (Fuson et al., 1992; Murata, 2004). Students were also more likely to notice the structure between the number lines if they chose those jumps. However, fewer than one-third of students used the common structure of the number lines to help them solve the problems, suggesting an increased need to focus on the connections among number combinations. Three factors appeared to detract students' attention to the structure. First, the context may have been distracting for some students. For students who did not think it was feasible for a bunny to jump a certain distance, comparing distances between the two number lines may not have been likely. Interestingly, some students who talked about one choice being better than another did talk about jumps being longer or shorter, suggesting they did notice the differences between number lines to some extent and reinforcing that some contexts may work for some students and not for others. Second, and related to the first, the number line model was unfamiliar to some students, or they focused on the number of jumps instead of the distance of the jumps.

While we maintain that the number line is an important model for students to interpret, we could explore additional ways of presenting the distances (perhaps as bars stretching from one point to the next instead of jumps) to support learners who are less familiar with these models. Third, the difficulty of the numbers played a role in students' attention to the structure. Students (typically third graders) with strong number sense did not necessarily benefit from noticing structure among the jumps; on the other hand, students (such as most of the first graders) who used more on guessing or had inaccurate counting were not likely to notice the structure. Therefore, to help students notice structure the specific numbers in the items need to be chosen carefully. For example, the first graders might have been more successful if the bunny was starting at 1 and jumping to either 5, 10, or 11 before jumping to 15. That said, encouraging reflection on the visuals of the jumps did promote some students' curiosity about the relations between the two sub-items.

Even if they did not use the structure of one number line to interpret distances on the second number line, students' use of known facts without explanation provides some evidence that they noticed structure within the numbers presented (i.e., adding a ten or making a ten). Regarding those who did notice structure, their answers suggest that pairing problems with number lines in this way could help students check and reason about the accuracy of their answers and could help students think about the relation between number combinations, especially if paired with discussion. Helping them make connections about common distances on the number line could later support a focus on distance when using rulers.

Acknowledgments

This work was supported by NSF DRL ITEST Grant #1759254.

References

Bobis, J. (2007). The empty number line: A useful tool or just another procedure? *Teaching Children Mathematics, 13*(8), 410-413.

Bobis, J., & Bobis, E. (2005). The empty number line: Making children's thinking visible. In M. Coupland, J. Anderson, & T. Spencer (Eds.), *Making Mathematics Vital: Proceedings of the twentieth Biennial Conference of the Australian Association of Mathematics Teachers* (pp. 66-72). The Australian Association of Mathematics Teachers, Inc.

Cain, C., Faulkner, V., & Fanelli, K. (2020). Developing number sense through an exploration of subtraction. *Australian Primary Mathematics Classroom, 25*(2), 26-30.

Ernest, P. (1985). The number line as a teaching aid. *Educational Studies in Mathematics, 16*(4), 411-424. http://www.jstor.org/stable/3482448

Fuson, K. C., & Fuson, A. M. (1992). Instruction supporting children's counting on for addition and counting up for subtraction (Brief Report). *Journal for Research in Mathematics Education, 23*(1), 72-78. https://www.jstor.org/stable/749165

Fuson, K. C., Wearne, D., Heibert, J. C., Murray, H. C., Human, P. G., Olivier, A. I., Carpenter, T. P., & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for Research in Mathematics Education, 28*(2), 130-162.
Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics, *Mathematical Thinking and Learning, 1*(2), 155-177. DOI: 10.1207/ s15327833mtl0102_4
Klein, A. S., Beishuizen, M., & Treffers, A. (1998). The empty number line in Dutch second grades: Realistic versus gradual program design. *Journal for Research in Mathematics Education, 29*(4), 443-464. http://www.jstor.org/stable/749861?origin=JSTOR-pdf
Lambert, R., Imm, K., & Williams, D. A. (2017). Number strings: Daily computational fluency. *Teaching Children Mathematics, 24*(1), 48-55. https://www.jstor.org/stable/10.5951/teacchilmath.24.1.0048
Murata, A. (2004). Paths to learning ten-structured understandings of teen sums: Addition solution methods of Japanese grade 1 students. *Cognition and Instruction, 22*(2), 185-218. DOI: 10.1207/s1532690xci2202_2
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* The National Council of Teachers of Mathematics, Inc.
National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics*. Washington DC: Author. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.
National Research Council. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.
Saxe, G. B., Earnest, D., Sitabkhan, Y., Haldar, L. C., Lewis, K. E., & Zheng, Y. (2010). Supporting generative thinking about the integer number line in elementary mathematics. *Cognition and Instruction, 28*(4), 433-474. http://dx.doi.org/10.1080/07370008.2010.511569
Schwartz, D. L., Tsang, J. M., & Blair, K. P. (2016). *The ABCs of how we learn: 26 scientifically proven approaches, how they work, and when to use them*. WW Norton & Company.

PLAYFUL MATH: MODELING STUDENTS' ENGAGEMENT IN PLAY-BASED ALGEBRA ACTIVITIES

Amy B. Ellis
University of Georgia
amyellis@uga.edu

Dru Horne
University of Georgia
dhorne8@uga.edu

Anna Bloodworth
University of Georgia
adominey@uga.edu

Annelise Nielsen
University of Idaho
annelise@uidaho.edu

Robert Ely
University of Idaho
ely@uidaho.edu

Interest-driven activities, such as mathematical play, can support student agency, motivation, and engagement, and can foster dispositions that reflect authentic disciplinary engagement. However, the bulk of research on mathematical play investigates the mathematics that emerges in young children's natural play or in informal spaces such as video games. We introduce the term "playful math" to highlight the potential of playifying classroom-based activities, and we explore the nature of students' activity when engaged in playful math tasks in a teaching experiment. Our findings show that playful math tasks increased students' agency, authority, investment, and goal selection, as well as encouraged the development of creative, challenging ideas. We present a case of two students' playful engagement in the form of an Explore-Strategize Cycle and discuss implications of playful math for student engagement.

Keywords: Algebra and Algebraic Thinking, Cognition, Problem-Based Learning

Motivation and engagement are critical factors in supporting students' abilities to understand and persist in mathematics (e.g., Durksen et al., 2017). Students' experiences of self-efficacy can foster motivation, which in turn can predict persistence in STEM fields (Simon et al., 2015). There remain, however, ongoing challenges with student motivation and engagement in mathematics (Martin & Marsh, 2006). These challenges are particularly salient in algebra, which represents a critical transition to secondary mathematics, and can act as a gatekeeper, with algebra performance often serving as the main criterion to determine a student's readiness for more advanced courses (Riegle-Crumb, 2006).

One contributing factor to these challenges is that the mathematics taught in algebra can emphasize routinized procedures, resulting in students reporting decreased motivation and engagement (Herzig, 2004). In contrast, focus on interest-driven activities such as mathematical play invites student agency and can increase equitable access to algebra (Widman et al., 2019). In fact, much of authentic disciplinary engagement involves many of the same features as play (Gresalfi et al., 2018; Jasien & Horn, 2018), and professional mathematicians have been shown to engage in mathematical play as part of their disciplinary practice (Lockhart, 2009).

Mathematical play has the potential to support a productive environment for conjecturing and exploring by centering student voices and by offering opportunities to investigate novel ideas (Gresalfi et al., 2018). It can also offer important engagement, motivation, and conceptual benefits, with studies suggesting positive effects for enjoyment, attitudes, and learning outcomes (e.g., Barab et al., 2010; Plass et al., 2013; Wager & Parks, 2014). However, the bulk of existing research on mathematical play is situated either in early childhood learning, or in informal spaces such as video games. Mathematical play can certainly occur in classroom settings, but less is understood about how to incorporate play into the school mathematics that students and teachers navigate in classroom settings, particularly for adolescents. Therefore, this study investigates what occurs when incorporating play-based elements in algebra problem-solving tasks. We use the term "playful math", rather than "mathematical play", to highlight the potential of "playifying" classroom mathematical activity. In particular, we address the following research questions: (1) What characterizes students' mathematical activity when investigating rates of change within play-based activities? (2) How does playifying mathematical tasks affect the nature of students' engagement, if at all?

Background Literature: The Potential Benefits of Playful Math

Mathematics is an area for which interest-driven engagement is not always a consideration in pedagogical design, and yet, focus on interest-driven activities can invite student agency and create spaces for students to bring in more of their whole identities (Widman et al., 2019). Playful math can offer multiple points of access into mathematical ideas, providing opportunities for students to create new challenges for themselves and experiment with ideas that go beyond familiar operations and connections (Featherstone, 2000). Playful engagement can also free students from the stigma of traditional assessment, encouraging a disposition of exploration and innovation (Barab et al., 2010). Research investigating children's and adolescents' playful math suggests both affective and conceptual benefits. Playful math has been shown to support increased enjoyment (Plass et al., 2013), increased engagement (Barab et al., 2010), positive social interaction and communication (Edo et al., 2009), and to engender positive attitudes towards mathematics (Holton et al., 2001). Playful math has also been shown to offer some cognitive benefits, supporting children's geometric thinking (Levine et al., 2005), spatial skills (Casey et al., 2008; Levine et al., 2012), and number development and numeracy (Siegler & Ramani, 2008; Wang & Hung, 2010). Some studies even suggest increased learning efficacy in play-based environments (Barab & Gresalif, 2010; Bodrova, 2008).

Some benefits afforded by play may occur because playful math offers avenues for students to "conjecture and explore in disciplinarily authentic ways" (Jasien & Horn, 2018, p. 624). In fact, there are a number of features of play that mirror the forms of engagement seen in the work of mathematicians, including open exploration, the use of imagination, being voluntary, being ordered and rule governed, and exercising personal agency to determine and pursue goals (Gresalfi et al., 2018; Featherstone, 2000). These parallels point to the importance of intellectual play in learning mathematics, and suggest a need for research examining ways to make mathematics tasks more playful. In particular, the field needs more research exploring playful math with older students in middle-school, high-school, and college, particularly in terms of incorporating playful elements into classroom mathematics in critical domains such as algebra.

Theoretical Frameworks: Defining Playful Math and Quantitative Reasoning

Definitions of mathematical play vary, but all emphasize students' agency in exploration, self-selection of goals, and self-direction in how to accomplish them (Jasien & Horn, 2018). For instance, Williams-Pierce (2019) defines mathematical play as "voluntary engagement in cycles of mathematical hypotheses with occurrences of failure" (p. 591), and Holton et al. (2001) describe mathematical play as the playful exploration that emerges when learners find themselves in mathematical contexts with an open goal. One challenge in characterizing play is that it describes both a form of activity (such as playing a video game) and a stance or orientation towards an activity (Malaby, 2009). We address the second aspect to define playful math as a particular form of engagement in mathematics, one that entails (a) agency in exploration, (b) self-selection or investment in mathematical goals, (c) self-direction in goal accomplishment, and (d) a state of immersion and/or enjoyment.

For the purposes of this study, we leveraged situations involving covarying quantities that students can manipulate and investigate in order to reason flexibly about function as a representation of dynamically changing events (Carlson et al., 2002). By quantities, we mean schemes composed of a person's conception of an object or event, such as a rectangle; a quality of the object or event, such as the rectangle's length or area; an appropriate unit, such as centimeters or square centimeters; and a process for assigning a numerical value to the quality (Thompson, 1994). By covariation, we refer to the visualization of two quantities' values simultaneously, uniting the quantities' magnitudes in order to understand that at every instance, both quantities have a corresponding value (Saldanha & Thompson, 1998; Thompson & Carlson, 2017). Researchers argue that students naturally attend to coordinated changes (e.g., Blanton & Kaput, 2011), which is supported by studies showing that students typically first analyze functional situations from a coordinated change perspective (e.g., Confrey & Smith, 1995; Madison et al., 2015). Furthermore, we hypothesized that the visualization opportunities afforded by a covariation approach could potentially support the type of open exploration and agency that occurs in playful engagement. As we describe in the next section, we built on students' natural ways of reasoning to modify a set of existing covariation activities in order to playify them.

Methods

Setting and Participants

We conducted two teaching experiments (TEs) (Steffe & Thompson, 2000), which both met weekly for 60-75 minutes a session for 5 consecutive weeks. The first author was the teacher-researcher (TR). Both teaching experiments were video and audio recorded, with the exception of Day 2 for each, when a technical error prevented video recording. After each session, we collected all student written work. The first teaching experiment was a paired TE with two rising 7th-grade students, Stewie and GJ. The second TE was with three participants, Artemis (a rising 7th-grade student) and Apollo and Francis (rising 6th grade students). (Participants chose their own pseudonyms.) Due to schooling interruptions from Covid-19, the students' experiences with graphing and functions was largely limited to plotting points. The rising 7th graders also had some experience with graphing lines, but none of the students had experienced a unit on linear functions. For the purposes of this paper, we report on data from the first TE with Stewie and GJ.

Task Design Principles

Our aim was to playify existing covariation problems in order to investigate students' mathematical activity when exploring rates of change within play-based tasks. By "playify", we mean increasing the potential for playful math engagement, while also acknowledging that a task cannot dictate how students will engage with it. We drew on a set of established research-based activities to support understanding of linear and quadratic growth with continuously covarying quantities (Ellis et al., 2020; Matthews & Ellis, 2018). In these activities, students investigate dynamically growing shapes, determining the rate of change of a shape's area compared to its changing length as it sweeps out from left to right, and then graph that relationship (Figure 1).

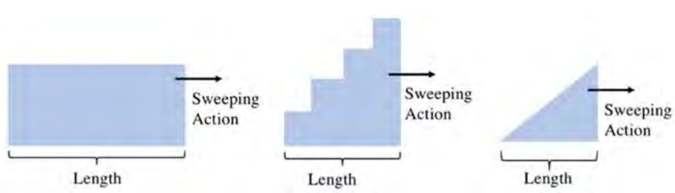


Figure 1: Growing Rectangle, Stairstep, and Triangle

We followed four design principles to playify the tasks (Plaxco et al., 2021): (1) allow for free exploration within constraints; (2) allow the student to act as both designer and player; (3) engender anticipating within the task; and (4) provide a method for authentic feedback. These led to the *Guess My Shape* activity, in which the students created secret shapes of their choice (design principles #1, #2), constructed graphs comparing length and area (design principles #2, #3), and then challenged one another or the TR to determine the shape based on the graph alone (design principles #2, #3, #4). We hypothesized that the playified tasks would encourage playful engagement, without assuming that they would guarantee such engagement.

For Session 1, we only used the extant covariation tasks (Figure 1), which we call standard tasks. For Sessions 2 - 5, we used both the standard tasks and the playified tasks. For each of those sessions, we spent the first two-thirds of the session working with the standard tasks and the last one-third implementing the playified tasks. Figure 2 shows an example of the types of shapes the students encountered and their associated graphs.

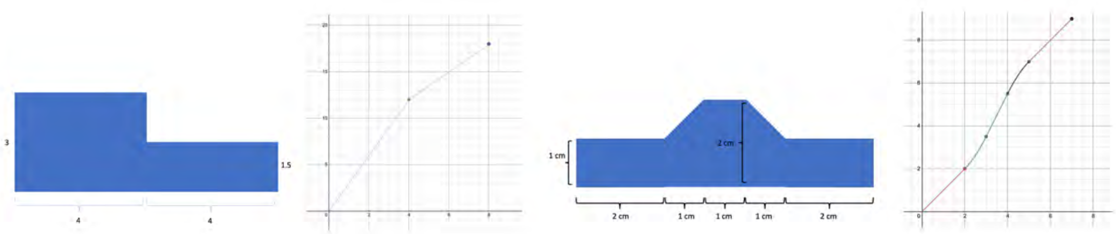


Figure 2: Shapes and associated length-area graphs

Analysis

We employed retrospective analysis (Steffe & Thompson, 2000) in order to characterize the students' conceptions throughout the teaching experiment. We first transcribed each session and then produced a set of enhanced transcripts that included verbal utterances, images of student work, and descriptions of non-verbal actions. We then analyzed the data with two lenses. With the first lens, we identified students' conceptions of graphs, covariation, and rates of change, drawing on Thompson and Carlson's (2017) framework of variational and covariational reasoning, Ellis et al.'s (in press) conceptual acts for constructing linear and quadratic growth through covariation, and Moore and colleagues' graphing actions (Liang & Moore, 2020; Moore et al., 2019; Tasova & Moore, 2020). With the second lens, we identified aspects of playful mathematics, relying on a combination of a priori codes from our definition of playful math, as well as emergent codes that occurred during the coding process through the constant comparative method (Strauss & Corbin, 1990). For the purposes of this paper, we focus on the analysis conducted through the second lens of playful math, and we describe those codes in the next section. The first three authors coded each transcript independently, and then the project team met weekly to refine and adjust codes and resolve discrepancies; this iterative process continued through eight rounds of code adjustments, until all codes had stabilized.

Results: The Explore / Strategize Cycle

Recall that our second research question considered whether and how playifying tasks may affect the nature of students' mathematical engagement. In addressing this question, we turned to our playful math codes to assess whether there was any difference in code frequencies across the two task types. We developed six playful math codes: (a) self-selection or investment in mathematical goals, (b) agency in exploration, (c) investment and/or enjoyment, (d) taking on authority, (e) creative / unusual, and (f) harder math. The first three codes were drawn from our definition of mathematical play. *Self-selection of goals* occurred when students chose their own goals or showed evidence of being invested in a goal. *Agency in exploration* refers to instances of students demonstrating agency in how they explored a task or its goals, and *investment and/or enjoyment* occurred when students showed evidence of investment or immersion, and/or demonstrated enjoyment of their activity. The remaining three codes were emergent. *Taking on authority* refers to instances of students demonstrating mathematical authority with a task or in their engagement with the TR. *Creative/unusual* applies when students have developed an idea, representation, or task that we perceived as creative or novel, and *harder math* occurred when students created a goal or task that introduced challenging mathematics that surpassed the TR's intended set of topics or concepts.

Across the five sessions we implemented 18 standard and 5 playified tasks. In our initial rounds of coding, we applied the relevant code any time we observed a student demonstrating evidence of playful engagement, resulting in a total of 130 code occurrences. We then revisited the codes to determine how many times they occurred during the standard tasks compared to the playified math tasks. We found that our first three codes occurred across both tasks, but with greater frequency in the playified tasks, supporting our hypothesis that the playified tasks would in fact elicit more playful engagement. The last three codes only occurred during playified tasks (Figure 3). The difference in code frequencies is striking given that 78% of the tasks enacted during the teaching experiment were standard tasks.

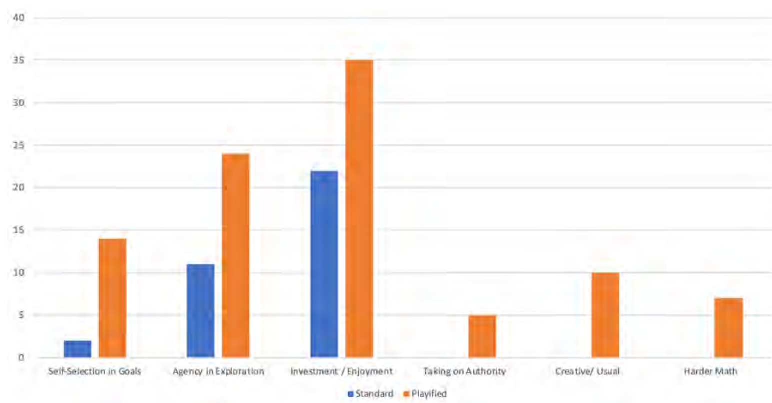


Figure 3: Playful math codes across standard and playified tasks

The difference in code frequencies led us to consider in more detail the nature of students' activity within the playified tasks, addressing our first research question. We found that during those tasks, the students demonstrated a novel form of engagement, which we call the *Explore-Strategize Cycle* (Figure 4). Within this cycle, students shift back and forth between exploration and strategizing. Exploration occurs when students either (a) engage in an action with little or no anticipation of an associated outcome, or (b) notice properties of outcomes and wonder about the connection between those outcomes and the actions that led to them. For example, in the Guess My Shape game, students engaged in exploration when they invented new shapes or when they plotted points arbitrarily to try to develop an unusual graph. In contrast, strategizing occurs when students engage in actions tied to an anticipated outcome. This could include modifying a shape to create a graph that will have a specific property, such as symmetry, or recognizing that a particular shape will be difficult to graph. Once we identified the Explore-Strategize Cycle in students' activity, we revisited the entire data corpus to determine whether this cycle occurred across both task types. We found no instances of the cycle in the standard tasks; instead, during those tasks, students engaged in repeated acts of strategizing but without associated exploration.

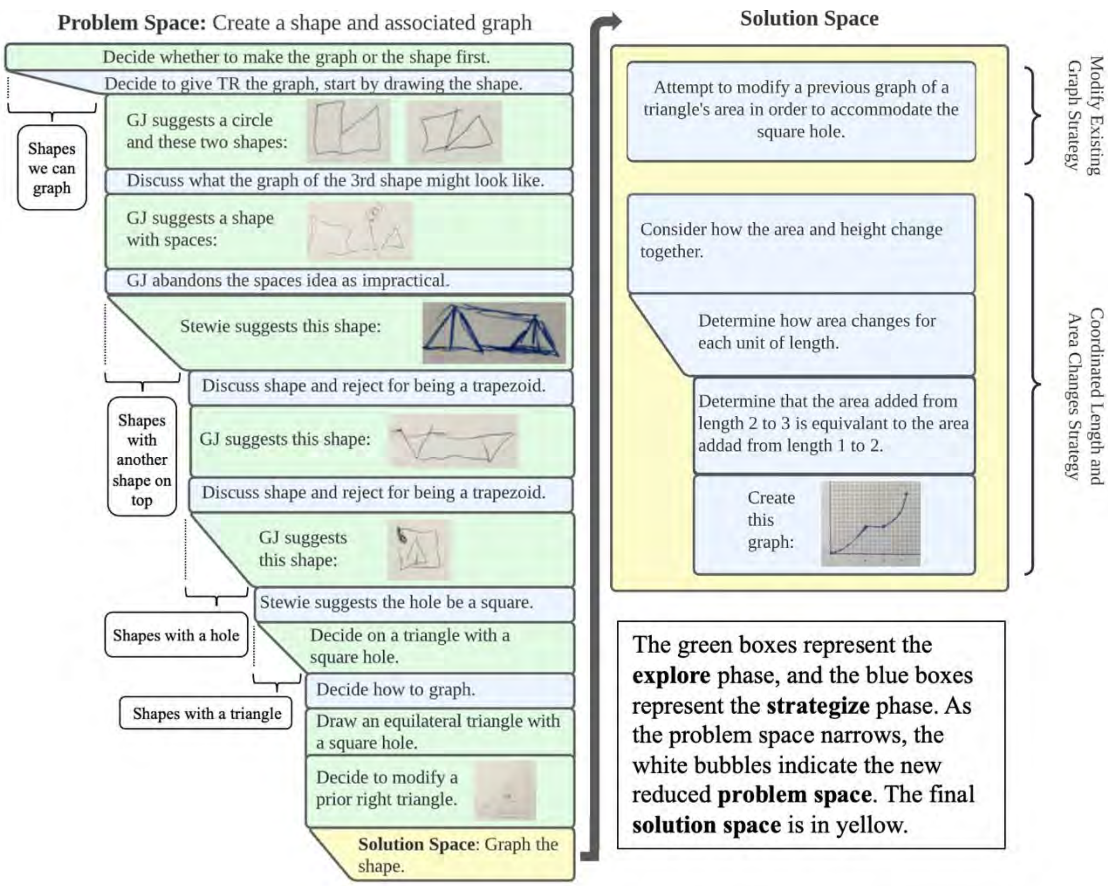


Figure 4: The Explore-Strategize Cycle for A Triangle with a Square Hole

In order to exemplify the nature of students' playful engagement, we present one Explore-Strategize cycle, in which Stewie and GJ collaborated on the Guess My Shape game to create a task that would stump the TR. Within any cycle, there are both problem spaces and solution spaces. As students engage in exploring and strategizing, they narrow the problem space. For instance, when playing Guess My Shape, the set of possible problems are determined by the shapes that the students draw and graph. As they consider what shape to create and how to graph it, they progressively narrow the problem space by exploring characteristics of their invented shape (such as symmetry, the inclusion of a hole, and so forth), or by strategically selecting aspects of the graph or shape to examine. In doing so, the students exclude shapes that do not meet the desired criteria. This can be seen in Figure 4 in the narrowing of the initial problem space, "create a shape and associated graph", down to the final problem they chose to solve, "graph the area and length of this right triangle with a square hole". Once the students have narrowed down to the final problem, they shift into the solution space.

As GJ and Stewie narrowed the problem space, they set implicit and explicit goals to guide their actions. The first decision they had to make was whether to start with a shape or a graph. Stewie suggested that they begin with the graph, "because that would be harder", but GJ wanted to begin by drawing the shape. Together, the students had an implicit goal of creating a shape that was both feasible for them to construct and still challenging for the TR. Stewie strategically negotiated these goals by suggesting that they create the shape first but give the TR the graph. In doing so, he anticipated that determining the shape from the graph would be more difficult than determining the graph from the shape. From this strategizing action, the students have now narrowed the problem space to shapes within their ability to graph.

GJ and Stewie then began to explore a series of potential shapes. Each time, their exploration phase was followed by a strategizing phase in which they tried to anticipate what the associated area-length graph would be. During this process, they rejected multiple shapes as being too difficult to graph, including a circle, composite shapes, and shapes with spaces. They also rejected shapes as being too simple, such as composite shapes that would have area-length graphs identical to trapezoids. GJ then suggested a square with a triangular hole. However, in shifting to considering the associated graph (strategizing phase), the students decided that it would be easier to graph a shape with a square hole. They then explored a triangle shape with a square hole, and then when strategizing how to graph it, anticipated that they could create a graph by partitioning their chosen shape into squares. In exploring final shape options, GJ and Stewie settled on a right triangle, similar to a prior shape they had graphed, but this time with a square hole.

Once the students reached the solution space, they used two different strategies. Their first strategy was to modify a prior graph they had made of a similar triangle without a hole. Stewie said, referring to the constantly-increasing area, "It's going to be a curved line the whole way, and the triangle, but then, what do you do?" Stewie was anticipating that the graph would look different where the hole occurred, but he trailed off, becoming uncertain about how to account for the hole. He then shifted to the second strategy, suggesting that they consider how the area and length change together. In doing so, he quantified the length and area segments, writing "4 in." for both the length and the height of the triangle. He then suggested they determine the area and its change for each unit of length: "At the first point, it was half, all right. For the second one, was one and a half." As he spoke, Stewie began to graph a curve for the first two points. In determining the area from $x = 2$ in. to $x = 3$ in., the portion of the triangle with a 1-in.^{2} hole, both students decided that the area added would be the same as the area added for the prior increment, from $x = 1$ in. to x = 2 in:

Stewie:	So, third one is the same as the second one pretty much.
GJ:	It is.
Stewie:	So, so it would just be like, here. [Draws a horizontal line segment.]

Stewie and GJ anticipated, incorrectly, that the graph would be horizontal if the amount of area added from $x = 2$ to $x = 3$ was the same as the amount of area added from $x = 1$ to $x = 2$. They were correct that the amount of area added was the same across the two increments, but they conflated total accumulated area with added area. When next considering the final increment, from $x = 3$ in. to x = 4 in., Stewie returned to a curve, recognizing that the amount of area constantly increased throughout the increment. It was later in their conversation with the TR that the students realized that the portion of the graph from $x = 2$ in. to $x = 3$ in. would be another curve, rather than a horizontal line segment.

The students demonstrated all six playful math codes during this Explore/Strategize Cycle. They self-selected multiple goals in deciding what types of shapes to pursue, and they also demonstrated agency in exploring those shapes and their associated graphs, freely shifting across different potential shapes. In suggesting novel shapes such as circles, composite shapes, and shapes with spaces and holes, the students demonstrated ideas that were creative and unusual; many of their shapes were markedly different from those that had been previously introduced.

Furthermore, their decision to introduce a hole was not only creative, it also represented a new challenge (harder math), one that took the TR by surprise and brought up a set of mathematical ideas that the TR had not anticipated addressing. In particular, once holes are allowed, area-length graphs no longer uniquely determine a shape, which raises new questions about the set of shapes with holes that can be determined by an area-length graph.

The students also demonstrated immersion, investment, and enjoyment in the task. Their engagement was sustained; they spent 27 minutes on this task alone, and they considered and abandoned many shapes before ultimately settling on a final version. When bringing the TR in to determine the shape from their graph, they actually changed their minds about the accuracy of their graph in the moment. In doing so, the students decided to send the TR away again so that they could re-evaluate. Stewie told the TR, "I think I have an idea. I'm, I'm not totally confident. Yeah, go away." In this moment, the students evidenced investment in their graph, as well as mathematical authority, deciding not only what the graph should be, but also dictating how the process of sharing their work should proceed.

Discussion

Playifying tasks did result in more playful engagement during those tasks; in fact, three forms of playful engagement (taking on authority, creative / unusual, and harder math) occurred only during the playified tasks. However, we still found evidence of playful engagement in the standard tasks, which underscores that playful math as a construct describes a form of engagement, rather than a type of task. Furthermore, although we only observed explore-strategize cycles during the playified tasks, this may be due to the fact that the playified tasks were also more open-ended tasks. More research is needed to tease out the effects of playifying tasks that begin as open-ended tasks. The presence of playful engagement across all tasks suggests that playifying mathematics might support a more playful disposition in general. In our data, we found that one form of playful engagement, investment and/or enjoyment, increased throughout the sessions (occurring six times across the first two days, compared to 34 times on the last two days). It may be that sustained engagement in playful math supported students' increased investment in subsequent tasks, regardless of their form.

One affordance of playful math is that it has the potential to center student voices, particularly those voices less likely to be taken up in more traditional classroom settings. Two of our design principles were particularly important in this role: allow free exploration within constraints, and allow the student to act as both player and designer. By shifting the student's role to designer and encouraging exploration in designing, we opened a space for students to introduce their own ideas in ways that were less constrained by pre-determined topics. Their resulting ideas were novel, creative, and mathematically challenging, and the students demonstrated autonomy and authority. We know that mathematicians, in their own disciplinary practice, also experience this type of autonomy and shift back and forth between exploration and strategizing in a manner similar to what we observed in our Explore / Strategize Cycles (e.g., Lockwood et al., 2016). By playifying mathematics, we set up spaces for more students to adopt productive mathematical dispositions. But more importantly, playful math has the potential to increase overall engagement and enjoyment of mathematics, particularly for students who may not experience traditional mathematics as engaging or enjoyable.

References

Barab, S.A., Gresalfi, M., & Ingram-Goble, A. (2010). Transformational play: Using games to position person, content, and context. *Educational Researcher, 39*(7), 525 – 536.

Blanton, M., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Springer.

Bodrova, E. (2008). Make-believe play versus academic skills: A Vygotskian approach to today's dilemma of early childhood education. *European Early Childhood Education Research Journal, 16*(3): 357–69.

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). *Applying covariational reasoning while modeling dynamic events: A framework and a study*. *Journal for Research in Mathematics Education*, *33*, 352–378.

Casey, B., Andrews, N., Schindler, H., Kersh, J.E., Samper, A., & Copley, J. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26*(3), 269-309.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66–86.

Durksen, T., Way, J., Bobis, J., Anderson, J., Skilling, K., & Martin, A. (2017). Motivation and engagement in mathematics: a qualitative framework for teacher-student interactions. *Mathematics Education Research Journal, 29*, 163-181.

Edo, M., Planas, N., & Badillo, E. (2009). Mathematical learning in a context of play. *European Early Childhood Education Research Journal*, *17*(3), 325-341.

Ellis, A.B., Ely, R., Tasova, H., & Singleton, B. (2020). Scaling continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*, *104*(1), 87 – 103.

Ellis, A.B., Özgür, Z., & Dogan, M.F. (In Press). A conceptual analysis of early function. In G. Akar, I. Zembat, S. Arslan, & P. Thompson (Eds.), *Quantitative Reasoning in Mathematics and Science Education*.

Featherstone, H. (2000). "-Pat+ Pat= 0": Intellectual Play in Elementary Mathematics. *For the Learning of Mathematics*, *20*(2), 14-23.

Gresalfi, M., Horn, I., Jasien, L., Wisittanawat, P., Ma, J.Y., Radke, S.C.,…,Sfard, A. (2018). Playful mathematics learning: Beyond early childhood and sugar-coating. In J. Kay & R. Luckin (Eds.), *13th International Conference of the Learning Sciences* (pp. 1335 – 1342). London, UK: University College London.

Herzig, A. H. (2004). Becoming mathematicians: Women and students of color choosing and leaving doctoral mathematics. *Review of Educational Research*, *74*(2), 171-214.

Holton, D., Ahmed, A., Williams, H., & Hill, C. (2001). On the importance of mathematical play. *International Journal of Mathematical Education in Science and Technology, 32*(3), 401 – 415.

Jasien, L., & Horn. I.S. (2018). "Ohhh, Now I Can Do It!": School-age Children's Spontaneous Mathematical Sensemaking in Construction Play. *Proceedings of the International Conference of the Learning Sciences, ICLS, 1*(June), 624 – 631.

Levine, S.C., Huttenlocher, J., Taylor, A., & Langrock, A. (1999). Early sex differences in spatial skill. *Developmental Psychology, 35*(4), 940 – 949.

Levine, S.C., Ratliff, K.R., Huttenlocher, J., & Cannon, J. (2012). Early puzzle play: A predictor of preschoolers' spatial transformation skill. *Developmental Psychology, 48*(2), 530 – 542.

Liang, B. & Moore, K. C. (2020). Figurative and operative partitioning activity: Students' meanings for amounts of change in covarying quantities. *Mathematical Thinking and Learning*, *23*(4), 291-317.

Lockhart, P. (2009). A mathematician's lament. *Devlin's Angle March, 1*(March), 1 – 25.

Lockwood, E., Ellis, A.B., & Lynch, A.G. (2016). Mathematicians' example-related activity when exploring and proving conjectures. *International Journal of Research in Undergraduate Mathematics Education*, 1 – 32.

Madison, B. L., Carlson, M., Oerthman, M., & Tallman, M. (2015). Conceptual precalculus: Strengthening students' quantitative and covariational reasoning. *Mathematics Teacher, 109*(1), 54- 61.

Malaby, T. M. (2009). Anthropology and play: The contours of playful experience. *New Literary History, 40*(1), 205-218.

Martin, A. J., & Marsh, H. W. (2006). Academic resilience and its psychological and educational correlates: a construct validity approach. *Psychology in the Schools, 43*(3), 267–282.

Matthews, P.M., & Ellis, A.B. (2018). Natural alternatives to natural number: The case of ratio. *Journal of Numerical Cognition, 4*(1), 19 – 58.

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*.

Plass, J.L., O'Keefe, P.A., Homer, B.D., Case, J., Haward, E.O., Stein, M., & Perlin, K. (2013). The impact of individual, competitive, and collaborative mathematics game play on learning, performance, and motivation. *Journal of Educational Psychology, 105*(4), 1050 – 1066.

Plaxco, D., Reimer, P.N., Williams-Pierce, C., Ellis, A.B., Molitoris-Miller, S., Simpson, A., Zandieh, M., Mauntel, M., & Dogan, M.F. (2021). Mathematical play: Across ages, context, and content. In Olanoff, D., Johnson, K.,

& Spitzer, S. (Eds.), *Proceedings of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1913 – 1915), Philadelphia, PA.
Riegle-Crumb, C. (2006). The path through math: Course sequences and academic performance at the intersection of race-ethnicity and gender. *American Journal of Education*, *113*(1), 101-122.
Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the annual meeting of the Psychology of Mathematics Education - North America* (vol. 1, pp. 298–303). North Carolina State University.
Siegler, R. S. and Ramani, G. B. (2008). Playing linear numerical board games promotes low-income children's numerical development. *Developmental Science, 11*(5): 655–61.
Simon, R. A., Aulls, M. W., Dedic, H., Hubbard, K., & Hall, N. C. (2015). Exploring student persistence in STEM programs: a motivational model. *Canadian Journal of Education*, *38*(1), n1.
Steffe, L., & Thompson, P. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A., Kelly, & R., Lesh (Eds.), *Handbook of research design in mathematics and science education*. Lawrence Erlbaum Associates.
Strauss, A., & Corbin, C. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage Publications.
Tasova, H. I., & Moore, K. C. (2020). Framework for representing a multiplicative object in the context of graphing. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). Mathematics Education Across Cultures: *Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico (pp. 210–219). Cinvestav/PME-NA.
Thompson, P.W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). SUNY Press.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.
Wager, A., & Parks, A. N. (2014). Learning mathematics through play. *Handbook of play and learning in early childhood*, 216-227.
Wang, Z., & Hung, L.M. (2010). Kindergarten children's number sense development through board games. *The International Journal of Learning, 17*(8), 19 – 31.
Widman, Chang-Order, Penuel, & Wortman (2019). Using evaluation tools toward more equitable youth engagement in libraries: Measuring connected learning and beyond. *Young Adult Library Services, 17*(4), 37 – 43.
Williams-Pierce, C. (2019). Designing for mathematical play: failure and feedback. *Information and Learning Sciences*.

“THIS ONE IS THAT”: A SEMIOTIC LENS ON QUANTITATIVE REASONING

Allison L. Gantt	Teo Paoletti	Steven Greenstein
University of Delaware	University of Delaware	Montclair State University
agantt@udel.edu	teop@udel.edu	greensteins@montclair.edu

Despite significant research exploring students' quantitative reasoning, few studies have explored the semiotic processes that mediate its development. In this report, we present a case study to show how one student constructed a semiotic chain for a quantity as he worked with a mathematical task. Importantly, we connect frameworks for quantitative reasoning and semiotics to make sense of this process. Our findings show how our case student constructed a sign for a chunk of change in a triangle to support his later construction of the quantity of amount of change of area. We also describe how the case student leveraged these signs to bolster his development of the quantity of total area. We emphasize the role of artifacts, such as physical manipulatives, a digital applet, and a diagram, in this process. Finally, we discuss the implications of this analysis for future studies that explore students' constructions of quantity.

Keywords: Algebra and Algebraic Thinking, Cognition, Mathematical Representations

Students' construction of quantities is a non-trivial process (Smith & Thompson, 2008; Thompson, 2011). According to Thompson (2011), “Too often quantities, such as area and volume, are taken as obvious, and hence there is no attention given to students' construction of quantity through the dialectic object-attribute-quantification” (p. 34). In this paper, we seek to investigate this dialectic by bridging two perspectives—namely, quantitative reasoning and semiotics. Specifically, we investigate the following research question: *What can a semiotics lens reveal about a learner's constructions of quantities?*

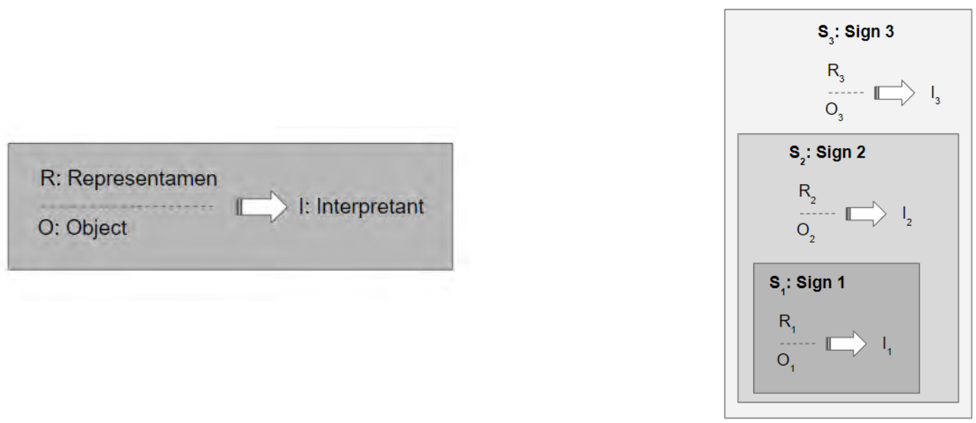
In this report, we first briefly present the frameworks of quantitative reasoning and semiotics, with attention to the ways we intend to use these perspectives concurrently. Then, we describe the method and findings from a revelatory case study (Yin, 2018) of one student's construction of a semiotic chain related to the quantity of an *amount of change* in a dynamic task. We discuss the implications of these findings for researchers and practitioners intending to support students to engage in quantitative reasoning.

Quantitative Reasoning and Semiotics

We follow Smith and Thompson (2008) in conceptualizing quantities as mental constructions of measurable attributes of objects or phenomena. Quantities establish the “conceptual content” (Smith & Thompson, 2008, p. 10) students may represent mathematically (Paoletti & Moore, 2017; Thompson, 1994). Quantitative reasoning, then, entails constructing and reasoning about relationships between conceived quantities.

Individuals can conceive of a quantity that changes its measure (or *varies*). Consider, for instance, the quantity of changing area in a growing triangle (https://ggbm.at/t6d63cun). An individual may develop a smooth image of change for the area of the triangle such that it “[takes] on values in the continuous, experiential flow of time,” (Castillo-Garsow et al., 2013, p. 34). For instance, they can imagine an animation of the triangle growing continuously. An individual may also develop a chunky image of change, entailing a visualization of completed intervals without attending to variations within an interval. For example, one can imagine the triangle's growth in snapshots over time. Although both smooth and chunky images are important for students to develop, chunky images of change can also enable the conceptualization of another important quantity—an *amount of change* (hereafter AoC) between intervals (Carlson et al., 2002). Researchers have highlighted the importance of students' constructing amounts of change, for example, to differentiate between linear and non-linear growth (Paoletti & Vishnubhotla, in press), conceive of rates of change (Carlson et al., 2002), and represent quadratic (Wilkie, 2019, 2021) and trigonometric (Moore, 2014) relationships.

Given the importance of imagery in conceptualizations of quantity, we conjecture that one's mental representations necessarily meditate a student's quantity construction and corresponding meanings. Presmeg's (2006) model for semiotics in mathematics education (adapted from Peirce, 1998) provides a way to describe this process. Presmeg argued that connecting ideas in mathematics involves the negotiation of *signs*, which is the focus of semiotics. Learners construct signs as they strive to make sense of relationships between representations and their associated objects, ultimately developing interpretations for the object and representation. Presmeg follows Peirce (1998) in identifying signs as composed of these three components—*object*, *representamen* (or representation), and *interpretant* (or interpretation)—which operate together in triadic relationship as meaning is negotiated. Figure 1a shows a model for a sign.



(a) (b)

Figure 1: (a) The triadic relationship of a sign (Peirce, 1998) and (b) the process of semiotic chaining (Presmeg, 2006).

To illustrate and define these terms, consider an animation of a continuously growing triangle. One may conceive of the changing triangle (i.e., the *object*, or what a sign could stand for) and construct this object in conjunction with a mental image of the changing triangle (i.e., their *representamen* of the object, or the perceivable part of a sign). In the process of relating object and representamen, they can interpret the changing triangle in relation to a quantifiable attribute such as its area. This would be regarded as an *interpretant* for the individual, which explains the relationship they constructed between the object and the representamen. This process of arriving at an interpretant is an act of *meaning making* for the individual.

As individuals continue to engage with new ideas, they can call to mind their previous signs (i.e., triads of object-representamen-interpretant) and construct new ones. Once established, the whole of a sign can be regarded as an object onto which new representations and meanings can be built (e.g., Sfard, 1991). Presmeg (2006) refers to this process as *semiotic chaining*, whereby previous signs are nested within a new sign which "comprises everything in the entire chain to that point" (p. 169). Semiotic chains can thus explain how learners' meanings are enriched as they continuously connect the signs they make. We present a recreated visual of Presmeg's conceptualization of this process in Figure 1b.

As suggested in our example, objects, representations, and interpretations may be motivated by the artifacts with which one engages (e.g., animations, diagrams, manipulatives). As learners interact with and reconcile their interpretations of multiple artifacts, each artifact can add nuance to associated signs the learner possesses (Radford, 2010). Maffia and Maracci (2019) define this process as *semiotic interference*, which is "an enhancing of signs emerging from the contexts of use of different artifacts and referring one to the other" (p. 3-58). That is, as a learner engages a new artifact, they may translate their interpretations of "old" signs from one context in relation to new ones in another, furthering the development of meanings associated with the sign. In this way, semiotic interference describes how signs become decontextualized and generalized, and thereby more meaningful.

Method

We present a revelatory case study (Yin, 2018) to elucidate how one student's quantitative reasoning developed through a process of semiotic chaining and semiotic inference. The case study was drawn from a larger teaching experiment (Steffe & Thompson, 2000), where a pair of students engaged several tasks designed to elicit and support their covariational reasoning, detailed in Paoletti and Vishnubhotla (in press). We chose to focus on the case of a single student's activity in the beginning stages of one such task, as this account revealed how artifacts played a role in his development of meanings for a quantity via semiotic chaining.

Participant, Context, and Task

Our case student, Vicente, attended sixth grade at a public middle school in the Northeastern United States at the time of data collection. He engaged with the tasks of the teaching experiment with a partner, Lajos. One teacher-researcher (TR, second author on this paper) conducted the teaching experiment with the two students over 12 sessions.

The students engaged the *Growing Triangle Task*, which asked them to consider the relationship between the length of the base segment and area of a growing triangle, presented to them through a dynamic applet (seen in Figure 2a). The ultimate goal of the *Growing Triangle Task* was for students to graph the covariational relationship between these two quantities. However, in the three 40-minute sessions that Vicente and Lajos spent working through the task, only about half of the time involved graphing activity. The TR first sought to develop students' meanings for key quantities in the scenario (e.g., base segment length, total area, AoC of area). We focus this report on Vicente's pre-graphical work as it allows us to provide evidence of his process of semiotic chaining as he constructed the AoC of area.

To support the goals of developing meanings for situational quantities, the TR designed multiple artifacts to mediate students' engagement with the task. Students had access to a digital applet that could change to display the triangle's growth as smooth or chunky (Figure 2a). They also had access to physical manipulatives of instantiations of the growing triangle (Figure 2b) as well as trapezoidal pieces intended to represent the quantity of AoC of area between instantiations (Figure 2c). We note that, in addition to these prepared artifacts, the students and TR also generated artifacts (e.g., diagrams) in the moment to support their work.

Data Collection and Analysis

We audio- and video-recorded Vicente and Lajos's activity using a free-standing camera and screen capturing software. We also collected and scanned written activity from each session. We then transcribed the recordings, including verbal utterances, gestures, and the artifacts students engaged (both prepared and actor-generated) to support our investigation of the artifacts' mediating role in Vicente's construction of quantity.

As we began to respond to our research question through our analysis of the data, we identified transcript segments of interest and the emerging meanings associated with them. First, to ensure we captured Vicente's activity and signs related to AoC of area in their context, we organized our transcript into analytical units bounded by the TR's central prompts (e.g., "How are they changing together?"). Then, through open and axial analysis (Corbin & Strauss, 2015), we examined the data for the types of meanings and related signs Vicente expressed around AoC. After this process, we analyzed each segment to consider how Vicente used or referenced artifacts in his expressions of meanings, allowing us to conjecture the occurrence of semiotic interference. We used these analyses to create models of Vicente's semiotic chain related to AoC of area in connection with his engagement with artifacts. Throughout the analysis we maintained a chain of evidence connecting our findings to data to establish trustworthiness (Yin, 2018).

(a) (b) (c)

Figure 2: Prepared artifacts from the *Growing Triangle Task*.

Results

In this section, we highlight the development of Vicente's semiotic chain involving the quantity AoC of area in the *Growing Triangle Task*. Vicente began his activity conceptualizing a chunk of change in the triangle (Sign 1). He built upon this sign to later elaborate this object as a quantity—an AoC of area (Sign 2). Finally, we show how Vicente connected these objects in his elaboration of the total area of the triangle (Sign 3). Throughout, Vicente's activity promoted semiotic interference across artifacts, which functioned to trigger, enhance, and connect signs in his semiotic chain.

Sign 1: Chunk of Change in Triangle

The activity we present in this section shows how Vicente developed a sign for *chunk of change* in the triangle. This involved his understanding that a chunk could be added after an iteration of each triangle to reach the next (e.g., move from TA1 to TA2 in Figure 2b). This interpretant emerged through Vicente's active meaning making as he negotiated the relationship between the growing triangle (an object) and the chunked compositions he came to see in that triangle (his representamen). Vicente enhanced his meanings for this sign through semiotic interference as he shifted across digital applet and physical manipulative artifacts.

To begin the *Growing Triangle Task*, the TR displayed the initial applet (smoothly and then in chunks) to Vicente and Lajos. We posit Vicente established the growing triangle as a dynamically growing object in the context of the digital applet. Furthermore, the chunky representation introduced the possibility for Vicente to conceptualize another object—namely a chunk of change.

The TR next introduced a new artifactual context to the students through physical manipulatives. He stacked two triangle manipulatives (e.g., TA1 and TA2 in Figure 2b) on top of one another (see Figure 3a) and asked Vicente, “Alright so I’m going from this one [*taps the smaller triangle*] to this one [*taps the larger triangle*]. How much has the area changed by?” Initially, Vicente responded attempting to determine a value: “Maybe by 2…like 2 of these [*taps smaller triangle twice*].” To explore how Vicente was mentally representing a change in the triangle, the TR prompted Vicente to color in “how much the area changed by from the first to the second.” Vicente proceeded to shade the darker area in Figure 3a. For Vicente, this was an instance of semiotic interference; he reinterpreted his growing triangle object and representation of a chunked composition from the digital applet to the context of the physical manipulatives.

(a) (b) (c)

Figure 3: Displays of *Growing Triangle Task* in Vicente’s construction of chunk of change.

Next, the TR prompted Vicente to consider his change in area meaning in the digital applet. The TR displayed the applet in chunks, showing the smallest triangle and then its second iteration (producing the same visual as Figure 3a, on a computer screen). He asked Vicente about “what shows the change in area,” and Vicente again pointed to the grey area in Figure 3a. Given the digital applet’s unique affordance for introducing a growing triangle object, we conjecture that this action may have supported Vicente to connect his representation of the chunked composition of the growing triangle with a notion of dynamic change.

Indeed, Vicente’s subsequent activity indicated that he was beginning to develop a meaning for this change as a chunk added to previous instantiations of the triangle. Importantly, this became salient when the TR prompted him to again revisit the physical manipulatives, furthering the semiotic interference. The TR asked Vicente, “Do you see any pattern… with respect to how the changes in area are growing?” The TR arranged overlapping triangle manipulatives (e.g., TA1, TA2 and TA3 in Figure 2b) in a display like Figure 3b. Vicente’s explanation suggests that chunks of change emerged through additive operations. He stated:

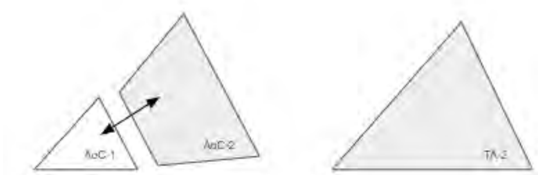
> I think that it’s just, copying off each other… you can see if you counted like it, this [*points to the darkest grey area in Figure 3b*], this would be a bigger one, so you can just add it over [*gestures back toward smallest triangle in Figure 3b*].

As Vicente finished his explanation alluding to meanings for addition, Lajos picked up a new manipulative piece—a trapezoid, as shown in Figure 2c—and placed it against the longest side of the current display of stacked triangles. Vicente explained to the TR, “You can just add another one of the same size… you can kind of add that like there [in Figure 3c].” This interaction shows how the introduction of this new artifact (a trapezoidal manipulative) coincided with even greater emphasis on addition in Vicente’s language about the chunk of change. Taken together, this suggests that semiotic interference (across the digital applet, physical triangle manipulatives, and physical trapezoidal manipulatives) supported Vicente’s development of a sign for a chunk of change in the triangle: that is, a chunk of change had become a new object that could be added after an iteration of each triangle to reach the next.

Sign 2: Amount of Change (AoC) of Area

As described above, Vicente had an interest in numerically describing the magnitude of the chunk of change object he was forming once he first characterized the object (e.g., "Maybe [the area changed] by 2 [of the smallest triangles]"). At the end of the first session, the pair returned to the physical manipulatives, stacking copies of the smallest triangle on top of the trapezoidal pieces (now representing chunks of change) to measure the chunks. They recorded their findings in a diagram (Figure 4a). Thus, Vicente established two artifacts that later became central to his semiotic interference in the development of an AoC of area in his semiotic chain: the physical manipulatives (both triangular and trapezoidal) and the diagram.

(a)



(b)

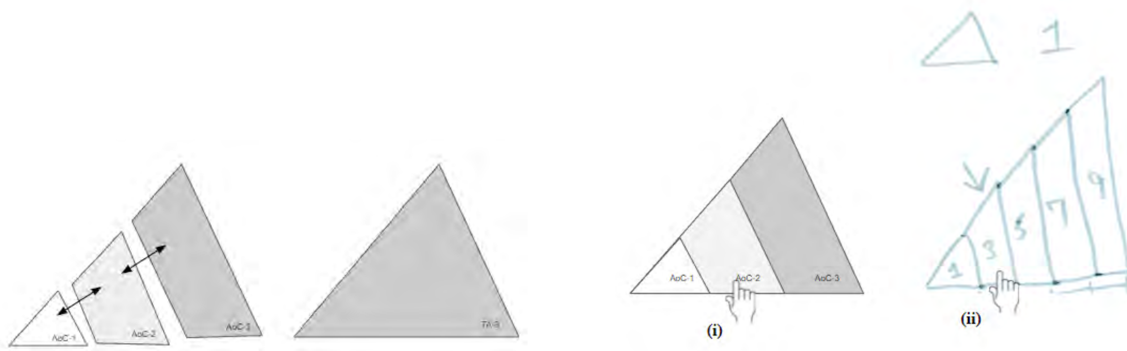
Figure 4: Displays of *Growing Triangle Task* in Vicente's construction of AoC of area.

When the pair of students returned for the second day of the *Growing Triangle Task*, the TR intended to have them to renegotiate their sign for an AoC of area, and he employed both the physical and diagrammatic artifacts to do so. First, referring to their diagram (Figure 4a) the TR asked, "What were these numbers representing, what were you talking about with these?" As Vicente swiped his finger over each corresponding trapezoidal component in the diagram, he described that "in here you could fit 3, then here you could fit 5, then here you could 7… [of] the little triangles." We interpret that Vicente understood the numeric values as representing measures of AoC of area in terms of the first triangle (e.g., AoC 1 in Figure 2c). Moreover, the diagrammatic artifact provided a recorded, concretized representation of chunks of change and their values he could draw upon in the construction of his sign for AoC of area.

The TR, however, encouraged Vicente to extend his meanings back again to the physical manipulative context of the scenario. The manipulative context was important in Vicente's sign for AoC of area due to its previous interference with the digital applet. Arranging the shapes as in Figure 4b, the TR asked Vicente, "So when you say what fits 3 of what?" Vicente at first picked up the larger triangle (i.e., TA2 in Figure 2b) and attempted to arrange copies of the smallest triangle on top. However, he quickly decided "not this one," picked up the trapezoidal piece (i.e., AoC2 in Figure 2c), and began to arrange the small triangles on top of there. He was able to fit 3 small triangles and affirmed, "Yeah, this one." We note that Vicente's activity was effortful, indicative of the semiotic interference involved in his meaning making for AoC that extended across diagrammatic and manipulative contexts. Collectively, Vicente's negotiations across artifacts indicate the development of a new sign in his semiotic chain: an AoC of area. For him, the AoC of area was an object with magnitude that could be determined using the small triangle's area as a unit, establishing a quantity constructed from his sign for a chunk of change.

Sign 3: Total Area of the Triangle

Vicente's chaining of signs for chunks of change and AoC of area supported him in developing new meanings for a related object: total area in the growing triangle. As described in the previous section, the TR had now re-introduced total area manipulatives of the growing triangle (e.g., TA3 in Figure 2b). In this interaction, the TR arranged trapezoids and a triangle piece on the table in front of the two students (see Figure 5a). He asked the pair, "This [smallest triangle] is 1; how big is that [*gesturing to large triangle displayed in Figure 5a*]?"



(a) (b)

Figure 5: Displays of *Growing Triangle Task* in Vicente's construction with total area.

Vicente explained his reasoning to Lajos in a way that first expressed a meaning for the relationship he understood to exist between the largest triangle and trapezoidal pieces in the arrangement of Figure 5a. He argued that "this [*tapping on the largest triangle in Figure 5a*] is the same as that [*gesturing to the trapezoids pushed together in Figure 5a*]." He then picked up the largest triangle piece and placed it on top of the trapezoidal arrangement to show it "fits in perfectly." We note Vicente's argument established a sign for total area as an object related to artifacts that had established his sign for AoC, suggesting a new link in his semiotic chain.

Vicente's next statements demonstrated the salience of his sign for AoC of area as he found a measurement of total area. At first, he gestured to each progressively larger trapezoid in the manipulatives, declaring that they had values of 3 and 5, respectively. When explaining to Lajos, he revealed how his own meanings for the AoC object had been supported through the semiotic chaining he had established with the diagrammatic artifact. Specifically, Vicente explained, "this one [*pointing to manipulatives as in Figure 5b(i)*] is that [*pointing to diagram as in Figure 5b(ii)*], so you can fit 3 [small triangles] in there." Here, Vicente directly related the area of the trapezoidal physical manipulatives to the areas noted in their diagram.

Finally, Vicente called upon his initial interpretations of chunks of change to determine the total area of the triangle. He concluded that "1 plus 3 is 4, and then… 4 plus 5 is 9," identifying that 9 would be the total area of the given triangle. We note that, in this explanation, Vicente does not express his conclusion for total area as simply an addition of consecutive bands in the composite triangle (that is, 1 + 3 + 5 = 9). Instead, Vicente considers each intermediate total area in a way that might reflect an AoC of area as adding on to previous total areas in each step (i.e., 1 + 3 = 4, then 4 + 5 = 9). This further suggests the critical role of Vicente's previously constructed signs in his reasoning. Viewing Vicente's argument holistically, AoC of area (which was based on his sign for chunk of change) formed the basis of his exploration of magnitude with a related quantity (and sign) of total area. We summarize our hypothesis about Vicente's enacted semiotic chain in Figure 6. This figure shows the process by which his sign (1) for a chunk of change in the triangle became an object for his sign (2) AoC of area, and then these subsequently became objects for a sign (3) for total area of the triangle.

Conclusion and Discussion

In this paper, we brought together quantitative reasoning and semiotics frameworks to present a case study that highlighted the emergence of a semiotic chain (Presmeg, 2006) of a student's construction of quantity through his engagement with multiple artifacts. Specifically, we anchored our analysis in Vicente's development of signs related to the quantity of AoC of area. This included his initial constructions of a chunk of change in the triangle and his later connection to the total area of the triangle. Moreover, our evidence showed the processes of semiotic interference (Maffia & Maracci, 2019) that account for the enhancement of Vicente's signs (e.g., building a sign for the quantity of AoC of area by connecting notation in a diagram to his work with physical manipulatives). Addressing our research question, a semiotic lens on Vicente's quantitative reasoning helped us to make sense of the mediated processes that supported him as he developed associated meanings in conjunction with his representational activity through both the processes of semiotic interference and semiotic chaining.

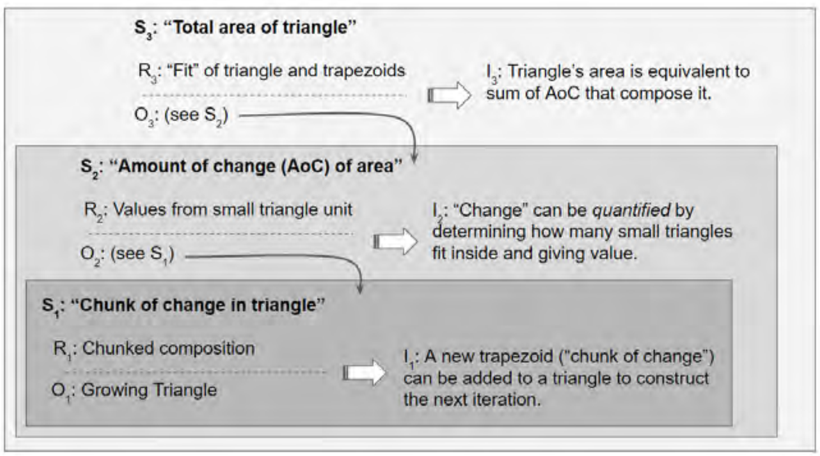


Figure 6: Model of Vicente's construction of a semiotic chain involving AoC of area

By identifying and describing the signs Vicente related to his sign for AoC of area, we charted how conceptualization of this specific quantity may develop. These findings suggest that scaffolded engagement with multiple artifacts such as applets, manipulatives, and diagrams may support learners' development of AoC in other types of tasks, particularly those involving non-linear relationships (e.g., Wilkie, 2021). They may also suggest features that could be productive to investigate beyond the scope of AoC (e.g., a quantity's relationship within and across other quantities), thus contributing to new potential lines of inquiry in the field of quantitative reasoning (e.g., Smith & Thompson, 2008).

Although not described in this paper, we also note that Vicente's signs relating to AoC of area in the *Growing Triangle Task* did not end with the semiotic chain in Figure 6. In fact, he continued to engage with these signs and construct more sophisticated meanings as he interacted with new artifacts (e.g., constructing and interpreting graphs) and as he anticipated smooth change in area (see Paoletti & Vishnubhotla, in press). By maintaining a semiotic lens on Vicente's activity, this work has offered new understandings of students' images and meanings for changing quantities (Carlson et al., 2002; Castillo-Garsow et al., 2013). Through a combined focus on quantities, artifacts, and the meaning-making process that relates them, we intend for this work to open new directions for research into how multiple artifacts can support students' conceptions of quantity.

Acknowledgments

This material is based upon work supported by the Spencer Foundation under Grant No. 201900012 and NSF Award No. 1812887.

References

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352–378. https://doi.org/10.2307/4149958

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics, 33*(3), 31–37.

Corbin, J. M., & Strauss, A. L. (2015). *Basics of qualitative research: techniques and procedures for developing grounded theory*. (Fourth edition). SAGE.

Maffia, A., & Maracci, M. (2019). Multiple artifacts in the mathematics class: Tentative definition of semiotic interference. *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education*. Pretoria, South Africa.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education*, *45*(1), 102–138. https://doi.org/10.5951/jresematheduc.45.1.0102

Paoletti, T., & Moore, K. C. (2017). The parametric nature of two students' covariational reasoning. *The Journal of Mathematical Behavior, 48*, 137–151. https://doi.org/10.1016/j.jmathb.2017.08.003

Paoletti, T., & Vishnubhotla, M. (in press). Leveraging covariational reasoning and emergent shape thinking to distinguish nonlinear and linear relationships. In G. K. Akar, I. O. Zembat, S. Arslan, & P. W. Thompson (Eds.), *Quantitative reasoning in mathematics and science education*. Springer.

Peirce, C. S. (1998). *The essential Peirce, Volume 2: Selected philosophical writings, 1893-1913* (Peirce Edition Project, Ed.). Indiana University Press.

Presmeg, N. (2006). Semiotics and the "connections" standard: Significance of semiotics for teachers of mathematics. *Educational Studies in Mathematics*, *61*(1–2), 163–182. https://doi.org/10.1007/s10649-006-3365-z

Radford, L. (2010). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education*, *12*(1), 1–19. https://doi.org/10.1080/14794800903569741

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*(1), 1–36. https://doi.org/10.1007/BF00302715

Smith, J., & Thompson, P. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 95–132). Erlbaum.

Steffe, L., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Erlbaum. https://asu.pure.elsevier.com/en/publications/teaching-experiment-methodology-underlying-principles-and-essenti

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. *CBMS Issues in Mathematics Education, 4*, pp. 21-41. https://doi.org/10.1090/cbmath/004/02

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L. L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e* (pp. 33–57).

Wilkie, K. J. (2019). Investigating secondary students' generalization, graphing, and construction of figural patterns for making sense of quadratic functions. *The Journal of Mathematical Behavior*, *54*. https://doi.org/10.1016/j.jmathb.2019.01.005

Wilkie, K. J. (2021). Seeing quadratics in a new light: Secondary mathematics pre-service teachers' creation of figural growing patterns. *Educational Studies in Mathematics*, *106*(1), 91–116. https://doi.org/10.1007/s10649-020-09997-6

Yin, R. K. (2018). *Case study research and applications: Design and methods* (Sixth edition). SAGE.

PROBLEM SOLVING WITH UNDERSTANDING

Siqi Huang
University of California, Berkeley
shuang123@berkeley.edu

It is not well understood when, how, or why students seek a deep understanding of the underlying mathematics relevant to the problems they try to solve. This study of five pairs of UC Berkeley students (undergraduate n=4, graduate n=6) solving challenging mathematics problems indicated consequential differences in the degree to which they worked to understand underlying principles. Drawing on Hiebert and Carpenter's (1992) framework on understanding and Schoenfeld's (2010) model of decision making, I developed an observation system with a coding scheme to investigate students' orientations with respect to deep understanding. Preliminary results show that the system captures students' divergent choices of moves and illuminates aspects of their understanding-related orientations that are not otherwise evident. This approach holds further potential as both a teaching and a learning tool in mathematics classrooms.

Keywords: Problem Solving; Mathematical Processes and Practices; Measurement

Helping students understand is one of the most important goals in mathematics education (Sierpinska, 1994; Duffin & Simpson, 2000; Common Core State Standards, 2010), as is problem solving (NCTM, 1980, 2014; Schoenfeld, 1992). Literature shows that understanding can be viewed as recognizing relationships and making connections between pieces of information (Davis, 1984; Greeno, 1977; Hiebert, 1986; Michener, 1978). According to Hiebert and Carpenter's (1992) Representation and Connection framework, the degree of understanding is determined by the number, accuracy, and strength of connections, and that a richly connected network allows students to draw on and apply knowledge flexibly (Hiebert et al., 1997). But understanding "is not an all or none phenomenon" (Hiebert & Carpenter, 1992, p. 69); it can be rather limited if only some of the potentially-related ideas are connected, and that a deep understanding necessarily implies students' rich, accurate, and robust networks of connections.

For the purpose of this study, I define problems as unfamiliar, perplexing, and challenging tasks that one attempts to solve; they should be distinguished from routine exercises of which one knows the solution or with which one practices what has just been taught. Building on Schoenfeld's (1985) framework of *what* and *how* factors affect problem-solving outcomes, this study contributes to the understanding of *when*, *how*, and *why* students seek a deep understanding of the underlying principles when they solve challenging mathematics problems. It might seem obvious that students should always try to understand the problem and the underlying mathematics when they solve a problem; yet according to my data, that practice does not seem natural — in fact, the first principle in Pólya's (1945) groundbreaking book *How to solve it* is to understand the problem. Drawing from Schoenfeld's model of how people think and act in goal-oriented activities (2010), I hypothesize that an underlying major factor in successful problem solving is the extent to which students are oriented towards deep understanding. The main goal of this paper is to build an observation system with a coding scheme that captures students' choices of moves and that sheds some light on their hidden orientations with respect to deep understanding.

Methods

Think-aloud protocols (Schoenfeld, 1985; Leighton, 2017) were the main data sources. Five self-selected pairs of friends (defined by feeling comfortable discussing math together) from UC Berkeley participated in my study: two pairs were STEM undergraduates and three pairs were Education Ph.D. students. The pairs participated in the study sequentially at a reserved conference room at UC Berkeley, where I emphasized that the correctness of their solution was of no interest to the study; their thinking process was what mattered. To ease their anxiety, I gave them some warm-up exercises to practice thinking aloud and I stressed that the problem required nothing more than high-school knowledge. When they felt ready, I handed out the problem sheet with scratch paper and stood behind their seats to focus my camera on their writing. After each pair finished, I collected their writing artifacts and transcribed the videos. The problem-solving time ranged from 18 minutes to 77 minutes, with a median of 28 minutes.

Problem-Solving Task. A teacher wrote a large number on the board and asked her 30 students to tell about the divisors of the number. The 1^{st} student said, "The number is divisible by 2." The 2^{nd} student said, "The number is divisible by 3." The 3^{rd} student said, "the number is divisible by 4." ... The 30^{th} student said, "The number is divisible by 31." The teacher then commented that exactly two students, who spoke consecutively, were wrong. Who was wrong?

The Observation System. The goal of the observation system was to highlight *when* and *how* students showed evidence of working to develop a deep understanding of the underlying mathematics when they solved the given problem. In formulating the system, I reviewed the videos, developed transcripts, identified emerging themes, applied preliminary codes to the transcripts, and repeatedly reflected and revised the system. As shown in Figure 1, the current version highlights four different aspects of the students' orientations with respect to deep understanding: Examine the Problem Statement (ES); Examine the Proposed Ideas (EI); Suggest Solving with Multiple Approaches (MA); and Problematize the Ongoing Work (PW). Figure 2 provides descriptions and examples for each of the four dimensions of the observation system.

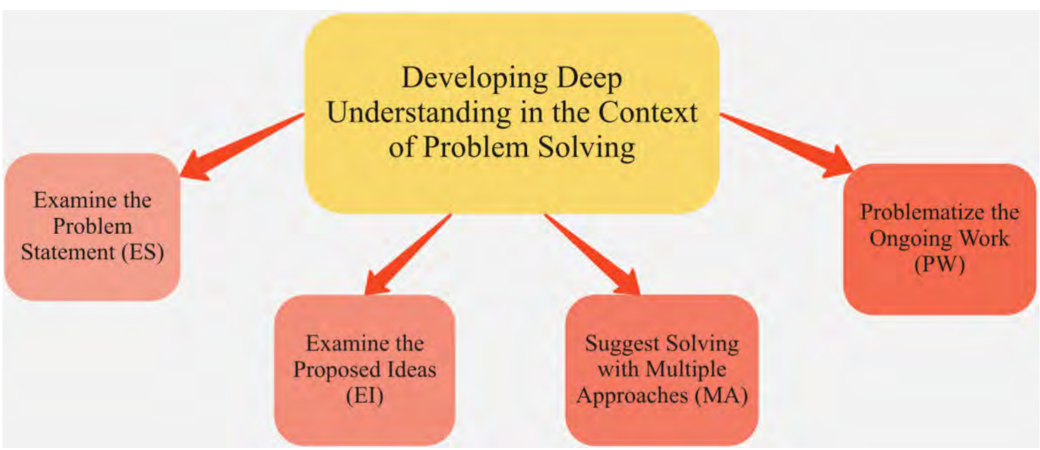


Figure 1: Observation System

Dimension	Description	Examples
ES: Examine the Problem Statement	Clarify, reread, or rephrase the problem statement.	(Pair #3) E: The teacher then commented that exactly two students, exactly, that means all the other students were right?
EI: Examine the Proposed Ideas	Explicitly engage with partners' proposed ideas by (a) rephrasing what has just been said, (b)	(Pair #2) S: I think that we *can*, because we use the 5 and the 4 thing. J: Okay, so we have to *choose* a pair that doesn't overlap.

	using their own reasoning to justify their partners' ideas, (c) suggesting immediate consequences, or (d) asking follow-up questions that probe for further clarification or justification.	(Pair #2) J: We can cross out 21. S: Oh yea, cuz 21, I mean this is the same as 3 and 7, yeah. (Pair #4) N: 7, 14. And every... You are cross[ing] out every single... (Pair #3) E: Do you do 20 to 25, I'll do 26 to 31? L: Do we, what do we do with the numbers? (Pair #2) S: Wait, hold on a second, tell me why? Cuz I thought 27 was a candidate.
MA: Suggest Solving with Multiple Approaches	Explicitly suggest solving the problem in more than one way.	(Pair #2) S: So... Err, let's do... We can maybe just solve using both ways. We can do your way first, and then if we have time, we can do my way.
PW: Problematize the Ongoing Work	Make problems of their ongoing work; suggest doing something more than finding the answer.	(Pair #2) S: Yea I think it is [the answer], but I think we should just spend more time talking about why it is that 16, like why it is that the logic does work for 24 but doesn't work for 16...

Figure 2: Coding Scheme

Results & Analysis

Figure 3 summarizes (ES) when and how many times each pair examined the problem statement, (EI) how many times each pair examined the proposed ideas, (MA) how many times each pair suggested solving with multiple approaches, and (PW) how many times each pair problematized their ongoing work.

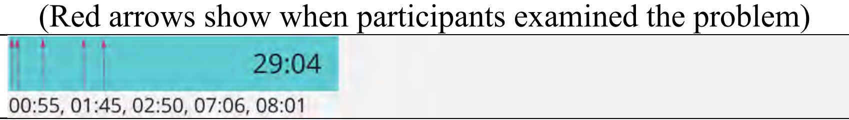
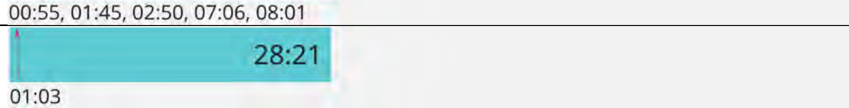
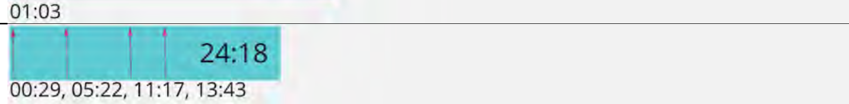
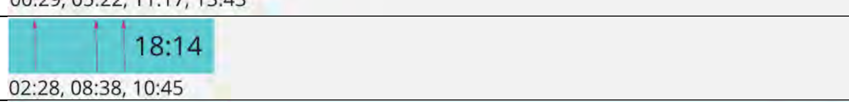
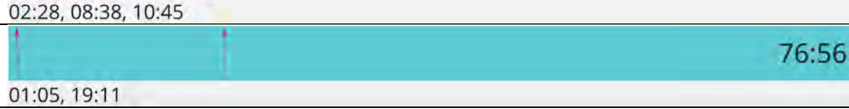
Pair	ES (Red arrows show when participants examined the problem)	EI	MA	PW
#1		19	1	1
#2		34	1	2
#3		11	0	0
#4		16	0	0
#5		0	0	0

Figure 3: Results

Examine Problem Statement (ES) & Proposed Ideas (EI). Every pair examined the problem statement at some point. However, only pair #2 examined the problem constraints to their satisfaction at the beginning — all other pairs went back and clarified the problem setup in the middle of their problem-solving process. Every pair except pair #5 examined the proposed ideas; one possible explanation is that a participant in pair #5 simply pursued his approach without any serious consultation with his partner, despite his partner showing clear evidence of not being able to follow along. Unfortunately, the acts of ignorance and of murmuring without communication seemed to result in an unproductive joint problem-solving space and even the failure in pair #5's problem solving.

Suggest Multiple Approaches (MA) & Problematize Ongoing Work (PW). Only pairs #1 and #2 showed evidence of attempting to solve in more than one way and problematizing established ideas or solutions; the other three pairs did not. Why did some students *want* to spend time doing extra things that were not expected from them and that would not earn them any external rewards, while others did not? It seemed plausible that their dominant goals were different: Some might aim for making more connections and developing a deeper understanding of the underlying principles, while others might have a stronger answer-getting motivation. On the assumption that orientations fundamentally shape the way students engage with problems, investigating the extent to which they work to develop a deep understanding sheds light on *what* and h*ow much* they gain from their problem-solving process.

Discussion

Pairs #1 and #2 showed much evidence along all four dimensions, suggesting that they were very likely to have a strong orientation towards deep understanding. This aligns well with my observation that they appeared to care much about connecting various ideas and developing deeper understandings of the underlying principles. In contrast, pair #5 showed evidence of only examining the problem statement. To the extent that the four proposed dimensions are sufficient to capture students' attitudes, this result suggests that pair #5 were likely to have a weak orientation towards deep understanding. Pairs #3 and #4 seemed to be intermediate cases, showing evidence along the first two dimensions (only).

These results give rise to clear profiles of the students' orientations with respect to deep understanding, and appear to capture meaningful differences among groups that also align with my impressions as an observer. While much research is needed to test, refine, and problematize the proposed system, there is reason to believe that this is a step forward into highlighting and investing students' understanding-related orientations in problem solving. Of course, some may argue that showing no evidence of exploring different approaches under this laboratory setting may not imply that the students were not oriented towards deep understanding; it could simply be that they were not aware of this expectation — and I agree. But what if we are in a classroom with a norm of challenging oneself to look for alternative solutions and respectfully discussing different approaches with others? The proposed observation system holds great potential as a teaching tool for teachers to diagnose students' current states of orientations and model desirable problem-solving behaviors, and as a learning tool for students to self-evaluate and self-guide their development of productive mathematical practices.

References

Common Core State Standards. (2010). *Mathematics standards*. http://www.corestandards.org/the-standards/mathematics. Accessed May 14, 2022.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Norwood, NJ: Ablex.

Duffin, J. M., & Simpson, A. P. (2000). A search for understanding. *The Journal of Mathematical Behavior*, *18*(4), 415-427.

Greeno, J. G. (1977). Process of understanding in problem solving. In N. J. Castellan, Jr., D. B. Pisoni, & G. R. Potts (Eds.), *Cognitive Theory* (Vol. 2, pp. 43-83). Hillsdale, NJ: Lawrence Erlbaum.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65–97). Macmillan Publishing Co, Inc.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making Sense: Teaching and Learning Mathematics with Understanding.* Portsmouth, NH: Heinemann.
Leighton, J. P. (2017). *Using think aloud interviews and cognitive labs in educational research.* Oxford, UK: Oxford University Press.
Michener, E. R. (1978). Understanding understanding mathematics. *Cognitive science, 2*, 361-383.
NCTM. (1980). *An agenda for action.* Reston, VA: National Council of Teachers of Mathematics, Inc.
NCTM. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston: National Council of Teachers of Mathematics, Inc.
Pólya, G. (1945). *How to solve it.* Princeton: Princeton University Press.
Schoenfeld, A. H. (1985). *Mathematical problem solving.* New York: Academic Press.
Schoenfeld, a. h. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In Grouws, D. A. (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334-370). New York: Macmillan.
Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications (1st ed.).* Routledge. https://doi.org/10.4324/978020384000.
Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.

MANIPULATIVES, AFFORDANCE AND THE LEARNING OF FRACTION: THE TWELFTH TASK

MATÉRIEL DE MANIPULATION, AFFORDANCE ET APPRENTISSAGE DES FRACTIONS: LA TÂCHE DOUZIÈME

Doris Jeannotte	Claudia Corriveau
UQAM	Université Laval
jeannotte.doris@uqam.ca	claudia.corriveau@fse.ulaval.ca

This study investigates the use of manipulatives by elementary students working on a fraction task. Extending previous work on the role played by the manipulatives in students' activity, we aim at describing how the choices made for the task design disrupt students' activity, creating opportunities to learn. The theoretical underpinnings allow envisioning the students' activity through the concept of routine and the manipulatives through the concept of affordance. The analysis of the students' mathematical activity allows us to better understand how manipulatives can serve as breaching elements, leading students to modify their mathematical activity, and thus, creating opportunities to learn.

Keywords: Elementary School Education, Rational Numbers, Instructional Activities and Practices, Mathematical Representation

In this paper, we draw on our previous research (Jeannotte & Corriveau, 2020; Corriveau et al, forthcoming) about the role played by manipulatives in students' activity when solving arithmetical tasks. More specifically, we are studying the relation between didactic choices linked to manipulatives and the students' mathematical activities in a task about fractions. We aim to describe how those choices disrupt students' activity, creating opportunities to learn. To do so, we first describe why it is still important to research the use of manipulative in mathematics classrooms. We then define mathematical activity from a commognitive perspective. The concepts of routines and affordance are theoretically explicated to frame our methodological and analytical work. The result section presents the main routines that emerged from our analysis of a mathematical task called the "Twelfth task". We illustrate with examples how those routines are bound to the manipulatives used in this specific task and how different didactic choices may create different opportunities to learn.

Manipulatives and Mathematical Learning

The use of manipulatives for teaching arithmetic at the elementary level is encouraged and even prescribed by several curricula around the world (OCDE, 2019). Thus, it is often seen as an efficient teaching practice and is rarely questioned (authors, 2020). In particular, the learning of fractions, considered as one of the most difficult areas in elementary school mathematics, seems to benefit from this use (Carbonneau et al., 2013). However, research on manipulatives mainly focuses on whether manipulatives support student achievement in mathematics (e.g., Moyer, 2001), without articulating the uses of manipulatives and the different choices made when designing the educational setting. Some research sheds light to a certain extent on some didactic variables. For example, the use of familiar manipulatives may mislead students (McNeil & Jarvin, 2007) or, for the same task, two different manipulatives can lead students to engage in different mathematical activities (Authors, 2020). Familiarity and types of manipulatives are two examples of variables among others that can shape the students' mathematical activity. In this paper, we present a fraction task that students solve using pattern blocks. We aim to show how different choices, especially the choice of limiting the number of Pattern blocks available to students, introduce breaching elements that can disrupt the students' activity and thus create opportunities to learn.

Conceptual Framework

From a commognitive point of view, mathematics is defined as a particular form of communication, with its vocabulary, its visual mediators, its distinct routines, and its narratives generally accepted by a mathematical community (Sfard, 2008). By positioning ourselves in this sociocultural and non-mentalist perspective, manipulatives, and therefore their use, become part of the mathematical activity developed in the culture of the classroom. The use of manipulatives, when defined in commognitive terms, is viewed as participation in the mathematical discourse of the classroom through the manipulation of physical apparatus. In other words, using manipulatives is not a concrete version of mathematical activity, but a mathematical activity with its specific vocabulary, visual mediators, and narratives generally accepted by the classroom. For example, we often hear students say, "I am breaking down the tens into units" (referring to base-ten blocks) instead of "I am dividing the tens into units."

The Concept of Routine

Sfard (2008) defines a mathematical routine as a repetitive pattern of discourse discernable by the vocabulary and visual mediator used, and the narratives produced. Moreover, a routine is linked to the concept of 'task situation.' A task situation is "any setting in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159). For example, a 4th-grade student may refer to number fact to calculate mentally twelve divided by three to answer the question: "What is the third of twelve?". Another may draw a rectangle, divide the rectangle into four horizontal lines and three vertical lines to form twelve equal parts, and color four of them. Thus, "a routine performed in a given task situation by a given person *is the task, as seen by the performer, together with the procedure she executed to perform the task*" (Lavie et al., 2019, p. 161, italic added). Manipulatives can play distinct roles in routines. We can recognize manipulatives through vocabulary. Indeed, when using Pattern blocks, students will refer to a kind of block by its color or its shape. We can also observe students act upon the manipulatives, by moving them, stacking them, regrouping them in certain ways. Manipulatives can also play the role of visual mediator, representing mathematical objects.

Learning is defined as a change of discourse, and more specifically as a change of routine (Lavie et al., 2019). Discourse, therefore, progresses in an individualization-(re)communication direction (i.e., learning) through changes in routines, thus in vocabulary, visual mediators, and accepted narratives, what Lavie et al. (2019) name routinization. Therefore, because we aim to describe how manipulatives can create opportunities to learn, we are particularly interested in moments when routines seem disrupted, i.e., when students are bound to act differently.

The Concept of Affordances

We call affordances the relational properties of an object within a certain environment that are bonded to certain actions. "An affordance relates attributes of something in the environment to an interactive activity by an agent who has some ability, and an ability relates attributes of an agent to an interactive activity with something in the environment that has some affordance" (Greeno, 1994, p. 338). For example, a pen will generally be seen as an object to write with, but it could be seen as a tool to unlock a door. For some people with long and thick hair, it could be seen as an object to tie hair. Thus, "[w]hether or not the affordance is perceived or attended to will change as the need of the observer changes, but being invariant, it is always there to be perceived" (Gibson, 1977, as cited in Brown et al., 2004, p. 120).

In a given task situation, the use of manipulatives by students generates different affordances. In other words, the affordance realized in a task situation relies not only on the tool itself but rather on the interactions between the manipulatives, the students, and the teachers, in a particular educational setting (Drijvers, 2003, in Brown et al., 2004).

To try to disrupt students' routine, we can play with didactic variables to subtly change the affordance, thus the task situation by adding some breaching elements. Didactic variables are parameters, linked to the design of a teaching situation, which can vary according to the teacher's choices, and which can lead to a change in the students' mathematical activity (Brousseau, 1981). The concept of breaching (from ethnomethodology) refers to a subtle breakage of what is familiar. By varying several aspects related to the use of manipulatives, it is possible to place students in conditions familiar enough for them to engage in a task but sufficiently unfamiliar that it resonates as a conflictual use of the manipulatives. In a way, it plays into how students perceive the affordances of the manipulative and leads them to explicit their usual routines and renew them.

Methodology

In collaboration with teachers from grades 4 to 6, we created a task (titled "Twelfth") that aims to develop different routines to represent and compare fractions using Pattern blocks. Pattern blocks were familiar manipulatives for the students as they already had worked with them before the experimentation. We experimented "Twelfth" into two fourth-, two fifth-, and one sixth-grade classes of about twenty students each. The task was implemented with different choices according to teachers' rationales (mainly based on students' abilities and familiarity with fractions) during a 60-minute period.

The Task "Twelfth" and the Didactic Choices

"Twelfth" is, at first sight, a common task in which students are asked to represent a certain fraction with pattern blocks. We chose to place students in pairs. Each pair received a kit of limited Pattern blocks (Figure 1) and were asked to do several subtasks. These can be divided into three categories:

1. The given fraction is a twelfth and students must represent a third (the piece representing one-twelfth varies): e.g., the green triangle (blue rhombus, red trapezoid, etc.) is worth 1/12, we want 1/3 of the same whole.
2. The given fraction is one-twelfth and students must represent other varied fractions (unit and non-unit): e.g., the blue rhombus is worth 1/12, we want 1/24 of the same whole; the red trapezoid is worth 1/12, we want 5/6 of the same whole.
3. The given fraction varies, and students must represent varied fractions: e.g., the yellow hexagon is worth 2/5, we want 1/15 of the same whole.

Figure 1: kit of pattern blocks available to each pair of students

The subtasks were projected on the whiteboard, one or two at a time. There was a whole class discussion after each subtask. In the two 4th grade classes, the students were allowed to use scrap paper to take notes. In one 5th grade class, the students were not allowed to use any paper at all. In the two other classes (5^{th} and 6^{th} grade), the students were given a document to work on. On each sheet, they had two subtasks with a space to draw and record their answers.

Data Analysis

In each class, a camera with a primary focus on the front of the class filmed the whole group and other cameras filmed pairs of students with a primary focus on their hands (10 in 4th grade, 14 in 5th grade, and 7 in 6th grade). For this paper, we focus our analysis on the resolution of the first subtask by the students working in pairs: "Find what is 1/3 of a whole, if the triangle is 1/12." Following Powell, Francisco, and Maher (2003), each video excerpt was watched multiple times and described; critical events were identified. We transcribed, coded, constructed the storyline, and composed the narratives. Descriptions of the videos and codes helped us compose the narrative to report the results. Conversely, composing the narrative forced us to revisit some critical events. Critical events are moments when students' routines seem disrupted, i.e., when they seemed to doubt, when the routine implemented did not lead to solving the task and was challenged by someone, or when they suddenly changed their routines.

Routines and Affordances

By analyzing the data, we were able to highlight three intertwined routines that could be linked to the interpretation of the subtask: 1) 'comparing-and-naming', 2) 'forming and partitioning' a whole, and 3) 'covering'. Those routines had distinct functions throughout the resolution of the task but were all bound to certain affordances of the manipulatives in this specific setting.

The Routine of Comparing-and-Naming

Several pairs used a routine we called "comparing-and-naming" at different moments during their resolution. They attributed a name to each kind of block, or group of blocks by comparing pieces and relying on number facts. Some used fraction-names (e.g., a hexagon is ½ [of the whole], or a rhombus is 1/3 [of a hexagon]) and other whole number names (e.g., a rhombus is 2 [triangles]). Adam and Phil, a pair of 4^{th} graders in Heloise's class, help us illustrate how this "comparing-and-naming" routine was bound to the manipulatives, the didactic variables, and the students' discourse. Right at the beginning, Adam associated some blocks with a fraction-name by referring to the given value of different blocks: "yellow is ½ because 6+6", "red is ¼ because it's half of the half." Like 1/3, those unit fractions are familiar to the students as they start working with them in 1^{st} grade. However, because of our didactic choices, it was impossible to associate one and only one block to 1/3. Indeed, 6 triangles are equivalent to a hexagon and a hexagon to two trapezoids, but 4 triangles could not be replaced by a single block. Phil proposed to name the rhombus 1/3 of a hexagon. This proposition combined with the accepted narrative that the whole was not the hexagon led them to realize that more than one piece was requisite. If we had chosen 1/6 as the given value of the triangle, their routine could have led them to solve the subtask. Here, the given value combined with how the Patterns blocks are designed created a need to change the routine, and thus created an opportunity to develop a new discourse about fractions, that is to accept a visual mediator of a unit fraction composed of more than one piece, a necessary narrative to solve this task.

The Routine of Forming-Then-Partitioning a Whole

The routine of forming-then-partitioning a whole appeared in three different variations linked to the function of the routine: solving the subtask, validating the calculation, and justifying the answer prompted by a request from an adult or a peer. This routine was recognizable by the action of composing a whole with a certain number of blocks followed by its partitioning in a certain number of equivalent (mostly identical) groups. In a way, this routine was the expected one. Indeed, the curriculum (MEQ, 2001) prescribes the learning of fractions through manipulatives and schemas use. Thus, we expected them to use this routine. The blocks available to students for this first subtask allowed them to compose a whole, but not with twelve identical blocks, leading to the impossibility to partition it into three identical groups of four triangles. Figure 2 presents possibilities of different wholes with their partitioning. Except for the last example (Figure 2d) where each group is formed of one rhombus and two triangles, the students needed to form three equivalent but not identical groups. The shape of the groups is the same, but neither the orientation nor the kind of blocks (colors and shapes) are the same.

Figure 2: Partitioning of the whole with three equivalent groups

Solving the subtask. John and Owen, a pair of 5th graders in Christophe's class, engaged in the routine of forming-then partitioning the whole to solve the task, after reformulating the question as "1/3 of twelve is what?":

Owen: We say the whole is that [taking one hexagon in his hand]
John: No, this is six! […]
Owen: Oh, ok, so how many there are [counting the triangles]. Ok, six, so this [pointing a hexagon] is six. So, that [sliding two hexagons on the side], means, it's our whole. And this [sliding a triangle beside the two hexagons, Figure 3a] is one-twelfth.
John: No! This [the triangle] is one-sixth. We have to find out the half of it. Impossible... Oh, wait, wait, I know what it is! Look, [regrouping six triangles]
Owen: it's the half.
John: [regrouping a hexagon to the six triangles]. This is twelve. So, this [separating one triangle, Figure 3b] is one-twelfth.
Owen: It's what I've been saying since earlier!
John: we've found the twelfth, we have to divide by four... by three. [Separating 3 triangles from the whole and covering it with a trapezoid, Figure 3c].

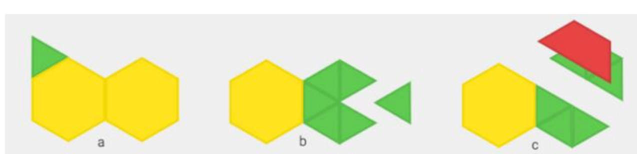


Figure 3: Forming-then-Partitioning a Whole

In this short excerpt, we observe how Owen and John struggle to make sense of all the information with the manipulatives at their disposition, juggling between one hexagon as the whole/one triangle as one-sixth and two hexagons as the whole/one triangle as one-twelfth, suggesting that the routine of "naming" interfered with the routine of "forming-then-partitioning" the whole. Moreover, after convincing themselves that they succeeded at representing the whole, they intended to partition it. But the interpretation of the division led them to the wrong partitioning.

Validating. Some pairs used manipulatives to validate an answer obtained through calculation. Indeed, some students interpreted the task as one of mental calculation. At some point, usually early in the resolution, they calculated 12 divided by 3 or searched for the number that multiplied by 3 gave 12. Tim and Nick, two 5th graders, used this method. Tim seemed to calculate mentally and then added two rhombi on top of the two hexagons. Nick wanted to validate. To do so, he counted "one-third, two-thirds, and three-thirds" pointing with his fingers holding imaginary blocks over the whole but without moving any pieces. Tim and Nick were able to perceive the twelve triangles inside the two hexagons and the three groups formed of four triangles. The Pattern Blocks design facilitated this identification for Tim and Nick.

Justifying the answer. Adults and peers sometimes challenged an answer by asking why it was the good one. Tim and Nick had already found the solution through calculation and validated through forming-then-partitioning the whole. When an adult asked them to explain their solution, they argued that one-twelfth means there are twelve triangles in the whole. "The third of twelve is four." Therefore, the answer is four triangles. Their explanation relied on calculation, but they also overlaid four triangles on top of the two hexagons (see Figure 4). They answered that four triangles "fit" three times in two hexagons. They completed their justification by effectively making three-thirds "fit" on top of the two hexagons pointing fingers to show the three distinct groups as Nick did when validating the answer earlier. For Tim and Nick, a calculation was not an acceptable justification. They used the manipulative to visually back their claim in the context of the task, showing the whole, the part, and the fact that it fits three times.

If the students did not use the wanted whole, the teacher could use the manipulatives to challenge not only the answer but the students' routines. For example, Mike and Edith, two 5th graders, attributed the name 1/3 to the hexagon and then formed a whole with three of them. Incidentally, they had placed the manipulatives as positioned in figure 1 in front of them when they received the bag, the three hexagons were a bit separate from the other blocks. They reproduced three hexagons on a sheet of paper and wrote 1/3 in one of them. An adult asked them to explain their answer: "[After they answer that 1/3 was a hexagon]. Ok, if this [showing a triangle] is one twelfth, can you show me the whole in which this triangle represents one twelfth?" By reintroducing the triangle and its value in the conversation, the students were able to determine the wanted whole [two hexagons] and modified their routine accordingly:

Mike: Look, it can be those two [touching two rhombi]
Edith: Ha, yes, it works... mmm, but you cannot split it in three.
…
Mike: Yes, look in this [taking the two hexagons], there are six like that [covering the hexagons with two rhombi]
Edith: but we are looking for the third
Mike: exactly! [splitting the whole in three by pointing rhythmically]

After the adult intervention, Mike and Edith were able to modify their routine by using new visual mediators (the whole and the part) that considered the given value of the triangle (one-twelfth) and Mike convinced Edith that two rhombi were an adequate answer by simulating a partitioning through gesture.

The (Sub)-Routine of Covering.

As illustrated in the examples presented above, we observed students in each grade using a "covering routine" (see Figure 4). This routine has at least two functions. First, it allows comparing blocks or groups of blocks to determine that they fit exactly on top of each other. For example, by covering the trapezoid with three triangles, we can figure out that a trapezoid is equivalent to three triangles. However, as said before, it was impossible to cover two hexagons with twelve triangles, forcing students to rely on imaginary blocks, as Tim and Nick did, or to use equivalent combinations as in Figure 2. The breaching element brought to bear by the limited manipulatives pushed the students to slightly modify their routine.

Second, it allows showing not only the part representing one-third but the whole in which it represents one-third. This second function was used more in 5th and 6th-grade classes, leading us to hypothesize that it was part of the classes' discursive rules, i.e., that, in those specific classes, students were expected to justify their answers. Indeed, to be able to judge the validity of the answer, you need to refer to the whole.

Figure 4: Example of an answer obtained through the covering routine

Interpretations

Through the lens of routine and affordance, we aim to describe how different choices disrupt students' activity, creating opportunities to learn. The analysis exposes how the students' mathematical activity is bound to the manipulatives used in this task and how different didactic choices may create different opportunities to learn. First, we hypothesize that the manipulatives led students to different routines than a paper-and-pencil task would. With no manipulatives, students may have reproduced the triangle twelve times (as a set or a whole) and circled four of them, or circled three groups of four. The comparing-and-naming nor the covering routine would have been needed.

Second, the limited quantity of pattern blocks is an important breaching element that created different opportunities to learn, here, through three routines. In this task situation, students were bound to represent different fractions with manipulatives. They could have represented the given fraction, the whole, the sought fraction, or other specific fractions. The 'representation-able' quality of the manipulative is linked to one of the main affordances. Indeed, the diverse ways that pattern blocks can be used as visual mediators are specific to this type of manipulatives, leading to breaches of routines.

A first breach was observed when students were not able to associate one-third to only one block, this permitted us to note a breach in their routine of 'comparing-and-naming'. This choice of limiting the blocks used created an opportunity to enrich visual mediators associated with unit fractions and to challenge the rule that "a unit fraction should be represented by one and only one block." A second breach was observed for the routine of 'forming-then-partitioning' a whole. Students could not represent the whole with twelve triangles. They had to adapt their routine. One way to do so was to use two hexagons. This breaching element led to work on equivalent surfaces. The third breach is also linked to this routine and to the importance of equivalence. Indeed, using only one kind of block to represent a fraction was a common routine for students.

The limited manipulatives introduced a breaching element that created opportunities to develop new narratives about partitioning a whole, focusing on equivalence rather than on the identity of the parts. A fourth breach was observed when students lost sight of the given value one-twelfth, leading them to associate the whole with one or three hexagons. In both cases, the manipulatives available led to the perception of a whole partitioned into three identical parts. Clearly for some students, in this task situation, one-third was associated with the narrative "one of three identical parts." Thus, they were looking for a whole constituted of three identical parts.

Even though it was not the focus, this analysis also put to light other affordances linked to the qualities of the manipulative. The 'representation-able' quality of the manipulatives did not only depend on the number of blocks available but on the design of the pattern blocks. Indeed, depending on the fraction given, Pattern blocks do not allow the representation of any fractions. A second affordance can be linked to the "move-able" quality of manipulatives. Students can move the blocks, stack them, group them, separate them. This quality helps students perceive and validate the equivalence of distinct groups, an element that cannot be avoided in this task partly because the manipulatives are limited. A third affordance can be linked to the "communication-able" quality of Pattern Blocks. Indeed, these manipulatives are tied to certain keywords as each piece has a specific color and a specific form. Other layers of analysis are required to push further the interpretation here.

Conclusion

The limited manipulatives, as a breaching element, created opportunities to learn by introducing nuances in what is mathematically important when working with fractions (e.g., the use of identical parts when partitioning a whole to represent a fraction is not necessary, the use of equivalent parts is). The didactic choices made helped us identify shared narratives and routines in those specific communities of students and how those choices created opportunities to learn, i.e., created changes in routine. Referring to Nachlieli and Tabach (2019), we could define those opportunities to learn as exploration-requiring opportunities to learn, i.e., to complete the task or meet the teacher's expectations, students cannot apply an already familiar single procedure. They needed to create a new combination of actions, a new routine. While Nachlieli and Tabach (2019) linked the creation of those opportunities to the kind of question asked (e.g., presenting request with words like what, why, find, explain), we add that those breaching elements can also play a key role.

Those findings also point out some implications for the classroom. Because breaching elements are possible only when disrupting familiar ways of doing (Corriveau, 2013), introducing such elements in tasks is not trivial. To go further than the question of whether to use manipulatives or not, one must attend to the implicit culture of the class. In other words, it is to be able to realize what is usually taken for granted and articulate it with the targeted mathematical learning to make clever didactic choices. It also brings us back to the fact that the use of manipulatives is not a magic solution (Ball, 1992).

Acknowledgments

This research received the support of (name of the organization, number). We would also like to warmly thank the teachers who collaborated on this project.

References

Ball, D. L. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator, 16*(2), 14–18, 46–47

Brousseau, G. (1981). Problèmes de didactique des décimaux. *Recherches en didactique des mathématiques, 2*(1), 37-127.
Brown, J., Stillman, G., & Herbert, S. (2004). Can the notion of affordances be of use in the design of a technology enriched mathematics curriculum? In I. Putt, R. Faragher, & M. McLean (Eds.), *Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 119–126). Sydney: MERGA.
Corriveau, C., Jeannotte, D. et Michot, S. (forthcoming). Didactic variables related to the use of manipulatives and their impact on students' activity: an illustrative case around fractions. *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education.* Bolzano, Italy.
Corriveau, C. (2013). *Des manières de faire des mathématiques comme enseignants abordées dans une perspective ethnométhodologique pour explorer la transition secondaire collégial* [Unpublished Dissertation], UQAM.
Carbonneau, K. J., Marley, S. C. et Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*(2), 380–400.
Greeno, J. G. (1994). Gibson's Affordances. *Psychological R*eview, 101(2), 336–342.
Jeannotte, D. & Corriveau, C. (2020). Interactions Between Pupils' Actions and Manipulative Characteristics when Solving an Arithmetical Task. Proceedings of the 11th Congress of the European Society for Research in Mathematics Education. Utrecht, NL. Lavie, I., Steiner, A., & Sfard, A. (2019). Routines we live by: From ritual to exploration. *Educational Studies in Mathematics, 101*(2), 153–176.
MEQ (2001). *Programme de formation de l'école québécoise, Éducation préscolaire*, Enseignement primaire. Québec: Gouvernement du Québec.
Moyer, P. S. (2001). Are we having fun yet? how teachers use manipulatives to teach mathematics. *Educational Studies in Mathematics, 47*(2), 175–197.
OECD (2019), PISA 2018 Mathematics Framework, In PISA 2018 Assessment and Analytical Framework, OECD Publishing, Paris.
Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. The journal of mathematical behavior, 22(4), 405–435
Sfard, A. (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing.* New York: Cambridge University Press.

STEM UNDERGRADUATES' STRUCTURAL CONCEPTION OF RATE OF CHANGE

Sindura Subanemy Kularajan
Texas State University
s_k252@txstate.edu

Jennifer A.Czocher
Texas State University
Czocher.1@txstate.edu

Using data from teaching experiments and theories from quantitative reasoning, we built second-order accounts of students' mathematics with regards to how they conceived rate of change through operating on existing quantities. In this report, we explain three different ways STEM undergraduates structurally conceive rate of change as they constructed mathematical models for real-world scenarios.

Keywords: Quantitative Reasoning, Mathematical Modeling, Rate of Change

Understanding *rate of change* (RoC) is central to the understanding of mathematical concepts such as the fundamental theorem of calculus (Thompson, 1994b; Thompson, 2008), derivative (Zandieh, 2000), limits and function (Tall, 1986), and differential equations. Beyond mathematical concepts, an understanding of RoC is necessary to model dynamic situations (e.g., Arleback et al, 2013; Ellis, 2007). Generally, the main goal of studies surrounding RoC is to understand students' mathematical conceptions of RoC. That is, the field is reporting accounts of students' understanding of RoC as they are working on tasks that help them to understand RoC as a mathematical concept. We still need an account of how students are conceiving RoC as they are developing mathematical models for real world scenarios that aren't necessarily set out teach the concept of RoC. This is important because educators of the field will get an idea of how students apply their understanding of RoC into practice. Accordingly, we ask: how do STEM undergraduates conceive RoC while constructing mathematical models for real-world scenarios?

Theoretical Framing and Background

Our research lies within the cognitive perspective of mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling is the cognitive processes involved in constructing a mathematical model for real-world scenarios. We define a mathematical model to be a representation of the relations among conceived quantities.

A *quantity* is a mental construct of a measurable attribute of an object. It consists of three inter-dependent entities: an object, a measurable attribute, and a *quantification*. Quantification involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between an attribute's measure and the unit of measure (Thompson, 2011). Quantification takes place in the mind of an individual and we can only use an individual's externalized actions to infer whether she has quantified an attribute. Czocher & Hardison (2021) presented eight quantification criteria that can be taken as indication that a student has conceived measurement process for a measurable attribute of an object.

Quantitative reasoning is the mental operations involved in conceiving a situation through quantities and relation among those conceived quantities (Thompson, 2011). A relation among conceived quantities is established through operation on quantities. Thompson (1994) defines *quantitative operation* as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a *relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994). Examples of quantitative operations include combining two quantities additively or multiplicatively and comparing two quantities additively or multiplicatively. A *quantitative structure* is a network of quantitative relationships (Thompson, 1990).

Reasoning about quantities may also entail reasoning about how the quantities can vary. While *variational reasoning* involves reasoning about varying quantities independently (Thompson & Carlson, 2017), *co-variational reasoning* involves "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al, 2002, p.354). Scholars have extended the work of covariational reasoning to *multivariational reasoning*, which is reasoning about more than two quantities changing in conjunction with each other (Jones 2018; Jones & Jeppson,2020). Recently, Ellis et al (2020) defined *scaling continuous covariation* as "imagining the continuum as infinitely zoomable, couple with the understanding that one can re-scale to any arbitrarily small increment for x and coordinate that scaling with associated values for y" (p. 88). When an individual engages in covariational reasoning she constructs a *multiplicative object* in her mind. Constructing a multiplicative object entails coupling of two quantities (Saldanha & Thompson, 1998) that yields a new conceptual object "formed by mentally uniting the attributes of two quantities, so that the new conceptual object is simultaneously one and the other" (Thompson & Carlson, 2017, p. 433). For example, the rate at which the size of a bank account changes with respect to time is a multiplicative object that may be formed by coupling *change in size of the account* and the *length of time elapsed*.

Many scholars have posited ways in which students may develop a conceptual understanding of RoC based on *rates*. For example, using a quantitative reasoning orientation, Thompson (1994a) defined *rate* as a reflectively abstracted constant ratio, where a ratio is a result of comparing two quantities multiplicatively. Thompson (1994b) argued that developing images of rate starts with student imagining a change in some quantity, next coordinating the changes two quantities, and finally developing images of the two quantities covarying so that they remain in constant ratio. Confrey & Smith (1994) used a covariational reasoning approach as a way to learn the concept of function through exploring the concept of RoC. They discovered that students employed three different ways to describe RoC in the context of cell splitting: an additive RoC, a multiplicative RoC, and a 'proportional new to old' RoC. Through these investigations, they proposed an analytic approach to RoC as a "unit per unit comparison" (p. 37) where a unit is a "invariant relationship between a successor and its predecessor" (p. 142), as means to instill a multiplicative unit approach viewing RoC for exponential functions. Johnson (2015) investigated how students quantify *rate* through "forming and interpreting relationship between varying quantities" (p. 66). Johnson discovered that students quantified rate by associating amounts of change in extensive quantities, using a single extensive quantity that represents an association of two quantities, and coordination of intensive and extensive quantity through the quantitative operations *coordination* and *comparison*. Through *coordination,* rate is quantified as an intensive quantity and through *comparison,* rate is quantified as an extensive quantity. While Johnson reported distinct ways students quantify RoC, it excludes ways in which students may conceive RoC that do not necessarily involve quantitative operations as how Thompson defined. We postulate one such operation in this report.

Borrowing ideas from the aforementioned constructs we define *structural conception* of *RoC* as conceiving RoC as a measurable attribute of an object through forming a relation among constituent quantities (i.e., operating on quantities). We use the term *operating on quantities* to include Thompson's quantitative operations, mental operations involved in reasoning about varying quantities, and other operations on quantities that could yield a new quantity. The research question addressed in this report is: In what ways do STEM undergraduates structurally conceive RoC while constructing mathematical models for real-world scenarios?

Methods

We present data from a pair of 10hr individual teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors at a large university. The overall goal of the teaching experiment was to investigate how students conceive real-world situations through quantities and relations among quantities. Our students, Szeth and Pai were both enrolled in differential equations at the time of the interviews. Throughout the teaching experiment, Pai and Szeth worked on 9 and 10 tasks, respectively, that were based on real-world scenarios. In this report we present data from the *CI8 Account Task* and *The Cats & Birds Task*. We focus on these tasks because they exemplify how Pai and Szeth operated differently on similar quantities, that they had previously constructed, to conceive RoC.

CI8 Account: The competing Amtrak Trust has introduced a modification to City bank's SI8, which they call the CI8 account. Like the SI8 account, the CI8 earns 8% of the "initial investment". However, at the end of each year Amtrak Trust recalculates the "initial investment" of the CI8 account to include all the interest that the customer has earned up to that point. Create an expression that gives the value of the CI8 account at any time t (Castillo-Garsow, 2010).

Cats & Birds: Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian's Migratory Bird Center and the U.S. Fish and Wildlife Service concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral "unowned" cats but noted that their domestic peers "still cause substantial wildlife mortality" (Raasch,2013). Consider a backyard habitat, where cats are the natural predators of birds. Model the rate of decrease of the population of birds due to predation by cats.

The interviews were retrospectively analyzed to construct second-order accounts (Steffe & Thompson, 2000) of students' reasonings via inferences made from students' observable activities such as verbal descriptions, language, written work, discourse, and gestures. The retrospective analysis consisted of multiple passes of through the data to arrive at examples that illustrate the different ways Szeth and Pai structurally conceived RoC. First, we watched the videos in MAXQDA in chronological order and paraphrased each interview by chunks. Next, we created accounts of students' mathematics and the reasons they attributed to their mathematics. Next, we went over the accounts we created and the videos at the same time and refined our accounts by adding details using quantitative and covariational reasoning lenses. We credited a student to have instantiated a quantity if we were able to infer from his reasonings that he had conceived an object, attribute, and a measurement process for the attribute. As evidence of student to have conceived a measurement process, we checked if at least one of the quantifications criteria (see Czocher&Hardison, 2021) was met. We used segments of transcripts, where the students engaged in quantitative reasoning along with inscriptions and gestures as evidence for our claims. Next, from these accounts, we selected instances where the students engaged in conceiving RoC. Next, while watching the videos and going over our accounts of students' RoC, we further refined our accounts by adding details of how Pai and Szeth operated on constituent quantities to form a relation in order to conceive RoC. Next, we went through our accounts of Pai's and Szeth's conception of RoC and observed any patterns, in terms of operation on constituent quantities, that were consistent throughout the sessions. We made note of them by summarizing the pattern. We watched the videos in their entirety again and made sure all of the different ways of conceiving RoC were recorded and refined our second-order accounts by seeking clarification on utterance and gestures to support our claims on Pai's and Szeth's structural conception of RoC. We present some of these second-order accounts below.

Findings

Pai and Szeth structurally conceived RoC in the following ways: (1) as how quickly change is happening, (2) as a multiplicative object, (3) as the derivative, (4) as the net change during a time interval, and (5) as the additive comparison of rate of increase and rate of decrease. Due to space constraints, we share examples only for (1)-(3).

Rate of Change as How Quickly Change is Happening

In the CI8 Account task**,** Szeth constructed the following models for the value of the account at the end of 1st, 2nd, and nth years: $S_1 = \frac{8}{100} \cdot S_0 \cdot t + S_0$; $S_2 = \frac{8}{100} \cdot S_1 \cdot t + S_1$;$S_n = \frac{8}{100} \cdot S_{n-1} \cdot t + S_{n-1}$, where t is the length of the compounding period. We then started to discuss how the account would grow when the compounding periods gets smaller. Szeth reasoned:

> Szeth: gut reaction is it [rate] should increase faster because the compounding is more frequent. Then giving it some thought, I'd agree. If we did one month, make this calculation [Figure 1], it's slightly less than like a year, but you're doing that calculation 12 times over the year. Yeah, yeah, if you decrease the time interval, your rate should increase quicker … I mean, the final amount of money in the account should increase quicker, which I guess means that our rate would be of a greater value over a shorter interval. Yeah, that makes sense.

In the excerpt above Szeth was coordinating three inter-dependent quantities: the length of the time interval, the amount of money in the account, and the rate at which the account grows. Through coordinating the length of the time interval with the rate at which the account grows, Szeth was engaging in *scaling continuous co-variation*. When asked to elaborate further, Szeth coordinated the value of the account for compounding periods 1 year and 1 month (Figure 1). He started with \$500 and showed how the value of the account would increase each compounding period when the length of the compounding period is 1 year. He indicated that the increase in the account's value would be the *difference* between the sizes of the account for consecutive years. Next, he considered the increase in the size of the account when the compounding periods are of length 1 month. He reasoned: "Then actually they should all be the same numbers, but the point being that's one year… But that's one month" (Figure 1). Here Szeth is *comparing* the values of the account for compounding periods of different lengths to discuss how quickly the value of the account changes. We infer that, for Szeth, a bank account in which the size of the account changes over a small time period grows at a greater rate over time than one for which the same change in the size of the account happens over a larger time period. In this scenario, Szeth constructed a mental image of how quickly the size of the account changes in order to reason about the rate at which the account would grow.

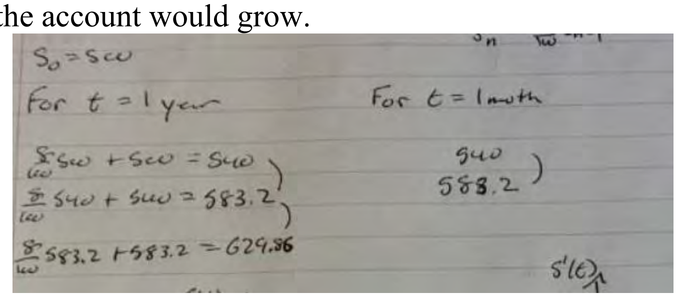


Figure 1: Szeth evaluates the size of the account for different compounding periods

Rate of Change as a Multiplicative Object

In the CI8 Account Task, Pai constructed the following models to indicate the size of the account at the end of 1, 2, and 3 years: $S_1 = S_0 + 0.08(S_0)(t)$; $S_2 = S_1 + 0.08(S_1)(t)$; $S_3 = S_2 + 0.08(S_2)(t)$, were $t = 1$ year. He drew the graph in Figure 2, and we proceeded have a discussion on how during each year the account grows linearly at a rate proportional to the size of the account at the beginning of that year.

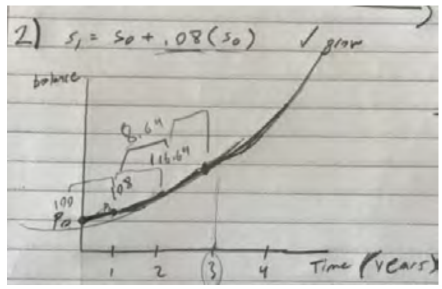


Figure 2. Pai's graph of account value vs time in years

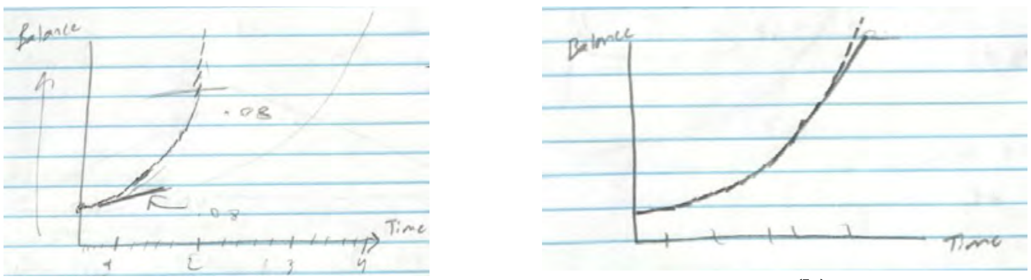
We asked Pai if his expression for S_i would work for any time period of length Δt and not just $t = 1$ year. Pai indicated that it would work as long as all of the S_i's are the account's value at the beginning of each of the compounding period. He rewrote the expression as $S = S_0 + (0.08)(S_0)(\Delta t)$, where Δt was the length of the compounding period and S_0 was the size of the account at the beginning of the compounding period. We asked him to interpret his graph as Δt gets smaller. For example, when we discussed the possibility of having four compounding periods instead of one in a year, Pai gave the following reasoning:

> Pai: "It would change, not the overall direction, but it would change the number of straight pegs, I guess, straight increases, because there's now 16 compounded periods in our time period [4 years]. So it'd be more round, but still not perfectly round, but it'd be my thinking of a circle, circle has infinite edges. It would be like that [tracing a curve along his graph (Figure 3b)] …that's what it would be. It'd be more like a curve"

In the above excerpt, Pai was thinking of the graph taking a curve like shape as Δt becomes smaller and smaller. We next asked Pai the effect smaller Δts would have on the RoC of the account's value, specifically its proportional relationship with the money at the beginning of the compounding period. During this discussion Pai reasoned:

> Pai: That should be proportional still. It's still proportional, but time is increasing so much that, or so fast, or the compounded periods are so small that it's going to change instantly…Like S is going to change. Like S_0 is going to be extremely different than the first... S_0 without… a very shortchange… like a very rapid change. It's still proportional to what was initially there, which just happening much, much faster because the t is much smaller than when we started"

Pai thought the rate at which the account's value changes would still be proportional to the account's value at the beginning of the compounding period. However, this time, he argued that the account's value at the beginning of the compounding period would change rapidly because Δt was much smaller. Pai was *coordinating* the quantities Δt, RoC of the account's value, and account's value at the beginning of the compounding period. We interpret Pai's reasoning as indicative of Pai imagining the quantities change in the size of the account and Δt to be happening simultaneously. We infer that Pai formed a multiplicative object of (change in size of the account, length of time elapsed). We also infer that Pai's formation of a multiplicative object had consequences beyond enabling him to coordinate the quantities change in size of the account and time taken for the change. He also used this imagery to help him in reasoning about the rate at which the size of the account changes. This was further evidenced when we asked him to write an expression for the RoC of the size of the account. Pai wrote $\frac{dS}{dt} = 0.08(S_0)$, where S_0 is the starting value at any compounding period. He reasoned as "[S_0] is just any starting period. Not just the literal start, but any starting period of these pegs, as t [Δt] gets smaller, it's 0.08 of that starting period, so the derivative will change."



(a) (b)

Figure 3: Pai's graph for as Balance vs time as Δt gets smaller

Rate of Change as the Derivative

In the Cats & Birds task, Pai and Szeth both conceived RoC as a measurable attribute that can be measured by taking the derivative of the function. For example, both Pai and Szeth conceived RoC of the bird population due to cats as a measurable attribute that could be measured by taking the derivative of the function bird population at time t.

Taking the derivative. Szeth arrived at the expression the $I(t) = \alpha \cdot \beta \cdot C(t) \cdot B(t)$ as the number of birds killed due to cat-bird interactions. We asked he could use the expression to model the rate of decrease of bird population due to predation by cats. Szeth Solved the above expression for $B(t)$ as $B(t) = \frac{I(t)}{\alpha \cdot \beta \cdot C(t)}$. Since he intended model the rate of decrease of the bird population, he mentioned that he first wanted to isolate $B(t)$ from the expression and thus solved for it. Szeth said, to find the rate he "would take the derivative [of $B(t)$]". Even though we had asked him to construct an expression for the rate of decrease of bird population due to just cats, Szeth tried to evaluate $B'(t)$, which the RoC of bird population due to all causes, and not just cats - a different attribute of the object bird population. Szeth proceeded with taking the derivative of $B(t)$ using the quotient rule.

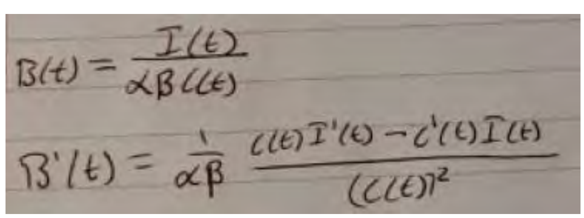
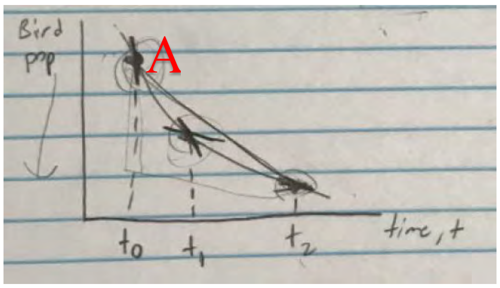
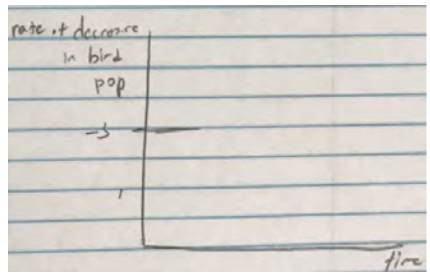


Figure 4: Szeth takes the derivative of B(t)

Imagining the slope of the tangent. In the Cats & Birds Task, After Pai constructed an expression for "the number of bird-cat interaction that resulted in a kill" (which for Pai, is also the “number of birds dead at time t”) as $f_k(t) = [\alpha B(t)C(t)]\beta$, we asked what the rate of decrease of the bird population due to cat predation be. Pai’s first reaction to this prompt was to draw a graph of bird population vs time [Figure 5]., where he *grossly coordinated* the quantities the population of bird and time.



(a)



(b)

Figure 5: (a) bird population vs time and (b) rate of decrease of bird population vs time

According to the graph above, for Pai, as time goes on, the bird population decreases due to cat predation. After drawing the graph Pai reasoned:

> Pai: I think this question is trying to plot the function for the derivative of my model, because it wants the rate of decrease. I think we're asking for decrease in bird pop would be the y axis…over time t. The rate would be whatever this slope is [pointing at A], called, I don't know. Negative three. It would be like that [Figure 5b].

We infer that Pai is imagining the slope of the tangent line at different points in time ($t = t_0, t = t_1, t = t_2$), would give him the rate of decrease of bird due to cat predation. Here Pai, like Szeth, conceives the RoC of bird population due to cats by considering the derivative of the bird population. However, he wasn’t very certain what the outcome of evaluating the slope of the tangent at different points in time would look like. Since Pai needed a “a model for the derivative” he thought $f_k'(t)$ would represent the tangent lines in figure 5a. He proceeded to take the derivative of $f_k(t)$ using the product rule and arrived at $f_k{}'(t) = \beta\alpha[B'(t)C(t) - B(t)C'(t)]$. However, he expressed that the result didn’t make sense to him. When asked why it didn’t make sense, Pai reasoned:

> Pai: $B(t)$ is birds at time t. I don't really know what $B'(t)$ would be. The change in birds at time t? The increase or decrease of their population? That's just a fact of, for bird, $B'(t)$, the birds, for that specific case, it would just be change in the natural bird population, not due to cats killing them? Because the bird population is changing, decreasing, because cats are constant killing them. I guess birds are also just die. They're birds... So, B' would be that change. Cats too, cats get run over and whatever, so it would be the change in their population [pointing at $C'(t)$].

We believe the biggest reason it wasn’t making any sense to him was because he wasn’t sure how to distinguish among the quantities $f_k'(t)$ and $B'(t)$.While reasoning out loud, he settled on $f_k'(t)$ as the rate of decrease of bird population due to predation by cats which he expressed in terms of $B'(t)$, where he attributed $B'(t)$ as change in bird population due to all causes. Even though f_k, for Pai, was “number of birds dead at time t”, Pai’s conception of RoC as a derivative was so strong, he overlooked the attribute $f_k{}'$ would be measuring.

Discussion

We presented three examples of differing ways that Pai and Szeth structurally conceived RoC in terms of operating on constituent quantities. Conceiving RoC as how quickly change is happening entails considering the differences in consecutive amount-values and coordinating it with the amount of time it took for the change to happen. Conceiving RoC as a multiplicative object entail uniting the attributes of two quantities so that their chnages both happen simultaneously. In both Pai's and Szeth's cases, conceiving RoC in these manners arose while they were both grappling with the thought: *How would the account grow as the compounding periods get smaller?* However, the ways in which they chose to operate on existing quantities differed. In Szeth's case, he first engaged in the *additive comparison* of the quantities amount of money at $t + 1$ and amount of money at t, and next considered how long it took for the change to happen. In Pai's case, he *coordinated* the change in the amount of money in the account and length of time duration for the change and envisioned those changes to be happening at the same time.

Conceiving RoC as a derivative entails considering the derivative of a function as a measurable attribute of an object. This way of conceiving the RoC is different from the first two examples presented here (and ways (4) and (5) not exemplified in this report). This is because envisioning the derivative or constructing an image of a derivative may or may not fall within the scope of quantitative operation. A derivative may be attributed as a quantitative operation if the meaning student has for derivative entailed the coordination of two quantities. For both students, there was no clear evidence that taking the derivative or envisioning the slope of the tangent carried situationally relevant quantitative meaning. Conceiving of RoC as the derivative might limit a modeler in terms of what he can model. For both students, the phrase RoC acted as a cue to take the derivative, dropping the situationally relevant meaning the derivative being taken on was measuring. Both Szeth and Pai, produced quantities that weren't measuring the attribute that was in question: *RoC of bird population due to cats*. However, for both students, considering derivative resulted in a quantified attribute, namely, the bird population. That is, Both Pai and Szeth conceived RoC as a measurable attribute that can be measured by considering the derivative with respect to time of the function bird population.

By adhering to the definition of quantitative operation and quantitative relation, instances like these, where a student may or may not have quantitative meanings for RoC as a derivative, would be missed. In our data, we found instances where the operation performed does not have clear evidence of a quantitative operation (as defined above), but the outcome of the operation is clearly a new quantity. We ask, *where ought these borderline instances be placed?* We do believe that even though taking the derivative in and of itself may not be credited as a quantitative operation, envisioning the derivative as yielding a new quantity is an image that is prevalent in undergraduate learners and the result of that action is often times useful for mathematical modeling. For example, in a simple interest context, Pai took the derivative of the size of the account at time t to describe the behavior of the growth of the account. Although the field has reported on students' conception of derivative through lenses of quantitative reasoning and covariational reasoning, we still need research on what it means when a student takes a derivative, as a procedure, through a quantity-oriented lens.

Acknowledgements

This material is based upon work supported by the national Science Foundation under Grant No. 1750813.

References

Ärlebäck, J. B., Doerr, H. M., & O'Neil, A. H. (2013). A modeling perspective of interpreting rates of change in context. *Mathematical Thinking and Learning, 15*,314-336.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378.

Castillo-Garsow, C. C. (2010). *Teaching the Verhulst model: A teaching experiment in covariational reasoning and exponential growth.* Unpublished Ph.D. Dissertation, Arizona State University, Tempe, AZ.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics*, 26, 135-164.

Czocher, J. A., & Hardison, H. L. (2021). Attending to Quantities through the Modelling Space. In F. Leung, G. A. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West*. Springer.

Ellis, A. B. (2007a). Connections between Generalizing and Justifying: Students' Reasoning with Linear Relationships. *Journal for Research in Mathematics Education, 38*(3), 194-229.

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: supporting students' algebraic reasoning. *Educational Studies in Mathematics*, 104,87-103.

Johnson, L. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*. 17(1), 64-90.

Jones, S. R. (2018). Building on covariation: Making explicit four types of "multivariation". In A. Weinberg, C. Rasmussen, J. Rabin, & M. Wawro (Eds.), *Proceedings of the 21st annual Conference on Research in Undergraduate Mathematics Education.* San Diego, CA: SIGMAA on RUME.

Jones, S. R., Jeppson, H. P. (2020). Students' reasoning about multivariational structures. In. Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Mexico

Kaiser, G. (2017). The teaching and learning of mathematical modeling. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 267–291). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Raasch, C. (2013, January 29). USA Today. Cats kill upto 3.7B birds annually. https://www.usatoday.com/story/news/nation/2013/01/29/cats-wild-birds-mammals-study/1873871/

Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillside, NJ: Erlbaum.

Tall, D. (1986). *Building and testing a cognitive approach to the calculus using interactive computer graphics*. Unpublished doctoral dissertation, University of Warwick, Coventry, England.

Thompson, P. W. (2008). Conceptual analysis of mathematical ideas: Some spadework at the foundations of mathematics education. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sépulveda (Eds.), Plenary Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education, (Vol 1, pp. 45-64). Morélia, Mexico.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education.* WISDOMe Mongraphs (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium for research in mathematics education*, 421-456.

Thompson, P. W. (1994a). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (1994b). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26, 229-274.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.). *Research in Collegiate Mathematics Education*, (4th ed.,Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.

UNDERGRADUATE PERSPECTIVES ON THE NATURE OF MATHEMATICS THAT ARISE THROUGH EXPLORATION OF UNSOLVED CONJECTURES

Jeffrey Pair
California State University Long Beach
Jeffrey.Pair@csulb.edu

Gabe Calva
California State University Long Beach
Gabe.Calva@student.csulb.edu

We report on a research study conducted within a transition-to-proof course for mathematics majors at a large public university. Within the course, students explored famously unproven conjectures and reflected on how their perspectives of mathematics changed through this exploration (if at all). In this report we share students' takeaways from the project. For instance, some students experienced mathematics as a creative subject for the first time, as they tried their own methods to solve the conjectures; other students reflected on developing a greater understanding of the behind the scenes work of mathematicians that goes into mathematical creation. We also report on the subjective emotional experiences of the students, which ranged from frustration from being unable to find patterns to enjoyable exploration.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Reasoning and Proof; Undergraduate Education

Within mathematics education scholarship, authors have lamented that students do not experience mathematics in ways similar to those experienced by practicing mathematicians (Boaler 2015; Lampert, 1990; Lockhart, 2009; Hersh, 1997). After describing the creative activity of research mathematicians, Boaler (2016) wrote, "I strongly believe that if school math classrooms presented the true nature of the discipline, we would not have this nationwide dislike of math and widespread math underachievement" (pp. 22-23). What can a mathematics instructor do so that students understand the nature of the discipline of mathematics that Boaler refers to?

If mathematics classrooms are to provide a space for students to experience mathematics in ways similar to mathematicians, then mathematics teacher education programs must provide space for future teachers to experience mathematics in novel ways. But curriculum change is a slow-moving process—the curriculum of the undergraduate mathematics major has remained relatively constant for decades. Mathematics majors frequently take Calculus, Introduction-to-Proof, Number Theory, Geometry, Abstract Algebra, Analysis, etc… How can we improve mathematics instruction in these existing courses so that future teachers are better prepared to support their students in engaging in authentic mathematical practices? One approach that has been used in content courses is to design novel curriculum so that future teachers can collaboratively develop their mathematical knowledge for teaching (Lischka, Lai, Strayer, & Anhalt, 2020). These approaches have been successful in courses for geometry, algebra, statistics, and modeling. But what about traditional proof courses? Inquiry-based and collaborative pedagogical approaches within proof courses are another way to provide future teachers opportunities to experience mathematics in ways more aligned to the discipline (Bleiler-Baxter & Pair 2017, Bleiler-Baxter, Pair, & Reed 2021, Pair & Calva, 2021). In this paper we report on a new approach aimed at enriching students' perspectives of mathematics within a transition-to-proof course—engaging students in the exploration of unsolved mathematical conjectures.

Purpose

Our goal for this research project was to engage students in a novel activity—the exploration of unsolved mathematical conjectures. Unbeknownst to the general human population, mathematics is a dynamic field in which new results are constantly being discovered and reported (Pair, 2017). Unanswered questions and unsolved problems drive mathematical research. Two such unsolved problems, the Twin Primes Conjecture and the Collatz Conjecture have remained unsolved for decades (Bairrington & Okano, 2019; Rezgui, 2017). Our goal with this project was to expose these conjectures to students and provide them the opportunity to develop their own methods to try and prove the conjectures and then reflect on what they learned about mathematics through the activity. Our hypothesis was that engaging in these activities would provide the undergraduates novel opportunities to enrich their perspectives on the nature of mathematics. Our research question is: How are students' perspectives on the nature of mathematics enriched through their exploration of unsolved conjectures (if at all)?

Theoretical Perspective

We designed the study and corresponding unsolved conjectures project in efforts to promote the humanistic vision of mathematics as captured by the four characteristics of mathematics as expressed in the *IDEA* Framework for the Nature of Mathematics (Pair, 2017). This framework was developed through a dissertation study that involved immersive experiences in mathematics (collaboration with a research mathematician, teaching a transition-to-proof course). The purpose of the framework is to provide a list of possible goals for students' understandings of the nature of mathematics. This framework posits that, in order for students to possess a refined humanistic view of mathematics, they should understand that:

I - Mathematical ideas are part of our *I*dentity.
D - Mathematical knowledge is *D*ynamic and ever-changing.
E - Doing mathematics involves an *E*motional *E*xploration of ideas.
A - Mathematical knowledge is socially validated through *A*rgumentation.

In order to foster an understanding of these characteristics of mathematics, students must be engaged in learning activities for which these characteristics of mathematics become apparent. We hypothesized that by engaging in an activity that research mathematicians also engage in (trying to prove an unsolved conjecture), students would have the opportunity to reflect on mathematics in ways that align with these four characteristics.

Methods

Participants, Context and Data Collection

The research took place at a large public university in the United States. The university is a Hispanic-serving institution, and has very diverse student body. This diversity is represented in the students who participated in this study, which was approved by our university's institutional review board. Participants included volunteers enrolled in a transition-to-proof course. There were 16 female participants and 9 male participants. All names in this paper are pseudonyms.

As an added component of the course, students kept a mathematician's notebook in which they were scaffolded into exploring unsolved conjectures and reflecting on the process. These notebooks served as the primary source of data for the project. This exploration was conducted through several assignments over the course of the semester. For the first assignment, students reflected on questions such as 1) What is mathematics all about? And 2) What is mathematical proof and how is it used by mathematicians? Students were then introduced to both the Twin Primes Conjectures and the Collatz Conjectures. After doing some relatively simple explorations, they described how the assignment was similar to or different from their previous mathematical activity. For subsequent assignments, students were assigned to research teams to work on either conjecture according to their preferences. For the second assignment, students were scaffolded into relatively difficult tasks to explore the conjectures—for instance looking for patterns in the twin primes or Collatz sequences; or articulating what it would mean to attempt a proof by contradiction to prove the conjectures. For the third assignment, students invented their own methods and approaches to try and solve the conjectures; there was then time for in-class sharing of approaches and methods. For the fourth and final assignment, students continued to use their own invented approaches to try and prove the conjectures; they also watched videos in which mathematicians described their own work on the conjectures; and finally, they reflected upon the process—this reflection included revisiting the questions: 1) What is mathematics all about? And 2) What is mathematical proof and how is it used by mathematicians? Additionally, students also responded to two more questions: 3) How has your thinking regarding mathematics and mathematical proof developed and changed during this semester? Which changes were the result of studying the Twin Primes and Collatz conjectures? And 4) What were the challenges and successes of your experience with mathematical research this semester?

Data Analysis

The researchers analyzed the notebooks through a thematic analysis process (Ryan & Bernard, 2003). This analysis process took place through two main stages. During the first stage, two researchers individually read through each participants' notebook, one at a time, making note of 1) Student quotations that aligned with the nature of mathematics as expressed in the IDEA framework[1]; 2) Evidence of change in perceptions of the nature of mathematics; 3) Other substantial reflections about mathematics and the research process; 4) Interesting mathematical ideas. Subsequently, the researcher's discussed their findings from each notebook, describing their observations. After this task was completed for the individual student notebooks, each researcher created a holistic summary of what they saw as the major trends in student reflections. The researchers also used student quotations as evidence for the trends they observed. This process resulted in a list of several recurring themes in student reflections (e.g. seeing mathematics creatively for the first time; frustration in being unable to find patterns). During the second stage of analysis, the researchers took the list of recurring themes developed in phase 1, and then reviewed each notebook again, making note of which ideas were expressed by each individual student. We then tallied these recurrences, resulting in a quantitative summary of the ideas expressed by the students. For a more detailed description of the methodology, see (Pair & Calva, 2021).

Results

Overall, we found that exploring conjectures provided an opportunity for students to experience and see mathematics in new ways. Students recognized that they had the opportunity to engage in a process similar to mathematicians, and felt they had a better understanding of the process of how mathematical knowledge is developed. They also experienced first-hand the emotional aspects of mathematics, and recorded a wide variety of emotions from frustration to joy. We now present our results, generally organized according to the four categories of the IDEA Framework.

[1] We also used de Villiers (1990) roles of proof framework in our analysis. Those results are shared in (Pair & Calva, 2020).

Identity

In terms of identity, Figure 1 shows the tallies of students whose notebook reflections expressed related ideas. We identified ten total students whose reflections showed that they either took ownership of their ideas using personal (I/my) language, showed that they grew in their own mathematical identity, or identified with mathematicians to a greater degree than they did before (the total is more than ten as some students were categorized in more than one category).

Identity	10 (total)
Take owernship of ideas (using I/my language)	9
Grew in own mathematical identity (right major choice).	4
Identify with mathematicians	4

Figure 1: Recurring Themes Related to Identity

For instance, we can see that Isabella uses the language "my work" and "things I figured out" in this excerpt from her notebook as she references a concept she invented, "pink numbers":

> Overall, I liked this process of communicating with the class and working together. I got excited when I figured 'the first pink number' out […] I'm even more excited to figure stuff out, or simply just share my work with my classmates. My most important work are on pages 12 and 18 because they show the 2 things I figured out.

See Figure 2 for an image that shows an excerpt from Isabella's work that she referred to:

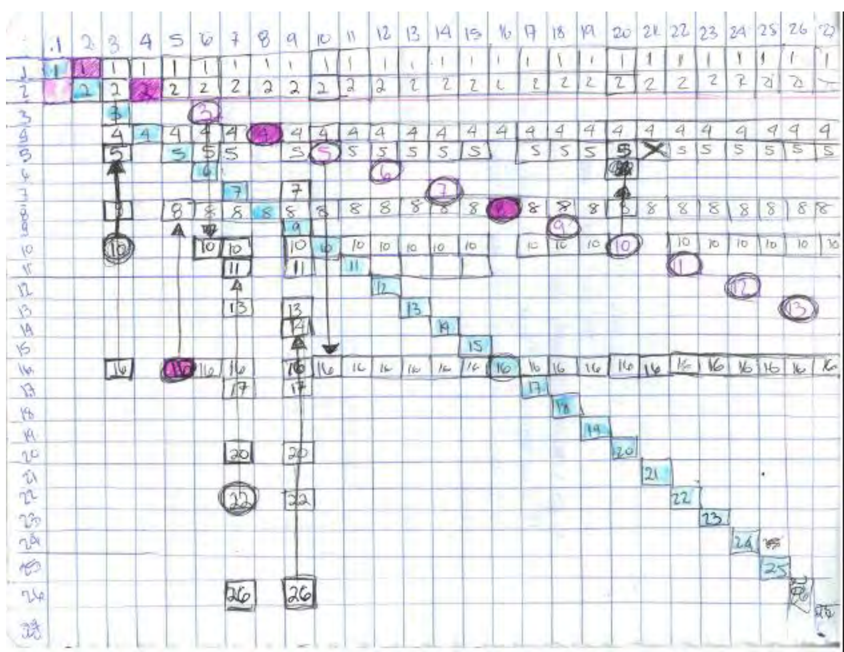


Figure 2: Excerpt from Isabella's Notebook

Four students grew in their mathematical identity as the conjecture exploration helped them see that they enjoyed proofs and/or mathematical research. For instance, Brent wrote,

> This notebook assignment has made me realize that even the simplest arithmetic can lead to problems that even modern mathematics isn't ready for. But it's also grown my passion for

> proofs and the subject itself. I want to dive deeper into this and see what I can learn! Oh, it also made me realize that I made the right choice in majoring in math since I love proofs so much ☺.

Four students wrote that they believed the assignment helped them to identify with mathematicians as they got to engage in the same activities as mathematicians. Vincent wrote,

> I had a lot of fun working on this second assignment because we are working on an unsolved proof like many other mathematicians. It is pretty cool to try and give one's own spin and perspective on the twin primes conjecture and if not prove it, at least come to the understanding of the conjecture and have some hands-on experience.

Dynamic

In terms of the dynamic nature of mathematical knowledge, see Figure 2 for the tallies of how many students reflected upon a particular theme. We found that eleven students either described how they learned about conjectures for the first time, understood better how mathematical knowledge is built, conjectured that new mathematics may be required to solve conjectures, and/or referenced discovery (some students were identified for more than one category).

Dynamic	11 (total)
Learning about conjectures for first time	5
Described building up of knowledge in mathematics	5
Believed new mathematics required to solve conjectures	4
Reference Discovery	6

Figure 3: Recurring Themes Related to Dynamic Nature of Mathematics

Vanessa and at least four other students learned about conjectures for the first time through the assignments. Early in the semester she wrote, "I enjoyed learning about conjectures. I didn't know what conjectures were, but now I do." Near the end of the semester, after watching a video of a mathematician describing their work on their conjectures, Vanessa wrote "Mathematicians use these proofs to help prove other conjectures. As Maynard's proof was influenced by Zhang's proof. Eventually, Maynard's proof will be used to help prove other conjectures."

We found that four students hypothesized that new mathematics may be required to solve the conjectures. Lorenzo wrote, "The Collatz conjecture is unprovable… for now. There has to be some future mathematics that will prove it. However right now, in 2019, the Collatz conjecture remains unsolved." Five students referenced mathematical discovery in their final reflections. Daniel concluded, "Proofs are used by mathematicians to assist them in creating other proofs to eventually have a breakthrough that is groundbreaking in mathematics as well as the world."

Emotional Exploration

Almost every student in the course reflected on significant aspects of mathematical exploration. Five students experienced mathematics creatively for the first time and seven recognized the need to think outside the box in exploring the conjecture. Eight students reflected that the process allowed them to get a glimpse of the discipline of mathematics and what goes into the making of mathematics. Students also described a range of emotions from frustration to joy. We found that five students described frustrations but no positive emotions, 9 students described both frustration and positive emotions, whereas seven students described only positive emotions. See Figure 4 for a summary of themes related to exploration (some students were counted for multiple categories).

Exploration	24 (total)
View of Math as creative	5
Need to think outside the box or think critically	7
Looking at conjecture from different perspectives	5
Getting a glimpse of the discipline.	8
Contrasted conjectures with prior math experiences	11
Frustrated	14
Enjoyed/Excited/rewarding (positive emotions)	16

Figure 4: Recurring Themes Related to Exploration

Students were frustrated when they could not find patterns, or when they were at a loss for how to proceed in working on the conjectures. For example, consider this excerpt from Charity: "I feel lost. Working on this assignment feels like a road with no end. I don't feel confident in what I'm doing unlike when I'm working on what we're working on in class." Other students described similar challenges, as the assignment was different from their previous mathematical experiences. Cassandra found that she needed to look at math more creatively to meet the challenge of the assignment.

> My experience working on this conjecture thus far has been challenging. It is difficult to shift how you approach mathematics from just learning and applying formulas and theorems that define what you do to then try and figure it out yourself. This has made me look at math more creatively than I have before. This conjecture has also made me try and see more patterns in number sequences and how they relate to one another.

After a period of frustration, most students productively engaged in the conjecture by trying a variety of approaches. The quote from Carlos below illustrates how he found enjoyment in the experience and also developed a new perspective on mathematics.

> For the most part, I was able to get a solid foundation about number theory, primes, "3n+1" and all its uses, and also going down the rabbit hole of several given solutions on Collatz or twin primes. This assignment was fun. Really fun. Being able to think of math in a more analytical sense was like a breath of fresh air. Being able to break down, and to be able to show each part of a statement to be true or false. It's kind of eye opening that this is how to "read math" and "talk math."

In addition to considering how to read and talk math, other students described how they got a picture of what goes into the making of mathematical knowledge. Jennifer wrote,

> This assignment was very interesting to me because it had me learn more about why Collatz sequences have not been proven yet. This was a different experience for me compared to my prior mathematical experiences because usually I work with problems that have already been proven. So it was definitely interesting getting a glimpse of what professional mathematicians struggle with when trying to prove something for its first time.

Argumentation

While we have evidence that students had opportunities to enrich their perspectives on the nature of mathematics in ways that aligned with the first three categories of the IDEA framework, we did not find much evidence that students reflected on the fourth category—argumentation. Perhaps this is because the students only had one day in class for in-person sharing of our findings related to the conjecture. That day was a show and tell experience rather than a critiquing experience—the students did not have the opportunity for social validation of mathematical knowledge through the conjecture assignment.

Discussion

We are encouraged by the results of our study. We have some evidence that providing students a scaffolded opportunity to explore unsolved conjectures, and time to reflect on this activity, is a productive means to enrich students mathematical experiences/perspectives. While we chose the Twin Primes and Collatz conjectures due to our familiarity with them, we believe that any accessible conjecture could be used fruitfully with undergraduate students. This activity was implemented in a transition-to-proof course, but a similar activity could be used as a project in courses on analysis, number theory, abstract algebra, etc… It is especially important for future teachers in such courses to have opportunities to experience mathematics in non-traditional, collaborative activities that more closely align with the work of practicing mathematicians. By doing so, they will be better positioned to provide their future students with opportunities for authentic mathematical practice.

In this exploratory study, we used the IDEA framework as a conceptual framework in data analysis for the first time. In our analysis, we carefully looked for student reflections that expressed ideas that aligned with the categories of the IDEA framework. This was a lengthy qualitative process. Future studies may improve upon this methodology, perhaps creating quantitative assessments that can be used to measure the changes in students' perspectives of the nature of mathematics through novel classroom interventions.

We note that of the four categories in the IDEA framework, students had the opportunity to learn about each aspect of the nature of mathematics except that mathematics is socially validated through *A*rgumentation. Future research studies are needed to discover classroom interventions that help students experience and understand this aspect of the nature of mathematics.

References

Bairrington, B., & Okano, A. (2019). New experimental investigations for the 3x+ 1 problem: The binary projection of the Collatz map. *Rose-Hulman Undergraduate Mathematics Journal*, *20*(1), 4.

Boaler, J. (2015). *What's math got to do with it? How teachers and parents can transform mathematics learning and inspire success.* New York, NY: Penguin.

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching.* San Francisco, CA: Jossey-Bass.

Bleiler-Baxter, S. K., & Pair, J. D. (2017). Engaging students in roles of proof. *The Journal of Mathematical Behavior, 47,* 16-34.

Bleiler-Baxter, S. K., Pair, J. D., & Reed, S. D. (2021). Let's Create, Not Replicate: Supporting Students as Proof Creators through the Group Proof Activity. *PRIMUS, 31*(7), 760-774.

De Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24,*17-24.

Hersh, R. (1997). *What is mathematics, really?* New York, NY: Oxford University Press.

Lampert, M. (1990). *When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching.*American Educational Research Journal, 27(1), 29-63.

Lischka, A. E., Lai, Y., Strayer, J. F., & Anhalt, C. (2020). Developing mathematical knowledge in and for teaching in content courses. In G. W. Martin, B. R. Lawler, A. E. Lischka, & W. M. Smith (Eds.) The mathematics teacher education partnership: The power of a networked improvement community to transform secondary mathematics teacher preparation (pp. 119-141). Information Age Publishing.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form*. Bellevue literary press.

Pair, J. D. (2017). *The nature of mathematics: A heuristic inquiry.* (Doctoral dissertation, Middle Tennessee State University).

Pair, J. D. & Calva, G. (2020) Understanding the Roles of Proof through Exploration of Unsolved Conjectures. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* Mexico.

Pair, J.D., & Calva, G. (2021). Mathematics majors' exploration of famous conjectures in a transition-to-proof course. *PRIMUS*. https://doi.org/10.1080/10511970.2021.1983902

Rezgui, H. (2017). Conjecture of twin primes (still unsolved problem in number theory) an expository essay. *Surveys in Mathematics and its Applications*, *12*, 229-252.

Ryan, G. W., & Bernard, H. R. (2003). Techniques to identify themes. *Field methods*, *15*(1), 85-109.

JUSTIFICATIONS STUDENTS USE WHEN WRITING AN EQUATION DURING A MODELING TASK

Elizabeth Roan
Texas State University
eaw109@txstate.edu

Jennifer Czocher
Texas State University
czocher.1@txstate.edu

Literature typically describes mathematization, the process of transforming a real-world situation into a mathematical model, in terms of desirable actions and behaviors students exhibit. We attended to STEM undergraduate students' quantitative reasoning as they derived equations. Analysis of the meanings they held for arithmetic operations ($+, -, \cdot, \div$) *provided insight into how participants expressed real-world relationships among entities with arithmetic relationships among values. We extend the findings from K-12 literature (e.g., using multiplication to instantiate a rate) to STEM undergraduates and found evidence of new ways of justifying the usage of arithmetic operations (e.g., using multiplication to instantiate an amount).*

Keywords: Modeling, Undergraduate Education, Mathematical Representations

Mathematical modeling (hereon called modeling) provides a venue within which to promote STEM education, a stance taken by the US government (Committee on STEM Education, 2018, p. 17; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and researchers alike. Modeling is typically defined as the cyclical process of taking a real-world phenomenon, transforming that phenomenon into a mathematical representation (called mathematizing), working with that mathematical representation, and interpreting that work back into the real-world. The idea to use modeling activities to enrich STEM education is well established. Modeling helps students develop general competences and attitudes towards creative problem-solving while building feelings of self-reliance and competence, prepare to live and act as informed socially conscious citizens, use math to describe extra-mathematical situations, see a richer more comprehensive picture of mathematics, and acquire, learn, and keep mathematical concepts by providing motivation for and relevance of mathematical studies (Blum & Niss, 1991).

Literature on modeling competencies, and specifically mathematizing, typically frame the research as identifying some sort of "blockage" or that an action associated with a competency was "difficult" to perform (Brahmia, 2014; Galbraith & Stillman, 2006; Jankvist & Niss, 2020; Stillman & Brown, 2014). While these descriptive actions are an important step in fully understanding the complexity of mathematizing, we do not yet know why certain actions, such as writing an equation, is difficult for students. In particular, it is often unclear how or why a participant chooses to represent a real-world relationship with a given arithmetic operation ($+, -, \cdot, \div$). Understanding the justifications students use when writing down their equations will inform facilitator scaffolding of mathematizing and may also provide insight into when and how students ensure their resulting equations are adequate. We target this one aspect of mathematizing in order to make strides towards developing rich, empirically- and theoretically-grounded descriptions of students' mathematical reasoning during mathematizing.

Theoretical Framework

Previous studies have indicated that quantitative reasoning promotes building of models (Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Mkhatshwa, 2020) and is a lens with which to understand STEM undergraduates' reasoning while mathematizing during a modeling task (Carlson, Larsen, & Lesh, 2003; Czocher & Hardison, 2021; Larson, 2013). Czocher and Hardison (2019, 2021) reported findings of a task-based interview with a participant working on a modeling problem. Their retrospective analysis documented how the participant's model changed through the course of the interview by attending to quantities imposed onto the task situation, the relationships among those quantities, mathematical inscriptions, and changes to each of those elements. An interesting implication of this analysis is that the models that students could potentially make during a modeling task depend on (and are constrained by), the quantities the participant imposes onto the situation. Given the arguments and implications of the literature above, it is appropriate to use quantitative reasoning as a lens to study students' reasoning while mathematizing.

In the remainder of this section, we put together the theoretical constructs from quantitative reasoning in order to describe the justifications students used for certain arithmetic operations $(+, -, \cdot, \div)$ when writing an equation during a modeling task. "Quantitative reasoning is the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships" (Thompson, 1993). A quantity is a mental construct, whose creation is (often) effortful (Thompson, 2011). That effort is characterized by the act of someone conceptualizing an object that has some attribute and intending to measure that attribute. (Thompson & Carlson, 2017, p. 425). This means a quantity is a triple consisting of an object, attribute, and quantification (Thompson, 2011). Quantification in this instance means to conceptualize some object with an attribute that has a measure, and that measure has a proportional relationship with its unit (Thompson, 2011). Further, quantification can be operationalized as the set of operations an individual could enact on the attribute (Hardison, 2019). According to Ellis (2007), length, area, volume, cardinality, speed, temperature, and density are all attributes of some object that can undergo quantification. A quantity is made from an individual's conceptions of objects within the situation, rather than the objects or situations themselves (Ellis, 2007), further, each individual could quantify an attribute using differing sets of operations, therefore a quantity is idiosyncratic to the individual. Because a quantity is idiosyncratic to the individual, quantities are (cognitively) distinct from (mathematical) variables.

A quantitative operation is a conceptual operation where an individual creates a new quantity in relation to one (or more) already created quantities (Ellis, 2007; Thompson, 2011). Enacting a quantitative operation upon two quantities can be thought of as composing or combining two quantities (which could be denoted through arithmetic operations) to yield the new one. Thompson (2011) outlined the arithmetic operations associated with the quantitative operation as seen in his data with students k-12, see **Error! Reference source not found.**1.

Table 1 Arithmetic operations done to construct a new quantity (Thompson, 2011, p. 43).

Structure	Arithmetic Operation to Evaluate the Resultant Quantity
A quantity is the result of an additive combination of two quantities	Addition
A quantity is the result of an additive comparison of two quantities	Subtraction
A quantity is the result of a multiplicative combination of two quantities	Multiplication
A quantity is the result of a multiplicative comparison of two quantities	Division
A quantity is the result of an instantiation of rate	Multiplication
A quantity is the result of a composition of ratios	Multiplication

The left side of Table 1 describes the type of relationships between two quantities that results in a new quantity. In other words, this left-hand side is the schema of action used to apply arithmetic operations to symbols that represent quantities. A schema of action is an organized pattern of thoughts or behaviors (actions) that can be applied to different cognitive objects in different situations (Nunes & Bryant, 2021). For example, in k-12 literature, two schemas of action for addition are: putting together (i.e., quantity is the result of additive combination), and part-whole relations (i.e., a comparison of two quantities) (Nunes & Bryant, 2021). Additionally, a schema of action for multiplicative relationships could be a one-to-many correspondence for quantities with a fixed ratio relationship (i.e., a multiplicative combination of two quantities). There might be other schemas of action participants are using to apply structure to a real-world phenomenon. Structure here is defined to be the schema of action, and the arithmetic operations are the result of the imposition of the schema of action in the mathematical representation.

Research has shown that some schemas of action are not advantageous for combining quantities. As pointed out by Schwartz (1988), the notion of multiplication being a one-to-many correspondence does not work for cases such as (miles/hours * hours = miles), because iterating the relationship between miles and hours "number of hours times" cannot be done. Similarly, Brahmia (2014) argued that students who conceptualize multiplication as repeated addition are not prepared to conceptually understand products of quantities. Additionally, students who see division only as an operation that creates parts from a whole do not have the mental operations available to conceptualize ratio quantities (e.g., density, velocity). That is, students being able to effectively construct quantities in applied contexts must have strong conceptualizations of multiplication and division that differ from the schema of action of "repeated addition" and "creating parts from a whole" (Brahmia, 2014).

In order to describe the justifications participants hold for the equations they write down, we address the question: *What schema of action do STEM undergraduates use to justify their choices of arithmetic operations while mathematizing during a modeling task?*

Methods

This study was part of a larger study of facilitator scaffolding moves that foster participants' modeling competencies. For the present study, 11 STEM undergraduates participated in sets of individual cognitive task-based interviews through Zoom. Each participant saw at least six tasks over ten sessions. The tasks were modeling problems designed to scaffold participants' modeling activities by attending to quantitative reasoning and appealing to similarities in structures. We report findings from three participants' work on the *Cats and Birds* task (see below). Pattern, Neturo, and Khriss were all enrolled in, or had already taken, differential equations (DE) at the time. Pattern majored in civil engineering while both Neturo and Khriss majored in physics with a minor in mathematics. These three participants were chosen from the larger study to showcase schema of action that differ from those described in previous literature.

> *Cats & Birds.* Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian's Migratory Bird Center and the U.S. Fish and Wildlife Service (Raasch, 2013) concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral "unowned" cats but noted that their domestic peers "still cause substantial wildlife mortality." In this problem, we will build a model (step-by-step) that predicts the species' population dynamics, considering the interaction of the two species.

In this task, participants were asked to consider a back-yard habitat where birds are the primary prey for cats. The participants were then asked a series of questions with the aim of having them write an equation for the instantaneous rate of change of the bird and cat population. Here we report the students' work to build a differential equation of the bird population due only to cat predation. It is appropriate to study students' mathematization while modeling in dynamic tasks like Cats and Birds because dynamic tasks elicit dynamic reasoning, which is connected to quantitative reasoning (Keene, 2007).

Data were analyzed in four phases. First, the transcripts and written work were segmented according to changes in discussion topic (e.g, the interviewer asked a new question). Second, we identified the quantities the participant imposed onto the task scenario by describing the object, attribute and how the participant exhibited quantification for that attribute according to the quantification criteria developed by Czocher and Hardison (2021). In phase three, we noted instances of arithmetic operations used on the quantities the participant constructed and documented the participants' reason (or inferred reason when the participant's reason was not stated) for using that specific arithmetic operation and if the instance is a quantitative operation, that is, the arithmetic operation is done to create a new quantity. In phase four, we compared the reason we inferred the participant was using to the schema of action documented in Thompson (2011) and Nunes and Bryant (2021) and noted if the schema of action matched previous literature or was not present in previous literature. For example, Neturo stated that he used multiplication because one cat will meet many birds. Neturo's explanation was comparable to a one-to-many correspondence as described by Brahmia (2014) and Nunes and Bryant (2021).

Results

We report the schema of action participants employed when expressing arithmetic operations with the quantities they imposed onto the Cats and Birds task as they built their model for the rate of change of the bird population due only to cat predation. According to our theoretical perspective, arithmetic operations are the result of imposing the schema of action in the mathematical representation and thus tell us how the participant was justifying (either explicitly or implicitly) their usage of arithmetic operations. First, we provide an overview of all of the inferred schema of action exhibited by our participants in Table 2. This table reports the quantities the participant combined, the resultant quantity, the inferred schema of action, and if this schema of action was present or not present in previous literature.

Table 2 Schema of action exhibited by all three participants

Quantity 1	Quantity 2	Resultant quantity	Schema of action	In previous literature?
Number of birds at time t	Number of cats at time t	Number of total possible cat-bird interactions that could occur in the back yard habitat.	Multiplication- One-to-many correspondence (Brahmia, 2014; Nunes and Bryant, 2021,)	Yes
Rate of encounters between cats and birds per time interval	Number of total possible cat-bird interactions	Number of encounters per unit time	Multiplication- Instantiation of a rate. (Thompson, 2011)	Yes
Percentage of encounters that are realized	Number of total possible cat-bird interactions	Number of encounters that are realized	Multiplication- Subsetting	No
Number of encounters that result in a death	Duration of time	Number of birds that die during Δt	Multiplication- Instantiation of an amount.	No
Total accumulated dead birds during a time segment	Total number of observations of the population made during a time segment	Average change in the bird population	Division - Separating into equal parts. (Brahmia, 2014)	Yes
Change in the bird population due to cats	Number of birds that die during Δt	Change in the bird population due to cats per change in time	Division - Evaluate a quantity that is an operand of a quantitative operation. (Thompson, 2011)	Yes
Number of birds present at the beginning of a time segment Δt	Number of birds at the end of a time segment Δt	Change in the bird population due to cats	Subtraction - Additive comparison between two quantities (Thompson, 2011)	Yes

Quantity 1	Quantity 2	Resultant quantity	Schema of action	In previous literature?
Number of lethal encounters between a cat and a bird on day t_1	Number of lethal encounters between a cat and a bird on day t_n	Total accumulated dead birds during a time segment	Addition - Additive combination of multiple quantities	No

Overall, our participants did exhibit schema of action described in previous literature. This indicates that schemas of action present in k-12 literature are also exhibited by undergraduate STEM students. We next illustrate the three schemas of action not present in the previous literature.

Schema of action not present in the literature

Additively combining multiple quantities. Khriss was asked how he could model the decrease in the bird population due to cats during some arbitrary passage of time $\Delta\tau$. Khriss initally reponded by writing down the symbol Σ, indicating he was thinking of constructing some form of summation. The interviewer suggested he find the decrease in the bird population due to cats after five days, instead of an arbitrary duration of time. In Khriss' reasoning, the number of birds dying each day was variable. He used $LE(t)$ to represent the number of lethal encounters between a cat and a bird at a specific time t. In response to the prompt, Khriss wrote $\int_0^5 LE(t)dt$, denoting summation. Khriss understood the decrease in bird population during five days as an accumulation of the non-constant values of the number of lethal encountes on a specific day. Khriss used this schema of action again when he was asked to consider the average rate of change of the bird population during some number of days D. In reponce to this, Khriss labeled multiple instances of time that the number of lethal encounters were recorded that occured during the number of days D. He labeld these instances of time $t_1, t_2, t_3, \dots, t_n$. Khriss then added the number of lethal encounters for each day to yeild the total accumulated dead birds during some number of days D. The equation written for this calculation was $\sum_{t=1}^{n} LE(t)$. A snapshot of Khriss's work is provided to showcase how Khriss thought of this sumation (see Figure 1), and below we provide a quote of Khriss explaning his summation.

Khriss: So during a segment of time D, to find an average I guess we would need multiple readings of what the population is throughout that time. I guess we could we use t_1, t_2, t_3, t_n. Yeah, I guess that would be it. Just to make sure what question I'm answering here, I have lethal encounters at time t adding up all of those for each different t all the way to my last t and dividing that by the number of measurements gives me an average. So that gives me my average number of dead birds within t.

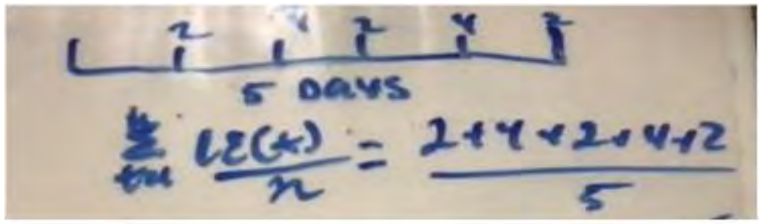


Figure 1: Average change in bird population over five days according to Khriss.

His schema of action for both of these instances was additively combining multiple quantities to arrive at a new quantity. The schema of action described in Thompson (2011) occurs strictly between two quantities. Because Khriss is adding more than one quantity at a time, Khriss' schema of action is different. We infer that Khriss' schema of action is a step up in complexity from the schema described in Thomspon (2011), because Khriss is able to implicetly construct each measure of dead birds between $LE(t_1)$ and $LE(t_n)$ in order to combine an unknown number of quantites additevely.

Multiplication for subsetting. Pattern had previously constructed a quantity that represented the number of total encounters, which is calculated by $B(t) \cdot C(t)$. In order to account for the fact that not all birds encounter all cats, he constructed two new quantities. The first quantity represented the percentage of encounters that are realized, labeled α, and the second quantity represented the number of encounters that are realized, calculated by $\alpha \cdot C(t) \cdot B(t)$. Pattern said that he multiplied α and $C(t) \cdot B(t)$ to take a percentage of the total possible encounters, indicating the schema of action for this instance of multiplication was different than a one-to-many correspondence, and was more like subsetting from a larger amount.

> Pattern: Okay. So now that we have a percentage, then you just do $C(t)$ times $B(t)$ times α equals encounters. Because you're going to take… So this is the total possible encounters that could possibly happen if perfect conditions are met for each cat to meet each bird, and then you're going to take a percentage of that total, and that would be your total...

Multiplication for instantiating an amount. Pattern had previously constructed a quantity that represents the number of birds that have died, calculated by $\beta \cdot C(t) \cdot B(t)$. Pattern then constructed a new quantity that represented the number of birds that die during Δt, labeled D_t, which was calculated by $\left(C(t) \cdot B(t)\right) \cdot \beta \cdot \Delta t$. Pattern multiplied the amount of birds that have died, which he did not view as connected to any specific duration of time, by some duration of time to yield the number of birds that have died during Δt. The schema of action used for multiplication was combining two quantities, however this was not a one-to-many correspondence and was also different from the subsetting schema of action. We infer that Pattern was multiplying by Δt in order to instantiate the number of birds that have died specifically during Δt. Pattern did not evidence multiplying a rate and a duration of time to yield an amount (per-time rate $\cdot\ \Delta t =$ amount) because Pattern did not associate a duration of time with $\beta \cdot C(t) \cdot B(t)$. For Pattern, the quantity calculated by $\beta \cdot C(t) \cdot B(t)$ was an amount, not a rate. The quote below shows Pattern explaining that multiplying by Δt transforms $\beta \cdot C(t) \cdot B(t)$ into the quantity that represents mortality (the number of birds that die) during Δt,

> Pattern: I honestly just know that I have to add [append] this Δt, and it kind of makes sense in my head because this mortality doesn't have a specific time attached to it. It (referring to $C(t) \cdot B(t)$) has already the t, but that's just telling you the amount of cats and birds at that time. It doesn't tell you how much time has passed. So if you multiply all of that by

this number (pointing at Δt),… this is time that has passed….It's not encounters anymore, now it's.. What this is telling me the mortality during Δt.

Discussion

Collectively, our participants exhibited schema of action discussed previously in the literature. Pattern constructed a quantity by additively comparing between two quantities. Khriss used division when finding the average by separating a quantity into equal parts. Pattern and Khriss used division to evaluate a quantity that is an operand of a quantitative operation. Neturo, Pattern, and Khriss used multiplication to combine to quantities in a one-to-many correspondence fashion. Lastly, Khriss used multiplication to instantiate a rate. All of these schemas of action are documented in previous literature (Brahmia, 2014; Nunes & Bryant, 2021; Thompson, 2011). Our participants also exhibited schema of action not present in the previous literature. Khriss used addition to additively combine multiple quantities instead of just two quantities. Neturo used a minus sign to indicate a negative magnitude for a quantity. Khriss showcased a new schema of action where he simultaneously additively combined multiple quantities in order to arrive at a new quantity. Khriss' schema of action is a step up in complexity from the schema described in Thompson (2011) because Thompson's definition is restricted to combining two quantities and Khriss is implicitly combining an unknown number of quantities. Pattern constructed a quantity that indicated a fraction or percentage and then multiplied by a quantity that represents an amount to gain some fraction of the original quantity. This schema of action does resemble the schema of action described in Thompson (2011) as a multiplicative combination of two quantities, however the schema of action for multiplication was different from a "one-to-many" correspondence described by Brahmia (2014) and Nunes and Bryant (2021). Pattern used multiplication in order to take a subset from a larger whole. Lastly, Pattern exhibited a new schema of action where multiplication was used to combine two quantities in order to instantiate an amount.

Overall, our results document justifications STEM undergraduates used when carrying out arithmetic operations ($+, -, \cdot, \div$) on the quantities while mathematizing that were not previously in the literature. We extended what is known about schema of action present in k-12 literature by confirming the presence of schema of action already described, and by introducing three new schemas of action. We postulate that these new schemas of action were observable because our participants are working on a modeling task focused on a predator-prey relationship. This conjecture is based on findings that the models participants make depend on (and are constrained by), the quantities the participant imposes onto the situation (Czocher & Hardison, 2019, 2021), which inherently impacts the types of relationships the participants were able to express using arithmetic operations. Attending to these new schemas of action reveals how students are expressing real-world relationships among entities with arithmetic relationships among quantities, specifically in a predator-prey context. We speculate additional schema of actions could be observed for arithmetic operations in other task scenarios that call for advanced mathematics. Once we know more about the justifications students use when utilizing arithmetic operations, we can then design task and scaffolding moves that directly address, preempt, or build upon, those justifications.

Acknowledgments

Research that will be reported in this paper is supported by National Science Foundation Grant No. 1750813 with Dr. Jennifer Czocher as principal investigator.

References

Blum, W., & Niss, M. (1991). Applied Mathematics Problem Solving, Modelling, Applications, and Links to Other Subjects: State, Trends and Issues in Mathematics Instruction. *Educational Studies in Mathematics, 22*(1), 37-68.

Brahmia, S. M. (2014). *Mathematization in Introductory Physics.* (Doctor of Philosophy). The State University of New Jersey,

Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating a models and modeling perspective with existing research and practice. *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*, 465-478.

Committee on STEM Education. (2018). *Charting A Course For Success: America's Strategy For STEM Education*. The White House

Czocher, J., & Hardison, H. (2021). Attending to Quantities through the Modelling Space. In F. Leung, G. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West* (pp. 263-272): Springer.

Czocher, J., & Hardison, H. (2019). *Characterizing Evolution of Mathematical Models.* Paper presented at the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, St. Louis, MO.

Ellis, A. (2007). The Influence of Reasoning with Emergent Quantities on Students' Generalizations. *Cognition and Instruction, 25*(4), 439-478.

Ellis, A., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the Jactus. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context, 2*, 93-112.

Galbraith, P. L., & Stillman, G. (2006). A Framework for Identifying Student Blockages during Transitions in the Modelling Process. *ZDM: Mathematics Education, 38*(2), 143-161.

Hardison, H. L. (2019). *Four Attentional Motions Involved in the Construction of Angularity.* Paper presented at the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Jankvist, U. T., & Niss, M. (2020). Upper secondary school students' difficulties with mathematical modelling.

Larson, C. (2013). Modeling and quantitative reasoning: The summer jobs problem. In R. Lesh, P. L. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 111-118). Boston, MA: Springer.

Mkhatshwa, T. P. (2020). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical thinking and learning, 22*(2), 139-161.

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common Core State Standards Mathematics*. Washingtion, DC: National Governors Association Center for Best Practices

Council of Chief State School Officers.

Nunes, T., & Bryant, P. (2021). *Using Mathematics to Understand the World: How Culture Promotes Children's Mathematics*. London and New York: Routledge.

Schwartz, J. L. (1988). Intensive quantity and referent transforming arithmetic operations. *Research Agenda for Mathematics Education Number Concepts and Operations in the Middle Grades, 2*, 41-52.

Stillman, G., & Brown, J. P. (2014). Evidence of implemented anticipation in mathematising by beginning modellers. *Mathematics Education Research Journal, 26*, 763-789.

Raasch, C. (2013, January 29). Cats kill up to 3.7B birds annually. *USA Today*. https://www.usatoday.com/story/news/nation/2013/01/29/cats-wild-birds-mammals-study/1873871/

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

Thompson, P. W. (2011). Quantitative Reasoning and Mathematical Modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 3357). Laramie, WY: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). *Variation, Covariation, and Functions: Foundational Ways of Thinking Mathematically*. Restion, VA: National Council of Teachers of Mathematics.

EXAMINING HOW UNDERGRADUATE STUDENTS DESCRIBE THE STANDARDS FOR MATHEMATICAL PRACTICE

Elyssa Stoddard
SUNY Oneonta
elyssa.stoddard@oneonta.edu

Rebekah Elliott
Oregon State University
rebekah.elliott@oregonstate.edu

While the Common Core State Standards for Mathematical Practice are a focal point of K-12 mathematics education, there is limited research examining how future teachers' (e.g., undergraduate students, teacher candidates) develop their conceptions of these standards. We investigate how opportunities within a mathematics-focused bridge course within a teacher education program provided opportunities for undergraduate students to develop their conceptions of the Standards for Mathematical Practice. Specifically, we explore how undergraduate students drew upon the Common Core provided descriptions to describe key practice ideas. This study contributes to the scholarship on mathematics teacher education and how teacher educators can support students in developing their understanding of mathematical practice.

Keywords: mathematical practices, mathematics teacher education

The Common Core State Standards for Mathematical Practice (SMPs) describe eight ways of thinking and doing mathematics that parallel the ways that mathematicians engage with mathematics (National Governors Association for Best Practices, 2010). These standards reflect a broader goal in K-12 mathematics education focused on moving beyond K-12 learners *acquiring* mathematical content knowledge, towards developing learners capable of actively *doing* something with mathematics (Kilpatrick et al., 2001; National Council of Teachers of Mathematics, 2018). Implicit in the creation and adoption of the SMPs is the assumption that if K-12 learners are to develop these skills and ways of thinking, their mathematics teachers should provide appropriate opportunities and support to develop and engage in these practices.

Initial teacher education offers a space where mathematics teacher educators can provide opportunities for undergraduate students to develop their conceptions about the SMPs including what they are (Bostic & Matney, 2014; Kruse et al., 2017), what doing an SMP looks like (Max & Welder, 2020), and how to support K-12 learners in doing them as well (Cheng, 2017; Gurl et al., 2016). While there are studies of how initial teacher education provides opportunities for undergraduate students to develop their conceptions of the SMPs, these studies are limited and there remain questions regarding these opportunities. In particular, because of the limited literature, it is unclear how different initial teacher education contexts (e.g., content courses, methods courses, bridge courses, student teaching) provide opportunities for undergraduate students to develop their conceptions of the SMPs. Understanding the differences, affordances, and limitations of the opportunities in these contexts is essential if teacher educators are to effectively support (future) teachers in providing K-12 learners with the appropriate opportunities and supports.

This paper shares how a mathematics-focused bridge course within a teacher education program provided opportunities for undergraduate students (hereto called 'students') to develop their conceptions of the SMPs. We discuss how students described the SMPs, including what ideas from the Common Core SMP descriptions students attended to in their own descriptions. We then discuss questions for further research based on these findings. We add to the scholarship on mathematics teacher education and how teacher educators can support their students in building their understanding of mathematical practices.

Framework

To understand how a bridge course provided opportunities for students to develop their conceptions of the SMPs, this study draws upon situative perspectives on learning (Greeno, 1998; Peressini et al., 2004) and sensible belief systems (Hoyles, 1992; Leatham, 2006). Specifically, these perspectives are used to examine how course learning opportunities supported the development of students' knowledge and beliefs about the SMPs. The following sections provide an overview of situative perspectives on learning, including situated knowledge and beliefs, as well as the notion of sensible belief systems.

Situative perspectives on learning

Situative perspectives conceptualize learning as both a social and individual process (Greeno, 1998; Peressini et al., 2004). Learning is social because one learns by actively participating within a context, and through this participation comes to learn the knowledge and accepted social practices of that context. Learning is also an individual process because it is changes in how an individual participates that indicate learning, as one's actions begin to reflect the accepted ways of participating in a given context. Scholars such as Putnum and Borko have argued that due to the variety of contexts within which initial teacher education occurs (e.g., content courses, methods courses, bridge courses, student teaching), applying a situative perspective to teacher education of offers a way of, "disentangling – without isolating - the complex contributions of these various contexts to novice teachers' development" (2000, p. 71).

In addition to knowledge, teachers' beliefs are also situated (Green, 1971; Hoyles, 1992; Leatham, 2006). In particular, the idea of *clustering* describes how beliefs can be connected to or isolated from one another based on the clusters within which they are held (Green, 1971), with clusters being based on the contexts within which the beliefs were formed. The idea of clustering is essential to making sense of beliefs because it describes how someone may hold beliefs that appear contradictory because they are held in different clusters. This clustering allows for the contextualization of beliefs where "a person may believe one thing in one instance and the opposite in another" (Leatham, 2006, p. 95) In other words, beliefs, like knowledge, are situated.

Sensible Systems of Beliefs

Due to the situated nature of beliefs, Leatham (2006) argues that how teachers' beliefs (and actions) are positioned in the literature needs to shift. He argues that if beliefs are situated then teachers' beliefs and actions are sensible with respect to the context within which they are working. Furthermore, any perceived 'inconsistencies' between teachers' beliefs and actions are indications that there are *other* beliefs at play in that particular context that are taking precedent over others; these beliefs may be held consciously or unconsciously and may be difficult for teachers to articulate. However, regardless of whether a belief is made explicit or not, teachers' actions are "fundamentally sensible" (Hoyles, 1992, p. 37) with respect to their beliefs.

Applying a sensible systems lens to teachers' beliefs pushes researchers to better understand the beliefs that are actually influencing teachers' actions rather than what we want to be the influence (Leatham, 2006). By understanding the actual influences on teachers' actions, teacher educators can better provide opportunities for students in teacher education courses to explore their beliefs, as well as provide learning opportunities that can shape students' beliefs so that what is sensible reflects the broader goals of K-12 mathematics education policy.

Conceptions

Finally, while knowledge and beliefs can be discussed separately, some scholars use the notion of conceptions to capture knowledge and beliefs together (Lesseig & Hine, 2021; Philipp, 2007; Thompson, 1992). This is because while knowledge and beliefs can be considered two distinct ways of knowing (beliefs are held with varying degrees of conviction while knowledge is held with certainty), distinguishing between them can be difficult. For example, the level of conviction with which one holds beliefs and knowledge can vary between people (Philipp, 2007). Moreover, what is considered knowledge to one person may be belief for another. Therefore, this study uses the notion of conception to capture students' knowledge and beliefs about the SMPs without trying to disentangle them. More specifically, this study investigates how a bridge course within a teacher education program provided opportunities for undergraduate students to develop their conceptions of the SMPs.

Context and Methods

Data for this study were collected from two sections of a mathematics-focused bridge course focused on familiarizing undergraduate students interested in becoming teachers with K-12 mathematics education policy. In five two-week modules, students in this course were introduced to the Common Core SMPs and content standards (NGA, 2010), the five strands of mathematical proficiency (Kilpatrick et al., 2001), and other key instructional ideas such as multiple representations or classroom discourse (see Table 1). To support students' developing understanding of the SMPs, the course provided opportunities for students to solve mathematical tasks, analyze their own mathematical work and the work of others for evidence of the SMPs, and reflect on their growing understanding of the SMPs and their relation to other course topics. Through activities such as these, course learning opportunities acted as a *bridge* between students thinking about *mathematics* and thinking about *teaching and learning mathematics*.

Table 1: Bridge Course Module Overview

Module	SMP	Course Topic	Strand of Proficiency
1	CCSSM & SMP Overview	Mathematical Identity	Productive Disposition
2	SMP 2: Reasoning SMP 6: Precision	Multiple Representations	Conceptual Understanding
3	SMP 7: Structure SMP 8: Repeated reasoning	Discourse	Procedural Fluency
4	SMP 3: Argumentation SMP 5: Tools	Justification	Adaptive Reasoning
5	SMP 1: Problem-solving SMP 4: Modeling	Cognitive Demand	Strategic Competence

Eight students consented to participate from the two sections of the course. Section one took place online, asynchronously during Winter 2021, with two of 12 enrolled students consenting to participate; Section two took place online, synchronously during Spring 2021 with six of 35 enrolled students consenting to participate. Of the eight participants, two expressed interest in becoming secondary mathematics teachers, five in becoming elementary teachers, and one a K-12 guidance counselor; one student was employed as an elementary classroom assistant while another was completing their elementary student teaching practicum. While limited, the consenting students capture the range of backgrounds and interests enrolled in this course.

Methodology

The primary data sources for this study are two course assignments, one completed at the end of module 3 and another completed at the end of module 5 (*n=16*; two per student); these assignments were a part of the course and would have been completed irrespective of the study. Through assignment prompts, students were asked to describe and provide evidence of their developing understanding of the SMPs including a) how they would describe key SMP ideas, b) why the SMPs are important, c) connections between the SMPs and other course topics, d) examples of how they saw themselves or others doing the SMPs when completing mathematical tasks, e) how class activities have shifted their understanding of the SMPs, and f) remaining questions they had about the SMPs.

To analyze these data, we first read through all student assignments and segmented the data by structural codes that corresponded to the assignment prompts (Saldaña, 2013). This first phase of segmenting and coding indicated that of the assignment prompts, students most frequently provided *descriptions* of key SMP ideas (65 of 231 coded segments). Based on these frequencies, second phase coding focused on the *description* text segments identified in phase one. Second phase coding used Toulmin's model of argumentation (1958) and an adaptation of Nardi et al.'s (2012) classification of warrants to capture the sources students drew upon when describing the SMPs. Due to space, we only provide a brief overview of these frameworks and how they guided analysis; more detail will be provided in our presentation.

Briefly, Toulmin's model (1958) identifies six different components of an argument: 1) a *conclusion/claim*, 2) the *data* upon which the conclusion is based, 3) the *warrant* that connects the conclusion to the data, 4) a *backing* that further supports the warrant through additional reasoning or evidence, 5) *qualifiers* to express one's confidence in the argument, and 6) *rebuttals* that address possible exceptions or refutations to the argument. While many scholars have used Toulmin's model to analyze teachers' and students' mathematical and pedagogical arguments (Krummheuer, 2015; Steele, 2005; Yackel, 2001, 2002), this model also has limitations. In particular, while Toulmin offers a way to make sense of an argument's *structure*, it does not offer a way to make sense of an argument's *quality*. To address this, scholars have proposed classifying warrants to identify the sources of influence one may draw upon when making and supporting an argument (Freeman, 2005; Nardi et al., 2012). For example, Nardi and colleagues (2012) offer seven different types of warrants mathematics educators use in their arguments: 1) *a priori epistemological*, 2) *a priori pedagogical*, 3) *institutional curricular*, 4) *institutional epistemological*, 5) *empirical personal*, 6) *empirical professiona*l, and 7) *evaluative*. These categories capture the scope of influences on mathematics teachers' arguments including their own personal or professional experiences (*empirical*), personal views or beliefs (*evaluative*), curricular resources (*institutional curricular*), shared disciplinary practices (*institutional epistemological*), and established definitions and pedagogical principles (*a priori epistemological* and *pedagogical*).

This study uses an adaptation of Nardi et al's (2012) warrant classification to which we added *policy* classifications to capture the influence of the CCSSM content standards (*policy-content*) and SMP descriptions (*policy – SMP*) (NGA, 2010), and the strands of mathematical proficiency (*policy – strand*) (Kilpatrick et al., 2001). These additional classifications were created to explicitly capture students' attention to different K-12 mathematics education policies, as well as recognize the ongoing debate about the eight SMPs in the literature. Specifically, scholars have argued the SMPs provide an inaccurate picture of mathematical practices due to the methods used to identify them (e.g., self-reported autobiographical data) or because the wide range of practices used by mathematicians cannot be captured in such a small number of standards (Moschkovich, 2013; Weber et al., 2020). Therefore, while *institutional epistemological* warrants refer to shared disciplinary practices, we determined it would be inappropriate to classify the SMPs as 'shared' due to the ongoing debate and created this new categorization. In using this adapted categorization of warrants to analyze students' SMP descriptions, our analysis allowed us to see which sources students were drawing upon as they described the SMPs, including how they related to course learning opportunities and resources.

Findings and Discussion

Within the 65 text segments coded as *description* during first-phase coding, second-phase codes were applied to 165 smaller segments, with each segment representing a different idea within students' SMP descriptions. As illustrated in Table 2, the CCSSM SMP descriptions were overwhelmingly the most frequent influence on students' SMP descriptions (*policy – SMP*; *n=114*). As almost 70% of text segments were coded in this category, further analysis focused on which components or ideas from the CCSSM students drew upon in their descriptions.

Table 2: Frequency of Second-Level Code Application

Influence	Frequency
Policy – SMP	*114*
Institutional Curricular	32
Institutional Epistemological	7
Evaluative	7
A priori Pedagogical	3
Policy – Content	1
Empirical Personal	1

Influence of the CCSSM

Overall, students attended to a range of different ideas from the CCSSM when describing the SMPs, ranging from four different ideas for SMP 5: Tools to 15 different ideas for SMP 3: Argumentation. Closer analysis indicates that how students attended to these ideas differed across SMPs. Specifically, students' descriptions of SMP 1: Problem-solving and SMP 4: Modeling focused on the overarching idea(s) of the SMP, with less attention given to specific actions related to doing the SMP. Alternatively, students' descriptions of SMP 3: Argumentation and SMP 6: Precision focused on one particular way of doing these SMPs, even though the CCSSM describes each of these SMPs as involving multiple actions. Finally, descriptions of the remaining four SMPs generally focused on a smaller range of ideas, including overarching idea(s) as well as specific actions for doing the SMPs.

Overarching idea(s)

Table 3 illustrates the different ideas from the CCSSM descriptions of SMP 1: Problem-solving and SMP 4: Modeling that were attended to in students' descriptions. For SMP 1: Problem-solving, students most frequently attended to the overarching ideas of *make sense of problems* (*n=7*) and *planning* (*n=7*). The CCSSM description also includes *how* one may do this sense making, such as *creating representations, considering analogous problems*, or *looking for entry points,* however, these actions were some of the ideas least frequently included in students' descriptions. This is similar to students' descriptions of SMP 4: Modeling which primarily focused on the overarching idea of *connecting math and the real world* (*n=9*), including *solving real world problems* (*n=7*) and *using math in real world contexts* (*n=2*). However, as with problem-solving, students attended less to the actions involved in carrying out this overarching idea, such as *making assumptions* or *identifying important quantities*. Taken together, these findings suggest students' conceptions of SMPs 1 and 4 are largely focused on the overarching idea(s) of these SMPs, with less focus on specific actions involved in doing them.

Table 3: Ideas from the CCSSM included in students' SMP 1 and SMP 4 descriptions

SMP 1: Problem-solving		**SMP 4: Modeling**	
Make sense of problems	7	Connecting math & the real world	9
Planning	7	Solve real world problems	7
Evaluate own work	6	Apply or use math in real-world contexts	2
Revise/change plan	3	Create representation	3
Create representation	2	Make assumptions	2
Look for entry points	1	Interpret results with respect to context	1
Analogous problems	1	Identify quantities	1
Persevere	1	Make sense of relationships to draw conclusions	1
		Example of modeling	1

Specific actions

Students' descriptions of SMP 3: Argumentation and SMP 6: Precision primarily focused on *particular ways of doing* each of these SMPs. For example, the CCSSM describes SMP 3 as the ability to "construct viable arguments and critique the reasoning of others" (NGA, 2010). In other words, argumentation in mathematics includes both "give and take" where one builds and shares an argument but also makes sense of the arguments of others. From Table 4 we can see that students primarily focused on the "give" aspect of doing SMP 3, largely describing SMP 3 as justifying one's own solution and communicating that solution or justification to others; listening to, discussing, and critiquing the arguments of others were some of the least frequently raised ideas. Overall, these findings illustrate how students generally described SMP 3: Argumentation as one-directional, focusing on students constructing and sharing their own arguments.

Students' descriptions of SMP 6: Precision also focused on a particular way of doing this SMP. Students primarily described SMP 6 as precision with respect to communication, both in general and how definitions, mathematical language, labels, and calculations can support communication (see Table 4). Precise communication, including the formulation of clear explanations and use of definitions, are explicitly included in the CCSSM SMP 6 description. Interestingly, the CCSSM also describes SMP 6 as the ability to "calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the context of the problem" (NGA, 2010). Despite this explicit attention to precise calculations, only two students referred to calculations in their descriptions, with one student including accurate calculations as a *component of* communication. Furthermore, no students touched on the notion of the appropriateness of precision for the context of the problem. These findings suggest that students' conceptions of SMP 6 are largely focused on precision with respect to communication *about* mathematics, with less attention to precision when carrying out mathematical calculations or processes.

Table 4: Ideas from the CCSSM included in students' SMP 3 and SMP 6 descriptions

SMP 3: Argumentation		SMP 6: Precision	
Communicate solution or justification to others	11	Communication	14
Grade-level appropriateness of arguments	6	Definitions	4
Justify	4	Mathematical language	3
Example of engagement	3	Explain reasoning	3
Concrete referents	2	Examine claims	1
Create diagrams	1	Correct calculations	1
Listen to other	3	Units	1
Identify mistakes/flaws	2	General	1
Critique others' ideas, solutions, or justifications	2	Label graphs	2
Compare arguments	1	Units	2
Ask clarifying questions	1	Correct calculations/answer	1
Make conjectures	1	Clear work	1
Discuss others' solutions or justifications	1	Contextual meaning	1
Use definitions and established information	1		
Analyze problem	1		
Respond to others	1		

Remaining SMPs

Students' descriptions of SMP 2: Reasoning, SMP 5: Tools, SMP 7: Structure, and SMP 8: Repeated reasoning included both overarching ideas and actions related to doing each SMP; these descriptions also generally focused on a smaller range of ideas overall. For example, while the average number of different SMP ideas raised throughout students' assignments for SMPs 1, 3, 4, and 6 was 9, students raised an average of 4.75 different ideas for SMPs 2, 5, 7, and 8.

When describing SMP 2: Reasoning, students' descriptions included multiple ideas from the CCSSM including: contextualizing, decontextualizing, reasoning about quantitative relationships, and creating representations. Similarly, students' descriptions of SMP 5: Tools included both the ability to use tools and the ability to determine which tools are available and appropriate for a given problem, with each of these ideas occurring almost equally in students' descriptions (see Table 5).

As previously discussed, students sometimes focused on a particular action in their descriptions even if multiple actions were included in the SMP title itself (e.g., SMP 3: Constructing **and** critiquing arguments). Interestingly, this was not the case for SMP 7: Structure or SMP 8: Repeated reasoning, which both include looking for *and* making use of structure/ repeated reasoning. While the "look for" aspect of SMPs 7 and 8 was the most frequently raised by students (see Table 5), students included how to "make use" almost as frequently. For example, students' descriptions of SMP 7: Structure included *identifying underlying structures* *and patterns (n=5)* and how to make use of this structure via the *(de)composition* of mathematical objects almost equally *(n=4)*. This is similar to students' descriptions of SMP 8: Repeated reasoning which focused on *identifying patterns and repetition (n=10)*, as well as how to use that repetition to *generalize (n=6)* and *determine shortcuts (n=2)*.

Table 5: Ideas from the CCSSM included in students' SMP 2, SMP 5, SMP 7, and SMP 8 descriptions

SMP 2: Reasoning		**SMP 5: Tools**	
Decontextualize	4	Use tools	6
Contextualize	3	Identify appropriate tools	5
Make sense of quantitative relationships	3	Examples of tools	4
Create representations	3	Purpose of tools	2
Units	1		
Flexible use of operations	1		
SMP 7: Structure		**SMP 8: Repeated reasoning**	
Identify underlying structures or patterns	5	Identify patterns/repetition	10
Examples of using structure	5	Generalize	6
(De)composition	4	Reasonableness of results	3
Shift perspectives	2	Examples of looking for or using repeated reasoning	2
		Determining shortcuts	2

Interestingly, while students' descriptions of both SMP 7: Structure and SMP 8: Repeated reasoning included the idea of patterns, the word 'pattern' is only included in the CCSSM description for SMP 7; SMP 8 refers to noticing repeated calculations or regularity. Therefore, while students' descriptions suggest the CCSSM helped students understand what making use of structure and repeated reasoning entails, the frequent use of the word 'pattern' to describe both does raise questions about the similarity between SMPs 7 and 8.

Conclusion

This study explores how a mathematics-focused bridge course provided opportunities for undergraduate students to develop their conceptions of the SMPs. The influence of the CCSSM on students' descriptions suggests that course opportunities to read and discuss the CCSSM SMP descriptions supported students in developing their SMP conceptions. Furthermore, the ideas that students *did* include provides insight into which ideas students are holding onto and can act as a foundation upon which to build in future learning opportunities. In doing so, we hope this study will inform future research examining how different teacher education contexts provide learning opportunities for students to develop their SMP conceptions, as well as how later learning opportunities can build off conceptions established in previous courses.

References

Bostic, J., & Matney, G. T. (2014). Role-playing the Standards for Mathematical Practice: A professional development tool. *NCSM Journal*, 3–10.

Cheng, J. (2017). Learning to attend to precision: The impact of micro-teaching guided by expert secondary mathematics teachers on pre-service teachers' teaching practice. *ZDM*, *49*(2), 279–289. https://doi.org/10.1007/s11858-017-0839-7

Freeman, J. B. (2005). Systematizing Toulmin's Warrants: An Epistemic Approach. *Argumentation*, *19*(3), 331–

346. https://doi.org/10.1007/s10503-005-4420-0
Green, T. F. (1971). *The activities of teaching*. McGraw-Hill.
Greeno, J. G. (1998). The situativity of knowing, learning, and research. *American Psychologist*, 5–26.
Gurl, T. J., Fox, R., Dabovic, N., & Leavitt, A. E. (2016). Planning questions and persevering in the practices. *The Mathematics Teacher*, *110*(1), 33–39. https://doi.org/10.5951/mathteacher.110.1.0033
Hoyles, C. (1992). Mathematics teaching and mathematics teachers: A meta-case study. *For the Learning of Mathematics*, *12*(3), 32–44.
Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding It up: Helping children learn mathematics*. National Academy Press.
Krummheuer, G. (2015). Methods for reconstructing processes of argumentation and participation in primary mathematics classroom interaction. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to Qualitative Research in Mathematics Education* (pp. 51–74). Springer, Dordrecht.
Kruse, L., Schlosser, M., & Bostic, J. (2017). Shifting perspectives about the standards for mathematical practice. *Ohio Journal of School Mathematics*, *77*, 34–43.
Leatham, K. R. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education*, *9*(1), 91–102. https://doi.org/10.1007/s10857-006-9006-8
Lesseig, K., & Hine, G. (2021). Teaching mathematical proof at secondary school: An exploration of pre-service teachers' situative beliefs. *International Journal of Mathematical Education in Science and Technology*, 1–17. https://doi.org/10.1080/0020739X.2021.1895338
Max, B., & Welder, R. M. (2020). Mathematics teacher educators' addressing the common core standards for mathematical practice in content courses for prospective elementary teachers: A focus on critiquing the reasoning of others. *The Mathematics Enthusiast*, *17*(2 & 3), 843–881.
Moschkovich, J. N. (2013). Issues regarding the concept of mathematical practices. In Y. Li & J. N. Moschkovich (Eds.), *Proficiency and Beliefs in Learning and Teaching Mathematics* (pp. 257–275). SensePublishers. https://doi.org/10.1007/978-94-6209-299-0_15
Nardi, E., Biza, I., & Zachariades, T. (2012). 'Warrant' revisited: Integrating mathematics teachers' pedagogical and epistemological considerations into Toulmin's model for argumentation. *Educational Studies in Mathematics*, *79*(2), 157–173. https://doi.org/10.1007/s10649-011-9345-y
National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. National Council of Teachers of Mathematics.
National Governors Association for Best Practices. (2010). *Common Core State Standards Mathematics*. National Governors Association for Best Practices, CCSSO. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf
Peressini, D., Borko, H., Romagnano, L., Knuth, E., & Willis, C. (2004). A conceptual framework for learning to teach secondary mathematics: A situative perspective. *Educational Studies in Mathematics*, *56*(1), 67–96. JSTOR.
Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. *Second Handbook of Research on Mathematics Teaching and Learning*, *1*, 257–315.
Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher*, *29*(1), 4–15.
Saldaña, J. (2013). *The coding manual for qualitative researchers* (2. ed). SAGE Publ.
Steele, M. D. (2005). Comparing Knowledge Bases and Reasoning Structures in Discussions of Mathematics and Pedagogy. *Journal of Mathematics Teacher Education*, *8*(4), 291–328. https://doi.org/10.1007/s10857-005-0854-4
Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In *Handbook of research on mathematics teaching and learning* (pp. 127–146).
Toulmin, S. (1958). *The uses of argument*. Cambridge University Press.
Weber, K., Dawkins, P., & Mejía-Ramos, J. P. (2020). The relationship between mathematical practice and mathematics pedagogy in mathematics education research. *ZDM*, *52*(6), 1063–1074. https://doi.org/10.1007/s11858-020-01173-7
Yackel, E. (2001). Explanation, justification, and argumentation in mathematics classrooms. In M. v. d. Heuvel-Panhuizen (Ed.), *Proceedings of the Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 9–24).
Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *The Journal of Mathematical Behavior*, *21*(4), 423–440. https://doi.org/10.1016/S0732-3123(02)00143-8

OPPORTUNITIES FOR REASONING-AND-PROVING IN MATHEMATICAL TASKS: A DISCURSIVE PERSPECTIVE

Merav Weingarden
University of New Hampshire
Merav.weingarden@unh.edu

Orly Buchbinder
University of New Hampshire
orly.buchbinder@unh.edu

Jinqing Liu
University of New Hampshire
jinqing.liu@unh.edu

In this paper, we offer a novel framework for analyzing the Opportunities for Reasoning-and-Proving (ORP) in mathematical tasks. By drawing upon some tenets of the commognitive framework, we conceptualize learning and teaching mathematics via reasoning and proving both as enacting reasoning processes (e.g., conjecturing, justifying) in the curricular-based mathematical discourse and as participation in the meta-discourse about proof, which is focused on the aspects of deductive reasoning. By cluster analysis performed on 106 tasks designed by prospective secondary teachers, we identify four types of tasks corresponding to four types of ORP: limited ORP, curricular-based reasoning ORP, logic related ORP, and fully integrated ORP. We discuss these ORP and the contribution of this framework in light of preparing beginning teachers to integrate reasoning and proving in secondary mathematics classrooms.

Keywords: Reasoning and Proof, Instructional Activities and Practices, Classroom Discourse, Preservice Teacher Education

Introduction

Reasoning and proving are mathematical *processes* such as identifying patterns, generalizing, conjecturing, and justifying, which are at the heart of mathematics (Ellis et al., 2012; Jeannotte & Kieran, 2017; Stylianides, 2008). The role of reasoning and proving in mathematics classrooms is to support students' sense-making and meaningful learning of mathematics (Hanna & deVillers, 2012; NGA & CCSSO, 2010; NCTM, 2009; 2014). This emphasis is embodied in the notion of Proof-Based Teaching (Reid, 2011) and in Buchbinder & McCrone's (in press) Teaching Mathematics via Reasoning and Proving (TMvRP) framework. The three guiding principles of TMvRP are: (a) integration of reasoning and proving within the mathematics curriculum; (b) emphasis on deductive reasoning for producing and validating mathematical results, and (c) use of precise mathematical language but within the conceptual reach of the students. These principles seem straightforward but require operationalizing. For example, how can reasoning and proving be integrated within the curriculum? And how an emphasis on deductive reasoning can look like in student learning? In particular, teaching mathematics via reasoning and proving requires instructional materials and tasks enabling teachers to engage students with reasoning and proving. Mathematical tasks constitute one of the main sources of student learning (Watson & Ohtani 2015). Teachers design learning opportunities for students by choosing, adapting, or creating mathematical tasks (Brown, 2009; Remillard, 2005), therefore opportunities for students to engage with reasoning and proving vary across tasks. This leads to a question: How can a task's potential for engaging students with reasoning and proving be assessed and characterized? Studies that examined the opportunities for reasoning and proving in tasks tend to focus on the processes students can enact while engaging with the task, such as identifying patterns or making conjectures (Davis, 2012; Thompson et al., 2012). Jeannotte and Kieran (2017) systematized these mathematical reasoning processes (e.g., generalizing, validating, proving) into a taxonomy, carefully defining each concept. However, when considering *how* students engage with such processes, research has identified persistent difficulties in understanding and applying deductive reasoning, which often deviates from reasoning outside mathematics (Harel & Sowder, 2007; Stylianides et al. 2017). Thus, it is important for teachers to expose students to deductive reasoning. This entails a different genre of mathematical tasks, whose learning opportunities for reasoning and proving are not captured by Jeannotte and Kieran's (2017) framework. This different genre of tasks aligns with the Pedagogical Framework for Teaching Proof proposed by Cirillo and May (2020), which delineates logic-related processes that focus on the nature of proof, its structure, rules of logic, the structure of theorems and their use.

These two frameworks provide valuable but different insights into important aspects of teaching mathematics via reasoning and proving, yet there seems to be no unified framework. Our objectives in this paper are two-fold. First, we offer a novel framework for analyzing, classifying, and characterizing tasks and the *Opportunities for Reasoning-and-Proving (OPR)* embedded in them. We rely on the tenets of the commognitive perspective (Sfard, 2008) to combine into a unifying *ORP Task Analysis Framework* the mathematical reasoning processes described by Jeannotte and Kieran (2017), and the logical aspects of deductive reasoning offered by Cirillo and May (2020). The discursive perspective of commognition allows characterizing ORP embedded in a task by identifying the type of discourse to which a task belongs: curricular-based mathematical discourse or meta-discourse about proof. Second, we illustrate the utility of *ORP Task Analysis Framework* by analyzing a corpus of tasks designed by prospective secondary teachers (PSTs) enrolled in a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020a). We show the variation in ORP afforded by different types of tasks and discuss its potential application to broader contexts.

The Opportunities for Reasoning-and-Proving (ORP) Task Analysis Framework

The commognitive framework views learning as participating in discourse while learners communicate about objects (Sfard, 2008). Working within the commognitive perspective, Jeannotte and Kieran (2017) developed a conceptual model of mathematical reasoning for school mathematics. The model describes nine *mathematical reasoning processes* such as identifying a pattern, generalizing, conjecturing, justifying, validating, and proving. These meta-discursive processes "derive narratives about mathematical objects or relations by exploring the relations between objects" (Jeannotte & Kieran, 2017, p. 9). For example, generalizing is "a process that infers narratives about a set of mathematical objects or a relation between objects of the set from a subset of this set" (Jeannotte & Kieran, 2017, p. 9). The mathematical reasoning processes can be enacted on mathematical objects (e.g., numbers, equations, geometric figures, etc.), which are at the core of various mathematical discourses. For example, tasks in which students are asked to solve an equation and explain their answer focus on *algebraic expression* mathematical object and thus belong to the discourse on algebra (Caspi & Sfard, 2012). The object at the core of a task can be identified by examining what the task is about and what the task asks to find. As students engage with the task they operate on mathematical objects and enact different types of processes (routines in Sfard's terms), which are the "patterns of action that appear when people participate in discourse" (Lavie & Sfard, 2019, p. 6). For example, in the discourse on algebra, students formalize, distribute and group like terms. Identifying the objects at the core of a task and the processes students can enact while engaging with the task, can determine the discourses to which the task belongs. In this paper we define *curricular-based mathematical discourse* as a discourse that focuses on curricular-based mathematical objects. Students participating in a curricular-based mathematical discourse can engage with reasoning and proving by enacting mathematical reasoning processes (e.g., conjecturing, justifying, validating) on the curricular-based mathematical objects, such as linear functions or complementary angles.

However, there are other ways to provide students ORP. Justifying, validating, and other mathematical reasoning processes are rooted in deductive reasoning, which poses multiple difficulties to students (Stylianides et al., 2017). Therefore, it is important to provide students with opportunities to engage with the important elements of deductive reasoning such as analyzing the structure of a theorem or a conjecture, or discussing the generality of proof. These can involve activities such as identifying the hypothesis and the conclusion of a conditional statement, writing the converse of a conditional statement, and identifying or constructing counterexamples (Buchbinder & McCrone, 2020b; Cirillo & May, 2020). Such logic-related processes are not captured by Jeannotte and Kieran's model of reasoning processes since they are not directly related to the deriving of a new mathematical narrative about mathematical objects. Rather, these logic-related processes are enacted on logic-related objects such as conditional statements that are at the core of what we define as *meta-discourse about proof.*

The object at the core of the task and the processes students can enact while engaging with the task determine the type of discourse the task belongs to. By drawing upon the discourses to which the tasks belong, we conceptualize the ORP provided by the task. These ORP can be embedded in the meta-discourse about proof, where students can enact logic-related processes on logic-related objects, or in the curricular-based mathematical discourse, where students can enact mathematical reasoning processes. This classification is illustrated in Figure 1, which summarizes the *ORP Task Analysis Framework*.

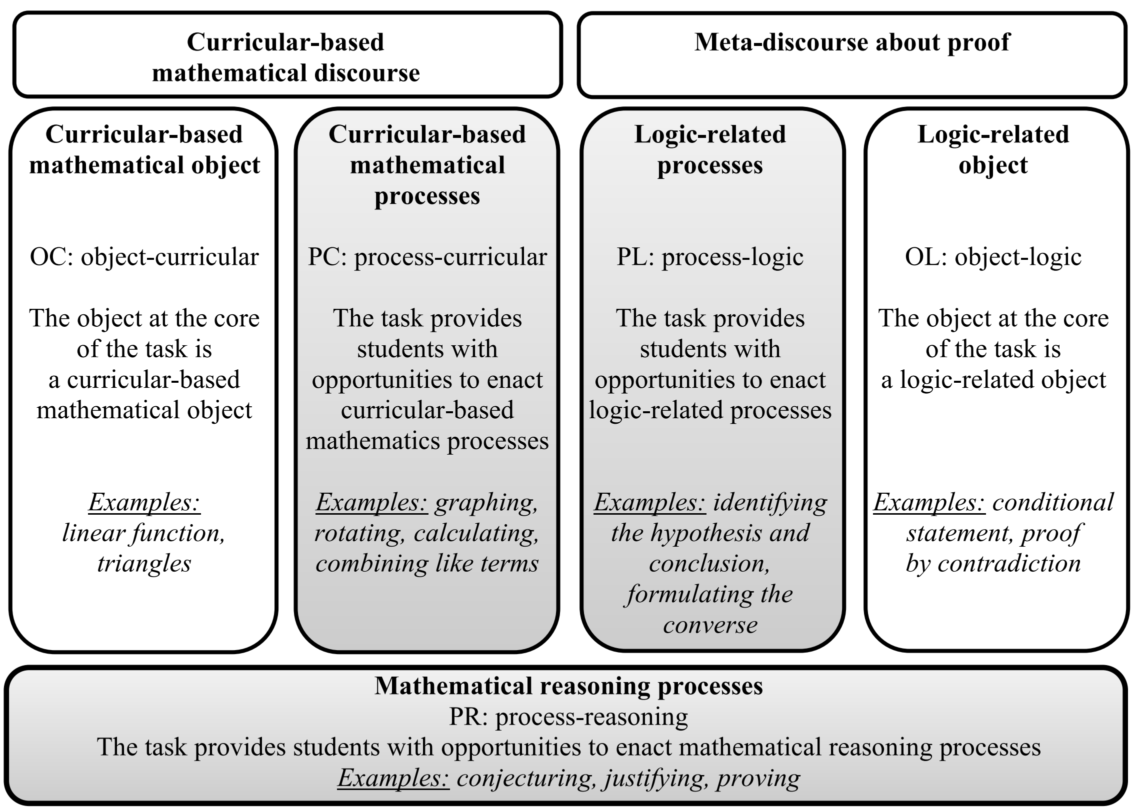


Figure 1: The OPR Task Analysis Framework

In what follows, we illustrate the application of the ORP Framework to identify types of ORP in tasks designed by prospective secondary teachers (PSTs) enrolled in a capstone course *Mathematical Reasoning and Proving for Secondary Teachers*. Our exploration was guided by the following research question: W*hat types of ORP can be identified in tasks designed by PSTs who learn how to teach mathematics via reasoning and proving?*

Application of the ORP Framework

Context: PSTs Learning to Teach Mathematics via Reasoning and Proving

This study is part of a larger research project which investigates beginning teachers' expertise to teach mathematics via reasoning and proving (Buchbinder & McCrone, in press). The first stage of this project focus on PSTs participating in a university-based capstone course: *Mathematical Reasoning and Proving for Secondary Teachers*. The course comprised four modules, which focused on: (1) direct proof and argument evaluation, (2) conditional statements, (3) roles of examples, and (4) indirect reasoning. Each module began with activities belonging to the meta-discourse about proof and contained activities engaging PSTs in integrating the proof themes with the regular curriculum. At the end of each module, each PST designed and taught in a local school a lesson that integrated a particular proof theme with the ongoing mathematical topic from the secondary school curriculum.

Data Corpus and Analysis

We analyzed 12 lesson plans designed by three PSTs who participated in the capstone course in 2021 Fall and consented to patriciate in the study. The analysis focused on mathematical tasks in these lesson plans. Since some tasks included one question while others had multiple questions, we chose the unit of analysis to be a single question – the smallest unit in which students were asked to come up with any type of answer. Overall, we analyzed 106 questions.

The analysis proceeded in several stages. First, we identified the object at the core of the question – namely, what is the question about? Questions that focused on curricular-based mathematical objects (e.g., isosceles triangles) were coded as OC (object-curricular), and questions that focused on logic-related objects (e.g., a counterexample) were coded as OL (object-logic). Second, we identified the different types of processes afforded by the question: curricular-based processes, e.g., graphing, formulating an equation, rotating (PC - process-curricular), logic-related processes, e.g., formulating the converse of a statement (PL - process-logic), and mathematical reasoning processes, e.g., justifying, conjecturing (PR - process-reasoning). For each question, we recorded its core object and the specific process.

We performed cluster analysis on the five variables: OC, OL, PC, PL, PR to identify types of opportunities for reasoning and proving. Cluster analysis is a statistical algorithm that groups a set of objects by identifying similar patterns, which are examined by the distance of the object from each other (Kaufman & Rousseeuw, 2009; Weingarden & Heyd-Metzuyanim, in press). Cluster analysis allowed us to identify similar patterns in the way questions were characterized by logic-related objects and processes, curricular-based mathematical objects and processes, and mathematical reasoning processes.

Types of ORP in Mathematical Tasks

The analysis revealed four clusters, which differ in the combinations of the five variables that characterize the questions designed by the PSTs. Table 1 describes the four clusters, the number of questions in each cluster and the frequencies of each variable in the cluster. The quality of this clustering according to Silhouette measure of cohesion and separation (Sarstedt & Mooi, 2014) was found to be good (between 0.5 to 1). In addition, a Chi-Square test between each of the five variables and the four clusters showed that there is a significant difference between the clusters (OC, OL, PL, PR: $\chi^2(3,106) = 106$, $p<0.001$. PC: $\chi^2(3,106) = 81.042$, $p<0.001$). The identified clusters correspond to four types of ORP that characterize the PSTs' questions (Table 1). In what follows, we describe these types of ORP, by depicting questions from each cluster.

Table 1: The four clusters describing the four types of ORPs

Variables	**Cluster 1:** Curricular-based mathematical discourse (OC-PC)	**Cluster 2:** Reasoning processes in the curricular-based mathematical discourse (OC-PC-PR)	**Cluster 3:** Meta-discourse about proof (OL-PL)	**Cluster 4:** Integrated curricular-based and logic-related discourse (OC-PC-PR-OL-PL)
	N = 37	N = 12	N = 36	N = 21
OC	**100%**	**100%**	0%	**100%**
PC	**100%**	**75%**	0%	**76.2%**
PR	0%	**100%**	0%	**100%**
OL	0%	0%	**100%**	**100%**
PL	0%	0%	**100%**	**100%**

Note. OC: Object-Curricular, PC: Process-Curricular, PR: Process-Reasoning, OL: Object-Logic, PL: Process-Logic

Cluster 1: Curricular-based Mathematical Discourse (OC-PC)

The first cluster includes questions belonging to a curricular-based mathematical discourse where both the object at the core of the question and the processes students can enact while engaging with the question are purely curricular-based.

During this activity, use the given quadrilateral and steps to reflect, rotate, or translate on the grid area.

1. Label the starting quadrilateral P
2. Rotate P'' 90 degrees clockwise about any corner, label your final shape P'''
3. Translate P', 2 inches in any direction, label the new shape P''
4. Reflect over line the x-axis, label the shape P'

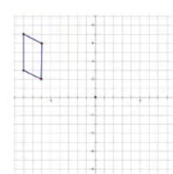


Figure 2: Questions that belong to curricular-based mathematical discourse

Figure 2 shows four questions from this cluster. The curricular-based mathematical object is geometrical shapes and the processes students are expected to perform are rotating, reflecting, and translating. These types of questions, as all questions in this cluster, provide only limited opportunities, if at all, for students to engage with reasoning and proving.

Cluster 2: Reasoning Processes in Curricular-based Mathematical Discourse (OC-PC-PR)

The second type of ORPs includes questions that belong to the curricular-based mathematical discourse but, in contrast to the questions in cluster 1, the questions in cluster 2 also involve mathematical reasoning processes (See Figure 3 for an example).

Create an equation that one can use to find the dimensions, area, number of smaller triangles for any given number triangle.

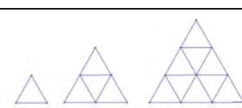


Figure 3: A question involving reasoning processes in a curricular-based mathematical discourse

Figure 3 shows the last question in a task which had students explore the growth of a triangular pattern. The four preceding questions were: (1) what would the next triangle look like? (2) What would its dimensions be? (3) How many triangles would be in the larger triangle? and (4) how many sticks would the next triangle be made up of? These questions belong to the curricular-based mathematical discourse (cluster 1) since the processes involved are drawing and counting sides and triangles, which are performed on the growing series of figures (mathematical objects). The last question of this task (Figure 3) involved curricular-based processes – creating an equation that describes the growing series of figures. However, this question also involved mathematical reasoning processes of identifying a pattern, generalizing, and conjecturing. Although the first four questions of this task provide limited ORP, they can be seen as scaffolding toward the last question (Figure 3), and specifically for the reasoning processes of identifying a pattern and generalizing.

Cluster 3: Meta-Discourse about Proof (OL-PL)

In contrast to the first two types of ORP that include questions that belong to the curricular-based mathematical discourse, the third type of ORP includes questions that belong to the meta-discourse about proof. Both the object at the core of the question and the potential processes provided are logic-related. Figure 4 presents two questions from this cluster.

Using the given conditional statements, circle the given and underline the conclusion. At the end, write the form of these statements in terms of P and Q (These statements are in the form If P, then Q):

- Tim knows that if he misses the practice before the game, then he will not be able to be a starting player in the game.
- If a shape has 4 right angles, then it is a square.

Figure 4: Questions characterized by OPR in the meta-discourse about proof

The logic-related object of these questions is a conditional statement, where the logic-related processes embedded in these questions are identifying the hypothesis and conclusion in the conditional statement and writing a conditional statement in the form of "if-then." Such questions focus on the nature and the logical aspects of proof. Note that although the second question relates to squares and angels, in contrast to the first question that does not refer to mathematical objects at all, both questions focus on conditional statements and their structure rather than the content of the statements.

Cluster 4: Integrated Curricular-based and Logic-related discourse (OC-PC-PR-OL-PL)

The fourth type of ORP includes questions characterized both by opportunities for enacting reasoning processes in the curricular-based mathematical discourse (similar to cluster 2) and by logic-related objects and processes (as cluster 3).

Now, let's make a conditional statement about this [an acute triangles that students were asked to construct in the former section] and equilateral triangles. (You can look back at your previous drawing for reference). Discuss with your partner what claim could be made that relates an equilateral triangle to an acute triangle.

Figure 5: A question characterized by curricular-based reasoning and logic-related ORP

As such, these types of questions provide rich opportunities for students to engage with reasoning and proving. See Figure 5 for an example. This question belongs to the meta-discourse about proof since it deals with a logic-related object – conditional statement ("let's make a conditional statement") and a logic-related process: writing a conditional statement. In addition, this question involves curricular-based mathematical objects – equilateral and acute triangles, and curricular-based processes, such as drawing triangles and recognizing the properties of different triangles. By integrating the two types of discourses, this question, like the other questions in cluster 4, provides students opportunities both to derive narratives about curricular-based mathematical objects, that of isosceles and equilateral triangles, while using mathematical reasoning processes such as conjecturing, and to deepen their deductive thinking about logic-related objects of conditional statements.

Discussion and Contributions of this Study

In this theoretical paper we offer a novel framework – *Opportunities for Reasoning-and-Proving (ORP) Task Analysis Framework*, for examining, characterizing, and classifying mathematical tasks according to the ORP provided by them. Cluster analysis performed on 106 questions revealed four different clusters corresponding to different types of ORP.

The first type represents *limited* OPR, as it corresponds to types of tasks focused solely on mathematical objects and includes curricular-based processes, as illustrated in Figure 2. These tasks belong to the curricular-based mathematical discourse without any specific demands on students to enact reasoning processes, such as conjecturing, justifying, or proving.

In contrast, *curricular-based reasoning* ORP corresponds to tasks that involve the enactment of reasoning processes on curricular-based mathematical objects (see example in Figure 3). Tasks characterized by this type of ORP are underrepresented in mathematical textbooks (Davis, 2012; Stylianides, 2009; Thompson et al., 2012), even in high-school geometry (Otten et al., 2014). However, curricular-based reasoning ORP are essential for supporting students' mathematical sense-making (Ellis et al., 2012) and for teaching mathematics via reasoning and proving (Buchbinder & McCrone, in press). Thus, teachers have an important role in designing tasks with curricular-based reasoning ORP, and teacher educators have an important role in preparing PSTs to design such tasks for students (Arbaugh et al., 2018; NCTM, 2014).

The third type of ORP – *logic related* ORP – appears in tasks characterized by logic-related objects (e.g., conditional statement) and can engage students with logic-related processes, such as identifying the hypothesis and conclusion, determining what needs to be done for proving or refuting a statement (see example in Figure 4). These types of tasks rarely occur outside high-school geometry chapter on proof (Otten et al., 2014), but they are essential for student engagement with reasoning and proving due to their focus on the nature of proof, the logical structure of theorems and how they are written and used (Cirillo & May, 2020).

The fourth type of ORP– *fully integrated* ORP – involves opportunities to enact reasoning processes on the curricular-based mathematical object with participating in the meta-discourse about proof. This type of task engages students with deductive reasoning and logical inferences, which are fundamental for learning mathematics (Harel & Sowder, 2007) while operating with mathematical objects (Figure 5). The fully integrated ORP can be contrasted with the 3rd cluster questions with logic related ORP where the mathematical objects often serve as a mere background for logic-related processes, such as identifying the structure of statements (see Figure 4 for an example). Tasks with fully integrated ORP require students to apply the logic-related processes on the mathematical objects. The questions in this cluster sensibly integrated both objects – the curricular and logic, so that students must operate on both, applying two types of processes: curricular-based and logic-related.

The four types of ORP described above came out of analyzing a specific corpus of data based on 12 lesson plans of three PSTs participating in a university-based course. Despite this methodological limitation of our study and the need to further validate its outcomes, our study offers several theoretical and practical contributions.

First, the *ORP Framework* combines two fundamental perspectives underlying learning and teaching mathematics via reasoning and proving (Buchbinder & McCrone, in press). One perspective focuses on reasoning and proving as a set of processes, such as identifying a pattern, conjecturing, and proving involved in deriving narratives about mathematical objects (Ellis et al., 2012; Jeannotte & Kieran, 2017; Stylianides, 2008). The second perspective focuses on the structure of theorems and the logical aspect of proof (Cirillo & May, 2020). The establishment of the *ORP Framework* strengthens previous studies that pursue the importance of engaging students with reasoning and proving and provides a unique tool for examining various discourses – curriculum-based and meta-discourse about proof – within a unified framework.

The second contribution of our study is identifying the four types of ORPs. While theoretical considerations suggest two types of objects at the core of the task and three types of processes (Figure 1), only some combinations of the five variables showed up empirically. Moreover, only two clusters, curricular-based mathematical discourse and meta-discourse about proof (clusters 1 and 3), contain tasks that fall under a single object-process characterization. Clusters 2 and 4 comprise tasks with multiple characterizations (Table 1).

Collectively, the *ORP Framework* and the OPR identified in this study contribute to further conceptualizing and operationalizing the notion of learning and teaching mathematics via reasoning and proof. While Buchbinder and McCrone (in press) suggested principles for *teaching* mathematics via reasoning and proving, we maintain that *learning* mathematics via reasoning and proving can be conceptualized discursively (Sfard, 2008) as students participating in the meta-discourse about proof and in a curricular-based mathematical discourse while enacting mathematical reasoning processes. This operationalization can also be used in the context of preparing PSTs to teach mathematics via reasoning and proving. PSTs need to develop expertise both in the logical aspect of proof (i.e. different types of proofs, valid and invalid modes of reasoning, the roles of examples in proving, logical relations), and in the pedagogical aspect of integrating reasoning and proof in curricular-based materials (Buchbinder & McCrone, 2020a). The *ORP Framework* can be used by teacher educators as a pedagogical learning tool for teachers, both to design tasks with various types of ORP and to discuss the potential of mathematical tasks in engaging students with reasoning and proving. The framework can enable teacher educators and PSTs to communicate about these OPR by operating with definite characteristics such as objects and processes. This may contribute to more explicit, and unambiguous communication among teachers and teacher educators about mathematical and pedagogical ideas (Weingarden, 2021).

In addition, the *ORP Framework* can be applied to various research settings which examine tasks and the ORP provided by them. These may include tasks designed by PSTs, beginning teachers, in-service teachers who participate in a professional development program, or tasks that appear in textbooks and other resources. In a related, ongoing study, we use the *ORP Framework* to examine how beginning teachers who participated in the capstone course integrate reasoning and proving in the tasks they design in their classroom. By promoting teacher competencies in designing tasks that embed rich ORP, teacher educators can contribute to the goal defined by NCTM (2014) as creating “systemic excellence” and providing “mathematics education that supports the learning of *all* students at the highest possible level” (NCTM, 2014, p. 2).

Acknowledgments

This research was supported by the National Science Foundation, Award No. 1941720. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Arbaugh, F., Smith, M. P., Boyle, J., Stylianides, G. J., & Steele, M. (2018). *We Reason & We Prove for ALL Mathematics: Building Students' Critical Thinking, Grades 6-12*. Corwin Press.

Buchbinder, O. & McCrone, S. (2020a). Preservice Teachers Learning to Teach Proof through Classroom Implementation: Successes and Challenges. *Journal of Mathematical Behavior, 58*, 100779.

Buchbinder, O. & McCrone, S. (2020b). Secondary Prospective Teachers' Strategies to Determine Equivalence of Conditional Statements. In Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds.). *Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education* (pp. 62 – 69). Boston, MA.

Buchbinder, O., & McCrone, S. (in press). Guiding principles for teaching mathematics via reasoning and proving. To appear in the *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education*, Bolzano, Italy.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17-36). New York: Routledge.

Caspi, S., & Sfard, A. (2012). Spontaneous meta-arithmetic as a first step toward school algebra. *International Journal of Educational Research, 51–52*, 45–65. https://doi.org/10.1016/j.ijer.2011.12.006

Cirillo, M., & May, H. (2020). Decomposing proof in secondary classrooms: A promising intervention for school geometry. *Proceedings of the 14th International Congress on Mathematical Education*, July 2020.

Davis, J. D. (2012). An examination of reasoning and proof opportunities in three differently organized secondary mathematics textbook units. *Mathematics Education Research Journal, 24*(4), 467–491. https://doi.org/10.1007/s13394-012-0047-2

Ellis, A. B., Bieda, K., & Knuth, E. (2012). *Developing essential understanding of proof and proving for teaching mathematics in grades 9–12*. Reston, VA: National Council of Teachers of Mathematics.

Hanna, G., & deVillers, M. D. (2012). *Proof and proving in mathematics education: The 19th ICMI study*. New York: Springer.

Harel, G., & Sowder, L. (2007). Toward Comprehensive Perspectives on the Learning and Teaching of Proof. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 6179, Issue 619, pp. 805–842). National Council of Teachers of Mathematics.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–16. https://doi.org/10.1007/s10649-017-9761-8

Kaufman, L., & Rousseeuw, P. J. (2009). *Finding groups in data: An introduction to cluster analysis*. John Wiley & Sons.

Lavie, I., & Sfard, A. (2019). How Children Individualize Numerical Routines: Elements of a Discursive Theory in Making. *Journal of the Learning Sciences, 28*(4–5), 419–461. https://doi.org/10.1080/10508406.2019.1646650

National Council of Teachers of Mathematics (2009). *Focus in high school mathematics: Reasoning and sense making*. Reston, VA: NCTM.

National Council of Teachers of Mathematics (2014). *Principles to Actions: Executive Summary*. Reston, VA: NCTM.

National Governors' Association Center for Best Practices and Council of Chief State School Officers, (2010). *Common core state standards: Mathematics.* Washington DC: Council of Chief State School Officers.

Otten, S., Males, L. M., & Gilbertson, N. J. (2014). The introduction of proof in secondary geometry textbooks. *International Journal of Educational Research, 64*, 107-118. https://doi.org/10.1016/j.ijer.2013.08.006

Reid, D. A. (2011). Understanding Proof and Transforming Teaching. In Wiest, L., & Lamberg, T. (Eds.), *Proceedings of the 33rd Annual Meeting of the PME-NA* (pp. 15–30). Reno. NV: University of Nevada.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*, 211-246.

Sarstedt, M., & Mooi, E. (2014). *A Concise Guide to Market Research*. Springer Heidelberg Dordrecht London New York.

Sfard, A. (2008). *Thinking as communicating*. Cambridge University Press.

Stylianides, G. J. (2008). Investigating the guidance offered to teachers in curriculum materials: The case of proof in mathematics. *International Journal of Science and Mathematics Education, 6*(1), 191–215.

Stylianides, G. J. (2009). Reasoning-and-proving in school mathematics textbooks. *Mathematical Thinking and Learning, 11*(4), 258–288. https://doi.org/10.1080/10986060903253954

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.). *Compendium for research in mathematics education* (pp. 237–266). Reston, VA: National Council of Teachers of Mathematics.

Watson, A. & Ohtani, M. (Eds.) (2015). *Task design in mathematics education: The 22nd ICMI study* (New ICMI study series). New York: Springer.

Weingarden, M. (2021). *The Realization Tree Assessment (RTA) tool as a visual mediator for communicating about teaching mathematics for explorative participation* [Unpublished doctoral dissertation]. Technion – Israel Institute of Technology, Haifa.

Weingarden, M., & Heyd-Metzuyanim, E. (in press). Seeing the forest for the trees: What can the Realization Tree Assessment tool (RTA) reveal about explorative classroom discussions? *Journal for Research in Mathematics Education*.

Thompson, D. R., Senk, S. L., & Johnson, G. J. (2012). Opportunities to learn reasoning and proof in high school mathematics textbooks. *Journal for Research in Mathematics Education, 43*(3), 253–295. https://doi.org/10.5951/jresematheduc.43.3.0253

MODELING COVID-19 VACCINE HESITANCY USING RESEARCH ON PROBABILISTIC THINKING

Hyunkyoung Yoon	Mina Gong	Cameron Byerley
Arizona State University	University of Georgia	University of Georgia
Hyunkyoung.Yoon@asu.edu	Mina.Gong@uga.edu	cbyerley@uga.edu

Min Sook Park	Dru Horne	Ulfa Aulyah Idrus
University of Wisconsin Milwaukee	University of Georgia	University of Georgia
minsook@uwm.edu	dhorne8@uga.edu	UlfaAulyah.Idrus@uga.edu

A number of factors contribute to vaccine hesitancy including citizens' determination of the risks and benefits of vaccination. In this study, we use interviews and surveys to understand how United States and South Korean citizens quantify the risks of COVID-19 infection and vaccination. Many citizens used benchmark values such as 0%, 50%, 100% to estimate their risks of COVID-19 infection and COVID-19 vaccination. Although neither infection nor vaccination has a 50% risk of severe outcome, the citizens' thinking is consistent with the outcome approach described by Konold (1989, 1991). In his work, a 50% risk often implied uncertainty about if an outcome would happen in a single trial. It is important to support citizens' understanding of risk because in our sample the citizens who thought COVID-19 vaccination was riskier than COVID-19 infection were often unvaccinated.

Keywords: Probability, Vaccine Hesitancy, Outcome Approach, Cognition

Introduction

The development of the COVID-19 vaccine has been a critical strategy to battle the pandemic. To mitigate the economic and personal impacts of COVID-19, sufficient population vaccination is critical. However, even vaccine availability does not directly lead to a higher vaccination rate because of pervasive COVID-19 vaccine hesitancy across the world (Troiano & Nardi, 2021). The WHO identified vaccine hesitancy as one of the top 10 global health threats in 2019 (World Health Organization, 2019). Many people have concerns over the safety and efficacy of COVID-19 vaccines (Callaghan et al., 2020).

COVID-19 vaccines are widely recommended because the risks of side effects are smaller compared to the reduction in personal and collective risk from COVID-19 infection (World Health Organization, 2020). However, many citizens are unwilling to take a new vaccination solely because authorities recommend it, and they want to understand the risks and benefits for themselves first (Machingaidze & Wiysonge, 2021). Weighing the risks and benefits of vaccination involves comparing percentages less than one, analysis of data from clinical trials, computing expected values, mathematical modeling of measuring quantities such as infection fatality rates, and more (Tenforde, 2021). Given that much of the mathematics and statistics involved in forming recommendations for vaccination is beyond that commonly taught in secondary or tertiary education (Kollosche & Meyerhöfer, 2021), it is important to understand how citizens are making sense of vaccination and infection risks. Prior studies argue the necessity of evidence-based vaccine education campaigns (Chou & Budenz, 2020; Dror et al., 2020; Loomba et al., 2021), and building a successful education campaign is contingent on understanding citizens' thinking about vaccination risks and benefits. We look for harmony between vaccine hesitancy research to understand the dissonant perspectives on COVID-19 vaccines.

We use constructs from mathematics education research on probabilistic thinking to model how citizens quantify the risk of COVID-19 infection and COVID-19 vaccination. Thus, we investigate the questions:

1. How do citizens estimate their risk of the COVID-19 infection and vaccination?
2. How do citizens' estimates of the relative risk of COVID-19 infection and vaccination correlate to their vaccination status?

Many citizens estimated risks using reasoning that is consistent with the outcome approach, so we focus on our answer to question one using that construct (Konold, 1989, 1991).

COVID-19 Vaccine Hesitancy

Vaccine hesitancy refers to delay in acceptance or refusal of vaccination despite the availability of vaccination services (MacDonald, 2015). Vaccine hesitancy is not a new issue globally and many researchers are interested in understanding "the reasons why a considerable number of people do not receive recommended vaccinations" for the purpose of designing interventions to increase vaccine acceptance (Betsch et. al, 2018, p. 1). A common cause of general vaccine hesitancy is a lack of confidence in the safety and efficacy of vaccines and the systems that deliver them (Betsch et. al, 2018). Other important factors identified in the literature include "complacency (not perceiving diseases as high risk), constraints (structural and psychological barriers), calculation (engagement in extensive information searching), and aspects pertaining to collective responsibility (willingness to protect others)" (Betsch et. al, 2018, p. 1).

COVID-19 specific vaccine hesitancy has also arisen because of the vaccine's novelty (Callaghan et al., 2020; Machingaidze & Wiysonge, 2021; Schwarzinger et al., 2021; Ward et al., 2020) even though recent studies show COVID-19 vaccines have similar safety and efficacy with other previously approved vaccines (Polack et al., 2020). Many think the vaccine development and production have occurred too quickly (Troiano & Nardi, 2021).

Incorrect or misleading information about COVID-19 vaccines also lowers citizens' intent to be vaccinated (Loomba et al., 2021). Information such as, "Vaccine trial participants have died after taking a candidate COVID-19 vaccine" (p.337), contributes to people's fears, doubts, or cynicism over new COVID-19 vaccines. Even among medical staff, there is vaccine skepticism due to fears of the safety and rapid development of vaccines (Dror et al., 2020).
Modeling thinking related to vaccination is complex because it involves social, emotional, scientific, statistical, and mathematical thinking. Despite our acknowledgment that approaching the problem from a mathematics education lens will miss other important factors contributing to vaccine hesitancy, we want to do our part to understand this global issue. Mathematics educators are positioned to model citizens' understandings of the risk of COVID-19 infection (which is related to complacency), and their understandings of the risks and benefits of vaccination (which is related to confidence in safety and efficacy).

Theoretical Perspectives on Probabilistic Thinking

The probability of an event occurring can be understood in multiple ways. Konold (1991) describes both formal theories and beliefs of probability that are held by experts such as classical, frequentist, and subjectivist, and informal theories which are the ways that people actually think about and make sense of probabilities. Our focus is on an informal theory: the outcome approach (Konold, 1989, 1991).

Konold (1989, 1991) introduced the outcome approach construct as an informal model of probability for binary decisions. People using the outcome approach interpret probability as predicting the outcome of the next trial and often evaluate their prediction as being correct or incorrect after one trial. In interpreting probabilities with an outcome approach, Konold (1989, 1991) found that students often use benchmark values of 0%, 50%, and 100% to represent “No, it will not occur”, “I don’t know if it will occur”, and “Yes, it will occur”, respectively. These three benchmark percentages help someone decide what will happen given a probability to interpret. When Konold (1989, 1991) asked students to explain a weather prediction of 70% chance of rain tomorrow, the students using the outcome approach interpreted this to mean that it will rain tomorrow since 70% is close to their 100% benchmark. A 30% chance of an event occurring could be interpreted similarly to 0% meaning its likelihood of occurrence is low (Konold, 1989). When using an outcome approach people interpret 50% as an indicator that “anything could happen” instead of that it on average happens 5 times out of 10 trials. This is because 50% is the mid-point of the decision continuum and implicates equally possible results of yes or no. In general, people using an outcome approach often ignore frequency information and instead focus on other features such as causality and physical features when interpreting probability (Konold, 1991).

Methods

We use Zoom interviews and an online survey to understand how U.S. and South Korean citizens quantify their risk of COVID-19 infection and vaccination. We do not analyze the interviews in depth in this study and instead explain how they served as inspiration for the online survey. We conducted task-based clinical interviews with 11 U.S. and 7 SK citizens between Oct 26th, 2020 and May 19th, 2021. Each interview lasted approximately one hour and included one researcher and one citizen. Interviews were recorded using Zoom and we paid participants 50 dollars per hour. We intentionally recruited a diverse group of citizens with respect to race, education level, political beliefs, geographic location, and views toward COVID-19 and vaccination. During the interviews, we asked citizens to estimate risks associated with COVID-19 infection for themselves and well-known public figures, asked for their opinions about COVID-19 vaccination, and asked them to interpret the Relative Risk Tool (Joshua et al., 2022). We noticed that interview participants (even those with an advanced mathematics background) gave surprisingly high estimates of the risk of COVID-19 infection. This is consistent with other surveys of US citizens’ assessment of COVID-19 hospitalization risk. Rothwell and Witters (2021) reported that 41% of Democrats and 22% of Republicans estimated the risk of hospitalization for an unvaccinated person as 50% or higher. Further, interview participants were more worried about side effects of vaccination than many of their everyday activities that had greater risk such as driving. We noticed that the way many interview participants responded was consistent with the outcome approach described by Konold (1989, 1991), and decided to survey a random sample of U.S. citizens to investigate this idea.

We conducted a survey with a random sample of 151 U.S. adults between Jan 31st, 2022 and Feb 8th, 2022 using CloudResearch Prime Panel formerly known as TurkPrime (CloudResearch, 2022). CloudResearch supports collecting random human subjects data in multiple fields, including social and behavioral sciences (Chandler et al., 2019; Litman et al., 2017). Using CloudResearch allowed us to obtain a diverse and random sample. All participants were informed of the goals of the study and consented to be involved in research before completing the survey.

Survey questions included:1) Have you had a COVID-19 vaccine?; 2) If you were infected with COVID-19, what do you think your percent risk of dying from COVID-19 would be? (Enter number between 0 and 100 without % sign.) Why?; 3) If you receive a COVID-19 vaccine in the future, what do you think your percent risks of a serious adverse reaction would be? An example of a serious adverse reaction is an allergic reaction requiring treatment in a hospital. Do not include your risk of common side effects such as fatigue. Why?; and 4) In your opinion, is COVID-19 infection or a COVID-19 vaccination more risky for you? We included other questions not reported in this paper such as demographic information, items from prior risk literacy studies (Lipkus et al., 2001), and their reasons for choosing or delaying vaccination.

Results

Our goal is to understand citizens' decisions regarding vaccines by using the outcome approach, so we focus on citizens' use of benchmark values (0%, 50%, 100%) when estimating their risks of COVID-19 infection and vaccination. We then discuss the relationships between citizens' estimates of risk and vaccination status.

COVID-19 Death Risk Estimates and Risk of COVID-19 Vaccination Estimates

COVID-19 death risk estimates in interviews. We first present the interviews because they inspired the online survey. Minji, a 32-year-old South Korean who holds an undergraduate degree in mathematics education, was interviewed on May 19th, 2021. She said that she wanted to receive either Pfizer or Moderna vaccines instead of AstraZeneca vaccine. Other Korean citizens also had negative perceptions of AstraZeneca's COVID-19 vaccines due to its side effects, especially of blood clotting (Seo, 2021). In May 2021, COVID-19 vaccine refusal rate in South Korea had reached 33% among citizens who were eligible for COVID-19 vaccination (Lee et al., 2021). When Minji was asked about her estimate of her COVID-19 death risk, she said "honestly, it is really hard for me to predict it, the death risk." The interviewer encouraged her to give a rough estimate and she said:

> Minji: Rough estimate of the death risk… it's 50% though. I think it's about 50%? Because I've never had the virus before, I cannot say for sure whether I would die or not from COVID-19. I don't have any numerical information that allows me to think about how sick I will be.

The excerpt shows that Minji did not know whether she would die from COVID-19 or not, so she estimated her risk of dying from COVID-19 was roughly 50% which is consistent with the outcome approach because she used 50% to indicate uncertainty. We hypothesize Minji did not want to give a value above or below 50% because that would indicate a prediction of what would happen that she did not feel comfortable making.

Ian, a 27-year-old U.S. citizen who holds a PhD in chemistry, was interviewed on October 26th, 2020. He estimated that his risk of death due to COVID-19 infection was less than 1% for a variety of reasons. For example, he said "I have a balanced micronutrient and macronutrient intake and I also exercise regularly with both resistance training and cardiovascular training." We asked him to estimate Donald Trump's risk of dying from COVID-19 on the day he was diagnosed. Ian said "50 percent. Yeah, based off his risk factors". Ian did not seem to be attending to frequencies of infections and deaths in his estimate of 50% for Trump. We knew Ian was capable of estimating how many citizens would die if large numbers of people similar to Trump were infected. Many others picked values close to 0% to estimate risk for a young, healthy person and values greater than or equal to 50% to estimate the risk of an older person. The outcome approach construct is a plausible explanation for why citizens might pick widely spaced risk estimates for young and old. They are trying to convey the outcome for a young person will likely be good and the outcome for an old person will likely be bad.

Across many other interviews, citizens gave surprisingly high estimates for the risk of COVID-19 infection and also expressed substantial concern about extremely rare side effects of vaccination. Many responses made more sense when we interpreted the numbers as predictions of the single next outcome as opposed to citizens' thinking about frequencies.

COVID-19 infection death risk estimates in the online survey. The mean of the survey participants' ages is 45.13 and its standard deviation is 14.93. In our sample, there were 52 citizens who had not received a COVID-19 vaccine, 14 citizens who were partially vaccinated for COVID-19 (one dose of Pfizer or Moderna), 30 citizens who were fully vaccinated for COVID-19 (two-dose of Pfizer of Moderna, or single dose of J&J vaccine) without a booster shot, and 55 citizens who were fully vaccinated for COVID-19 with a booster shot.

We first present the results from citizens' estimates of percent risks of dying from COVID-19 infection and discuss the difference between citizens who received no vaccine and citizens who received at least one vaccine (Figure 1). There are 52 citizens who received no vaccine and 99 citizens who received at least one vaccine.

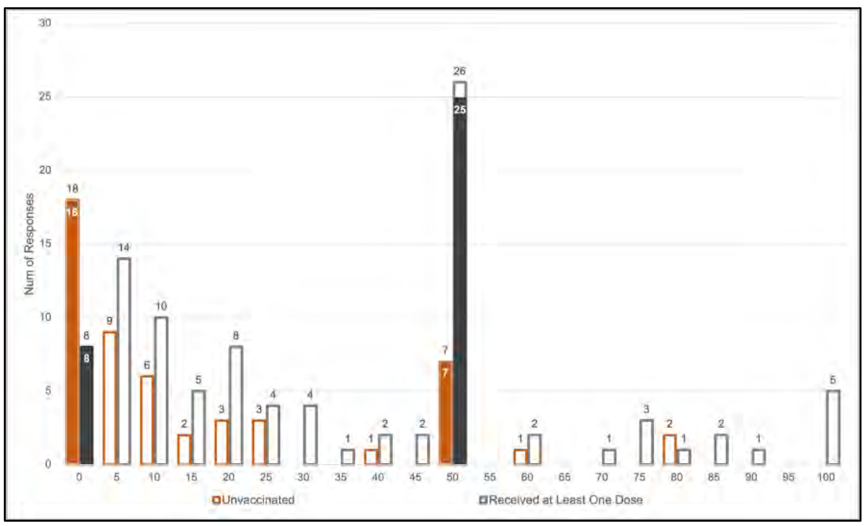


Figure 1. Citizens' estimates of COVID-19 death risk by vaccination status.

In Figure 1, citizens' estimated values of 0 are graphed above the number 0, estimated values that are greater than 0 and less than or equal to 5 are graphed above the number 5, estimated values that are greater than 5 and less than or equal to 10 are graphed above the number 10, etc. The red and blue outline bars above 50% represent the number of citizens that estimated their risk as greater than 45% and less than or equal to 50%. The red and blue solid bars represent the number of citizens who estimated their risk at exactly 50%. Among unvaccinated citizens, these counts were the same, but for the citizens receiving at least one vaccine, the counts differ (one citizen estimated their risk at 49% and so 25 out of 26 citizens estimated their risk at exactly 50%). Both citizens who received no vaccine and citizens who received at least one vaccine overestimated their risks of dying from COVID-19 infection. Figure 1 shows that about 38% (58 of 151) citizens estimated their death risk from COVID-19 at exactly the benchmark values of 0% and 50%.

Approximately 35% of citizens (18 of 52) who received no vaccine used 0% as a benchmark value. Many, but not all of the explanations associated with 0%, conveyed that the person was sure that he or she would not die from COVID-19 and associated the number 0% with something that is not likely to happen. Examples include, “I am healthy and take vitamins”, “I can fight it off”, and “COVID-19 is nothing but a common cold”. Some citizens wrote “I never had symptoms” and “I personally know no one that died from it”.

About 25% of citizens (25 of 99) who received at least one dose of the vaccine, estimated their risk of death as 50%. Many but not all of the explanations associated with 50% conveyed that the person was uncertain about what would happen and associated the number 50% with uncertainty. Examples include, “Because I think it’s something you can never know”, “Do not know until I get it”, and “I don’t know either way.” Another wrote, “I have a really good immune system but I have heart failure so it’s a gamble.” Some citizens did not mention uncertainty in their responses and instead wrote about personal risk factors. For example, “I am a smoker and a little overweight” or “because I’m 76 yrs old.” It is possible that citizens who wrote about their personal risk factors are reasoning using the outcome approach, but the evidence is inconclusive.

The citizens who had not received a COVID-19 vaccine (mean 16.73%) gave lower estimates of COVID-19 death risk compared to citizens with at least one vaccine (mean 33.78%). Citizens who received no vaccine tended to use 0% as a benchmark value whereas citizens who received at least one dose of the vaccine tended to use 50% as a benchmark value when estimating their risk of COVID-19 infection death.

Many citizens who estimated their risks between 0% and 50% also appeared to be using the outcome approach instead of focusing on frequencies of outcomes. For example, 12 out of 41 citizens who estimated their risk of death between 10% and 25% said they were “somewhat” or “moderately concerned” about COVID-19 infection. If they are considering 10% as meaning that the next time they are infected they will not die, this makes sense. Based on interviews, we do not think that the citizens who said they were slightly concerned about a 10% risk of death are focused on frequencies. In interviews, once citizens computed 1% of 50 million (as an example) they started to think of 1% risk of death as substantial.

Risks of COVID-19 vaccination estimates in the online survey. We report the results from citizens’ estimates of percent risks of severe adverse reactions to COVID-19 vaccination and discuss the difference between vaccinated and unvaccinated citizens’ estimates. Again, we identify two clusters in 0% (43 of 151) and 50% (27 of 151) that could be explained by the citizens’ use of the outcome approach.

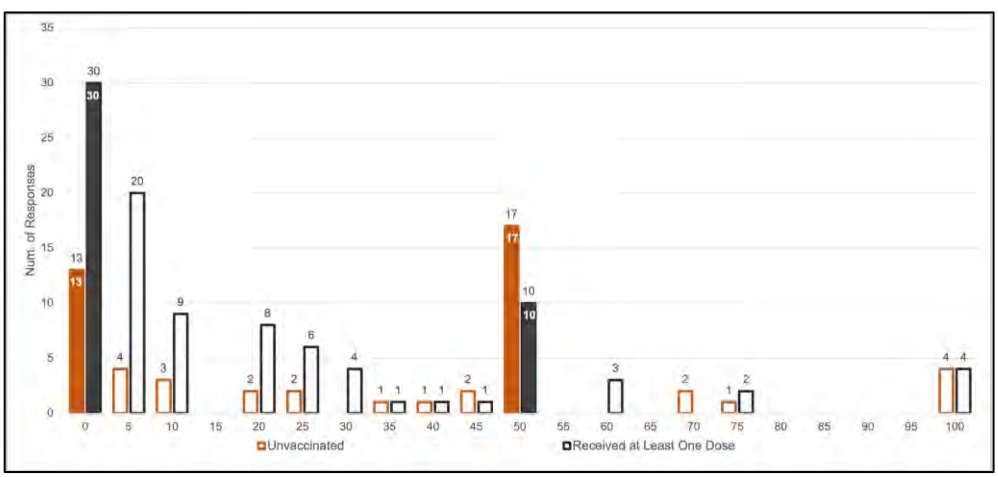


Figure 2. Citizens' estimates of COVID-19 vaccination risk by vaccination status

A number of people who estimated 0% risk had already been vaccinated without an allergic reaction or had no history of allergies. Their estimate of 0% could be explained with the outcome approach because their prediction of the outcome of a single next trial is that they would not have a reaction. The estimates of 0% are also consistent with more normative ways of thinking about probability. The general risk of severe allergic reaction is around four in 1.8 million (CDC, 2021), and the risk is likely even lower for people who have already been vaccinated with no reactions. The citizens' estimates are close to the actual risk. There is a wide variety of ways of thinking that could result in those values including normative probabilistic thinking. Even vaccinated citizens tended to dramatically overestimate the risks of vaccination. We found in interviews that people who had been vaccinated were often still frightened of vaccination because they thought there was substantial potential for long-term side-effects.

The majority of citizens who estimated a risk of 50% of severe side effects indicated uncertainty in their explanations of their risk. For example, they wrote "because I'm not sure what's in this vaccine", "It effects everyone different and you don't know until you have a reaction", and "Do not know until it happens." One citizen estimated at a 50% risk for all questions in the survey including a question about getting in a serious car accident over the course of one year. He explained, "because like I said always a 50/50 chance." The citizens with at least one vaccine typed lower numbers compared to unvaccinated citizens. Unvaccinated citizens often used 50% (17 of 52) as a benchmark value and citizens with at least one vaccine often used 0% (30 of 99).

Relationship between Estimates of Risk and Vaccination Status

Thinking about estimates of risk using the outcome approach is a viable strategy for many areas of life. For example, if someone interprets a 50% chance of rain as meaning that the weatherman does not really have information about whether or not it will rain, they might make a reasonable decision to bring an umbrella. In our interviews, we found that citizens who overestimated their risk of COVID-19 infection were not necessarily constrained by their estimates. Many who overestimated their risks were not stopping their lives due to fear of COVID-19 and were still socializing but just taking precautions such as wearing masks. We are concerned that using the outcome approach is not adequate to weigh the relative risk of COVID-19 infection and COVID-19 vaccination. It is more important to think about the relative risk of infection and vaccination than to focus on the uncertainty of any medical outcome. In fact, medical professionals compare risks and benefits to decide whether or not to approve a vaccination. If citizens estimate the risk of both infection and vaccination at 50% because they are uncertain of the outcome of either event, it does not help them make a productive decision about the benefits of vaccination. In fact, we found that citizens who are unsure about the relative risk of infection and vaccination are less likely to be vaccinated (See Table 1).

Table 1. The relative risk of infection and vaccination by vaccine status

	No vaccine	Partially vaccinated	Fully vaccinated	Fully vaccinated with a booster shot	Total
Equally risky[1]	19	4	12	6	41
Vaccine risky[2]	30	2	2	3	37
Infection risky[3]	3	8	16	46	73
Total	52	14	30	55	151

[1]COVID-19 infection is equally risky as a COVID-19 vaccination for me.

[2]COVID-19 infection is less risky than a COVID-19 vaccination for me.
[3]COVID-19 infection is more risky than a COVID-19 vaccination for me.

Table 1 shows that about 46% (19 of 41) people who think COVID-19 infection is equally risky as a COVID-19 vaccination for them did not receive a vaccine. Table 1 also demonstrates that people who think COVID-19 infection is less risky than a COVID-19 vaccination for them tended not to receive a vaccine (30 of 37). If a person thinks COVID-19 infection is riskier than a COVID-19 vaccination for them, the person tended to be either boosted or fully vaccinated (62 of 73). The relationship between a citizens' response about risk and vaccination status is statistically significant $\chi^2(6,151) = 78.9$, $p < .001$.

We also noticed that many vaccinated citizens without a booster (partially and fully vaccinated) are often unsure about whether or not infection or vaccination is riskier. We were somewhat surprised by the number of citizens who received a vaccine despite thinking vaccination might be as risky as COVID-19 infection. We acknowledge again that there are a number of factors such as social or emotional thinking influencing people's decisions to be vaccinated beyond their assessments of risks. We think that our part is to use a mathematics education lens to model citizens' thinking about COVID-19 infection and vaccination risks.

Discussion

The results demonstrated that using Konold's outcome approach can provide plausible explanations for citizens' thinking about their estimates of COVID-19 risks. Many citizens used benchmark values "0%" to represent "I will not die from COVID-19" and "50%" to represent "I don't know if I will die from COVID-19 or not". Citizens who have at least one COVID-19 vaccine showed a tendency to use "50%" as their estimate of COVID-19 death risk and "0%" as their estimate of COVID-19 vaccination risk. Citizens without a COVID-19 vaccine tended to use "0%" as their estimate of COVID-19 death risk and "50%" as their estimate of COVID-19 vaccination risk. As in Konold's (1991) data, citizens appeared to move up or down from benchmark values to indicate varying degrees of certainty. For example, many citizens who were somewhat concerned about COVID-19 infection listed a death risk of 10%. We think that mathematics educators could help students develop formal interpretations of probability such as understanding percent risks based on the relative frequency of outcomes.

Variations in citizens' estimates of risk explain a fair amount of variation in vaccination status. Of course, many other factors are also involved, but convincing citizens that vaccination is less risky than infection seems like a potentially important target for public health education. Many vaccinated citizens who do not yet have a booster are unsure about whether or not infection is riskier than vaccination and might be reassured by well-crafted messaging about relative risks. Kollosche and Meyerhöfer (2021) argue that mathematics related to COVID-19 is more complicated than what is commonly taught in school, and mathematics education could help a layperson understand health recommendations. Our results show that it is important to support citizens in understanding their risk of COVID-19 infection and vaccination. We think giving citizens opportunities to see their risks based on their ages would help them make informed decisions regarding COVID-19. We designed the Relative Risk Tool (https://www.covidtaser.com/relativerisk) to help citizens easily compare the relative risk of COVID-19 infection and COVID-19 vaccination without needing sophisticated mathematical or statistical knowledge (Joshua et al., 2022; Lipkus et al., 2001; Wong & Yang, 2021).

Acknowledgments

Research reported in this study was supported by NSF Grant No. DUE #2032688. We would like to thank you to the COVID-TASER team including Kevin Moore, Surani Joshua, Heather Lavender, Laura Valaas, Sukjin You, Anne Waswa, and Brian Rickard.

References

Callaghan, T., Moghtaderi, A., Lueck, J. A., Hotez, P. J., Strych, U., Dor, A., Franklin Fowler, E., & Motta, M. (2020). Correlates and disparities of COVID-19 vaccine hesitancy. *Available at SSRN 3667971*.

CDC. (2021). Allergic reactions including anaphylaxis after receipt of the first dose of Pfizer-BioNTech COVID-19 vaccine—United States, December 14–23, 2020. *Morbidity and Mortality Weekly Report, 70*(2), 46.

Chandler, J., Rosenzweig, C., Moss, A. J., Robinson, J., & Litman, L. (2019). Online panels in social science research: Expanding sampling methods beyond Mechanical Turk. *Behavior research methods, 51*(5), 2022-2038.

Chou, W.-Y. S., & Budenz, A. (2020). Considering emotion in COVID-19 vaccine communication: addressing vaccine hesitancy and fostering vaccine confidence. *Health communication, 35*(14), 1718-1722.

CloudResearch. (2022). *From the Classroom to the Boardroom.* https://www.cloudresearch.com/why-cloudresearch/

Dror, A. A., Eisenbach, N., Taiber, S., Morozov, N. G., Mizrachi, M., Zigron, A., Srouji, S., & Sela, E. (2020). Vaccine hesitancy: the next challenge in the fight against COVID-19. *European journal of epidemiology, 35*(8), 775-779.

Joshua, S., Drimalla, J., Horne, D., Lavender, H., Yon, A., Byerley, C., Yoon, H., & Moore, K. (2022). Exploring Relative Size with Relative Risk. *Mathematics Teacher: Learning and Teaching Pre-K-12*.

Kollosche, D., & Meyerhöfer, W. (2021). COVID-19, mathematics education, and the evaluation of expert knowledge. *Educational studies in mathematics, 108*(1), 401-417.

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6*(1), 59-98.

Konold, C. (1991). Understanding students' beliefs about probability. In *Radical constructivism in mathematics education* (pp. 139-156). Springer.

Lee, H., Noh, E. B., Park, S. J., Nam, H. K., Lee, T. H., Lee, G. R., & Nam, E. W. (2021). COVID-19 Vaccine Perception in South Korea: Web Crawling Approach. *JMIR public health and surveillance, 7*(9), e31409.

Lipkus, I. M., Samsa, G., & Rimer, B. K. (2001). General performance on a numeracy scale among highly educated samples. *Medical decision making, 21*(1), 37-44.

Litman, L., Robinson, J., & Abberbock, T. (2017). TurkPrime. com: A versatile crowdsourcing data acquisition platform for the behavioral sciences. *Behavior research methods, 49*(2), 433-442.

Loomba, S., de Figueiredo, A., Piatek, S. J., de Graaf, K., & Larson, H. J. (2021). Measuring the impact of COVID-19 vaccine misinformation on vaccination intent in the UK and USA. *Nature human behaviour, 5*(3), 337-348.

MacDonald, N. E. (2015). Vaccine hesitancy: Definition, scope and determinants. *Vaccine, 33*(34), 4161-4164.

Machingaidze, S., & Wiysonge, C. S. (2021, 08/01/). Understanding COVID-19 vaccine hesitancy. *Nature Medicine, 27*(8), 1338. https://doi.org/10.1038/s41591-021-01459-7

Polack, F. P., Thomas, S. J., Kitchin, N., Absalon, J., Gurtman, A., Lockhart, S., Perez, J. L., Marc, G. P., Moreira, E. D., & Zerbini, C. (2020). Safety and efficacy of the BNT162b2 mRNA Covid-19 vaccine. *New England Journal of Medicine*.

Rothwell, J., & Witters, D. (2021). *Gallup*. https://news.gallup.com/opinion/gallup/354938/adults-estimates-covid-hospitalization-risk.aspx

Schwarzinger, M., Watson, V., Arwidson, P., Alla, F., & Luchini, S. (2021). COVID-19 vaccine hesitancy in a representative working-age population in France: a survey experiment based on vaccine characteristics. *The Lancet Public Health, 6*(4), e210-e221.

Seo, H. (2021). S. Korea reports first blood clot case linked to AZ vaccine. *The Hankyoreh*. https://english.hani.co.kr/arti/english_edition/e_national/997560.html

Tenforde, M. W. (2021). Effectiveness of Pfizer-BioNTech and Moderna vaccines against COVID-19 among hospitalized adults aged≥ 65 years—United States, January–March 2021. *MMWR. Morbidity and mortality weekly report, 70*.

Troiano, G., & Nardi, A. (2021). Vaccine hesitancy in the era of COVID-19. *Public Health*.

Ward, J. K., Alleaume, C., Peretti-Watel, P., Seror, V., Cortaredona, S., Launay, O., Raude, J., Verger, P., Beck, F., Legleye, S., L'Haridon, O., & Ward, J. (2020, 11/01/). The French public's attitudes to a future COVID-19 vaccine: The politicization of a public health issue [Report]. *Social Science & Medicine, 265*. https://doi.org/10.1016/j.socscimed.2020.113414

Wong, J. C. S., & Yang, J. Z. (2021). Comparative Risk: Dread and Unknown Characteristics of the COVID-19 Pandemic Versus COVID-19 Vaccines. *Risk Analysis*.
World Health Organization. (n.d.). *Ten health issues who will tackle this year*. World Health Organization. Retrieved February 14, 2022, from https://www.who.int/news-room/spotlight/ten-threats-to-global-health-in-2019
World Health Organization (2020). Behavioural considerations for acceptance and uptake of COVID-19 vaccines: WHO technical advisory group on behavioural insights and sciences for health, meeting report, 15 October 2020.

MATHEMATICIANS' UNDERSTANDING: AN ARGUMENT FOR THE VALUE OF PSEUDO-OBJECTS

Kyle Flanagan
Virginia Tech
flanaga5@vt.edu

Keywords: Advanced Mathematical Thinking, Mathematical Processes and Practices

In academia, mathematicians frequently engage in solving difficult mathematical problems. For these mathematicians, these problems often involve highly formal and abstract concepts. Still, professional mathematicians are successful at completing new mathematical research and "understanding" new abstract mathematical concepts that they encounter. This theoretical poster aims to present the beginnings of a theoretical framework addressing this phenomenon, which grates against commonly held beliefs in the mathematics education literature.

Various process-object theories have been developed to help explain mathematical understanding and thinking. Davis et al (1997) summarize these various theories of encapsulation and reification, which includes the building blocks of APOS theory. A commonplace of most of these theories is that an individual often (or must) progress from process-level to object-level understanding. Dubinsky and McDonald (2001) state about the four components of APOS theory, "each conception in the list must be constructed before the next step is possible" (p. 277). In the literature, having less than a process-object-level understanding is often considered unvaluable. Related to this, syntactic proof strategies, namely strategies involving symbol-pushing and unpacking definitions, are often seen as unproductive. Weber (2001) mentions these strategies "are not capable at solving complex problems" (p. 114).

Sfard (1992) defines a pseudostructural conception as an object that is not conceptualized as a process, which will be referred to as a pseudo-object following Zandieh (2000). Though a full process-object-level understanding is preferable, I argue that a pseudo-object-level understanding can be a productive place to operate, and hypothesize that, for a time, mathematicians likely function with that level of understanding when encountering new advanced concepts. It is also plausible that mathematicians may not have the time to always develop a process-object-level understanding for the new concepts they encounter, yet they still productively do mathematics with these concepts. To share a personal example, when first encountering inverse limits of modules in commutative algebra, I successfully used the inverse limit as a pseudo-object to learn subsequent material and write proofs (sometimes with syntactic proof strategies) but did not fully understand the underlying process other than the mathematical objects it consisted of. Only after revisiting the concept about a year later, did I begin to gain a process-level understanding too.

In support of this viewpoint, Weber and Alcock (2004) state that syntactic proof strategies are valuable and can be used to build intuition about a concept. Additionally, Weber (2019) provides evidence that set theorists can gain epistemic benefits from just knowing syntactic representations of concepts, rather than knowing the full derivation. This aligns closely with the belief that mathematicians may be content and successful with only having a pseudo-object-level understanding of certain concepts. If this hypothesis is accurate, then it would have significant implications for the teaching of advanced mathematics. If mathematicians often productively leverage a pseudo-object-level understanding, then teachers should not be quick to discount the value of the learning and progress that can take place in students with this level of understanding, when encountering new mathematics, especially at a more advanced mathematical level.

References

Davis, G., Tall, D., & Thomas, M. (1997). What is the object of the encapsulation of a process? *Proceedings of MERGA, Rotarua, New Zealand, 2*, 132-139.

Dubinsky, E., & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In D. Holton (Ed.), *The teaching and learning of mathematics at university level* (pp. 275-282). Kluwer Academic Publishers.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-The case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (MAA notes, no. 25, pp. 59-84). Mathematical Association of America.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101-119.

Weber, K. (2019). The role of syntactic representations in set theory. *Synthese*.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56(2)*, 209-234.

Zandieh, M. J. (2000). A theoretical framework for anlayzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education, 8*, 103-127.

FOSTERING MATHEMATICAL THINKING USING DYNAMIC APPLETS: INTERCEPT'S ROLE IN THE LINEAR REGRESSION MODEL

PROMOVIENDO EL PENSAMIENTO MATEMÁTICO USANDO APPLETS DINÁMICOS: EL PAPEL DEL INTERCEPTO EN EL MODELO DE REGRESIÓN LINEAL

Diana Elizabeth Hernández-Ceja
Universidad de Guadalajara
diana.hernandez2181@alumnos.udg.mx

Claudia Orozco-Rodríguez
Universidad de Guadalajara
claudia.orozcor@academicos.udg.mx

Carlos Valenzuela García
Universidad de Guadalajara
carlos.valenzuela@academicos.udg.mx

Keywords: Advanced Mathematical Thinking, Data Analysis and Statistics, Problem Solving

Introduction

When technology supports problem-solving activities, it promotes strategy implementation and heuristics extension (Santos-Trigo, 2014), it also can influence the development of certain aspects of the mathematical thinking, such as the way information is represented and the way the results are communicated by the students (Barrera-Mora & Reyes-Rodríguez, 2018).

The informal fitting of the line is considered to be an educational experience towards understanding the linear regression model (Bargagliotti et al., 2012 cited in (Casey, 2015)). This study examines the mathematical thinking developed by a group of students taking a Statistics course, in a problem-solving activity about linear regression and using the Rossman/Chance applet "Two Quantitative Variables" (Chance & Rossman, 2015).

Methods

A qualitative panel study was carried out in a group of 19 students from the Logistics and Transportation Engineering program at University of Guadalajara, all the students were 20 years old. After receiving instruction about the topic, the small-group problem-solving activity was worked over a hybrid session, due to COVID-19 restrictions. The data about the student's work was collected with Google Forms questionnaires and Google Meet recordings.

The context of the problem-solving activity was automated warehouses, the data provided had 11 entries and two variables: Number of vehicles and Time Spent in Congestion (in seconds). Using the data, the students were asked to propose "the best line" that better represents the relationship between variables using the movable line from the applet which also provides the line's algebraic representation.

Results

As students were exploring possible lines, questions about the intercept and the slope, in the context of the problem, arose among the teams. By answering these questions, the students were able to give meaning to the model's coefficients after placing different lines over the scatterplot. For the role of the intercept, they were able to communicate that it made no sense to have a time different than zero, if no vehicles were operating. This led some students to propose a model with no intercept. It was later shown in class that the intercept of the linear regression model for the data was not significant, and thus the best model had no intercept.

References

Barrera-Mora, F. , & Reyes-Rodríguez, A. (2018). Elementos del pensamiento matemático que emergen al resolver problemas, en contextos hipotéticos, con Excel. En *Actas, Tópicos Selectos de Educación en CITeM* (págs. 37-54). Ecorfan.

Casey, S. A. (2015). Examining Student Conceptions of Covariation: A Focus on the Line of Best Fit. *Journal of Statistics Education*, 23(1). doi:10.1080/10691898.2015.11889722

Chance, B., & Rossman, A. (2015). *Investigating Statistical Concepts, Applications, and Methods.* Brooks/Cole Thomson Learning.

Santos-Trigo, L. M. (2014). *La resolución de problemas matemáticos. Fundamentos cognitivos* (2a ed.). México: Trillas.

LINEAR OR NONLINEAR? RELATING COLLEGE ALGEBRA STUDENTS' COVARIATIONAL REASONING AND GRAPH SELECTION

Robert Knurek
University of Colorado Denver
robert.knurek@ucdenver.edu

Heather Lynn Johnson
University of Colorado Denver
heather.johnson@ucdenver.edu

Keywords: Assessment, Mathematical representations, Reasoning and proof

We investigate the following problem: How does students' covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017) relate to their graph selection on a fully online assessment? In particular, we focus on students' distinction between linear and nonlinear graphs representing the same direction of change in variables. Figure 1 shows two such graphs.

Figure 1: Two graphs relating diameter and height for a fishbowl filling with water

When a student engages in covariational reasoning, they can conceive of relationships between attributes that they view to be capable of varying and possible to measure (Carlson et al., 2002; Thompson & Carlson, 2017). Per the framework from Thompson and Carlson (2017) an early level of covariational reasoning is the "gross coordination of values," which refers to a loose connection between the direction of change in attributes. For either graph in Figure 1, a student reasoning this way can conceive of the diameter increasing then decreasing while the height continues to increase (see Figure 1).

We situate this case study (Stake, 1994) in a larger interview-based validation study of a fully online covariation assessment (Johnson et al., 2021). The assessment contains six items; students view a video animation, then select one of four graphs that best represents a relationship between variables. We report on a case of an early undergraduate student, Maya, who spontaneously wondered how to select between graphs such as the ones shown in Figure 1.

On three items, Maya's covariational reasoning was at least at a level of gross coordination of values, as evidenced by her explanations. Appealing to the direction of change in attributes or a value of an attribute at a single instance, Maya narrowed down the graphs to two choices. Both represented the same direction of change in related attributes. She then selected a graph based on physical aspects of the situation (e.g., "the motion of the cone seems more linear than, than like a wave"). For Maya, either graph was satisfactory. The assessment forced her to make a choice.

Maya's case illustrates how students' covariational reasoning can intertwine with their interpretation of physical aspects of the situation. She demonstrates how students can engage in both figurative and operative thinking (Moore et al., 2019) when interpreting graphs, in ways that are compatible from a student perspective.

Acknowledgement. This research was supported by U.S. National Science Foundation award 2013186. Opinions, findings, and conclusions are those of the authors.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378. https://doi.org/10.2307/4149958

Johnson, H. L., Olson, G., Smith, A., Gardner, A., Wang, X., & Donovan, C. (2021). Validating an assessment of students' covariational reasoning. In Olanoff, D., Smith, K., and Spitzer, S (Eds.), *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 63-67). Philadelphia, PA.

Moore, K. C., Stevens, I. E., Paoletti, T., Hobson, N. L. F., & Liang, B. (2019). Pre-service teachers' figurative and operative graphing actions. *The Journal of Mathematical Behavior*, *56*, Article 100692. https://www.sciencedirect.com/science/article/pii/S0732312318300269

Stake, R. E. (1994). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 236–247). Sage Publications, Inc.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Chapter 8:

Number Concepts and Proportional Reasoning

SENTIDO NUMÉRICO ACERCA DE LOS NÚMEROS REALES: UN ESTUDIO INICIAL CON ALUMNOS DE 15-16 AÑOS

NUMBER SENSE ABOUT REAL NUMBERS: AN INITIAL STUDY WITH STUDENTS AGED 15-16

Virginia Garrido	Olimpia Figueras	Minerva Martínez
Cinvestav	Cinvestav	Cinvestav
virginia.garrido@cinvestav.mx	figuerao@cinvestav.mx	mmartinezo@cinvestav.mx

El sentido numérico ha sido estudiado desde diferentes perspectivas. Definirlo no ha sido una tarea sencilla y la mayoría de las definiciones que existen en la literatura versan sobre los números naturales, enteros y racionales. En esta comunicación se dan a conocer los avances de un estudio cuyo objetivo es caracterizar el sentido numérico acerca de los números reales. A partir de la revisión de la literatura se propusieron 15 habilidades, éstas se consideraron como indicadores del uso de un buen sentido numérico acerca de los números reales. Los indicadores fueron puestos a prueba al analizar las respuestas a preguntas de un cuestionario. Con el mismo test fue posible observar dificultades que enfrentan los estudiantes al usar tales números, y cómo en este estudio, el uso de la calculadora no fue determinante para valorar aspectos relacionados con el sentido numérico.

Palabras clave: High School Education, Number Concepts and Operations

En los Estándares Curriculares y de Evaluación para la Educación Matemática de los Estados Unidos (NCTM, 1989) se menciona que la expresión *sentido numérico* se refiere a la comprensión general que tiene una persona sobre los números, junto con la habilidad de usarla de forma flexible para hacer juicios matemáticos e implementar estrategias numéricas. Desarrollar un *buen sentido numérico* implica entender el sistema decimal, ser consciente de las múltiples relaciones que se dan entre los números (tanto gráficas, como simbólicas), reconocer su magnitud relativa, considerar el efecto relativo de las operaciones numéricas y disponer de puntos de referencia para las mediciones de objetos comunes y situaciones en el entorno.

Psicólogos e investigadores de educación matemática han propuesto definiciones de sentido numérico de acuerdo con el contexto y época en la que desarrollaron su investigación. Whitacre, Hening y Atabaş (2017) hicieron un estudio teórico en el que revisaron 124 artículos y argumentaron que las complicaciones o inconsistencias para definir el sentido numérico se originan porque se trata de un caso de polisemia, se usa la misma palabra para tres constructos: sentido numérico innato (*innate number sense*), sentido numérico temprano (*early number sense*) y sentido numérico maduro (*mature number sense*). Es decir, algunos investigadores reportan hallazgos sobre un constructo y otros sobre uno distinto usando el mismo término; esto genera una confusión acerca de lo que se va a entender por sentido numérico. Así que es necesario aclarar que en la investigación de la que forma parte este informe la expresión *sentido numérico* se considera como el constructo *sentido numérico maduro (mature number sense)* que se refiere a algo que no es innato y se puede desarrollar mediante la enseñanza en grados intermedios o posteriores a la educación básica.

Greeno, en 1991, afirmó que el sentido numérico es para la aritmética como el alfabeto para la escritura, por lo que promover su desarrollo en los primeros años escolares es indispensable. Sin embargo, Nguyen (2016) y Salinas (1985) pusieron de manifiesto que no sólo en la educación básica se requiere de una buena comprensión de los números, pues estos objetos matemáticos van cambiando y siendo más complejos a medida que se avanza en la educación posterior.

El desarrollo del sentido numérico ha sido estudiado en diferentes países, la mayoría de las investigaciones se han llevado a cabo con estudiantes de educación básica –de 5 a 14 años de edad– (ver, por ejemplo, Reys y Yang, 1998; Alajmi, 2009; Corso y Dorneles, 2017; Long y Melnikova, 2019). Los sistemas numéricos estudiados en este nivel educativo son los números naturales, enteros y racionales. Sin embargo, en el nivel medio superior –alumnos de 15 a 18 años de edad– además del dominio de esos sistemas numéricos, también se requiere emplear los números irracionales. En otras palabras, esos estudiantes necesitan usar los números reales, pero ¿han construido los conocimientos y desarrollado las habilidades necesarias para usar esos números con suficiente flexibilidad? Scalia (2000) y Castañeda (2011) dan un panorama sobre la respuesta a esta pregunta, pues documentaron dificultades que enfrentan alumnos de esas edades para comparar, ordenar y hacer operaciones con los números reales.

La investigación descrita en este documento forma parte de un estudio más amplio, en el cual se pretende desarrollar el sentido numérico de estudiantes del nivel medio superior acerca de los números reales y fortalecer aquel relacionado con los otros sistemas numéricos. Para ello es necesario analizar y reflexionar sobre semejanzas y diferencias que existen entre el sentido numérico definido por otros investigadores y el sentido numérico que se define en esta investigación. En este informe, se expone un primer acercamiento para responder la pregunta ¿Cómo caracterizar el sentido numérico acerca de los números reales? y el objetivo es determinar si dicha caracterización permite obtener evidencias de cómo usan el sentido numérico aquellos estudiantes que recién ingresaron al bachillerato.

Marco de referencia

Varios investigadores han propuesto definiciones que fueron modificadas por otros con la intención de esclarecer qué se debe entender por sentido numérico. Aun cuando tenían diferentes perspectivas, todos coinciden en la importancia de desarrollar un ‘buen sentido numérico’.

Greeno (1991) lo definió como una ‘expertez’ cognitiva, es decir, como el conocimiento que resulta de una actividad extensa a través de la cual las personas aprenden a interactuar exitosamente en diversos dominios conceptuales. Él describe al sentido numérico como un conocimiento situado dentro de un dominio conceptual y plantea metáforas con las cuáles es posible entender que el sentido numérico es diferente a los temas que regularmente se enseñan; pertenece a otra dimensión, si se imagina que las operaciones básicas están en un plano, el sentido numérico debería estar en una tercera dimensión; como un dron desde el cual se pueden ver y seleccionar las operaciones más convenientes para lograr un propósito.

A su vez Sowder (1992a) define el sentido numérico como una red conceptual bien organizada que le permite a uno relacionar las propiedades de los números y las operaciones, así como resolver problemas numéricos de formas flexibles y creativas. Para Marshall (1989), el sentido numérico es la riqueza de conexiones del conocimiento matemático.

Las tres definiciones anteriores en su momento tuvieron gran aceptación por parte de la comunidad científica, empero son muy generales. Un investigador puede imaginar la ‘expertez’, la red conceptual o la riqueza de conexiones; pero observar ese tipo de cosas en el aula no es tarea fácil, por ello las autoras de esta investigación decidieron buscar más definiciones o características que permitieran notar el sentido numérico que han desarrollado los estudiantes.

Como resultado de esa búsqueda se observó que el sentido numérico había sido definido de muchas formas (ver Reys y Barger, 1991; Reys, 1994; Löwenhielm, Marschall, Sayers y Andrews, 2017) pero invariablemente se identificaba la presencia de dos componentes: conocimientos y habilidades. Los conocimientos comprenden la información que se va a usar para desarrollar habilidades y ambos, en conjunto dan origen al sentido numérico.

Para caracterizar el sentido numérico acerca de los números reales, se analizó si alguna de las definiciones planteadas en estudios previos se adecuaba a las necesidades de la investigación global que lleva a cabo. La conclusión fue que lo más conveniente era formular otra definición que permitiera incluir actividades en las que se requirieran números reales, la propuesta es la siguiente:

> El sentido numérico es el conjunto de conocimientos y habilidades acerca de los números reales que usa una persona para hacer juicios matemáticos y desarrollar estrategias numéricas al resolver problemas.

Como la investigación es sobre el sentido numérico acerca de los números reales, los conocimientos son las propiedades de dicho sistema numérico y las habilidades son sobre su uso y las diferentes formas de representarlos. A continuación, se enlistan las habilidades que se han propuesto para la investigación que se lleva a cabo:

1. Habilidad para componer, descomponer y recomponer números
2. Habilidad para identificar cuál representación de un número es más conveniente que otra
3. Habilidad para comparar números
4. Habilidad para ordenar números
5. Habilidad para lidiar con el orden de magnitud de un número en situaciones concretas
6. Habilidad para usar puntos de referencia
7. Habilidad para vincular símbolos de operación y relación de manera significativa
8. Habilidad para reconocer los efectos de las operaciones en los números
9. Habilidad para hacer cálculos mentales mediante estrategias propias
10. Habilidad para hacer estimaciones
11. Habilidad para asociar los números con el contexto en el que aparecen
12. Habilidad para localizar números en la recta numérica
13. Habilidad para reconocer hechos dados
14. Habilidad para distinguir diferencias y semejanzas entre los sistemas numéricos
15. Habilidad para autorregularse

Las primeras diez habilidades tienen como base los comportamientos que demuestran la presencia del sentido numérico sugeridos por Sowder (1992b). Las siguientes cuatro habilidades fueron propuestas por las autoras de esta comunicación y están relacionadas con conocimientos específicos de los números reales, tales como la densidad, el proceso de racionalización y la potenciación que tienen características distintas a las de los otros sistemas numéricos. La última habilidad fue propuesta por Resnick (1989), se refiere a que una persona debe reconocer cuando en un procedimiento es mejor detenerse e iniciar otro camino para encontrar la solución de algo.

Las habilidades anteriores se pueden considerar como indicadores que permiten valorar el uso del sentido numérico por parte de los estudiantes. Para verificar si todas las habilidades propuestas son necesarias o si se requiere de alguna otra, se aplicó un test que se describe en la sección de método.

Método

Participantes. Fueron 56 estudiantes mexicanos, cuya edad oscila entre 15 y 16 años. En agosto de 2021 iniciaron su educación media superior en una institución pública ubicada en una zona rural.

Recolección de los datos. Los datos se obtuvieron a través de un cuestionario inicial. Para diseñar el test se tomaron en cuenta ítems sobre sentido numérico usados en otras investigaciones con adaptaciones y preguntas construidas exprofeso. Después de una validación con pares y un estudio piloto, el instrumento para la toma de datos quedó compuesto por 16 preguntas, de las cuales 8 son de elaboración propia. Además, se especificó cuáles habilidades y conocimientos que forman parte del sentido numérico están relacionadas con cada ítem. En la Tabla 1 se muestran dos ejemplos.

Tabla 1: Ejemplos de habilidades y conocimientos relacionadas con cada ítem

Ítem	Habilidad	Conocimiento	Intención
6. ¿Cómo se puede usar el resultado $156 - 100$ para calcular $156 - 99$? Adaptado de Hope (1989)	Habilidad para hacer cálculos mentales mediante estrategias propias.	Valor posicional de los números.	Observar la flexibilidad al hacer operaciones.
8. En la recta ubica los números -0.85, $-\frac{11}{8}, -\frac{9}{8}, -\sqrt{2}$	Habilidad para localizar números en la recta numérica.	Los números racionales tienen dos representaciones: una decimal y otra fraccionaria. Los números irracionales se pueden aproximar con números decimales.	Observar si las estrategias usadas para ubicar a los números positivos son congruentes al localizar números negativos

La aplicación del cuestionario inicial se llevó a cabo de forma presencial en octubre de 2021 en dos sesiones, en días diferentes pero consecutivos durante el horario de la asignatura Álgebra. Los estudiantes previamente fueron notificados de las fechas para este cuestionario y el material que necesitarían: papel, lápiz, colores y calculadora. Aunque el tiempo planeado para contestar cada parte del test era de 60 minutos no se dio esta información a los alumnos para reducir la presión o angustia. La mayoría lo entregó a los 50 minutos, en la última hoja del instrumento se anotó el tiempo de entrega de cada uno.

Los estudiantes estaban separados en dos grupos de 28 alumnos; a un grupo se le permitió el uso de calculadora y al otro no. Esta variante se propuso porque en investigaciones como las de Bobis (1991), Alajmi (2009) y Lyublinskay (2009) se plantea la posibilidad de usar recursos tecnológicos para desarrollar el sentido numérico. Se decidió iniciar la exploración con la calculadora por ser un recurso de fácil acceso.

Con previa autorización de los tutores de los alumnos, se grabó un video de la sesión del grupo que usó calculadora. Como no fue posible enfocar la cámara y grabar qué tecleaban o qué cálculos hacían los estudiantes con ella, al final se les pidió que en la mitad de una hoja blanca escribieran cómo usaron la calculadora y qué tipo de operaciones habían hecho con ella.

Análisis de los datos. El análisis de los datos se llevó a cabo en dos etapas, en la primera se hizo una distinción sobre cuáles fueron las preguntas en las que se tuvo mayor o menor éxito. En la Tabla 2 se muestra un ejemplo de este análisis.

Tabla 2: Análisis de la pregunta 1

Número de estudiantes	Con calculadora	17	Contestaron	17	Correctamente	9
					Incorrectamente	8
			No contestaron	0		
	Sin calculadora	39	Contestaron	37	Correctamente	0
					Incorrectamente	37
			No contestaron	2		

Dado que cada pregunta estaba relacionada con al menos una de las habilidades sobre el uso de los números reales y las diferentes formas de representarlos, en la segunda etapa se analizó la pertinencia de los indicadores propuestos. Durante este análisis, surgió la oportunidad de identificar algunas dificultades que tenían los estudiantes al emplear estos números. Por cuestiones de espacio, sólo se explican las observaciones hechas para los ítems 1 y 3.

Con respecto al ítem 1, que está relacionado con la *habilidad para componer, descomponer y recomponer números*. La pregunta fue si $\sqrt{40}$ era igual a $2\sqrt{10}$, esperando que los estudiantes se percataran que $\sqrt{40}$ se puede descomponer como $\sqrt{4(10)}$ además de tomar en cuenta que la raíz cuadrada de un producto es el producto de las raíces, es decir, $\sqrt{4(10)} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$. En la Figura 1 se puede notar que algunos estudiantes por el hecho de ver números distintos aseguran que las expresiones numéricas son diferentes, no recurren a la descomposición del número. Entonces la *habilidad para componer, descomponer y recomponer números* es necesaria para desarrollar estrategias numéricas. Buscar comportamientos asociados a esta habilidad permitirá encontrar evidencias sobre el uso del sentido numérico por parte de los estudiantes. Por otro lado, el que no desarrollen esta habilidad provoca que tengan dificultades para escribir los números reales de diferentes formas.

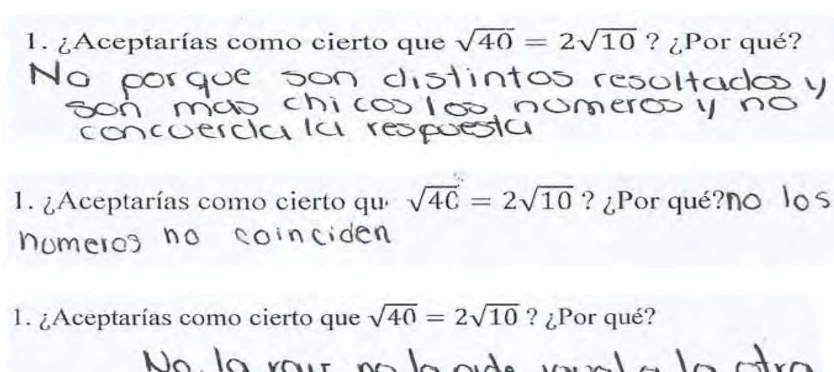
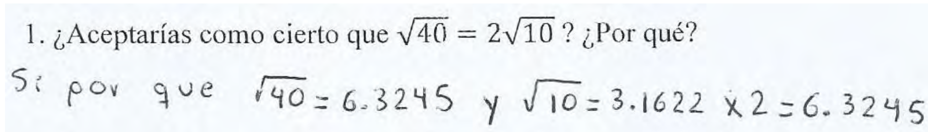


Figura 1. Respuestas al ítem 1, dadas por estudiantes sin uso de calculadora

El grupo de estudiantes que usó calculadora hizo las operaciones correspondientes y un poco más de la mitad dio una respuesta correcta, aunque no hubo evidencia de que los estudiantes usaran la *habilidad para componer, descomponer y recomponer números*, ver Figura 2.

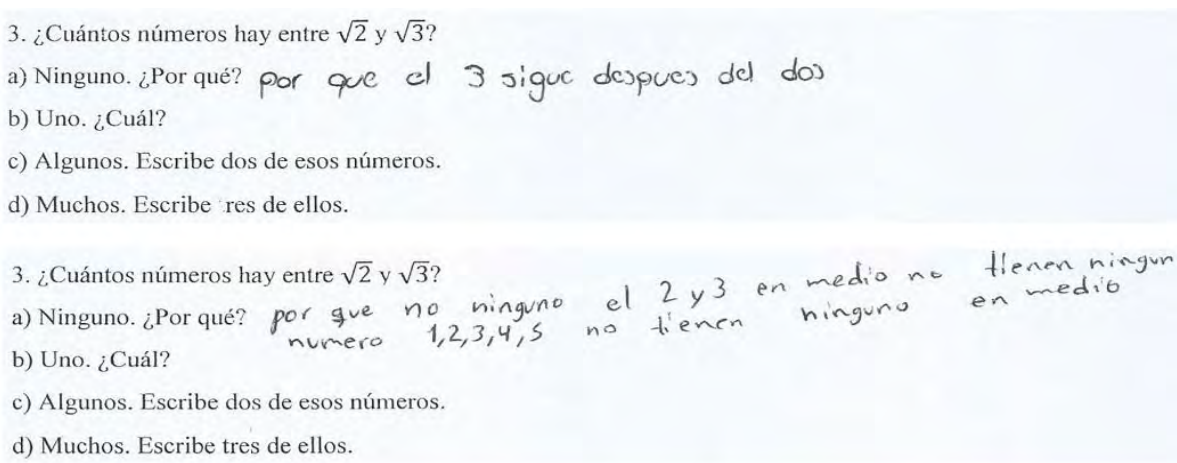

1. ¿Aceptarías como cierto que $\sqrt{40} = 2\sqrt{10}$? ¿Por qué?

Si por que $\sqrt{40} = 6.3245$ y $\sqrt{10} = 3.1622 \times 2 = 6.3245$

Figura 2. Respuesta al ítem 1, un estudiante con uso de calculadora

El ítem 3 (ver Figura 3) fue adaptado de Reys y Yang (1998), el propósito de la pregunta original era evaluar el conocimiento que tenían los estudiantes sobre la densidad de los números decimales. En este caso el ítem está relacionado con la *habilidad para distinguir diferencias y semejanzas entre los sistemas numéricos*. Es posible que en la educación básica los estudiantes no hayan tenido la necesidad de distinguir a qué sistema numérico pertenecían los números que estaban usando; sin embargo, en bachillerato hay asignaturas como cálculo diferencial, cálculo integral y probabilidad en las cuáles se requiere considerar las propiedades de los números que se están empleando (ver Secretaría de Educación Media Superior, 2022).

En la Figura 3 se puede ver como los estudiantes hacen referencia a una característica de los números naturales sin percatarse de que la pregunta involucra dos números irracionales. Los alumnos que podían utilizar la calculadora tampoco reflexionaron sobre las propiedades de los números que usaron porque aun cuando tenían la aproximación en decimales de los números $\sqrt{2}$ y $\sqrt{3}$ no lograron pensar en 3 números entre los extremos del intervalo, ver Figura 4.


3. ¿Cuántos números hay entre $\sqrt{2}$ y $\sqrt{3}$?

a) Ninguno. ¿Por qué? por que el 3 sigue despues del dos

b) Uno. ¿Cuál?

c) Algunos. Escribe dos de esos números.

d) Muchos. Escribe tres de ellos.

3. ¿Cuántos números hay entre $\sqrt{2}$ y $\sqrt{3}$?

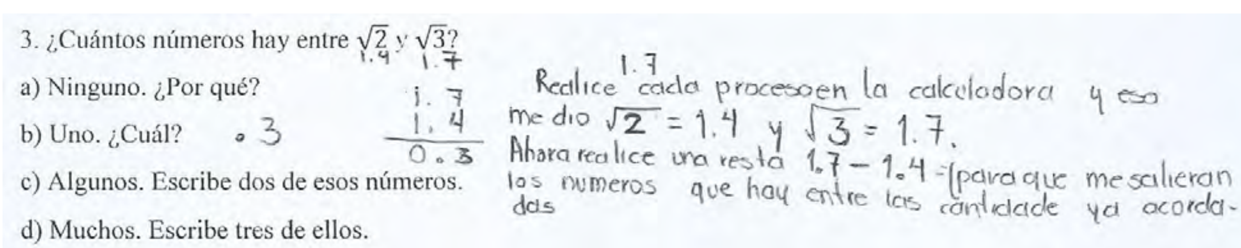
a) Ninguno. ¿Por qué? por que no ninguno numero 1,2,3,4,5 no tienen el 2 y 3 en medio no ninguno tienen ningun en medio

b) Uno. ¿Cuál?

c) Algunos. Escribe dos de esos números.

d) Muchos. Escribe tres de ellos.

Figura 3. Respuestas al ítem 3, dadas por estudiantes sin uso de calculadora


3. ¿Cuántos números hay entre $\sqrt{2}$ y $\sqrt{3}$?

a) Ninguno. ¿Por qué?

b) Uno. ¿Cuál? .3

c) Algunos. Escribe dos de esos números.

d) Muchos. Escribe tres de ellos.

1.4 1.7

1.7 − 1.4 = 0.3

Realice cada proceso en la calculadora y eso me dio $\sqrt{2} = 1.4$ y $\sqrt{3} = 1.7$. Ahora realice una resta 1.7 − 1.4 = (para que me salieran los numeros que hay entre las cantidade ya acorda-das

Figura 4. Respuesta al ítem 3, dada por un estudiante con uso de calculadora

El que para los estudiantes pasen desapercibidas las propiedades de los sistemas numéricos repercute en cómo hacen juicios matemáticos; por lo tanto, se puede asegurar que la *habilidad para distinguir diferencias y semejanzas entre los sistemas numéricos* es un indicador del uso de sentido numérico por parte de los estudiantes. El no haber desarrollado esta habilidad, puede estar relacionada con la dificultad que enfrentan los estudiantes al calcular la raíz cuadrada de un número o el que no se den cuenta que 0.3 no está entre los números 1.4 y 1.7 como lo asegura el estudiante con uso de calculadora, ver Figura 4.

Un análisis similar se hizo con las respuestas a cada ítem del cuestionario inicial, buscando evidencia sobre la pertinencia de los indicadores propuestos e identificando las dificultades que enfrentaban los estudiantes. En la sección de resultados se enuncian los hallazgos más importantes.

Resultados

Existen tres vertientes para los resultados: 1) sobre la pertinencia de los indicadores, 2) sobre el uso de la calculadora, y 3) con respecto a las dificultades que enfrentan los alumnos al ingresar al nivel medio superior.

Las habilidades que permiten analizar el desarrollo del sentido numérico propuestas por Sowder (1992b), que al parecer sólo se habían considerado para números naturales, enteros y racionales, también parecen servir como indicadores para evaluar el sentido numérico acerca de los números reales. Por ejemplo: *la habilidad para usar puntos de referencia* es necesaria para aproximar números reales, decidir qué valor es más grande $sen(60°) = \frac{\sqrt{3}}{2}$ o $\cos(45°) = \frac{\sqrt{2}}{2}$.

Por otro lado, los indicadores que se agregaron tienen pertinencia. La *habilidad para ubicar números en la recta* podría ser una herramienta para que los estudiantes comparen y ordenen números; la *habilidad para lidiar con el orden de magnitud de un número en situaciones concretas* ayudaría a que juzguen la razonabilidad de sus respuestas, es decir, reflexionen sobre la relación entre números y cantidades.

En el cuestionario inicial se planteó un ítem para validar la habilidad 15; sin embargo, la actuación asociada a dicha habilidad no es fácil de observar ya que está relacionada con funciones ejecutivas superiores. Por lo que será necesario analizar la pertinencia de esta habilidad o una mejor manera de valorarla. Un indicador que no estaba considerado, pero que parece ser necesario es la *habilidad para reconocer patrones*; en un ítem del cuestionario, para encontrar la solución, los estudiantes deberían notar que la suma de un número par de números nones es par, en otras palabras, reconocer un patrón. Aunque en el ítem la habilidad se observó para números naturales, ésta también podría ser un indicador del desarrollo del sentido numérico acerca de los números reales por parte de los estudiantes, pues les permitiría reconocer que si π es un número irracional, los números como $-2\pi, 3\pi, \frac{\pi}{4}$ también lo son.

Sobre el uso de calculadora, no hubo una diferencia significativa en las respuestas correctas e incorrectas dadas por los estudiantes (ver Tabla 2). Esto pudo haber sucedido porque previamente no se dio una instrucción sobre el funcionamiento o ventajas que ofrecía este artefacto, ya que en bachillerato sólo en temas específicos se permite el uso de la calculadora (Figueras, Valenzuela y Martínez, 2021). Por otro lado, se puede asegurar que el uso de la calculadora no sesga la información sobre el sentido numérico que han desarrollado los estudiantes porque habilidades *como descomponer, componer y recomponer números* o *distinguir diferencias y semejanzas entre los sistemas numéricos* deberían poderse observar con o sin calculadora. De hecho, si vemos el sentido numérico como una red conceptual con la que se entretejen conocimientos matemáticos como propuso Sowder (1992a), dentro de estos conocimientos pudiera estar el uso inteligente de la calculadora.

Con respecto a las dificultades que enfrentan los estudiantes al ingresar al bachillerato, al analizar sus respuestas se identificó que se les complica leer expresiones numéricas, escribir números de diferentes formas, compararlos, ordenarlos, usar puntos de referencia, hacer estimaciones y cálculos de forma flexible, así como identificar a qué sistema numérico pertenecen los números que están usando.

El que los estudiantes tengan las dificultades mencionadas en el párrafo anterior genera una disonancia crítica. Los profesores de educación media superior, inmersos en los tiempos y contenidos marcados en los planes y programas de estudio, asumen que los alumnos conocen y pueden usar los números reales sin complicaciones así que continúan con las actividades correspondientes a los temas del curso; mientras que algunos estudiantes no logran entender por qué una expresión numérica es igual a otra o siguen pensando que entre dos números decimales no hay otros números. Esto pone en desventaja a muchos jóvenes, que de acuerdo con Nguyen (2016) al no desarrollar un buen sentido numérico su desempeño en matemáticas se ve disminuido.

Una de las autoras es profesora en la institución educativa en la que se lleva a cabo la investigación y durante varios años ha observado como esta disonancia crítica abre una brecha entre los estudiantes que han desarrollado un buen sentido numérico y los que no. Fortalecer el sentido numérico de los estudiantes contribuye a que todos tengan mejores oportunidades para concluir el bachillerato, incluso de seguir su preparación académica ya que en los exámenes de admisión a la universidad casi el 40% de los reactivos son de matemáticas.

A manera de conclusión, la caracterización del sentido numérico acerca de los números reales se hizo a través de un conjunto de habilidades y conocimientos para que fuera operativa y de esta manera poder valorar el sentido numérico que los estudiantes han desarrollado. Las dificultades identificadas alertan sobre focos de atención para diseñar actividades que coadyuven al desarrollo del sentido numérico, una segunda fase de la investigación que se está llevando a cabo.

Referencias

Alajmi, A. (2009). Addressing computational estimation in the Kuwaiti curriculum: teachers' views. *Journal of Mathematics Teacher Education. 12,* 263–283.

Bobis, J. (1991). Using a calculator to developing number sense. *The Arithmetic Teacher*, 38(5), pp. 42 – 45.

Castañeda, Y. (2011). Introducción de prerrequisitos de Cálculo en un curso de Geometría Analítica de Bachillerato (Tesis de maestría) Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional. México.

Corso, L. & Dorneles, B. (2017). Working memory, number sense and arithmetical performance: Relations between these domains. CERME 10, Feb 2017, Dublin, Ireland.

Figueras, O., Valenzuela, C. y Martínez, M. (2021). ¿Debemos usar calculadoras en un examen? *Revista de Didáctica de las Matemáticas*, *92 (1)*, 45-53.

Greeno, J. (1991). Number Sense as Situated Knowing in a Conceptual Domain. *Journal for Research in Mathematics Education*. 22 (3), 170-218.

Long, V.N. & Melnikova Y. (2019). Number sense: The case of Sara. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 218-219). St Louis, USA.

Löwenhielm, A., Marschall, G., Sayers, J. & Andrews, P. (2017). Opportunities to acquire foundational number sense: A quantitative comparison of popular English and Swedish textbooks. CERME 10, Feb 2017, Dublin, Ireland.

Lyublinskay, I. (2009). Developing number sense with technology-based science experiments: Reflections on classroom practice in primary grades and pre-service education. *Mathematics Teaching – Research Journal* 3(2), pp. 1-14.

Marshall, S. P. (1989). Retrospective paper: Number sense conference. In J. T. Sowder & B. P. Schappelle (Eds.), Establishing foundations for research on number sense and related topics: Report of a conference (pp. 40-42). San Diego: San Diego State University Center for Research in Mathematics and Science Education.

National Council of Teachers of Mathematics (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA. NCTM.

Nguyen, V. L. (2016). Number Sense in High School Mathematics Students (Requirements for the Degree of Master). University of Texas. USA.
Resnick, L. B. (1989). Defining, assessing, and teaching number sense. In J. T. Sowder & B. P. Schappelle (Eds.), Establishing foundations for research on number sense and related topics: Report of a conference (pp. 35-39). San Diego: San Diego State University Center for Research in Mathematics and Science Education.
Reys, B. (1994). Promoting Number Sense in the Middle Grades. *National Council of Teachers of Mathematics, 1(2),* 114-120.
Reys, B. y Barger, R. (1991). Developing number sense in the middle grades. USA. National Council of Teachers of Mathematics.
Reys, R.E. & Yang D.C. (1998). Relationship between Computational Performance and Number Sense among Sixth- and Eighth-Grade Students in Taiwan. *Journal for Research in Mathematics Education*, 29(2), 225-237.
Salinas, P. (1985). Obstrucciones e imágenes conceptuales en el aprendizaje de los números reales. (Tesis de doctorado). Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional. México.
Scaglia, S. B. (2000). Dos conflictos al representar números reales en la recta (Tesis de doctorado). Universidad de Granada. España.
Secretaría de Educación Media Superior (2022). Nuevo currículum de la educación media superior. Ciudad de México, México. http://www.sems.gob.mx/curriculoems/programas-de-estudio
Sowder, J. T. (1992a). Estimation and number sense. In Grouws D. (National Council of Teachers of Mathematics). Handbook of research on mathematics teaching and learning. (pp. 371-395). San Diego, USA.
Sowder, J. T. (1992b). Making sense of numbers in school mathematics. In Leinhardt G., Putnam R. y Hattrup R.A. (Lawrence Erlbaum Associates). Analysis of arithmetic for mathematics teaching. (pp. 1-51). New Jersey, USA.
Whitacre, I., Hening, B. & Atabaş S. (2017). Disentangling the research literature on “number sense”: three constructs, one name. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 203-210) Indianapolis, USA.

NUMBER SENSE ABOUT REAL NUMBERS: AN INITIAL STUDY WITH STUDENTS AGED 15-16

Number sense has been studied from different perspectives. Defining it has not been an easy task and most of the definitions that exist in the literature deal with natural, integer and rational numbers. In this communication, the advances of a study whose objective is to characterize the number sense about real numbers are disclosed. From the review of the literature, 15 skills were proposed, these were considered as indicators of the use of a good number sense about real numbers. The indicators were put to the test by analyzing the responses to questions in a questionnaire. With the same test it was possible to observe difficulties that students face when using such numbers, and how in this study, the use of the calculator was not decisive to assess aspects related to number sense.

PROMOTING PROPORTIONAL REASONING WITH THE SUPPORT OF DIGITAL TECHNOLOGY

Erasmo Islas Ortiz
Cinvestav
erasmo.islas@cinvestav.mx

Carlos A. Cuevas-Vallejo
Cinvestav
ccuevas@cinvestav.mx

José Orozco-Santiago
Benemérita Universidad Autónoma de Puebla
jose.orozco@fcfm.buap.mx

In this report, we present an advance of doctoral research. We explore teaching alternatives that promote proportional reasoning in Mexican students between 14 and 15 years old with the support of digital technology. We designed a sequence of activities that pretends to signify the concepts of ratio and proportion in their diverse representations and from the perspective of linear functions. For the design of the tasks, elements of the Realistic Mathematics Education framework and Cuevas-Pluvinage Didactics were used. The learning objectives and proposed activities were organized through a hypothetical learning trajectory. Subject to presenting the group results, we randomly selected a student to present the analysis of the results achieved. We found that the tasks promoted proportional reasoning.

Keywords: High School Education, Technology, Rational Numbers & Proportional Reasoning

Introduction

The concepts associated with proportional reasoning, such as fraction, ratio, and proportion, are essential in mathematics education at basic levels since they are pillars of higher mathematics because their application is transversal to other disciplines (Lamon, 2007). Given the importance of the problem, we agree with several studies (Lesh et al., 1988; Adjiage & Pluvinage, 2007) that the lack of success in the handling of ratios and proportions is caused by poor development of proportional skills, as well as an excessive arithmetization of the processes involved. As a consequence of the lack of sense in dealing with proportional situations in the classroom, a tendency of students to apply the proportional model in situations where it is not applicable has been observed, which has been called the illusion of linearity (De Book et al., 2002).

The problem facing the community is how to teach students to reason proportionally. The most frequent school approach to proportionality is from an arithmetic perspective, separating it from its links with functional relationships and their multiple representations. In our proposal, we rescue the results of research in this line and test a teaching design based on a functional approach through realistic contexts and supported by digital technology, with which we hope to obtain ideas that contribute to the design of activities that promote proportional reasoning.

Theoretical framework

The Realistic Mathematics Education (RME) teaching approach emphasizes that real situations are fundamental for learning mathematics since mathematics is part of a historical and cultural construction (Freudenthal, 2002). In this current, students are active participants in developing their learning and constructing models that mathematize reality from an everyday context (Gravemeijer, 2004). In RME, two types of mathematization are identified: horizontal mathematization, where students move from the real to the symbolic of responding to problems in their context; and vertical mathematization, where students make conceptual connections and create strategies to solve problems within the mathematical system, which may or may not refer to the initial context. That is, they detach from the context towards the path of abstraction and generalization (Van den Heuvel-Panhuizen & Drijvers, 2020).

As for proportional reasoning, we start from the definition of Lesh et al. (1988):

> Proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves both qualitative and quantitative methods of thought. (p. 93)

According to this rationale, we assume as valid the proportional reasoning model proposed by Modestou & Gagatsis (2010) and recognize the need for research invoked by the studies on the illusion of linearity by De Bock et al. (2002). In our study, we identify proportional skills as those faculties considered necessary for a person to possess sound proportional reasoning. Lobato et al. (2010) and Weiland et al. (2021) indicate that these skills should include: 1) Attending to and coordinating two quantities that vary dependently, 2) Recognizing and using the structures of proportional situations (unit ratio, constant of proportionality, linearity), 3) Identifying the invariant in a proportion, 4) Understanding proportionality from multiple representations, 5) Distinguishing linear from nonlinear situations. We consider these five skills as key to our study, and they are the ones we invoke with the term proportional skills.

To encompass the theoretical framework and structure of the intervention, we rely on the Design-Based Research (DBR) approach proposed by Bakker (2018). Since DBR design and innovation in the classroom are vital aspects, digital technology is proposed to allow students to interact with multiple representations, facilitate the simulation of realistic situations, and test their results in Virtual Interactive Didactic Scenarios (VIDS).

To organize the design of the tasks that guide students in the mathematization process, we rely on the Hypothetical Learning Trajectory (HLT) (Simon, 1995). For the design of the activities that integrate each task, we rely on the Cuevas-Pluvinage didactic framework, which provides didactic engineering that rescues several didactic principles of the active school and Piaget and adapts them to mathematics education (Cuevas & Pluvinage, 2003).

We agree with Sierpinska (2004) that "reporting tasks is necessary to probe the effectiveness of the teaching approach" (p. 13). According to the author, the problematization, justification, and discussion of the tasks begin with the a priori analysis of the design. However, the problematization is even more substantial and fruitful in the a posteriori analysis, that is, after the experimentation in the classroom. In this order of ideas, in this report, we show the problematization both in the design phase and the retrospective analysis phase since it provides guidelines for redesigning our tasks.

One of the most complex tasks for a teacher is implementing constructivist didactic principles in a traditional classroom with the usual elements. For example, introducing a mathematical concept through the problematization of an everyday situation in a conventional classroom is almost impossible. A possible solution is to have a virtual scenario that simulates a real phenomenon, in which students can interact with various contexts dynamically, and, in addition, its use allows the construction of mathematical knowledge (Moyer et al., 2016). In this way, students are endowed with the opportunity for action, where each student can learn at their own pace. Digital technology provides the student and the teacher with a portable laboratory using various devices, such as cell phones, tablets, or computers.

The above considerations lead us to the research question: How to design a didactic sequence, in realistic contexts mediated by technology, that helps students to develop their proportional reasoning and allows them to progress in the mathematical levels of RME?

Methodology

Based on our methodological framework, the DBR approach consists of the following phases: preparation and design, teaching experiment, and retrospective analysis.

Preparation and design phase

The following were designed and developed in this phase: 1) A pretest to collect students' prior knowledge. 2) A HLT who leads the teaching process and sets the learning objectives. 3) Three sequences of didactic activities with their respective VIDS: "Naranjada", "Zoom Totoro" and "Autos". 4) A survey to inquire about the students' perspectives on their experience using the VIDS and the Guided Learning and Exploration Sheets (GLES) activities. The pretest and final survey were hosted on Google forms to answer at home to save classroom time.

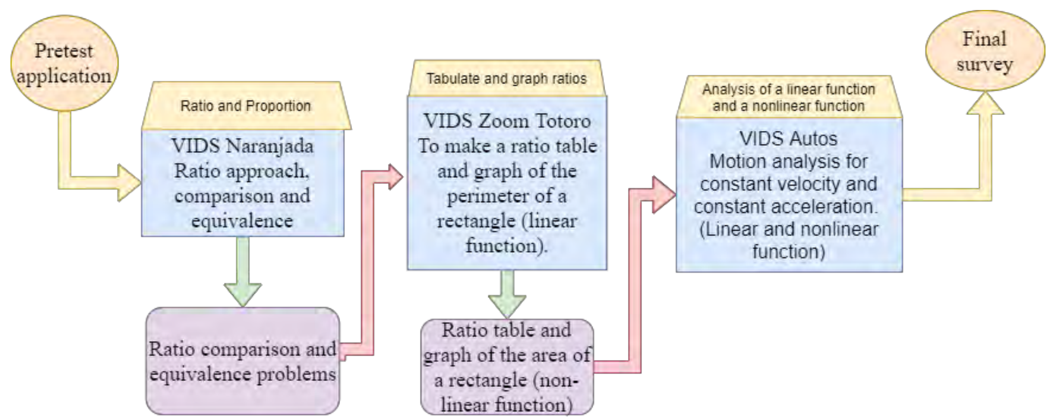


Figure 1: Design of the didactic pathway guiding the instructional sequence

Task 1. The task's context is to compare proposals for mixing orange juice and water (see Figures 2a and 2b). The objective is for students to learn: to pose, compare and determine equivalence between ratios, as well as to generate tables of equivalent ratios and identify the constant of proportionality. Finally, an extra activity is performed to apply the knowledge learned in a different context. The activities are: 1) Expressing a ratio in different notations, e.g., three glasses of juice for every five glasses of water: 3 to 5, 3:5 and 3/5. 2) Coordinate the covariation between two variables; given a ratio of A glasses of orange to every B glasses of water, the taste is more intense orange if A increases or B decreases. 3) Comparison of two ratios, given two random orangeade proposals with orange to water ratios A:B and C:D, determine which will have an intense orange flavor.

Task 2. The task consists of performing a "zoom" effect to reduce or increase an image of Totoro figures according to a similarity ratio entered in an input box (see Figure 2c). The objective is for students to perform similarity activities using the constant of proportionality. Also, it is intended to address the problem of the illusion of linearity by tabulating and graphing the ratio-perimeter (linear) and ratio-area (quadratic) relationships to lead students to compare both models.

Task 3. The task's context is to visualize a car moving at a constant speed above the allowed limit and a patrol car initiating a chase with constant acceleration as the car passes (see Figure 2d). The objective is for students to interact with the VIDS as they describe, analyze, and compare the characteristics of the two motions, i.e., uniform rectilinear motion (URM) and uniformly accelerated rectilinear motion (UARM). Students must identify the models present (linear and quadratic) and compare them as they transit between their representations.

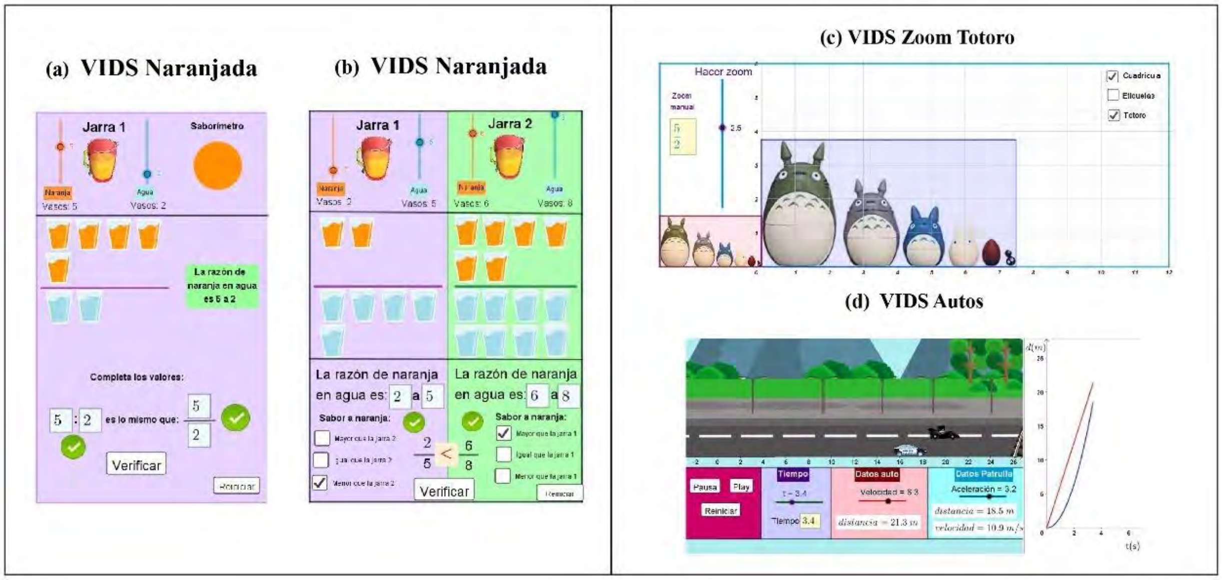


Figure 2. VIDS used in the teaching sequence

Teaching experiment phase

The study was performed through a face-to-face intervention in two groups in a secondary school in Mexico. Thirty-five students (14-15 years old) participated in the study in the winter of 2021. The instruction was divided into three 90-minute sessions in a computer room at the school. Each student had a personal computer that had VIDS preloaded, and each student had their respective GLES. The first author conducted the intervention, supported by the group's teacher and a research assistant. During the sessions, collaborative learning was promoted, and the answers were discussed as a group.

Results and retrospective analysis

According to Freudenthal (2002), the context and didactic design should allow students to move from horizontal mathematization to vertical mathematization. Gravemeijer (2004) identifies four levels towards vertical mathematization: N1) Situational level. Reality is interpreted and organized through informal and context-dependent mathematical reasoning (horizontal mathematization). N2) Referential level. Schemes and models that make sense within the initial context are created, and vertical mathematization begins with the emergence of "models of...". N3) General level. Concepts are related, strategies separated from the context are generated, reasoning occurs in the mathematical world, and "models for..." emerge. N4) Formal level. Concepts are understood by employing their mathematical symbolism, support from a real context is no longer needed, and reflection has moved to the mathematical world.

Subject to presenting the whole group's results, we randomly selected one student, whom we will call Lucy. Based on this student's findings, we will present a detailed analysis of the results of the tasks and the evaluation of the levels of vertical mathematization proposed in the RME.

Results of Task 1. "Naranjada"

Lucy uses her previous knowledge to perform the activities, i.e., the activities lead her to a horizontal mathematization, correctly assimilating the three notations for ratio. In addition, she found that in a random proposal of orange and water mixture, the intensity of flavor depends on a covariation process. At this stage, Lucy is at the situational level because she moves within the numerical world only to interpret the context and make a judgment. The horizontal mathematization is coming and going from the context to the mathematical world.

One finding occurred when the virtual scenario showed Lucy the following pair of orange to water ratios: 2:7 and 3:8 (see Figure 4a). Lucy mentally performed a 1-to-1 bijection between the glasses of orange and the glasses of water. Once the bijection was done, when the same number of glasses of water remained unassociated in both scenarios, Lucy intuitively responded that the two have the same taste, i.e., the same ratio. This erroneous way of comparing ratios, analogous to the erroneous additive principle, she extended to all situations, and it worked for her when the remaining glasses of water were different. However, when the remaining glasses of water were the same, the virtual scenario pointed out the error, which was incomprehensible to Lucy. At the close of the session, through group discussion, methods for comparing ratios were specified, and it was found that most students adopted the reasoning described. Some students were even surprised that the method failed. Since this exact situation was present in most students, we will call it a "proportionality illusion".

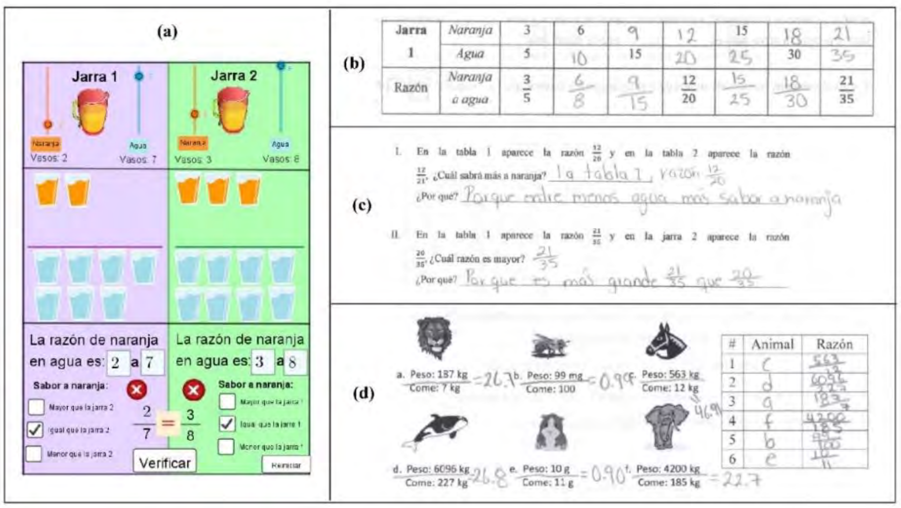


Figure 3. Evidence of the activities corresponding to Task 1.

The activity of completing tables of equivalent ratios is subject to the context (orange-water mixtures in the same proportion), but the multiplicative pattern that Lucy uses to fill the table is given in the mathematical domain; it is also observed that at the beginning, the additive reasoning persists (see Figure 3b). One way to compare equivalent ratios in the tables is to consider pairs of ratios that have equal numerators or denominators, and Lucy uses that fact (see Figure 3c) to make numerical judgments and gives a covariation argument based on the denominator of the ratios, so a referential level is evident.

Lucy uses the unit ratio method in application exercises in a context other than mixtures. Figure 3d shows a ratio ordering exercise (adapted from Lamon, 2020), which consists of ordering a set of animals about their weight. Lucy applies the unit ratio method to solve these problems. It is inferred that she has acquired a "model for" and progress from the referential level to the general level is glimpsed.

In this task, Lucy shows skills in posing ratios in different notations, comparing ratios, applying the multiplicative principle to obtain equivalent ratios, and using the unit ratio method to solve practical problems. At this level, students already possess proportional reasoning skills.

However, the data show that Lucy had a rootedness in additive reasoning in her knowledge structure, which she managed to modify and even consolidate the unit ratio method.

Results of task 2. "Zoom Totoro"

In the initial stage, processes of observation, measurement, and multiplicative principle are involved. The activity of calculating the dimensions of the image for different ratios is anchored to the context (situational level); however, following the pattern marks the way to the referential level since Lucy notices that to obtain the measurements of the replicated image, the measurements of the original image are multiplied by the similarity ratio (see table in Figure 4a).

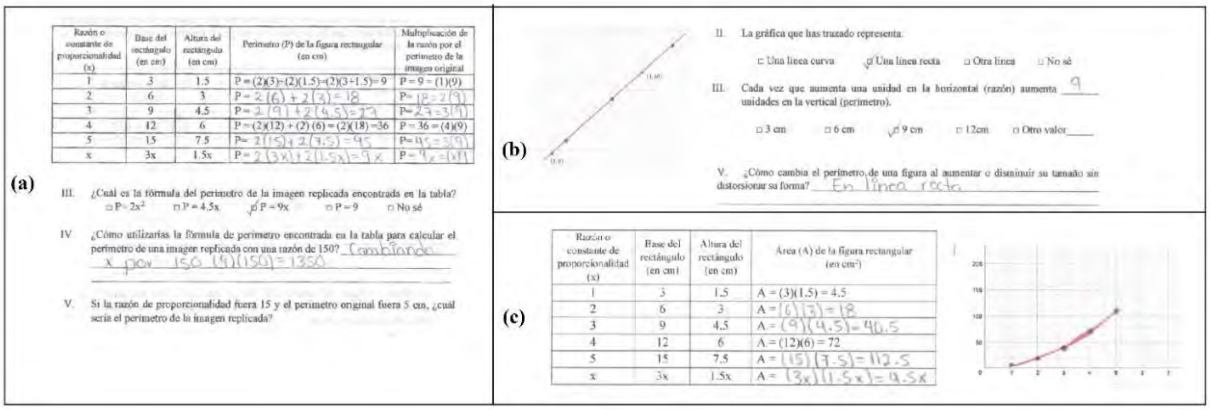


Figure 4. Evidence of the activities corresponding to Task 2.

In the table for calculating the perimeter of magnified images according to a similarity ratio, Lucy relies on the dimensions table. She departs from the context because the initial concept is related, in the first instance with the perimeter and the second with the area. Lucy calculates the perimeters for the proposed ratios by following the initial example in the table until she arrives at the linear expression P=9x and applies the "model of" found (see Figure 4a). To plot the ratio-perimeter relationship, Lucy transits between the tabular-graphical representations and manages to see the linear variation (see Figure 4b). As for the ratio-area relationship, she completes the table and makes the graph, describing it as curved (see Figure 4c). However, she does not conclude in the "model of" sought (a quadratic relationship) and does not get to the contrast between perimeter and area.

In performing this task, Lucy showed the ability to move between tabular, graphical and algebraic representations of proportionality; identifying the value of the constant of proportionality in the case of perimeter, and also obtaining a model that describes the particular case. However, Lucy is not able to arrive at a general model.

Results of task 3. "Autos"

Initially, the activities in this VIDS, are of instrumentation and contextualization. Lucy identifies the characteristics of movements, both at a constant speed and constant acceleration. Her previous and informal ideas about such concepts were modified after interacting with the VIDS and observing the change in the variables, their relationship, and numerical observations.

As shown in Figure 5c, Lucy transitions correctly between tabular and graphical representations and identifies linear and nonlinear features in both the graph and tables. In this task, Lucy sees relationships between variables that go beyond the initial context by identifying a "model of", using it, and concluding that the variation is proportional (in the car) and quadratic (in the patrol car); also, she distinguishes linear and nonlinear behaviors in the representations. However, Lucy failed to propose a model for constant acceleration.

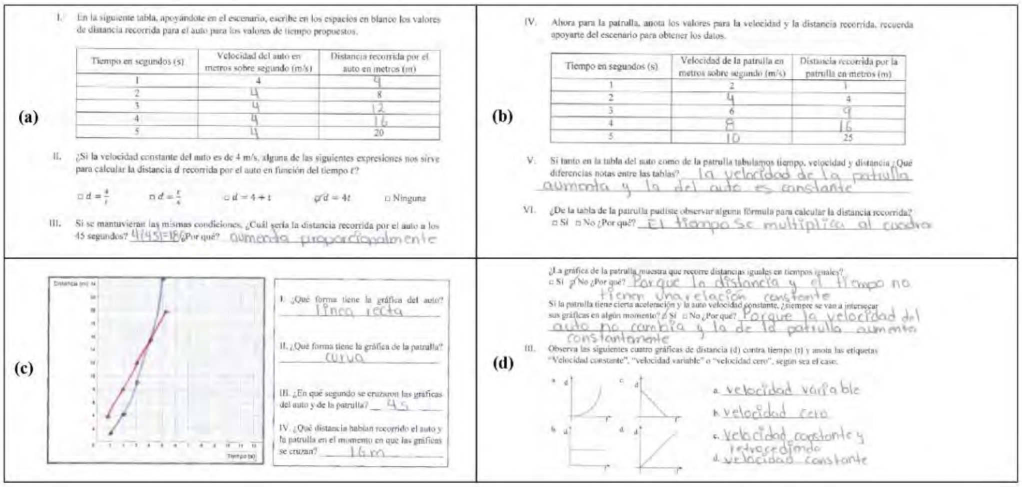


Figure 5. Evidence from Task 3 activities.

By visualizing and comparing the graphs of the movements for various parameters of speed and acceleration, Lucy interprets growth and variation, arguing: "if equal distances are traveled in equal times, distance and time have a constant relationship", and that the growth of acceleration will always exceed that of speed because "the speed of the car does not change and that of the patrol car is constantly increasing" (see Figure 5d). In Lucy's reasoning, we observe a conceptual enrichment that transcends the context, and although they are not numerical or algebraic manipulations, they are of a general nature. They are general reasonings that apply to any URM and UARM context. In the words of Freudental (2002), "The relationship between constant ratio and linearity is a feat of vertical mathematization" (p.43).

In performing this task, Lucy shows the ability to distinguish a linear situation from a nonlinear situation and identify the distinctive features of each in different representations. Thus, she shows clear progress in her proportional reasoning.

Discussion and conclusion

In Task 1, Lucy had difficulties comparing two ratios whose difference between numerator and denominator was the same, 2/7 and 3/8; that idea changed with the task. We infer that there are difficulties in students' understanding that emerge with this activity. According to Sierpinska (2004), "tasks should be able to reveal students' most hidden misconceptions" (p. 14). In Task 2, the objective is to confront the illusion of linearity regarding the confusion students present when assuming that area and volume have a linear relationship with the similarity ratio. However, the activities do not prove to be sufficient and should be redesigned. It is necessary to add geometric figures so that students distinguish the linear relationship of the perimeter with the quadratic relationship of the area and the cubic relationship of volume in a more playful geometric way. In Task 3, the detailed analysis of Lucy's responses sets the tone for more in-depth activities that bring out the "models for" and generalization.

The detailed analysis of Lucy's answers in each task shows us the learning she acquires to work with ratios, move between proportional representations and contrast them with non-linear phenomena in different contexts. The above gives us elements to answer our research question and gives us sufficient guidelines for redesigning the activities organized by the HLT. On the other hand, it provides us with enough evidence to discern which of the tasks should be deepened in order to achieve that students show better progress in the levels of vertical mathematization. That is to say, the diversity of contexts favors horizontal mathematization. Considering the times in the curriculum, the designers of didactic sequences must discern what is most convenient in each situation since reducing the number of contexts could increase the opportunity to achieve greater depth in the mathematization process. In this sense, we consider that there is also a need for fully mathematized activities without reference to any context that deals with the formal level.

In this development research, we try to find teaching ways to address the concepts involved in proportional reasoning through approaches to linear functions and their representations (symbolic, tabular, and graphical). In this sense, the first axis of the proposal is to prioritize the concept of ratio as a precursor to accessing equivalence, comparison, and ratio tables, which leads inexorably towards linear functions; the second axis is that the student faces diverse contexts to explore proportional situations in their different representations and the third axis consists of considering the phenomenon of the illusion of linearity. It is intended that students experience realistic situations, both linear and nonlinear, and question linearity (or the absence of it) each time they face a problem, reacting in a reasoned manner according to the characteristics of the situation.

Concerning the influence of technology in the study, it is clear that its use makes it possible to simulate and replicate everyday situations that can be problematized and provide pragmatic and epistemic advantages. For example, the use of VIDS makes it possible to present random data to the student, interact with contexts dynamically, show mathematical objects in different representations and validate results. As described in the analysis, the students' experience with VIDS made it possible to observe misconceptions that would go unnoticed, probably in traditional forms of teaching. On the other hand, interacting with the VIDS provoked an interest that went beyond the mathematical context posed, involving them with problems derived from the tasks.

The results are not conclusive; however, they encourage us to continue this work. It is necessary to mention that we had an external variable to consider since the students were returning to face-to-face classes after a long pandemic period.

References

Adjiage, R. & Pluvinage, F. (2007). An Experiment in Teaching Ratio and Proportion. *Educational Studies in Mathematics*, *65*(2), 149–175. https://doi.org/10.1007/s10649-006-9049-x

Bakker, A. (2018). *Design Research in Education: A Practical Guide for Early Career Researchers* (1st ed.). Routledge. https://doi.org/10.4324/9780203701010

Cuevas, C. A. & Pluvinage, F. (2003). Les projets d'action practique, elements d'une ingeniere d'ensigment des mathematiques. *Annales de didactique et de sciences cognitives*, *8*, 273-292.

De Bock, D., Van Dooren, W., Janssens, D. & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*(3), 311-334. https://doi.org/10.1023/a:1021205413749

Freudenthal, H., (2002) Revisiting mathematics education. China lectures. Kluwer, Dordrecht

Gravemeijer, K. (2004). Local Instruction Theories as Means of Support for Teachers in Reform Mathematics Education. *Mathematical Thinking and Learning*, *6*(2), 105–128. https://doi.org/10.1207/s15327833mtl0602_3

Lamon, S. J. (2007). Rational Numbers and Proportional Reasoning: Towards a Theoretical Framework for Research. En F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629-667). Information Age Publishing.

Lamon, S. J. (2020). *Teaching Fractions and Ratios for Understanding: Essential Content 83 and Instructional Strategies for Teachers*. Routledge, 2020.
Lesh, R., Post, T. & Behr, M. (1988). Proportional reasoning. En J. Hiebert y M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). National Council of Teachers of Mathematics.
Lobato, J. E., Ellis, A. B. & Charles, R. I. (2010). *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics in Grades 6-8*. National Council of Teachers of Mathematics.
Modestou, M. & Gagatsis, A. (2010). Cognitive and Metacognitive Aspects of Proportional Reasoning. *Mathematical Thinking and Learning*, *12*(1), 36-53. https://doi.org/10.1080/10986060903465822
Moyer-Packenham, P. S., Bolyard, J. J. (2016). Revisiting the Definition of a Virtual Manipulative. En Moyer-Packenham, P. (Eds.) *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives. Mathematics Education in the Digital Era*, vol 7. Springer, Cham. https://doi.org/10.1007/978-3-319-32718-1_1
Sierpinska, A. (2004). Research in Mathematics Education through a Keyhole: Task Problematization. *For the Learning of Mathematics*, *24*(2), 7–15. http://www.jstor.org/stable/40248450
Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145. https://doi.org/10.2307/749205
Van den Heuvel-Panhuizen M (2003) The didactical use of models in realistic mathematics education: an example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*(1):9–35
Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic Mathematics Education. In *Encyclopedia of Mathematics Education* (pp. 713–717). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_170
Weiland, T., Orrill, C.H., Nagar, G.G., Brown, R. & Burke, J. (2021) Framing a robust understanding of proportional reasoning for teachers. *J Math Teacher Educ*, *24*, 179–202. https://doi.org/10.1007/s10857-019-09453-0

PROMOVIENDO EL RAZONAMIENTO PROPORCIONAL CON APOYO DE LA TECNOLOGÍA DIGITAL

Erasmo Islas Ortiz
Cinvestav
erasmo.islas@cinvestav.mx

Carlos A. Cuevas-Vallejo
Cinvestav
ccuevas@cinvestav.mx

José Orozco-Santiago
Benemérita Universidad Autónoma de Puebla
jose.orozco@fcfm.buap.mx

En este artículo presentamos un avance de investigación doctoral en donde exploramos formas alternativas de enseñanza que promuevan el razonamiento proporcional en estudiantes mexicanos entre 14 y 15 años, con apoyo de la tecnología digital. Diseñamos una secuencia de actividades que pretende significar los conceptos de razón y proporción, en sus diversas representaciones y desde la perspectiva de las funciones lineales. Para el diseño de tareas se emplearon elementos del marco de la Educación Matemática Realista y de la Didáctica de Cuevas-Pluvinage. Los objetivos de aprendizaje y las actividades propuestas fueron organizadas mediante una trayectoria hipotética de aprendizaje. A reserva de presentar los resultados del grupo, seleccionamos aleatoriamente a una estudiante para presentar el análisis de los resultados obtenidos. Encontramos que las tareas promovieron un razonamiento proporcional.

Palabras clave: Educación secundaria, tecnología, números racionales y razonamiento proporcional

Introducción

Los conceptos asociados al razonamiento proporcional, tales como fracción, razón y proporción son clave en la educación matemática en los niveles básicos, ya que son pilares de las matemáticas superiores y su aplicación es transversal a otras disciplinas (Lamon, 2007). Dada la importancia de la problemática, coincidimos con diversos trabajos (Lesh et al., 1988; Adjiage y Pluvinage, 2007), en que la falta de éxito en el manejo de las razones y proporciones es causada por un pobre desarrollo de las habilidades proporcionales, así como una excesiva aritmetización de los procesos involucrados. Como consecuencia a la carencia de sentido al abordar situaciones proporcionales en el aula, se ha observado una tendencia de los estudiantes a aplicar el modelo proporcional en situaciones donde no es aplicable, lo que se ha denominado ilusión de la linealidad (De Book et al., 2002).

El problema al que se enfrenta la comunidad es cómo enseñar a los estudiantes a razonar proporcionalmente. El acercamiento escolar más frecuente hacia la proporcionalidad se da desde una perspectiva aritmética, separándola de sus vínculos con las relaciones funcionales y sus múltiples representaciones. En nuestra propuesta rescatamos los resultados de investigación en esta línea y ponemos a prueba un diseño de enseñanza basado en un acercamiento funcional mediante contextos realistas y apoyado en la tecnología digital; con lo cual, se espera obtener ideas que aporten al diseño de actividades que promuevan el razonamiento proporcional

Marco teórico

El enfoque de enseñanza de la Educación Matemática Realista (EMR) destaca que las situaciones realistas son fundamentales para el aprendizaje de las matemáticas, ya que las matemáticas son parte de una construcción histórica y cultural (Freudenthal, 2002). En esta corriente, los estudiantes son participantes activos en el desarrollo de su propio aprendizaje y en la construcción de modelos que matematizan la realidad a partir de un contexto cotidiano (Gravemeijer, 2004). En la EMR se identifican dos tipos de matematización: la matematización horizontal, donde los estudiantes transitan de lo real a lo simbólico para dar respuesta a problemas del propio contexto; y la matematización vertical, donde los estudiantes realizan conexiones conceptuales y crean estrategias para resolver problemas dentro del sistema matemático, que pueden o no referirse al contexto inicial. Es decir, se separan del contexto hacia el camino de la abstracción y la generalización (Van den Heuvel-Panhuizen y Drijvers, 2020).

En cuanto al razonamiento proporcional, partimos de la definición de Lesh et al. (1988):

> El razonamiento proporcional es la habilidad que permite trabajar con situaciones que impliquen variación, cambio, sentido de covariación y comparación múltiple, además de la capacidad de procesar y almacenar mentalmente varias piezas de información; también está ligado a la inferencia y a la predicción e involucra tanto métodos cualitativos como cuantitativos. (p. 93)

Acorde a este fundamento, asumimos como válido el modelo de razonamiento proporcional propuesto por Modestou y Gagatsis (2010) y reconocemos la necesidad de investigación a la que invocan los estudios sobre la ilusión de la linealidad de De Bock et al. (2002). En nuestro estudio identificamos a las habilidades proporcionales como aquellas facultades que son consideradas necesarias para que una persona posea un razonamiento proporcional sólido. Lobato et al. (2010) y Weiland et al. (2021) indican que esas habilidades deben incluir: 1) Atender y coordinar dos cantidades que varían de forma dependiente, 2) Reconocer y utilizar las estructuras de las situaciones proporcionales (razón unitaria, constante de proporcionalidad, linealidad), 3) Identificar el invariante en una proporción, 4) Comprender la proporcionalidad desde múltiples representaciones, 5) Distinguir las situaciones lineales de las no lineales. Consideramos a estas cinco habilidades como clave para nuestro estudio y son las que invocamos con el término habilidades proporcionales.

Para englobar el marco teórico y estructurar la intervención nos apoyamos del enfoque de Investigación Basada en Diseño (IBD) propuesto por Bakker (2018). Dado que en la IBD el diseño y la innovación en el aula son aspectos clave, se propone el uso de tecnología digital como medio que permita a los estudiantes interactuar con múltiples representaciones, facilitar la simulación de situaciones realistas y comprobar sus propios resultados en Entornos Didácticos Virtuales Interactivos (EDVI).

Para organizar el diseño de las tareas que guían a los estudiantes en el proceso de matematización, nos basamos en la Trayectoria Hipotética de Aprendizaje (THA) (Simon, 1995). Para el diseño de las actividades que integra cada tarea nos apoyamos del marco didáctico Cuevas-Pluvinage, que aporta una ingeniería didáctica que rescata principios didácticos de la escuela activa y Piaget adaptados a la educación matemática (Cuevas y Pluvinage, 2003).

Coincidimos con Sierpinska (2004) en que “reportar las tareas es necesario para probar la eficacia del enfoque de enseñanza” (p. 13). Según la autora, la problematización, justificación y discusión de las tareas inicia con el análisis a priori del diseño, sin embargo, la problematización es aún más sustancial y fructífera en el análisis a posteriori; es decir, después de la experimentación en el aula. En este orden de ideas, en este informe mostramos la problematización tanto en la fase de diseño como en la fase de análisis retrospectivo, ya que da pautas para el rediseño de nuestras tareas.

Una de las labores más complejas para un docente es tratar de implementar principios de una didáctica de corte constructivista en un aula tradicional con los elementos usuales. Por ejemplo, introducir un concepto matemático mediante la problematización de una situación cotidiana, en un aula tradicional, resulta casi imposible. Una posible solución es contar con un escenario virtual que simule un fenómeno real, en el que los estudiantes puedan interactuar con diversos contextos de forma dinámica y, además, su uso permita la construcción de conocimiento matemático (Moyer et al., 2016). De este modo, se dota a los estudiantes de la oportunidad de acción, en donde cada estudiante puede aprender a su propio ritmo. La tecnología digital brinda al estudiante y al profesor de una especie de laboratorio portátil, al utilizar diversos dispositivos, como móviles, tabletas, etc.

Las consideraciones anteriores nos conducen a la pregunta de investigación: ¿Cómo diseñar una secuencia didáctica, en contextos realistas mediados por la tecnología, que ayude a los estudiantes a desarrollar su razonamiento proporcional y les permita progresar en los niveles de matematización de la EMR?

Metodología

Con base en nuestro marco metodológico, el enfoque de IBD utilizado consta de las siguientes fases: preparación y diseño, experimento de enseñanza y análisis retrospectivo.

Fase de preparación y diseño

En esta fase se diseñaron y desarrollaron: 1) Un pretest para recabar conocimientos previos de los estudiantes. 2) Una THA que guía el proceso de enseñanza y marca los objetivos de aprendizaje. 3) Tres secuencias de actividades didácticas con sus respectivos EDVI "Naranjada", "Zoom Totoro" y "Autos". 4) Una encuesta para indagar acerca de la perspectiva de los estudiantes en su experiencia con el uso de los EDVI y las actividades de las hojas de exploración y aprendizaje guiado (HEAG). El pretest y la encuesta final se alojaron en formularios de Google para responder en casa con el fin de ahorrar tiempo de aula.

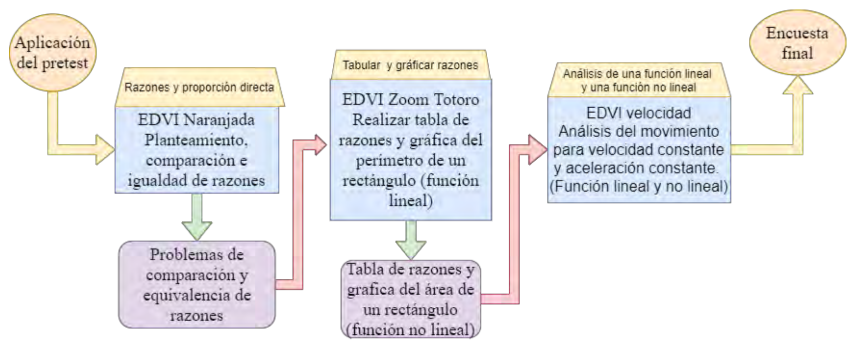


Figura 1: Diseño de la ruta didáctica que guía la secuencia de instrucción

Tarea 1. El contexto de la tarea consiste en comparar propuestas de mezcla de zumo de naranja y agua (ver figura 2a y 2b). El objetivo es que los estudiantes aprendan: a plantear, comparar y determinar equivalencia, entre razones; así como generar tablas de razones equivalentes e identificar la constante de proporcionalidad. Finalmente se realiza una actividad extra para aplicar el conocimiento aprendido en un contexto diferente. Las actividades que se realizan son: expresar una razón en notaciones diferentes, por ejemplo, 3 vasos de jugo por cada 5 de agua: 3 a 5, 3:5 y 3/5). 2) Coordinar la covariación entre dos variables, dada una razón de A vasos de naranja por cada B de agua, el sabor es más intenso a naranja si A aumenta o si B disminuye. 3) Comparación de dos razones. Dadas dos propuestas aleatorias de naranjada con razones de naranja en agua A:B y C:D, determinar cuál tendrá un mayor sabor a naranja.

Tarea 2. La tarea consiste en realizar un efecto "zoom" para reducir o aumentar una imagen de figuras Totoro acorde con una razón de semejanza que se introduce en una casilla de entrada (ver figura 2c). El objetivo es que los estudiantes realicen actividades de semejanza utilizando la constante de proporcionalidad. También, se pretende afrontar el problema de la ilusión de la linealidad al, tabular y graficar, las relaciones razón-perímetro (lineal) y razón-área (cuadrática), con el fin de conducir a los estudiantes hacia la comparación de ambos modelos.

Tarea 3. El contexto de la tarea consiste en visualizar un auto que se mueve a una velocidad constante superior al límite permitido y una patrulla que inicia una persecución con aceleración constante en el momento que el auto pasa a su lado (ver figura 2d). El objetivo es que los estudiantes interactúen con el EDVI mientras describen, analizan y comparan las características y las representaciones de los dos movimientos, es decir, el movimiento rectilíneo uniforme (MRU) y el movimiento rectilíneo uniformemente acelerado (MRUA). De este modo, los estudiantes deben identificar los modelos presentes (lineal y cuadrático) para comparar ambos comportamientos y transitar en sus representaciones.

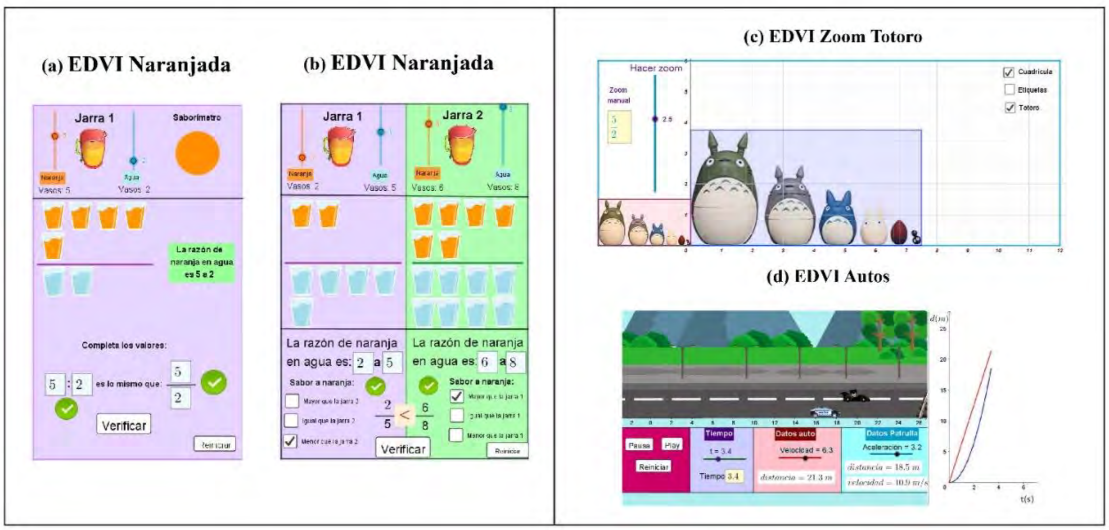


Figura 2. EDVI´s utilizados en la secuencia de enseñanza

Fase de experimento de enseñanza

El estudio se realizó mediante una intervención presencial en dos grupos de una Escuela Secundaria en México. Participaron en el estudio 35 estudiantes (14-15 años) en el invierno de 2021. La instrucción se dividió en tres sesiones de 90 minutos en un salón de cómputo de la escuela. Cada estudiante contaba con una computadora personal que tenía precargados los EDVI, también cada estudiante contaba con sus respectivas HEAG. La intervención estuvo a cargo del primer autor, apoyado por el profesor del grupo y un asistente de investigación. En las sesiones se promovió un aprendizaje colaborativo y se discutieron las respuestas en grupo.

Resultados y análisis retrospectivo

De acuerdo con Freudenthal (2002), el contexto y el diseño didáctico deben permitir a los estudiantes transitar de la matematización horizontal hacia la matematización vertical. Gravemeijer (2004) identifica cuatro niveles hacia la matematización vertical: N1) Nivel situacional. Se interpreta y organiza la realidad mediante razonamientos matemáticos informales y dependientes del contexto (matematización horizontal). N2) Nivel referencial. Se crean esquemas y modelos que tienen sentido dentro del contexto inicial, inicia la matematización vertical al surgir “modelos de...”. N3) Nivel general. Se relacionan los conceptos, se generan estrategias que se separan del contexto, el razonamiento se da en el mundo matemático y surgen “modelos para...”. N4) Nivel formal. Se comprenden los conceptos mediante su simbolismo matemático, ya no se necesita apoyo de algún contexto real, la reflexión ha transitado al mundo matemático y se puede prescindir de los modelos.

A reserva de presentar los resultados del grupo, seleccionamos aleatoriamente a una estudiante, que llamaremos Lucy, para presentar un análisis detallado de los resultados de las tareas y su evaluación mediante los niveles de matematización vertical propuestos en la EMR.

Resultados de la Tarea 1. “Naranjada”

Para realizar las actividades de la naranjada, Lucy utiliza sus conocimientos previos, es decir, las actividades la conducen a una matematización horizontal, asimilando correctamente las tres notaciones para razón. Además, encontró que a una propuesta aleatoria de naranjada la intensidad de sabor depende de un proceso de covariación. En esta fase Lucy se encuentra en el nivel situacional, porque se mueve dentro del mundo numérico solo para interpretar el contexto y emitir un juicio, la matematización horizontal es un ir y venir del contexto al mundo matemático.

Un hallazgo se dio cuando el EDVI mostró a Lucy el siguiente par de razones de naranja a agua: 2:7 y 3:8 (ver figura 4a), Lucy Realizó mentalmente una biyección 1 a 1 entre los vasos de naranja y los vasos de agua. Una vez hecha la biyección, cuando en ambos escenarios quedó el mismo número de vasos de agua sin asociar, Lucy respondió, intuitivamente, que las dos tienen el mismo sabor, es decir, la misma proporción. Esta forma errónea de comparar razones, análoga al erróneo principio aditivo, la extendió a todas las situaciones, y le funcionó cuando los vasos restantes de agua fueron distintos. No obstante, cuando los vasos restantes de agua fueron iguales el EDVI señaló un error, incomprensible para Lucy. Al cierre de la sesión, mediante discusión grupal, se precisaron métodos para comparar razones y se comprobó que la mayoría de los estudiantes adoptaba el razonamiento descrito, e incluso algunos estudiantes se mostraron sorprendidos de que el método fallara. Como esta misma situación se presentó en la mayoría de los estudiantes la llamaremos “ilusión de la proporcionalidad”.

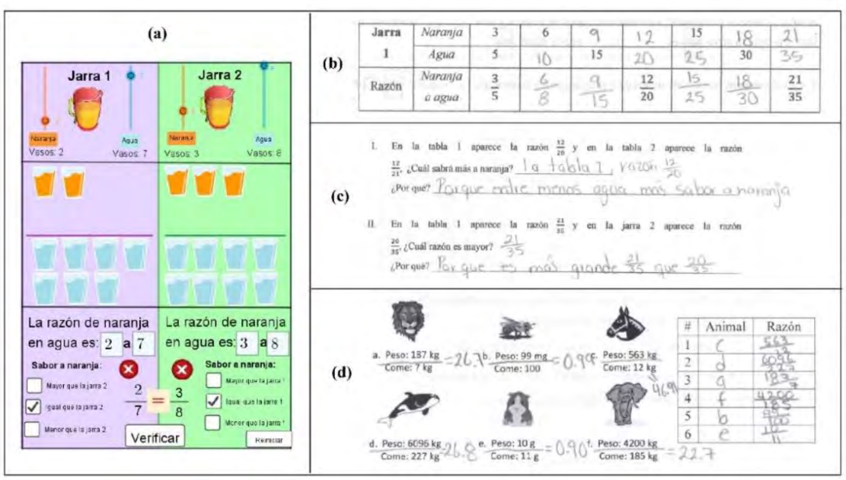


Figura 3. Evidencia de las actividades correspondientes a la Tarea 1

La actividad de completar tablas de razones equivalentes está sujeto al contexto (mezclas naranja-agua en la misma proporción), pero el patrón multiplicativo que Lucy usa para llenar la tabla se da en el ámbito matemático, también se observa que al inicio persiste el razonamiento aditivo (Ver figura 3b). Una manera de comparar las razones equivalentes de las tablas es considerar pares de razones que tengan numeradores o denominadores iguales, Lucy se vale de ese hecho (ver figura 3c) para hacer juicios numéricos y da un argumento de covariación basado en el denominador de las razones, por lo que es claro un nivel referencial.

En los ejercicios de aplicación, posteriores a la actividad con el EDVI, Lucy utiliza eficientemente el método de la razón unitaria. En la figura 3d se muestra un ejercicio de ordenamiento de razones (adaptado de Lamon, 2020), que consiste en ordenar a un conjunto de animales en relación con su peso. Lucy aplica el método de la razón unitaria para resolver problemas en contextos diferentes, se infiere que ha adquirido un “modelo para” y se vislumbra un progreso del nivel referencial al nivel general.

En esta tarea, Lucy muestra habilidades para plantear razones en distintas notaciones, comparar razones, aplicar el principio multiplicativo para obtener razones equivalentes y utilizar el método de la razón unitaria para resolver problemas prácticos. En este nivel, los estudiantes ya poseen habilidades de razonamiento proporcional. Sin embargo, los datos muestran que Lucy tenía, en su estructura de conocimientos un arraigo en el razonamiento aditivo; el cual se logró modificar, e incluso se consolidó el método de la razón unitaria.

Resultados de la tarea 2. “Zoom Totoro”

En la etapa inicial se involucran procesos de observación, medida y principio multiplicativo. La actividad de calcular las dimensiones de la imagen para distintas razones está anclado al contexto (nivel situacional), sin embargo, al seguir el patrón se marca el camino al nivel referencial, ya que Lucy nota que para obtener las medidas de la imagen replicada se multiplican las medidas de la imagen original por la razón de semejanza (ver tabla en la figura 4a).

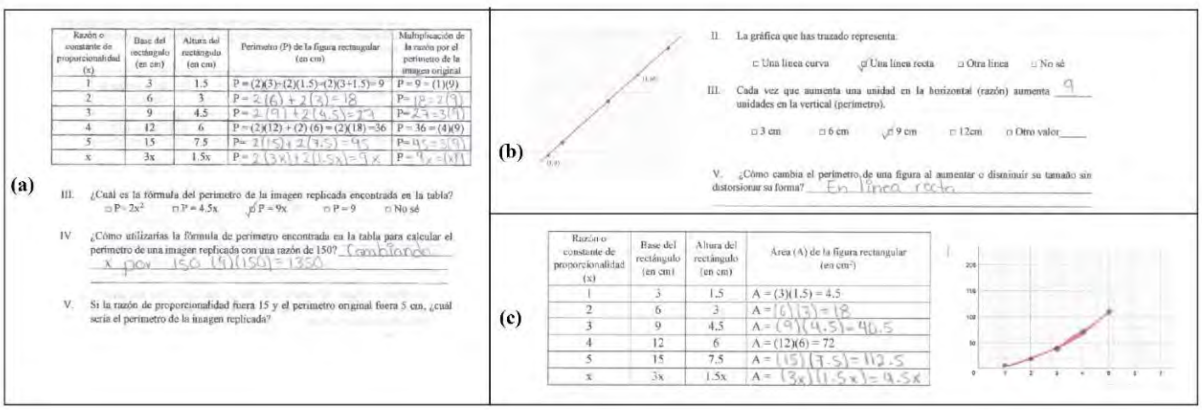


Figura 4. Evidencia de las actividades correspondientes a la Tarea 2

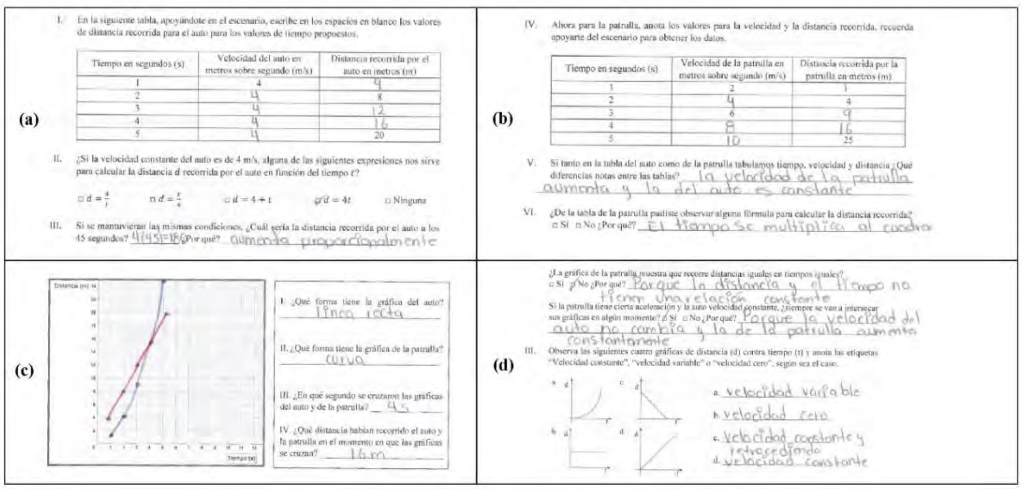
En la tabla para calcular el perímetro de imágenes aumentadas según una razón de semejanza, Lucy se basa en la tabla de dimensiones y se aleja del contexto debido a que el concepto inicial se relaciona, en primera instancia con el perímetro, y en segunda con el área. Lucy calcula los perímetros para las razones propuestas siguiendo el ejemplo inicial de la tabla hasta llegar a la expresión lineal P=9x y aplica el “modelo de” encontrado (ver figura 4a). Para graficar la relación razón-perímetro, Lucy transita entre las representaciones tabular-gráfica y logra ver la variación lineal (ver figura 4b). En cuanto a la relación razón-área, completa la tabla y realiza la gráfica, describiendo que es curva (ver figura 4c). Sin embargo, no concluye en el “modelo de” buscado (relación cuadrática) y no se llega a contrastar entre perímetro y área.

Al realizar esta tarea, Lucy mostró la habilidad para transitar entre las representaciones tabular, gráfica y algebraica de la proporcionalidad; identificando el valor de la constante de proporcionalidad en el caso del perímetro, y además obteniendo un modelo que describe el caso particular. Sin embargo, no se logra avanzar hacia el nivel general.

Resultados de la tarea 3. "Autos"

De inicio, las actividades en este EDVI, son de instrumentación y contextualización. Lucy identifica las características de los movimientos, tanto a velocidad constante como a aceleración constante. Sus ideas previas e informales sobre tales conceptos se modificaron después de interactuar con el EDVI y observar el cambio en las variables, su relación y al realizar observaciones numéricas.

Como se puede observar en la figura 5c, Lucy transita correctamente entre las representaciones tabulares y gráficas, y también identifica características lineales y no lineales, tanto en la gráfica como en las tablas. En esta tarea Lucy ve relaciones entre las variables que van más allá del contexto inicial, al identificar un "modelo de", utilizarlo y concluir que la variación es proporcional (en el auto) y cuadrático (en la patrulla); también, distingue los comportamientos lineales y no lineales en las representaciones. Sin embargo, Lucy no logró proponer un modelo para la aceleración constante.


(a)

I. En la siguiente tabla, apoyándote en el escenario, escribe en los espacios en blanco los valores de distancia recorrida para el auto para los valores de tiempo propuestos.

Tiempo en segundos (s)	Velocidad del auto en metros sobre segundo (m/s)	Distancia recorrida por el auto en metros (m)
1	4	4
2	4	8
3	4	12
4	4	16
5	4	20

II. ¿Si la velocidad constante del auto es de 4 m/s, alguna de las siguientes expresiones nos sirve para calcular la distancia d recorrida por el auto en función del tiempo t?

□ $d = \frac{4}{t}$ □ $d = \frac{t}{4}$ □ $d = 4 + t$ ☑ $d = 4t$ □ Ninguna

III. Si se mantuvieran las mismas condiciones, ¿Cuál sería la distancia recorrida por el auto a los 45 segundos? 4(45)=180 ¿Por qué? aumenta proporcionalmente

(b)

IV. Ahora para la patrulla, anota los valores para la velocidad y la distancia recorrida, recuerda apoyarte del escenario para obtener los datos.

Tiempo en segundos (s)	Velocidad de la patrulla en metros sobre segundo (m/s)	Distancia recorrida por la patrulla en metros (m)
1	2	1
2	4	4
3	6	9
4	8	16
5	10	25

V. Si tanto en la tabla del auto como de la patrulla tabulamos tiempo, velocidad y distancia ¿Qué diferencias notas entre las tablas? la velocidad de la patrulla aumenta y la del auto es constante

VI. ¿De la tabla de la patrulla pudiste observar alguna fórmula para calcular la distancia recorrida? □ Sí □ No ¿Por qué? El tiempo se multiplica al cuadro

(c)

I. ¿Qué forma tiene la gráfica del auto? línea recta

II. ¿Qué forma tiene la gráfica de la patrulla? curva

III. ¿En qué segundo se cruzaron las gráficas del auto y de la patrulla? 4 s

IV. ¿Qué distancia habían recorrido el auto y la patrulla en el momento en que las gráficas se cruzan? 16 m

(d)

¿La gráfica de la patrulla muestra que recorre distancias iguales en tiempos iguales? □ Sí ☑ No ¿Por qué? Porque la distancia y el tiempo no tienen una relación constante

Si la patrulla tiene cierta aceleración y la auto velocidad constante, ¿siempre se van a intersecar sus gráficas en algún momento? ☑ Sí □ No ¿Por qué? Porque la velocidad del auto no cambia y la de la patrulla aumenta constantemente

III. Observa las siguientes cuatro gráficas de distancia (d) contra tiempo (t) y anota las etiquetas "Velocidad constante", "velocidad variable" o "velocidad cero", según sea el caso.

a. velocidad variable
b. velocidad cero
c. velocidad constante y retrocediendo
d. velocidad constante

Figura 5. Evidencia de las actividades correspondientes a la Tarea 3

Al visualizar y comparar las gráficas de los movimientos para diversos parámetros de velocidad y aceleración, Lucy interpreta el crecimiento y la variación, al argumentar: "si se recorren distancias iguales en tiempos iguales, distancia y tiempo tienen una relación constante", y que el crecimiento de la aceleración siempre superará al de la velocidad porque "la velocidad del auto no cambia y la de la patrulla aumenta constantemente" (ver figura 5d). En los razonamientos de Lucy se observa un enriquecimiento conceptual que trasciende al contexto y, aunque no son manipulaciones numéricas o algebraicas, son de carácter general. Es decir, son razonamientos generales que aplican para cualquier contexto de MRU y MRUA. En palabras de Freudental (2002), "La relación entre la razón constante y la linealidad es una hazaña de matematización vertical" (p.43).

Al realizar esta tarea, Lucy muestra la habilidad de distinguir una situación lineal de una situación no lineal, e identificando los rasgos distintivos de cada una en distintas representaciones. Con lo cual, muestra claros progresos en su razonamiento proporcional.

Discusión y conclusión

En la Tarea 1 Lucy tuvo dificultades para comparar dos razones cuya diferencia entre numerador y denominador era la misma, 2/7 y 3/8, esa idea cambió con la tarea, inferimos que hay dificultades en la comprensión de los estudiantes que emergen con esta actividad, de acuerdo con Sierpinska (2004), "las tareas deben ser capaces de revelar las concepciones erróneas más ocultas de los estudiantes" (p. 14). En la Tarea 2 el objetivo es confrontar la ilusión de la linealidad, respecto a la confusión que presentan los estudiantes al suponer que el área y el volumen guardan una relación lineal con la razón de semejanza. Sin embargo, las actividades no dan muestra de ser suficientes por lo que deben rediseñarse. Es necesario añadir figuras geométricas, para que los estudiantes distingan la relación lineal del perímetro con la cuadrática del área y la cúbica del volumen, de una manera geométrica más activa. En la Tarea 3, el análisis detallado sobre las respuestas de Lucy nos marca la pauta para profundizar en las actividades que hagan surgir los "modelos para" y se logre la generalización.

El análisis detallado de las respuestas de Lucy en cada tarea da muestras del aprendizaje que adquiere para trabajar con razones, transitar en representaciones proporcionales y contrastarlas con fenómenos no lineales en diferentes contextos. Lo anterior nos da elementos para responder nuestra pregunta de investigación y nos da pautas suficientes para el rediseño de las actividades que organiza la THA en cada una de las tareas propuestas. También, nos aporta evidencia suficiente para discernir en cuál de las tareas, y de qué forma, se debe profundizar para lograr que los estudiantes muestren mejores progresos en los niveles de matematización vertical. Es decir, la diversidad de contextos favorece la matematización horizontal (nivel situacional), en consecuencia, considerando los tiempos en el currículo, los diseñadores de secuencias didácticas debemos discernir qué es lo más conveniente en cada situación, ya que al reducir el número de contextos podría aumentar la oportunidad de lograr mayor profundidad en el proceso de matematización. En este sentido, consideramos que también son necesarias actividades totalmente matematizadas sin referencia a ningún contexto que den cuenta del nivel formal.

En esta investigación en desarrollo, tratamos de encontrar vías de enseñanza para abordar los conceptos involucrados en el razonamiento proporcional. Primero con acercamientos hacia la razón y la proporción con las funciones lineales y sus representaciones (simbólica, tabular y gráfica). En este sentido, un primer eje de la propuesta es priorizar el concepto de razón como precursor para acceder a la equivalencia, la comparación y las tablas de razones, lo cual conduce inexorablemente hacia las funciones lineales; un segundo eje es que el estudiante se enfrente a diversos contextos para explorar situaciones proporcionales en sus distintas representaciones y el tercer eje consiste en considerar el fenómeno de la ilusión de la linealidad. Se intenta que los estudiantes experimenten situaciones realistas tanto lineales como no lineales y cuestionen la linealidad (o la ausencia de ella) cada vez que se enfrenten a un problema, reaccionando de manera razonada de acuerdo con las características de la situación.

En cuanto a la influencia de la tecnología en el estudio, es claro que su uso permite simular y replicar situaciones cotidianas susceptibles de problematizarse, además de brindar ventajas pragmáticas y epistémicas. Por ejemplo, el uso de EDVI posibilita: presentar al estudiante datos aleatorios, interactuar con los contextos de forma dinámica, mostrar los objetos matemáticos en distintas representaciones y validar los resultados. Como se ha descrito en el análisis, la experiencia de los estudiantes con los EDVI permitió observar concepciones erróneas que, probablemente en formas de enseñanza tradicionales, pasarían inadvertidas. Por otro lado, interactuar con los EDVI provocó en los estudiantes un interés que iba más allá del contexto matemático planteado, involucrándose con problemas que se derivaban de las tareas.

Referencias

Adjiage, R. & Pluvinage, F. (2007). An Experiment in Teaching Ratio and Proportion. *Educational Studies in Mathematics*, *65*(2), 149–175. https://doi.org/10.1007/s10649-006-9049-x

Bakker, A. (2018). *Design Research in Education: A Practical Guide for Early Career Researchers* (1st ed.). Routledge. https://doi.org/10.4324/9780203701010

Cuevas, C. A. & Pluvinage, F. (2003). Les projets d'action practique, elements d'une ingeniere d'ensigment des mathematiques. *Annales de didactique et de sciences cognitives*, *8*, 273-292.

De Bock, D., Van Dooren, W., Janssens, D. & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*(3), 311-334. https://doi.org/10.1023/a:1021205413749

Freudenthal, H., (2002) Revisiting mathematics education. China lectures. Kluwer, Dordrecht

Gravemeijer, K. (2004). Local Instruction Theories as Means of Support for Teachers in Reform Mathematics Education. *Mathematical Thinking and Learning*, *6*(2), 105–128. https://doi.org/10.1207/s15327833mtl0602_3

Lamon, S. J. (2007). Rational Numbers and Proportional Reasoning: Towards a Theoretical Framework for Research. En F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629-667). Information Age Publishing.

Lamon, S. J. (2020). *Teaching Fractions and Ratios for Understanding: Essential Content 83 and Instructional Strategies for Teachers*. Routledge, 2020.

Lesh, R., Post, T. & Behr, M. (1988). Proportional reasoning. En J. Hiebert y M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). National Council of Teachers of Mathematics.

Lobato, J. E., Ellis, A. B. & Charles, R. I. (2010). *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics in Grades 6-8*. National Council of Teachers of Mathematics.

Modestou, M. & Gagatsis, A. (2010). Cognitive and Metacognitive Aspects of Proportional Reasoning. *Mathematical Thinking and Learning*, *12*(1), 36-53. https://doi.org/10.1080/10986060903465822

Moyer-Packenham, P. S., Bolyard, J. J. (2016). Revisiting the Definition of a Virtual Manipulative. En Moyer-Packenham, P. (Eds.) *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives. Mathematics Education in the Digital Era*, vol 7. Springer, Cham. https://doi.org/10.1007/978-3-319-32718-1_1

Sierpinska, A. (2004). Research in Mathematics Education through a Keyhole: Task Problematization. *For the Learning of Mathematics*, *24*(2), 7–15. http://www.jstor.org/stable/40248450

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145. https://doi.org/10.2307/749205

Van den Heuvel-Panhuizen M (2003) The didactical use of models in realistic mathematics education: an example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics*, *54*(1):9–35

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic Mathematics Education. In *Encyclopedia of Mathematics Education* (pp. 713–717). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0_170

Weiland, T., Orrill, C.H., Nagar, G.G., Brown, R. & Burke, J. (2021) Framing a robust understanding of proportional reasoning for teachers. J Math Teacher Educ, 24, 179–202. https://doi.org/10.1007/s10857-019-09453-0

MIDDLE SCHOOL STUDENTS' MATURE NUMBER SENSE IS UNIQUELY ASSOCIATED WITH GRADE LEVEL MATHEMATICS ACHIEVEMENT

Patrick K. Kirkland
University of Notre Dame
pkirklan@nd.edu

Claire Guang
University of Notre Dame
cguang@nd.edu

Ying Cheng
University of Notre Dame
Ycheng4@nd.edu

Christine Trinter
University of Notre Dame
ctrinter@nd.edu

Saachi Kumar
University of Notre Dame
skumar22@nd.edu

Sofia Nakfoor
University of Notre Dame
snakfoor@nd.edu

Tiana Sullivan
University of Notre Dame
Tsulli23@nd.edu

Nicole M. McNeil
University of Notre Dame
nmcneil@nd.edu

Students exhibiting mature number sense make sense of numbers and operations, use reasoning to notice patterns, and flexibly select the most effective and efficient problem-solving strategies (McIntosh et al., 1997; Reys et al., 1999; Yang, 2005). Despite being highlighted in national standards and policy documents (CCSS, 2010; NCTM, 2000, 2014), students' mature number sense and its nomological network are not yet well specified. For example, how does students' mature number sense relate to their knowledge of fractions and their grade-level mathematics achievement? We analyzed 129 middle school students' scores on measures of mature number sense, fraction and decimal computation, and grade-level mathematics achievement. We found mature number sense to be measurably distinct from their fraction and decimal knowledge and uniquely associated with students' grade-level mathematics achievement.

Keywords: Number Concepts and Operations; Cognition; Assessment; Rational Numbers

Over the last three decades, "number sense" has emerged as a central goal of mathematics education. While high quality mathematics teachers were certainly already teaching in ways that helped students make sense of number and operations before then, "number sense" was not yet a commonly used term in mathematics education. In the late 1980's, government commissioned reports (e.g., National Research Council, 1989) and national curriculum frameworks (e.g., National Council of Teachers of Mathematics (NCTM), 1989) highlighted "number sense" as a core objective of K-12 mathematics education. Students showing strong mature number sense exhibit the disposition to *make sense of numerical situations* and use a rich conceptual understanding of number and operations to flexibly solve problems.

Despite being a central objective in mathematics education, there is little evidence in the literature on how students' mature number sense relates to other important psychological constructs in mathematics education. For example, how do students' levels of mature number sense relate to their grade-level mathematics achievement or their understanding of fractions? Without an understanding of the nomological network, it is harder for researchers to explain how students construct mature number sense, leaving educators without rigorous evidence on how best to address it within their own classrooms. In this study, we aimed to begin to address this gap in the literature by examining how middle school students' mature number sense relates to other theoretically related constructs, testing the hypothesis that mature number sense is both distinct from fraction and decimal computation and grade-level mathematics achievement and uniquely associated with students' grade-level mathematics achievement.

Background and Theoretical Framework

Number sense is a difficult construct to define (e.g., Berch, 2005). How the construct is operationalized and assessed looks very different across the disciplines of cognitive and developmental psychology, mathematics education, and special education (summarized in Whitacre et al., 2020). In this project, our construct of interest is what Whitacre et al. (2020) have termed *mature number sense*, which we define in line with McIntosh et al. (1992) as "a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways." This focus is distinct from research on approximate number sense (e.g., Dehaene, 2001) or early number sense (e.g., Jordan et al., 2009).

Perhaps to help facilitate operationalizing it in empirical research, mature number sense has often been subdivided into components (McIntosh et al., 1997; NCTM, 1989; Reys et al., 1999; Yang & Lin, 2015). Building from this prior work and in consultation with expert mathematics teachers, teacher educators, and mathematics education researchers, we specified a four-component framework of mature number sense (Kirkland et al., *under review*) as follows:

1. Understanding the effect of operations on numbers: Students demonstrating strong mature number sense recognize how the four core operations affect whole numbers as well as fractions and decimals. They understand that patterns observed with operations with whole numbers may not hold true for fractions and decimals between 0 and 1. Students use this understanding to efficiently estimate computational results and ensure the results make sense given the relationship between operations and rational numbers.
2. Using multiple representations of a number: Students demonstrating strong mature number sense are able to translate among multiple representations of rational numbers efficiently and flexibly to solve problems. Students recognize that fractions are numbers, for example, rather than separately perceiving the numerator and denominator. Students are able to use this understanding to translate between representations to make sense of numerical situations.
3. Understanding mathematical equivalence: Students demonstrating strong mature number sense understand the equal sign as a relational symbol, reflecting that the two sides of an equation are equal and interchangeable. Rather than approaching equations with an "operational" approach, students understanding equivalence recognize patterns across the equal sign and use this relational thinking to flexibly solve problems (c.f. Jacobs et al., 2007).
4. Understanding basic number concepts and number magnitude: Students demonstrating strong mature number sense have a rich conceptual understanding of fractions, decimals, and whole numbers. Students use their understanding of place value and rational number magnitude to efficiently estimate results using concepts such as unit fractions.

We hypothesized, similar to Yang (2019), that mature number sense would be an overarching hierarchical latent construct, with the four components each theoretically related to mature number sense. That is, a student's understanding of the effect of operations on a number reflects their mature number sense as well as a more specific understanding of how operations affect computational results. In our initial development of a brief assessment of mature number sense (Kirkland et al., *under review*), the structure of the response data reflected this hypothesized framework. A bifactor model with each component as a specific factor best fit the response data over other theoretically related models. We now aimed to use this brief assessment to further knowledge about mature number sense's nomological network, including other important psychological constructs in mathematics education.

Rationale for Current Study

Mature number sense has been measured and studied internationally over the last twenty-five years (summarized in Whitacre et al., 2020), most frequently in a single time point. Most of these studies comment on the "low" levels of students' mature number sense observed. However, few studies have analyzed its potential overlap with other related psychological constructs in mathematics education. Understanding how mature number sense fits together with constructs such as fraction computation and school mathematics achievement is important for establishing the construct validity of measures of mature number sense (Cronbach & Meehl, 1955). Throughout the literature, mature number sense is often contrasted with standardized school mathematics achievement and the algorithmic, overly procedural mathematics instruction common in schools (Whitacre et al., 2020). On the other hand, it is reasonable to hypothesize that students with strong mature number sense are flexible problem solvers with a deep understanding of number and operation and, thus, that they might do well on more "traditional" school assessments. Yang et al. (2008) is the only study we are aware of that examined this association. They found 5^{th} grade Taiwanese students' number sense were positively associated with school achievement, but there was no detail on how math achievement was measured (e.g., school grades, standardized tests, etc.). In addition, given Taiwanese students' strong performance on international standardized mathematics tests (ranked 5^{th} in world in PISA 2018) and its centralized education system (Lin et al., 2021), Taiwan may be a unique context to compare students' mature number sense and school math achievement.

To begin to address this gap in the literature, we aimed to study how US middle school students' mature number sense related to their grade-level mathematics achievement, controlling for a variety of other potentially related factors, including fraction knowledge and addition fluency. Our research questions were as follows: 1) Is the mature number sense of middle school students measurably distinct from their fraction and decimal computation skills as well as their grade-level mathematics achievement? and 2) Is mature number sense uniquely associated with students' standardized grade-level mathematics achievement, even after controlling for their fraction and decimal computation?

Methods

Participants

One hundred twenty-nine middle school (6^{th}-8^{th} grade) students from schools surrounding a university in the midwestern United States participated (45% identified as female; 67% identified as White, 13% Multiracial, 8% Black, and 4% Hispanic). Sessions took place after school hours in a local middle school *(N = 36)* or university building *(N = 93)*. Students were participants in timepoint 1 of a larger (still ongoing) longitudinal study. They completed eight measures over the course of two 45-minute sessions, scheduled about a week apart (median of 7 days between sessions). For this analysis, we focused on the four measures described below, but note that the results reported hold up to robustness checks that include the additional measures.

Measures and Procedure

Brief Assessment of Mature Number Sense (Kirkland et al., *under review*). This is an electronic, 24 item multiple-choice test of students' mature number sense, aligned with our theoretical framework detailed above. Items differ from a traditional curriculum and are designed to specially assess students' number sense. Each item has a time limit of 60 seconds and students are not allowed to use paper and pencil to discourage the use of traditional algorithms. Student's total sum score is used in the analyses. Figure 1 includes an example item for each of the four components in our mature number sense framework. In the validation analysis (Kirkland et al., *under review*), student scores on the brief assessment were reliable over time (r = 0.83) and had evidence from expert reviews, factor analyses, student think-alouds, and item response theory analyses to support our validity argument.

Massachusetts Comprehensive Assessment System (MCAS) Grade-Level Mathematics Test (2019). Students completed the released 2019 MCAS paper test appropriate for their grade level. This is a freely available standardized test designed to assess student proficiency with grade-level mathematics standards. Student scores are converted to percent correct because the maximum possible correct differs by grade level. Students had no time limit to complete each section of the test and could use scratch paper but not calculators.

Rational Numbers Measure (Powell, 2014). This is a 35-item paper and pencil test of students' computational skills with fractions, decimals, and percent. Students are asked to perform the four operations with both fractions and decimals, find common denominators of fractions, generate equivalent fractions, and connect representations of fractions, decimals, and percent (e.g., "Convert 2.08 to a percentage"). Students had 20 minutes to work. They could use scratch paper but not calculators. Students received 1 point for each correct response.

Addition Fluency Task (Geary et al., 1996). This measure includes all combinations of randomly presented single digit addition facts with the numbers 1-9. Students were tasked with solving as many correctly as they could in 1 minute. The order of the facts was predetermined randomly and then kept standard for all participants. Students received 1 point for each correct response.

Understanding the Effect of Operations on Numbers	**Number Concepts and Magnitude**
What is the best estimate for 26 * ¾ ? O Less than 26 O Equal to 26 O Greater than 26 O Can't know	Which fraction represents the largest amount? O 5/6 O 5/7 O 5/8 O 5/9
Multiple Representations of a Number	**Mathematical Equivalence**
What is true about the number 2/5 (two-fifths)? O It is greater than ½ (one half). O It is the same as 2.5. O It is equivalent to 0.4 O It is less than 1/6 (one sixth).	47 + 16 + ___ = 16 + 8 + 47 O 8 O 63 O 71 O 4716

Figure 1: Sample Items Assessing Mature Number Sense Organized by Component

Data Analyses

To address the research question on whether mature number sense is measurably distinct from students' general mathematics achievement, we first calculated the zero-order correlations between the constructs. We predicted mature number sense to be most closely related to students' grade-level math achievement on the MCAS and scores on the Rational Numbers Measure. We expected mature number sense to be least closely related to addition fluency. We then ran a series of partial correlation analyses. We analyzed the correlation between mature number sense and both other constructs after controlling for students' addition fluency scores. We then analyzed the associations between students' grade-level math achievement, fraction and decimal computation, and mature number sense, controlling for each in turn. In each case, we predict the relationship between mature number sense and the other construct to remain significant, even after controlling for a strongly related third variable.

For additional evidence that mature number sense and fraction and decimal computation are distinct, we then conducted a series of factor analyses (Shaffer et al., 2016). We used a common method where we tested a constrained model with 2 latent factors set with a covariance equal to 1 and an unconstrainted model where the 2 latent factors are allowed to freely covary with each other. If the unconstrained model provides a better fit through a RMSEA test that is significant, then the two measures of interest can be said to be distinct. We tested these two models using a chi-squared, $\chi 2$, difference test. Due to each grade's different test for the MCAS, we are unable to perform this same analysis on grade-level mathematics achievement.

To address research question 2, we ran a partial correlation test on the relationship between grade-level mathematics achievement and fraction and decimal computation, controlling for mature number sense. We then ran a linear regression model, regressing mature number sense and other predictors measured in the study on students' grade-level mathematics achievement.

Findings

We first begin with the overall student performance on the measures in the study. Students solved on average 14.3 (*60%)* items correctly on our brief assessment of mature number sense (*SD* = 5.65). Overall, students performed slightly better on the brief assessment than in the original validation study (*M* = *56%* correct). Across the three grade levels, students solved on average 50% of the items correctly on the MCAS. Out of a maximum possible of 35 correct, students answered on average 9.74 items correct on the Rational Numbers measure. While this number may appear very low as a percentage correct (*28%*), this measure was designed by Powell to include many problems to help differentiate students' fraction and decimal computation up to the college level. Finally, on average, students answered correctly 19.48 (*SD* = 6.72) single-digit addition problems on the addition fluency task.

To begin to address research question 1, the zero-order correlations between the measured constructs are summarized in Table 1. As predicted, students' mature number sense scores correlated very highly with their fraction and decimal computation skills (*r* = 0.80) and grade-level mathematics achievement (*r* = 0.76), and this was significantly higher than the correlation with addition fluency (*r* = 0.49).

Table 1: Correlation Between Measures in Study

Construct	*M*	*SD*	1	2	3	4
1. Mature Number Sense	14.34	5.65	-			
2. MCAS – Grade-Level Achievement	50.00%	24%	0.76	-		
3. Fraction and Decimal Computation	9.74	7.21	0.80	0.66	-	
4. Addition Fluency	19.48	6.72	0.49	0.36	0.53	-

Controlling for students' addition fluency scores, students' mature number sense still correlated highly with their grade-level mathematics achievement (r =0.72, $t(126)$ = 11.52, p < .001) and their fraction and decimal computation (r =0.73, $t(126)$ = 11.90, p < .001). We then further examined the associations between each pair of these three mathematics-specific constructs. Controlling for students' fraction and decimal computation, their mature number sense scores still were significantly positively related with their grade-level achievement (r =0.52, $t(126)$ = 6.77, p < .001). The same was true when controlling for grade-level achievement and examining mature number sense's relationship with fraction and decimal computation (r =0.61, $t(126)$ = 8.59, p < .001). We would predict that if mature number sense and grade-level mathematics achievement are redundant constructs, this relationship would no longer be significant. Together, this all provides initial evidence that mature number sense is distinct from their grade-level mathematics achievement.

As discussed above, we then ran a series of confirmatory factor analyses for evidence on if mature number sense and fraction and decimal computation are distinct (Shaffer et al., 2016). These analyses provided evidence that the Rational Numbers Measure and the Brief Assessment of Mature Number Sense measure distinct constructs, χ^2 difference = 56.65, p < 0.001; *CIF* difference = 0.02 compared to benchmark of 0.002. This suggests the constructs, as measured in this study, are highly related, but not redundant.

To further examine the unique importance of mature number sense, we first tested the partial correlation between fraction and decimal computation and grade-level achievement. Interestingly, when controlling for students' mature number sense, there is no longer a significant correlation between the two other constructs (r =0.13, $t(126)$ = 1.51, p =0.13). We then ran a linear regression on students' grade-level mathematics achievement (Model $R^2 = 0.59$). Results from the model are summarized in Table 2. Students' mature number sense was significantly positively related ($B_{NS} = 0.03, p$ < .001) to students' grade-level mathematics achievement. However, students' fraction and decimal computation ($B_{RN} = 0.005, p$ =0.11) and addition fluency ($B_{Add} = -0.002, p$ =0.54) were not significantly related to achievement. To compare the relative importance of each regressor, we examined the semipartial correlation of each predictor in the model (Darlington & Hayes, 2016; Hayes & Rockwood, 2017). The semipartial correlation coefficient for mature number sense was significant (sr = 0.39, $t(126)$ = 4.74, p < .001). However, the semipartial correlation for fraction and decimal knowledge (sr = 0.09, $t(126)$ = 0.98, p = 0.33) was not significant. We can interpret the ratio of the two semipartial correlations (0.39/0.09 = 4.33) as a measure of the relative importance of each in explaining students' grade-level achievement. As another way to quantify mature number sense's importance, we ran an additional linear regression model with mature number sense removed from the model. The model R^2 dropped from 0.59 to 0.43 (adjusted R^2 from 0.58 to 0.42).

Table 2: Summary of Regression Model Predicting Grade-Level Mathematics Achievement

Variable	*B*	*SE*	*t*	*p*
(Intercept)	0.08	0.051	1.645	0.103
Mature Number Sense	0.03	0.004	6.78	<0.001
Fraction and Decimal Computation	0.005	0.003	1.61	0.11
Addition Fluency	-0.002	0.002	-0.62	0.54

Discussion and Implications

From the evidence presented here, students' mature number sense is distinct from their general grade-level mathematics achievement and fraction and decimal computation skills. It is not the same to say, for example, that a student has "strong number sense" and is "good with fraction operations." We also found students' mature number sense to be predictive of their mathematics achievement, above and beyond their computation skills with fractions and decimals. Mature number sense explained a significant portion of that association, suggesting that mature number sense is uniquely associated with students' math achievement.

It is perhaps surprising that mature number sense was a stronger predictor of grade level mathematics achievement than students' fraction and decimal computation, given the significance fraction knowledge in particular has been shown to play in students' mathematics achievement (e.g., Barbieri et al., 2021; Booth et al., 2014; Booth & Newton, 2012; Siegler et al., 2013). Fractions are often seen as a "gatekeeper" to success in algebra or later mathematics courses (e.g., Powell et al., 2019; Siegler & Pyke, 2013). However, we see in this study that middle school students' mature number sense was a more important factor than their knowledge of fractions and decimals. Perhaps students' ability to make sense of numbers and operations and use that knowledge flexibly to solve problems is a better indicator of future mathematics achievement than fraction or decimal computation skills. Further research is needed to understand the nomological network more fully, especially in a longitudinal setting where the potential mediating role of mature number sense can be accurately tested.

In reviewing the item level data from the brief assessment of mature number sense, students performed better overall on whole number items than those involving fractions and decimals. For example, 79% of students solved this whole number distributive property item (4 x 36 = 4 x (___ + 6)) correctly. However, only 36% of students correctly identified that the answer to $12 \div \frac{1}{5}$ would be greater than 12 whereas 50% of students said it would be less than 12. Across this and other items, students struggled estimating the effect of operations on fractions and decimals less than one, displaying a continued bias toward "rules" learned with whole number operations, such as multiplication always makes the answer bigger, division makes the answer smaller, and each number has a unique predecessor and successor (Fazio et al., 2016; Siegler, 2016). Greater attention is required to address these misconceptions that are often learned as "rules" in early elementary classrooms (Karp et al., 2015). Encouragingly, a greater number of students displayed accurate understanding of fraction and decimal magnitude. For example, 72% of students correctly identified $\frac{5}{6}$ as the largest among the following fractions: $\frac{5}{6}, \frac{5}{7}, \frac{5}{8}, \frac{5}{9}$. Such "common numerator" problems have been shown to be significantly more difficult for students than the much more frequently seen "common denominator" comparisons (Fazio et al., 2016).

Overall, we have initial evidence here to start to build a nomological network for mature number sense and its associated psychological constructs in mathematics education. This is an important step forward to future research more fully examining how mature number sense might lay a foundation for future mathematics achievement or whether improving mature number sense might be a more important target for instruction than, say, computational skills with fractions. This is, however, only one initial study with a sample of students that may not be wholly representative of middle school students more broadly. In addition, we only have evidence from a single time-point rather than longitudinal data to test if mature number sense precedes and predicts grade level mathematics achievement. By continuing to characterize mature number sense's nomological network, researchers can more fully explain how students develop understanding of important psychological constructs in mathematics education and provide rigorous evidence to educators on how to help ensure all students have the disposition to *make sense of numerical situations*.

Acknowledgements

This research was supported by National Science Foundation Grant #MASKED.

References

Barbieri, C. A., Rodrigues, J., Dyson, N., & Jordan, N. C. (2020). Improving fraction understanding in sixth graders with mathematics difficulties: Effects of a number line approach combined with cognitive learning strategies. *Journal of Educational Psychology*, *112*(3), 628–648. https://doi.org/10.1037/edu0000384

Barbieri, C. A., Young, L. K., Newton, K. J., & Booth, J. L. (2021). Predicting Middle School Profiles of Algebra Performance Using Fraction Knowledge. *Child Development*, *n/a*(n/a), 1–22. https://doi.org/10.1111/cdev.13568

Berch, D. B. (2005). Making Sense of Number Sense: Implications for Children With Mathematical Disabilities. *Journal of Learning Disabilities*, *38*(4), 333–339. https://doi.org/10.1177/00222194050380040901

Booth, J. L., & Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, *37*(4), 247–253. https://doi.org/10.1016/j.cedpsych.2012.07.001

Booth, J. L., Newton, K. J., & Twiss-Garrity, L. K. (2014). The impact of fraction magnitude knowledge on algebra performance and learning. *Journal of Experimental Child Psychology*, *118*, 110–118. https://doi.org/10.1016/j.jecp.2013.09.001

CCSS. (2010). *Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers*.

Cronbach, L. J., & Meehl, P. E. (1955). Construct validity in psychological tests. *Psychological Bulletin*, *52*(4), 281–302. https://doi.org/10.1037/h0040957

Darlington, R. B., & Hayes, A. F. (2016). *Regression analysis and linear models: Concepts, applications, and implementation*. Guilford Publications.

Dehaene, S. (2001). Précis of the number sense. *Mind & Language*, *16*(1), 16–36.

Fazio, L. K., DeWolf, M., & Siegler, R. S. (2016). Strategy use and strategy choice in fraction magnitude comparison. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *42*(1), 1–16. https://doi.org/10.1037/xlm0000153

Geary, D. C., Bow-Thomas, C. C., Liu, F., & Siegler, R. S. (1996). Development of arithmetical competencies in Chinese and American children: Influence of age, language, and schooling. *Child Development*, *67*(5), 2022–2044.

Hayes, A. F., & Rockwood, N. J. (2017). Regression-based statistical mediation and moderation analysis in clinical research: Observations, recommendations, and implementation. *Behaviour Research and Therapy*, *98*, 39–57.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional Development Focused on Children's Algebraic Reasoning in Elementary School. *Journal for Research in Mathematics Education*, *38*(3), 258–288. JSTOR. https://doi.org/10.2307/30034868

Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: Kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, *45*(3), 850–867. https://doi.org/10.1037/a0014939

Karp, K. S., Bush, S. B., & Dougherty, B. J. (2015). 12 math rules that expire in the middle grades. *Mathematics Teaching in the Middle School*, *21*(4), 208–215.

Kirkland, P.K., Cheng, Y., Trinter, C., & McNeil, N.M. (*under review*). Developing a Validity Argument for a Brief Assessment of Mature Number Sense.

Lin, S.-W., Tzou, H.-I., Lu, I.-C., & Hung, P.-H. (2021). Taiwan: Performance in the Programme for International Student Assessment. In N. Crato (Ed.), *Improving a Country's Education: PISA 2018 Results in 10 Countries* (pp. 203–226). Springer International Publishing. https://doi.org/10.1007/978-3-030-59031-4_10

Maxwell, S. E., & Cole, D. A. (2007). Bias in cross-sectional analyses of longitudinal mediation. *Psychological Methods*, *12*(1), 23–44. https://doi.org/10.1037/1082-989X.12.1.23

McIntosh, A., Reys, B. J., & Reys, R. E. (1992). A Proposed Framework for Examining Basic Number Sense. *For the Learning of Mathematics*, *12*(3), 2–44.

McIntosh, A., Reys, B., Reys, R., Bana, J., & Farrell, B. (1997). Number sense in school mathematics: Student performance in four countries. *ECU Publications Pre. 2011*. https://ro.ecu.edu.au/ecuworks/6819

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school*

mathematics. National Council of Teachers of Mathematics.
National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics*. NCTM.
National Council of Teachers of Mathematics (NCTM). (2014). *Principles to Actions: Ensuring Mathematical Success for All: National Council of Teachers of Mathematics*. NCTM.
National Research Council. (1989). Everybody counts. A report to the nation on the future of mathematics education. *National Academy Press*.
Powell, S. R. (2014). *Rational numbers measure*. Available from SR Powell.
Powell, S. R., Gilbert, J. K., & Fuchs, L. S. (2019). Variables influencing algebra performance: Understanding rational numbers is essential. *Learning and Individual Differences*, *74*, 101758. https://doi.org/10.1016/j.lindif.2019.101758
Preacher, K. J., Zhang, Z., & Zyphur, M. J. (2011). Alternative methods for assessing mediation in multilevel data: The advantages of multilevel SEM. *Structural Equation Modeling*, *18*(2), 161–182.
Reys, R., Reys, B., Emanuelsson, G., Johansson, B., McIntosh, A., & Yang, D. C. (1999). Assessing Number Sense of Students in Australia, Sweden, Taiwan, and the United States. *School Science and Mathematics*, *99*(2), 61–70. https://doi.org/10.1111/j.1949-8594.1999.tb17449.x
Rosseel, Y., Oberski, D., Byrnes, J., Vanbrabant, L., Savalei, V., Merkle, E., Hallquist, M., Rhemtulla, M., Katsikatsou, M., & Barendse, M. (2017). Package 'lavaan.' *Retrieved June*, *17*, 2017.
Shaffer, J. A., DeGeest, D., & Li, A. (2016). Tackling the Problem of Construct Proliferation: A Guide to Assessing the Discriminant Validity of Conceptually Related Constructs. *Organizational Research Methods*, *19*(1), 80–110. https://doi.org/10.1177/1094428115598239
Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science*, *19*(3), 341–361. https://doi.org/10.1111/desc.12395
Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17*(1), 13–19. https://doi.org/10.1016/j.tics.2012.11.004
Siegler, R. S., & Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, *49*(10), 1994–2004. https://doi.org/10.1037/a0031200
Whitacre, I., Henning, B., & Atabas, S. (2020). Disentangling the Research Literature on Number Sense: Three Constructs, One Name. *Review of Educational Research*, 0034654319899706. https://doi.org/10.3102/0034654319899706
Yang, D.-C. (2005). Number sense strategies used by 6th-grade students in Taiwan. *Educational Studies*, *31*(3), 317–333. https://doi.org/10.1080/03055690500236845
Yang, D.-C., Li, M., & Lin, C.-I. (2008). A Study of the Performance of 5th Graders in Number Sense and its Relationship to Achievement in Mathematics. *International Journal of Science and Mathematics Education*, *6*(4), 789–807. https://doi.org/10.1007/s10763-007-9100-0
Yang, D.-C., & Lin, Y.-C. (2015). Assessing 10- to 11-year-old children's performance and misconceptions in number sense using a four-tier diagnostic test. *Educational Research*, *57*(4), 368–388. https://doi.org/10.1080/00131881.2015.1085235

THE PRIME NUMBER THEOREM AS A MAPPING BETWEEN TWO MATHEMATICAL WORLDS

Anderson Norton
Virginia Tech
norton3@vt.edu

Kyle Flanagan
Virginia Tech
flanaga5@vt.edu

This paper frames children's mathematics as mathematics. Specifically, it draws upon our knowledge of children's mathematics and applies it to understanding the prime number theorem. Elementary school arithmetic emphasizes two principal operations: addition and multiplication. Through their units coordination activity, children construct two mathematical worlds: an additive world and a multiplicative world. Understanding how children might map between their additive and multiplicative worlds provides insights into the prime number theorem. It also helps us appreciate the power of children's mathematics, constructed through the coordination of their own mental actions.

Keywords: Cognition; Elementary School Education; Learning Theory; Number Concepts and Operations

Prime numbers fascinate mathematicians of all ages. When we learn to count, we learn that we can build numbers additively, without end, by continuing an iteration of 1s (e.g., 6 as 1+1+1+1+1+1). With prime numbers, we can build those same numbers multiplicatively, as a product of primes (e.g., 6 as 2×3). Confrey and Smith (1995) have described these constructions as two different worlds: an additive world and a multiplicative world. The prime number theorem relates those two worlds in a surprisingly simple way. It states that the number of primes less than or equal to any natural number, n, is approximately n/ln(n). In this way, the theorem relates the additive building blocks of number (n 1s) to its multiplicative building blocks (the π(n) prime numbers less than or equal to n). It states π(n)≈n/ln(n).

Aside from its simplicity, the prime number theorem is surprising in that it describes the relative distribution of primes without discerning a pattern of primes. Attempts to generate primes algorithmically have never failed to fail (e.g., the first five Fermat numbers are prime but, as far as we know, none of the others are). In fact, our national security depends on such failures (though quantum computing might change the game). Moreover, simple conjectures about primes, such as whether every even number can be written as the sum of two primes (e.g., 24=11+13; Goldbach's conjecture), remain unproved.

The prime number theorem is also surprising because, in spite of the simplicity of the theorem itself, its proofs are notoriously complex. Attempts to understand the distribution of primes and prove the prime number theorem led to the invention of the Riemann zeta function—a function defined through an infinite sum of complex-valued terms. Indeed most proofs of the prime number theorem rely on that function. The apparent disconnect between the simplicity of the theorem and its proofs led mathematicians like Atle Selberg (1949) and Paul Erdös (1949) to generate "elementary proofs." Clever as they may be, these proofs are at least as convoluted as the complex ones, but we cannot blame them for trying. After all, prime numbers are natural numbers, and their definition relies on a sieving process that remains contained within the natural numbers. So, why should investigations of prime numbers appeal to complex numbers? One clue appears in the theorem itself, in the form of the natural logarithm.

Confrey and Smith (1995) framed logarithms as mappings between multiplicative and additive worlds. Logarithmic functions convert the formal operation of multiplication into one of addition, thus potentially transforming multiplicative building blocks into additive ones. Indeed, that's precisely what the prime number theorem implies. Formally, this implication appears in the claim (equivalent to the prime number theorem) that the second Chebyshev function—the psi function—approaches n asymptotically. We return to this function later.

Although this paper ostensibly concerns the prime number theorem, it is fundamentally about the nature of mathematics itself: an epistemology of mathematics that we can learn through interactions with children. In tracing the roots of mathematics to a coordination of actions, we can find in children's actions the primary source of all mathematics. We can learn mathematics from children, not only as a source of content knowledge for improving our instruction (Ball, Thames, & Phelps, 2008), but also as a trace of the historical development of formal mathematics. In the case of the prime number theorem, research on children's numerical development teaches us how mathematicians construct additive and multiplicative worlds and how they might navigate between those two worlds. This perspective stands in contrast to Platonist views that have become crystallized in formal mathematics (Lakatos, 1976). As such, this theoretical paper addresses three questions raised in this year's conference theme:

1. How does your work improve learning conditions for each and every mathematics learner?
2. How does your work challenge a settled mathematics learning status quo?
3. Whose voice does your work center in the mathematics learning process? What can be learned from reflecting on this question?

The purpose of this paper is to review research on the additive and multiplicative worlds that children construct and to theorize how our understanding of children's mathematics elucidates our understanding of mathematics as a whole. Along the way, the paper also theorizes why addition and multiplication appear as the two principal operations in formal arithmetic. The paper frames children's arithmetic as the construction, transformation, and coordination of units. In doing so, it privileges students' units coordinating activity as the psychological basis for formal arithmetic, complex analysis, and the prime number theorem. In addition to the framing offered by Confrey and Smith's (1995) mapping between worlds, the paper draws heavily from Ulrich's (2015, 2016) framing of additive and multiplicative units coordination, as well as Boulet's (1998) characterizations of multiplicative reasoning as a transformation of units.

Preserving Units in the Additive World

Researchers who build models of students' mathematics privilege students' mathematics over school mathematics. They explicitly recognize students' ways of operating as the focus of mathematics education: "If, as teachers, we want to foster understanding, we will have a better chance of success once we have more reliable models of students' conceptual structures, because it is precisely those structures upon which we hope to have some effect." (von Glasersfeld & Steffe, 1991, p. 102). Specifically, Steffe and colleagues began a research program to build models of the ways that students construct and transform units to solve arithmetic problems. These ways of operating become the basis for counting and arithmetic. Ulrich (2015, 2016) has exquisitely synthesized this knowledge gained from children in a pair of articles published in *For the Learning of Mathematics*.

According to Euclid, "a *unit* is that by virtue of which each of the things that exist is called one" (*Elements*, Book VII). This definition fits well enough but overlooks the human activity that goes into constructing units of 1. Children begin to construct units through their counting activity. At first, they count by pointing to figurative items (e.g., chips), building a one-to-one correspondence between their number words and their acts of pointing. A number word, say "five," might then stand in place of that activity (Steffe, 1992).

Arithmetic units emerge from the child's reflection on such activity, when they can take any of the counting acts as an identical unit of 1 (Steffe, 1992). The child can then iterate that unit of 1 to produce collections of 1's, such as a 5. This 5 might then become a composite unit, composed of five 1s, as well as a unit in itself (one 5). Reasoning additively amounts to partitioning composite units into other units—ultimately units of 1—and continuing an iteration of those units. This activity produces nested sequences of numbers (e.g., the explicitly nested number sequence; Steffe, 1992), wherein 5 contains 4, 4 contains 3, and so on.

Since Plato, Western philosophers have marveled at the absolute truth of mathematical equalities, such as 2+2=4. However, if we appreciate the human activity of constructing mathematical objects, like 2 and 4, the mystery fades away. If 2 is two iterations of 1, the addition of another 2 continues the iteration of identical units of 1, so that we have 1+1+1+1, also known as 4. One, two, and four do not have an existence apart from such activity, and that same activity establishes their various relationships. In the case of addition, we continue the iteration of a common unit.

Addition preserves units. If a and b are natural numbers, defined by the iteration of a unit of 1, then a+b is defined through the iteration of that same unit of 1. However, if a and b are fractions, we have to find a common unit for measuring each of them. Usually we refer to a common denominator, but implicitly this refers to a unit fraction that we could iterate to produce both a and b (Steffe, 2003). Moreover, the result, a+b, is expressed as an iteration of this same unit fraction; again, units are preserved.

Now consider the case of repeated addition, say determining the value of 4×5 through the addition 5+5+5+5. In this case, we have four composite units of 5, and we can think about the product as four iterations of an identical unit of 5. Alternatively, we can think about each 5 as five 1s, so that 4×5 becomes (1+1+1+1+1)+(1+1+1+1+1)+(1+1+1+1+1)+(1+1+1+1+1)=20. Either way, units (5s and 1s) are preserved. Ostensibly, we have completed a multiplication task via addition: four 5s is twenty 1s. However, multiplicative reasoning does not operate like additive reasoning. It does not preserve units; it transforms them. As we will see, additive and multiplicative reasoning are distinct, though related through units coordinating activity (Ulrich, 2015, 2016).

Transforming Units in the Multiplicative World

In another article published in *For the Learning of Mathematics*, Boulet (1998) set out to "formulate a uniform concept of multiplication that is not simply another arithmetical operation in disguise" (i.e. repeated addition; p. 12). Her approach stands in contrast to that of Vergnaud (1983) who required three subtypes to address multiplication across the domains of natural numbers, integers, and fractions. Following Davydov (1992), Boulet (1998) promoted multiplicative reasoning as a transformation of units. Boulet extended this perspective to include the multiplication of negative numbers, fractions, and irrational numbers. Consider the example illustrated in Figure 1.

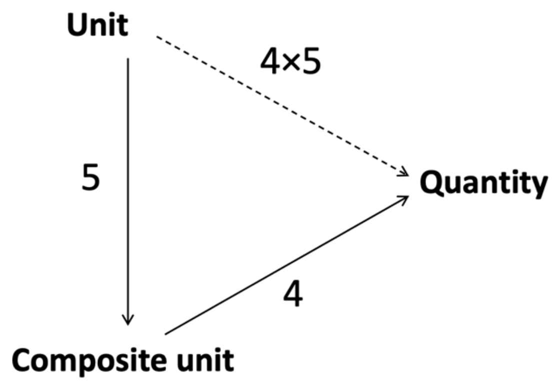


Figure 1: Multiplication as a Transformation of Units

When assimilated as a multiplicative task, 4×5 does not represent the repeated addition of 5s but, rather, a transformation from units of 5 to units of 1 (or vice versa). The product 4 times 5 generates a quantity that measures four units of 5, but we want to know its measure in units of 1, so we project five units of 1 into each unit of 5, transforming each unit of 5 into five units of 1. In terms of units coordination, we have distributed five units of 1 into each of the four composite units of 5 (Steffe, 1992). We could compute the result by counting by 5s (as in repeated addition), but the transformation itself produces the product. Rather than working within a single number sequence, we have transformed numbers across two number sequences: one measured in units of 1 and the other measured in units of 5, with a 5-to-1 transformation between those units.

Boulet's (1998) framing of multiplication places special emphasis on the distinction between roles of multiplicand and multiplier. The multiplicand is the composite unit; it is related to the unit of 1 through a unit transformation (i.e., a distribution of units). The multiplier describes the number of iterations of that unit. With repeated addition, we would take the composite unit (the multiplicand, say 5) and iterate it (say four times). With multiplicative reasoning, we take the four units of 5 and transform them into 20 units of 1, all at once, even if we have to determine their numerosity through skip counting. The focus of activity is on the multiplicand instead of the multiplier.

From this perspective, the fundamental theorem of arithmetic states that we can uniquely produce every natural number in either of two ways: through the iteration of units (a multiplier applied to a multiplicand), or through a transformation of units (transforming the multiplicand). The atoms of addition are units of 1; the atoms for multiplication are primes. Prime factorization of a number, then, represents a sequence of transformations of units, from units of 1, to units of n, where $n=p_1^{R1}p_2^{R2}\ldots p_k^{Rk}$. This idea is generalized in linear algebra, wherein vectors represent units, scalars represent iterations of a unit, and matrices represent unit transformations, transforming vectors (including the basis for a vector space) into other vectors. Additionally, matrix multiplication allows for a recursive transformation of units M×M without reference to any particular scalar (multiplier) or any particular unit (vector). In this way, we can understand the product (-1)×(-1) as a transformation from units of 1 to units -1, and back (as if composing two reflections; Norton, 2022).

Mapping between Worlds

Buttressed by a four-year longitudinal study of upper elementary school students, Lamon (2007) produced the most comprehensive review of literature regarding advanced numerical reasoning. Lamon concluded her report with a list of outstanding questions, beginning with the following: “what are the links between additive and multiplicative structures?” Ulirch (2015, 2016) responded in terms of composite units and ways that students operate with those units: iterating composite units, disembedding units from a composite unit, distributing the units of one composite unit across the units of another composite unit. Essentially, both additive reasoning and multiplicative reasoning rely on the construction of composite units, so they have a common cognitive basis, but as we have seen, students operate on composite units differently in additive and multiplicative reasoning. Specifically, we have made a distinction between ways students preserve (composite) units when engaged in additive reasoning, and ways they transform units when engaged in multiplicative reasoning.

Lamon’s (2007) review included Confrey’s (1994) concept of a split, through which students might transform a unit of 1 into a unit of n. Steffe (2002) proposed a similar concept, describing ways students could partition a whole into five equal parts and, reversibly, reconstruct the whole by iterating any one of those parts five times. For both Confrey and Steffe, splitting describes a multiplicative way of operating. However, in anticipating a mapping between additive and multiplicative worlds, Confrey (with Smith, 1995) included the possibility of recursive splits.

Confrey and Smith (1995) used “basic splits” to refer to splits by prime numbers (e.g., a one-to-seven split). They used basic splits recursively to consider how students might map between additive and multiplicative worlds. In the additive world, we begin at 0 and iterate by units of 1 to produce other natural numbers. In the multiplicative world, we being at 1 and perform basic splits to produce those same numbers. To map between the worlds, we rely on logarithms and exponential functions.

We can define exponential and logarithmic functions as mappings between additive and multiplicative structures. Formally, we say they are isomorphisms between the group of integers under the operation of addition, and the group of positive real numbers, under multiplication (e.g., $\log_2(1/4\times8)=-2+3$). Pedagogically, they relate children’s counting activity to their splitting activity. Following Confrey and Smith (1995), we can use the activity of paper folding to exemplify. In the example of $\log_2(1/4\times8)$ we might say that, starting from a folded sheet of paper, if we fold it in half twice more (-2; or 1/4) and then unfold it three times (+3, or ×8), we end up with a sheet that has twice the area we started with (+1, or 1/4×8).

Proofs of the Prime Number Theorem

Two and a half millennia after Plato, Platonist views of mathematics still dominate the field (Lakatos, 1976). When we make and prove conjectures about prime numbers, we tend to think about those numbers as pre-existing, lying in wait for their discovery; this despite the fact that Eratosthenes demonstrated, through his sieve, that prime numbers are the result of a process of filtering out multiples of units, beginning with multiples of 2. Indeed, the clearest existing explanation of the prime number theorem follows quickly from the fact that, to find primes up to n, we need only apply the sieving process up to the square root of n (Marshall & Smith, 2017). However, this proof is not generally accepted as rigorous because it relies on what professional mathematicians consider to be probabilistic arguments about the density of prime numbers. Neither will professional mathematicians trust empirical arguments, and rightly so, but again, such arguments suggest that the prime number theorem is not about prime numbers at all. Specifically, in computer simulations where a random sieve is applied to the natural numbers (randomly eliminating half of the numbers, taking the smallest remaining number, k, and randomly removing 1/k of the remaining numbers, and so on) the distribution of these “random primes” behaves the same way (Brown, 1978, citing a study by Hawkins, 1958). There are approximately n/ln(n) of them less than or equal to n, as n gets large.

To prove the prime number theorem, mathematicians are more comfortable moving into the complex plane, which comes complete with a geometry of numbers and all kinds of computational power, including infinite sums, complex roots, and calculus. The first proofs of the prime number theorem relied on the Riemann zeta function. Interestingly, and in line with more intuitive arguments introduced here, the infinite sum that defines the zeta function can also be expressed as an infinite product of terms involving distinct primes. However, the complexity of existing proofs has left mathematicians wondering whether a simpler proof could be found. After all, and to echo a question first raised by David Hilbert (Kreisel, 1984), why should the proof of a theorem about natural numbers rely on complex analysis?

Driven by such questions, Selberg (1949) and Erdös (1949) developed proofs that relied only on arithmetic functions. Paul Erdös was a Hungarian mathematician and a prolific mathematical collaborator. The subject of *The Man Who Loved Only Numbers* (Hoffman, 1998), Erdös believed in a Book of perfect proofs held by the supreme fascist (i.e., God), who withholds from us knowledge of the Platonic realm. No doubt, the convoluted proofs that he and Selberg developed—though clever—are not in the book. Without the computational power of complex analysis, they relied on several specialized functions, such as the Chebyshev functions, which were designed to investigate prime numbers. For example, Chebyshev's psi function ($\psi(n)$) takes a natural number, n, and produces the sum of natural logarithms of new prime factors introduced by numbers less than or equal to n (e.g., $\psi(6)=0+\ln(2)+\ln(3)+\ln(2)+\ln(5)+0$). Note that the value for the fourth term in the sum is ln(2) because 4 introduces a second 2 in its prime factorization; and the value for the sixth term is zero because both prime factors of 6 (2 and 3) were introduced earlier in the sum.

Learning Mathematics from Children

Children have a lot to teach us, especially if we want to teach them. For teachers, learning from children is critical because, if the goal is to promote the development of students' mathematical structures, we must first understand how their mathematical structures operate. Hackenberg (2010) has framed the issue as one of "mathematical caring relations." Children feel cared for when we attempt to harmonize with their thinking. Beyond teaching and learning, the present focus is on an epistemology of mathematics and what children might teach us about mathematics itself.

Consider the prime number theorem in light of what we know about the additive and multiplicative worlds that children construct. In the additive world, children produce natural numbers, starting from 0, through an iteration of 1s. When 1s become iterable for the child, each number contains all the numbers that precede it, as a nested sequence (e.g., as five units of 1, 5 contains 4, which contains, 3, and so on). In the multiplicative world, children produce numbers, starting from 1, through a transformation of units. As the multiplicative analogue of a nested sequence, the least common multiple of a sequence of numbers contains all the numbers that precede it. For example, 60 is the least common multiple of 2, 3, 4, and 5, and contains each of those numbers within its prime factorization: $60=2^2\times3\times5$.

As Confrey and Smith (1995) suggested, we can understand logarithms and exponentials as mappings between those additive and multiplicative worlds. The prime number theorem pertains to such a mapping, so we should not be surprised at the appearance of a logarithm within it. Still, we might wonder why it is the natural logarithm, in particular, that appears. The natural number, e, is defined by a limit. Corresponding to this limit, e^x is defined as follows, and from this definition, we can derive a definition for its inverse function, ln(x):

- $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$
- $ln(x) = \lim_{n\to\infty} n(\sqrt[n]{x} - 1)$

Thinking about units and unit transformations, we can read a lot into this definition of the natural logarithm. We have a multiplier, n, applied to the difference of two units: the nth root of x, and 1. In an additive world, we would produce n through n iterations of the unit of 1. In a multiplicative world, we would produce x through a sequence of n unit transformations, each with the n^{th} root of x as its ratio.

Now consider what happens when x is the least common multiple of the first k natural numbers. The prime factorization of x would be the union of the prime factors of 1, 2, 3, …, k (including multiples of the same prime factor), and ln(x) would be the sum of the logarithms of those prime factors. Because the logarithm function maps multiplicative worlds to additive worlds, we should expect this sum to be k: the multiplicative world containing the numbers 1 through k would get mapped to the additive world containing the numbers 1 through k. In fact, that is exactly what the natural logarithm function does as k gets large, and that fact is equivalent to the prime number theorem.

The least common multiple of the first k numbers is the product of the union of prime factors up to k. So, the natural logarithm of this least common multiple is the sum of the natural logarithms of those prime factors. This sum is Chebyshev's psi function, as Table 1 illustrates.

Table 1: The Prime Number Theorem

Iterations of 1	k	Product of Primes	New Unit Transformations	New Terms in Sum	Value of Sum
+1	1	p^0	×1	0	0
+1	2	p_1	$\times p_1$	$+\log(p_1)$	0.693
+1	3	p_2	$\times p_2$	$+\log(p_2)$	1.792
+1	4	p_1^2	$\times p_1$	$+\log(p_1)$	2.485
+1	5	p_3	$\times p_3$	$+\log(p_3)$	4.094
+1	6	p_1p_2	×1	+0	4.094
+1	7	p_4	$\times p_4$	$+\log(p_4)$	6.040
+1	8	p_1^3	$\times p_1$	$+\log(p_1)$	6.733
+1	9	p_2^2	$\times p_2$	$+\log(p_2)$	7.832
+1	10	p_1p_3	×1	+0	7.832
+1	11	p_5	$\times p_5$	$+\log(p_5)$	10.300
			$(p_1^{R1})\ldots(p_{\pi(k)}^{R\pi(k)})$	$\psi(k) \rightarrow \log(k)\pi(k)$	$\rightarrow k$

The left two columns in Table 1 show the production of natural numbers, k, through the iteration of a unit of 1, beginning from 0. The third column shows the production of each of those natural numbers through a transformation of units (a product of primes), beginning from 1. The fourth column shows new prime numbers introduced in the prime factorization of each natural number (×1 indicates no new primes). The fifth column shows the natural logarithms of new prime factors, which are the terms in the sum that define Chebyshev's psi function. The rightmost column shows this sum, up to each k. Note that because the iteration of 1s begins at 0 and the sequence of unit transformations begin at 1, the value in this right column would better approximate k if we added 1 to it.

The bottom line is that each number, from 1 to k, can be expressed as the product of primes contained within the least common multiple of those numbers. This is expressed in the bottom row of the third column as $(p_1^{R1})(p_2^{R2})\ldots(p_{\pi(k)}^{R\pi(k)})$, where Ri is the highest power to which each prime p_i can be raised without exceeding n. Note that there are π(k) factors, p_i^{Ri}, and as k gets large, each of the factors approximates k. So, the entire product is approximately k raised to the π(k). Correspondingly, ψ(k) is approximately π(k)×ln(k). As suggested by the definition of natural logarithm given above (as a mapping from nested products to nested sums), and as shown in the right column, ψ(k) is also approximately k, which implies the prime number theorem. Moreover, this definition of natural logarithm, along with the prime number theorem, suggests an alternate definition of e. As proved below, we can define e as the n^{th} root of the least common multiple of the first n natural numbers, as n gets large:

- $\Psi(k) = \ln(\mathrm{lcm}(1,2,\ldots,k)) = \lim_{n\to\infty} n(\sqrt[n]{\mathrm{lcm}(1,2,\ldots,k)} - 1)$
- In units of k, $\lim_{n\to\infty} nk(\sqrt[nk]{\mathrm{lcm}(1,2,\ldots,k)} - 1) = \lim_{n\to\infty} nk(\sqrt[n]{\mathrm{lcm}(1,2,\ldots,k)^{1/k}} - 1)$
- So, $\frac{\Psi(k)}{k} = \lim_{n\to\infty} n(\sqrt[n]{\mathrm{lcm}(1,2,\ldots,k)^{1/k}} - 1) = \ln\sqrt[k]{\mathrm{lcm}(1,2,\ldots,k)}$
- Because $\frac{\Psi(k)}{k} \to 1$, $\sqrt[k]{\mathrm{lcm}(1,2,\ldots,k)} \to e$, so $e = \lim_{n\to\infty} \sqrt[n]{\mathrm{lcm}(1,2,\ldots,n)}$.

Summary

To summarize the heuristic argument about the prime number theorem, if we knew that the natural logarithm mapped the nested product of the first n natural numbers (their least common multiple) to their nested sum (n), the prime number theorem would follow. The ideas of nested number sequences and logarithms as maps between multiplicative worlds were informed by research on children's mathematics (e.g., Steffe, 1992; Ulrich, 2015a, 2015b). Related constructs, such as units and unit transformations, provide additional insights into why we might call the natural logarithm natural.

Aside from arguments about the prime number theorem, this paper makes an argument about the epistemology of mathematics, particularly regarding questions related to the conference theme. The first question asks about improving learning conditions for all children. In response, we frame children's mathematics as genuine mathematics. This mathematics is important to teachers as they support children's continued development as mathematicians. This idea is in line with calls for teachers to develop pedagogical content knowledge (e.g., Ball, Thames, & Phelps, 2008), as well as Hackenberg's (2010) descriptions of mathematical caring relations as "harmonizing with students' schemes" (p. 242). Additionally, we argue that children's mathematics is mathematics that can inform research in mathematics, even at advanced levels.

In framing children's mathematics as mathematics, we answer the second question (regarding the status quo) by challenging Platonist perspectives that still the dominate the field. For instance, there is a tendency to think about prime numbers as pre-existing as if the prime number theorem pertains to discovering some kind of pattern in the fabric of the universe. However, our heuristic argument buttresses prior arguments (e.g., investigations of random primes; Hawkins, 1958) that suggest that the theorem pertains to processes of producing prime factors (e.g., sieving) and has little to do with a pre-determined subset of the natural numbers. In response to the third question, we privilege the voices of children, from whom we have much more to learn.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Boulet, G. (1998). On the essence of multiplication. *For the Learning of Mathematics*, *18*(3), 12-19.

Brown, S. (1978). *Some prime comparisons*. Reston, VA: NCTM.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-331). Albany, NY: SUNY Press.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for research in mathematics education*, *26*(1), 66-86.

Davydov, V. V. (1992). The Psychological Analysis of Multiplication Procedures. *Focus on Learning Problems in Mathematics*, *14*(1), 3-67.

Erdös, P. (1949). On a new method in elementary number theory which leads to an elementary proof of the prime number theorem. *Proceedings of the National Academy of Sciences of the United States of America*, *35*(7), 374-384.

Hackenberg, A. J. (2010). Mathematical caring relations in action. *Journal for Research in Mathematics Education*, 236-273.

Hawkins, D. (1958). Mathematical sieves. *Scientific American*, *199*(6), 105-114.

Hoffman, P. (1998). *The man who loved only numbers: The story of Paul Erdos and the search for mathematical truth*. New York: Hyperion.

Kreisel, G. (1984). Frege's Foundations and Intuitional Logic. *The Monist*, *67*(1), 72-91.

Lakatos, I. (1976). *Proofs and refutations*: *The logic of mathematical discovery.* Cambridge, United Kingdom: Cambridge University Press.

Lamon, S. J. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. *Second handbook of research on mathematics teaching and learning*, *1*, 629-667.

Marshall, S. H., & Smith, D. R. (2017). Feedback, control, and the distribution of prime numbers. *Mathematics Magazine*, *86*(3), 189-203.

Norton, A. (2022). *The psychology of mathematics: A journey of personal mathematical empowerment for educators and curious minds*. London: Routledge.

Selberg, A. (1949). An elementary proof of the prime-number theorem. *Annals of Mathematics*, 305-313.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259-309.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *The Journal of Mathematical Behavior*, *20*(3), 267-307.

Steffe, L. P. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 5. *The Journal of Mathematical Behavior*, *22*(3), 237-295.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (Part 1). *For the Learning of Mathematics*, *35*(3), 2-7.

Ulrich, C. (2016). Stages in constructing and coordinating units additively and multiplicatively (Part 2). *For the Learning of Mathematics*, *36*(1), 34-39.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), Acquisition of math concepts and processes (pp. 127–174). London: Academic Press.

von Glasersfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *The Journal of Educational Thought, 25*(2), 91-102.

DIFFERENT WAYS OF LEARNING NUMBER DENSITY: A HYPOTHETICAL TRAJECTORY WITH HIGH SCHOOL STUDENTS

DIFERENTES FORMAS DE APRENDER LA DENSIDAD NUMÉRICA: UNA TRAYECTORIA HIPOTÉTICA CON ESTUDIANTES DE BACHILLERATO

Mayra Suárez-Rodríguez
Cinvestav
mayra.suarez@cinvestav.mx

Ana Isabel Sacristán
Cinvestav
asacrist@cinvestav.mx

We present a study where we designed a Hypothetical Learning Trajectory (HLT) for promoting the learning of the property of numerical density in high-school students. The tasks of the HLT used various school mathematics topics that can provide opportunities for thinking about that property and were framed in several contexts and semiotic representation registers. We present the sequence of the HLT, as well as some results of its implementation with four high-school students. We analyze the participants' data in terms of their thinking regarding discreteness and density. We observed that at the end of the HLT tasks, the participants recognized more easily an infinite quantity of intermediate rational or real numbers in an interval, although they didn't fully comprehend that only natural numbers have successors.

Keywords: Learning trajectories and progressions, mathematical representations, number concepts and operations, rational numbers

Introduction and Research Questions

In many countries, it is common for high school students to learn about representations of numbers on the number line, classes of intervals, as well as the continuity of real numbers, but not about the property of density (Vamvakoussi & Vosniadou, 2010; McMullen & Van Hoof, 2020). The density property of a number set is defined as that given any two numbers in that set, there is a number between them. If a number set satisfies this property, it implies that, in that set: 1) No number can have an (immediate) successor; 2) Any interval contains an infinity of numbers. Likewise, a completely dense set is not discrete, understanding discrete, as stated by Vamvakoussi and Vosniadou (2010), in terms of order relations.

Some students arrive at the university level education, with poor conceptions about this property. Among the difficulties that are observed are the misconceptions that: 1) In any number set, beyond the sets of natural or integers, a successor exists for any number; and 2) there is a finite quantity of numbers in an interval of the rational numbers. For example, the second misconception was observed with preservice teachers in Israel by Tirosh et al., (1999). Also, Merenluoto and Lehtinen (2004) observed that 10% of students studying Mathematics at a university in Finland believed that there was a finite number of rational numbers in an interval. Likewise, discrete thinking stemming from thinking related to natural numbers is problematic: Vamvakoussi and Vosniadou (2010), in a study in Greece, found that the idea of the discrete was dominant in their observed students and that the symbolic representation of the extremes of the interval had an effect in their judgments regarding, not only the number of intermediate numbers in an interval, but also what kinds of numbers are in it (i.e., some of these students considered that there can only be decimals in an interval whose endpoints are decimal numbers, but there cannot be fractions).

Based on the above and on the difficulties that students have to learn and understand the property of number density, in the research presented here, we developed a Hypothetical

Learning Trajectory (HLT), as part of a design-based research (Cobb and Gravemeijer, 2008), with the aim of promoting the learning of this property in high school students. In the HLT, we use different contexts, situations and representations, inspired by McMullen and Van Hoof (2020), who indicate that there are moments in school mathematics that can provide, with additional instruction, opportunities to reason about number density.

This report is an extension of the research study carried out by Suárez-Rodríguez and Sacristán (2021), where we addressed the following research questions :

a) How can we design a sequence of tasks with a specified HLT, to promote the learning of the density property in the set of real numbers?

b) How do high school students understand the property of density in real numbers using semiotic representations produced during our proposed tasks?

Theoretical and Methodological Frameworks

Thinking about the discrete and the dense

As Vamvakoussi and Vosniadou (2004, 2010) indicate, the properties of natural numbers tend to be transferred to other number sets and this affects students' conceptions of the property of density. These authors (Vamvakoussi and Vosniadou, 2004) defined certain categories of thinking about the discrete, the dense and its relationship with the number of numbers in an interval, to identify ways of reasoning in students (Table 1 shows a reinterpretation by us of these categories).

Natural numbers, unlike real numbers, have the property of discreteness as established in the Peano axioms where it is stated that between a natural number and the following one, there is no other number (Ni and Zhou, 2005). Furthermore, all natural numbers have a successor (Ni and Zhou, 2005) – since it is a well-ordered set.

Thus, from the study of natural numbers, a thinking linked to the discrete is formed. Vamvakoussi and Vosniadou (2004) used the term *naive*, coined by Carey (1987) to define two of their categories of thinking that refer to "naive ideas"(see Table 1): The category *naive thinking about the discrete* refers to a "naive idea" that rational numbers are consecutive. Likewise, naive *thinking about the dense* is when it is considered that there is an infinity of numbers in an interval, but it is not possible to justify why the existence of this infinity. Contrasting these naive thinking are the categories of forward thinking: The category of *discrete forward thinking* is when it is believed that there are a finite number of numbers in an interval; while in the category *sophisticated thinking about the dense*, the formal definition of density is understood.

Table 1: Categories of thinking about the discrete and the dense

Naive discreteness [Naive thinking about the discrete]	The thinking is that there is no number between two *consecutive false* rational numbers. Vamvakoussi and Vosniadou (2004) coined this expression to refer to the idea that a successor of a rational number exists.
Advanced discreteness [Advanced thinking about the discrete]	The thinking is that there is a finite quantity of numbers between two consecutive false rational numbers.
Discreteness–density [Mixed thinking between discrete and dense]	In some cases, the thinking is that between two rational numbers there is an infinity of numbers; and in other cases, that there is a finite number.

Naive density [Naive thinking about density]	There is an understanding that there is an infinity of numbers in an interval, but this situation is not justified by using the density property. The symbolic representation of the extremes of an interval influences this way of thinking; it is believed there can only be an infinite number of decimal numbers between decimals and an infinity of fractions between fractions, but not an infinity of fractions between decimals or otherwise.
Sophisticated density [Advanced thinking about density]	There is a sophisticated understanding of the density property; that is, it is understood that there is an infinite number of numbers between two rational numbers, regardless of their symbolic representation and this is justified through the use of the density property.

Description of a Hypothetical Learning Trajectory (HLT)

Simon and Tzur (2004) define a HLT as a series or sequence of teaching activities that comprise conjectures or hypotheses about how the student can learn certain mathematical concepts. Thus, a HLT is a prediction of a learning path that includes didactic interventions through the sequence of activities (Simon, 1995). Thus, in a THA it is essential to develop and propose hypotheses on how to support the student in the construction of new knowledge (Simon, 1995). In fact, Simon (1995) establishes three components in the design of a HLT: 1) The *learning goal* that defines the direction of the HLT so that the student builds new knowledge; 2) the *learning activities* that follow a sequence; and 3) *the hypothetical learning process*, where a prediction is made of how the students' thinking and understanding will evolve, and that defines how to support their learning process. During the activities, the researcher (or teacher) observes how the student is learning in the scenarios proposed in the HLT to determine if the design is adequate and changes need to be made (Simon, 1995).

Semiotic registers of representation

One way that we consider important to support the learning of the property of density is the use of several registers of representation. Mathematics deals with different registers of representation –be it geometric, algebraic (symbolic expressions), arithmetic, even colloquial language (linguistic expressions)– and each of these registers constitutes a different semiotic form (Duval, 1995/2004); that is, it communicates meaning differently. Semiotic representations are fundamental for the development of mathematical activity itself, which depends on a writing system (decimal, fractional, binary, etc.) (Duval, 1995/2004). This is also particularly relevant because when working with real numbers (as is our case), it is necessary to take into account that the real number has different representations (Apostol, 1976/2006).

Methodology

Design and development of the HLT activities

As we already mentioned, in our research we developed a HLT according to the guidelines of design-based research (Cobb and Gravemeijer, 2008), a methodological approach in which a researcher carries out a detailed analysis of how to test and improve the conjectures that are outlined. Cobb and Gravemeijer (2008) recommend the design and elaboration of more than one cycle of activities in order to refine or modify the conjectures raised during the first cycle. In our research, two cycles of activities were planned; in this report, we present some results from the first cycle, from data by four students from Colombia, between 15 and 17 years of age: Angie, Paola and Néstor (pseudonyms), who were high-school students; and Violeta (pseudonym), who was in her last year of middle-school. These students solved some tasks proposed by the HLT in person, while other tasks were carried out online, due to the COVID-19 pandemic.

The preparation and implementation of the first cycle of the HLT activities consisted of three phases. The first phase was the application of a questionnaire as a pre-test; in the second, the activities of the HLT were carried out; and in the last phase, the questionnaire was applied again as a post-test. We describe below the phases of the HLT, then we present some results of the first two phases and give an example of what happened in the last phase.

First and third phase. *Pre-test and post-test*. The same questionnaire (to be answered in approximately 30 min) was used for the pre- and post-tests, in order to determine the possible evolution of the students' thinking. The questionnaire consists of four questions related to the number of intermediate numbers in an interval –those questions were based on those by Suárez-Rodríguez and Figueras (2020), as well as others by Vamvakoussi and Vosniadou (2004). The aim of the questionnaire was to find out how much the participants knew about the property of number density; the answers were classified according to the categories proposed by Vamvakoussi and Vosniadou (2004) (see Table 1). The questions included in the test are: *How many numbers are there between 0 and 1? Can you find decimal numbers and/or fractions between 0.49 and 1/2? How many numbers are there between 0.899 and 0.90?* In the fourth question, the following situation was presented:

> Pedro wants to draw a line where he can place decimal numbers. He starts with the number 0.1. Then he places 0.2. At that point, Laura interrupts him and says that the number, "next" to 0.1, is 0.11. Who is right, Pedro or Laura? Or neither? Or both?

Second phase. *Implementation of the HLT activities*. The HLT was designed to be carried out in four activity sessions, each with a corresponding learning hypothesis (see Table 2; similar to the one that appears in Suárez-Rodríguez and Sacristán, 2021, p. 1223). The hypotheses were defined taking into account various topics of school mathematics that we supposed can guide the student to a learning and understanding of the property of density. It is not necessary for the student to follow the sequence of the sessions, although a first approach to density is through decimal numbers, and the sequence ends with irrational numbers. Each session lasted about 1h30'.

Table 2: HLT sessions with their corresponding hypotheses

Levels	*Hypotheses*
Level 1. Initial approaches to number density.	From two situations: one related to everyday life and the other related to a hypothetical scenario, it is thought that the student can have his first approaches to the property of density.
Level 2. Approach to number density through the similarity of triangles.	It is considered that by using triangle similarity students can learn about the density property of rational numbers.
Level 3. Approach to number density through arithmetic progressions and geometric progressions.	It is possible that by finding arithmetic and geometric halves in an interval, students can understand the density property of rational numbers in the set of real numbers.
Level 4. Approach to numerical density through the property of continuity.	Using the continuity property, students are believed to understand the density property of irrational numbers in the set of real numbers.

Through this HLT students are expected to achieve a metaconceptual awareness, realizing that their suppositions and beliefs are not unquestionable and that they constrain the way in which new knowledge is acquired (Vosniadou, 1994). At the end of the HLT activities, students are expected to be able to distinguish between a discrete set and a dense set.

Results

The categories of Vamvakoussi and Vosniadou (2004) (Table 1) were used for the analysis of the responses to the questionnaire, as well as for the data from the HLT activities.

Pre-test results (phase 1)

Figure 1 shows Angie's response to the question about the quantity of numbers between 0.899 and 0.90: *"[t]here is only one, because it is in the middle of the two and because it leads to the next number".* A *naive thinking about the discrete* in Angie, that comes from her knowledge that the natural numbers are discrete, is inferred when she says that "there is only one" and that, for her, a decimal number has a successor ("it leads to the next number").

solo ay uno, porque esta en la mitad de los dos y porque da paso al siguiente numero *There is only one, because it is in the middle of the two and because it leads to the next number*

Figure 1: Example of naive thinking about the discrete (Angie – pre-test)

In Figure 2, we observe that the answers of Violeta, Angie and Néstor to the question of whether there is a number that follows 0.1, all coincide in that 0.2 is the number that follows 0.1. In Paola's answer: "*Yes, there is a number that follows 0.1[:] 0.11, 0.111, 0.111...,*" we observe that although, in her colloquial language, she affirms the existence of a successor, and names several: "0.11, 0.111, 0.111[1]…", for her, the fact of adding the number 1 each time in the decimal part of the number, indicates that there are more "successors" for 0.1. We consider that these four students exhibited in this question, *a naive thinking about the discrete* since, apparently, they believe that, for decimals, there exists a successor (or several in Paola's case).

4. b. Is there a number that follows 0.1? If your answer is yes, write the number that you think follows 0.1. But if your answer is not affirmative, explain why.	
Angie's response 0.1 Le sigue 0.2 *0.1 follows 0.2*	Violeta's response Si existe es el 0.2 y asi sucesivamente con lo demas *Yes, it exists; it is 0.2 and so on with the others*
Néstor's response Rta. Para mi creo que es 0.2 *For me I think it's 0.2*	Paola's response Si, existe un número que le sigue a 0,1. 0,11, 0,111, 0,111... *Yes, there is a number that follows 0.1, 0.11, 0.111, 0.111…*

Figure 2: Other examples of naive thinking about the discrete (Pre-test)

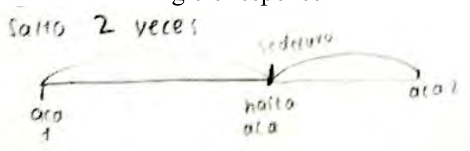
Results of the HLT activities (phase 2)

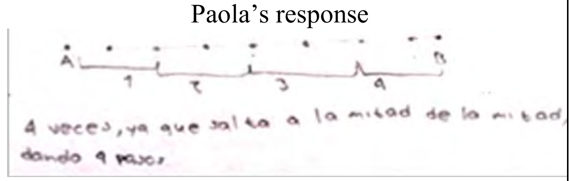
We present some results from the four students' data during the development of the HLT.

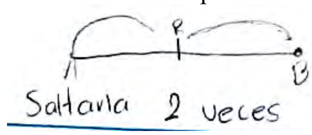
Session 1. In the first session, the students solved two learning activities that were meant as a first approach to density. The second of those activities involved the following:

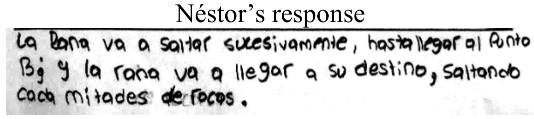
> Imagine that you see that a frog starts to jump from point A and tries to reach point B in the following way: 1st step: The frog jumps half the distance [from A to B]. 2nd step: It then jumps half the distance that remains. 3rd step: Again, it jumps half of what remains. And so on.
> 1. After imagining the situation, now describe it, or explain it...
> 2. How many times will the frog jump between points A and B? Explain your answer.

In solving this activity, we consider that Angie, Violeta and Paola (see Figure 3) are approaching *advanced thinking about the discrete*, when they indicate that the frog has jumped a finite number of times and use as a representation a segment of points in a graphic log. On the other hand, Néstor says *"The Frog is going to jump successively until it reaches point B; […] it will reach its destination, jumping each [time] halves"*; one might think that he conceives of a potential infinity of jumps (also in the previous activity he had said that there were "infinite numbers" between two decimal places); we could say that he shows a *naive thinking about the dense*.

<table>
<tr><td>Angie's response


It jumped twice</td><td>Paola's response


4 times, since it jumps half of a half, taking 4 steps</td></tr>
<tr><td>Violeta's response


It would jump 2 times</td><td>Néstor's response


The frog will jump successively, until reaching point B; and the frog will reach its destination, jumping over each [time] halves [between the] rocks.</td></tr>
</table>

Figure 3: Responses of the four students in the activity "Frog Jumps" (Session 1)

Session 2. The activity of this session is based on tasks done by Tovar (2011). The hypothesis is that the density property of fractions and decimals can be learned by working with constructions of similar triangles using the number line, both in GeoGebra (geometric register) and with paper and pencil (arithmetic or algebraic register). Students are asked to locate the points A = 0 and B = 1 on the line and construct a triangle ABD. Then they are asked to create another similar triangle AXC, with C as midpoint of $\overline{AD}$ (Figure 4). They then have to find the numerical value corresponding to X. Figure 5a shows that Angie found X = 0.5 by applying proportions. They then repeat the process for a third similar triangle AGF, with F as midpoint of $\overline{AC}$ (Figure 4), and find the numerical value corresponding to G. Figure 5b shows that Angie found G = 0.25 applying again proportions. So, Angie found the numbers 0.25 and 0.5 between 0 and 1, using semiotic representations in an algebraic register, writing fractions and decimals.

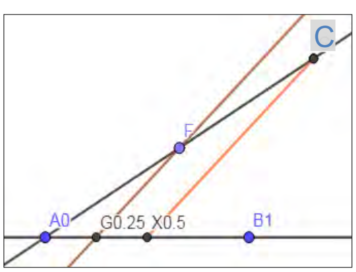
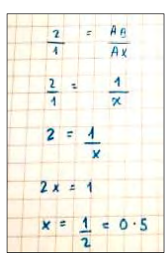
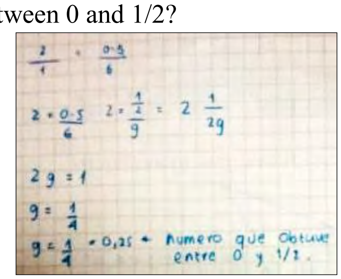


Figure 4: Angie's graph in GeoGebra, locating points between A=0 and B=1 using similar triangles (Session 2)

a. Using triangle similarity find the intersection point (x)



b. Do the same procedure for the interval between 0 and 1/2. What number did you get between 0 and 1/2?



0,25← the number I got between 0 and 1

Figure 5: Answers from Angie (Session 2) for:
a. finding the value of point x between 0 and 1
b. finding the number between 0 and 1/2

Subsequently, the students have to find another point between 0 and 1, such as 3/4 (0.75), and they are asked: *What happens if you continue to carry out the same procedure between 3/4 and 1? And so on, successively? How many rational numbers would be obtained?* Figure 6 shows the responses of the four students to these questions. Angie and Néstor seem to show *advanced discrete thinking* because they refer to a finite number of numbers (they say that there are "several" rational numbers between 3/4 and 1, and that more can be obtained). On the other hand, Violeta's response *("the number of numbers continues to increase; many rational numbers will be obtained")* could perhaps indicate a potential infinity, which would correspond to a *naive thinking about the dense*. As for Paola, she says: *"We would continue to obtain half between 3/4 and 1 and so on"*, which may indicate that she considers a potential infinity of rational numbers, which would correspond to a *sophisticated thinking about the dense*, since she justifies it with the process of removing halves successively.

Angie	Se podrán ver más de un numero racional ya que si de 0 a 1 hay varios, de $\frac{3}{4}$ a 1 tendrían más *You can see more than one rational number because, if there are several from 0 to 1, from 3/4 to 1 there would be more.*
Néstor	Rta: Pueden obtener varios números racionales, pero no se puede pasar de 1. *You can get multiple rational numbers, but you can't go past 1.*
Violeta	Siguen aumentando la cantidad de numeros Se obtendrian muchos numeros racionales. *Numbers keep increasing. Many rational numbers would be obtained*
Paola	Se seguiría obteniendo la mitad entre $\frac{3}{4}$ y 1 y así sucesivamente Infinitos *You would still get half between 3/4 and 1 and so on. Infinite.*

Figure 6: Responses of the four students in the activity on proportions (S2)

Session 3. In this session, three activities related to arithmetic progressions and geometric progressions were carried out so that students could see another way of finding numbers in an interval (in order to learn about the density of rationals in the real number set). Initially, students had to find five arithmetic means between 4 and 22 using the nth term of a sequence: $u = a + (n - 1)d$, where a is the first term, n the number of terms, and d the difference between one term and another. They were then asked what the new arithmetic means would be if the difference, d, were halved (see Figure 7, Violeta).

If d were halved, that is, $d=1.5$, or $d=3/2$, what would be the new arithmetic means between 4 and 22?

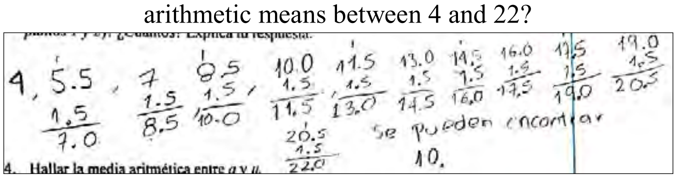


Figure 7: Violeta's Response (Session 3)

Can you find more numbers between 4 and 22 (other than the numbers found in the previous questions)? How many? Explain your answer

Rta: La Respuesta es, son infinitos.

The answer is, they are infinite.

Figure 8: Néstor's response (S3)

Figure 7 shows how Violeta finds the new arithmetic means –5.5, 7, 8.5, 10, 11.5, 13, 14.5, 16, 17.5, 19, 20.5–, and answers the question of whether more numbers can be found between 4 and 22 (different from the numbers found in the previous questions of the activity), how many and why. She replies that *"10 can be found"* (although it was 11 that she found), that is, a finite quantity of numbers between 4 and 22, so she shows *advanced discrete thinking*. Violeta uses decimal writing as a representation and adds to an arithmetic register. Néstor (Figure 8) exhibits a *naive thinking about the dense*, when he affirms that an infinity of numbers can be found.

Session 4. In this session, another activity based on exercises proposed by Tovar (2011) was used. The hypothesis is that students can learn about the density of the irrational numbers in the real numbers, through the continuity of the line in relation to the non-correspondence between rational numbers and the points on the line. They were asked to trace a square of side 1, in the cartesian plane, with a vertex on the origin, and rotate its diagonal so that it would lie on the line of the x-axis (see segment $\overline{DF}$ in Figure 9 –Angie's example). Students noted that the “right endpoint” of the diagonal did not coincide with any rational number on the line when they zoomed in several times with GeoGebra. They wrote then various intervals that would enclose that point: Figure 10a shows two intervals given by Néstor that contain the point. Paola (Figure 10b) gave four increasingly smaller intervals. With this, the students could note that between two real numbers, an irrational number can be found (here, $\sqrt{2}$). No further analysis was done here.

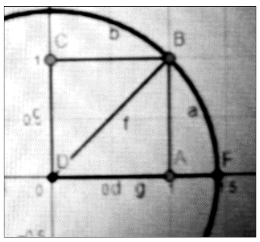


Figure 9: Angie's graph in GeoGebra (S4)

a. Néstor´s response **b.** Paola’s response

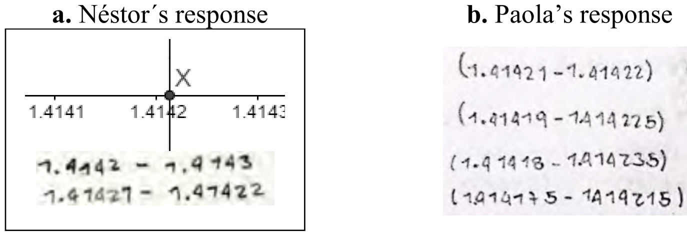


Figure 10: Intervals written by Néstor and Paola that enclose the value of the point corresponding to the length of the square’s diagonal (S4)

Post-test results (phase 3)

In their responses to the post-test, the students showed evidence of having changed their thinking in their answers to the questions related to the quantity of numbers in an interval, with respect to those given in the pre-test (phase 1). For example, for the question *Can you find decimal numbers or fractions between 0.49 and 1/2? How many numbers?* Néstor changed from *naive thinking about discreteness* in the pre-test (when he said: *“1/2 = 0.5 follows 0.49; there are no [numbers]”*) to *naive thinking about dense* (when, in the post-test, he said: *“there are infinite numbers”*).

Concluding remarks

The activities proposed in the HLT aimed for students to learn about the property of number density when solving tasks in various contexts and semiotic registers. Topics such as constructions of similar triangles using the number line, arithmetic and geometric progressions, the property of continuity, make it easy to find different ways to locate numbers in an interval. Likewise, the students had the opportunity to use several semiotic registers of representation, such as the algebraic and the geometric, using fractional and decimal writing, as well as colloquial language to express their ideas during the implementation of the HLT. In some situations, the students were able to understand that there was an infinity of numbers in an interval in the reals. In others, however, they could not overcome the belief that any number has a successor (or perhaps they understand “successor” to refer to any number that is greater). Therefore, we suggest to develop activities focused on overcoming this difficulty.

Acknowledgments

To Conacyt and Cinvestav for financing this research study.

References

Apostol, T. (2006). *Análisis Matemático* (2a ed.). Editorial Reverté, S.A.

Carey, S. (1987). Conceptual change in childhood. Cambridge, MA: MIT Press.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes, In A.E Kelly, R.A. Lesh, & J.Y. Baek (Eds.). *Handbook of design research methods in education. Innovations in Science, Technology, Engineering and Mathematics Learning and Teaching*, (pp. 68-95). Mahwah, NJ: Lawrence Erlbaum Associates.

Duval, R. (2004). *Semiosis y Pensamiento Humano. Registros Semióticos y Aprendizajes Intelectuales*. Universidad del Valle.

McMullen, J., & Van Hoof, J. (2020) The role of rational number density knowledge in mathematical development. *Learning and Instruction*, 65, 101228. https://doi.org/10.1016/j.learninstruc.2019.101228

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Outlines for new teaching strategies. *Learning and Instruction*, *14*, 519-534. https;//doi.org/10.1016/j.learninstruc.2004.06.016

Ni, Y., & Zhou, Y-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40, 27-52. https://doi.org/10.1207/s15326985ep4001_3

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145. https://doi.org/10.2307/749205

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91–104.

Suárez-Rodríguez, M., & Figueras, O. (2020). An approach to density in decimal numbers: a study with pre-service teachers. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proc. 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 803-819). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020-122

Suárez-Rodríguez, M., & Sacristán, A. I. (2021). A hypothetical learning trajectory for the understanding of number density in high school students. In D. Olanoff, K. Johnson & S.M. Spitzer, (Eds). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1220-1237). Philadelphia, PA.

Tirosh, D., Fischbein, E., Graeber, A. O., & Wilson, J. W. (1999). Prospective elementary teachers' conceptions of rational numbers. http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html

Tovar, J. (2011). *Un acercamiento al concepto y completitud de los números reales* (Tesis de maestría). Universidad Nacional de Colombia. https://repositorio.unal.edu.co/bitstream/handle/unal/10182/johnedinsontovargil.2011.pdf?sequence=1&isAllowed=y

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, 14, 453-467. https://doi.org/10.1016/j.learninstruc.2004.06.013

Vamvakoussi, X. y Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, 28(2), 181-209. https://doi.org/10.1080/07370001003676603

Vosniadou, S. (1994). Capturing and modelling the process of conceptual change. *Learning and Instruction*, 4, 45-69.

DIFERENTES FORMAS DE APRENDER LA DENSIDAD NUMÉRICA: UNA TRAYECTORIA HIPOTÉTICA CON ESTUDIANTES DE BACHILLERATO

DIFFERENT WAYS OF LEARNING NUMBER DENSITY: A HYPOTHETICAL TRAJECTORY WITH HIGH SCHOOL STUDENTS

Mayra Suárez-Rodríguez
Cinvestav
mayra.suarez@cinvestav.mx

Ana Isabel Sacristán Rock
Cinvestav
asacrist@cinvestav.mx

Se presenta un estudio donde se diseñó una Trayectoria Hipotética de Aprendizaje (THA) para promover el aprendizaje de la propiedad de densidad numérica en estudiantes de bachillerato. Las actividades de la THA se inspiraron en varios temas de la matemática escolar que se pueden aprovechar para hablar de esta propiedad y se enmarcan en diversos contextos y registros semióticos. Mostramos la secuencia de la THA y algunos resultados de una implementación de esta con cuatro estudiantes de bachillerato. Se analiza el pensamiento de los participantes en términos de sus pensamientos sobre lo discreto y lo denso. Se observa que al final de la THA se acercan a reconocer una infinidad de números racionales o reales intermedios en un intervalo, aunque no logran comprender del todo que solamente los números naturales tienen sucesores.

Palabras clave: Trayectorias de aprendizaje y progresiones, representaciones matemáticas, conceptos y operaciones numéricas, números racionales

Introducción y Preguntas de Investigación

En muchos países es común que estudiantes de bachillerato aprendan sobre representaciones de números en la recta numérica, clases de intervalos, así como la continuidad de los reales, pero no la propiedad de densidad (Vamvakoussi y Vosniadou, 2010; McMullen y Van Hoof, 2020). La propiedad de densidad de un conjunto numérico es que dados cualesquiera dos números de ese conjunto, existe un número entre ellos. Si un conjunto numérico cumple dicha propiedad eso implica que, en ese conjunto: 1) ningún número puede tener un sucesor (inmediato); 2) cualquier intervalo contiene una infinidad de números. Asimismo, un conjunto completamente denso no es discreto, entendiendo discreto, en palabras de Vamvakoussi y Vosniadou (2010), en términos de relaciones de orden.

Algunos estudiantes llegan al nivel de educación superior (universidad), con concepciones deficientes sobre esta propiedad. Entre las dificultades que se observan están las creencias érroneas de que: 1) la existencia de un sucesor en cualquier conjunto de números, más allá de los conjuntos de los naturales o enteros; y 2) hay una cantidad finita de números en un intervalo de números racionales. Por ejemplo, la segunda creencia fue observada con profesores en formación en Israel por Tirosh et al., (1999). Por su parte, Merenluoto y Lehtinen (2004) observaron que 10% de los estudiantes que estudiaban la carrera de Matemáticas en una universidad en Finlandia creía que había una cantidad finita de números racionales en un intervalo. Asimismo, el pensamiento discreto proveniente del pensamiento relacionado con los números naturales es problemático: Vamvakoussi y Vosniadou (2010), en un estudio en Grecia encontraron que la idea de lo discreto era dominante en los estudiantes observados y la representación simbólica de los extremos del intervalo afectaba para decidir no solamente la cantidad de números intermedios en un intervalo, sino también qué tipo de números se encuentran en él (i.e., algunos de esos estudiantes consideraban que solo puede haber decimales en un intervalo cuyos extremos son números decimales, pero no puede haber fracciones).

Con base en lo expuesto anteriormente y las dificultades que tienen los estudiantes para aprender y comprender la propiedad de densidad numérica, en la investigación que aquí presentamos, elaboramos una Trayectoria Hipotética de Aprendizaje (THA), como parte de una investigación basada en diseño (Cobb y Gravemeijer, 2008), con el objetivo de promover el aprendizaje de esa propiedad en estudiantes de bachillerato. Para la THA tomamos diferentes contextos, situaciones y representaciones, inspiradas en McMullen y Van Hoof (2020) quienes indican que hay momentos de la matemática escolar que, con instrucción adicional, pueden proporcionar oportunidades para razonar sobre la densidad numérica.

El presente informe es una extensión del estudio de investigación hecho por Suárez-Rodríguez y Sacristán (2021) donde se plantearon dos preguntas de investigación:

1. ¿Cómo diseñar una secuencia de actividades con una THA para para promover el aprendizaje de la propiedad de densidad en el conjunto de los números reales?
2. ¿Cómo estudiantes de bachillerato comprenden la propiedad de densidad en los números reales usando representaciones semióticas generadas durante la secuencia propuesta?

Marcos Teórico y Metodológico

Pensamiento sobre lo discreto y lo denso

Como señalan Vamvakoussi y Vosniadou (2004, 2010), las propiedades de los números naturales se tienden a transferir a otros conjuntos numéricos y esto afecta las concepciones de los alumnos en relación con la propiedad de densidad. Estas autoras (Vamvakoussi y Vosniadou, 2004) definieron ciertas categorías de pensamiento sobre lo discreto, lo denso y su relación con la cantidad de números en un intervalo, para identificar formas de razonar en los alumnos (la Tabla 1 muestra una reinterpretación de esas categorías explicadas por nosotras).

Los números naturales, a diferencia de los números reales, tienen la propiedad de lo discreto como se establece en los axiomas de Peano cuando se define que entre un número natural y el que sigue no existe otro número (Ni y Zhou, 2005). Más aún, todos los números naturales tienen sucesor (Ni y Zhou, 2005) –es un conjunto bien ordenado.

Así, a partir del estudio de los números naturales, se va formando un pensamiento vinculado con lo discreto. Vamvakoussi y Vosniadou (2004) usaron el término *ingenuo,* acuñado por Carey (1987) al referirse a "ideas ingenuas", para definir dos de sus categorías de pensamiento (ver Tabla 1): La categoría *pensamiento ingenuo sobre lo discreto* se refiere a una "idea ingenua" de que los números racionales son consecutivos. Asimismo, *el pensamiento ingenuo sobre lo denso* es cuando se considera que sí existe una infinidad de números en un intervalo, pero no se logra justificar por qué la existencia de esta infinidad. En contraposición a esos pensamientos ingenuos, están las categorías de pensamiento avanzado: La categoría *pensamiento avanzado sobre lo discreto* es cuando se cree que hay una cantidad finita de números en un intervalo; mientras que en la categoría *pensamiento sofísticado sobre lo denso*, se comprende la definición formal de la densidad.

Tabla 1: Categorías de pensamiento sobre lo discreto y lo denso

Pensamiento ingenuo sobre lo discreto	Se considera que no hay otro número entre dos números racionales *consecutivos falsos* (expresión acuñada por Vamvakoussi y Vosniadou, 2004, para decir que existe un sucesor en los números racionales.)
Pensamiento avanzado sobre lo discreto	Se cree que hay un número finito de números intermedios entre dos números racionales consecutivos falsos.
Pensamiento entre lo discreto y lo denso	En algunas situaciones se piensa que entre dos números racionales hay una cantidad infinita de números, y en otros, que hay una cantidad finita.

Pensamiento ingenuo sobre lo denso	Se contempla que hay una infinidad de números en un intervalo, pero no se justifica la situación usando la propiedad de densidad. La representación simbólica de los extremos de un intervalo influye en la forma de pensar; se cree que solo puede haber una infinidad de decimales entre decimales, pero no fracciones; de igual modo sucede con fracciones entre fracciones.
Pensamiento sofísticado sobre lo denso	Hay una comprensión bastante sofisticada de la propiedad de densidad, es decir, se pone de manifiesto que se entiende que entre dos números racionales hay una infinidad de números, independientemente de su representación simbólica, y se justifica con la propiedad de la densidad.

Descripción de una Trayectoria Hipotética de Aprendizaje (THA)

Simon y Tzur (2004) definen una THA como una serie o secuencia de actividades de enseñanza que comprenden unas conjeturas o hipótesis de cómo el estudiante puede aprender ciertos conceptos matemáticos. Así, una THA es una predicción de una ruta de aprendizaje que incluye intervenciones didácticas por medio de la secuencia de actividades (Simon, 1995). Así, en una THA es fundamental la elaboración y planteamiento de hipótesis de cómo apoyar al estudiante en la construcción de conocimientos nuevos (Simon, 1995). De hecho, Simon (1995) establece tres elementos fundamentales para el diseño de una THA: 1) *los objetivos*, entendidos como los propósitos que se quiere lograr para que el estudiante construya nuevos conocimientos, 2) *la ruta de aprendizaje*, que se conforma de actividades instruccionales siguiendo una secuencia, y 3) *planteamiento de hipótesis*, en el que el investigador elabora conjeturas para apoyar el proceso de aprendizaje del estudiante. Durante las actividades, el investigador (o profesor) observa cómo va aprendiendo el estudiante en los escenarios propuestos en la THA para determinar si el diseño es el adecuado y cambios a realizar (Simon, 1995).

Registros semióticos de representación

Una manera que consideramos importante para apoyar el aprendizaje de la propiedad de densidad es el uso de varios registros de representación. La matemática trata con diferentes registros de representación –ya sea el geométrico, el algebraico (expresiones simbólicas), el aritmético, incluso el lenguaje coloquial (expresiones lingüísticas) – y cada uno de estos registros constituye una forma semiótica diferente (Duval, 1995/2004); es decir, comunica el significado de manera diferente. Las representaciones semióticas son fundamentales para el desarrollo de la actividad matemática misma que dependen de un sistema de escritura (decimal, fraccionaria, binaria, etc.) (Duval, 1995/2004). Esto además es particularmente relevante porque cuando se trabaja con números reales (como es nuestro caso), es necesario tener en cuenta que el número real tiene diversas representaciones (Apostol, 1976/2006).

Metodología

Diseño y elaboración de las actividades de la THA

Como ya mencionamos, en nuestra investigación trazamos una THA de acuerdo con los lineamientos de una investigación basada en diseño (Cobb y Gravemeijer, 2008), un enfoque metodológico en el que un investigador realiza un análisis minucioso de cómo probar y mejorar las conjeturas que se bosquejan. Cobb y Gravemeijer (2008) recomiendan el diseño y elaboración de más de un ciclo de actividades con el fin de refinar o modificar las conjeturas planteadas durante primer ciclo. Para esta investigación se contemplaron dos ciclos de actividades, y en el presente informe documentaremos algunos resultados del primer ciclo por cuatro estudiantes de Colombia, de entre 15 y 17 años de edad: Angie, Paola y Néstor (pseudónimos), quienes cursaban en su momento la educación media (bachillerato); y Violeta (pseudónimo), quien cursaba el último año de educación básica (secundaria). Estos estudiantes resolvieron algunas tareas propuestas de la THA de manera presencial, mientras que otras fueron desarrolladas de manera virtual, debido a la pandemia causada por el COVID-19.

La elaboración e implementación del primer ciclo de las actividades de la THA se conforma de tres fases. La primera fase corresponde a la aplicación de un cuestionario como pretest, la segunda corresponde a las actividades de la THA, y en la última se vuelve aplicar el cuestionario como postest. A continuación describimos las fases de la THA, luego presentamos resultados de las dos primeras fases y damos un ejemplo de lo ocurrido en la última fase.

Primera y tercera fase. *Pretest y postest.* Se utilizó el mismo cuestionario para el pre- y postest (con duración aproximada de 30 min), para observar la posible evolución del pensamiento de los alumnos. El cuestionario se conforma de cuatro ítems de preguntas vinculadas con la cantidad de números intermedios en un intervalo –preguntas inspiradas en las hechas por Suárez-Rodríguez y Figueras (2020), así como en otras planteadas por Vamvakoussi y Vosniadou (2004). La finalidad del cuestionario es conocer qué tanto saben los participantes respecto a la propiedad de densidad numérica, clasificando las respuestas de acuerdo a las categorías (ver Tabla 1) de Vamvakoussi y Vosniadou (2004). Las preguntas incluidas en el cuestionario son: *¿Cuántos números hay entre 0 y 1?*, *¿Puedes encontrar números decimales y/o fracciones entre 0.49 y 1/2?*, *¿Cuántos números hay entre 0.899 y 0.90?* En el cuarto ítem se presentó la siguiente situación:

> Pedro quiere dibujar una línea donde pueda ubicar números decimales. Inicia con el número 0.1. Después coloca el 0.2. En ese momento, Laura lo interrumpe y dice que el número que "sigue" de 0.1 es 0.11. ¿Quién tiene la razón, Pedro o Laura?, o ¿ninguno de los dos?, o ¿ambos?

Segunda fase. *La implementación de las actividades de la THA.* La THA se diseñó para llevarse a cabo en cuatro sesiones de actividades (ver Tabla 2; parecida a la que aparece en Suárez-Rodríguez y Sacristán, 2021, p. 1223), cada una con una hipótesis de aprendizaje correspondiente. Las hipótesis se definieron teniendo en cuenta varios temas de la matemática escolar que supusimos pueden guiar al estudiante a un aprendizaje y comprensión de la propiedad de densidad. No es necesario que el estudiante siga la secuencia de las sesiones, aunque se hace un primer acercamiento a la densidad con la de los números decimales, y se finaliza con los números irracionales. Cada sesión tuvo una duración de alrededor de 1h30'.

Tabla 2: Sesiones de la THA con sus hipótesis correspondientes

Sesión	*Hipótesis*
Sesión 1. Primeros acercamientos a la propiedad de densidad.	A partir de dos situaciones: una relacionada con la cotidianidad y otra vinculada con un escenario hipotético, se piensa que el estudiante puede tener sus primeros acercamientos a la propiedad de densidad.
Sesión 2. Acercamiento a la propiedad de densidad a través de la semejanza de triángulos.	Se contempla que usando semejanza de triángulos los estudiantes puedan aprender sobre la propiedad de densidad de los números racionales.
Sesión 3. Aproximación a la propiedad de densidad a partir de progresiones aritméticas y progresiones geométricas.	Es posible que hallando medios aritméticos y geométricos en un intervalo el estudiante comprenda la propiedad de densidad de los números racionales en el conjunto de los reales.
Sesión 4. Aproximación a la propiedad de densidad por medio de la propiedad de continuidad.	Se cree que usando la propiedad de continuidad los estudiantes comprenden la propiedad de densidad de los números irracionales en el conjunto de los reales.

Mediante esta THA se busca que los estudiantes logren una conciencia metaconceptual, al percatarse que sus concepciones no son incuestionables y que restringen la manera en que se adquiere un nuevo conocimiento (Vosniadou, 1994). Al final de las actividades de la THA se espera que el estudiante pueda distinguir entre un conjunto discreto y uno denso.

Resultados

Se usaron las categorías de Vamvakoussi y Vosniadou (2004) (Tabla 1) para el análisis de las respuestas del cuestionario, así como también de las respuestas en las actividades de la THA.

Resultados del pretest (primera fase)

La Figura 1 muestra la respuesta de Angie a la pregunta sobre la cantidad de números entre 0.899 y 0.90: *"solo [h]ay uno, porque está en la mitad de los dos y porque da paso al siguiente número".* Se deduce un *pensamiento ingenuo sobre lo discreto* en Angie derivado de sus conocimientos de los números naturales que son discretos, al decir "sólo hay uno" y que para ella un número decimal tiene sucesor ("da paso al siguiente número").

solo ay uno, porque esta en la mitad de los dos
y porque da paso al siguiente numero

Figura 1: Ejemplo de pensamiento ingenuo sobre lo discreto (Angie en pretest)

En la Figura 2 observamos que las respuestas de Violeta, Angie y Néstor a la pregunta de si existe un número que le sigue a 0.1, coinciden en que 0.2 es el número que le sigue a 0.1. En la respuesta de Paola: "*Sí existe un número que le sigue a 0.1[:] 0.11, 0.111, 0.111…,*" observamos que aunque, en su lenguaje coloquial, afirma la existencia de un sucesor, y nombra varios: "0.11, 0.111, 0.111[1]…", para ella, el hecho de añadir cada vez el número 1 en la parte decimal del número, le indica que existen más "sucesores" para 0.1. Consideramos que esos cuatro estudiantes exhibían allí un *pensamiento ingenuo sobre lo discreto* ya que, al parecer, creen en la existencia de un sucesor (o varios en el caso de Paola) en los decimales.

4. b. ¿Existe un número que le sigue a 0.1? Si tu respuesta es afirmativa, escribe el número que crees que sigue a 0.1. Pero si tu respuesta no es afirmativa explica por qué.

Respuesta de Angie

0.1 le sigue 0.2

Respuesta de Violeta

Si existe es el 0.2 y asi sucesivamente con los demas

Respuesta de Néstor

Rta. Para mi creo que es 0.2

Respuesta de Paola

Si, existe un número que le sigue a 0,1.
0,11, 0,111, 0,111...

Figura 2: Otros ejemplos de pensamiento ingenuo sobre lo discreto (Pretest)

Resultados de las actividades de la THA (segunda fase)

Se narran algunas actuaciones de los cuatro estudiantes durante el desarrollo de la THA.

Sesión 1. En la primera sesión los estudiantes resolvieron dos actividades de aprendizaje para tener un primer acercamiento a la densidad. En la segunda actividad se planteaba lo siguiente:

> Imagina que puedes observar a una rana que empieza a saltar desde el punto A y trata de llegar al punto B de la siguiente forma: 1er paso: La rana salta a la mitad. 2do paso: Luego salta a la mitad de lo que le quedó. 3er paso: Nuevamente, salta la mitad de lo que quedó, y así sucesivamente.
> 1. Después de que hayas imaginado la situación, ahora descríbela, o explícala…
> 2. ¿Cuántas veces saltará la rana entre los puntos A y B? Explica tu respuesta.

Al resolver esta actividad, consideramos que Angie, Violeta y Paola (ver Figura 3) se acercan a un *pensamiento avanzado sobre lo discreto*, cuando indican que la rana ha saltado una cantidad finita de veces, y usan como representación un segmento de puntos en un registro gráfico. Por otro lado, Néstor dice "*La Rana va a saltar sucesivamente hasta llega al punto B; […] va a llegar a su destino, saltando cada [vez] mitades*"; se podría pensar que concibe una infinidad potencial de saltos (también en la actividad previa había dicho que había "infinitos números" entre dos decimales); podría decirse que allí exhibe un *pensamiento ingenuo sobre lo denso*.

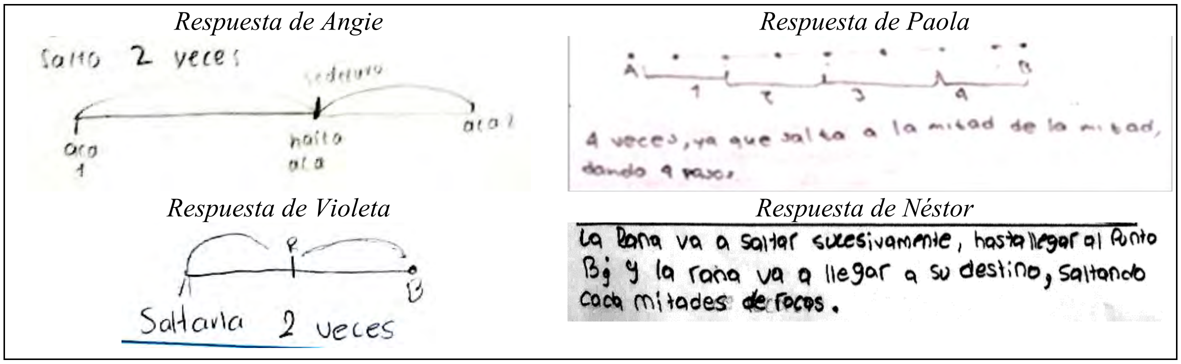


Figura 3: Respuestas de los cuatro estudiantes en la actividad "Saltos de Rana" (Sesión 1)

Sesión 2. La actividad de esta sesión se basa en tareas hechas por Tovar (2011). Se plantea la hipótesis de que se puede aprender sobre la propiedad de densidad de fracciones y decimales al trabajar con construcciones de triángulos semejantes sobre la recta numérica, tanto en GeoGebra (registro geométrico) como con lápiz y papel (registro aritmético o algebraico). Se pide ubicar los puntos A = 0 y B = 1 en la recta y construir un triángulo ABD. Luego se pide construir otro triángulo semejante AXC, con C punto medio de $\overline{AD}$ (Figura 4). Se pide hallar el valor numérico correspondiente a X. En la Figura 5a se observa que Angie halla X = 0.5 aplicando proporciones. Después se pide repetir el proceso para un tercer triángulo semejante AGF, con F punto medio de $\overline{AC}$ (Figura 4), y hallar el valor numérico correspondiente a G. En la Figura 5b se observa que Angie halla G = 0.25 aplicando nuevamente proporciones. Así Angie halla los números 0.25 y 0.5 entre 0 y 1, empleando representaciones semióticas en un registro algebraico como las escrituras fraccionaria y decimal.

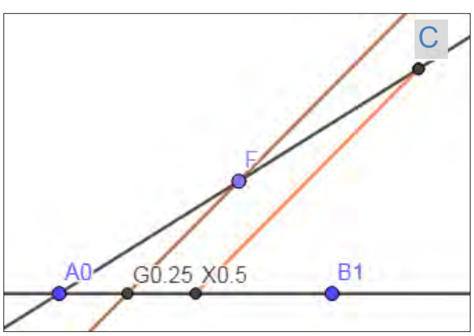
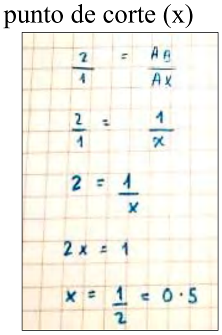


Figura 4: Gráfica de Angie en GeoGebra, localizando puntos entre A=0 y B=1 mediante semejanza de triángulos (Sesión 2)

a. Usando semejanza de triángulos halla el punto de corte (x)



b. Realiza el mismo procedimiento para el intervalo entre 0 y 1/2. ¿Qué número obtuviste entre 0 y 1/2?

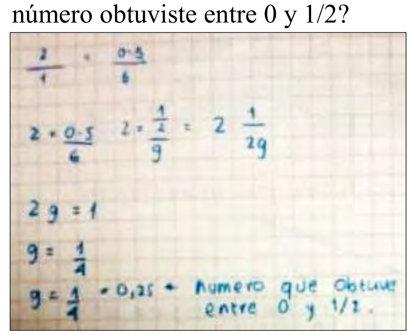


Figura 5: Respuestas de Angie (S2) para:
a. encontrar el valor del punto x entre 0 y 1
b. encontrar el número entre 0 y 1/2

Posteriormente, los estudiantes hallan otro punto entre 0 y 1 como es el 3/4 (0.75), y se les pregunta: *¿Qué sucede si continúas realizando el mismo procedimiento entre 3/4 y 1?,¿y así, en forma sucesiva?, ¿Cuántos números racionales se obtendrían?* La Figura 6 muestra las respuestas de los cuatro estudiantes a estas preguntas. Se puede pensar que Angie y Néstor exhiben un *pensamiento avanzado sobre lo discreto* porque hacen referencia a una cantidad de números finita (dicen que hay "varios" números racionales entre 3/4 y 1, y que se pueden obtener más). Por otro lado, la respuesta de Violeta ("*siguen aumentando la cantidad de números; se obtendrán muchos números racionales*") podría tal vez indicar un infinito potencial, lo que correspondería a un *pensamiento ingenuo sobre lo denso*. En cuanto a Paola, ella dice: "*Se seguiría obteniendo la mitad entre 3/4 y 1 y así sucesivamente*", lo que puede indicar que considera una infinidad potencial de números racionales, lo que correspondería a un *pensamiento sofisticado sobre lo denso,* ya que lo justifica con el proceso de sacar mitades en forma sucesiva.

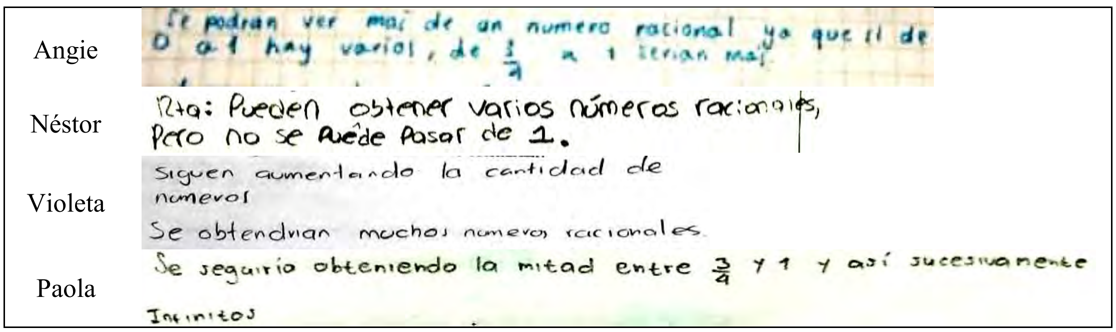


Figura 6: Respuestas de los cuatro estudiantes en la actividad sobre proporciones (S2)

Sesión 3. En esta sesión se llevaron a cabo tres actividades relacionadas con progresiones aritméticas y progresiones geométricas para que el estudiante pudiera ver otra forma de encontrar números en un intervalo (para aprender sobre la densidad de los racionales en los reales). Inicialmente, los estudiantes debían hallar cinco medios aritméticos entre 4 y 22 utilizando el término n-ésimo de una sucesión: $u = a + (n - 1)d$, donde a es el primer término, n el número de términos y d la diferencia entre un término y otro. Después se les preguntó cuáles serían los nuevos medios aritméticos si la diferencia, d, se redujera a la mitad (ver Figura 7, Violeta).

Si d se redujera a la mitad, es decir, d=1.5, o d=3/2, ¿cuáles serían los nuevos medios aritméticos entre 4 y 22?

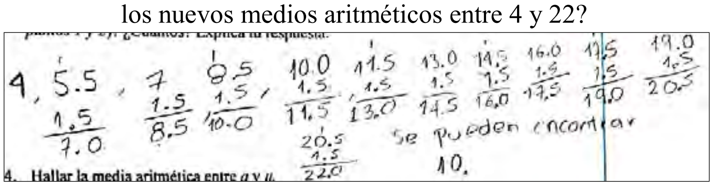


Figura 7: Respuesta de Violeta (Sesión 3)

¿Puedes hallar más números entre 4 y 22 (diferentes a los números hallados en las anteriores preguntas)? ¿Cuántos? Explica tu respuesta

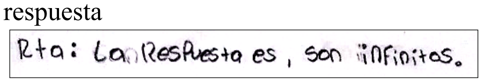


Figura 8: Respuesta de Néstor (S3)

La Figura 7 muestra cómo Violeta halla los nuevos medios aritméticos –5.5, 7, 8.5, 10, 11.5, 13, 14.5, 16, 17.5, 19, 20.5–, y responde a la pregunta de si se pueden hallar más números entre 4 y 22 (diferentes a los números hallados en las preguntas anteriores de la actividad), cuántos y por qué. Ella responde que *"se pueden encontrar 10"* (aunque fueron 11 los que halló), es decir, una cantidad finita de números entre 4 y 22, por lo que exhibe un *pensamiento avanzado sobre lo discreto*. Violeta usa la escritura decimal como representación y realiza adiciones en un registro aritmético. Néstor (ver Figura 8) exhibe un *pensamiento ingenuo sobre lo denso,* cuando afirma que se puede hallar una infinidad de números.

Sesión 4. En esta sesión se utilizó otra actividad basada en ejercicios propuestos por Tovar (2011). La hipótesis es que el estudiante puede aprender sobre la densidad de los irracionales en los reales, a partir de la continuidad de la recta con relación a la no correspondencia entre los números racionales y los puntos de la recta. Se pedía construir un cuadrado de lado 1, en el plano cartesiano, con un vértice en el origen, y rotar la diagonal del cuadrado para quedar sobre la recta del eje horizontal (ver el segmento $\overline{DF}$ en la Figura 9 – ejemplo de Angie). Los estudiantes observaron que el "extremo derecho" del segmento de la diagonal no coincidía con ningún número racional de la recta cuando realizaban varios zooms en GeoGebra. Luego ellos escribieron varios intervalos que encerraran a ese punto: la Figura 10a muestran dos intervalos dados por Néstor que contienen a ese punto. Paola (ver Figura 10b) dio cuatro intervalos cada vez más pequeños. Con esto los estudiantes observaron que entre dos números reales se puede hallar un número irracional (en este caso $\sqrt{2}$). Aquí no se hizo un análisis de los pensamientos.

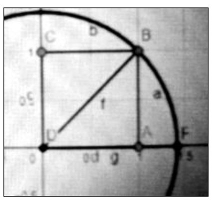


Figura 9: Gráfica de Angie en GeoGebra (S4)

a. Respuesta de Néstor

b. Respuesta de Paola

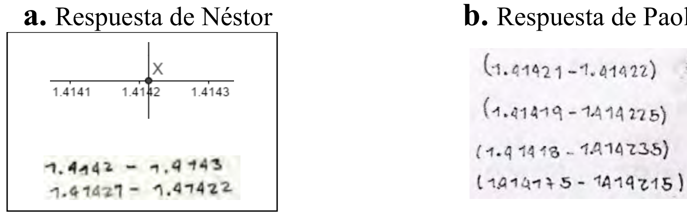


Figura 10: Intervalos escritos por Néstor y Paola que encierran al punto cuyo valor corresponde a la longitud de la diagonal del cuadrado (Sesión 4)

Resultados del postest (tercera fase)

En sus respuestas al postest, los estudiantes mostraron evidencia de haber cambiado sus formas de pensamiento en sus respuestas a las preguntas relacionadas con la cantidad de números en un intervalo con respecto a las dadas en el pretest (primera fase). Por ejemplo, para la pregunta *¿Puedes encontrar números decimales o fracciones entre 0.49 y 1/2? ¿cuántos números?* Néstor cambia de un *pensamiento ingenuo sobre lo discreto* en el pretest (donde él dice: "*1/2 = 0.5 sigue a 0.49. No hay* [*números*]") a un *pensamiento ingenuo sobre lo denso* (cuando en el postest cuando dice: "*hay infinitos números*").

Conclusiones

Las actividades propuestas en la THA tenían como objetivo que los estudiantes aprendieran sobre la propiedad de densidad numérica al resolver tareas en diversos contextos y registros semióticos. Temas como construcciones de triángulos semejantes sobre la recta numérica, progresiones aritméticas y geométricas, propiedad de continuidad, facilitan el encontrar diferentes formas de hallar números en un intervalo. Asimismo, los estudiantes tuvieron la oportunidad de usar varios registros semióticos de representación, como el algebraico y el geométrico, empleando las escrituras fraccionaria y decimal; incluso el lenguaje coloquial para expresar sus ideas durante la puesta en marcha de la THA. En algunas situaciones los estudiantes lograron comprender que existe una infinitud de números en un intervalo real. Sin embargo, en otras, no pudieron superar la creencia de que cualquier número tiene un sucesor (o tal vez entienden por "sucesor" a cualquier número mayor). Por ello, se sugiere la realización de actividades enfocadas a superar de esta dificultad.

Agradecimientos

A Conacyt y a Cinvestav por el financiamiento de este estudio de investigación.

Referencias

Apostol, T. (2006). *Análisis Matemático* (2a ed.). Editorial Reverté, S.A.

Carey, S. (1987). Conceptual change in childhood. Cambridge, MA: MIT Press.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes, In A.E Kelly, R.A. Lesh, & J.Y. Baek (Eds.). *Handbook of design research methods in education. Innovations in Science, Technology, Engineering and Mathematics Learning and Teaching*, (pp. 68-95). Mahwah, NJ: Lawrence Erlbaum Associates.

Duval, R. (2004). *Semiosis y Pensamiento Humano. Registros Semióticos y Aprendizajes Intelectuales*. Universidad del Valle.

McMullen, J., & Van Hoof, J. (2020) The role of rational number density knowledge in mathematical development. *Learning and Instruction*, 65, 101228. https://doi.org/10.1016/j.learninstruc.2019.101228

Merenluoto, K., & Lehtinen, E. (2004). Number concept and conceptual change: Outlines for new teaching strategies. *Learning and Instruction*, *14*, 519-534. https;//doi.org/10.1016/j.learninstruc.2004.06.016

Ni, Y., & Zhou, Y-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40, 27-52. https://doi.org/10.1207/s15326985ep4001_3

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114–145. https://doi.org/10.2307/749205

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning*, 6(2), 91–104.

Suárez-Rodríguez, M., & Figueras, O. (2020). An approach to density in decimal numbers: a study with pre-service teachers. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). *Mathematics Education Across Cultures: Proc. 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico* (pp. 803-819). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020-122

Suárez-Rodríguez, M., & Sacristán, A. I. (2021). A hypothetical learning trajectory for the understanding of number density in high school students. In D. Olanoff, K. Johnson & S.M. Spitzer, (Eds). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1220-1237). Philadelphia, PA.

Tirosh, D., Fischbein, E., Graeber, A. O., & Wilson, J. W. (1999). Prospective elementary teachers' conceptions of rational numbers. http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html

Tovar, J. (2011). *Un acercamiento al concepto y completitud de los números reales* (Tesis de maestría). Universidad Nacional de Colombia. https://repositorio.unal.edu.co/bitstream/handle/unal/10182/johnedinsontovargil.2011.pdf?sequence=1&isAllowed=y

Vamvakoussi, X., & Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: A conceptual change approach. *Learning and Instruction*, 14, 453-467. https://doi.org/10.1016/j.learninstruc.2004.06.013

Vamvakoussi, X. y Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction*, 28(2), 181-209. https://doi.org/10.1080/07370001003676603

Vosniadou, S. (1994). Capturing and modelling the process of conceptual change. *Learning and Instruction*, 4, 45-69.

CONCEPTUAL REORGANIZATION, FROM COUNT-UP-TO TO BREAK-APART-MAKE-TEN: A CASE OF A 6TH GRADER STRUGGLING IN MATHEMATICS

Ron Tzur
University of Colorado Denver
ron.tzur@ucdenver.edu

Cody Harrington
University of Colorado Denver
cody.harrington@ucdenver.edu

Dennis DeBay
University of Colorado Denver
dennis.debay@ucdenver.edu

Alan Davis
University of Colorado Denver
alan.davis@ucdenver.edu

Through a constructivist teaching experiment, we studied how a 6th-grade student (Adam, pseudonym) struggling in mathematics may reorganize his available additive scheme (count-up-to) into a more advanced scheme involving the decomposition of composite units (break-apart-make-ten, or BAMT). First, we posed a task that led us to infer Adam was yet to construct the BAMT scheme at the anticipatory stage (solving a task without prompting). We thus turned to promote reorganization of his anticipatory, count-up-to scheme used to solve missing-addend tasks. Through reflection on the relationships between his goal, count-up-to actions, and effect of those actions, Adam independently brought forth what he called "easy number-pairs" (i.e., 10+X = X-teen). This seemed to afford his reorganization of count-up-to into the BAMT scheme. We discuss implications of this reorganization for theory building and practice.

Keywords: conceptual reorganization, concept of number, struggling student

We address the research question: How may a child, who constructed count-on for adding or count-up-to for solving a missing addend task (see Fuson, 1982, 1986) at the anticipatory stage (see below), advance to constructing the break-apart-make-ten (BAMT) strategy? The former two strategies indicate a concept of number that does not yet include disembedding units, whereas the latter does (Ulrich, 2015; see next section). Addressing this question is important in two ways. First, in additive reasoning, count-on (e.g., 8 + 5 = ?) and count-up-to (e.g., 8 + ? = 13) involve (a) recognition of the first addend as a unit of its own right and (b) *operating on 1s* of the second addend. In count-on, the child may use a sequential activity of concurrently uttering a number while putting up a finger for it (e.g., 8; 9-10-11-12-13). For count-up-to, the child may use a sequential activity of concurrently uttering numbers with fingers and stopping at the given total, then looking at their finger pattern to determine the answer. Critically, decomposing units by disembedding a part without "losing sight of a given total" is needed for subtraction to become the opposite of addition. For example, 13 is recognized as a unit made of two sub-units, 8 and 5, that could compose 13 or one unit be disembedded from 13 to figure out the other unit as the difference. Such a concept serves as a foundation for BAMT (e.g., 10 = 8 + 2 and 5 = 3 + 2, so "I can give 2 (from 5) to 8 to make a ten, then add the remaining 3 to 10).

Second, recent studies relating children's spontaneous additive strategies to multiplicative reasoning demonstrated that BAMT provides a stronger conceptual basis for the latter (Tzur et al., 2021; Zwanch & Wilkins, 2020). We further explain this linkage in the next section. Here, we only note that, to us, BAMT constitutes a direly needed 'conceptual key' to open the 'gate' for advancing any student, let alone those struggling in mathematics, to multiplicative reasoning. A study of how a child may construct BAMT as reorganization in their less advanced additive schemes can contribute to explaining, as well as promoting, this advance.

Conceptual Framework

Our study is informed by general and content specific notions of a constructivist theory. The former derive from Glasersfeld's (1995) scheme theory, which portrays the cognitive 'apparatus' a child uses in assimilation of mathematical problem situations (tasks). Drawing on the 3-part definition of a scheme (goal –> activity –> result/effect), Tzur & Simon (2004) postulated two stages in which a new scheme is being constructed. The first, *participatory* stage, is marked by the learner needing prompts to regenerate and use abstractions they have constructed. Without prompts, the learner brings forth and uses schemes available to them that are less advanced than the evolving (prompt-dependent) scheme. Available schemes are those at the *anticipatory* stage, which are marked by a learner's spontaneous use of those schemes to make sense of and solve given tasks (Tzur et al., 2021). To articulate how a learner reorganizes available schemes into more advanced ones we draw on Simon's (1995) notion of *hypothetical learning trajectory* (HLT). That is, we link goals specified for learners' construction with a hypothesized change process they may go through and with tasks that may foster that change process.

The content specific constructs revolve around Steffe and Cobb's (1988) core notion of *number as a composite unit*. They postulated a progression of schemes in children's numerical development. In the first, *Initial Number Sequence* (INS), a child anticipates a number to be the result of counting activities. The INS affords count-on because a child takes for granted both the first addend (e.g., uttering "8") and the second addend as shown in keeping track and anticipating where to stop the count (e.g., 9-10-11-12-13). However, a child reasoning with the INS is limited to operating on 1s and is yet to consider the nested nature of numbers – a scheme that evolves with the *Tacitly Nested Number Sequence* (TNS). Here, in activities that involve counting of 1s, a child can implicitly think of a given number (e.g., 8) as nested (embedded) within 9, or within 10, or within 13, etc. Whereas the TNS affords using count-up-to for solving missing addend tasks, a child reasoning with the TNS is yet to disembed composite units from a given number (e.g., 10) – a more advanced scheme known as the *Explicitly Nested Number Sequence* (ENS). Here, in situations that involve operating on 1s and composite units, a child can intentionally decompose given numbers into smaller numbers without losing sight of the given number. Thus, the ENS affords a child's use of BAMT, which requires intentionally decomposing the second addend in a way suitable for composing 10 with the first addend, then going back to the remaining part of the second addend (as explained in the previous section). Due to the new ability for composing/decomposing units at will, the child can also coordinate count-on and count-up-to with the use of "easy number pairs" in the second decade (i.e., $10 + 3 = 13$).

Methodology: A Constructivist Teaching Experiment

This study was part of a larger research and professional development project, focusing on teaching mathematics conceptually (grades K-8) and funded by a small school district in the USA southwest region (see Acknowledgment). We conducted a teaching experiment (Cobb & Steffe, 1983) with seven of the nine, struggling grade-6 students at the school who provided parent consent and a student assent. A teaching experiment is a qualitative methodology designed to build models of how learners construct (reorganize) schemes.

Participant and Context

Two main reasons led us to focus on Adam (pseudonym, age 11, grade 6, identified by teachers as struggling in mathematics), who worked with a partner (Gary). First, at the study start, Adam demonstrated spontaneous use of count-on and count-up-to (INS and TNS, respectively) considered a conceptual basis for constructing the BAMT strategy and thus advance to the ENS scheme. Second, despite our repeated efforts to promote BAMT during the first four episodes, using a task and manipulatives (see Double Decker Bus below) designed and successfully used in prior research, Adam was yet to construct BAMT. As our data show, BAMT was not yet available to Adam (or Gary) at the start of the episode reported in this study.

Accordingly, Adam can serve as a case for articulating an HLT from his anticipatory schemes to BAMT (and ENS). Consistent with literature on case studies (Creswell, 2013), we stress that the case is not Adam but rather the process of change we could infer from our work with him. That is, we designed our study to provide an explanatory account of the conceptual change process – not to demonstrate the extent to which it characterizes (many) other children's learning. We thus contend that the HLT explained here is likely to apply to other children who have an anticipatory count-up-strategy and spontaneously use "easy number pairs" as did Adam.

Data Collection

In a teaching experiment researchers serve as teachers of students whose schemes they intend to model. The lead researchers (denoted R1 and R2) conducted weekly teaching episodes with Adam and Gary, 20-40 minutes each, because they seemed conceptually near to one another. Due to the COVID-19 pandemic, we conducted the video recorded teaching episodes in a hybrid mode. R2 worked with the students on site. R1 participated virtually (Zoom) and created a back-up recording on that software. Our work required that we all wear masks, which limited the ability to observe the child's lip movements when operating silently. The presence of R2 allowed her to hear nearly everything the students whispered. To ascertain validity of our data, we thus constantly asked the students to first solve tasks as they wanted, then – using what we could observe – either repeat their work out loud or tell if a researcher's account of it is accurate.

In teaching Adam and Gary, we used a game called the *Double-Decker Bus* (DDB). The first author designed DDB to promote an advance to BAMT in children inferred to have an anticipatory stage of count-on or count-up-to. The students are introduced to the game with a picture of a double-decker bus and a rule made to constrain their operations on units so 10 becomes a special anchor. Each deck on the DDB has 10 seats; the rule is that all seats on the lower-deck must be filled before passengers can move to the upper-deck. In each round of the game, a task is posed using the following story line. The bus leaves main station empty, then a few passengers get on it at the first stop (e.g., 9) and at the second stop (e.g., 4). The child's goal is to figure out the total number of passengers on the DDB after the second stop.

To support the child's reasoning and reflection on their goal-directed activities, a two-row rekenrek, with 10 beads each (grouped by color as 5 + 5), is provided for them to use as they deem appropriate. In our example, a typical solution both Adam and Gary initially used was to move 5 + 4 beads on the lower-deck for 9 passengers, then count-on to add 1 more bead to that deck and up to the given four (hence, 3) to the upper deck. Then, they glean the answer from the manipulative as 10 + 3 = 13. The DDB can foster reflection on the decomposition of 10 into 9 + 1 as a step in the child's activity to find the total. For example, the researcher may ask students: "I asked you to add 4, so why are there only 3 on the upper-deck?" We note that numbers used in those tasks typically begin with 9 + x. Because 10 is the number after 9 (Baroody, 1995), a child is hypothesized to focus on decomposing just one unit of 1. Then, game variations proceed to 8 + x and to 7 + x. Children can reflect on their operation across different tasks (e.g., 9 + 3, 9 + 6, 9 + 4) to abstract the intended, invariant anticipation known as BAMT.

Data Analysis

To analyze our video records and field notes from each episode, we used three iterations. First, after each episode, we held an ongoing analysis (debrief) to discuss major events. These debriefs led to planning next episode tasks. Second, team members created and read transcripts of the episode on which we focus in this paper and the episode just prior to it, highlighting segments with critical events (Powell et al., 2003), such as changes in a student's strategy, or beneficial teaching moves. This iteration included hypotheses we raised about why Adam behaved in the ways he did. Third, we discussed (while re-observing) the highlighted segments to identify compelling evidence for our inferences and conceptual claims. We organized all segments in a story line (next section) that conveys the HLT from count-up-to into BAMT.

Results

We begin with data analysis indicating that, at the start of the episode in which Adam reorganized his concept of number, he was still using an anticipatory scheme involving count-up-to, that is, a concept of number characteristic of TNS students. Next, we shift to data analysis indicating his advance from that early numerical way of operating to the more sophisticated way of BAMT, characteristic of ENS students.

Adam's Anticipatory Count-Up-To Scheme (TNS)

During the first four episodes with Adam and Gary, they both appeared to use the count-on strategy spontaneously and independently. For example, when asked how many total candies they would have if adding 8 and 5, they both used their fingers to keep track of the second addend items (8; 9-10-11-12-13). During the episode prior to the one focused on in this paper, they also solved missing addend tasks using the count-up-to strategy (e.g., to solve how many more cubes are needed to create a tower of 12 if you already have 9 cubes, they both counted on their fingers: 9; 10-11-12 and then looked at the fingers they raised and respond, "3").

During those four episodes, we engaged Adam and Gary in solving Double Decker Bus (DDB) tasks. Using the manipulatives in activity, they showed some initial progress toward BAMT. For example, to add 9+4 they first moved one bead on the lower deck to complete the required number of 10 passengers. Then, they moved 3 more beads to complete the task and responded: "13" (pointing to 10+3 as the two "adjusted" addends). Furthermore, in reflection on their activity, they explained why the upper deck has only 3 passengers whereas the problem asked about 4 passengers added to 9 and responded: "Because you have to put 1 passenger here [lower deck] and 4 minus 1 is 3." In fact, at the fourth episode they also began solving tasks asking about 8 + 5 passengers (e.g., "we put 2 more on the bottom and 3 go here [upper deck]"). However, critically, when we presented a task at the start of each episode without any hints (e.g., How many in all are 9 cubes and 4 more cubes?), they reverted to spontaneously and independently using count-on to solve it.

After the 1.5-month winter break, we again presented a missing addend task for them to solve – not in the DDB context but rather in a context (towers and cubes) we never used with them before: "Pretend you made a tower of 9 cubes. Write 9 on your paper so you remember ... If I wanted to make the tower taller and have 16 cubes, how many more cubes would each of you need?" Excerpt 1 shows how Adam solved this task (R1 and R2 stand for the researchers).

Excerpt 1: Anticipatory count-up-to

R1: (After Gary explained his answer of 7) Adam, how did you get it?

Adam: I counted on my fingers; because 9; [then] 10 (puts up thumb on left hand), 11 (puts up index finger), 12 (puts up middle finger), 13 (puts up ring finger), 14 (puts up pinky), 15 (puts up thumb on right hand), 16 (puts up index finger; shows a full hand plus two more fingers). That's 7, so I know there's 7 [cubes needed].

Excerpt 1 indicates that, at the episode start, Adam has independently and spontaneously used the count-up-to strategy. As Tzur et al. (2021) explained, a spontaneous strategy for a task involving no hints serves as an indicator of a child's anticipatory scheme and, likely, the most advanced strategy (and concept) the child has available at the time. Because count-up-to involves operating on 1s while reasoning numerically about the missing addend, but not yet disembedding a composite unit, we attributed to them the TNS stage. We thus turned to further attempts of fostering their use of BAMT. Excerpt 2 presents a segment of those attempts. We emphasize that Adam's use of count-up-to was used as his answer-checking strategy, whereas his explanation of how he solved the task indicated a shift to BAMT.

Excerpt 2: Adam's shift to BAMT

R1: Now, you still have the tower of 9. [But] I said, you know what, 16 is too tall of a tower. I only want you to have 14. You already have 9, but you want to have 14, not 16. How many cubes would you need to ask for?

Adam: (First, immediately and with no indication of using his fingers, writes down "5" on the page. Then, seemingly to check his answer, he quietly uses count-up-to on his left-hand fingers, putting them up one at a time. After three times of checking his answer, he raises his hand to indicate, "I am done.")

R1: What is the answer?

Adam: Five (5).

R1: How did you get it?

A: I got it by taking away; because 9 + 1 is 10, then all you have to do is add 4 to it.

R1: This is really cool. I like it. (Turns to Gary) Did you understand what Adam said, or would you like him to repeat [his explanation]?

Gary: (To Adam) Could you repeat it?

Adam: I added 1 to 9, then I had 4 left; so, 14. [It is another] easy number-pair.

R1: Here's what I heard Adam say. I am at 9. I only need 1 more to make 10. Is that correct so far?

Adam: Yes.

R1: Now, if I take one to make 10, and I need 14, to get from 10 to 14, I need 4 more. Is that correct Adam?

Adam: Yes.

R1: Then, if I need to take 1 to make 10, and 4 more to make 14, the 1 and the 4, make 5. Adam, is that what you did?

Adam: (Nods head, "yes.")

R1: Adam, I then saw you also using your fingers like you did before. Did you do that to check?

Adam: (Nods head, "yes") Mm-hmm.

R1: So, first you did the 9+1 and 4 more, then you checked with your fingers?

Adam: (Nods head, "yes.")

R2: (A bit later) Adam, you said, "easy number-pair." Can you tell me what you mean by that?

Adam: Like 10+4, 10+3, 4+4, and different numbers like that.

R2: So, are you saying they're like easy numbers to add?

Adam: Yes.

R2: You're trying to create an easy number [to] add by making the 9 [into] a 10?

Adam: Yes.

R1: (A little later, asking about other easy number pairs) What if I wanted 19?

Adam: 10 + 9. Anything in the teens can be paired up with a 10.

R1: What about 13?

Adam: 10 + 3.

Excerpt 2 indicates the first time we witnessed Adam's use of BAMT independently, spontaneously, and not in the DDB context. Importantly, we neither taught Adam about "easy number pairs" nor have knowledge when and how he was taught to use it. Key here, however, is that for the first time he independently and spontaneously brought forth this idea in what, for him, seemed a novel situation. We explain his shift to BAMT as rooted in reflecting on his count-up-to strategy and its effect after solving the first task (9 + ? = 16).

Specifically, when solving and then explaining that first task, Adam repeatedly showed how he raised his thumb for 1 item added while uttering 10 as the number after 9. Upon completion of the count, he held up and looked at 7 fingers while uttering "16." We infer he noticed the effect of his count-up-to as a number in the teens, which brought forth his notion of easy number pairs. In turn, he likely also coordinated "6" in the easy number pair (10 + 6) as being 1 less than 7 given to him by the second addend (which he showed on his fingers). Our explanation resonates with how Adam explained to Gary why this (BAMT) strategy worked for finding how many more cubes were needed to make a tower of 14 cubes ("I added 1 to 9, then I had 4 left; so, 14. [Another] easy number-pair."). It was further corroborated by Adam's response ("Yes") to R2's question, "You're trying to create an easy number [to] add by making the 9 [into] a 10?" and to R1's follow-up question about 19: "10 + 9. Anything in the teens can be paired up with a 10."

We emphasize that Adam's work led us to explain his shift to BAMT differently than what would be a trajectory from count-on to BAMT. In such a typical shift, a child would have to first notice the need to decompose the second addend into two numbers, 1 and the number before the second addend (e.g., to add 9 + 7 they would decompose 7 into 6 + 1, then add 1 to 9 to make 10, and complete with adding 6). Adam's case presents a somewhat reversed order of the reflective process. That is, he did not begin by decomposing 7 into 6 + 1. Rather, he first reflected on his work to solve a missing addend task by noticing that a total in the teens (e.g., 16) could be thought of (decomposed) into 10 and another number in the easy pair (e.g., 6). In turn, this reflection seemed to lead to his consideration of that easy number pair addend as a constituent of the task's second addend (here, 7), which led to decomposing that given second addend (e.g., 7 = 6 + 1, and that 1 is used to "make 9 [into] a 10.") We contemplate that Adam's shift might be a more accessible one for students than the one requiring decomposing of the second addend because such a shift requires setting a sub-goal of disembedding 9 from 10 as a strategic first step (sub-goal) for a yet-to-be-foreseen decomposition of the second addend.

Following this realization of Adam's shift to BAMT, to further promote his and Gary's decomposition of 10 into composite units, we engaged them in creating a tower of 10 (each). Then, we asked one of them to "chop off" a few cubes and write an equation for their quantities. For example, we asked Adam to take the top 2 cubes and place them near the tower. To symbolize this situation involving his and Gary's cubes, he wrote: "10 = 8 + 2." Similarly, after Gary chopped off 3 from his tower of 10 cubes, Adam wrote the equation: "10 = 7 + 3." It is in those tasks that we realized the difference in the two students' reasoning. For Gary, writing an equation proved to be a tremendous challenge, mostly done after hearing Adam's explanation. For Adam, writing an equation symbolizing his decomposition activity seemed straightforward. We emphasize that his equations did not show an addition problem on the left and an answer on the right but rather what seemed to be an equivalence of the two quantities. At this point, we decided to turn back to tasks presented in the DDB context. The first task involved figuring out how many passengers need to get on the bus in the second bus-stop if there were already 9 passengers on it and no one went to the upper deck. Adam knew, and explained to Gary, why just 1 passenger would get on the bus at the second stop. We followed this task with one requiring to go beyond 10 passengers on the bus in all (Excerpt 3).

Excerpt 3: Adam's transfer of BAMT to the DDB context

R1: Let's go back to the start. First stop, 9 people get on the bus. Now, I want you to find the number of people who will get on at the second stop so in the end we will have 12 people on the bus in all. How many people should get on at the second stop, so we have 12 people?

Adam: (Immediately and with no hesitation uses the rekenrek to add 1 more bead to the lower level and 2 more to the upper level.)

R1: Adam, tell us how many you had to put in order to get to where you are.

Adam: Three (3), because there is one in the bottom deck. That makes 10, then 2 up here (points to the upper deck), makes 12.

R1: Why is it 3?

Adam: There's a rule saying the bottom deck has to be full before you go to the top one. So, 1 (shows moving again one bead on the lower level), 2 (moves a bead over on the top deck), 3 (moves another bead over on the top deck).

Excerpt 3 indicates that, in this new missing addend task, Adam transferred his evolving anticipation of the goal-directed activity in the towers and cubes context to a context (DDB) in which we never saw him using BAMT on his own. From his immediate actions we inferred that at this point he anticipated the activity sequence of first using an easy number-pair (12 = 10 + 2), leading to then concluding that 3 passengers would be added to 9 for that total to be accomplished because, for him, 1 would be necessary to compose 10 as the start of the easy number-pair. Based on this transfer, and on the fact that Adam spontaneously brought forth the easy number pair component of a BAMT strategy, we conjectured that he had constructed an anticipatory stage of an ENS concept of number, indicated by disembedding and decomposing two addends (i.e., 10 + x = x-teen and 10 = 9 + 1). We tested this conjecture with a no-hint task at the start of the following week's episode. It turned out we preemptively attributed the anticipatory stage to Adam. For example, he used count-on to solve the task: How many cookies are there in all if a baker placed 9 in one pan and 7 in another pan. We thus attributed to Adam an initial abstraction of BAMT at the participatory stage.

Discussion

In this study, we articulated an advance in a child's concept of number as a composite unit, from count-up-to (TNS) to the BAMT strategy (ENS). Being a case study of the phenomenon of such a conceptual advance, we contend that the reorganization articulated here is likely to pertain to other students at different ages, grade-bands, or mathematical aptitudes. In and of itself, this reorganization is key to the child's additive reasoning, that is, to decomposing and disembedding numbers as composite units when adding or subtracting whole numbers. For example, Adam's BAMT indicated he thought of 12 as a unit that could be decomposed in two related ways: (a) into 10+2 and (b) into 9+x, with x being composed of 1+(x-1). Such an understanding underlies thinking of "fact-families" (e.g., 9+3=12, 3+9-12, 12-3=9, and 12-9=3). Just as important, this conceptual advance opens the way to developing multiplicative reasoning with whole numbers, which are highly limited or impossible for INS or TNS students (Steffe & Cobb, 1988).

For research and theory, this study contributes a plausible HLT for advancing to BAMT. Importantly, the advance we studied did not begin with a child's use of the anticipatory count-on strategy (INS) and a corresponding attempt to promote decomposition of the second addend. In fact, our data indicated that this path, used with Adam and Gary during the first four episodes, failed to promote BAMT. Instead, capitalizing on what Adam taught us, our HLT builds on the child's count-up-to strategy (TNS) and focuses on disembedding and decomposing two addends that make up an "easy number-pair" in the second decade, or "x-teens" (e.g., 14 = 10 + 4). In cases of adding 9 + x (or 8 + x), the child would then bring forth a decomposition of 10 into 9 + 1 (or 8 + 2) and learn to use it strategically by subtracting the needed complement-to-10 (e.g., 1) from the given second addend (e.g., 4 – 1 = 3). In turn, the child would complete the process by adding that decomposed/disembedded unit to 10 (e.g., 3; so, 13).

For practice, our study can inform teachers' efforts to effectively foster what, arguably, is the most foundational concept in mathematics, namely, number as composite unit (Steffe & Cobb, 1988). The HLT we proposed would likely support children's construction of addition and subtraction of whole numbers as two sides of the same coin. Said differently, someone who can compose and decompose given units at will, could come to "see" that any whole number (e.g., 14) may be constituted as a part-part-whole relationships (14 = 10 + 4 = 9 + 5, etc.). Thus, this study supports a way of thinking about and carrying out teaching of BAMT to students, a concept that proved an asset in students' learning to reason multiplicatively (Tzur et al., 2021).

Acknowledgments

This study was supported by a grant from the Sheridan School District #2. The opinions expressed do not necessarily reflect the views of the funders.

References

Baroody, A. J. (1995). The role of the number-after rule in the invention of computational shortcuts. Cognition and Instruction, 13(2), 189-219.

Cobb, P., & Steffe, L. P. (1983). The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14(2), 83-94.

Creswell, J. W. (2013). Qualitative inquiry & research design (3rd ed.). Sage.

Fuson, K. C. (1982). An analysis of the counting-on solution procedure in addition. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), Addition and subtraction: A cognitive perspective (pp. 67-80). Lawrence Erlbaum.

Fuson, K. C. (1986). Teaching children to subtract by counting up. Journal for Research in Mathematics Education, 17(3), 172-189.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. Journal of Mathematical Behavior, 22(4), 405-435.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. Journal for Research in Mathematics Education, 26(2), 114-145.

Steffe, L. P., & Cobb, P. (1988). Construction of arithmetical meanings and strategies. Springer-Verlag.

Tzur, R., Johnson, H. L., Norton, A., Davis, A., Wang, X., Ferrara, M., Harrington, C., & Hodkowski, N. M. (2021). Children's Spontaneous Additive Strategy Relates to Multiplicative Reasoning. Cognition and Instruction(online first), 1-26.

Tzur, R., & Simon, M. A. (2004). Distinguishing two stages of mathematics conceptual learning. International Journal of Science and Mathematics Education, 2, 287-304.

Ulrich, C. (2015). Stages in constructing and coordinating units additively and multiplicatively (part 1). For the Learning of Mathematics, 35(3), 2-7.

von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Falmer. (1995)

Zwanch, K., & Wilkins, J. L. M. (2020). Releasing the conceptual spring to construct multiplicative reasoning. Educational Studies in Mathematics.

PROPORTIONAL REASONING: VISUALIZING A KNOWLEDGE RESOURCES FRAMEWORK

Aaron Brakoniecki
Boston University
brak@bu.edu

David Glassmeyer
Kennesaw State University
dglassme@kennesaw.edu

Julie M. Amador
University of Idaho
jamador@uidaho.edu

Keywords: Rational Numbers & Proportional Reasoning, Teacher Knowledge, Preservice Teacher Education

Proportional reasoning is a key concept in school mathematics. However, research on the teaching and learning of proportional reasoning is still lacking (Lamon, 2007). There has been wider recognition recently of the importance of proportional reasoning and new frameworks have emerged to help researchers make sense of how teachers draw upon their own knowledge of proportional reasoning when responding to classroom tasks about the subject. Among these is the Framework for Teachers' Robust Understanding of Proportional Reasoning for Teaching (Weiland, et al., 2020). This framework consists of 19 knowledge resources teachers could draw upon in responding to proportional reasoning tasks. Although, these resources are a contribution to the field of mathematics education, they are presented as a list, without structure, hierarchy, or relation, calling to question the relation between the knowledge resources.

In our recent use of this framework to code beginning teacher proportional reasoning responses (Glassmeyer et al., 2021), the research team documented patterns in coding which caused us to look beyond individual categories. We turned our attention to look for relationships that might exist, and if there was a hidden structure among these 19 knowledge resources that could emerge to help us make better sense of the framework. We began a process of grouping knowledge resources by topic and theme, noting that all the categories seemed to exist on a spectrum of proportional reasoning. Some knowledge resources put a primary focus on an individual ratio (e.g. creating the ratio, identifying the quantities within the ratio, abstracting a measure that the ratio described). Other knowledge resources put their focus on proportions that are created (e.g. equivalence, covariance, and preventing distortion). Between these tentpoles, other knowledge resources focused on the altering of a ratio into something equivalent, setting the stage for the creation of a proportion. Knowledge resources were placed along the spectrum in groupings with other related categories. A couple knowledge resources were broad in their description and covered a greater portion of the spectrum. The end result was an organized framework that delineates how the original knowledge resources (Weiland et al., 2020) coexist in relation to one another. To factcheck our framework, we sought the input of Weiland et al. (2020), who agreed that we maintained the original integrity of the knowledge resources.

This poster presents the visual organization resulting from the work done classifying and sorting the 19 knowledge resources of the Framework for Teachers' Robust Understanding of Proportional Reasoning for Teaching. We show how this visual organization, our revised framework, was used to help in the coding of proportional reasoning data. Additionally, we also show how the idea of the groupings along the spectrum allowed for new patterns to emerge when trying to make sense of coding results and how this may inform both researchers and teacher educators as they conceptualize how to support students with learning proportional reasoning. This poster would be helpful not just for those focused on proportional reasoning, but also for those looking to add new layers of understanding to existing frameworks in use.

References

Glassmeyer, D., Brakoniecki, A., & Amador, J. M. (2021). Identifying and supporting teachers' robust understanding of proportional reasoning. The Journal of Mathematical Behavior, 62. https://doi.org/10.1016/j.jmathb.2021.100873

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester Jr. (Ed.), Second handbook of research on mathematics teaching and learning (Vol. 1, pp. 629–667). Charlotte, NC: Information Age.

Weiland, T., Orrill, C. H., Nagar, G. G., Brown, R. E., & Burke, J. (2020). Framing a robust understanding of proportional reasoning for teachers. Journal of Mathematics Teacher Education, 1-24.

Chapter 9:

Policy, Instructional Leadership, and Teacher Education

MASTER TEACHING FELLOWS' PERCEPTIONS OF TEACHER LEADERSHIP

Mahtob Aqazade
Rice University
ma119@rice.edu

Adem Ekmekci
Rice University
ae16@rice.edu

Anne Papakonstantinou
Rice University
apapa@rice.edu

Teacher leadership is essential to meet the needs of students, teachers, and schools in an effective way. To provide a better support system in high-need schools, the Rice University Robert Noyce Master Teaching Fellowship (RU-MTF) program further developed and supported 14 secondary mathematics teacher leaders. In this paper, we described two of the master teaching fellows' (MTF) reflections on and perceptions of teaching mathematics in high-need schools and teacher leadership throughout this program. To do so, we applied a deductive approach in thematic data analysis and relied on conceptual framework of teacher leadership and domains of teacher leader model standards. The emerged themes reflected the ways in which these two MTFs attempted to (a) promote student success, (b) address concerns around diversity and equity, and (c) facilitate uncertain challenges during COVID-19 pandemic.

Keywords: Instructional Leadership, Professional Development, Teacher Beliefs, Equity, Inclusion, and Diversity.

Teacher shortage and turnover have been a continuous concern in the United States particularly for mathematics teachers and in high-poverty and high-need schools (Carver-Thomas & Darling-Hammond, 2019). Effective teacher leadership can support developing and sustaining highly qualified teachers in the profession and positively affect student achievement (Darling-Hammond, 1998; Elmore, 2002; York-Barr & Duke, 2004). Thus, developing K-12 mathematics teacher leaders can support a system for teachers' persistence and retention (Green & Kent, 2016). Over a five-year period, Rice University Robert Noyce Master Teaching Fellowship (RU-MTF) program developed secondary mathematics teacher leaders with mathematical content and research-based pedagogical knowledge, leadership, and mathematics advocacy skills. As a part of exploring the effects of the RU-MTF program, in this paper, we describe two master teaching fellows' (MTF) perceptions of teaching mathematics in high-need schools and teacher leadership throughout this program.

Teacher Leadership

Over the past few decades, many constructs of teacher leadership have been discussed in which teachers play a central role in ways schools operate and in the core functions of teaching and learning (Danielson, 2007; York-Barr & Duke, 2004). Leadership by teachers is essential to serving the needs of students, schools, and the teaching profession (Teacher Leadership Exploratory Consortium [TLEC], 2011). Teacher leadership may be broadly defined as the active involvement of teachers in the improvement of school culture and instruction and ultimately student learning through their participation in school-wide decision-making, and promotion of their teaching and learning expertise (York-Barr & Duke, 2004). According to York-Barr and Duke (2004), the dimensions of teacher leaders' practices include coordination and management; district or school curriculum work; professional development of teacher peers; participation in school improvement efforts; parent and community involvement; professional contributions; and building partnerships with pre-service teacher education programs. Similar to these dimensions, the teacher leader model standards present seven domains: "fostering a collaborative culture to support educator development student learning;" "assessing and using research to improve practice and student learning;" "promoting professional learning for continuous improvement;" facilitating improvements in instruction and student learning;" "promoting the use of assessments and data for school and district improvement;" "improving outreach and collaboration with families and community;" and "advocating for student learning and the profession" (TLEC, 2011, p. 9).

To address the critical need for recruiting and supporting teachers to become teacher leaders in high-need schools as set forth by National Science Foundation's (2021) Robert Noyce Program and to build upon prior efforts for developing teacher leadership practices (e.g., York-Barr & Duke, 2004), RU-MTF aimed to engage MTFs in customized professional learning experiences that include university graduate-level mathematics coursework and Advancement Via Individual Determination (AVID) Path Training. In this paper, we focused on two MTFs and reported on their reflections and perceptions of teaching mathematics in high-need schools and teacher leadership within the RU-MTF program.

Methods

MTFs and Context

Fourteen secondary mathematics teachers (seven female and seven male) with exceptional academic merit reflecting the diversity of the student population and teachers (seven White, three Hispanic, two African American, and two Asian) from high-need schools in Houston Independence School District were selected as MTFs to participate in the five-year (2016-2021) RU-MTF program. In 2016, MTFs' teaching experience ranged from four to 40 years.

RU-MTF

In the first two years of the program, MTFs completed two one-week long summer courses in contemporary topics in secondary mathematics (e.g., equity, content, pedagogy, leadership skills in mathematics teaching) and two one-week long AVID Path Trainings to advance culturally relevant teaching practices for both educators and students. In addition, MTFs got involved in leadership activities (e.g., plan and co-teach with other mathematics teachers, work with colleagues to enhance students' self-efficacy, support and collaborate other MTFs, etc.) in their schools, school district, and at Rice University. Throughout each school year, MTFs also attended meetings to discuss the goals and expectations of their leadership work.

Data Sources and Analysis

In each year of the program, MTFs wrote their reflections for the Summer, Fall, and Spring semesters. In this paper, we focused on only two MTFs' reflections about teaching mathematics in high-need schools and teacher leadership. We chose Chad and Mark (pseudonyms) because as two high-school mathematics teachers with different years of teaching experience (17 and four respectively at the beginning of the RU-MTF program) and with different ethnic and educational backgrounds, these two teachers provided the most insightful reflections that are also representative of the whole group. We analyzed Chad's and Mark's written reflections (15 reflections in total per person and 2–5 pages each reflection) by applying a deductive approach in thematic data analysis (Saldana, 2013) and using York-Barr and Duke's (2004) conceptual framework of teacher leadership and domains of teacher leader model standards (TLEC, 2011).

Findings and Discussion

Before the RU-MTF program, Mark described "mentoring new teachers," serving as "content lead," and "shared decision-making committee" and Chad identified "unofficial mentor[ing] for several teachers" as part of their leadership, mentoring, and adult education. Across the five-year RU-MTF program, Chad's and Mark's reflections encompassed challenges and successes of teaching mathematics in high-need schools as well as ways of incorporating and sustaining leadership roles to: (a) promote student success, (b) address issues concerning diversity and equity; and (c) facilitate uncertainties during times like the COVID-19 pandemic. These emerged themes highlighted important components of York-Barr and Duke's (2004) conceptual framework such as "supportive culture," "supportive principal and colleagues," "resources," and "establish trusting and constructive relationship" (p. 289). Looking at these themes in-depth also illuminated domains of teacher leadership within teacher leader model standards (TLEC, 2011). To Mark, an effective mathematics instruction that prepares high-school students for college "involves gradually giving the student more autonomy." Chad's perspective is to prepare students to "become more capable of thinking abstractly…and model real-life phenomena."

Providing a Path to Student Success

"I'm frustrated with the inflexibility of our public school system… I'm concerned about kids that don't succeed in our very inflexible system."

Improving student success within the current school system, unsurprisingly, was one of the most common topics discussed in both Chad's and Mark's reflections, which involved the roles of school culture, administrations and colleagues, and testing.

School culture. Chad described his school culture as "a culture of teachers giving the students assignments, and the students complying with the assignments" and he wanted to introduce some of his teachings: "I simply cannot teach math as a set of rules to be followed, and for the students to believe those rules because I, the authority, told them so." Thus, Chad took the lead of the pre-calculus team to implement some of his visions. However, he continuously faced challenges because of the testing priority, school inflexibility, and unproductive collaboration.

> The reason that this isn't a HUGE problem is that [*my school*] has this high achievement culture where students are expected to memorize, memorize, memorize. So, they do. And they pass the tests. I'm really the only one that has a problem with it (well, me and the kids who don't memorize so well). My problem is that I feel we are missing the spirit of math.

Chad's challenge reflects the importance of establishing a school culture that advocates for collaboration, "shared leadership, differentiated roles for teachers, and mutual accountability for student learning" (TLEC, 2011, p. 28).

Administration and colleagues. Although Chad's department chair supported him to implement the changes in the pre-calculus course as a teacher leader, "[my] teammates basically withdrew from me, and became extremely critical of anything that I had to offer." As team members developed a climate of trust and critical reflection, their meetings became more collaborative and drew from backward lesson planning. Mark also encountered "a lot of skepticism and pushback from my appraiser" when incorporating technology into his teaching.

> There is a lot of pushback with this because there is an unrealistic expectation from administers to focus solely on specific objectives and not on the skills necessary to achieve those objectives…It's very frustrating to deal with this mentality and complete lack of understanding about the situation that we are dealing with at our campus.

In line with these examples are facilitating "skills to create trust among colleagues" and using "knowledge of existing and emerging technologies to guide colleagues in helping students skillfully and appropriately" from teacher leader model standards (TLEC, 2011, p. 16).

Testing. Mark alluded to "don't teach to those tests" in relation to student success:

> With administrators often only concerned with testing results or data, many teachers often panic and fall to pressure to push our kids on something that they can't easily fix. This creates a source of friction and to some degree distrust and resentment between those teachers and their students.

Although teaching for testing is not a path to student success, teachers and teacher leaders can work "individually and collaboratively to examine test and other performance data to understand each learner's progress" (TLEC, 2011, p. 53)—a crucial step in data-driven changes in school culture and "promoting the use of assessments and data for schools and district improvement" (TLEC, 2011, p. 18).

Diversity and Equity

As a white middle-class teacher in a predominately African American school, Chad discussed the "systemic racism and white supremacy" in the school culture.

> White teachers trying to "save" the minority students by getting them to buy into our culture instead of the culture of their family and their neighborhoods…I didn't realize at the time was that school culture wasn't just middle class, it was white middle class…I get very uncomfortable when I see us educators pushing particular values on students…I want to open people's minds and...teach students to think logically and rationally.

As Mark emphasized "that different cultures have conventions that are very different from American conventions," prompting instructional strategies that address diversity and equity issues (TLEC, 2011) is central to leadership standards.

The Good and Bad of the COVID-19 Pandemic

"Especially with everything that has been going on in this country, we have to set formulas and calculators aside sometimes, and reach their hearts if we want to reach their minds."

In the last two years of the program, MTFs shared lessons learned from transitioning to remote learning during the pandemic and leadership roles they took to facilitate this transition. Their leadership roles become even more critical as the uncertain condition called for "supportive culture," "resources," and "supportive principal and colleagues" (York-Barr & Duke, 2004, p. 289). Chad perceived both the good and bad of the pandemic:

> The main problem I have with online teaching and online meetings is just how tired they make me feel afterward… [during RU-MTF summer professional development], we had quite the international group. Not only were the facilitators from points scattered all over the globe, but the participants as well…It was interesting to hear people talk about the math education that they had on the other side of the world.

Mark shared how online teaching encouraged some of his students to participate more in classes but overall "a very low level of participation, and a large amount of inflation by teachers and administration." Acknowledging the limits, a virtual format contains, Mark's experiences with virtual professional development were overall positive, "I feel that many people felt more comfortable sharing in a virtual format, than in an in-person." These positive experiences with virtual professional development opportunities are important to consider even post-pandemic.

Acknowledgment

This research was supported by National Science Foundation – DUE #1556006.

References

Carver-Thomas, D., & Darling-Hammond, L. (2019). The trouble with teacher turnover: How teacher attrition affects students and schools. *Education Policy Analysis Archives*, *27*(36), 1–27.

Elmore, R. F. (2002). Hard questions about practice. *Educational Leadership, 59*(8), 22–25.

Danielson, C. (2007). *Enhancing professional practice: A framework for teaching* (2nd ed.). ASCD.

Darling-Hammond, L. (1998). Teacher learning that supports student learning. *Educational Leadership, 55,* 6–11.

Green, A. M., & Kent, A. M. (2016). Developing science and mathematics teacher leaders through a math, science & technology initiative. *Professional Educator*, *40*(1), 1–19.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372–400.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* NCTM.

National Science Foundation (2021, October 4). *Robert Noyce teacher scholarship program.* https://beta.nsf.gov/funding/opportunities/robert-noyce-teacher-scholarship-program

Saldaña, J. (2021). *The coding manual for qualitative researchers*. Sage.

Teacher Leadership Exploratory Consortium (2011). *Teacher leader model standards*. Centre for Teaching Quality.

York-Barr, J. & Duke, K. (2004). What do we know about teacher leadership? Findings from two decades of scholarship. *Review of Educational Research, 74,* 255–316. https://doi.org/10.3102%2F00346543074003255

A CRITICAL LITERATURE SYNTHESIS ON METHODOLOGICAL AND DISSEMINATION CONSIDERATIONS AROUND MATHEMATICS SPECIALISTS

<u>Kristin E. Harbour</u>
University of South Carolina
kharbour@mailbox.sc.edu

<u>Margret A. Hjalmarson</u>
National Science Foundation
mhjalmar@nsf.gov

<u>Stefanie D. Livers</u>
Missouri State University
stefanielivers@missouristate.edu

Courtney K. Baker
George Mason University
cbaker@gmu.edu

Evthokia Stephanie Saclarides
University of Cincinnati
saclares@ucmail.uc.edu

Katherine C. Edwards
George Mason University
kedwar21@gmu.edu

Mathematics specialists are prominent figures in calls to advance the teaching and learning of mathematics. While the calls for mathematics specialists have gone on for decades, research in this area of study is still emerging. Because of this, there is a need for the field of mathematics education to better understand the research that has been conducted to move the field forward. As such, we present preliminary findings of a literature synthesis investigating the ways in which mathematics specialists are studied within empirical research between the years 1981 and 2018. Our findings document the quality of the research, the research methods used, and the journal outlets research is disseminated around mathematics specialists, as well as discussion and implications based on our results.

Keywords: instructional leadership, research methods

In the United States, a call for mathematics specialists (MS) has been prominent over several decades, with a rise in this call and research related to mathematics specialists occurring since the late 2000s (Baker et al., 2022; Fennell, 2017). However, despite this increase in research, in general, MS are still under-investigated (e.g., Herbst et al., 2021; Hjalmarson & Baker, 2020), leading to policy decisions being made without ample evidence. MS are considered “an important lever” for advancing teacher learning (Marshall & Buenrostro, 2021, p. 8) by supporting teacher development and facilitating district efforts (Cobb et al., 2018; NCTM, 2020), we believe it is imperative for the field of mathematics education to better understand the research situated around MS to advance this work.

Mathematics Specialists

MS are instructional leaders who work to advance effective mathematics instruction and student learning through their work with teachers (McGatha & Rigelman, 2017). MS often take on many roles within a school or district and may have various responsibilities depending on the context of their work (Baker et al., 2022; Harbour et al., 2021; McGatha & Rigelman, 2017). MS may be charged with facilitating the implementation of school, district, or state initiatives, the integration of research-based practices and curriculum materials, and the use of assessment data (e.g., Campbell & Malkus, 2011; Fennell, 2017). MS may also model lessons and instructional strategies, as well as provide professional development (e.g., Chval et al., 2010; Fennell, 2017; Polly et al., 2013). While the roles and responsibilities of MS vary, the ways in which these supports are provided also varies. MS may work on school-level initiatives (e.g., Cobb et al., 2018) or work with small groups of teachers (e.g., Livers, 2019; Lesseig et al., 2016). MS may also work with individuals or pairs of teachers (e.g., Saclarides & Harbour, 2020).

Purpose

Because of the need to understand prior research to know what future research is needed, the purpose of our work is to synthesize existing research, through 1981-2018, to develop an understanding of the methodological considerations around MS. Our goal is to present preliminary findings to raise questions around methodological decisions and dissemination of existing research. We first extend prior synthesis work around MS (Baker et al., 2022; Hjalmarson et al., 2020) by evaluating the quality of each published article using an established criterion, an important step in a literature synthesis (Cooper et al., 2019). Following the application of the appraisal, or quality, criteria, we then sought to answer the following research questions: (a) what type of overarching methodology are researchers using when studying MS; and (b) how are researchers who study MS disseminating their empirical work (i.e., peer-reviewed articles)?

Methods

We draw upon earlier work (Baker et al., 2022), wherein we explored empirical research to examine the positioning of MS (see Baker et al., 2022 for a detailed process of inclusion and exclusion criteria). The researchers employed in vivo coding (Saldaña, 2021) across 130 research articles in a three-layer process. First, the population in which the MS engaged with was determined (e.g., in-service teachers, pre-service teachers). Next, building upon McGatha and Rigleman's (2017) positions for MS (coach, teacher leader, interventionist) were assigned. As codes were applied, new positioning of MS emerged, resulting in six distinct positionings, wherein three were previously defined (McGatha & Rigelman, 2017) and three newly emerged: (1) coach, (2) teacher, (3) interventionist, (4) university stakeholder, (5) other P-12 stakeholder, and (6) learner. Lastly, a layer of coding allowed for specificity of context (i.e., school or district levels), thus providing positionings such as MS as Coach: District-Level.

For the present study, we homed in on six unique positionings of MS whose primary audience was *in-service* to pilot our exploration (see Table 1, initial frequency). As our goals were to extend previous work and to develop an understanding of the methodological considerations around MS, we engaged in three types of analysis: (a) article quality evaluation, (b) methodological approaches, and (b) journal outlets. The methodology we used, results, and implications follows. Articles may have included more than one role.

Table 1: Mathematics Specialists Positioning

Positioning	Initial Frequency	Final Frequency
Inservice: MS as Math Coach: School-level	26	23
Inservice: MS as Math Coach: District-level	1	1
Inservice: MS as Instructional Coach: School-level	6	4
Inservice: MS as Teacher Leader	21	15
Inservice: MS as Learner	10	6

Article Quality Evaluation

To evaluate the quality of the studies meeting inclusion and exclusion criteria, we used the appraisal criteria of Risko et al. (2008). This evaluation tool has seven quality criteria and has been applied in previous research (e.g., Miller et al., 2015). Unlike criteria which only address one methodology (e.g., Thunder & Berry, 2016), Risko and colleagues' (2008) criteria enables the appraisal of articles from various methodologies. Paralleling the process established (Risko et al, 2008), articles that received a score of 7 (i.e., meet all criteria), were used in analyses.

We coded 65 articles which prominently featured MS. Three researchers coded 23 articles to establish reliability with the rubric, and then pairs of those researchers coded the remaining 42. In reaching consensus of 0 or 1 scores for each criteria indicator, we approached the subjective terms in the descriptions (e.g., the term *several*) with leniency. For instance, rather than articles needing all listed descriptors in the indicator, if the article had 50% of them, we scored the indicator as a 1. Of the 65 articles, 47 met all the criteria (i.e., 73%; see Table 1, final frequency). This inclusion rate is higher than the 35% inclusion rate reported in Risko et al. (2008). We also identified five tool development papers that we only include in the analysis of journal outlets. We did not score these 5 papers using the Risko et al. (2008) criteria because those articles were not structured in a way that fit the criteria's goals or purpose.

Methodology Approach Coding

To determine the overarching methodological approach used in each of the 47 articles that meet quality criteria (e.g., using Risko et al., 2008), a team of three researchers distributed the articles to where two researchers coded the methodology for each article and the third researcher was used, as needed, to reach 100% consensus. We coded articles in one of three ways: (a) Qualitative, (b) Quantitative, or (c) Mixed Methods. We then determined the frequency in which each of the overarching methodological approaches appeared.

Dissemination of Research: Journal Outlet

To determine how research around MS was disseminated, we collected where each article was published, determined the frequency count for each journal, followed by applying a code of "math" or "general" to each journal to determine the type of journals that were publishing research involving MS.

Results

Research Question 1: Methodologic Approaches

The 47 articles that met all the quality criteria (Risko et al., 2008) included 26 qualitative studies, 8 quantitative studies, and 13 mixed methods studies. We are still analyzing the methods in more detail (e.g., HLM, case study, social network analysis). Notably, there are a significant number of qualitative case studies (e.g., of specialists as individuals) within the 47 articles. Data gathered often includes interviews, observations of classroom or professional development, surveys, and other artifacts (e.g., lesson plans, student work). There are also assessments of teacher knowledge and beliefs. The quantitative work included studies linking mathematics coaching and student outcomes (e.g., Campbell & Malkus, 2011) and studies of the influence of math coaching (e.g., Sun et al., 2014). Mixed methods studies often included qualitative evidence coupled with interviews or surveys. For instance, Hopkins et al., 2017 was coded as mixed methods as it employed social network analysis along with analysis of interviews.

Research Question 2: Dissemination of Work

To describe dissemination efforts, we included the 47 research articles and 5 articles on tool development. The tool development studies were primarily about mathematics coaching or leadership, however, were not written in a way that aligned well with the Risko et al. (2008) appraisal criteria. However, to represent the landscape of outlets that are publishing work about MS, we included the five tool development articles in this portion of the analysis. We found there were 30 peer-reviewed journals represented in our data set of 52 articles. For brevity, we coded them as *mathematics journals* if the title indicated either a singular focus on mathematics or a dual focus of mathematics and science or *general journals* if they focused on other more general education topics (e.g., teacher education). Twenty-six articles were published in mathematics focused journals, and twenty-seven were published in general journals. The top five journals where MS research was disseminated were *The Journal of Mathematical Behavior* (n=7), *Journal of Mathematics Teacher Education* (n=6), *ZDM Mathematics Education* (n=4), and *The Elementary School Journal* (n=4). We note that six of the *JMB* articles appeared in a special issue.

Discussion

As we sought to develop an understanding of the quality of research, methodological considerations, and dissemination of work around MS, some interesting findings emerged that necessitate discussion and hopefully implications on future research involving MS.

First, using the appraisal criteria of Risko et al. (2008) to determine the quality of empirical research, we found most articles in our data set met criteria at a much higher rate reported by the authors. We find this a promising, indicating high-quality research is being conducted related to MS. Of importance for future research, most of the articles were eliminated due to methodological issues; specifically, around following three indicators: (a) "Ensures that methods are presented in sufficient detail and clarity to visualize procedures", (b), "Relies on measurements or observational methods that provide reliability, credibility, and trustworthiness", and (c) "Describes participants" (Risko et al., 2008, p. 256). Within the Risko et al. (2008) criteria, more information is provided for each indicator (shortened for brevity here). Many articles not meeting criteria were often scored lowest in participant description. This makes sense within the MS research as there are often blurred lines in their positioning and definition of who is a mathematics specialist (National Mathematics Advisory Panel, 2008). However, as previous work has now provided a framework of positionings (Baker et al., 2022), we hope that future research can employ this positioning and provide more details on the participants of studies.

Second, as we explored the methodological approaches used to study MS, we see the majority of published, empirical work employs qualitative methods. By no means are we suggesting that qualitative methodological approaches should not continue; however, we do believe it is important to note this uneven approach among qualitative and quantitative methodologies. Quantitative methodologies are important within research involving MS to determine causal links and provide evidence that is generalizable on a larger scale.

Third, we find the lack of a particular journal that represents a "home" for MS research both encouraging and problematic. It is encouraging to see there is MS research being published in mathematics education, teacher education, education leadership, and other domains. However, there is a need for studies about the preparation, practice, and impact of MS on teaching and learning. We also find the lack of focus in premiere mathematics education journals troubling. For example, the only article in this collection focused on MS published in the *Journal of Research in Mathematics Education* was a highly influential tool development study by Munter (2014). Outside of the special issue in *JMB*, other mathematics education journals only published 1-2 articles across this long period of time (i.e., 1981-2018).

Conclusion

While there has been growth in research about MS as leaders of teacher learning and professional development, there is clearly more work to be done. There are many lenses and bodies of existing research that can inform studies of MS, and there are clearly different educational arenas that are beginning to examine MS as part of the educational system. We encourage the field to continue these investigations and continue expanding the methods and theories that are used in these studies.

References

Baker, C. K., Saclarides, E. S., Harbour, K. E., Hjalmarson, M. A., Livers, S. D., & Edwards, K. C. (2022). Trends in mathematics specialist literature: Analyzing research spanning four decades. *School Science and Mathematics*, 122, 24– 35. https://doi.org/10.1111/ssm.12507

Campbell, P. F., Ellington, A. J., Haver, W. E., & Inge, V. L. (2013). *The elementary mathematics specialist's handbook*. National Council of Teachers of Mathematics.

Campbell, P. F., & Malkus, N. N. (2011). The impact of elementary mathematics coaches on student achievement. *The Elementary School Journal, 111*(3), 430–454. https://doi. org/10.1086/657654

Chval, K. B., Arbaugh, F., Lannin, J. K., van Garderen, D., Cummings, L., Estapa, A. T., & Huey, M. E. (2010). The transition from experienced teacher to mathematics coach: Establishing a new identity. *The Elementary School Journal, 111*(1), 191–216. https:// doi.org/10.1086/653475

Cobb, P., Jackson, K., Henrick, E., & Smith, T. M. (2018). *Systems for instructional improvement: Creating coherence from the classroom to the district office.* Harvard Education Press

Cooper, H., Hedges, L. V., & Valentine, J. C. (Eds.). (2019). *The handbook of research synthesis and meta-analysis*. Russell Sage Foundation

Fennell, F. (2017). We need mathematics specialists now: A historical perspective and next steps. In M. B. McGatha & N. R. Rigelman (Eds.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (pp. 3– 18). Information Age Publishing Inc.

Harbour, K. E., Saclarides, E. S., Adelson, J. L., & Karp, K. S. (2021). Exploring relationships among responsibilities of mathematics coaches and specialists and student achievement. *International Electronic Journal of Mathematics Education, 16*(2), em0640. https://doi.org/10.29333/iejme/10907

Herbst, P., Chazan, D., Crespo, S., Matthews, P. G., & Lichtenstein, E. K. (2021). Considering the importance of human infrastructure in the apprenticing of newcomers in mathematics education research practices. Journal for Research in Mathematics Education, 52(3), 250–256. https://doi.org/10.5951/jresemathe duc-2021-0019

Hjalmarson, M. A., & Baker, C. K. (2020). Mathematics specialists as the hidden players in professional development: Researchable questions and methodological considerations. International Journal of Science and Mathematics Education, 18(1), 51–66. https://doi.org/10.1007/s10763-020-10077-7

Hjalmarson, M. A., Saclarides, E. S., Harbour, K. E., Livers, S. D., & Baker, C. K. (2020). Mathematics specialists and teacher leaders: An ongoing qualitative synthesis. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 1868-1872). Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Hopkins, M., Spillane, J. P., Jakopovic, P., & Heaton, R. M. (2013). Infrastructure redesign and instructional reform in mathematics: Formal structure and teacher leadership. *The Elementary School Journal*, *114*(2), 200– 224. https://doi.org/10.1086/671935

Lesseig, K., Elliott, R., Kazemi, E., Kelley-Petersen, M., Campbell, M., Mumme, J., & Carroll, C. (2017). Leader noticing of facilitation in videocases of mathematics professional development. Journal of Mathematics Teacher Education, 20(6), 591–619. https://doi.org/10.1007/s10857-016-9346-y

Livers, S. D. (2019). Facilitating teacher learning through leading: An elementary mathematics coach's story. *Tennessee Educational Leadership, 46(*2), 23–32.

Marshall, S. A., & Buenrostro, P. M. (2021). What makes mathematics teacher coaching effective? A call for a justice-oriented perspective. *Journal of Teacher Education*, *72*(5), 594-606. https://doi.org/10.1177/00224871211019024

McGatha, M., & Rigelman, N. R. (2017). *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (Vol. 2). Information Age Publishing.

Miller, D. M., McTigue, E. M., & Scott, C. E. (2015). The quality of recent studies in content-area writing in secondary classrooms. *Literacy Research: Theory, Method, and Practice, 64*(1), 461–477. https://doi.org/10.1177/2381336915617602

National Council of Teachers of Mathematics. (2020). *Catalyzing Change in Early Childhood and Elementary Mathematic*s. Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. U.S. Department of Education.

Polly, D., Mraz, M., & Algozzine, R. (2013). Implications for developing and researching elementary school mathematics coaches. *School Science and Mathematics, 113*, 297–307.

Risko, V. J., Roller, C. M., Cummins, C., Bean, R. M., Collins Block, C. Anders, P. L., & Flood, J. (2008). A critical analysis of research on reading teacher education. *Reading Research Quarterly, 43*. https://doi.org/10.1598.RRQ.43.3.3
Saclarides, E. S. & Harbour, K. E. (2020). A case of one-on-one coaching to differentiate mathematics instruction. *Professional Development in Education*, 1-24. https://doi.org/10.1080/19415257.2020.1756900
Saldaña, J. (2021). *The coding manual for qualitative researchers* (4th ed.). SAGE Publications.
Sun, M., Wilhelm, A. G., Larson, C. J., & Frank, K. A. (2014). Exploring colleagues' professional influence on mathematics teachers' learning. *Teachers College Record, 116*(6), 1–30. https://doi.org/10.1177/016146811411600604
Thunder, K., & Berry, R. Q. (2016). Research commentary: The promise of qualitative metasynthesis for mathematics education. *Journal for Research in Mathematics Education*, *47*(4), 318–337. https://doi.org/10.5951/jresematheduc.47.4.0318

Disclaimer

This work was completed while one author (Margret Hjalmarson) served as a program director at the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

SUPPORTING DISCUSSION PRACTICE IN MATHEMATICS METHODS: APPLICATIONS OF WHOLE-CLASS SCAFFOLDING

Signe E. Kastberg	Alyson E. Lischka	Susan L Hillman
Purdue University	Middle Tennessee State Univ.	Saginaw Valley State Univ.
skastber@purdue.edu	Alyson.Lischka@mtsu.edu	shillman@svsu.edu

Discussion is a practice used in mathematics methods courses to support prospective teachers' pedagogical ideas. Underexamined but central to the development of instructional activities, including discussions of teaching, are mathematics teacher educators' (MTEs') tacit and explicit theories of learning and teaching. We report findings from a self-study of three MTEs' discussion practice in methods courses. Data sources include transcripts of MTEs' dialogic analysis of their discussion practice and evidentiary maps based on instructional artifacts. We argue that whole-class scaffolding serves as a tacit theory informing MTE discussion practice. We support this argument using evidence that our discussion practice was driven by prospective teachers' move toward independence and by layering instructional activities.

Keywords: Discussion, Preservice Teacher Education, Teacher Educators

Learning theories (Casey et al., 2018) and professional experiences (Leikin, 2020) have been shown to inform mathematics teacher educator (MTE) practices. Furthermore, professional practices shape the "learning potential" (Grossman et al., 2009, p. 2090) of instructional activities. These findings link theories and experiences of MTEs to learning opportunities provided to prospective mathematics teachers (PTs). Calls for the study of MTE work (Grossman et al., 2009; Lee & Mewborn, 2009) have resulted in descriptions of MTE practices (e.g., van Es et al., 2014), instructional activities (e.g., Tyminski et al., 2021), and MTE knowledge (e.g., Beswick & Goos, 2018) with a few studies exploring MTEs' professional growth (Krainer et al., 2021). In this paper we address MTEs' professional growth of discussion practice, providing an example of how the learning potential of discussions of teaching are developed. Beyond providing descriptions of discussion practice as useful models for MTEs, unpacking theories that inform such practice links MTEs' learning to teach about teaching to PTs' opportunities to learn mathematics teaching.

In this paper we use self-study methodology to identify tacit theory that informs the discussion practice of three white MTEs' teaching mathematics methods at different institutions to elementary and secondary PTs. The three MTEs' are relational (Kitchen, 2005) constructive teachers (Steffe & D'Ambrosio, 1995) of mathematics teaching who utilize relationships with PTs and evidence of their pedagogical concepts to inform instructional decisions. Relational teacher educators view relationships as central to creating opportunities to learn about teaching. Constructivist teachers view learning as an intersubjective activity that includes learner discussions in a collaborative community (Kastberg, 2014; Steffe & D'Ambrosio, 1995). Taken together relational constructivist MTEs view whole class discussions of teaching as opportunities to theorize about teaching and problems from experience.

As relational constructivist MTEs, we use whole class discussions to support PTs' teaching practices. Self-study of our discussion practice resulted in new approaches to posing discussion questions (Kastberg et al., 2019) and creating phenomenological conditions (Lischka et al., 2020) supportive of discussions of teaching. Yet we continued to struggle to facilitate discussions with

PTs, instead often creating recitations during which PTs' shared ideas without addressing the ideas of others. To expand the domain of potential action (Brown & Coles, 2020) available in planning and facilitating discussions of teaching in mathematics methods courses we were guided by the question: *What tacit theories inform mathematics teacher educators' discussion practice in mathematics methods courses?*

Background and Literature

Loughran (2014) defines pedagogy as "two complementary aspects of knowledge and practice: teaching about teaching and learning about teaching" (p. 275) kept in relation. Loughran's definition situates pedagogy as a theory that maintains a relationship between teaching about teaching and learning about teaching. Such theories in mathematics teacher education have been used to design and implement instructional activities for PTs (Kastberg et al., 2018). Making theories that inform MTEs' practice explicit (Mewborn & Stanulis, 2000) is essential to expanding possibilities for instructional choices and engaging in discussions of those choices beyond simply modeling practices (Teuscher, et al., 2016). In this paper we focus on tacit theories that inform MTEs' whole class discussions of teaching.

Discussion is defined as human interaction to address "a question of common concern" (Dillon, 1994, p. 8) through an exchange of ideas (Alexander, 2019) and an examination of differing viewpoints (Kim & Wilkinson, 2019). Discussion differs from recitation in that recitation involves sharing ideas without engaging others in those ideas while discussion involves a sharing of ideas in which others take up those ideas and add to or counter them (Dillon, 1994). We define discussion as a talk strategy MTEs use to support development of PTs' pedagogical concepts (Simon, 2008). MTEs' use of discussion practice is informed by their "knowledge, theories, and understandings" (Pinnegar & Hamilton, 2009, p. 16) to develop "knowledge from practice" (Pinnegar & Hamilton, 2009, p. 17) as a way of "knowing to" (p. 18) engage PTs in a given practice.

Existing reports of MTEs' facilitation of discussions have identified practices associated online discussions (McDuffy & Slavit, 2002), ambitious teaching (Kazemi et al., 2016) and teaching videos (van Es et al, 2014). MTEs' facilitation of pedagogical discussions (Lischka et al., 2020; Kastberg et al., 2021) is also informed by phenomenological factors (Dillon, 1994) such as a sense of community and relevant common experiences. MTEs' discussion practices, including posing discussion questions, and supporting interpretation of the question, impacts the form (i.e., IRE, recitation, discussion) and content of the PTs' talk (Kastberg et al., 2021). Evidence teachers use to support claims made during discussions of teaching informs MTEs' discussion practice, yet findings are mixed. Steele (2005) identified teachers' use of experience to support claims in discussions of mathematics teaching while Dick et al. (2018) suggested teachers did not provide evidentiary support for claims made during discussions of their mathematics teaching. Such research illustrates that teachers' address of discussion prompts focused on teaching are conditional and respectful of the authority of experience (Munby & Russell, 1994), thus raising questions about how MTEs' can initiate and sustain discussions of teaching that move beyond sharing ideas to taking up and countering the ideas of others.

One theory that may support MTE's discussion practice draws from linking whole-class scaffolding and dialogic teaching (Bakker et al., 2015) including discussion. Smit et al. (2013) asserts that whole-class scaffolding involves interpreting and responding to learners' understandings and needs while fostering independence (Visnovska & Cobb, 2015). Whole-class scaffolding is "layered, distributed and cumulative" (Smit et al., 2013, p. 829) and takes place before, during, and after whole-class interactions. Bakker et al. (2015) illustrates ways whole-class scaffolding can inform dialogic teaching including discussions. Keys to whole-class scaffolding include understanding that instructional activities produce layers of opportunities for developing concepts while supporting movement toward engaging in activities independently. These two key components of whole-class scaffolding align with the existing reports of MTE work that illustrate the importance of PTs' movement toward independent teaching (Grossman et al., 2009) and the need to use pedagogies of practice in concert (Ghousseini & Herbst, 2016).

Methodology and Methods

Self-study methodology is a form of empirical practitioner research focused on improving practice in context. As practitioner research (Borko et al., 2007), self-study methodology supports inquiry into pedagogy of practice. Self-study is self-initiated, improvement-aimed, interactive, uses qualitative methods, and defines validity as based in trustworthiness (LaBoskey, 2007). Our study of tacit theories that inform discussion practices was initiated to inquire into how we planned for and enacted discussions of teaching in mathematics methods with a focus on improving our discussion practices. Beginning in 2015, our collaboration has included weekly meetings to discuss the development of instructional activities and practices. As a collaborative self-study group of MTEs from three different US institutions, we are critical friends (Schuck & Russell, 2005) who undertake scholarly inquiry (Lee & Mewborn, 2009) of our practices.

Our institutional missions range from teaching-focused to research-intensive and our program foci span elementary to secondary teacher certification. Signe's discussion study was guided by the question: How do children learn mathematics? Alyson's discussion study was guided by the question: What is the role of mathematics teachers related to social justice and equity? Susan's discussion study was guided by the question: How does cognitive demand of tasks and knowledge of children's mathematical thinking inform planning instruction? Studying our practice, we identify "living contradictions" (Whitehead, 1989, 41) between our intended and working models of practice.

Critical friendships serve as dialogic communities where dialogue serves as a process of coming to know. Pinnegar and Hamilton (2009) align the scientific method, action research cycle and dialogue as processes of coming to know in different methodologies. In self-study methodology dialogue involves expressing ideas in conversations to be "accepted and elaborated or rejected, rephrased, questioned, or ignored" (p. 87). Participants in the dialogue "may provide evidence, examples, representations, metaphors, or analogies in support of or opposition to the idea or as a way to synthesize and integrate the idea with others" (p. 87). Such dialogues in mathematics teacher education produce "theorizing" about MTE practice that "extends the range of possible behaviors, by dwelling in the details of the experience of teaching and considering the details on a more general level" (Brown & Coles, 2020 p. 99). Assessing the trustworthiness or quality of qualitative research (Grant & Lincoln, 2021) involves planning for and committing to making transparent authenticities that emerge in the study. In the case of self-study, ontological authenticity is central as researchers gather evidence of knowledge of self that illustrates what is learned through dialogic analysis and other analytical methods.

Within the methodology of self-study we used three qualitative analytic methods: analytical dialogues (Guilfoyle et al., 2007), evidentiary maps, and descriptive coding (Saldana, 2016). (1) We engaged in 8 analytic dialogues of our discussion practice in fall 2020 (Covid-19 hyflex teaching). Conversations focused on "coming to know" how we used discussion in our teaching that served as the basis for "action" (Guilfoyle et al., 2007, p. 1111) in our ongoing discussion practice. At the conclusion of this period we had formed a collection of categories that informed our discussion practices. (2) In spring 2021 we used course artifacts (recordings of whole class discussions, class summaries, assignments, and PT work samples) to create evidentiary maps (see Table 1 for an excerpt) of the "structure of events" (Jordan & Henderson, 1995, p. 57) in our fall 2020 discussion practice. Each map was analyzed using categories from the dialogic analysis. Confirming or contradicting evidence from the dialogic analysis was identified. (3) Descriptive coding (Saldana, 2016) of transcripts from analytical dialogues was used to triangulate results from analytic dialogues and evidentiary maps by linking the findings from the analytic dialogues to evidence in the transcripts of our dialogues. Analysis revealed evidence of commonalities across three MTEs' discussion practices. Movement in these analytical methods proceeded from dialogue to course artifacts and back to dialogue. These three analytic methods created an evidentiary basis for findings common across three MTEs' discussion practice and related contexts.

Findings

This section describes two components of whole-class scaffolding (Smit et al., 2013) that influenced our discussion practice: (1) the move toward independence in mathematics methods courses, and (2) layering learning activities driven by the move toward independence. Data from the three authors' practices were used to derive the findings. This paper uses examples from Susan's pedagogy of discussion practice drawn from transcripts of dialogic conversations and evidentiary maps to illustrate our findings for the research question: What tacit theories inform MTEs' discussion practice in mathematics methods courses? The findings are structured to first present an integrated view of Susan's practice in the form of a vignette. Based on the vignette and with example from our critical friend conversations we highlight each component of scaffolding, beginning with layering of instructional activities and turning to the move toward independence.

Susan's Vignette

One learning goal in Susan's elementary mathematics methods course was to understand cognitive demand of tasks to support PTs' lesson planning. She reasoned that distinguishing between demands of tasks, such as those focused on producing answers and those focused on sense-making, would support PTs' design of problem-based lessons. Susan planned for a discussion on cognitive demand of mathematics tasks for the third class.

The day before Susan's first class, Alyson identified the importance of anticipating in planning for discussions of teaching, just as we would for teaching mathematics.

> Alyson: So you are seeing these different perspectives that might come out from the PTs. Would it help if you thought through those different perspectives, and have some ideas of what you might want to draw out or probe a little more deeply? . . . It's like anticipating different directions that the discussion might go and being ok with different outcomes based on what they're bringing to the table. (Conversation 08-31-2020)

As a result, Susan considered how her first day activities, including eliciting PTs' memories of mathematics tasks, might inform the planned whole class discussion of cognitive demand (Table 1, Class 3). How might PTs associate tasks with children's mathematical thinking? Susan anticipated experiences PTs might share, and purposely planned instructional activities for the first two classes (Table 1) and the autobiography assignment to elicit PTs' ideas about tasks and mathematical thinking. As planned, PTs' shared experiences and perspectives on types of thinking involved in learning mathematics. Some were expected, such as timed facts tests associated with memorization. Others were unexpected necessitating modifications to support connections between PTs' experiences with mathematics tasks and mathematics learning.

Table 1: Excerpts from Evidentiary Map

Date	Event	Description
Sep 1	Class 1 Online	Think-Pair-Share: What are your memories of learning mathematics? First assignment: mathematics autobiography due two days later.
Sep 3	Class 2 In-Person	Small group activity: Examine K-5 mathematics curriculum standards. List verbs; classify levels and types of thinking.
Sep 15	Class 3 Online	From memories: identify thinking involved in learning mathematics. Introduce the Cognitive Demand framework (Smith et al., 1998). Small groups: examine tasks using the framework; whole class: share a claim about a task and *discuss level of cognitive demand*.
Sep 17	Class 4 In-Person	Reviewed verbs from Class 2 and mathematics tasks from Class 3. Small groups: What helps determine if a task involves lower- or higher- cognitive demand? Individual reflection prompt.
Sep 22	Class 5 Online	Revisit tasks from Class 3: Describe cognitive demand. Follow-up: choose a task and make a video explaining the cognitive demand.
Sep 23 – Nov 24	Individual Conferences	Consultations with PTs on lesson plans with attention to cognitive demand of tasks.

During Class 3 Susan initiated a discussion on cognitive demand of tasks by asking groups of PTs to make claims about a small collection of tasks. She knew this was risky given the limited time for the development of phenomenological characteristics (Dillon, 1994) needed for productive discussions. PTs shared claims about the cognitive demand of different tasks, but little evidence was offered. No PT challenged another PT's claim, a move that might have sparked a discussion. Instead, the PTs took turns sharing. Susan described the class activity as a recitation and turned to redesigning instructional activities for Class 4.

Our conversations focused on benefits and limitations of introducing PTs to professional language such as "cognitive demand." Alyson described the importance of introducing professional vocabulary, like cognitive demand, to support PTs' descriptions of teaching.

> Alyson: I still think that we share some of the best practices in different ways. As part of building that common experience and building opportunities for [PTs] to have language like the cognitive demand idea puts words to descriptions that they can then think about . . . I guess I feel like we are spending a lot of time developing that language.
>
> Susan: This whole thing about language and cognitive demand, . . . I don't think they [PTs] have thought about it [cognitive demand] even though I know they are introduced to levels of thinking [in previous classes] but when they come to math methods it's [cognitive demand of tasks] not connected to that [prior experience with levels of thinking]. I'm trying to connect to language other classes have been using. (Conversation, 09-21-20)

Susan describes her effort to connect language from other courses to cognitive demand.

Susan re-designed her instructional activity for Class 4 in response to PTs' association of mathematical tasks and cognitive demand. Susan had hoped to create a discussion of cognitive demand during Class 3 that would connect cognitive demand of tasks and student thinking. Instead PTs' shared tasks with rationales that exhibited limited understanding of the elements of cognitive demand. Susan adapted her next instructional activity to provide PTs with opportunities to explicitly describe task characteristics in relation to specific elements of cognitive demand.

Susan's initial lessons were designed to gather evidence of PTs' experiences and views of teaching and learning mathematics. Subsequent instructional activities were designed to support PTs to connect their experiences, cognitive demand of tasks, and types of thinking the tasks might elicit from children. Although Susan planned a discussion of cognitive demand of mathematics tasks for Class 3, the PTs had not yet developed connections between their experiences, demand of tasks, and opportunities for children's mathematical thinking. The PTs' responses allowed Susan to make sense of their ideas about cognitive demand. Susan used the PTs' ideas to re-design subsequent instructional activities to focus on connecting cognitive demand and task characteristics (Table 1). Susan did not intend the Class 3 discussion of cognitive demand to result in associating a level of cognitive demand for each task, but to relate levels of cognitive demand to task features and opportunities for children's mathematical thinking. Susan viewed these connections as essential in PTs' lesson planning later in the semester.

Moving Toward Independence

Our conversations about how to provide learning opportunities to support discussion focused on the importance of modeling PTs' ideas and designing instructional activities that would prepare PTs for planned discussions. We focused on developing meanings for key terms in the discussion questions and the significance of such questions in learning to teach. We reasoned from our experiences teaching mathematics, that we could use models of PTs' ideas about teaching (cognitive demand, learning, and social justice) to design instructional activities. Further, using PTs' ideas, we could facilitate the development of connections among ideas about teaching mathematics. This idea is illustrated as Signe describes her movement to using PTs' vocabulary, "way of talking," in instructional activities and discussions.

> Signe: When I finally did use the PTs' way of talking, they had something to say, and they knew they were going to be attended to when they were talking about it, because it was a significant idea. And, it matters when you're trying to do your planning for discussion. (Conversation 11-10-2020)

As we supported PTs to link their "way of talking" to key terms in mathematics teaching, such as cognitive demand in Susan's case, we noticed our assumptions about discussions and PTs' move toward independence. Signe described the need to move on to teaching other concepts, but our awareness that PTs would need to apply learned concepts in their teaching remained.

> Signe: For a while, it [conceptual understanding] was kind of a mystery, like they kind of used some of the words that I used ... but then the rubber meets the road and you get to the final discussion. . . . only that's not the final word, because I have their lesson plans, I have their concept summary . . . So there's not an end, because they're going to do more stuff with that idea, but I'm not going to focus on it anymore. I felt like I got to where I needed to get to, to be able to help them with facilitation of their lessons. (Conversation 11-10-2020)

Our conversations continued to focus on how layers of instructional activity related to planned discussion questions and supporting PTs' planning and teaching of mathematics lessons.

Layering to Support Discussions

Central in our conversations was how we could provide multiple opportunities for PTs to connect their knowledge and experience to key ideas during discussions of teaching. Susan's constructivist pedagogy informed her design and re-design of layers of instructional activity to support PTs' ideas about cognitive demand. Susan provided opportunities for PTs to create ideas about cognitive demand of mathematics tasks and opportunities for learners of mathematics from PTs' experiences. As she created models of PTs' ideas about cognitive demand, Susan designed layers of activity for PTs' to use their ideas in examining mathematics tasks.

Susan, Signe, and Alyson began the semester with dates for discussions on the syllabus. As these dates came, all three MTEs identified the need to build understanding of the concepts so that PTs could meaningfully engage in discussions of the concepts. Our conversations revealed how initial target dates for discussions were shifted back repeatedly to accommodate layering. Signe described trying to find the parameters for the discussion and understandings needed.

> Signe: The first stage of a discussion is to understand everyone's experience and futz around with the parameters of the discussion. Maybe I can't have 'where does knowledge for mathematics teaching come from' until they go into the field. But I can 'have how do students learn math' before that because they've been working on it.
>
> Alyson: moving that to the end, you'll have more shared experience to have that conversation, by the end of the semester. (Conversation 09-07-2020)

PTs needed time to build their ideas about key concepts through experiences with layers of instructional activities provided. We realized that often we planned whole class discussions too early before the PTs had developed ideas about teaching. Our premature rush to discuss as in Susan's Class 3 effort to connect experiences, task characteristics, cognitive demand, and children's mathematical thinking, positioned PTs as experts with knowledge of pedagogical concepts and ways to describe that knowledge. As Susan's evidentiary map illustrates, layers of instructional activities are created as MTEs' gather evidence of PTs sense making and re-design subsequent activities. These layers provide opportunities for PTs, but also suggest that MTEs' discussion practice is informed by the interpretation of PTs' pedagogical concepts.

Discussion

Findings confirm that components of whole-class scaffolding (Bakker et al., 2015; Smit et al., 2013), including movement toward independence and layering of instructional activities, informed our discussion practice. The significance of this finding lies in the potential of MTE theories (Casey et al., 2018), professional experiences (Leiken, 2020), and practices (Grossman et al., 2009) to shape the "learning potential" (p. 2090) of instructional activities. Explicit knowing of theories that inform discussion practice expands the possibility space (Brown & Coles, 2020) for MTE's decisions in planning and facilitating discussions of teaching. Findings from self-study of teaching provide evidence beyond descriptions of practice that serve as models of scholarly practice (Lee & Mewborn, 2009) to identify and describe factors that influence MTE's professional growth (Krainer et al., 2021).

Existing research exploring teacher discussions of practice provides evidence of the challenges MTEs' face in facilitating discussions. Of significance is whether (Dick et al, 2018) and how (Steele, 2005) teachers support claims about teaching during whole class discussions. This illustrates a central challenge in facilitating discussions where talk moves beyond sharing ideas to the consideration of those ideas by others who add to or counter them. Susan's effort to initiate a discussion of cognitive demand illustrates how two interrelated components of whole-class scaffolding informed her discussion practices. First, efforts to engage the class in discussions of cognitive demand were informed by the need for PTs to plan and teach lessons at the end of the term. In the practical work of methods teaching, dates for field experience and practicum are fixed in the schedule. PTs must be able to function independently when those dates arrive. This practical consideration in methods teaching informs decisions about when PTs must use ideas about teaching relevant to planning mathematics lessons. Susan knew that she needed to prepare the PTs to plan lessons (Table 1, Individual Conferences). Second and related to this practical press toward independence, is the layering of instructional activities as informed by evidence of PTs' pedagogical concepts. Susan knew almost immediately during the discussion of tasks and cognitive demand in Class 3 that a discussion would not be possible. PTs did as she asked and shared tasks and levels of cognitive demand, but there was no adding on or countering of PTs' claims. The evidence of PTs' sense making informed a redesign of instructional activities for Class 4 (Table 1) to provide additional opportunities for the development of connections between tasks, cognitive demand, and opportunities for children's mathematics thinking. Susan's discussion practice illustrates the interconnected nature of components of MTE discussion practice informed by whole-class scaffolding. Movement toward independence drives the development of instructional activities, yet evidence of PTs' thinking encourages MTEs to redesign instructional activities creating layers that provide opportunities for PTs to develop new versions of pedagogical concepts like cognitive demand of mathematics tasks.

Findings from our self-study support the claim that characteristics of whole-class scaffolding, including move toward independence and layering of instructional activities, were embedded in our planning for and facilitation of discussions. We do not claim that we use these components in particular ways or at particular times in our discussion practice. In addition, we do not make claims about coordination of whole-class scaffolding and other theories (e.g., Kitchen, 2005; Steffe & D'Ambrosio, 1995) used explicitly in planning for and implementing discussions of teaching. Instead, we claim that in the development of discussion practice, we attended to the practical work of teaching about teaching by providing opportunities for PTs' to learn about teaching. Additional work is needed to address how MTEs' use theories in concert in planning for and facilitating whole class discussions. For example, how do Susan's explicit theories of relational teacher education (Kitchen, 2005) and constructivist teaching (Kastberg, 2014; Steffe & D'Ambrosio, 1995) support and inform her development of instructional activities that would support PTs' concept of cognitive demand? Questions like this assume that theories are used in concert, but perhaps some theories or components of theories are used for planning, while others are used to facilitate PTs' discussions of teaching.

The two components of whole-class scaffolding, move toward independence and layering instructional activities, illustrate one tacit theory which informed our discussion practice. We claim these components contribute to the integrative concept of whole-class scaffolding (Bakker et al., 2015) that helps us maintain the complementarity between teaching and learning (Loughran, 2014) while informing our discussion practice. Structured this way, MTEs' discussion practice is informed by components of whole-class scaffolding (Bakker et al., 2015). Although we do not claim that all MTEs build discussions on a theory of whole-class scaffolding, we do claim that unpacking explicit and tacit theories that inform MTEs' discussion practice will contribute new knowledge of MTE growth of professional practice called for by Krainer et al. (2021). Findings from such studies have the potential to support MTEs beyond providing models of instructional activities described in studies of professional practice (Tyminski et al., 2021) to understanding the diversity of knowledge and experience that drives the decisions involved in professional practices. Such findings address in part how MTEs' tacit theories play a key role in the "learning potential" (Grossman et al., 2009, p. 2089) of instructional activities used in professional practice, while providing evidence of the "integration between knowledge and practice" (Krainer et al., 2021, p. S11).

References

Alexander, R. (2019). Whose discourse? Dialogic pedagogy for a post-truth world. *Dialogic Pedagogy: An International Online Journal, 7.* doi: 10.5195/dpj.2019.268

Bakker, A., Smit, J., & Wegerif, R. (2015). Scaffolding and dialogic teaching in mathematics education: Introduction and review. *ZDM, 47*(7), 1047-1065.

Beswick, K., & Goos, M. (2018). Mathematics teacher educator knowledge: What do we know and where to from here? *Journal of Mathematics Teacher Education, 21*(5), 417–427. doi:10.1007/s10857-018-9416-4

Borko, H., Liston, D., & Whitcomb, J. (2007). Genres of empirical research in teacher education. *Journal of Teacher Education, 58*(1), 3–11.

Brown, L., & Coles, A. (2020). Theorising theorising: About mathematics teachers' and mathematics teacher educators' energetic learning. In K. Beswick & O. Chapman (Eds.) *International Handbook of Mathematics Teacher Education: Volume 4* (pp. 81-102). Brill.

Casey, S., Fox, R., & Lischka, A. E. (2018). Interpretations and uses of classroom video in teacher education: Comparisons across three perspectives. In S. Kastberg, A. Tyminski, A. E. Lischka, & W. B. Sanchez (Eds.) *Building Support for Scholarly Practices in Mathematics* Methods (pp. 297 – 310). Information Age Publishing.

Dick, L., Sztajn, P., White, T., & Heck, D. (2018). Investigating sociopedagogical norms: Teachers' discussions about own and others' instruction. *Teaching and Teacher Education, 71*, 297-307.

Dillon, J. T. (1994). *Using discussions in classrooms*. Open University Press.

Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. *Journal of Mathematics Teacher Education, 19*(1), 79-103.

Grant, M. R., & Lincoln, Y. (2021). A conversation about rethinking criteria for qualitative and interpretive research: Quality as trustworthiness. *Journal of Urban Mathematics Education, 14*(2), 1-15.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers' College Record, 111*(9), 2055-2100.

Guilfoyle, K., Hamilton, M. L., Pinnegar, S., & Placier, P. (2007). The epistemological dimensions and dynamics of professional dialogue in self-study. In J. J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.) *International Handbook of Self-Study of Teaching and Teacher Education Practices* (pp. 1109-1168). Springer.

Jordan, B., & Henderson, A. (1995). Interaction analysis: Foundations and practice. *The Journal of the Learning Sciences, 4*(1), 39-103.

Kastberg, S. (2014). The power of what we know: Further directions for exploring constructivist model building. *Constructivist Foundations, 9*(3), 352-354.

Kastberg, S., Lishcka, A., & Hillman, S. (2019). Exploring mathematics teacher educator questioning as a relational practice: Acknowledging imbalances. *Studying Teacher Education, 15(1), 67-81.*

Kastberg, S. E., Lischka. A. E., & Hillman, S. L. (2020). Weaving Discussions with Questioning. In C. Edge, A. Standerford, & B. Bergh (Eds.), *Textiles and Tapestries: Self-Study for Envisioning New Ways of Knowing* (pp 623-632). EdTech Books. Retrieved from https://edtechbooks.org/textiles_tapestries_self_study/weaving_discussions_

Kastberg, S., Lischka, A, & Hillman, S. (2021). Revealing the pedagogy of discussion in mathematics methods. In Olanoff, D., Johnson, K., & Spitzer, S. M. (Eds). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 1494-1499) Philadelphia, PA.

Kastberg, S., Tyminski, A., Lischka, A., & Sanchez, W. (2018). *Building support for scholarly practices in mathematics methods*. Information Age Publishing.

Kazemi, E., Ghousseini, H., Cunard, A., & Turrou, A. C. (2016). Getting inside rehearsals: Insights from teacher educators to support work on complex practice. Journal of Teacher Education, 67(1), 18-31.

Kim, M. Y., & Wilkinson, I. A. (2019). What is dialogic teaching? Constructing, deconstructing, and reconstructing a pedagogy of classroom talk. Learning, Culture and Social Interaction, 21, 70-86.

Kitchen, J. (2005). Looking backward, moving forward: Understanding my narrative as a teacher educator. *Studying Teacher Education, 1(*1), 17-30. doi 10.1080/17425960500039835

Krainer, K., Even, R., Park Rogers, M., & Berry, A. (2021). Research on learners and teachers of mathematics and science: Forerunners to a focus on teacher educator professional growth. *International Journal of Science and Mathematics Education, 19(*1), 1-19. doi.10.1007/s10763-021-10189-8

LaBoskey, V. K. (2007). The methodology of self-study and its theoretical underpinnings. In J. Loughran, M. L. Hamilton, V. K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp. 817–869). Springer.

Lee, H. S., & Mewborn, D. S. (2009). Mathematics teacher educators engaging in scholarly practices and inquiry. In D. S. Mewborn & H. S. Lee (Eds.), *Scholarly practices and inquiry in the preparation of mathematics teachers,* (pp. 1-6). AMTE.

Leikin, R. (2020). How far is the horizon? Teacher educators' knowledge and skills for teaching high school mathematics teachers. In K. Beswick & O. Chapman (Eds.) *International handbook of mathematics teacher education: Volume 4* (pp. 81-102). Brill.

Lischka, A. E., Hillman, S. L., & Kastberg, S. E. (2021). Seeking Understanding of sustaining discussions in mathematics methods through self-study of teacher education practices. *Paper presented at American Education Research Association Conference,* April 2021.

Loughran, J. (2014). Professionally developing as a teacher educator. *Journal of Teacher Education, 65*(4), 271-283.

McDuffie, A. R., & Slavit, D. (2002). Utilizing online discussion to support reflection and challenge beliefs in elementary mathematics methods classrooms. *Contemporary Issues in Technology and Teacher Education, 2*(4), 447-465.

Mewborn, D. S., & Stanulis, R. N. (2000). Making the tacit explicit: Teacher educators' values and practices in a co-reform teacher education program. *Teacher Education Quarterly*, 5-22.

Munby, H., & Russell, T. (1994). The authority of experience in learning to teach: Messages from a physics methods class. *Journal of teacher Education, 45*(2), 86-95.

Pinnegar, S., & Hamilton, M. L. (2009). *Self-study of practice as a genre of qualitative research: Theory, methodology, and practice (Vol. 8).* Springer Science & Business Media.

Saldana, J. (2016). *The coding manual for qualitative researchers,* 3rd edition. Sage Publications.

Schuck, S. & Russell, T. (2005). Self-study, critical friendship, and the complexities of teacher education. *Studying Teacher Education, 1*(2), 107-121.

Simon, M. (2008). The challenge of mathematics teacher education in an era of mathematics education reform. In B. Jaworski & T. Wood (Eds.) *International handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (pp. 17-29). Sense.

Smit, J., AA van Eerde, H., & Bakker, A. (2013). A conceptualisation of whole-class scaffolding. *British Educational Research Journal, 39*(5), 817-834.

Steele, M. D. (2005). Comparing knowledge bases and reasoning structures in discussions of mathematics and pedagogy. *Journal of Mathematics Teacher Education, 8(*4), 291-328.

Steffe, L. P., & D'Ambrosio, B. S. (1995). Toward a working model of constructivist teaching: A reaction to Simon. *Journal for Research in Mathematics Education, 26(*2), 146-159.

Teuscher, D., Switzer, J. M., & Morwood, T. (2016). Unpacking the practice of probing student thinking. *Mathematics Teacher Educator, 5*(1), 47-64.

Tyminski, A. M., Simpson, A. J., Land, T. J., Drake, C., & Dede, E. (2021). Prospective elementary mathematics teachers' noticing of children's mathematics: A focus on extending moves. *Journal of Mathematics Teacher Education, 24(*6), 533-561. doi.10.1007/s10857-020-09472-2

van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education, 65*(4), 340-356.

Visnovska, J., & Cobb, P. (2015). Learning about whole-class scaffolding from a teacher professional development study. *ZDM, 47*(7), 1133-1145.

Whitehead, J. (1989). Creating a living educational theory from questions of the kind, 'How do I improve my practice?' *Cambridge Journal of Education, 19*(1), 41-52.

PROFESSIONAL LEARNING AT SCALE: DESIGNING A BOUNDARY OBJECT

Allison W. McCulloch
UNC Charlotte
allison.mcculloch@uncc.edu

Kathrine J. Mawhinney
Appalachian State University
mawhinneykj@appstate.edu

Cathy Holl-Cross
UNC Charlotte
chollcro@uncc.edu

P. Holt Wilson
UNC Greensboro
phwilson@uncg.edu

F. Paul Wonsavage
University of Florida
wonsavagef@ufl.edu

Education policies and innovations that aim to improve instructional quality often fail to produce any meaningful or sustained changes to teaching when implemented at scale because of the significant learning demands they place on the individuals, groups, and organizations that comprise an educational system. In this paper, we describe an implementation resource developed to promote professional learning and cross role discussions about new state mathematics standards and report on the ways educators at different levels of the state system used them. Results demonstrate how implementation resources designed to be a boundary object for educators at multiple levels of an educational system have the potential to support learning and create systemic conditions conducive of change.

Keywords: Policy, Systemic Change, Standards, Professional Development

In the US, education legislation, innovations, or policies aimed at improving instruction and student learning at scale have seldom led to even modest lasting improvements to teaching (Coburn et al., 2016). Critical scholars, policy researchers, and implementation scientists have offered various explanations for why large-scale educational reform remains elusive, one of the main ones being that it requires significant individual and organizational learning (Fullan & Pomfret, 1977). Initiatives are often implemented quickly, ignoring the time, structures, and resources needed to support the significant shifts required of teachers and teacher leaders. When working with smaller scale initiatives, teachers, curriculum leaders and/or mathematics teacher educators can come together and engage in professional learning experiences over time and in context. In contrast, when working with sweeping changes at scale we need to consider what structures and resources can support professional learning across all roles in ways that are aligned with what we know about effective professional learning experiences (PD).

In the context of this study, our focus was on the state-wide implementation of new high school mathematics standards. The State Board of Education adopted the standards in June with implementation expected in August. In response, our partnership of state and district leaders, mathematics education researchers, and classroom teachers quickly went to work designing structures and resources for state-wide, cross-role, professional learning experiences to support all stakeholders during the implementation. One of the key messages that accompanied the roll-out of the new standards was that they were based on research on teaching and learning. Through formal and informal feedback, stakeholders expressed that they wanted to know more about the research in which the standards were grounded. To address this need, a new set of resources were created and included in a multi-pronged professional learning structure—a collection of 20 two-page Research-Practice Briefs (R-P Briefs). These R-P Briefs were widely accessed and referenced by our stakeholders which left us wondering how the purposeful design of the R-P

Briefs contributed to their widespread use and in what ways the design features supported our stakeholders as they implemented the new standards.

Background

Much research has been done concerning the core features of effective professional learning experiences for teachers (e.g., Wei et al., 2009). Whether in person or online, effective PD provides opportunities that are responsive to the needs of participants, of sufficient duration, content focused, include active learning, driven by teachers' work with students, while also being focused around communities of practice and connected to solving a problem of practice (e.g., Desminone, 2009; Mantranga & Silverman, 2020; Wei et al., 2009). Implicit in these descriptions is that effective PD needs to be guided by research on teaching and learning (Loucks-Horsey et al., 2009). For example, PD focused on supporting students' developing number sense might be grounded in research on cognitively guided instruction (e.g., Carpenter et al., 2015) and learning trajectories (e.g., Clements & Sarama, 2009). While PD focused on supporting teachers' facilitation of whole class mathematical discussions might be grounded in the research on noticing mathematically significant pedagogical opportunities (Leatham et al., 2015; Stockero & VanZoest, 2013). Further, participants benefit from opportunities to interact with research through discussion, productive debate, and social interactions because research and its use are a social process (e.g., Nutley et al., 2007; Tseng, 2012). However, engaging with research in these ways is challenging as there are many barriers that make such engagement difficult to bring to fruition, especially in the context of addressing fast paced initiatives. Such barriers include addressing structural issues of access, time, and designing for engagement.

One of the most basic barriers to interacting around research is access (e.g., Hemsley-Brown & Sharp, 2003; Shkedi,1998). Since much research is hidden behind paywalls and protected by copyright laws, addressing a state-wide interaction around research is complicated by the reality that many stakeholders cannot access it (Shkedi, 1998). Even if it was accessible, there is the reality of the time it would take to read and make sense of primary resources (e.g., Behrstock et al., 2009; Hemsley-Brown & Sharp, 2003). It is widely acknowledged that the current structure of teachers' work day does not provide sufficient time for the everyday work of teaching, let alone time for professional learning. To address these concerns, researchers have created open-access research briefs as a way to disseminate research (Anderson et al., 2019); the idea being to present research in a format that is widely accessible, synthesized around a narrow topic, and easy to read in a short amount of time. The potential of this format to address accessibility led us to wonder how we might leverage it as a resource to support professional learning. To do so would not only require that we align its contents with the features of effective PD, but also that we consider how its design might encourage and support interactions around it.

Context & Theoretical Perspectives

This study took place in the context of a statewide research-practice partnership (Penuel et al., 2015) that focused on a shared problem of practice − improving the implementation process for new high school mathematics standards. Our first goal was to co-design and study a statewide professional learning initiative to support implementation efforts. Our co-design efforts follow from a theoretical perspective of communities of practice (Lave & Wenger, 1991; Wenger, 1998). Because practice is a defining characteristic of a community, communities are formed by collaboratively engaging with common resources toward a common goal. Though boundaries of practice distinguish communities across the social landscape, they are also a source of new learning. Boundary encounters allow for members of distinct communities to jointly negotiate meaning around boundary objects—artifacts that carry meaning in multiple communities and support the coordination of practices across them (Star & Griesemer, 1989; Wenger, 1998).

Many resources were co-developed in support of the partnership initiative and were intended to act as boundary objects. These included (but were not limited to) online PD modules, instructional frameworks, and the collection of R-P Briefs. Our hope was that each of the resources would become a boundary object, in that they would support knowledge exchange across the many different communities in our partnership (e.g., teachers, math leaders). The design principles for the R-P Briefs were initially informed by the literature on effective PD and refined over time through feedback from the statewide co-design network via surveys and focus group interviews. Ultimately, 20 R-P Briefs were developed, one for each unit of instruction across three different high school mathematics courses. Each R-P Brief was exactly 2 pages long and included: explanation of vertical alignment of the mathematics in the unit (both within the course and across courses), description of why the mathematics concepts included are important, at least one example task, research on students' mathematical thinking and learning related to the big mathematical ideas in the unit, research on effective pedagogy specific to the unit, explicit attention to connections to the Standards for Mathematical Practice (SMPs), and discussion questions to consider with colleagues.

The R-P Briefs were disseminated within online PD modules, on the partnership's website, and in many state, regional, and local meetings (e.g., sessions at state affiliate NCTM conferences). The number of downloads for each R-P Brief at the time of this study from just the online PD modules, are shown in Table 1. The download data, coupled with the other ways we were aware they were being accessed, indicated that the briefs were widely used, which prompted us to wonder if our intentional design played out like we expected—to support teacher and teacher leader learning about the standards themselves and the research that informed them, as well as their implementation of the standards.

Table 1: Number of Canvas Downloads as of July 2018

Math 1 Briefs (n=6)	Number of Downloads	Math 2 Briefs (n=6)	Number of Downloads	Math 3 Briefs (n=8)	Number of Downloads
1.1	416	2.1	593	3.1	693
1.2	455	2.2	105	3.2	152
1.3	111	2.3	302	3.3	212
1.4	82	2.4	84	3.4	72
1.5	262	2.5	30	3.5	218
1.6	68	2.6	9	3.6	45
				3.7	84
				3.8	78
Total	1394		1123		1554

Methods

This study used survey methods to build an understanding of how the R-P Briefs supported teacher learning about the implementation of new mathematics standards. To that end we aimed to address the following research questions: What aspects of the R-P Briefs do math teachers and math leaders say are helpful?; How do math teachers and math leaders say they use the R-P Briefs?; and What actions do math teachers and math leaders that use the R-P Briefs say they take as a result of reading them?

The partnership developed and administered a survey in Spring 2018 to inform its ongoing efforts to support standards implementation. Respondents were assigned to sets of questions specifically addressing the partnership's implementation resources and supports developed for their grade band. In this report, we focus on the set of questions addressing the R-P Briefs which was assigned to those identifying they worked in the high school grade bands.

The survey was distributed through the state agency's listservs to approximately 20,000 mathematics teachers, school administrators, and district mathematics leaders. A total of 1,768 educators from 96% of the state school districts accessed and completed at least 80% of the survey. Here, we report on responses from the 346 educators from 85 of 115 school districts who had an opportunity to respond to the set of questions focused on the R-P Briefs. The questions in this block asked participants if they were aware of the R-P Briefs. The numbers were about evenly split with 184 people (53%) said they were aware and 162 (47%) said they were not aware. The remaining analysis is based on the 184 people who were aware of the R-P Briefs. This includes responses to three survey items. The first two asked about how participants used the R-P Briefs and which aspects of them they found helpful. These were both "select all that apply" items. The third was an open response item that asked, "What are some of the actions you have taken (if any) after engaging in the research-practice briefs?"

The responses to the first two questions were analyzed by determining frequencies and percentages for each response option and then disaggregated by role group (e.g., teacher, math leader–i.e., school-based coaches, district curriculum leaders, school administrators). To analyze the responses to the open-ended, all members of the research team first read all of the responses to become familiar with their contents, making memos of common themes that emerged. This process and the ensuing team discussion led to the generation of five codes that captured the range of participants' actions with the R-P Briefs. Using our set of codes (e.g., planning, learning/reflection, shared with others, lead PD, guide curriculum decisions), each team member individually coded the participants' responses. Any coding disagreements were reconciled through group discussion. In the following sections we report the results of our analysis with respect to each of the research questions.

Findings

Useful Features of the R-P Briefs

The survey question that asked about the usefulness of particular features of the R-P Briefs was multiple response in format with answer choices aligned with the 7 design principles for the R-P Briefs described above. The design features that were selected as useful by more than half of the respondents were vertical alignment and tasks (51% and 53% respectively), with the description of why the topic is important being the least useful (9%). Disaggregating the data, revealed teachers and math leaders find some features similarly helpful, but respond quite differently to others. For example, while 48% of math leaders indicated that the discussion questions to consider with colleagues were useful, only 13% of the teachers selected this feature. While not as dichotomous, there are similar results for the research features and the SMPs. Such results suggest that usefulness of features may be aligned with how they are connected to one's daily work responsibilities as teachers and math leaders, while both are responsible for the implementation of the standards, they each have different roles in the implementation process.

Table 2: I find the following aspects of the R-P briefs useful [Select all that apply]

	All Participants (n=184)	Teachers (n=138)	Math Leaders (n=46)
Vertical alignment (grade/course distinctions)	51%	47%	63%
Description of why a topic is important	9%	9%	7%

Tasks/Examples	53%	52%	57%
Discussion questions to consider with colleagues	22%	13%	48%
Research on students' mathematical thinking and learning	29%	22%	52%
Research on effective mathematics pedagogy	27%	17%	57%
Explicit attention to the Standards of Mathematical Practice	27%	23%	37%
Other	2%	2%	0%

How the R-P Briefs are Being Used

The survey question that asked about how the R-P Briefs are being used was multiple response in format with answer choices that were aligned with not only the design elements of the R-P Briefs, but also the intention of sharing related research on teaching and learning in a usable way (See Table 3). High response rates on learning more about the mathematics of courses taught (54%), supporting instructional decisions (43%), and gaining more resources for instruction (57%) speak to the immediate needs of teachers in their daily work, but also indicate a valuing of what research says about teaching and learning by the teacher respondents. Math leaders also found the R-P Briefs useful for instructional decision making and resources (43% and 59%, respectively). This shows that as a boundary object, the two communities found common purpose for the R-P Briefs in addressing the overlapping aspects of the work of the two communities—supporting and implementing mathematics instruction.

Table 3: I use the R-P Briefs to … [Select all that apply]

	All Participants (n=184)	Teachers (n=138)	Math Leaders (n=46)
Learn about content for the math courses I teach	49%	54%	35%
Learn about pedagogy for the math courses I teach	36%	36%	35%
Learn about vertical alignment from previous math courses	42%	46%	30%
Participate in discussions with colleagues in informal settings	19%	15%	30%
Participate in discussions with colleagues in formal settings	32%	25%	52%
Support my instructional decisions	43%	43%	43%
Share information with my principal, colleagues, or district level personnel	16%	8%	41%
Get instructional resources	58%	57%	59%
I am familiar with the [blinded] briefs, but I do not find them useful	4%	5%	0%
Item left Blank	21%	21%	20%

Stark differences in the ways these two communities (i.e., teachers and math leaders) say they use the R-P Briefs appear in the collaboration-related response choices and vertical alignment. While some teachers used the R-P Briefs in formal or informal discussions with colleagues, a larger proportion of math leaders responded with the use of the R-P Briefs in this way. In particular over half of the math leaders responded that they use the R-P Briefs in formal settings. Further, only 8% of teachers indicated that they shared the tool with colleagues, compared with 41% of math leaders who reported that they had shared the R-P Briefs. Another distinction in how the tool was used by the different communities appeared when considering the learning about mathematics content choice alongside the vertical alignment choice. These response options speak to the usefulness of the R-P Briefs for teachers in situating these new courses within the K-12 continuum, an aspect also valued by math leaders but to a lesser extent.

Actions Taken as a Result of Engaging with the R-P Briefs

Of the 184 people that responded to the R-P Briefs question block, 107 of them (76 classroom teachers, 31 math leaders) responded to the open-ended item asking them to describe some of the actions they have taken (if any) after engaging with the R-P briefs. The results suggest that actions tended to be related to classroom level lesson planning, personal learning or reflection, sharing the R-P Briefs with others, using them when leading professional development, and using them to guide school or district-level curriculum decisions (see Table 4). In essence, this question ended up providing an opportunity for people to provide further detail regarding how they are using the R-P Briefs.

Table 4: Percent of Participants that Noted an Action by Theme

	Planning	Learning and/or Reflection	Shared with Others	Lead PD	Guide Curriculum Decisions
Teachers	70%	37%	20%	3%	4%
Math Leaders	1%	23%	52%	26%	13%
Total	52%	33%	29%	~1%	~1%

The most common actions described by teachers were different from those described by math leaders. Teachers overwhelmingly noted that they used what they gleaned from the R-P Briefs to inform their lesson planning and/or to reflect on their own learning. In contrast, math leaders most commonly noted they physically shared the R-P Briefs with others and/or used them when leading professional development. In the sections that follow we provide further description of the actions described in each of these categories.

Classroom level lesson planning. As we saw earlier, many respondents noted they used the R-P Briefs to support their instructional decisions and to get instructional resources. Consistent with that finding, almost three-quarters of the teachers that responded to this item described the ways they adjusted their classroom level lesson plans based on what they learned from the R-P Briefs. For example, T47 noted “I adjusted my lessons based on the research”, and T49 wrote “I have implemented some of the tasks suggested”. One of the most common planning actions was related to adjusting instructional emphasis based on the vertical alignment feature. For example, T22 wrote “Used for adjusting my instruction to better meet the curriculum without going too far”, T33 wrote, “Tweaked emphasis on units, placing less emphasis on some topics and increasing others.”, and T74 wrote, “Honing in on the exact type problems my students will have to pass. In the past it has been too broad and this helps streamline materials.” In fact, with the exception of the “discuss with your colleagues” feature, all of the R-P Brief features emerged as being used to inform classroom level planning in some way.

Personal learning or reflection. From the previous question, we know that teachers and math leaders used the R-P Briefs to learn about content, pedagogy, and vertical alignment. The open-ended responses that were focused on personal learning or reflection add to this picture because they highlighted participants’ attention to research on student thinking or pedagogy. For example, T16 notes using the R-P Briefs toward “Trying to better understand how students’ think”, and T43 noted “I have used these to gain a deeper understanding of what my students need to engage them in the mathematics outside of everyday calculations.” The most referenced aspect of the R-P Briefs was the vertical alignment of content. This seemed to support teachers and math leaders in understanding how the standards are situated both within and across courses. For example, T11 noted:

> For me, the briefs give me a better understanding of the depth of understanding the course is expected to achieve. In the past, I always went way too deep and students and parents would come back and say the student learned nothing in the next course or two that the learner took because too much had been covered. Knowing when to stop was a major challenge in the past. I can now see more clearly and confidently that concepts will be covered in due course and I did not have to feel responsible for getting the learners through too much material.

These responses indicate that not only did the teachers and math leaders go to the R-P Briefs with the intention to learn and reflect, but also that they found the features related to vertical alignment and research on teaching/learning to be the most impactful when they took this action.

Sharing the R-P Briefs with others. Across all roles, participants acted by sharing and discussing the R-P Briefs with others. There are examples of sharing and discussing the R-P Briefs with peers, administrators, and even parents. In some instances, it is unclear if sharing simply means giving the R-P Brief to someone else with no additional interaction (e.g., "Shared briefs with preservice teachers", T48). In other instances, it is clear that the sharing included interacting with each other around the R-P Brief in a meaningful way. For example, T46 responded, "I have had numerous planning meetings with the math 2/3 teacher at our small school to ensure we are using similar research based practices for teaching." and TL87 wrote, "Sharing information from the briefs with administrators and teachers. Encouraging use in Math PLCs." These responses give us a sense of how the R-P Briefs might land in the hands of people that might not have otherwise seen them and they illustrate how various communities (e.g., departments, professional learning teams, friendly colleagues) might be interacting around them.

Leading professional development. While we know the R-P Briefs are being shared and discussed, there is also evidence that math leaders are using them in their professional development work. This includes both formal professional development sessions and one-on-one coaching. For example, TL96 noted, "I have used them in PD with my district to try to get buy-in for using tasks on a regular basis." While TL103 explained, "I use them in facilitating discussion with the teachers I coach." These actions suggest that the briefs enabled connections between communities and were helpful in making meaning across boundaries.

Guide school or district level curriculum decisions. Among the many ways that participants said they acted on the R-P Briefs were actions to guide both school level and district level curriculum decisions. For example, TL85 noted,

> The briefs were used in each Math 1 and Math 2 Professional Learning session to discuss the vertical alignment of the mathematics and to understand the focus of the content for the unit for the course. The briefs are a part of the Math 1 Unit Framework that is currently being developed in our district.

Similarly, TL92 responded "Curriculum decisions - pacing, order of content, etc. for district-level documents." These actions are different from classroom level planning noted above in that such decisions affect teachers and students across a system.

Discussion & Conclusion

This study was motivated by a need for designed structures and resources that supported professional learning related to the implementation of large-scale educational innovations. In our case, this was the implementation of new state math standards. Given that our theory of learning is grounded in communities of practice (Wenger, 1998), any resources we designed were intended to be boundary objects–spanning the boundaries of the multiple communities of stakeholders in the system. Our findings suggest that the R-P Briefs not only supported professional learning as intended, but also acted as a boundary object in ways beyond those for which it was originally designed.

Our design principles were informed by the research on effective PD. Findings suggest that all but one of these features was seen as useful to at least 20% of the respondents, the only exception being the explanation of the usefulness of the mathematical concept. This exception is likely explained by the fact that the participants already know why the content is useful (they are math folks). Thinking about the R-P Briefs as both a resource to support learning and as a boundary object, the findings shed interesting light. First, there is evidence that the R-P Briefs physically crossed community boundaries. They were shared and/or discussed in communities of teachers, communities of math leaders, communities of teachers and math leaders together in a school, and even communities of teachers, parents, and students in a single class.

Second, Star & Greisemer (1989) describe boundary objects as, “objects which are both plastic enough to adapt to local needs and constraints of the several parties employing them, yet robust enough to maintain a common identity across sites” (p. 393). Looking at the two communities of focus, teachers and math leaders, the findings indicate that some of the R-P Brief features were found equally useful within each community, suggesting that these features mean something to both. Yet there were other features that seemed more useful to one community than the other. This coupled with the finding that different communities are using the resources in different ways, suggest that an important design feature of the R-P Briefs was including content that was recognizable and meaningful to both communities yet presented in a way that it was malleable enough to be used by the different communities in different ways. Ultimately, we designed the R-P Briefs to be boundary objects, and the findings indicate that they in fact are.

For others considering how such a resource might support work in a different context, our findings suggest that in addition to designing based on what is known about effective PD, design should attend to the needs of the communities you hope to span. Specifically, design principles should include some features that are recognizable and important to multiple communities to support the development of a common language and an avenue for important boundary encounters, while also including features directly connected to the work of the individual communities. At the same time, we caution you to be careful about what is and is not included in the design and content. Findings here suggest that when a resource travels across boundaries in a large system, it develops power and could be used in ways that were not originally intended.

To date, over 10,000 R-P Briefs have been downloaded, and we know from the results here that people are accessing them in other ways as well. We have come to see the R-P Briefs as one example of a group of designs that our partnership developed to support systemic coherence in the implementation of new innovations. Like the briefs, other implementation resources were also grounded in research on teacher and student learning, instruction, and implementation. Collectively they provide access to safe professional learning opportunities and represent the expertise of a diverse set of educators within the system. Our hope is that findings from studies like this one can support others in designing implementation resources as boundary objects that support professional learning in the context of implementing educational innovations at scale.

Acknowledgements

This work was supported by the National Science Foundation under grant DRL-2100903 awarded UNC Greensboro, DRL - 2100947 awarded UNC Charlotte, and DRL - 2100833 awarded to Appalachian State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the National Science Foundation.

References

Anderson, K., Podkul, T., Singleton, C., & D'Angelo, C. (2019). Summative evaluation of the Research+Practice Collaboratory: Final Report. SRI Education. https://www.informalscience.org/sites/default/files/RPC%20FInal%20Summative%20Report%20%20Collaboratory.pdf

Behrstock, E., Drill, K., & Miller, S. (2009). Is the supply in demand? Exploring how, when, and why teachers use research. Learning Point Associates.

Carpenter, T., Fennema, E., Franke, M., Levi, L., & Empson, S. (2015). Children's Mathematics: Cognitively Guided Instruction, 2nd Edition. Heinemann.

Clements, D. H., & Sarama, J. (2014). Learning and teaching early math: The learning trajectories approach. Routledge.

Coburn, C. E., Hill, H. C., & Spillane, J. P. (2016). Alignment and accountability in policy design and implementation: The Common Core State Standards and implementation research. Educational Researcher, 45(4), 243-251.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. Educational Researcher, 38(3), 181–199. https://doi.org/10.3102/0013189X08331140

Edelson, D. C. (2002). Design research: What we learn when we engage in design. The Journal of the Learning Sciences, 11(1), 105-121.

Fullan, M., & Pomfret, A. (1977). Research on curriculum and instruction implementation. Review of educational research, 47(2), 335-397.

Hemsley-Brown, J., & Sharp, C. (2003). The use of research to improve professional practice: A systematic review of the literature. Oxford Review of Education, 29(4), 449-471.

Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. Cambridge University Press.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. Journal for Research in Mathematics Education, 46(1), 88–124.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Love, N., & Hewson, P. W. (2009). Designing professional development for teachers of science and mathematics. Corwin press.

Matranga, A. & Silverman, J. (2020). An emerging community in online mathematics teacher professional development: an interactional perspective. *Journal of Mathematics Teacher Education, 25(1),* 63–89. https://doi.org/10.1007/s10857-020-09480-2

Nutley, S. M., Walter, I., & Davies, H. T. (2007). Using evidence: How research can inform public services. Bristol, UK. Policy press.

Penuel, W. R., Allen, A. R., Coburn, C. E., & Farrell, C. (2015). Conceptualizing research–practice partnerships as joint work at boundaries. Journal of Education for Students Placed at Risk (JESPAR), 20(1-2), 182-197.

Shkedi, A. (1998). Teachers' attitudes towards research: A challenge for qualitative researchers. International Journal of Qualitative Studies in Education, 11(4), 559-577.

Star, S. L., & Griesemer, J. R. (1989). Institutional ecology,translations' and boundary objects: Amateurs and professionals in Berkeley's Museum of Vertebrate Zoology, 1907-39. Social studies of science, 19(3), 387-420.

Stockero, S.L. & Van Zoest, L.R. (2013). Characterizing pivotal teaching moments in beginning mathematics teachers' practice. Journal of Mathematics Teacher Education, 16, 125 - 147. https://doi.org/10.1007/s10857-012-9222-3.

Tseng, V. (2012). The uses of research in policy and practice and commentaries. Social Policy Report, 26(2), 1-24.

Wei, R. C., Darling-Hammond, L., Andree, A., Richardson, N., & Orphanos, S. (2009). Professional Learning in the Learning Profession: A Status Report on Teacher Development in the US and Abroad. Technical Report. National Staff Development Council.

Wenger, E. (1998). Communities of practice: Learning, meaning and identity. Cambridge University Press.

DISSONANCE, HARMONY, AND CULTURAL CONFLICT IN INTERDISCIPLINARY CO-TEACHING

Aubrey Neihaus
Wichita State University
aubrey.neihaus@wichita.edu

Scott Kaschner
Butler University
skaschne@butler.edu

In this brief research report, we share our preliminary findings from a collaborative analytic autoethnography investigating our co-teaching across the disciplines of mathematics and mathematics education. We used identity to investigate sources of conflict and found that even successful interdisciplinary collaborations can give rise to conflict. This conflict may be attributable to contestation of identities stemming from disciplinary cultural differences.

Keywords: Teacher Educators; Professional Development; Autoethnography; Collaboration

The growing desire for interdisciplinary collaboration and the promise of its benefits are often blunted by fear of working beyond one's own familiar disciplinary community or even of conflict. What role does identity play in conflict that arises in interdisciplinary collaboration? Through this autoethnographic study, we illustrate how story as identity makes visible: 1) the causes of conflict in interdisciplinary collaboration, 2) the relation of conflict to identity, and 3) how understanding these aspects can make conflict resolution possible. These understandings provide interdisciplinary collaborators with both a language and means for minimizing conflict and resolving it as it arises. This study is a brief report on the emergent findings of a larger, on-going analytic autoethnography.

The need for collaboration between mathematicians and mathematics educators, particularly regarding the education of prospective mathematics teachers, has been well-documented (CBMS 2001, 2012; Graham & Fennell, 2001). This type of collaboration has taken many forms, including collaborative teaching partnerships between mathematicians and mathematics educators (Bleiler, 2015; Grassl & Mingus, 2007; Thompson et al., 2012). Co-teaching arrangements are often complicated by differences between the two fields in conventions, beliefs, language, and both research and instructional methods (Fried, 2014; Sultan & Artzt, 2005). While difficulty in overcoming these differences can result in either conflict or a passive collaboration, in which labor is distributed more than shared (Friend & Cook, 1993), there are examples of long-standing, successful teaching collaborations between mathematicians and mathematics educators (Bass & Ball, 2014; Goos & Bennison, 2018; Heaton & Lewis, 2011).

Arbaugh, McGraw, and Patterson (2019) note that the differences between mathematicians' and mathematics educators' disciplines can most accurately be understood as cultural differences. They suggest that co-teaching in which "members of different cultures and communities come together to design and implement experiences for teachers that are more impactful" are more successful than in which the groups designed and implemented them separately (p. 160). They also contend that a necessary condition for successful collaboration is that "the members of a collaborative mathematics teacher education team have an *ethnorelative* cultural sensitivity" (p. 163, italics in original). Differences in disciplinary culture can also produce conflict, and part of successful collaboration is working through that conflict by way of mutual cultural understanding.

We sought to better understand how conflict can arise in collaborations between a mathematician and mathematics educator (even long-standing, successful ones) and how identity as members in the respective disciplinary cultures contributes to this conflict.

Theoretical Frames

Autoethnography, our methodology for this study, is often employed to create meaningful research that is grounded in personal experience and has been used to speak to issues of identity (Ellis et al., 2011). It provides teacher educators "a means to understand themselves and enhance their practice" (Chapman et al., 2020) and has been used in education research as an insightful way to connect the personal to the cultural (Dyson, 2007). Guyotte and Sochacka (2016) found collaborative autoethnography to be particularly useful in investigating their own interdisciplinary collaborative teaching. This method of research can create empathy across different groups and thus, is pertinent to investigating interdisciplinary collaboration.

While there exist critiques of autoethnography as atheoretical or "navel-gazing" (Marshall & Rossman, 2016), it is notable that there is a range of approaches and analytic autoethnography is one that is rooted in traditional symbolic interactionism (Anderson, 2006a). Analytic autoethnography is ethnography in which the researcher: 1) is a full member of the research setting, 2) makes this membership visible in publication, and 3) uses an analytic framework focused on improving theoretical understandings of the broader phenomenon under study.

Sfard and Prusak (2005) define identity as a set of stories that are socially constructed (i.e., these stories can be told by others about the subject or by the subject themselves). Since identity is storied, it is not fixed but is continuously told and re-told through interactions with others and the self. Further, some of the "most significant stories are often those that imply one's memberships in, or exclusions from, various communities" (p. 17). When combined with analytic collaborative autoethnography, identity as story creates a powerful framework through which to investigate conflict in successful interdisciplinary collaboration, allowing us to consider both identity and cultural aspects of the conflict.

Methods

Participants and Context

We are positioned as both researchers and participants in this study. As a math education researcher, Aubrey's primary area of expertise is teaching mathematics; as a research mathematician, Scott's primary area of expertise is mathematics. Each has worked in their respective fields for over a decade. However, our expertise does overlap, as Aubrey's secondary area of expertise is mathematics and Scott's secondary area of expertise is teaching mathematics. As a person with expertise in teaching mathematics, Aubrey identifies as a supportive co-teacher; similarly, Scott identities as a good teacher of mathematics.

We have collaborated many times over the last eight years, from grant writing to research to professional development for both teachers and higher education faculty. We have served as references and letter writers for one another for professional advancement. We also reach out to each other for advice on working with and in one another's respective disciplines. We both view our various collaborations as successful and appreciate the professional insights of one another.

In the summer of 2018, we co-taught a two-week, 80-hour intensive mathematics content professional development program for practicing K-8 teachers. The program called for the course to be co-taught by a mathematician and mathematics educator and set the sequence of topics and tasks. This was the fourth consecutive summer that we co-taught this course but the first time at that location.

Data Collection and Analyses

During the professional development, we recorded audio of daily debriefing conversations. From nine separate sessions, there was a total of nearly four and a half hours of audio. Data analysis was conducted in four stages. First, we summarized the content of the recordings. Then using these summaries, we identified areas where conflict was a topic of discussion. Next, we reduced the data to one instance of conflict that was particularly compelling and clear. We then transcribed this section by hand and collaboratively hand coded it for themes (Guest et al., 2011; Saldaña, 2016). Last, we analyzed the data for identity (Sfard & Prusak, 2005), understood as a set of stories, and threats to identity.

We were intentional about two aspects of our data analyses: timing and co-construction. We allowed time to lapse between the data collection and the analysis of these data. This allowed us space to create distinction between ourselves as participants and ourselves as researchers. A year passed between recording the data and the summary analysis; another between the summary analysis and the data reduction to moments of conflict.

Co-construction was also an important aspect of our analyses, as it ensured that we created meaning from the data dialogically (Anderson, 2006b), and were in essence using member checking to ensure validity (Creswell, 2014).

Findings

The findings fall into two areas: evidence of the success of our collaboration and how identity contributed to conflict. First, the collaboration was largely successful and harmonious, and conflict was rare. This was seen in both early analysis (summaries) and in the identity analysis. The summaries showed that the content of the debriefs included multiple topics indicating the success of the collaboration (e.g., joint classroom management; victories and challenges; distribution of work; strategies for supporting learners). While our investigation here undertakes one of the conflicts experienced within our collaboration, it is worth noting that in general, conflict was only one of many topics and could not be said to characterize the debrief data overall. Further, in the later identity analysis of the conflict under investigation, there are multiple instances of endorsing one another's stories and of working through contested stories to find a story both could endorse.

Second, we found that in the specific instance of conflict under investigation here, the conflict arose when one of our stories threatened the other's identity. Further, the conflict arose when a particular interaction gave rise to multiple stories that were then contested. The resolution came from understanding 1) that identity was threatened by an interaction, 2) this caused the story to be contested, and 3) why it is important in the respective disciplinary culture. We use the following vignette to illustrate this phenomenon.

Treating for Mathematical Status and Hand Raising Protocols

In general, both of us recognize that within a teacher professional development course on mathematics content, there will be status differentials among the teachers, and that as the instructors, it is our responsibility to treat for status.[1] In the past, we have agreed to establish a hand-raising protocol early in the course to ensure that participation is more equitably distributed and that all the teachers have sufficient time to process and formulate responses when questions are posed to the group.

On the first day of the course when Scott taught, he did not initiate this hand raising protocol. He had an alternate plan for treating for status—to allow the over-participators to attempt to answer the posed question, but to focus on a lower status teacher when they notice the intended aspect(s) of the task at hand, thereby assigning competence to the lower status teacher.

Aubrey noticed that Scott had not initiated the hand-raising protocol. She knew that when he teaches undergraduate mathematics courses, he doesn't rely as heavily on this type of tool,

[1] See Cohen & Lotan, 2014.

preferring to instead develop a learning community with freely exchanged academic ideas and that he treats for status in alternate ways over the semester. Aubrey contemplated signaling to him to suggest initiating the hand-raising protocol, but instead decided to bring it up later, assuming that he had just forgotten and was using his usual techniques. When she did bring it up, she asked Scott if he had forgotten about the hand raising protocol. Scott interpreted this question as a judgment on his teaching and responded by asking whether Aubrey trusted him to make pedagogical decisions. Later, they were able to talk through this experience and understand one another's perspective and how their meanings were intended and misunderstood.

Contested Narratives and Threats to Identity

While the vignette above is brief, it provides an opportunity to consider how the stories from the interaction—and their contestation—explain the source of the conflict. In the vignette, there is an endorsed story shared by both Aubrey and Scott—that treating for status differentials in a mathematics class is important. This shared and endorsed story identified both Aubrey and Scott as "good mathematics teachers" in the sense that their teaching is informed by research on equitable mathematics teaching.

When Scott made an in-the-moment decision to treat for status in a different way, Aubrey assumed that he had forgotten the usual strategy and asked if he forgot. This event is where their individual stories (and the associated identities) diverge. Aubrey characterized that move as a reminder, thereby identifying herself as a supportive collaborator. Scott characterized that move as questioning his pedagogical acumen and thereby calling his identity as a "good mathematics teacher" into question. Scott's response, questioning whether Aubrey trusted him to make instructional choices, further clarifies that he is contesting the story that identified him as "not a good mathematics teacher." His response not only contests the story that he is not a good mathematics teacher, it also contests Aubrey's story that she is a supportive co-teacher. The resolution of the conflict comes only when these perspectives and cultural misunderstanding are made clear to one another.

Discussion and Conclusions

On the surface, this vignette displays a disagreement about how to best address status differentials in a mathematics professional development course. We found that an underlying source of tension is the way in which our identities are contested by one another, whether intentionally or not. Aubrey called Scott's identity as a "good teacher of mathematics" into question when she attempted to remind him of their usual pedagogical choice. Scott called Aubrey's identity as a "supportive co-teacher" into question when he questioned whether she trusted his choices. However, the conflict arose (and escalated) because of how these stories about pedagogical choices and trust signal something deeper about how we each identify the other in relation to our own primary expertise. This significance is compounded by the fact that in each case, one's primary expertise is the other's secondary expertise.

Owning to complementary expertise and disciplinary cultural differences, there is still opportunity for conflict. Despite the many years of collaboration across multiple aspects of our professional lives, and mutual respect for one another, we still find that conflicts arise. It seems unlikely that in our collaborations, we will be able to eliminate conflict completely, given that we are each firmly established in our respective disciplinary cultures. However, by identifying and managing the underlying causes–identity and disciplinary cultural differences–we are able to see what gives rise to conflicts, resolve them, and learn from them.

References

Anderson, L. (2006a). Analytic autoethnography. Journal of contemporary ethnography, 35(4), 373-395.

Anderson, L. (2006b). On apples, oranges, and autopsies: A response to commentators. Journal of contemporary ethnography, 35(4), 450-465.

Arbaugh, F., McGraw, R., & Patterson, C. L. (2019). Collaborations between Mathematics Educators and Mathematicians for Mathematics Teacher Education in the United States. In International Handbook of Mathematics Teacher Education: Volume 3 (pp. 153-181). Brill Sense.

Bass, H., & Ball, D. L. (2014). Mathematics and education: Collaboration in practice. In Mathematics & mathematics education: Searching for common ground (pp. 299-312). Springer, Dordrecht.

Bleiler, S. K. (2015). Increasing awareness of practice through interaction across communities: The lived experiences of a mathematician and mathematics teacher educator. Journal of Mathematics Teacher Education, 18(3), 231-252.

Chapman, O., Kastberg, S., Suazo-Flores, E., Cox, D., & Ward, J. (2020). Mathematics teacher educators' learning through self-based methodologies. In International Handbook of Mathematics Teacher Education: Volume 4 (pp. 157-187). Brill.

Coia, L., & Taylor, M. (2009). Co/autoethnography: Exploring our teaching selves collaboratively. In Research methods for the self-study of practice (pp. 3-16). Springer, Dordrecht.

Cohen, E. G., & Lotan, R. A. (2014). Designing groupwork: strategies for the heterogeneous classroom third edition. Teachers College Press.

Conference Board of the Mathematical Sciences. (2001). The mathematical education of teachers (Vol. 11). American Mathematical Soc.

Conference Board of the Mathematical Sciences. (2012). The mathematical education of teachers II. Providence, RI/Washington, DC: American Mathematical Society and Mathematical Association of America.

Creswell, J. W. (2014). Research design: qualitative, quantitative, and mixed methods approaches. SAGE Publications.

Dyson, M. (2007). My story in a profession of stories: Auto ethnography-an empowering methodology for educators. Australian Journal of Teacher Education (Online), 32(1), 36-48.

Ellis, C., Adams, T. E., & Bochner, A. P. (2011). Autoethnography: an overview. Historical social research/Historische sozialforschung, 273-290.

Fried, M. N. (2014). Mathematics & mathematics education: Searching for common ground. In Mathematics & mathematics education: Searching for common ground (pp. 3-22). Springer, Dordrecht.

Friend, M., & Cook, L. (1992). Interactions: Collaboration skills for school professionals. Longman Publishing Group, 95 Church Street, White Plains, NY 10601.

Gee, J. P. (2000). Chapter 3: Identity as an analytic lens for research in education. Review of research in education, 25(1), 99-125.

Goos, M., & Bennison, A. (2018). Boundary crossing and brokering between disciplines in pre-service mathematics teacher education. Mathematics Education Research Journal, 30(3), 255-275.

Graham, K. J., & Fennell, F. (2001). Principles and standards for school mathematics and teacher education: Preparing and empowering teachers. School Science and Mathematics, 101(6), 319-327.

Grassl, R., & Mingus, T. T. Y. (2007). Team teaching and cooperative groups in abstract algebra: nurturing a new generation of confident mathematics teachers. International Journal of Mathematical Education in Science and Technology, 38(5), 581-597.

Guest, G., MacQueen, K. M., & Namey, E. E. (2011). Applied thematic analysis. Sage Publications.

Guyotte, K. W., & Sochacka, N. W. (2016). Is this research? Productive tensions in living the (collaborative) autoethnographic process. International Journal of Qualitative Methods, 15(1), 1609406916631758.

Heaton, R. M., & Lewis, W. J. (2011). A mathematician-mathematics educator partnership to teach teachers. Notices of the AMS, 58(3), 394-400.

Marshall, C., & Rossman, G. (2016). B.(2006). Designing qualitative research. Sage: California.

Saldaña, J. (2016). The coding manual for qualitative researchers. SAGE Publications.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. Educational researcher, 34(4), 14-22.

Sultan, A., & Artzt, A. F. (2005). Mathematicians are from Mars, math educators are from Venus: The story of a successful collaboration. Notices of the AMS, 52(1), 48-53.

Thompson, D. R., Beneteau, C., Kersaint, G., & Bleiler, S. K. (1990). Voices of mathematicians and mathematics teacher educators co-teaching a mathematics course for prospective secondary teachers. Professional collaborations in mathematics teaching and learning: Seeking success for all (Seventy-fourth NCTM yearbook). Reston, VA: National Council of Teachers of Mathematics.

MATHEMATICS EDUCATION RESEARCHERS' PRACTICES IN INTERDISCIPLINARY COLLABORATIONS: EMBRACING WAYS OF KNOWING

Elizabeth Suazo-Flores
Purdue University
esuazo@purdue.edu

William S. Walker, III
Purdue University
wswalker@purdue.edu

Hanan Alyami
Purdue University
alyamih@purdue.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Mahtob Aqazade
Rice University
mahtob.aqazade@rice.edu

Mathematics education researchers (MERs) engage in interdisciplinary collaborations that contribute to the mathematics education discipline. MERs' learned forms of work and discourse, i.e., practices, are particular to their mathematics education discipline and might not align with practices needed to conduct interdisciplinary work. We interviewed four MERs who were leaders of interdisciplinary groups to learn about practices they reported using while collaborating with their groups. Using qualitative content analysis, we describe five practices commonly reported by the four MERs. We argue that these five practices are central ways of knowing and doing for MERs when working in interdisciplinary groups. Our study contributes to the mathematics education discipline by unpacking practices MERs use to engage in interdisciplinary groups that are influenced by interpersonal relationships.

Keywords: Mathematics education, interdisciplinary collaborations, practices, disciplines.

Mathematics Education Discipline and Work

Disciplines are defined by Williams et al. (2016) as a "phenomenon" involving "specialization" of work and discourse (p. 4). Disciplines are constantly evolving while increasing their specialization and differentiation. For instance, mathematics education as a discipline has evolved from being informed by the disciplines of psychology and mathematics (Kilpatrick, 2014; Stinson & Walshaw, 2017) to embracing disciplines such as neuroscience (e.g., Norton et al., 2019), science education (e.g., de Freitas & Palmer, 2016), and anthropology (González et al., 2001; Gutiérrez, 2013). Disciplinary practitioners are introduced to disciplinary ways of knowing and doing in their training programs. Mathematics Education Researchers (MERs) develop their professional identity by learning research practices within the discipline of mathematics education. In their career, practitioners gain status in their discipline by engaging in disciplinary forms of discourse. MERs further interact with practitioners from other disciplines and collaborate in interdisciplinary groups.

MERs have engaged in interdisciplinary work to address complex problems. For instance, Civil joined González from anthropology to explore funds of knowledge (González et al., 2001). Norton joined colleagues from computer science to develop software systems (Jones et al., 2015) and neuroscience to explore students' anxiety (Norton et al., 2019). Davis joined colleagues from various disciplines to explore spatial reasoning (Bruce et al., 2017). We can learn about MERs' interdisciplinary collaborations from publications, but because the collaborations are not the focus of the publications, descriptions about the process of interdisciplinary work are thin. Schön (1992) stated that practitioners of a discipline do their work by using more than "research-based technique" (p. 54). Conducting work involves aspects such as interpersonal relationships where people act and work towards solving important problems and finding common ground (Fletcher, 1998). Interpersonal relationships involve social exchanges that welcome and lift people involved academically and emotionally. We argue that working in interdisciplinary groups is more than contributing discipline specific expertise, it involves the negotiation of disciplinarity through interpersonal relationships. This includes paying attention to ways of being, interacting and doing within the groups. It also includes identifying disciplinary-based practices that are essential, can be modified or compromised, or do not align with the interdisciplinary work.

Because members of interdisciplinary groups have been trained in their respective disciplinary ways of knowing, communication and representation differences might emerge when coming together. Bruce et al. (2017) reported that members of an interdisciplinary group need to study "discipline-specific vocabularies and methodologies" and articulate problems so that all members can "identify and situate themselves" (p. 158). Goos and Bennison (2018) described the value of disciplinary knowledge to interdisciplinary team members. In addition, team members felt threatened when their disciplinary knowledge and ways of operating were not recognized professionally when "working outside their discipline" (p. 267).

Disciplinary ways of knowing and doing work are called practices (Hyland, 2004; Williams et al., 2016). In this paper, practices and practice categories are italicized to help the reader recognize when words or phrases are being used to describe practices. Building from existing research (e.g., Cobb & Yackel, 1996; MacIntyre, 1984; Schön, 1983, Wenger, 1998), Suazo-Flores et al. (2021a, para. 4) defined practices as "established ways of *being*, *operating*, and *interacting* with others." Suazo-Flores et al. (2021a, para. 9) found that in interdisciplinary groups, MERs engage in practices such as *working towards research interests*, *cultivating trust and open-mindedness*, and *understanding of institutional support*. Building from this work, we interviewed MERs who worked in interdisciplinary groups. Our initial analysis resulted in a revision of the categories of our definition of practices, where:

> *Being* […] refers to MERs describing their view of themselves and others in the interdisciplinary group including specific roles taken on by group members. *Operating* […] means members' ways of doing in the interdisciplinary group and acknowledging institutional policies and actions in order to complete the work. *Interacting* is […] developing communication standards, negotiating the meaning of ideas that allows the group to collaborate, and explaining work to people outside of the group. (Suazo-Flores et al., 2021b, p. 827, italics added)

With this revised definition, we conducted further research to address the question: What practices did MERs commonly report using in interdisciplinary research groups?

Methods and Analysis

This research is part of a larger project focused on describing MERs' lived experiences of working in interdisciplinary groups (Suazo-Flores, et al., 2021b; Suazo-Flores, et al., 2021c). Four MERs from three different projects who identified themselves as members of interdisciplinary research groups volunteered to participate in this research: Amelia, Ian and Alexis, and Iris (pseudonyms). Amelia was a faculty member as well as a primary developer and leader of her interdisciplinary project. Her project involved improving curriculum and pedagogy for university pre-service mathematics education programs. Ian and Alexis collaborated on a project to create engineering tasks that would allow students to learn new mathematics content. Ian was a graduate student and Alexis was a faculty member and Ian's advisor. Iris was a faculty member and leader of a project between mathematicians and mathematics teacher educators. The project developed curricular modules to help students learn mathematics content. We conducted three semi-structured interviews (Kvale, 1996) with the four MERs (Ian and Alexis were interviewed together) focused on practices that developed and were used in the interdisciplinary groups. MERs shared the story of the interdisciplinary group, ways of doing in the group, and reflections as a member of the group. For example, we asked MERs to describe situations when working in the group was an asset or a constraint.

To create codes and definitions for the categories of practices and to identify examples, we used grounded theory (Charmaz, 2005) to code Amelia's transcript (Suazo-Flores, et al., 2021b). Each practice was categorized as *being*, *operating*, or *interacting* and assigned a short descriptive phrase. Three phases of analysis resulted in a codebook with definitions of the practice categories, descriptions, and examples of different practices. For example, the following excerpt was coded as the practice *acknowledging a personal view of self* within the *being* category.

> Amelia: For me, it's been a wonderful growing experience. So, I don't feel like I've lost anything because I still have my life in my discipline, and I have a much enriched and expanded life as well by having these experiences that I didn't realize I was going to get.

Amelia referred to her *personal view of self* as an MER noting, "I still have my life in my discipline." She also recognized that her work in the interdisciplinary group had "enriched and expanded" her life. We identified this as a way of *being* because it provided evidence of her evolving identity as an MER through her work with people from other disciplines.

We then used the codebook to code all three interviews. Evidence of practices reported by the four participants in the form of interview transcripts constitutes our data. A deductive approach using the codebook allowed for analysis of the transcript data consistent with qualitative content analysis, such that the coding and analysis included two steps: "the first is a qualitative-interpretative step following a hermeneutical logic in assigning categories to text passages; the second is a quantitative analysis of frequencies of those assignments" (Mayring, 2015, p. 366).

For reliability, each transcript was coded by a member of the research team using the codebook and then checked by a second member of the research team. When coding differences between researchers were found, the coded practices were discussed by the entire research team to share reasoning and clarify codes. Once agreement was reached regarding the use of a code or its definition, the codebook was updated. Once this qualitative-interpretive step was completed, the codebook was updated and the coded items in each of the transcripts were reviewed for alignment with the updated codebook (c.f., Mayring, 2015).

Following the final coding of the three transcripts, frequency tables were created to identify how often each practice occurred in the data. Table 1 is the frequency table for the five practices described in this paper. The frequencies of the coded practices were disaggregated by project and show the amount of evidence from each of the projects used to make a claim. For example, there were 22 instances where the *being* practice *acknowledging a personal view of others' roles* was coded across three interview transcripts. Disaggregation showed that 6 of these codes were from Amelia's transcript, 9 of these codes were from Ian and Alexis' transcript, and 7 of these codes were from Iris's transcript. The frequency tables allowed for the identification of practices that were common between all of the participants. This research presents the five practices that were reported in each of the three transcripts at least three times.

Table 1: Frequency Table for Common Practices

Category	Practice	Amelia	Ian and Alexis	Iris	Total
Being	Acknowledging a personal view of self	4	6	20	30
Being	Acknowledging a personal view of others	6	9	7	22
Interacting	Developing ideas between partners	6	7	3	16
Operating	Working towards common goals	9	5	5	19
Operating	Using ways of working together	4	3	4	10

Findings

Our analysis identified five practices MERs commonly reported using in their interdisciplinary groups: *acknowledging a personal view of self*, *acknowledging a personal view of others*, *developing ideas between partners*, *working towards common goals*, and *using ways of working together*. The first two practices are in the *being* category, the third practice is in the *interacting* category, and the last two practices are in the *operating* category. Below, we describe each practice and use examples from the data to illustrate the category.

Being: Acknowledging a Personal View of Self

Acknowledging a personal view of self was coded as a type of *being* practice. This practice involved MERs using individual identity, including their dispositions, interests, sense of efficacy, and roles to engage with others in the group's work. Our participants showed evidence of recognizing their roles and dispositions in their interdisciplinary groups. For example, Ian indicated, he "was the central person" who at the beginning of the project "was trying to meet the needs of so many people," which we interpreted as him acknowledging his role in the group and others' expertise. In Iris' research group, she described how she was "not a mathematician," but brought her vast teacher education background to the project, "I have lots of experience in working with teachers."

Being: Acknowledging a Personal View of Others

Acknowledging a personal view of others was classified as a *being* practice because participants described how they perceived others as part of the interdisciplinary group. The code *acknowledging a personal view of others* was used when participants described other group members as members of a discipline, as taking on special roles in the group, or as influential members due to the added diversity of knowledge/experience. For instance, Iris described a member of her interdisciplinary group as having a "very strong mathematics background" with experience in the "high tech industry" and "a high school teacher." Iris acknowledged the member's expertise and experiences and appreciated that both were committed to "narrowing the gap between school mathematics and contemporary mathematics," which was the main goal of the interdisciplinary project. As a member of Ian's interdisciplinary group, Alexis described how she recognized roles and expertise in her colleagues. Alexis stated, "the primary role that the team members play was to bring their expertise to the table," and she noted that "Ian really was the lead in designing the material or coming up with the context and coming up with the activities."

Interacting: Developing Ideas Between Partners

Developing ideas between partners was coded as an *interacting* practice because MERs reported how group members negotiated the meaning of ideas, representations, or frameworks. When MERs described developing understandings and common definitions that allowed members of the group to collaborate and communicate to external audiences, we coded those instances as *developing ideas between partners*. For example, Iris indicated that she and a group member who was a mathematician had "arguments on the buildup, on the things that [the mathematician] thought would be very accessible." Iris and the mathematician had discussions regarding unpacking the mathematics content and making it accessible to students. She explained that such discussions allowed them to develop new ways to represent mathematics to external audiences. "I don't think I would be able to do it on my own, and I don't think he [the mathematician] would be able to do it on his own." In another example, Amelia indicated how it was important for the interdisciplinary group members to develop ways of communicating. Amelia described that given the different disciplinary expertise, people in the group needed "to be good listeners and very respectful and making it possible for questions to be asked and to be able to respond to those questions in a way that's serious and takes things seriously."

Operating: Working towards Common Goals

Working towards common goals was classified as an *operating* practice because the use of common goals was a factor in "ways of doing" for the groups. This practice was described by the participants when they shared purposes or common goals used by the group to focus the work being done. There were also times when MERs described working with the interdisciplinary group to develop common goals. Such instances were classified as an *interacting* practice like *developing ideas between partners* (see above) because developing goals involved exchanging ideas to build shared understandings. *Working towards common goals* was coded when (1) the group goals were described by a participant, (2) it was clear that the goals were established, and (3) it was clear that the goals were used to focus the work being done. For instance, Amelia's articulation of the initial purpose of her interdisciplinary group is that "it was a funded project" and that "the central purpose kept [them] together. So those two goals were always there." Ian described how the group's purpose was to create a curriculum for the "learning of new mathematics and new engineering at the same time."

Operating: Using Ways of Working Together

Similar to *working towards common goals*, *using ways of working together* is classified as an *operating* practice because it represents "ways of doing" for a group. *Using ways of working together* is how an interdisciplinary group worked as a team or a process that the group used to get work done. Iris described her work as "a true teamwork." She encouraged the mathematician to "set up a storyboard, so to speak," to tell the story around the targeted mathematics concept. Later, they worked together to add the "know-how" pieces. This is how the new story around the targeted mathematics concept began "with a question instead of starting with declarations." In Amelia's research group, she described how different group members already had different systems in place. The group shared these other systems, which created a menu of items, "and then we each picked something from that menu that we decided to try this approach next year, and then we swapped and learned from each other." The group developed a way of working together to learn from each other by picking different approaches from the menu.

Discussion

We have described five practices MERs commonly reported using in their interdisciplinary groups. Two practices are in the *being* category, one in the *interacting* category, and two in the *operating* category, respectively: *acknowledging a personal view of self*, *acknowledging a personal view of others*, *developing ideas between partners*, *working towards common goals*, and *using ways of working together*.

The practices identified in this research provide evidence that conducting interdisciplinary work is more than contributing disciplinary expertise, as asserted by Schön (1992). MERs used the *being*, *interacting,* and *operating* practices to navigate interpersonal relationships and conduct interdisciplinary work. The *being* types of practices reflect how MERs saw themselves in relation to others and how they perceived others in their groups. The practices of *acknowledging a personal view of self* and *others* allowed MERs to acknowledge disciplinary expertise and navigate work with others by finding common ground and contributing to the interdisciplinary group work. The *interacting* practice, *developing ideas between partners*, identifies the development and use of discourse norms within interdisciplinary groups to understand, listen to each other's perspectives, and build common understandings. The *operating* practices of *working towards common goals* and *using ways of working together* describe "ways of doing" in interdisciplinary groups. Members of interdisciplinary groups worked towards goals that met their interests and the interdisciplinary project; they also developed work systems that built from each other's expertise and backgrounds.

Because the members of interdisciplinary research projects come from different disciplines, that have different practices (Williams et al., 2016), MERs need to use *interacting* practices, which allow the group to negotiate and develop a common language among themselves and for external audiences. MERs in our study had discussions of meanings within the groups to conduct the interdisciplinary work and communicate the product of the collaboration to external audiences. For instance, Ian described how he acknowledged different peoples' expertise and then realized that his role and other team members' roles allowed the project to be more successful. Iris described the integral role of diversity in her team, highlighting the need to discuss ideas with colleagues from different disciplinary backgrounds to construct new knowledge. Amelia's description of team members' approach to interactions as "good listeners and very respectful" aligns with descriptions of dispositions of team members provided by Bruce et al. (2017). We also noted that the common language the members of the interdisciplinary group developed was unique to the members of the group and the project's final product. For instance, Iris reflected that it took many discussions for the group to narrow down ways of representing the product of their work to others. Iris' interaction with her colleagues and the consideration of how external audiences would use the project was instrumental in developing a way of communicating the work. Iris recognized that this important goal could not have been realized without each team member's contributions and expertise.

Once the group members have used *interacting* practices to cultivate a common language and framework, *operating* practices develop. We identified two operating practices: *working towards common goals* and *using ways of working together*. For instance, Iris and her group used the story system to start with the pure mathematics content to add know-how prompts that would allow the audience to have entry points to engage. Amelia and her group built from each of their experiences in teacher education to learn from each other instead of using mandating systems.

Our study exemplifies Williams et al.'s (2016) definition of disciplines for mathematics education. Mathematics education as a discipline has evolving forms of discourse and work. As

MERs are being asked to join interdisciplinary groups to expand the boundaries of mathematics education research (Cai et al., 2020), we advocate for professional development spaces where MERs reflect on ways of *being*, *interacting*, and *operating* in such groups. MERs need to learn how to productively work outside of their discipline and, with that, expand their expertise. MERs need to consider their role in the project and their personal goals in relationship to the project's goals and the group members from other disciplines. MERs also need to create spaces where members of interdisciplinary groups can openly discuss ideas. In such discussions, MERs need to acknowledge their colleagues' expertise and recognize that as a group in dialogue, new ideas will emerge that inform the final product of the interdisciplinary work.

The results of this research are influenced by the fact that each of our participants were leaders of their interdisciplinary groups. MERs as members of interdisciplinary groups led by others may contain practices not captured here. Similarly, more studies are needed to explore how power relationships influence work in interdisciplinary groups that include a MER. The study reported here did not capture the breadth of experiences that members of the research team have experienced as MERs on interdisciplinary projects, suggesting that additional work is needed. Similarly, studies on how MERs' race and ethnicity backgrounds influence their interpersonal relationships and practices in interdisciplinary groups are also needed.

References

Bruce, C. D., Davis, B., Sinclair, N., McGarvey, L., Hallowell, D., Drefs, M., . . . Woolcott, G. (2017). Understanding gaps in research networks: Using "spatial reasoning" as a window into the importance of networked educational research. Educational Studies in Mathematics, 95(2), 143-161.

Cai, J., Morris, A., Hohensee, C., Hwang, S., Robison, V., Cirillo, M., Kramer, S., & Hiebert, J. (2020). Improving the impact of research on practice: Capitalizing on technological advances for research. Journal of Research in Mathematics Education, 51(5), 518-529. https://doi.org/10.5951/jresematheduc-2020-0165

Charmaz, K. (2005). Grounded theory in the 21st century: Applications for advancing social justice studies. In N. K. Denzin & Y. S. Lincoln (Eds.), The Sage handbook of qualitative research (pp. 507–536). Sage Publications.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. Educational Psychologist, 31(3-4), 175-190.

de Freitas, E., & A. Palmer, A. (2016). How scientific concepts come to matter in early childhood curriculum: Rethinking the concept of force. Cultural Studies of Science Education. 11(4), 1201-1222. https://doi.org/10.1007/s11422-014-9652-6

Fletcher, J. K. (1998). Relational practice: A feminist reconstruction of work. Journal of Management Inquiry, 7(2), 163-186.

González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. Journal of Education for Students Placed at Risk, 6(1-2), 115-132. https://doi.org/10.1207/S15327671ESPR0601-2_7

Goos, M., & Bennison, A. (2018). Boundary crossing and brokering between disciplines in pre-service mathematics teacher education. Mathematics Education Research Journal, 30(3), 255-275. https://doi.org/10.1007/s13394-017-0232-4

Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. Journal for Research in Mathematics Education, 44(1), 37-68.

Hyland, K. (2004). Disciplinary cultures, texts and interactions. In Disciplinary discourses: Social interactions in academic writing (pp. 1–19). University of Michigan Press.

Jones, J. R., Balci, O., & Norton, A. (2015, December). A cloud software system for visualization of game-based learning data collected on mobile devices. In 2015 Winter Simulation Conference (WSC) (pp. 1080-1090). IEEE.

Kvale, S. (1996). Interviews: An introduction to qualitative research interviewing. Sage Publications.

Kilpatrick, J. (2014). History of research in mathematics education. In S. Lerman (Ed.), Encyclopedia of mathematics education (pp. 349-354). Springer. https://doi.org/10.1007/978-94-007-4978-8

MacIntyre, A. C. (1984). After virtue: A study in moral theory (2nd ed.). University of Notre Dame Press.

Mayring, P. (2015). Qualitative content analysis: Theoretical background and procedures. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), Approaches to qualitative research in mathematics education: Examples of methodology and methods (pp. 365-380). Springer. https://doi.org/10.1007/978-94-017-9181-6_13
Norton, A., Seok, Y., & Choi-Koh, S. (2019). Examining mathematics anxiety of undergraduates using a brain-based measurement, EEG. Journal of Behavioral and Brain Science, 9, 195-209. https://doi.org/10.4236/jbbs.2019.95017
Schön, D. A. (1983). The reflective practitioner: How professionals think in action. Basic Books.
Schön, D. A. (1992). The crisis of professional knowledge and the pursuit of an epistemology of practice. Journal of Interprofessional Care, 6(1), 49-63. http://doi.org/10.3109/13561829209049595
Stinson, D. W., & Walshaw, M. (2017). Exploring different theoretical frontiers for different (and uncertain) possibilities in mathematics education research. In J. Cai (Ed.), Compendium for research in mathematics education, (pp. 128-155). National Council of Teachers of Mathematics
Suazo-Flores, E., Alyami, H., Walker, W. S., III, Aqazade, M., & Kastberg, S. E. (2021a). A call for exploring mathematics education researchers' interdisciplinary research practices. Mathematics Education Research Journal. https://doi.org/10.1007/s13394-021-00371-0
Suazo-Flores, E., Walker, W. S., III, E., Alyami, H., Aqazade, M., & Kastberg, S. E. (2021b). Conceptualizing practices in interdisciplinary groups from a mathematics education researcher's perspective. In D. Olanoff, K., Johnson, & S. M., Spitzer (Eds.). Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Philadelphia, PA.
Suazo-Flores, E., Walker, W. S., III, E., Alyami, H., Aqazade, M., & Kastberg, S. E. (2021c). Understanding practices in an interdisciplinary group from a case study. In D. Kollosche (Ed.). Exploring new ways to connect: Proceedings of the eleventh International Mathematics Education and Society conference. Volume 3. Tredition.
Valero, P. (2010). Mathematics education as a network of social practices. In V. Durand-Guerier, S. Soury-Lavergne, & F. Arzarello (Eds.), Congress of the European Society for Research in Mathematics Education (pp. LIV-LXXX). Institut National de Recherche Pedagogique.
Wenger, E. (1998). Communities of practice: Learning, meaning, and identity. Cambridge University Press.
Williams, J., Roth, W. M., Swanson, D., Doig, B., Groves, S., Omuvwie, M., Borromeo Ferri, R., & Mousoulides, N. (2016). Interdisciplinary mathematics education. Springer. https://doi.org/10.1007/978-3-319-42267-1

DEVELOPMENT OF TEACHER LEADERS IN HIGH-NEED URBAN SCHOOLS

Adem Ekmekci	Mahtob Aqazade	Anne Papakonstantinou
Rice University	Rice University	Rice University
ae16@rice.edu	ma119@rice.edu	appa@rice.edu

Keywords: Equity, Inclusion, and Diversity; Instructional Leadership; Mathematical Knowledge for Teaching; Professional Development.

The Rice University Robert Noyce Master Teaching Fellowship (RU-MTF) was a five-year program designed to develop 14 exemplary secondary mathematics teachers into mathematics teacher leaders who are deeply grounded in sound mathematics content, pedagogical content knowledge (Hill et al., 2008), teacher leadership (York-Barr & Duke, 2004), adult education, and culturally relevant teaching (Ladson-Billings, 1994). RU-MTF's goals were to create an innovative, multi-faceted research-informed model for identifying, developing, and supporting mathematics leaders that can be replicated for different grade bands and in urban, suburban, and rural school districts.

In this paper, we report on lessons learned from implementing RU-MTF, particularly during COVID-19, and RU-MTF's impact on master teaching fellows' (MTF) leadership skills, mathematical knowledge for teaching (MKT), and diversity dispositions. Leadership skills (Mills et al., 2014) and diversity dispositions (i.e., Schulte et al., 2008) are measured through self-reported 5-point Likert-scale items. MKT is measured through learning mathematics for teaching assessment (Hill et al., 2008). We further highlight MTFs' focus group conversations and open-ended responses to, for example, their perceptions of effective mathematics instruction, effects of RU-MTF on beliefs about equity, and complications caused by the pandemic.

The COVID-19 pandemic created complications for teachers including "exacerbated learning gaps," "weakened community connections," "obstructed feedback cycles for colleagues," and "worsened social interactions." MTFs' perceptions of remote education during COVID-19 revealed the importance of developing equipped and energized teacher leaders who are devoted to improving mathematics instruction by supporting K-12 teachers, students, and parents. During this time of uncertainty, MTFs realized, even more than ever, the value of "community connections" and "leadership engagement." The quantitative data indicated that MTFs' leadership skills, MKT, and beliefs about diversity in education significantly improved from the onset to the end of RU-MTF. MTFs highlighted "student engagement," "hands-on learning," "learning by discovery," and "communication in flexible and variety of ways" as important parts of effective mathematics instruction. However, their concerns disrupting such approaches in effective teaching of mathematics included "tests being perceived as the most important element" and the "deficit understanding of effective math instruction." As MTFs' perspective of leadership roles "changed from individual to collaborative style" and their beliefs about equity in mathematics "evolved in to having a wider scope," we can argue that RU-MTF served as an effective model for developing teacher leaders.

Results include implications for both research and practice fields regarding teacher leader development (including whys and hows) and effective characteristics of a teacher leader development program.

Acknowledgment

This research was supported by National Science Foundation – DUE #1556006.

References

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. Journal for Research in Mathematics Education, 39(4), 372–400. https://doi.org/10.5951/jresematheduc.39.4.0372

Ladson-Billings, G. (1994). What we can learn from multicultural education research. Educational Leadership, 51(8), 22–26.

Mills, S. J., Huerta, J. J., Watt, K. M., & Martinez, J. (2014). AVID teacher leadership: Administrator and teacher perceptions of leadership attributes. Journal of School Leadership, 24(1), 146–163. https://doi.org/10.1177%2F105268461402400105

Schulte, L. E., Edwards, S., Edick, N. A. (2008). The development and validation of the diversity dispositions index. Teacher Education Faculty Publications, 5(3), 11–19. https://digitalcommons.unomaha.edu/tedfacpub/17

York-Barr, J. & Duke, K. (2004). What do we know about teacher leadership? Findings from two decades of scholarship. Review of Educational Research, 74, 255–316. https://doi.org/10.3102%2F00346543074003255

THE ROLE OF SELF-EFFICACY, LEADERSHIP, SCHOOL-WORK ENVIRONMENT, DIVERSITY BELIEFS, AND SOCIAL NETWORK IN TEACHER RETENTION

Adem Ekmekci
Rice University
ae16@rice.edu

Mahtob Aqazade
Rice University
ma119@rice.edu

Cindy Callard
University of Rochester
ccallard@Warner.Rochester.edu

David Gibson
Southern Illinois University-Carbondale
djgibson@siu.edu

Greg Rushton
Middle Tennessee State University
Gregory.Rushton@mtsu.edu

Rebecca McGraw
University of Arizona
rmcgraw@math.arizona.edu

Keywords: Instructional Leadership; Teacher Beliefs

Mathematics and science teacher turnover is a great and consistent problem in the U.S., particularly in high-need, high-minority, and urban schools (Cross, 2017). Most teachers leave the profession because of dissatisfaction, lack of support, lack of autonomy, and lack of collaboration opportunities (Carver-Thomas & Darling-Hammond, 2019). Thus, it is essential to understand the motivational, school-environmental, and social networking factors in relation to teacher retention. For instance, teachers' self-efficacy beliefs are important in fostering constructivist learning, student motivation, and higher academic performance, which impact job satisfaction and retention or attrition in the profession (Yost, 2006). Further, opportunities to develop leadership skills and engage in a collaborative school-work environment to improve school culture and instruction can support and sustain high-qualified teachers (Dauksas & White, 2010). Some features of teachers' social networks (e.g., density) support their persistence and are correlated with their self-efficacy (Polizzi et al., 2021). Lastly, positive diversity dispositions are associated with persisting in teaching in high-need schools (Williams et al., 2016).

In this paper, we explored the extent to which teachers' self-efficacy for teaching, leadership skills, diversity dispositions, school-work environment, and social network are related to their retention. From the Greater Houston area, about 250 teachers (27% male and 73% female; 30% elementary and 70% secondary teachers; 65% White and 35% from minoritized backgrounds) with, on average, 14 years of teaching experience (SD = 8.70) completed a survey responding to questions related to teaching, self-efficacy (Klassen et al., 2009), teacher leadership skills (Watt et al., 2010), person-organization fit (Pogodzinski et al., 2013), principal autonomy support (Baard et al., 2004), diversity dispositions (Schulte et al., 2009), and social network (e.g., size; Polizzi et al., 2021). Among these teachers, 14 were identified as shifters (who shifted to a leadership position) and 18 as leavers (who left or retired from K-12 teaching/education).

We conducted multinomial logistics regression analyses with retention as a three-level nominal outcome (Mayer et al., 2017). Findings indicated that secondary teachers were more likely to shift to a non-teaching position compared to stayers. The higher level of teacher leadership skills and lower degrees of person-organization fit were associated with shifting to a leadership position. Lastly, a higher level of teaching self-efficacy was observed in leavers compared to stayers. Reasons for shifting included teaching burnout, better pay, and having greater impact. Reasons for leaving included the pandemic, retirement, family, stress, and burnout. There are implications for practitioners, researchers, and administrators for supporting teachers to persist in teaching or shifting to a leadership position where they can have a greater impact for educational outcomes.

Acknowledgment

This research was supported by Robert Noyce NSF grants #1950019, 1949985, 1949925, 1949969, 1950001, 1950002, and 1949927.

References

Baard, P. P., Deci, E. L., & Ryan, R. M. (2004). Intrinsic need satisfaction: A motivational basis of performance and well-being in two work settings. *Journal of Applied Social Psychology, 34*(10), 2045–2068. https://doi.org/10.1111/j.1559-1816.2004.tb02690.x

Carver-Thomas, D., & Darling-Hammond, L. (2019). The trouble with teacher turnover: How teacher attrition affects students and schools. *Education Policy Analysis Archives*, *27*(36), 1–27.

Cross, F. (2017, June). *Teacher shortage areas nationwide listing 1990–1991 through 2017–2018.* U.S. Department of Education Office of Postsecondary Education.

Dauksas, L., & White, J. (2010). Should I stay or should I go? How teacher leadership can improve teacher retention. *AASA Journal of Scholarship and Practice, 7*(2), 27–32.

Klassen, R. M., Bong, M., Usher, E. L., Chong, W. H., Huan, V. S., Wong, I. Y., & Georgiou, T. (2009). *Contemporary Educational Psychology*, *34*(1), 67–76. https://doi.org/10.1016/j.cedpsych.2008.08.001

Meyers, L. S., Gamst, G., & Guarino, A. J. (2017). *Applied multivariate research: Design and interpretation* (3rd ed.). Sage Publications.

Pogodzinski, B., Youngs, P., & Frank, K. A. (2013). Collegial climate and novice teachers' intent to remain teaching. *American Journal of Education, 120*, 27–54. https://doi.org/10.1086/673123

Polizzi, S. J., Zhu, Y., Reid, J. W., Ofem, B., Salisbury, S., Beeth, M., Gillia, R., Mohr-Schroeder, M. Sheppard, K. & Rushton, G. T. (2021). Science and mathematics teacher communities of practice: social influences on discipline-based identity and self-efficacy beliefs. *International Journal of STEM Education*, *8*(1), 1–18. https://doi.org/10.1186/s40594-021-00275-2

Schulte, L. E., Edwards, S., Edick, N. A. (2008). The development and validation of the diversity dispositions index. *Teacher Education Faculty Publications, 5*(3), 11–19. https://digitalcommons.unomaha.edu/tedfacpub/17

Watt, K. M., Huerta, J., & Mills, S. J. (2010). Advancement Via Individual Determination (AVID) professional development as a predictor of teacher leadership in the United States. *Professional Development in Education*, *36*(4), 547–562. https://doi.org/10.1080/19415250903430122

Williams, D. L., Edwards, B., Kuhel, K. A., & Lim, W. (2016). Culturally responsive dispositions in prospective mathematics teachers. *Discourse and Communication for Sustainable Education, 7*(2), 17–33.

Yost, D. S. (2006). Reflection and self-efficacy: Enhancing the retention of qualified teachers from a teacher education perspective. *Teacher Education Quarterly*, *33*, 59–76

DEVELOPING ELEMENTARY SCHOOL PRINCIPALS' NOTICING TOWARDS A SHARED VISION OF HIGH-QUALITY MATHEMATICS INSTRUCTION

Patricia Fuentes Acevedo
UC Irvine
pfuente@uci.edu

Christina Kimmerling
UC Irvine
ckimmerl@uci.edu

Rossella Santagata
UC Irvine
r.santagata@uci.edu

Keywords: Instructional Leadership, Instructional Vision, Professional Development.

When trying to achieve instructional improvement, principal leadership plays a significant role in changing teachers' instructional practices (Supovitz et al., 2010). This study is part of a larger project, a research-practice partnership (RPP) with a school district that aims to develop school principals' instructional leadership practices so they can support their teachers in creating more socially just mathematics classrooms for their predominantly Latinx population.

We report on work conducted by our RPP team (district leaders, six school principals, and three researchers) towards one of the collaboratively defined goals for the 2021-2022 school year: building a shared vision of high-quality math instruction. We designed a learning experience inspired by Math Labs (Kazemi et al., 2018), consisting of five monthly visits to a fourth-grade math class to engage in observation and debriefing with the teacher on instructional practices and classroom interactions. In this context, our research examined in detail principals' learning process as they are involved in practice-based professional development. We attended to the following questions: How does the principals' noticing of student mathematical thinking evolve over the visits?; How is the noticing of each principal related to their instructional vision of high-quality mathematics instruction?

A vision of high-quality mathematics instruction involves the discourse that principals use to describe ideal classroom practices that are not necessarily mastered yet (Munter, 2014)--a first step for them to then support their teachers' learning effectively. High-quality mathematics teaching practices have been widely researched in mathematics education. We draw on observation tools that systematize them in specific dimensions that encompass rich and meaningful mathematics, ambitious practices (Boston, 2012), and culturally responsive teaching (Aguirre et al., 2013). We also draw on the construct of teacher noticing (Sherin et al., 2011) and hypothesize that developing the skills of attending to specific aspects of mathematics instruction, reasoning about them using frames of reference that characterize high-quality practices, and discussing the teacher's instructional decisions, will support principals in developing a shared vision of high-quality mathematics instruction. In this way, we seek to contribute to the limited existing research on principals' noticing and on how noticing relates to the development of a vision for high-quality math instruction. We collected data from four sources: audio transcripts of the sessions, a noticing task at the end of each session, field notes, and a final individual interview. Using dimensions from existing research (Aguirre et al., 2013; Boston, 2012; Munter, 2014), we developed a coding system to analyze and triangulate data, allowing emerging codes during the coding process.

Initial findings suggest that principals' noticing became more sophisticated, evolving from attention to general aspects of the lesson toward noticing specific elements of students' mathematical thinking and the instructional moves that led to and supported them. The data obtained in the final interview about the participants' vision will be analyzed alongside responses to the noticing task to examine relationships between the two constructs.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Boston, M. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal, 113*(1), 76–104. https://doi. org/10.1086/666387

Kazemi, E., Gibbons, L., Lewis, R., Fox, A., Hintz, A.B., Kelley-Petersen, M.,...Balf, R. (2018). Math Labs: Teachers, teacher educators and school leaders learning together with and from their own students. *NCSM Journal, 19*, 1.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584–635. https://doi.org/10.5951/jresematheduc.45.5.0584

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011). *Mathematics teacher noticing: Seeing through teachers' eyes.* New York: Routledge.

Supovitz, J., Sirinides, P., & May, H. (2010). How Principals and Peers Influence Teaching and Learning. *Educational Administration Quarterly, 46*(1), 31–56. https://doi.org/10.1177/1094670509353043.

DEVELOPING A SHARED VISION OF HIGH-QUALITY MATHEMATICS INSTRUCTION AMONG SCHOOL LEADERS

Christina Kimmerling
UC Irvine
ckimmerl@uci.edu

Patricia Fuentes Acevedo
UC Irvine
pfuente@uci.edu

Rossella Santagata
UC Irvine
r.santagata@uci.edu

Keywords: Instructional Vision, Instructional Leadership

Principals influence student achievement through setting school-wide goals, engaging in instructionally focused conversations with teachers, and providing opportunities for professional development and collaboration (Grissom et al., 2021; Supovitz & Sirinides, 2010). A well-developed vision of high-quality instruction will inform each of these activities. Further, in the context of district-wide instructional improvement, a shared vision will support coherence across professional development initiatives (Cobb et al., 2020). In this study, we explored how a shared instructional vision can be developed among district and school leaders in a research practice partnership (RPP) focused on improving elementary school students' mathematics learning, and in particular, the achievement of Latinx students within a district.

Through a collaborative process, RPP members (six principals, three district leaders, a mathematics coordinator and three university researchers) identified developing a shared vision as a key goal of improvement efforts. In service of this goal, RPP members participated in four Math Labs in a 4th grade classroom. The design of each Math Lab emphasized school leader learning in practice and was inspired by previous Math Labs designed by Kazemi et al. (2018). The Math Labs provided a context for district and school leaders to co-enact instructionally focused conversations with the teacher. In this study, we asked: how do Math Labs support the development of a shared vision of high-quality and equitable mathematics instruction?

Munter (2014) describes vision as an image of what is possible in a classroom space, though may not currently be enacted. He characterizes individual trajectories of a vision of high-quality mathematics instruction across four dimensions: mathematical tasks, the role of the teacher, classroom discourse, and student engagement. We extend this work in two ways. First, we explicitly integrate equitable practices as described by Aguirre et al., (2013) within a vision of high-quality instruction. Further, drawing from Goodwin's (1994) description of professional vision, we situate school and district leaders' vision of high-quality instruction within the shared discursive practices used to see and understand a classroom space. While elements of high-quality instruction can be observed during the Math Labs, how district and school leaders ultimately see those practices is informed by their roles within the context and larger society.

Transcripts of discussions and participant interviews and field notes from Math Labs and design sessions were coded by the team of researchers. Artifacts generated from the Math Labs were included in the data analysis. We explored the process by which a shared vision begins to take shape. Initial findings suggest that a shared experience, immersed in a classroom setting, provides a rich opportunity to make sense of aspects of high-quality and equitable instruction and to develop a shared discourse around teacher professional identity. Yet, we identified moments when what is seen is mediated by participant roles, suggesting how professional vision reflects and influences the image of high-quality instruction. Our study expands current literature on contexts that contribute to development of shared vision of high-quality mathematics instruction for school leaders.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices.* Reston, VA: National Council of Teachers of Mathematics.

Cobb, P., Jackson, K., Henrick, E., & Smith, T. M. (2020). *Systems for instructional improvement: Creating coherence from the classroom to the district office*. Harvard Education Press

Goodwin, C. (1994). Professional vision. *American Anthropologist*, 96, 606-633.

Grissom, J. A., Egalite, A.J., & Lindsay, C.A. (2021). How Principals Affect Students and Schools: A Systematic Synthesis of Two Decades of Research. *The Wallace Foundation.*

Kazemi, E., Gibbons, L. K., Lewis, R., Fox, A., Hintz, A. B., Kelley-Petersen, M., & Balf, R. (2018). Math Labs: Teachers, teacher educators, and school leaders learning together with and from their own students. *Journal of Mathematics Education Leadership,* 19(1), 23-36.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education*, 45(5), 584–635.

Supovitz, J., Sirinides, P., & May, H. (2010). How Principals and Peers Influence Teaching and Learning. *Educational Administration Quarterly,* 46(1), 31–56. https://doi.org/10.1177/1094670509353043

EXPLORING DISSONANCE AND HARMONY AMONG MENTEES AND MENTORS' CONCEPTUALIZATIONS OF EFFECTIVE PEER TEACHING MENTORSHIP

Melinda Lanius
Auburn University
melinda.lanius@auburn.edu

Leigh Harrell-Williams
The University of Memphis
leigh.williams@memphis.edu

RaKissa Manzanares
University of Colorado Denver
rakissa.manzanares@ucdenver.edu

Kelsey Funkhouser
University of Colorado Denver
kelsey.funkhouser@ucdenver.edu

Josias Gomez
The University of Memphis
jogomez@memphis.edu

Keywords: Professional Development; Doctoral Education; Instructional Leadership

Graduate student peer-mentoring programs benefit participants by providing unique academic, social, psychological, and career development opportunities (Lorenzatti et al., 2019). However, the positive effects of research-oriented peer-mentoring programs are much better understood than teaching-oriented ones. In our poster, we consider mentees and mentors' perceptions of effective mentoring in a teaching-oriented peer mentorship program.

Previous research provides possible frameworks regarding mentee-mentor interactions. Rose (2003) designed the Ideal Mentor Scale for doctoral students to consider the qualities they value most in a faculty mentor: *integrity, guidance*, and *relationship*. With regard to school-based youth mentoring, Brodeur et al. (2015) identified the dimensions of *structure, engagement, autonomy,* and *competency support* that governed mentee and mentor impressions of mentor behaviors. However, no existing theoretical or empirical framework could be identified to address the unique nuance of peer GTA teaching mentoring relationships.

As part of a multi-component program, *Promoting Success in Undergraduate Mathematics Through Graduate Teaching Assistant Training* (PSUM-GTT; Harrell-Williams et al., 2020), GTAs participated in a peer teaching mentoring program. This study focuses on two research questions: What are mentees and mentors' conceptualizations of "effective" peer teaching mentors? How do mentors' views of themselves compare to their conceptualizations?

Survey data was collected from mentees and mentors, with Likert-type and open-ended items relating to each groups' conceptualization of an effective mentor. For example, we asked mentors where they saw a mentor on a continuum of an authority figure (scored as "1") to a collaborator (scored as "5") and compared this to how they saw themselves as mentors using the same scale. According to a contingency table analysis, 10 of 15 mentors rated themselves as more collaborative (more "4"s or "5"s) than they imagine a mentor in general should be (more "2"s and "3"s). According to a psycholinguistic analysis in Linguistic Inquiry and Word Count (LIWC) 2022 (Boyd et al., 2022), 11 of the 15 mentors felt more confident, according to the LIWC clout score, when reporting about the nature of an ideal mentor than when they wrote about their own role as a mentor. Taken together, this suggests that our peer mentors didn't see themselves as being as much of an "authority" as they would like to in their new roles. Further results concerning mentees and mentors' conceptualizations and their alignment in conceptualizations will be presented in our poster.

Acknowledgements

This project is funded by collaborative NSF Awards DUE #1821454, 1821460, and 1821619.

References

Boyd, R. L., Ashokkumar, A., Seraj, S., & Pennebaker, J. W. (2022). The development and psychometric properties of LIWC-22. Austin, TX: University of Texas at Austin. https://www.liwc.app

Brodeur, P., Larose, S., Tarabulsy, G., Feng, B. & Forget-Dubois, N., (2015). Development and Construct Validation of the Mentor Behavior Scale, *Mentoring & Tutoring: Partnership in Learning*, DOI: 10.1080/13611267.2015.1011037

Harrell-Williams, L.M., Olson, G., Webb, J., Houston, S., & Gomez, J. (2020, February). How different is different? Examining institutional differences prior to scaling up a graduate teacher training program to improve undergraduate mathematics outcomes. Proceedings of 23rd Annual Conference on Research in Undergraduate Mathematics Education Conference (RUME), Boston, MA.

Lorenzetti, D. L., Shipton, L., Nowell, L., Jacobsen, M., Lorenzetti, L., Clancy, T., & Paolucci, E. O. (2019). A systematic review of graduate student peer mentorship in academia. *Mentoring & Tutoring: Partnership in Learning*, *27*(5), 549-576. DOI: 10.1080/13611267.2019.1686694

Rose, G. L. (2003). Enhancement of mentor selection using the ideal mentor scale. *Research in Higher Education*, *44*(4), 473-494.

CO-DESIGNING FOR STATEWIDE ALIGNMENT OF A VISION FOR HIGH QUALITY MATHEMATICS INSTRUCTION

Katherine J. Mawhinney
Appalachian State University
mawhinneykj@appstate.edu

Catherine S. Schwartz
East Carolina University
schwartzca@ecu.edu

P. Holt Wilson
UNC Greensboro
phwilson@uncg.edu

Michelle Stephan
UNC Charlotte
michelle.stephan@uncc.edu

Allison W. McCulloch
UNC Charlotte
Allison.mcculloch@uncc.edu

Olu Adefope
East Carolina
adefopeo@ecu

Christine Fisher
UNC Greensboro
cakierna@uncg.edu

Cathy Holl-Cross
UNC Charlotte
chollcro@uncc.edu

Olanrewaju Oriowo
UNC Charlotte
ooriowo@uncc.edu

Keywords: Instructional Vision, Systemic Change

This project aims to test the conjecture that developing a shared vision of high quality mathematics instruction (HQMI) is foundational to the successful implementation of STEM education innovations. Since 2016, our research practice partnership (RPP) of state, district, and school-based leaders, mathematics teachers, and researchers have engaged in design-based implementation research (Fishman, Penuel, Allen, Cheng, & Sabelli, 2013) to iteratively co-design instructional resources that promote a shared, state-wide vision of HQMI. We aim to build upon this work by investigating the visions of HQMI held by educators at different levels of a state educational system, the extent to which those visions are shared, and how the visions mediate and are mediated by the co-design and uptake of implementation resources.

HQMI aims for teachers to be intentional in supporting students, for example, by problematizing ideas (Munter, 2014), supporting students in developing mathematical authority (Lampert, 1990), and scaffolding classroom discussions in ways that formalize learning goals for students (Smith & Stein, 2011). Established and emerging research suggests that sharing a vision of HQMI can support successful implementation of new programs or policies (Gamoran, 2003), relates to improved instructional quality (Munter & Correnti, 2017), and can lead to improvements in students' academic outcomes (Chance & Segura, 2009). In addition, research points to the ways in which educators' visions are shaped by participating in different social contexts (Munter & Wilhelm, 2021) and informed by different or conflicting messages from both inside and outside schools (Ticknor & Schwartz, 2017). Teacher collaboration, PD, and productive collaborations across educator roles are rarely effective unless they are tied to a shared vision of instruction (Peterson et al., 1996).

Using a mixed methods design we are studying our design process with a goal of understanding the ways in which vision of HQMI and implementation resources interact across a state educational system and the extent to which a shared vision of HQMI leads to coherence. Our poster presentation will describe the project as a whole, share the initial iteration of our framework design, and report our initial findings.

Acknowledgements

This work was supported by the National Science Foundation under grant DRL - 2100895 awarded to East Carolina University, DRL-2100903 awarded UNC Greensboro, DRL - 2100947 awarded UNC Charlotte, and DRL - 2100833 awarded to Appalachian State University. Any opinions, findings, and conclusions or recommendations expressed herein are those of the principal investigators and do not necessarily reflect the views of the National Science Foundation.

References

Chance, P. L., & Segura, S. N. (2009). A rural high school's collaborative approach to school improvement. *Journal of Research in Rural Education (Online)*, *24*(5), 1-12.

Fishman, B. J., Penuel, W. R., Allen, A. R., Cheng, B. H., & Sabelli, N. O. R. A. (2013). Design-based implementation research: An emerging model for transforming the relationship of research and practice. *National society for the study of education*, *112*(2), 136-156.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: mathematical knowing and teaching. *American Educational Research Journal*, *27*, 29–63.

Munter, C., & Correnti, R. (2017). Examining relations between mathematics teachers' instructional vision and knowledge and change in practice. *American Journal of Education*, *123*(2), 171-202.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education*, *45*(5), 584-635.

Munter, C. & Wilhelm, A.G. (2020). Mathematics teachers' knowledge, networks, practice, and change in instructional vision. *Journal of Teacher Education*, 1-13. https://doi.org/10.1177/0022487120949836

Smith, M.S. & Stein, M.K. (2011). *5 practices for orchestrating productive mathematics discussions.* National Council of Teachers of Mathematics.

Ticknor, A. S., & Schwartz, C. S. (2017). "It just got real:" Navigating the affordances and constraints of school-based learning in a mathematics-specific induction program. *Action in Teacher Education*, *39*(2), 138-152.

Peterson, P. L., McCartney, S. J., & Elmore, R. F. (1996). Learning from school restructuring. *American Educational Research Journal*, *33*(1), 119-153.

EXPLORING ADMINISTRATORS' VISIONS OF EQUITABLE MATH INSTRUCTION

Lisa M. Siciliano
University of Illinois Chicago
Lsicil2@uic.edu

Enisa Selimbegovic Akgul
University of Illinois Chicago
Eakgul2@uic.edu

Keywords: Equity, Inclusion, and Diversity; Instructional Leadership; Instructional Vision

School administrators play an important role in influencing instruction (Nelson and Sassi, 2000) and building organizational capacity for equity (Galloway and Ishamaru, 2020). Previous research has focused separately on administrator content knowledge (Steele et al., 2015) and perceptions of equity (Nadelson et al., 2020). However, limited research details how administrators think about equity *within* the context of math instruction. Meanwhile, studies of teachers have shown a connection between their *visions* and their instructional practice (Munter & Correnti, 2017). Therefore, administrators' visions of equitable instruction may also affect math teaching and learning. This study examines this relationship, asking: *How do district administrators characterize their visions of equitable math instruction?*

Methods

This study is situated within a multi-year design-based implementation research initiative: the CASPIR Math Project. Leaders from two districts (Elm, n = 6; Maple, n = 5) were interviewed using the visions of high-quality mathematics instruction instrument (Munter, 2014) with the added question: *What would you look for to determine whether instruction is equitable?* The National Council of Teachers of Mathematics (NCTM) equity position statement (2014) provided an instruction-centered framework for coding, which included the following topics: access, students, curriculum, teachers, differentiation, progress monitoring, accommodations, and remediation/additional challenge. Authors looked for patterns within and across districts.

Initial Findings and Implications

No two individuals expressed the same vision. However, three categories emerged revealing patterns in how administrators spoke about equitable instruction: methods to achieve equitable math instruction, perception of students, and challenges to implementation. Administrators from Maple tended to mention grouping as the main form of differentiation, referring to students in three leveled groups. In contrast, Elm administrators mentioned more specific and varied forms of differentiation focusing on lesson planning and task modification, whilst maintaining high cognitive demand. They also tended to discuss students through a more individualized lens. On the other hand, challenges to equitable instruction in both districts were discussed and included teachers' ability and mindset, curriculum, and family support. These findings, while preliminary, have implications for researchers and practitioners. Alignment in visions cannot be assumed among administrators. Thus, conversations about what equitable mathematics instruction is, who should be working towards it, how it should be achieved, and for what reasons may be required to move towards a more common understanding.

Acknowledgements

Research reported in this paper is based on work supported by the National Science Foundation as part of the DRK-12 program. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Galloway, M. K., & Ishimaru, A. M. (2020). Leading equity teams: The role of formal leaders in building organizational capacity for equity. *Journal of Education for Students Placed at Risk (JESPAR)*, *25*(2), 107-125.

Munter, C., & Correnti, R. (2017). Examining relations between mathematics teachers' instructional vision and knowledge and change in practice. *American Journal of Education*, *123*(2), 17-202.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for research in mathematics education*, *45*(5), 584-635.

Nadelson, L. S., Albritton, S., Couture, V. G., Green, C., Loyless, S. D., & Shaw, E. O. (2020). Principals' Perceptions of Education Equity: A Mindset for Practice. *Journal of Education and Learning*, *9*(1), 1-15

National Council of Teachers of Mathematics. (2014). *Access and equity in mathematics education: A position of the National Council of Teachers of Mathematics.* https://www.nctm.org/uploadedFiles/Standards_and_Positions/Position_Statements/Access_and_Equity.pdf

Nelson, B. S., & Sassi, A. (2000). Shifting approaches to supervision: The case of mathematics supervision. *Educational Administration Quarterly*, *36*(4), 553-584.

Steele, M. D., Johnson, K. R., Otten, S., Herbel-Eisenmann, B. A., & Carver, C. L. (2015). Improving instructional leadership through the development of leadership content knowledge: The case of principal learning in algebra. *Journal of Research on Leadership Education*, *10*(2), 127-150.

Chapter 10:

Precalculus, Calculus, and Higher Mathematics

THE REASONING OF ENGINEERING STUDENTS WHEN THEY FACE PROBLEMS INVOLVING THE FUNDAMENTAL THEOREM OF CALCULUS

EL RAZONAMIENTO DE ESTUDIANTES DE INGENIERÍA FRENTE A PROBLEMAS QUE IMPLICAN EL TEOREMA FUNDAMENTAL DEL CÁLCULO

Omar Arenas Bonifacio
Departamento de Matemática Educativa, Cinvestav, IPN
omar.arenas@cinvestav.mx

Ernesto Alonso Sánchez Sánchez
Departamento de Matemática Educativa, Cinvestav, IPN
esanchez@cinvestav.mx

Mario Sánchez Aguilar
CICATA Legaria, Instituto Politécnico Nacional
mosanchez@ipn.mx

Francisco Sepúlveda Vega
Departamento de Matemática Educativa, Cinvestav, IPN
francisco.sepulveda@cinvestav.mx

La pregunta es: ¿Cómo razonan los estudiantes de ingeniería que han llevado un curso tradicional de cálculo ante problemas simples de flujo de líquido en los que el proceso de solución involucra el teorema fundamental del cálculo? Un cuestionario de tres preguntas se administró a 18 estudiantes de ingeniería de una universidad de la ciudad de México que habían llevado un curso ordinario de Cálculo. Desde una perspectiva de Teoría Fundamentada se codificaron sus respuestas y se generaron tres categorías generales que hemos denominado razonamientos procedimental, transicional o conceptual. Los resultados, al igual que estudios previos, muestran que la mayoría de los estudiantes utilizan un razonamiento procedimental, no obstante, el análisis permite proponer rasgos que indican la transición hacia un razonamiento conceptual. Además, se propone un marco para analizar el razonamiento de los estudiantes.

Palabras clave: Teorema Fundamental del Cálculo, Razonamiento procedimental, transicional y conceptual.

Thompson (1994, p. 142) se refiere al Teorema Fundamental del Cálculo (TFC) como “uno de los sellos intelectuales en el desarrollo del cálculo” y, por tanto, es uno de los objetivos importantes de aprendizaje de un curso anual de cálculo. El TFC conecta los dos grandes temas clásicos del cálculo: derivación e integración. En la enseñanza inicial y tradicional del cálculo cada uno de estos temas tiene un desarrollo propio y se motivan con modelos intuitivos (Fischbein, 1977) que no tienen que ver uno con el otro: por un lado, la pendiente de una línea tangente a la función que se deriva, por otro, el área bajo la curva de la función que se integra. No obstante, el curso anual de cálculo de una escuela de ingeniería suele incluir el TFC, ¿cómo lo procesan los estudiantes? En la investigación en educación matemática se han identificado las dificultades para entender el TFC y se han propuesto alternativas para superarlas, por ejemplo, el apoyo de modelos geométrico-visuales (Kirsch, 1976, 2014) o el modelo de la integral como acumulación asociada al modelo de derivada como razón de cambio (Thompson, 1994). Teniendo en perspectiva este modelo, nosotros nos preguntamos ¿Cómo razonan los estudiantes de ingeniería que han llevado un curso tradicional de cálculo ante problemas simples de flujo de líquido en los que el proceso de solución involucra del TFC? Responder a esta pregunta daría información sobre las tendencias de razonamiento hacia el TFC de los estudiantes actuales de ingeniería, al menos de la escuela en la que se hizo la investigación, y permitiría diseñar los cambios graduales del enfoque en los cursos de cálculo tradicionales sobre la base de dicha información y las nuevas propuestas que la investigación sugiere.

Marco conceptual

Se entiende por razonamiento a los procesos para obtener y verificar proposiciones (conclusiones) sobre la base de la evidencia o de conocimientos establecidos (teoremas) o suposiciones. El razonamiento puede tomar muchas formas que van desde argumentaciones informales hasta demostraciones deductivas (Mahir, 2009). Cuando los estudiantes responden a la pregunta ¿Por qué…?, ellos muestran aspectos importantes de su razonamiento. Las respuestas de los estudiantes las clasificamos en los siguientes tres niveles de acuerdo con el razonamiento que reflejan.

Razonamiento procedimental. Consiste en aplicar algoritmos o procedimientos típicos para responder a la pregunta sin utilizar los teoremas pertinentes para simplificar o evitar operaciones y llegar a la solución.

Razonamiento transicional. Ocurre cuando se identifican teoremas que permiten encontrar relaciones pertinentes para responder a la pregunta, pero tienen dificultades para su instrumentación.

Razonamiento conceptual. Consiste en utilizar una red de relaciones. Se identifican los teoremas pertinentes y se encuentran las relaciones con los datos del problema, los procedimientos y el uso de la notación simbólica.

Sfard (1991) desarrolló la idea de que la naturaleza de las entidades matemáticas es dual: se presentan como procesos y como objetos. Estas formas suelen captarse por los estudiantes en ese orden, primero como procesos y luego como objetos, aunque en ciertos casos pueden captarse a la inversa. Sfard señala que en la literatura se han formulado varias dicotomías que buscan dar cuenta de la naturaleza dual del conocimiento/pensamiento/comprensión matemática, por ejemplo, abstractas y algorítmicas (Halmos, 1985), declarativas y procedimentales (Anderson, 1976), proceso y producto (Kaput, 1979; Davis, 1975). En esta misma clase de dicotomías está la clásica distinción de Skemp (1976) de entendimiento procedimental y relacional.

En nuestro caso, en lugar de que el objeto de estudio sean las entidades matemáticas, el pensamiento o la comprensión hemos preferido elegir el razonamiento; este tiene la ventaja de que su calidad se puede evaluar en las respuestas de los estudiantes en las que intentan justificar un resultado. Por otro lado, a la dualidad procedimental-conceptual agregamos la categoría intermedia de razonamiento transicional que da cuenta de que el paso de uno a otro tiene rasgos que combina un cierto grado de desarrollo del primero y rasgos incipientes del segundo.

Antecedentes

Dos estados del arte recientes sobre aprendizaje y enseñanza del cálculo incluyen un panorama de la investigación en los temas de integral y TFC, estos son Bressoud (2016) y Larsen et al. (2018). En estas reseñas encontramos referencias que se relacionan con ideas de la presente investigación. Por ejemplo, Mahir (2009) y Rasslan y Tall (2002) sostienen que la mayoría de los estudiantes que han completado un curso del cálculo tienen una visión procedimental de la integración y son incapaces de interpretarla en otros contextos. Por otro lado, los estudios que tienden a enfatizar aspectos conceptuales de integración proponen introducir la integral con sumas de Riemann (Sealey, 2014) o desde el modelo de acumulación (Thompson, 1994); esto ayudaría a superar el enfoque puramente procedimental. Thompson y Silverman (2008) afirmaron que la noción de acumulación es sencilla en el sentido de que la mayoría de las personas entienden la idea de "acumular una cantidad a medida que obtenemos más" (p. 43). Un problema de estos acercamientos alternativos cuyo objetivo es destacar el aspecto conceptual de la integral es que no ponen énfasis en las relaciones de la integral con los procedimientos de integración y el manejo de la notación simbólica.

Varios autores han enfatizado en que el conocimiento conceptual puede visualizarse como una red de relaciones interconectadas (Skemp, 1978, Miller y Hudson, 2007; Rittle-Johnson y Schneider, 2015). Las conexiones de la integral, por ejemplo, abarcan la definición, la notación y los teoremas, además, por supuesto, su relación con las sumas de Riemann y los modelos de áreas y de acumulación. Es por esto que, en este trabajo, se exploran y caracterizan las relaciones que los estudiantes logran ver y utilizar exitosamente o no.

Método

El presente estudio explora el razonamiento que reflejan las respuestas de los estudiantes a los problemas del instrumento. Este se creó de manera que se relacionaran con el TFC y se planteara una situación de flujos. Las respuestas de los estudiantes se codificaron y se organizaron en los tipos de razonamiento presentados en el marco conceptual. Este análisis estuvo inspirado en los procedimientos de la teoría fundamentada (Birks y Mills, 2015).

Participantes

Dieciocho estudiantes de ingeniería (entre 18 y 21 años) de una universidad privada ubicada en la ciudad de México resolvieron individualmente los problemas diseñados para el presente estudio. Ellos habían finalizado su primer curso de cálculo antes de la aplicación del instrumento.

Instrumento

Consistió en tres problemas elaborados por los autores, dos ubicados en el contexto de flujos de agua y otro abstracto. La idea de los problemas es que pudieran resolverse con el TFC en un contexto en el que la derivada se interpreta como razón de cambio y la integral como acumulación. Otro rasgo de los problemas es que se pueden resolver mediante procedimientos algorítmicos, pero interpretando el TFC. Los problemas del instrumento se muestran en la figura 1.

1. Por un grifo sale agua a una razón variable $q(t) = \frac{1}{10}\sqrt{t+4} - \frac{2}{10}$ (dada en litros/segundo) que es colectada en un contenedor. $V(x)$ representa el agua que se ha colectado desde que se abre el grifo ($t = 0$) hasta el tiempo $t = x$
   1.1. ¿Cuál es la razón de cambio del volumen $V(x)$ respecto al tiempo, en los tiempos $t = 5\ s,\ t = 21\ s$ y $t = 32\ s$?
   1.2. Obtenga una expresión para: $V(x)$
   1.3. $V_a(x)$ representa el volumen que se ha almacenado desde $t = a$ hasta $t = x$ $(x > a)$. Obtenga una expression para: $V_a(x)$
   1.4. Obtenga $\frac{dV_a(x)}{dx}$
2. Se tiene un depósito de forma cilíndrica cuyo radio de la base es r y altura H; el depósito contiene agua hasta una altura h. $V(x)$ representa el volumen de agua contenido en tal depósito cuando h toma el valor x.
   2.1 Exprese $V(h)$ como una integral
   2.2 Obtenga $\frac{dV(x)}{dx}$
3. Si $F(x) = \int_a^x f(t)\,dt$, obtenga $F'(x)$, justifique su respuesta

Figura 1: Problemas del instrumento

Con relación al problema 1 anterior, consideramos que el razonamiento procedimental se presenta cuando se llevan a cabo las operaciones que indican las expresiones: $q'(t)$ y $\int q(t)dt.$ En cambio, el razonamiento conceptual se expresa cuando se plantea la expresión $V(x) = \int_0^x q(t)dt,$ y se obtienen las consecuencias del TFC sin necesariamente realizar muchas operaciones; es decir, se debe saber que si $q(x)$ es una función continua entonces V(x) es derivable y V'(x) = q(x). Como el flujo es continuo no se requieren otras versiones más generales del TFC.

Procedimiento

El profesor titular del grupo colaboró con los autores para aplicar el instrumento en su grupo; los estudiantes que accedieron a resolver los problemas lo hicieron voluntariamente a invitación expresa de su profesor. En una sesión de 45 minutos los estudiantes resolvieron sólo con lápiz y papel los problemas. Se les pidió que explicaran lo mejor posible y por escrito sus soluciones.

Procedimiento de análisis

Se codificaron las respuestas de los estudiantes mediante las comparaciones de unas con otras para determinar patrones de razonamiento que, mediante un proceso de abstracción, subsumimos en las categorías que se presentan en el marco conceptual. Cabe mencionar que tales categorías fueron emergentes del análisis y que no se tenían presentes antes de llevarlo a cabo.

Resultados

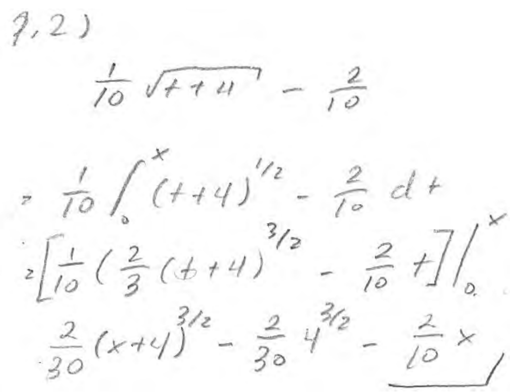
Las respuestas escritas que los estudiantes dieron a los problemas se analizaron, se codificaron y se agruparon en los tres tipos de razonamiento definidos en el marco. La tabla 1 muestra las frecuencias de respuestas a las preguntas 1.1, 1.2, 1.3, 1.4, 2.2 y 3 que cayeron en cada categoría. Después de la tabla se mostrarán ejemplos de soluciones de cada tipo de razonamiento.

Tabla 1: Frecuencia de respuestas por tipo de razonamiento

	Problemas						
Tipo de respuesta	1.1	1.2	1.3	1.4	2.2	3	Total
1. Procedimental	11	15	14	10	13	0	63
2. Transicional	5	0	3	1	0	8	17
3. Conceptual	1	0	0	1	0	7	9
4. Sin respuesta	1	3	1	6	5	3	19

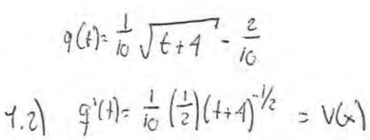
Como se puede observar en la tabla 1, el razonamiento procedimental es el que predominó en las soluciones que propusieron los estudiantes en el problema 1 y en el 2.2. La solución que se esperaba en el problema 1.2 era la expresión: $V(x) = \int_0^x q(t)dt$, sustituyendo $q(t)$ por la expresión que la define. No obstante, la totalidad de los estudiantes que respondieron no se conforman con lo anterior y entienden que deben llevar a cabo todos los cálculos. Por ejemplo, en la figura 2 se exhibe la solución de un estudiante al problema 1.2; al final de su solución el estudiante omite un exponente.

Dentro de las respuestas que reflejan un razonamiento procedimental, se observa la tendencia a creer que las respuestas a los problemas implican necesariamente llevar a cabo un procedimiento, sin imaginar que se les pide sólo indicar una relación pertinente. Por ejemplo, en el problema 1.4, que pedía sólo que se identificara que $\frac{dV(x)}{dx} = q(x)$, la mitad de los estudiantes respondieron derivando q(x), probablemente porque era la única función explícita que podían derivar. En la figura 3 se ilustra la respuesta de un estudiante.



$$" \frac{1}{10}\sqrt{t+4} - \frac{2}{10}$$
$$= \frac{1}{10}\int_0^x (t+4)^{\frac{1}{2}} - \frac{2}{10}dt$$
$$= \frac{1}{10}\left(\frac{2}{3}(t+4)^{\frac{3}{2}} - \frac{2}{10}t\right)\Big|_0^x$$
$$\frac{2}{30}(x+4)^{\frac{3}{2}} - \frac{2}{30}4^{\frac{3}{2}} - \frac{2}{10}x$$

Figura 2: Razonamiento procedimental. Respuesta al problema 1.2

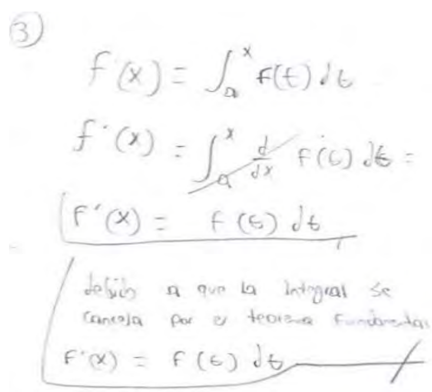


“$q(t) = \frac{1}{10}\sqrt{t+4} - \frac{2}{10}$

1.2) $q'^{(t)} = \frac{1}{10}\left(\frac{1}{2}\right)(t+4)^{-1/2} = V(x)$”

Figura 3: Razonamiento procedimental. Respuesta a la pregunta 1.4

Un razonamiento transicional lo detectamos con mucho menos frecuencia, del total de las 108 respuestas, clasificamos en transicional cerca del 16%. El estudiante 3, propone la respuesta de la figura 4. Este estudiante tiene una noción del TFC, no obstante, en varios aspectos importantes falla.



“$\mathrm{F}(x) = \int_a^x f(t)dt$

$$\mathrm{F}'(x) = \int_a^x \frac{d}{dx} f(t)\, dt$$

$$\mathrm{F}'(x) = f(t)dt$$

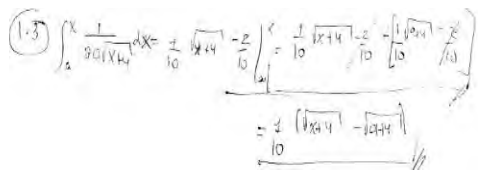
Debido a que la integral se cancela por el TFC:

$\mathrm{F}'(x) = f(t)dt$”

Figura 4: Razonamiento transicional. Respuesta de un estudiante al problema 3

Los estudiantes que manifiestan un razonamiento transicional en sus respuestas tratan de relacionar los problemas con el TFC, pero varios de ellos tienen una concepción imprecisa del él. Por ejemplo, algunos creen que la relación del TFC es la siguiente: $f(x) = \int_a^x \frac{d}{dx} f(x)dx$, la cual, de hecho, es falsa. Conviene observar en la respuesta al problema 3, cómo el estudiante en el segundo paso pretende haber obtenido esta expresión. Tampoco supo manejar la notación, manteniendo la variable muda t y el signo del diferencial dt.

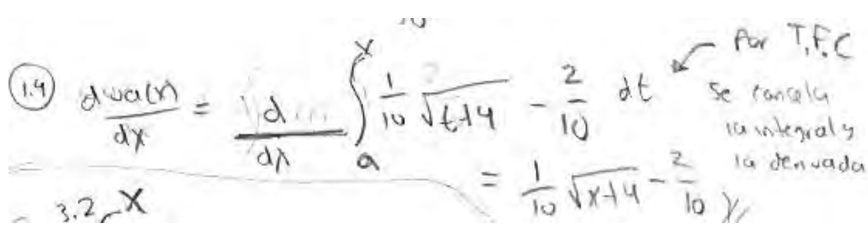
En la figura 5, vemos una respuesta al problema 1.3, que también clasificamos como transicional porque aparentemente se utiliza el TFC. En efecto, se puede notar que trató de reproducir la ecuación: $\int_a^x \frac{d}{dt} q(t)dt = q(x) - q(a)$. No obstante, en lugar de considerar que la integral indefinida del integrando es una constante cualquiera, utilizó 2/10; aunque este error intermedio al final no afecta. El problema es que no consideró que el flujo q(x) ya es una derivada y que la derivada de este flujo no es pertinente al problema.



$$\text{“}\int_a^x \frac{1}{20\sqrt{x+4}} dx = \frac{1}{10}\sqrt{x+4} - \frac{2}{10}\Big|_a^x = \frac{1}{10}\sqrt{x+4} - \frac{2}{10} - \left[\frac{1}{10}\sqrt{a+4} - \frac{2}{10}\right]$$
$$= \frac{1}{10}\left(\sqrt{x+4} - \sqrt{a+4}\right)\text{”}$$

Figura 5: Razonamiento transicional. Respuesta de un estudiante al problema 1.3

Finalmente, con relación al razonamiento conceptual, sólo 8% del total de las respuestas reflejaron este tipo de razonamiento. Un ejemplo es la respuesta del problema 1.4 de la figura 6, en la simpleza de dicha respuesta se esconden varias decisiones que el estudiante debió realizar, a saber, ha aplicado el modelo de acumulación al definir el volumen en términos de la integral de la función de flujo. Luego, sin apresurarse a calcular esta integral, deriva ambas partes de la ecuación integral que define el volumen, observando que al derivar una integral definida puede aplicar el TFC, menciona que se cancela las operaciones de derivada e integral, pero tiene el cuidado de sustituir la variable muda t, por la variable x (que representa el tiempo). Estas decisiones implican el conocimiento y coordinación de varias relaciones, por lo tanto, no se puede creer que es un procedimiento rutinario, sino que reflejan un nivel de razonamiento conceptual.



$$\text{“}\frac{dV_a(x)}{dx} = \frac{d}{dx}\int_a^x \frac{1}{10}\sqrt{t+4} - \frac{2}{10} dt \ = \frac{1}{10}\sqrt{x+4} - \frac{2}{10}$$

Por el TFC se cancela la integral y la derivada”

Figura 6: Razonamiento conceptual. Respuesta a la pregunta 1.4

Conviene aclarar que los 7 estudiantes cuyas respuestas se clasificaron como conceptuales en el problema 3 (abstracto), no necesariamente aplicaron el mismo tipo de razonamiento en los problemas en contexto, sino que en algunos casos sus respuestas fueron transicionales y en otros incluso procedimentales, por ejemplo, en las respuestas al problema 1.2.

Discusión y conclusiones

Es conocido que los estudiantes de ingeniería, en contraste con estudiantes de matemáticas, tienden a interesarse por los conocimientos que ellos creen que son prácticos, y evitan involucrarse con aquellos que consideran teóricos. El razonamiento procedimental se asocia más al conocimiento práctico, mientras que se piensa que el razonamiento conceptual es más afín al conocimiento teórico. Con esta tendencia, enfocan su conocimiento de los teoremas más en sus posibilidades de aplicación para fortalecer las herramientas prácticas, que en su función explicativa; por ejemplo, para ellos, suele no ser muy relevante la prueba de un teorema. No es sorprendente entonces verificar que la mayoría de los estudiantes del presente estudio reflejen en sus respuestas un razonamiento procedimental.

Un rasgo importante de las dos primeras situaciones/problema es que las funciones se presentaron en el contexto de flujos de agua; este contexto remite al modelo intuitivo de la integral como acumulación. En este sentido, cabe señalar, que en la mayoría de los casos los estudiantes no tuvieron dificultades para representar a la función volumen como una integral definida. Una vez identificado esto, la mayoría echaron mano de los procedimientos de derivación e integración para responder las preguntas. Sólo algunos estudiantes lograron percibir que el TFC podría ser utilizado para evitar procedimientos laboriosos y, de estos, sólo dos lograron tener control suficiente de la notación simbólica involucrada.

De manera más general, lo anterior nos lleva a reflexionar de que los teoremas permiten hacer más eficaces a los procedimientos, como lo ilustra claramente el TFC, pues sin él resulta muy laborioso el cálculo de integrales. Entonces el desarrollo de un razonamiento procedimental no es ajeno a un razonamiento conceptual. La interpretación y uso de los teoremas y el control de la simbología que implican requiere del dominio de muchas relaciones. En las respuestas que se clasificaron en transicional se puede ver que, aunque los estudiantes entienden que pueden aplicar el TFC, aparte de que requieren interpretar de manera adecuada éste, deben relacionarlo con los datos de la situación/problemas y con la notación simbólica correspondiente. Como conclusión, proponemos un marco para analizar el desarrollo del razonamiento conceptual en tareas de Cálculo de los estudiantes. Este consiste en las relaciones entre los siguientes cinco términos: 1) Situación/problema, 2) Modelo intuitivo, 3) Procedimientos, 4) Teoremas, 5) Notación simbólica.

En este modelo, el razonamiento procedimental es sólo uno de los términos de las relaciones que contiene el razonamiento conceptual, pues este requiere de la incorporación de otras relaciones. Los modelos intuitivos como el área bajo la curva, o las situaciones de acumulación para la integral, también representa un término de las múltiples relaciones que se requieren dominar. Pero lo anterior debe ser consistente con los teoremas que, aunque los estudiantes de ingeniería no los tengan que demostrar, los tienen que interpretar y usar para sus fines. Finalmente, el proceso del dominio de las anteriores relaciones debe ir en paralelo con el desarrollo del lenguaje matemáticos, que se refleja en el control de la notación simbólica. Este modelo puede servir para orientar la enseñanza del cálculo para desarrollar el razonamiento conceptual de los estudiantes de ingeniería sin que crean que es un enfoque muy teórico en detrimento de sus expectativas por un conocimiento práctico.

Referencias

Birks, M., & Mills, J. (2015). *Grounded Theory. A Practical Guide*. SAGE.

Bressoud, D., Ghedamsi, I., Martinez-Luaces, V., & Törner, G. (2016). *Teaching and Learning of Calculus.* Springer.

Carlson, M. P., Smith, N., & Persson, J. (2003). Developing and Connecting Calculus Students' Notions of Rate-of Change and Accumulation: The Fundamental Theorem of Calculus. In N. A. Pateman, B. J. Dougherty, & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PMENA* (Vol. 2, pp. 165-172). Honolulu, HI: CRDG, College of Education, University of Hawaii.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Children's Mathematical Behavior 1*(3), 7-35.

Fischbein, E. (1977). Image and concept in learning mathematics. *Educational Studies in Mathematics 8*, 153-165.

Halmos, P. R. (1985). Pure thought is better yet. *The College Mathematics Journal 16*(1), 14-16.

Kaput, J. J. (1979). Mathematics and learning: Roots of epistemological status. In J. Lochhead, and J. Clement (eds.), *Cognitive Process Instruction* (pp. 289-302). Philadelphia, PA: Franklin Institute Press.

Kirsch, A. (2014). The fundamental theorem of calculus: visually? *ZDM Mathematics Education*, *46*(4), 691-695.

Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2018). Understanding the concepts of calculus: frameworks and roadmaps emerging from educational research. In J. Cai. (Ed.). *Compendium for Research in Mathematics Education* (pp. 526-550). Reston, VA: National Council of Teachers of Mathematics, Inc.

Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology, 40*(2), 201–211.

Miller S. P. y Hudson, P. (2007). Using evidence-based practices to build mathematics competence related to conceptual, procedural, and declarative knowledge. *Learning Disabilities Research & Practice, 22*(1), 47–57.

Rasslan, S., & Tall, D. (2002). Definitions and images for the definite integral concept. In A. Cockburn & E. Nardi (Eds.), *Proceedings of the 26th conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 89–96). Norwich, United Kingdom: PME

Rittle-Johnson, B., & Schneider, M. (2015). Developing Conceptual and Procedural Knowledge of Mathematics. In R. Cohen Kadosh & A. Dowker (Eds.), *Oxford handbook of numerical cognition* (pp. 1102-1118). Oxford, UK: Oxford University.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Skemp, R. R. (1978). Relational understanding and instrumental understanding. *The Arithmetic Teacher, 26*(3), 9-15.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of Calculus. *Educational Studies in Mathematics, 26*(2&3), 229–274.

COMPRENSIÓN DEPENDENCIA LINEAL: UNA PERSPECTIVA DE LAS ESTRUCTURAS Y MECANISMOS MENTALES CON ESTUDIANTES UNIVERSITARIOS

UNDERSTANDING LINEAR DEPENDENCE: A PERSPECTIVE OF MENTAL STRUCTURES AND MECHANISMS WITH UNIVERSITY STUDENTS

Silvia Juliana Ballesteros Gualdrón
Universidad Nacional del Rosario
julianaballesterosg@hotmail.com

Solange Roa Fuentes
Universidad Industrial de Santander
roafuentes@gmail.com

Este artículo tiene como objetivo presentar el diseño de una descomposición genética del concepto de dependencia lineal a partir del análisis de los libros de textos, de investigaciones anteriores relacionadas con el concepto y de los resultados de una intervención con estudiantes de álgebra lineal de una universidad pública en Colombia. En donde se busca tener un modelo cognitivo que sea el resultado de la primera fase de la metodología propuesta por APOE.

Palabras clave: Educación superior, Álgebra y Pensamiento Algebraico.

Objetivos del estudio

El concepto de dependencia lineal se considera como un objeto abstracto en la asignatura de álgebra lineal. Saldanha (1995) menciona que comprender el concepto de dependencia lineal a partir de diferentes enfoques es difícil para el estudiante y por eso no solo se necesita de actividades que motiven su aprendizaje sino también que promuevan acciones mentales que pueden potenciar su construcción.

Trigueros y Possani (2013) por ejemplo, mencionan que es un concepto nuevo que se ve en un primer curso de álgebra lineal, con el cual los estudiantes no se han encontrado en años anteriores; además indican que, a pesar de ello, el concepto de dependencia lineal es un concepto básico para entender temas más avanzados del álgebra lineal (Trigueros y Possani, 2013). Otros investigadores como Stewart y Thomas (2010) dan cuenta de las dificultades que tienen los estudiantes al trabajar con el concepto de dependencia lineal. Los resultados muestran que un énfasis en los procesos matriciales no ayuda al estudiante a entender el concepto, es decir, se necesita además reflexionar sobre ellos.

Con base en lo anterior, las siguientes preguntas planteadas guían esta investigación:

¿Qué estructuras y mecanismos mentales desarrollan estudiantes de álgebra lineal cuando abordan el concepto de dependencia lineal desde su representación numérica, geométrica, y algebraica?

¿Qué mecanismos mentales evidencian estudiantes de álgebra lineal I de una universidad pública de Colombia cuando estructuran el concepto de dependencia lineal desde de sus interpretaciones concretas y abstractas?

Así, el objetivo general de esta investigación es:

Diseñar una descomposición genética validada del concepto de dependencia lineal que parta de la aplicación de Acciones sobre objetos concretos para la construcción de Objetos abstractos.

Marco teórico

La construcción de un concepto matemático, según la teoría APOE, consiste en la construcción de estructuras mentales necesarias para comprender ese concepto. Cada estructura mental se construye a través de un mecanismo mental o también llamado *abstracción reflexiva*, según Piaget.

En la tabla 1, se muestra cómo están clasificadas las estructuras y los mecanismos mentales.

Tabla 1: Estructuras y mecanismos mentales (Síntesis de Arnon, et al., 2014)

Estructuras mentales	Mecanismos mentales
Acción	Interiorización
Proceso	Reversión
Objeto	Encapsulación
Esquema	Coordinación
	Inversión
	Des-encapsulación
	Generalización

A continuación, se describen las cuatro estructuras mentales y como están relacionadas con los mecanismos, descritos anteriormente.

Acción: Se define como la transformación que se aplica a un Objeto, sin reflexionar sobre él. Dubinsky (1997) considera que la Acción que es realizada por un sujeto es externa a él. En este sentido, se puede considerar como una Acción, por ejemplo: una operación o un algoritmo matemático que se realiza de manera mecánica sin omitir pasos.

Proceso: Es una comprensión de las acciones aplicadas a un objeto. En términos de Asiala et al., (1996) cuando se repite una Acción, y el individuo reflexiona sobre ella, puede ser interiorizada como un Proceso. En este sentido, se puede considerar que un estudiante ha construido un Proceso, cuando omite pasos al realizar un algoritmo o no necesita del algoritmo para inferir sobre el resultado de una operación, como consecuencia de una reflexión de sus procedimientos. Un Proceso puede ser revertido para construir otro Proceso.

También dos Procesos pueden ser coordinados respectivamente y este resultado encapsulado para formar un nuevo Objeto. Cuando un Proceso ha sido encapsulado en un Objeto, puede volver a des-encapsularse cuando sea necesario.

Objeto: Se define como una estructura estática que fue transformada (encapsulada) de una estructura dinámica (Proceso) sobre la que el estudiante puede aplicar nuevas Acciones.

Esquema: Se define como una colección de Acciones, Procesos y Objetos y otros Esquemas que utiliza un individuo para resolver determinado problema.

En la figura 1 se sintetiza lo mencionado anteriormente, donde a partir de un Objeto (por ejemplo, dos vectores en $\mathbb{R}^2$) sobre el cual se le aplican Acciones (hallar el determinante) que se pueden entender como transformaciones sobre el objeto, en las cuales no se reflexiona, es decir, como aplicar un algoritmo sin interpretar el resultado (en el caso del determinante) se puede, por medio de interiorizaciones acerca dichas Acciones, llegar a una estructura mental Proceso (en donde el estudiante relaciona que si el determinante es cero es porque los vectores son linealmente dependientes), el cual se puede ver como un objeto dinámico que está en constante evolución, ya que se pueden generar nuevos Procesos por el mecanismo de *coordinación* y por medio del mecanismo de *encapsulación* puede ser transformado a una estructura mental Objeto, en donde ya se vuelve un objeto estático, el cual se puede utilizar para construir un nuevo concepto, como el concepto de base o transformación lineal.

El mecanismo de *desencapsulación* se utiliza cuando se necesita volver del Objeto al Proceso del cual dio origen.

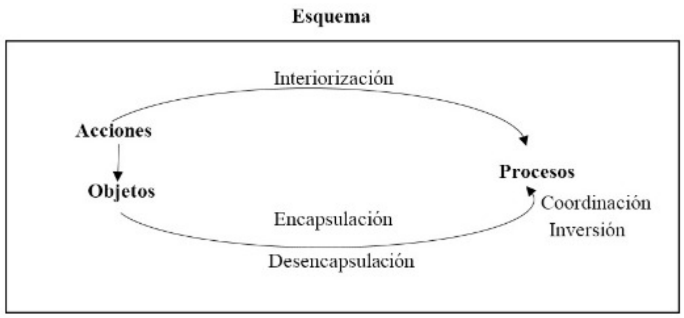


Figura 1: Estructuras y mecanismos mentales para la construcción del conocimiento matemático (Arnon et. al., 2014, p. 18)

Métodos de investigación

Contexto y participantes:

Este estudio fue realizado con un grupo de 28 estudiantes de álgebra lineal de primer semestre, de la carrera de Ingeniería Electrónica, en una universidad pública de Colombia.

Se realizaron 4 sesiones de clase donde se abordaron 17 problemas. Además se diseñó y realizó una entrevista didáctica, según Oktaç (2019) "didactic interviews give detailed information about the learning process" (p. 3), para lograr un análisis más fino de las estructuras y/o mecnaismos mentales, que los estudiantes lograron evidenciar en su razonamiento. Cuando se realiza la entrevista con el estudiante se pueden dar pistas o hacer preguntas a partir de las respuestas, para motivar razonamientos que el estudiante no ha realizado de manera previa.

Análisis Teórico:

A continuación, se presenta el modelo cognitivo propuesto para describir las estructuras y mecanismos mentales que a nuestro criterio, un estudiante necesita para construir el concepto de dependencia lineal a partir de la aplicación de Acciones sobre Objetos concretos que pueden dar paso a la construcción de Objetos abstractos.

Descomposición genética del concepto de dependencia lineal

Para iniciar un estudiante puede determinar las características de un conjunto de vectores representado geométricamente, esto es, pude encontrar cuándo son colineales o coplanares. Para esta investigación esto obedece a los Objetos concretos de los cuales partimos para construir el concepto de dependencia lineal. Entonces a partir de una gráfica como la que aparece a continuación (ver figura 2), el estudiante debe aplicar Acciones específicas que le permitan garantizar que los vectores colineales o coplanar.

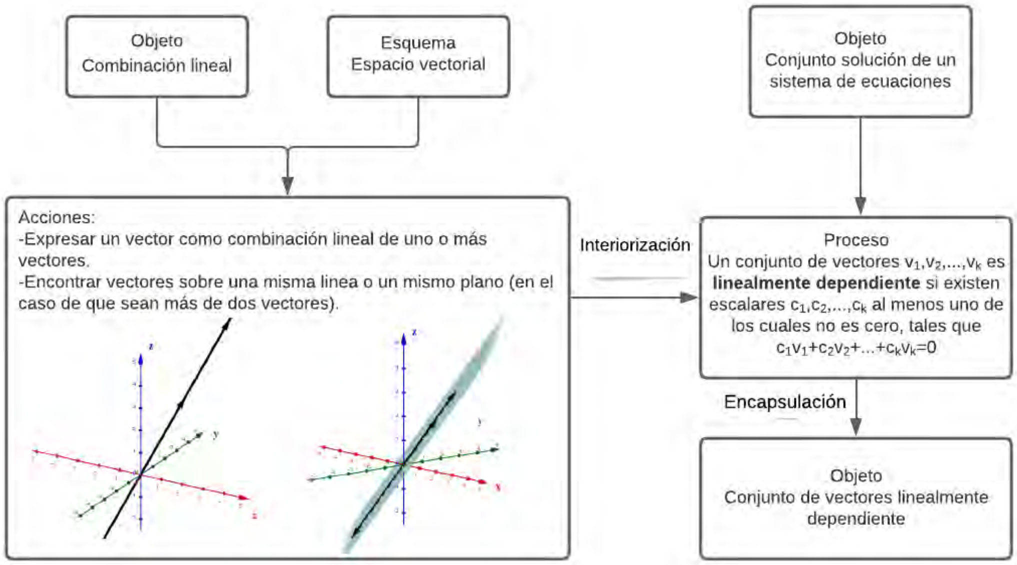


Figura 1: Estructuras y mecanismos en la construcción del concepto de dependencia lineal: Descomposición genética inicial

Las interpretaciones geométricas deben estar acompañadas de un análisis numérico que permita argumentar que los vectores son colineales más allá de la persepción visual del indivudo que observa la gráfica. Esto es una evidencia de que las Acciones que se aplican inicialmente sobre un Objeto concreto se están interiorizando en un Proceso abstracto.

Esto está determinado por la búsqueda del escalar, que permiten dado un vector u y a partir del producto por un escalar $\propto u$, detetminar qué valor de $\propto$ alarga, encoge o rota el vector u. Ahora la pregunta es:

¿Qué relación puede definirse entre los vectores u y $\propto u$?

A partir de una exploración sobre dichos vectores los estudiantes pueden empezar a construir el concepto de dependencia lineal, el vector $\propto u$ depende de u. Por tanto los vectores u y $\propto u$ son linealmente dependientes. Así la aplicaciones de Acciones para determinar si dos o más vectores son linealmente dependientes están asociadas a determinar cuándo uno de ellos es múltiplo escalar de otro o combinación lineal de otros vectores del conjunto.

Mediante la reflexión de las Acciones descritas anteriormente, el estudiante puede interiorizar dichas Acciones para describir de manera general el comportamiento de los vectores; es decir qué relación existen entre las interpretaciones algebraicas, geométricas y numéricas de los vectores, que dan cuenta de las características que tiene un conjunto de vectores linealmente dependiente.

Concepción Proceso: Para tener una concepción Proceso del concepto de dependencia lineal, un estudiante necesita una concepción Proceso de combinación lineal, para poder ver un vector en términos de uno o más vectores. También debe tener una concepción Objeto de conjunto solución de una ecuación, para considerar el conjunto de todas las posibles soluciones de un sistema de ecuaciones lineales. Mediante el mecanismo de coordinación el estudiante puede determinar todas las soluciones de las combinaciones lineales que dan el vector cero. En este caso el estudiante puede identificar que si la solución no es única entonces el conjunto es linealmente dependiente.

Concepción Objeto: Un estudiante evidencia una concepción Objeto, cuando ha encapsulado el Proceso de dependencia lineal, es decir, cuando realiza Acciones sobre el Objeto de un conjunto de vectores linealmente dependiente. También un estudiante da evidencia de una concepción Objeto, cuando puede desencapsular el Objeto para volver al Proceso del cual dio origen.

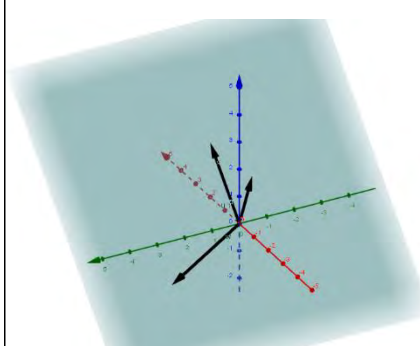
Resultados

A continuación, se presenta el análisis de un Caso de Estudio que llamamos Camilo. Camilo es un estudiante de ingeniería electrónica, que hace parte de un curso de álgebra lineal de estudiantes de ingeniería de primer año. Se propone a Camilo como caso de estudio ya que durante el desarrollo de las actividades (talleres y entrevista) realizadas en diferentes sesiones de clase, da evidencias de haber construido el concepto de dependencia lineal. A continuación, se presenta con detalle los razonamientos realizados por Camilo, analizados a la luz de la descomposición genética.

Aplicación de Acciones sobre Objetos concretos: El análisis de los procedimientos realizados por Camilo, evidencia la aplicación de Acciones sobre vectores como Objetos concretos, que representa de manera geométrica o que transforma en su mente a través de “movimientos” que relaciona con un vector y sus múltiplos escalares. Esto como se señala más adelante, no está determinado por la representación de los vectores en el enunciado del problema, sino que emerge como una manera de acceder a formas más abstractas relacionadas con el concepto de dependencia lineal.

Los procedimientos evidenciados por Camilo muestran que constantemente usa interpretaciones que abstrae de representaciones geométricas y/o numéricas, relacionadas con la colinealidad de los vectores y la relación múltiplo escalar, respectivamente. Por ejemplo, durante el taller se pide a Camilo trabajar con un conjunto de vectores definidos sobre un plano representado en GeoGebra (Tabla 2).

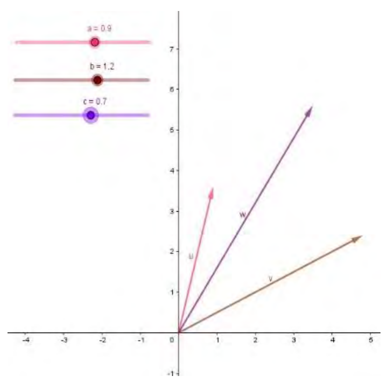
Tabla 2: Pregunta 5 taller 1

Pregunta 5 - Taller 1	Representación de los vectores dado en un archivo en GeoGebra
Los vectores $u = (1, -1, 1), v = (2, 0, 2)$ y $w = (-1, 3, -1)$ están sobre un mismo plano. ¿Existen $\alpha, \beta \in \mathbb{R}$ tales que $w = \alpha u + \beta v$?	
Si ya que w puede ser conbinacion lineal de ~~ø~~ u, v ? Esto por que estan en el mismo plano y u y v son LI entre ello	

A partir de la interpretación que Camilo estructura de la representación gráfica responde: “Si, ya que w puede ser combinación lineal de u, v. Esto por que están en el mismo plano y u y v son L.I. entre ellos” [esto es, u y v son Linealmente Independientes]. Es importante resaltar que a partir de la representación gráfica de los vectores Camilo puede determinar que los vectores u y v son linealmente independientes, ya que no están sobre una misma recta. De la actividad generada en el desarrollo del taller, se deduce que el razonamiento de Camilo está considerando que los vectores u y v generan cualquier vector sobre el plano y por lo tanto generan a w. Esto muestra que Camilo está interiorizando las Acciones que aplica sobre Objetos concretos en un Proceso abstracto que relaciona con la combinación lineal como una estructura dinámica. Para el problema presentado en la Tabla 2, Camilo no necesita determinar $\propto$ y β de manera específica para aceptar que existen.

Interiorización de Acciones en un Proceso: Durante el trabajo de Camilo en el Taller 2 sigue dando evidencias de que ha logrado una concepción Proceso de la dependencia lineal. Frente al problema que aparece en la Tabla 3, Camilo determina a partir de su interpretación geométrica, que, si tiene 3 vectores en $\mathbb{R}^2$, necesariamente son linealmente dependientes. Camilo escribe: “El conjunto de vectores $\{u, v, w\}$ son linealmente dependientes ya que puedo formar alguno con combinaciones lineales de los otros”.

Tabla 3: Pregunta 1 taller 2.

Pregunta 1 - Taller 2	Representación de los vectores dado en un archivo en GeoGebra
¿Los vectores u, v y w son linealmente dependientes o independientes?	
El conjunto de vectores {U, w, V} Son lineal mente dependientes ya que puedo formar alguno con combinaciones lineales de los otros	

Como se señaló en la descomposición genética de la sección anterior, el concepto de combinación lineal juega un rol fundamental en la construcción de la dependencia lineal. Los argumentos de Camilo a partir de la colinealidad (Acción sobre Objetos concretos) son reestructurados por medio de la combinación lineal. Esta estructura le permite reflexionar sobre dichas relación entre los vectores, sin pensar en sus coordenadas o en escalares específicos que determinan la combinación lineal.

Conclusiones

La presentación de las conclusiones las realizamos en tres secciones; i) Reflexiones sobre los resultados encontrados; ii) recomendaciones didácticas.

Reflexiones sobre los resultados encontrados

Consideramos que un primer curso de álgebra lineal debe tener una parte geométrica, ya que interpretar un concepto del álgebra abstracta de manera geométrica y algebraica ayuda al estudiante a comprender el concepto en cuestión, ya que le da herramientas para argumentar sus ideas, que posteriormente podrá generalizar.

A pesar de que en la mayoría de los libros de texto que se trabajan en algebra lineal se define la independencia lineal a partir de la dependencia lineal, creemos al igual que Mares (2016) que el tener la definición de dependencia lineal, no implica tener la definición de independencia lineal, por eso se deben definir por separado, para que un estudiante puede comprender las características que diferencia un concepto del otro.

Recomendaciones didácticas

El rediseño de la descomposición genética propuesta en esta investigación es una herramienta para profesores de álgebra lineal, ya que sirve de guía para la enseñanza del concepto de dependencia lineal, teniendo en cuenta los pre saberes que necesitan los estudiantes para construir el concepto. También en la forma como se desarrolla el concepto, ya que en la descomposición genética proponemos dos enfoques: algebraicos y geométricos. En este sentido, se vuelve indispensable realizar actividades que fomenten distintos tipos de Acciones y Procesos de forma tanto algebraica como geométrica.

Referencias

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. Maa Notes, 37-54.

Dubinsky, E. (1997). Some thoughts on a first course in linear algebra at the college level. MAA NOTES, 85-106.

Mares, C. (2016). Aprendizaje de los conceptos de independencia y dependencia lineal desde la teoría APOE. Tesis de maestría no publicada. Centro de Investigación y de Estudios Avanzados del Instituo Politecnico Nacional. México.

Oktaç, A. (2019). Mental constructions in linear algebra. ZDM, 51(7), 1043-1054.

Saldanha, L. A. (1995). The notions of linear independence/dependence: a conceptual analysis and students' difficulties (tesis de doctorado). Quebec, Canadá.

Stewart, S., & Thomas, M. O. (2010). Student learning of basis, span and linear Independence in linear algebra. International Journal of Mathematical Education in Science and Technology, 41(2), 173-188.

Trigueros, M., & Possani, E. (2013). Using an economics model for teaching linear algebra. Linear Algebra and its Applications, 438(4), 1779-1792.

UNDERSTANDING LINEAR DEPENDENCE: A PERSPECTIVE OF MENTAL STRUCTURES AND MECHANISMS WITH UNIVERSITY STUDENTS

This article aims to present the design of a genetic decomposition of the concept of linear dependence based on the analysis of textbooks, previous research related to the concept and the results of an intervention with students of linear algebra from the public university in Colombia. It is sought to have a cognitive model that is the result of the first phase of the methodology proposed by APOE.

STUDENTS' INTUITIVE MEANINGS FOR INFINITE SERIES CONVERGENCE AND CORRESPONDING IMPLICATIONS

Derek Eckman
Arizona State University
dceckman@asu.edu

Kyeong Hah Roh
Arizona State University
khroh@asu.edu

This paper describes our work to determine the naturalistic images that first-time second-semester university calculus students possess for series convergence. We found that the students we interviewed most frequently determined whether a series converged by imagining a process of appending summands into a running total and examining whether this running total appeared to approach an asymptotic value. We provide examples and three corresponding implications of this "asymptotic running total" that informed students' actions while determining series convergence or the value of convergence. Our paper adds to the research literature by confirming students' meanings for limits reported for other topics (e.g., limit of sequence, function, Taylor series) apply to infinite series and proposing relationships between previously reported meanings for series convergence.

Keywords: Calculus, Cognition, Undergraduate Education, Advanced Mathematical Thinking

Introduction and Literature Review

Infinite series convergence is a central mathematical topic applied across many branches of mathematics and other disciplines such as physics and engineering (Azevedo, 2021). The topic of infinite series also incorporates many other pivotal topics in advanced mathematics, such as sequence, limit, and infinity (Martin, 2013). Research related to infinite series has focused on the state of curricula and instruction (e.g., González-Martín et al., 2011; Lindaman & Gay, 2012); effective interventions to promote productive student thinking about limit or convergence (Martin et al., 2011; Roh, 2008, 2010b, 2010a; Swinyard & Larsen, 2012); and how students conceive of series convergence, the sequence of partial sums or the symbolic forms of representation students utilize to express series (Barahmand, 2021; Eckman & Roh, accepted; Kidron, 2002; Kidron & Vinner, 1983; Martin, 2013; Martínez-Planell et al., 2012; Strand et al., 2012; Strand & Larsen, 2013).

Most of the reports about students' meanings for infinite series describe students who have previously received (or are currently receiving) instruction on this topic. The purpose of this study is to examine how first-time university second-semester calculus students consider the concept of series convergence before receiving formal instruction on sequences and series. Documenting students' naturalistic images for infinite series before they receive formal instruction can (1) assist instructors in better anticipating students' initial perceptions of series convergence and (2) help researchers better describe students' acquisition of rigorous and productive meanings for convergence. We summarize the goals of this study with the following research questions: (1) *What meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?* (2) *How do the implications of these students' meanings for convergence inform their actions for determining whether a series converges and the value to which a convergent series converges?*

Theoretical Perspective

In this study, we attempt to describe individual students' meanings for series convergence and their utilization of these meanings to reason about various series. We adopt a radical constructivist (Glasersfeld, 1995; Thompson et al., 2014) approach for the term *meaning*. Thompson et al. (2014) described a *meaning* as the space of implications resulting from assimilating a situation. We consider assimilation, in this case, to refer to an individual's ability to consider her situation as analogous to previous experience (Glasersfeld, 1995) such that her subsequent actions are selected from the implications elicited by the meaning in her mind.

The concept of *meaning* can effectively describe student thinking and actions for reasoning about series convergence. Suppose that a student, while examining the series $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}+\cdots$, conceives a dynamic image of adding consecutive summands and decides to track the value of the "moving sum" generated through this process. We call this "moving sum" a *running total*. An implication of the student's evoked *meaning* for the series might involve a belief that if the *running total* approaches a particular value, then the series converges. An alternative implication of the *meaning* might include the notion that if the *running total* does not approach an asymptotic value, the series does not converge. The student's subsequent actions to determine whether the *running total* approaches an asymptotic value are grounded in the implication evoked as part of the student's *meaning* for series convergence. In the results section, we propose an overarching meaning for series that the students in this study appeared to possess and the implications that informed the students' actions to determine convergence.

Research Methodology

In this report, we detail the meanings and implications that students exhibited for series convergence during a set of individual 90-minute clinical interviews (Clement, 2000) that simultaneously functioned as a selection and pre-test interview for a constructivist teaching experiment (Steffe & Thompson, 2000). The purpose of the teaching experiment was to offer calculus students specific tasks designed for developing productive meanings for series convergence while documenting the evolution of their corresponding symbolization for series convergence. The virtual clinical interviews were conducted during the Fall 2021 semester, prior to formal coursework on sequences and series, at a large public university in the United States. Our analysis and results focus on two students, Monica and Sylvia (pseudonyms), who were first-time second-semester calculus enrollees at the university level.

We presented the students with six infinite series in expanded form (see Table 1). We chose not to present infinite series using summation notation to mitigate students' potential difficulties interpreting this notation (Strand et al., 2012; Strand & Larsen, 2013). Series 1-4 reflect various behaviors of sequences of partial sums (i.e., monotone convergent, monotone divergent, oscillating convergent, oscillating divergent). Series 5-6 reflect series types that have been reported as problematic for students, such as the decimal expansion of an irrational number (Kidron, 2002; Kidron & Vinner, 1983) and variations of Grandi's series (Bagni, 2005; Martínez-Planell et al., 2012).

For each series, we asked the student two questions adopted from the sequences and series unit in Larson and Edwards (2015): (1) *Does the series converge?* and (2) *If the series converges, what value does the series converge to?* After the student responded to the two questions for each series, we presented each question in a generalized form to determine what students believed about convergence and the value of convergence for any series.

After compiling and transcribing the interviews, we identified and analyzed individual moments of the interviews in the spirit of grounded theory (Strauss & Corbin, 1998). We considered a new moment to begin when a student either (1) encountered a new series, (2) changed their thinking about whether a series converged, or (3) proposed a particular value to which they believed the series converged. For each moment, we recorded whether a student believed the series converged, the convergent value (if applicable), the meaning that students appeared to exhibit for series convergence, and how the students' actions to determine series convergence might be viewed as implications of their meaning. We found that the two students reported in this study appeared to have a similar meaning for series convergence and that three distinct implications of this meaning emerged as the students interacted with various series.

Table 1: The six series presented to students during the interviews

Series	Rule	Expanded Form	Series type	Sequence of Partial Sums	Value of Convergence
1	$\sum_{n=0}^{\infty}\frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{1}}+\frac{3}{\sqrt{2}}+\frac{3}{\sqrt{3}}+\cdots$	p-series $(0<p<1)$	Monotone increasing divergent	
2	$\sum_{n=1}^{\infty}\frac{2(-1)^{n-1}}{n^5}$	$\frac{2}{1^5}-\frac{2}{2^5}+\frac{2}{3^5}-\cdots$	Alternating p-series $(p>1)$	Oscillating convergent	≈ 1.94
3	$\sum_{n=1}^{\infty}\sum_{i=1}^{99}\left[10^{-2n-1}-10^{-2(n+1)-1}i\right]$ $=\sum_{k=0}^{\infty}\frac{495}{10000}\left(\frac{1}{100}\right)^k$	$\frac{99}{10^3}+\frac{98}{10^3}+\cdots+\frac{1}{10^3}+$ $\frac{99}{10^5}+\cdots+\frac{1}{10^5}+$ $\frac{99}{10^7}+\cdots+\frac{1}{10^7}+\cdots$	Geometric	Monotone increasing convergent	$\frac{1}{20}$
4	$\sum_{n=0}^{\infty}\frac{(200-2n)(-1)^n}{n+1}$	$\frac{200}{1}-\frac{198}{2}+\frac{196}{3}-\cdots$	Alternating series	Oscillating divergent	
5	$\sum_{i=0}^{\infty}a_i$ (where a_i corresponds to the i^{th} decimal place of π and $a_0=3$.)	$3+.1+.04+\cdots$	Decimal expansion of irrational number	Monotone increasing convergent	π
6	$\sum_{n=0}^{\infty}(.07)\cdot(-1)^n$	$.07-.07+.07-\cdots$	Alternating series (Grandi's)	Oscillating divergent	

Results

The results comprise two sections. The first section explains the students' meaning of series convergence as analogous to a running total approaching an asymptotic value. The second section describes three distinct implications of this meaning that informed the students' actions to determine the convergence and convergence value of various types of series. We reflect on potential relationships between the implications in the discussion section.

Table 2: The evolution of students' responses to the question "*Does the series converge?*"

<table>
<tr><th colspan="2"></th><th colspan="3">Series 1</th><th colspan="2">Series 2</th><th>Series 3</th><th>Series 4</th><th colspan="4">Series 5</th><th colspan="2">Series 6</th></tr>
<tr><td rowspan="2">Monica</td><td>Converge</td><td colspan="2">Y</td><td>N</td><td colspan="2">Y</td><td>N</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td colspan="2">N</td></tr>
<tr><td>Value</td><td colspan="2">Unsure</td><td></td><td colspan="2">Unsure</td><td></td><td>0</td><td>π</td><td>4</td><td>3.2 and 4</td><td>Any upper bound</td><td colspan="2"></td></tr>
<tr><td rowspan="2">Sylvia</td><td>Converge</td><td>Y</td><td>N</td><td>Y</td><td>N</td><td>Y</td><td>N</td><td>Y</td><td colspan="4">Y</td><td>Y</td><td>N</td></tr>
<tr><td>Value</td><td>4</td><td></td><td>Unknown value larger than 4</td><td></td><td>2</td><td></td><td>200</td><td colspan="4">π</td><td>0</td><td></td></tr>
</table>

Students' Meaning for Convergence as a Running Total Approaching an Asymptotic Value

Monica provided normative final responses for each series' convergence except series 3 and 4, while Sylvia provided normative responses for series 2, 5, and 6 (see Table 2; "Y" and "N" stand for "Yes" and "No"). However, in instances where both students believed that the series converged (i.e., series 2, series 4, series 5), the students proposed differing convergence values.

Both students exhibited similar meanings for series convergence, which appeared to involve imagining a dynamic *running total* approaching an asymptotic value as additional summands are calculated into the running total. We call this meaning an *asymptotic running total* meaning. Monica's response to the final interview question regarding general series convergence provides an image of her *asymptotic running total* meaning. Note that in each of the following transcripts, the ellipses (…) refer to omitted text.

Monica: OK. So for the first question [*How can I tell whether a series converges?]*. The first thing I thought of was, kind of like with the limits thing, where are we approaching with every new, like, next iteration of this series? Are we approaching a value or infinity?... [In] the one where we were adding and subtracting and then adding and subtracting [series 4], but the numbers were decreasing, you still saw like a general trend towards, a number, which in that case was zero. So that's what I would say when I'm thinking about, how can I tell whether [the] series converges… Like I'm interested in, what it's approaching [i.e., the running total] and how adding on each next piece of the series is getting it closer to that value.

Int: OK.

Monica: So then the last one [series 6] where it [i.e., subsequent summand] was like essentially undoing what the previous one had done. That was why I said it [series 6] wasn't converging. Because there was not like one overall direction it was going.

Int: OK. Now, when you say one direction "it" was going. Just to clarify, what do you mean by the "it" that's going somewhere?

Monica: The series as a whole, or like the sum of the series as a whole [i.e., running total]. So like, if I were to take…like one plus two plus three plus four. And on this like, huge number line and I just go up all the way to like a very large number, and that number that was getting big, closer and closer to infinity. And then I went to the 200th part of that series. I can be like, Yeah, that's still getting close to infinity, and I could go to the 1000th part and it would be like, yeah, it still goes to infinity.

In Monica's initial comment she states that a convergent series will exhibit a "general trend" or "overall direction" towards a particular value. When the interviewer asked Monica to clarify the "it" moving towards the asymptotic value, Monica stated that she envisioned instantiations of the *running total* after summing various numbers of summands (i.e., 200 or 1000 summands). However, there is no indication that Monica coordinated these instantiations of the *running total* with an index to form the sequence of partial sums. For this reason, we use the terminology *summand* and *running total* in place of normative terminology (e.g., *term, partial sum, sequence of partial sums*) to make clear that we do not consider either student to have constructed explicit sequences through their reasoning about series convergence.

Implications of Asymptotic Running Total Meaning for Determining Series Convergence

In this section, we report three distinctive implications of the students' *asymptotic running total* meaning which contextualize students' actions while determining series convergence. The three implications we describe are *decreasing summands convergence*, *monotone running total divergence*, and *running total recreation through grouping*.

Implication 1: Decreasing summands convergence. One implication of an *asymptotic running total* meaning is that a student might believe that if each consecutive summand in an infinite series is smaller than the previous summand, the running total will eventually trend toward one specific value, suggesting that the series will converge. We call this implication *decreasing summands convergence*. In conventional mathematics, the notion of decreasing summands is a necessary but insufficient property of a convergent series (the most famous example of this principle is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$).

Both students' actions in various moments related to series 1 appeared to align with the *decreasing summands convergence* implication of the *asymptotic running total* meaning. We constructed series 1 to reflect a divergent p-series with a monotone increasing sequence of partial sums (see Table 1). We provide an example from Sylvia's reasoning about series 1 as evidence of the *decreasing summands convergence* implication.

Sylvia: OK, now I'm thinking that the series doesn't converge and like it [i.e., running total] keeps adding up and getting bigger and approaches like infinity or something like that.
Int: OK.
Sylvia: Wait…
Int: OK… can you say a little bit more about any of the stuff that you're thinking?
Sylvia: Um…if you picture like a perfect square root like three over the square root of nine is three over three, and that's one [$\frac{3}{\sqrt{9}} = \frac{3}{3} = 1$]. But then if you had like three over the square root of 81, that's one-third [$\frac{3}{\sqrt{81}} = \frac{1}{3}$]. So…each item in a series that you're adding on [*brings up hand and mimics placing summands of series sequentially*] is getting smaller. Yes, it is getting smaller. So it's going to… converge to something, but it's not going to be four. [*brief pause*] Yes, that's my view.

In Sylvia's initial conception of the convergence of series 1, she argued that series 1 does not converge because the *running total* will perpetually increase. She later determined that each subsequent summand is smaller than the previous summands, which lead to her declaration that the series converges to an unknown value. In this moment, Sylvia's asymptotic running total meaning implies that decreasing summands suggest convergence. However, we note that the *decreasing summands convergence* implication was insufficient for Sylvia to make a substantive claim regarding the value to which series 1 might converge.

Implication 2: Monotone running total divergence. Another implication of an *asymptotic running total* meaning is that a student might believe that since the *running total* perpetually increases (or decreases) in a monotone series, the *running total* will eventually surpass every potential upper bound (i.e., asymptotic value) that might indicate convergence. We call this implication *monotone running total divergence*.

Both students' actions in various moments related to series 3 aligned with the *monotone running total divergence* implication (Monica's actions related to series 1 also aligned with this meaning during certain moments). The authors constructed series 3, a convergent geometric series, by expanding each summand of the original geometric series into a finite arithmetic series that summed to the summand in the geometric series. Consequently, the expanded form of series 3 resembled a double summation (see Table 1). We provide an instance of Monica's reasoning about series 3 as an instance of the *monotone running total divergence* implication.

Monica: Um. [*pause*] So. In this case… we're adding very, very, very small fractions [i.e., summands], but they're always going to be positive numbers and then we're adding all of them together. This one [series 3]…the rate of growth would be significantly slower.

Int: OK.

Monica: But it would still, if you just did this forever [*makes a gesture to the right indicating extending series indefinitely*]…it [i.e., the running total] would just continue to increase.

Int: OK. So if I'm just summarizing what you've said, you're saying that this third series does not converge. And the reason why is once again, similar to the first series, all of the fractions [i.e., summands] are positive, they're all being added together. And so therefore, our value [i.e., running total] will never stop growing. And so as we continue on indefinitely, we'll approach infinity.

Monica: Right.

In Monica's comments, she noted that all of the summands in series 3 are positive and that although the rate at which the running total will grow is slow, the running total will never stop increasing. The interviewer then rephrased Monica's comment to determine whether Monica imagined that a monotone running total implies non-convergence, which Monica confirmed. In Sylvia's case, she recognized that although the summands for series 1 and series 3 both decrease in value, she claimed that series 1 converged and series 3 did not. Sylvia acknowledged the apparent contradiction in her reasoning but was unable to reconcile the issue.

Implication 3: Running total recreation through grouping. A final implication of an *asymptotic running total* meaning is that a student might believe that if she groups the terms in an alternating series to construct a uni-directional *running total*, the series will converge. We call this implication *running total recreation through grouping*.

Both students' actions related to various moments related to series 4 appeared to align with the *running total recreation through grouping* implication (Sylvia's actions related to series 2 were nearly identical to her actions for series 4). We constructed series 4 to reflect an alternating divergent series with a linearly decreasing numerator and linearly increasing denominator (see Table 1). Monica tenuously claimed that series 4 converged to zero and Sylvia confidently claimed the series converged to 200 (the value of the initial summand).

We provide Monica's rationale for claiming that series 4 converges to zero in the transcript below.

Monica: I think that if I were to say that it [series 4] does converge, then I would also say that it converges to zero.

Int: OK. Can you say a little bit more about that?

Monica: …So in this case, instead of seeing like each individual adding on a fraction is one thing [*places hands close together with small space between*], I'm kind of grouping them together, where we are subtracting and adding and that, I'm grouping that together in my head.

Int: OK, could you could you like, mark on the screen what it is that you're grouping together just so that I'm sure that I know?

Monica: Yes. So I would put these together [*draws a bracket above second and third summand*] and then I would put these together [*draws a bracket above the fourth and fifth summand*]… and I take the number 200 and I, do these two things do it [*cursor indicates second and third summands*], I'm going to have a number here [*moves cursor between third summand and minus sign separating third and fourth summand*] that's less than

200, but still very close to it. So basically, what I've decided is if you were to sum these two values [*moves cursor back to indicate second and third summands*], you would have a very small number and you subtract those… So that, that's what makes me think that this [i.e., the running total] is getting smaller and smaller and smaller and smaller.

In her response, Monica envisioned a *running total* that started at 200 (the value of the first summand) and noticed that by grouping each pair of consecutive summands after the first summand, she could construct a new series that had a distinctive pattern of $200 + (\text{small negative value}) + (\text{small negative value}) + \cdots$. Based upon her reconstruction of series 4 by grouping, Monica perceived a running total that approached an asymptotic value of zero.

In the transcript below, we provide Sylvia's rationale for claiming that series 4 converged to 200, which involved a different grouping action than Monica.

Sylvia: So I'm going to say that this one [series 4] converges. And I think I'm going to follow the same logic that I did with [series 2,] that it kind of…drops some and increases a little bit less than it dropped [*draws concave-up curve starting from (0, 200) that stops before reaching* $y = 200$], if that makes sense. And then it, like each wave gets smaller [*draws more waves with decreasing amplitudes that progressively move closer and closer to horizontal line at y = 200*]. And I would say my guess is that it converges to 200.
[omitted dialogue]
Int: So, can you explain a little bit more to me about how you're…seeing that come about [convergence to 200]?
Sylvia: So I guess like, if we start at 200, we subtract 99, we'll get 101, and then you add one ninety-six over three [$\frac{196}{3}$], and that number is smaller than 99. Yes. And so you're going to go back up and essentially cancel out some of the, the subtraction that you did. But not like all the way, like you're not going to get back up to 200.
Int: OK.
Sylvia: And then you're going to go down a little bit more, but not as great as just went up. And then, like, follow that pattern.
Int: So there. So it's like you're imagining that every time we jump up, we're jumping up farther than we drop down. And so over time, we're slowly moving back up towards 200.
Sylvia: Yes.

In her response, Sylvia's stated that the *running total* begins at 200, drops to 101, and then begins to move back toward 200 in progressively smaller "waves." Sylvia constructed her "waves" by recognizing that the sum of the third and fourth summand would be a positive value, and every subsequent pair of summands would yield positive—albeit decreasing—values. Sylvia then reconstructed series 4 as $101 + (\text{small positive value}) + (\text{small positive value}) + \cdots$ and envisioned that the *running total* would eventually approach the asymptotic value of 200, the initial summand of the series.

Discussion and Conclusion

Our findings provide unique and relevant contributions to the mathematics education literature. For instance, our description of the *asymptotic running total* meaning provides valuable insight into how students might initially perceive series convergence in university calculus courses. The three implications we described both emerge from and deepen the explanatory power of the *asymptotic running total* meaning. The first implication, *decreasing summands convergence*, emerges from an *asymptotic running total* meaning when a student imagines that if the summands in a series tend toward zero, the *running total* will eventually stabilize around a particular value. The second implication, *monotone running total divergence*, emerges when a student envisions that a never-ending string of non-zero summands imply that the *running total* will perpetually increase and eventually surpass all potential asymptotic values. The third implication, *running total recreation through grouping*, emerges when a student considers that she can regroup the terms in an alternating series to create a uni-directional *running total.* For the students in our study, recreating the series resulted in not only a uni-directional *running total,* but a plausible (in the students' minds) candidate for the asymptotic value to which the recreated *running total* might converge. In this manner, the three implications provide a connection between students' actions while considering the convergence of individual series and their overarching image of convergence as a *running total* approaching an asymptote.

Our findings both complement and extend work done by previous researchers. For example, the *asymptotic running total meaning* indicates that students consider series convergence as a dynamic process of approaching an asymptote, which has been previously reported about students' meanings for the limit of sequences (e.g., Roh, 2008) and functions (e.g., Swinyard & Larsen, 2012). Our findings also provide analogous results and additional insight into Martin's (2013) report on students' thinking for Taylor series convergence. For instance, Martin's (2013) *dynamic partial sum* image in the Taylor series case is analogous to the *asymptotic running total* meaning for the infinite series case. However, our report extends the findings of Martin (2013) by reporting new implications of the *asymptotic running total* meaning (e.g., *running total recreation through grouping*) and providing a meaning-implication relationship between images that were previously reported distinctly (i.e., *dynamic partial sum* and *termwise*).

Our data did not provide sufficient evidence to make rigorous claims about relationships between the three implications described in the results section. Still, we make two anecdotal comments about potential relationships. First, both students' actions tended to align with the *decreasing summands convergence* implication after constructing and reasoning about closed-form rules for the general summand in a series. In contrast, the *monotone running total divergence* implication often emerged during moments when the students could not construct a closed-form rule for the general summand or reasoned about the *running total* without referencing a closed-form rule for summand values. Second, the *monotone running total divergence* implication did not emerge during moments after students regrouped an alternating series. Rather, the students' responses indicated that although the regrouped series' running total would perpetually increase or decrease, the inherent oscillation of the original running total was sufficient to render the *monotone running total divergence* implication irrelevant.

We anticipate that our ongoing analysis of the teaching experiment data from which this report emerged will provide insight into effective instructional interventions to foster productive student meanings for series convergence. In particular, we hope to discern how to leverage students' naturalistic images of a series as a *running total* to construct a productive meaning for the sequence of partial sums, which Martínez-Planell et al. (2012) stated is pivotal to comprehending the formal definition of a series as the limit of its sequence of partial sums. We also hope to investigate possible relationships between the three implications we have described and uncover additional meanings and implications that first-time second-semester calculus students possess for series convergence.

References

Azevedo, D. (2021). An alternative integral test for infinite series. *International Journal of Mathematical Education in Science and Technology*, *52*(1), 131–139. https://doi.org/10.1080/0020739X.2020.1730458

Bagni, G. T. (2005). Infinite series from history to mathematics education. *International Journal for Mathematics Teaching and Learning [on-Line Journal]*, 12.

Barahmand, A. (2021). Exploring students' consistency in their notions: The case of comparing infinite series. *International Journal of Mathematical Education in Science and Technology*, *52*(7), 1049–1062. https://doi.org/10.1080/0020739X.2020.1736350

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Lawrence Erlbaum.

Eckman, D. C., & Roh, K. H. (accepted). A symbolizing activity for constructing personal expressions and its impact on a student's understanding of the sequence of partial sums. *Proceedings of the 25th Annual Conference on Research in Undergraduate Mathematics Education*.

Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

González-Martín, A. S., Nardi, E., & Biza, I. (2011). Conceptually driven and visually rich tasks in texts and teaching practice: The case of infinite series. *International Journal of Mathematical Education in Science and Technology*, *42*(5), 565–589. https://doi.org/10.1080/0020739X.2011.562310

Kidron, I. (2002). Concept Definition, Concept Image, and the Notion of Infinite Sum in Old and New Environments. In A. D. Cokburn & E. Nardi (Eds.), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education (PME)* (Vol. 3, pp. 209–216). PME.

Kidron, I., & Vinner, S. (1983). Rational numbers and decimal at the senior high level—Density and comparison. In R. Hershkowitz (Ed.), *Proceedings of hte 7th International Conference for the Psychology of Mathematics Education* (pp. 301–306).

Larson, R., & Edwards, B. (2015). *Calculus of a Single Variable: Early Transcendental Functions* (6th ed.). Cengage Learning.

Lindaman, B., & Gay, A. S. (2012). Improving the Instruction of Infinite Series. *Primus : Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *22*(5), 373–385.

Martin, J. (2013). Differences between experts' and students' conceptual images of the mathematical structure of Taylor series convergence. *Educational Studies in Mathematics*, *82*(2), 267–283. https://doi.org/10.1007/s10649-012-9425-7

Martin, J., Oehrtman, M., Roh, K. H., Swinyard, C., & Hart-Weber, C. (2011). Students' Reinvention of Formal Definitions of Series and Pointwise Convergence. In *Proceedings for the 14th Annual Conference on Research in Undergraduate Mathematics Education* (in S. Brown, S. Larsen, K. Marrongelle, and M. Oehtman (Eds.), Vol. 3, pp. 114–119).

Martínez-Planell, R., Gonzalez, A. C., DiCristina, G., & Acevedo, V. (2012). Students' conception of infinite series. *Educational Studies in Mathematics*, *81*(2), 235–249. https://doi.org/10.1007/s10649-012-9401-2

Roh, K. H. (2008). Students' images and their understanding of definitions of the limit of a sequence. *Educational Studies in Mathematics*, *69*(3), 217–233. https://doi.org/10.1007/s10649-008-9128-2

Roh, K. H. (2010a). An empirical study of students' understanding of a logical structure in the definition of limit via the ε-strip activity. *Educational Studies in Mathematics*, *73*(3), 263–279. https://doi.org/10.1007/s10649-009-9210-4

Roh, K. H. (2010b). How to help students conceptualize the rigorous definition of the limit of a sequence. *PRIMUS*, *20*, 473–287.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–307). Lawrence Erlbaum.

Strand, S., & Larsen, S. (2013). On the Plus Side: A Cognitive Model of Summation Notation. In S. Brown, G. Karakok, K. H. Roh, & Oehrtman, Michael (Eds.), *Proceedings of the 16th Annual Conferene on Research in Undergraduate Mathematics Education* (Vol. 2).

Strand, S., Zazkis, D., & Redmond, S. (2012). Summing Up Students' Understandings of Sigma Notation. In S. Brown, S. Larsen, K. Marrongelle, & M. Oehrtman (Eds.), *Proceedings of the 15th Annual Conference on Research in Undergraduate Mathematics Education* (Vol. 2, pp. 551–555).

Strauss, A., & Corbin, J. (1998). *Basics of Qualitative Research: Techniques and Procedures for Developing Grounded Theory* (2nd ed.). Sage Publications.

Swinyard, C., & Larsen, S. (2012). Coming to Understand the Formal Definition of Limit: Insights Gained From Engaging Students in Reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493. https://doi.org/10.5951/jresematheduc.43.4.0465

Thompson, P., Carlson, M., Byerley, C., & Hatfield, N. (2014). Schemes for Thinking with Magnitudes: A hypothesis about Foundational Reasoning Abilities in Algebra. In L. P. Steffe, K. C. Moore, L. L. Hatfield, & S.

Belbase (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing* (pp. 1–24). University of Wyoming. http://pat-thompson.net/PDFversions/2013MagsInAlg.pdf

TOWARDS THE CONSTITUTION OF THE MENTAL OBJECT SYSTEM OF LINEAR EQUATIONS IN PRE-UNIVERSITY STUDENTS

Xochitl Josefina García López
CINVESTAV-IPN
xgarcial@cinvestav.mx

Hugo Rogelio Mejía Velasco
CINVESTAV-IPN
hmejia@cinvestav.mx

The constitution of a mental object in a student implies their competence to perform an adequate reading and execute processes in the different contexts in which it is possible to find that related concept. As part of the constitution of the mental object system of linear equations, we consider that pre-university students must be competent in the use of a resolution method that allows recognizing a system of equations as a mathematical object and identifying the nature of its solution set. In addition, we consider necessary the competence to identify the elements of systems of linear equations in different mathematical sign systems.

Keywords: Cognition, Mathematical representations, Modeling, Problem solving.

The mastery of systems of linear equations (SLE) is fundamental in the learning of some disciplines in higher education (Oktaç & Trigueros, 2010). There is evidence that many university students are not competent in solving SLEs and working with their different forms of representation, as well as in understanding concepts such as the equivalence of SLEs and the existence of SLEs with no solution or with infinite solutions (DeVries & Arnon, 2004; Manzanero, 2007; Mares, 2016; Oktaç, 2018; Parraguez & Bozt, 2012). In learning linear algebra, university students have to understand a large number of concepts that do not seem to be related to previous mathematical knowledge (Mares, 2016). Students face problems with respect to learning SLE-related concepts and processes from the basic level (Filloy, Rojano & Solares, 2010) that are replicated at later levels. We consider the study of systems of linear equations at the pre-university level as an opportunity to detect and overcome students' difficulties in understanding and solving SLEs.

Purposes of the study

The objective of the study is to propose, analyze and interpret teaching activities focused on promoting the sense production of pre-university students around the concept of SLE and to strengthen the related mental object. Our hypothesis is that the systematic use of systems of equivalent equations mediated by the Gaussian elimination method favors the understanding of systems of equations as a mathematical object and allows determining and interpreting the nature of their solution. Through an intertextual analysis we seek to identify elements that allow students to acquire competence in modeling, solving and interpreting SLE solutions.

Theoretical framework

For the analysis of the development of competence in students we consider the concept of intertextuality. The concept of intertextuality is developed in contrast to the idea that the reading process is the extraction of a unique meaning deposited in a text by its author; it is considered that, in the search for meaning, a reader is immersed in a network of relationships between an infinite number of texts that are part of their previous experiences, which generates a unique and personal meaning of the text (Allen, 2011).

Rojano, Filloy and Puig (2014) point out that learning algebra is done by reading texts that are related to other texts and that, when reading, students are immersed in a network of intertextual relationships derived from their pool of linguistic and mathematical experiences. Once the student brings their intertextual network into play read or produce meaning, they generate a new text that will become part of their intertextuality. The tendencies, allusions or references in the students' productions allow us to observe in what ways their intertextuality facilitates or hinders the reading and production of texts. The use of the notion of intertextuality in this report is focused on the analysis and description of the development of student competencies in SLEs learning.

According to Puig (1997), when a person can correctly interpret the use of a mathematical object according to its context, they do not need its encyclopedic meaning, but their own personal semantic field, which is the result of the meanings they have managed to construct through the senses production. Hence the importance of ensuring that the students' semantic field is so rich that it allows them to adequately interpret mathematical objects in all possible situations. The objective is to build a personal intertextuality in mathematics composed of the concepts and mental objects related to that concept that allow the student to perform adequate readings and execute processes competently.

Methods

The experiment was carried out with three pre-university students whose real names have been protected. The data analyzed were obtained from the recordings of interviews with didactic intervention carried out on a videoconferencing platform. The analysis included the students' productions in worksheets and in a digital whiteboard application. The interviews were intended to promote discussion, reflection and the use of mathematical discourse. In some of the sessions, the dynamic geometry software (DGS) GeoGebra was used for the analysis of SLE representations in the plane and space of coordinate axes and in the tabular representation. This report presents the results of eight activities that made up the experimental phase. This is intended to exemplify some changes in the students' intertextuality in relation to SLEs. The activities began with the linear equation with one and more variables, analytical work for statement problems, and representations of SLEs. It concludes with the resolution of statement problems by applying the Gauss's method to the algebraic models proposed by the students.

Data Analysis and Findings

Given that one of the difficulties detected in high school students is the assumption that for each variable in a SLE there is a solution, we proposed in the third activity the numerical analysis of a problem statement divided into two conditions with two unknown quantities. Condition 1: There are two trucks. Truck A with a capacity of two tons and truck B with a capacity of four tons. You want to know how many trips each truck made given that 20 tons of material were transported to a construction site and that no truck traveled with less than its capacity, how many and which solutions are there? Condition 2: If it is known that for each trip of truck A the company charges 2 thousand pesos and for each trip of truck B 3 thousand pesos and the payment for the trips was 18 thousand pesos, what would the solution be?

From this activity we identify sense production with respect to the fact that the solution of an SLE can be a set of values and that at least two conditions are required to find two unknown quantities (Figure 1).

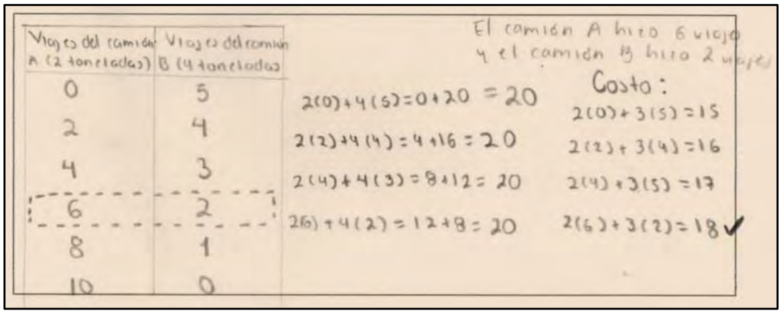


Figure 1: Vanessa´s numerical analysis of the statement problem

Before the didactic intervention, we identified that the concept of equivalent equation was not part of the students' intertextuality. During the development of the activity, there was evidence of the sense production with respect to the recognition of some of the transformations to obtain equivalent SLE. We observed evidence of competence in the use of the Gauss' method once the students incorporated the concept of equivalent scaled SLE into their intertextuality. A part of the activity for the use of Gaussian method was developed with the support of the DGS. The work that the students carried out was the identification of the operations per line necessary to find an equivalent scaled SLE and the analysis of the results obtained (Figure 2).

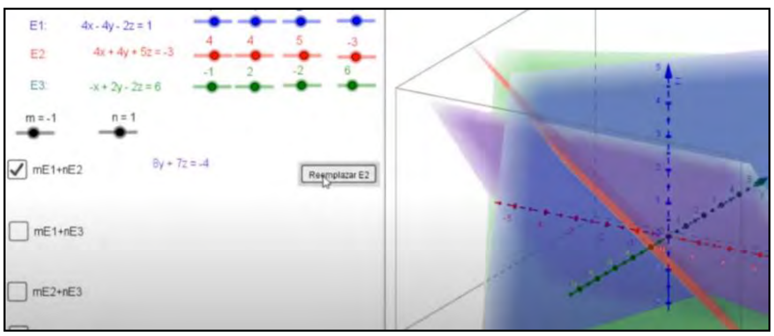


Figure 2: Solving a SEL using GeoGebra by Ana

The use of the DGS made it possible to complement the algebraic transformations and the results obtained with the visualization of the representation of SLEs and their transformations in graphical form and to produce meaning with respect to SLEs with no solution, with one and with infinite solutions. For example, by observing the SLE (Figure 3) the student identifies that it has infinite solutions.

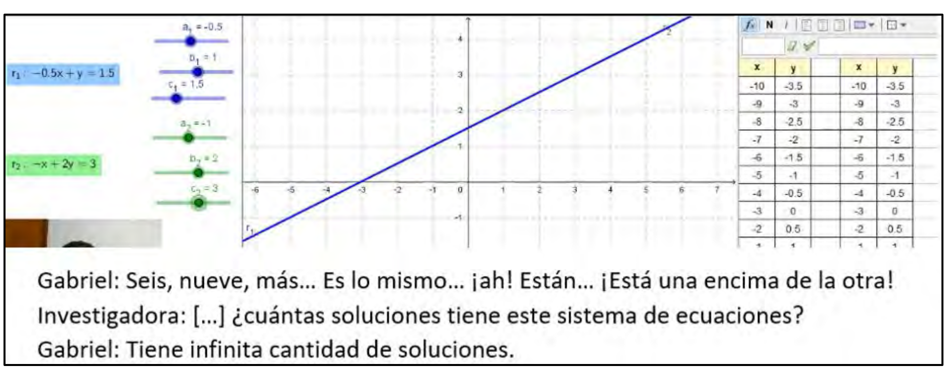


Figure 3. Representations of a SEL in interaction with Gabriel

With respect to solving word problems, students were competent in identifying unknown quantities and relating them to literals for their algebraic treatment. They also expressed the solution or results in terms of the problem. One of the major difficulties for the students was modeling the problem conditions in equations. However, through the use of concrete values, students were able to make sense of the relationships between known and unknown quantities, either to reflect and propose an equation or to revise their inferences (Figure 4).

> [Para plantear la condición de que la suma de respuestas correctas e incorrectas en un examen es 100]
> *Investigadora: Tú me dices que* $x-y=100$*, pero qué pasa si* x *vale 30, por ejemplo, entonces sería* $30-70=100$*. Y eso no es correcto, ¿cierto?*
> *Gabriel: Ok, entonces sería... 30 menos 70 no es... ¡Ah no! Es* $x+y=100$.

Figure 4: Interaction with Gabriel doing scans with specific values

Once the activities were carried out, the construction that the students achieved of the SLE mental object was influenced by the students' own intertextuality as related to first-degree equations with one unknown and the transformations necessary to solve them, as well as by the way the teaching situations were approached.

Discussion

There are significant differences in the students' personal intertextuality, which determines the senses production and competencies, not only before, but also during and after the development of the activities. However, we identified some coincidences in the constitution of mental objects. The students were able to incorporate elements to their intertextuality that allowed them to competently read systems of up to three equations with three unknowns without solution, with one or with infinite solutions in the plane, in the corresponding tables of numerical values and in algebraic form.

Some difficulties were encountered with systems of equations with three unknowns. The analysis in its different representations by means of the DGS was important for their understanding. The incorporation in the students' intertextuality of the equivalent SLE mathematical object and the recognition of the transformations to obtain them was evidenced, as well as the advantage of having a stepwise SLE to find their solution. Students showed competence in identifying unknown quantities and relating them to literals for their algebraic treatment. Finally, they showed difficulties in modeling problem conditions in equations that they overcame by using concrete values.

References

Allen, G. (2011). *Intertextuality.* Routledge.

DeVries, D., & Arnon, I. (2004). Solution-what does it mean? Helping linear algebra students develop the concept while improving research tools. *Proceedings of the 28th Conference of the International, 2*, 55-62.

Dubinsky, E., & McDonald, M. A. (2002). APOS: A constructivist theory of learning in undergraduate mathematics education research. *The teaching and learning of mathematics at university level*, 275-282.

Filloy, E., Rojano, T., & Solares, A. (2010). Problems dealing with unknown quantities and two different levels of representing unknowns. *Journal for Research in Mathematics Education*, 41(1), 52-80.

Manzanero, L. (2007). *Sistemas de Ecuaciones Lineales: Una perspectiva desde la Teoría APOE*. Tesis de Maestría, CINESTAV-IPN, México.

Mares, C. (2016). *Dependencia e independencia lineal. Un análisis con la teoría APOS*. México: CINVESTAV-IPN.
Oktaç, A. (2018). Conceptions about system of linear equations and solution. In S. Stewart, C. Andrews-Larson, A. Berman, & M. Zandieh (Eds.), *Challenges and strategies in teaching linear algebra* (pp. 71–101). Berlin: Springer.
Oktaç, A., & Trigueros, M. (2010). ¿Cómo se aprenden los conceptos de álgebra lineal? *Revista Latinoamericana de Investigación en Matemática Educativa*, 13(4-11), 373-385.
Parraguez, M. G., & Bozt, J. (2012). Conexiones entre los conceptos de dependencia e independencia lineal de vectores y el de solución de sistemas de ecuaciones lineales en R3 desde el punto de vista de los modos de pensamiento. *Revista electrónica de investigación en educación en ciencias*, 49-72.
Puig, L. (1997). Análisis fenomenológico. La educación matemática en la enseñanza secundaria, 61-94.
Rojano, T., Filloy, E., & Puig, L. (2014). Intertextuality and sense production in the learning of algebraic methods. *Educational Studies in Mathematics*, 83(3), 389-407.

STUDENT CHOICES: ONLINE VERSUS IN-PERSON TUTORING OPTIONS

Nicole Infante
University of Nebraska at Omaha
ninfante@unomaha.edu

Lori Ogden
West Virginia University
Lori.Ogden@mail.wvu.edu

Keith Gallagher
University of Nebraska at Omaha
keithgallagher@unomaha.edu

Renee LaRue
West Virginia University
Renee.LaRue@mail.wvu.edu

Deborah Moore-Russo
University of Oklahoma
dmr@ou.edu

Christine Tinsley
University of Oklahoma
cjoy@ou.edu

Ashley Berger
University of Oklahoma
berger@ou.edu

Access to free, high-quality, institutionally provided tutoring services can be critical to the success of university students. When the pandemic forced university tutoring operations to close or move online, many chose to move operations online. The Fall 2021 semester saw the return of in-person tutoring at many institutions; however, online options remained in place to serve students who may not be able to participate in-person. This created more equitable access to a critical student support. We collected data on which format students chose at two research-focused institutions. At one institution, visits were split at 47% online and 53% in-person. In contrast, visits at the other institution were split at 5% online, and 95% in-person. We will discuss the courses with the highest usage in each format and explore why students may be choosing one format over the other.

Keywords: Undergraduate education; Instructional Activities and Practices; Online and Distance Education; Equity, Inclusion, and Diversity

Tutoring services supply essential support for students, helping them to succeed in college mathematics classes. Although professional tutoring is available outside of the university, such services are not financially accessible to all students. University-based tutoring centers offer services to students at no additional cost. Access to high-quality online tutoring services can be critical to the success of students, especially those who are first-generation, are from low-socioeconomic backgrounds, or are facing other challenges to making in-person meetings. In this paper, we discuss students' usage of in-person and online tutoring options at two large research-focused institutions. Further, we will examine how this data may inform future tutoring structures.

Background

Institution-based mathematics tutoring services play a necessary role in preserving equitable access to resources among students. Student use of tutoring resources has been correlated with an increase in final course grades (Byerley et al., 2018; Rickard & Mills, 2018; Xu et al., 2001) and with improvements in persistence, retention, and degree completion (Astin, 1993; Pascarella & Terenzini, 1991, 2005; Rheinheimer et al., 2010; Rheinheimer & Mann, 2000; Rouche & Snow, 1977). First-generation college graduates have reported needing to work harder than their peers to attain the same level of achievement, and minoritized students have reported that lacking a sense of belonging presented a significant challenge to persistence in college (Richardson &

Skinner, 1992). Mathematics tutoring centers act as an academic scaffold for struggling students, allowing instruction to be tailored to the needs of the individual students (Anghileri, 2006) while also helping students develop their mathematical identities through social interaction (Bjorkman & Nickerson, 2019). Access to quality tutoring services is likely to play a significant role in student achievement in the coming years, as it is projected that secondary school students' performance in reading and mathematics will suffer as a result of interrupted instruction due to school closures (e.g., related to the pandemic or weather events), particularly in less affluent areas (Chetty et al., 2020; Kuhfeld et al., 2020).

Existing research points to the importance of tutor training both for in-person and online tutoring (Fitzmaurice et al., 2016; Johns & Mills, 2021; Simão et al., 2008; Turrentine & MacDonald, 2006). Lepper and Woolverton (2002), in a comparison of the practices of expert tutors, identified characteristics of and techniques used by those tutors during tutoring sessions. The behaviors identified in these expert tutors include cognitive, metacognitive, and affective pedagogical strategies and considerations which focus not only on the academic content under consideration but also emphasize the importance of study skills and student mindsets. Research on students' success in school (Blackwell et al., 2007; Boaler, 2013; Cohen & Sherman, 2014; Dweck, 2007; Good et al., 2003; Moser et al., 2011) has shown that non-cognitive or motivational factors including students' mindsets and beliefs about themselves as learners, their feelings about school, and their ability to self-regulate "can matter even more than cognitive factors for students' academic performance" (Dweck et al., 2014, p. 2). Students who work with a tutor tend to use prior knowledge, monitor their progress, utilize effective learning strategies, and seek help more often than those who do not – practices which lead to improved performance on assessments and development of mental models (Azevedo et al., 2008). The role played by tutors is much more than simple presentation of content; effective tutors also provide emotional support and help students learn how to self-regulate and develop productive mindsets about themselves as learners. Tutoring practices, like those discussed above, are known to be effective in the traditional, face-to-face tutoring environment, but in light of the COVID-19 pandemic, many tutoring centers moved operations online and will likely continue to provide their services either entirely or partially in an online environment to meet the needs of their students.

Theoretical Framing

We view tutoring from the perspective of Weisbord's Six-Box Model of how organizations function (Weisbord, 1976). The six boxes are: 1) Purpose, 2) Structure, 3) Relationships, 4) Rewards, 5) Leadership, and 6) Helpful Mechanisms. For this report, we focus on the dimensions of *purpose*, what "business" we are in, and *structure*, how we divide up the work. The purpose of our university tutoring centers is to provide students access to free, high-quality tutoring, and our structure consists of dividing the work between in-person and online offerings. Each of these dimensions is influenced by the environment (i.e., forces from the outside that cannot be controlled) in which it exists.

Weisbord suggested collecting data through observations, surveys, interviews, and readings (of written records) to assist in determining discrepancies between what people say and what people do as well as how the organization exists and how it ought to exist.

An organization should have *helpful mechanisms* that monitor, identify and revise any of the other boxes in Weisbord's model, as needed, to make the organization more effective. In this study, we report on the deliberate use of institutional data to study the university students' usage of and satisfaction with their mathematics tutoring centers in a pandemic era. We will consider if using this data is a helpful mechanism to identify how university tutoring centers might optimally fulfill their purposes considering structural changes implemented to benefit students. In terms of Weisbord's model, our research question is: How might student attendance at in-person and online tutoring sessions be used as a helpful mechanism to support the purposes and structures of tutoring centers? We then speculate on other mechanisms that would be helpful in determining the optimal structures that might help two different universities achieve their organizational purposes.

Context and Data Collection

The data presented here is part of a larger project examining the online tutoring experience. As part of this larger study, training materials for online tutors are being developed and tested with the goal of providing an online tutoring experience that is more comparable to in-person tutoring. Here, we are examining how students' choices of tutoring experience (in-person or online) may inform an institution of how well their structures are fulfilling their purposes with respect to tutoring offerings. Data for this paper include the number of visits to in-person and online tutoring sessions and a student experience satisfaction survey sent to those who attended online sessions.

We tracked the number of student visits to tutoring services provided by the mathematics departments at two large, research-focused universities, U1 and U2, during the Fall 2021 semester. Student visits to the mathematics tutoring centers were tracked via EAB's Navigate and TutorTrac, respectively, for U1 and U2. For in-person visits, students' university ID cards were swiped when they entered and exited to capture information about the mathematics course in which they were enrolled, and the time spent in the tutoring center. Online visits captured this data through a manual entering of the student's ID number by the tutor on duty. At the conclusion of the semester, data were downloaded, summarized, and analyzed.

It should be noted that students who attend in-person tutoring are not necessarily working with a tutor for the entirety of their visit. It is common for students to sit in the tutoring center working on problems independently and only work with a tutor as questions arise. In contrast, in the online environment, students are more likely to be working with a tutor for the duration of their tutoring center visits. Hence, while time spent in the tutoring center (in-person and online) is tracked, we only consider the number of visits in our analysis here. Tracking time spent in the tutoring center is valuable for determining operating hours and physical space considerations.

All online tutors were provided with a Samsung Galaxy Tab S6 Lite tablet to use for their tutoring sessions. Students who used the online tutoring services at U1 and U2 received a 3-question satisfaction survey. Students were asked: 1) Please rate the overall quality of your tutoring session (on a scale of 1 to 5), 2) Please indicate whether the technology allowed for effective interactions with your tutor (on a scale of 1 to 5), and 3) Is there anything else we should know to help serve you better in the future?

The satisfaction survey was sent only to users of the online tutoring services. In-person tutoring has a proven track-record of helping students succeed, and large numbers of students at U1 and U2 used these services pre-pandemic. The purpose of the current study was to determine how the online tutoring environment differed from the in-person, with a focus on how technology facilitates tutor-student interactions.

U1 Context

U1 is a research-focused institution in the mid-Atlantic region of the U.S. with a total enrollment of about 21,000 undergraduate students. It is approximately 50% residential, with mostly freshmen living on campus. U1 has two campuses, Campus A and Campus B, about 2 miles apart. The mathematics tutoring center is located on Campus A in the same building as the mathematics department and offers drop-in in-person tutoring. Although most math courses are offered on Campus A, only about 9% attend their mathematics class in the same building as the mathematics tutoring center and 25% of students enrolled in a math class attend their class on Campus B.

Most tutors were undergraduate students. All tutors (in-person and online) participated in tutor training with sessions occurring both synchronously and asynchronously online. This training consisted of an initial one-on-one meeting with the tutor coordinator to discuss tutoring basics, regular written self-reflections on the tutor's own practice, and the analysis of recorded mock tutoring sessions. Online tutors were encouraged to enroll in a separate one-credit hour course so that those who were working in the online environment received additional training on how to tutor online and how to effectively use technology to facilitate interactions between students and tutors.

At U1, in-person and online tutoring was offered throughout the semester. In-person tutoring was offered all semester, Monday – Friday from 10am-3pm for a total of 375 hours (63% of available tutoring hours). Online tutoring was offered all semester Sunday – Thursday from 6pm-9pm for 225 total hours (38% of available tutoring hours). Online tutoring sessions were conducted via Zoom. All students, regardless of whether they were enrolled in in-person or online classes, could attend either in-person or online tutoring.

U2 Context

U2 is a research-focused institution in the southwest region of the U.S. with a total enrollment of about 23,000 undergraduates. It is 29% residential, with most freshmen residing on campus. U2's mathematics tutoring center offers drop-in tutoring. U2's in-person tutoring center is located on the main floor of the building where about 75% of its math classes are offered and where all mathematics instructors' offices are located.

The tutors are a mix of faculty, graduate students, and undergraduate students. The graduate and undergraduate tutors undergo training. All tutors attend 6 hours of general training on tutoring prior to their first semester, of which 1 hour is dedicated to online tutoring. Graduate students take 30 hours of content training throughout their first year to qualify to tutor for the different courses served by U2's center. The graduate student tutors have an opportunity at the beginning of the year to reduce the number of hours of training if they pass one or more of the content exams. Undergraduate tutors earn raises for each additional content training they attend and exam they pass. U2's online tutors were selected from experienced undergraduate tutors who were also working, or had worked before, as in-person tutors. Online tutors receive additional training on how to tutor online, including how to use digital devices to display written or digital renderings of graphs, equations, tables, and writing for students to see.

Like U1, U2 offered both in-person and online tutoring options throughout the semester. At U2, in-person tutoring was offered on weekdays (Monday-Thursday, 10am-5pm and Fridays, 10am-2pm) for 479 hours (67% of available tutoring hours) during the semester while online tutoring was offered on weekday evenings (Monday-Thursday, 5pm-9pm) and Sunday afternoons (3pm-7pm) for 232 hours (33% of available tutoring hours). Online tutoring sessions were conducted via Zoom. All students, regardless of whether they were enrolled in in-person or online classes, could attend either in-person or online tutoring.

It should be noted that during the first year of the pandemic, when tutoring was only offered online, both U1 and U2 saw a significant decrease in the number of students seeking university provided tutoring services. Johns and Mills (2021) reported a similar phenomenon at 25 other universities.

Findings

We report on student enrollment and tutoring usage for the courses in two tracks: STEM intending and non-STEM intending. Within these tracks, we examine the data for the course of: College Algebra (STEM vs. non-STEM), Precalculus/Trigonometry, Calculus I, Calculus II, Calculus III, and Applied Calculus. While the general student populations at the two campuses are very similar, they make different choices regarding how they attend tutoring.

In-person Versus Online Usage

Table 1 displays the number of students enrolled in each course and the number of tutoring visits by course and modality. Note that the number of students enrolled in the courses under consideration at U1 (3,986) is almost twice as many as the number of students enrolled in those same courses at U2 (2,171). We observe that even though U2 has fewer students enrolled in these courses, the number of tutoring visits is significantly higher than those of U1. This equates to about 0.5 visits per student at U1, and 2.3 visits per student at U2. Further, we observe that the number of in-person tutoring visits at U1 (1,084) is approximately one-fourth that at U2 (4,679). For online tutoring visits, U1 had about two-and-a-half times as many visits (969) as U2 (372). Hence, we see that U2's students more frequently availed themselves of tutoring services and students heavily favored the in-person tutoring experience. In contrast, U1's students did not use tutoring services as often but used online and in-person tutoring services almost equally.

Table 1: Population and Tutoring Attendance for Algebra, Precalculus/Trigonometry and Calculus Courses

Track	Course	U1				U2			
		Course Population		Tutoring Visits		Course Population		Tutoring Visits	
		in-person	online	in-person	online	in-person	online	in-person	online
		n=3535	n=451	n=1084	n=969	n=1950	n=221	n=4679	n=372
STEM	Coll Algebra	466	14	44	118	344	31	551	43
	Precalc/Trig	159	75	136	24	421	23	1020	61
	Calculus I	731	118	187	282	542	57	2028	143
	Calculus II	240	50	120	89	194	63	678	87
	Calculus III	368	39	29	36	449	47	402	38
Non-STEM (Applied)	Coll Algebra	589	35	471	264	*	*	*	*
	Calculus	982	120	42	44	*	*	*	*

*Since U2 had no regular online offerings for non-STEM classes; such data were not considered.

Table 2 shows the percentages of students enrolled in each course modality and the percentage of tutoring visits utilized by those students. The majority of U1's students (88.7%) were taking math courses in person, and the majority (62.5% of possible hours) of U1's tutoring was offered in-person, while U1's in-person tutoring accounted for about half (52.8%) of its tutoring attendance.

The majority of U2's students (89.8%) were taking College Algebra, Precalculus/Trigonometry, and Calculus courses in person, and the majority (67.4% of possible hours) of U2's tutoring was offered in-person, while U2's in-person tutoring accounted for 92.6% of its normal tutoring attendance.

Table 2: Percentage of Tutoring Visits by Population for Algebra, Precalculus/Trigonometry and Calculus Courses

Track	Course	U1	U2

		Course Population		Tutoring Visits		Course Population		Tutoring Visits	
		in-person	online	in-person	online	in-person	online	in-person	online
		n=3535	n=451	n=1084	n=969	n=1950	n=221	n=4679	n=372
STEM	Coll Algebra	13.2	3.1	4.1	12.2	17.6	14.0	11.8	11.6
	Precalc/Trig	4.5	16.6	12.5	2.5	21.6	10.4	21.8	16.4
	Calculus I	20.7	26.1	17.3	29.1	27.8	25.8	43.3	38.4
	Calculus II	6.8	11.1	11.1	9.2	9.9	28.5	14.5	23.4
	Calculus III	10.4	8.6	2.7	3.7	23.0	21.3	8.6	10.2
Non-STEM (Applied)	Coll Algebra	16.7	7.7	43.5	27.2	*	*	*	*
	Calculus	27.8	26.6	3.8	4.5	*	*	*	*

U2 also offered special tutoring opportunities either one or two evenings before coordinated exams (for both STEM and non-STEM mathematics classes). For each exam, there were two types of review opportunities: an online evening facilitator-centered review session that was followed by an in-person “After Dark” tutoring session. Over the course of the Fall 2021 semester, the online review sessions were offered for 20 hours, and the After Dark in-person tutoring was offered for 30 hours. The attendance at these review opportunities (which is in addition to the numbers for regular tutoring provided in Table 1) was 1,058 for the online reviews and 1,507 for the in-person tutoring. The majority of U2’s review opportunities (60% of possible hours) was offered in-person and accounted for 58.8% of the student visits to special tutoring opportunities. U2’s online reviews accounted for 41.2% of its attendance at review opportunities and accounted for 40% of the possible hours to special tutoring opportunities.

Satisfaction Survey

We received 130 responses to the satisfaction survey, over 90% of which were from U1. The results of the survey suggest that students were generally pleased with their online tutoring experiences. Overall, 79% (24% and 55%, respectively) of survey responses rated the tutoring session as “very effective” or “extremely effective.” 86% (24% and 62%, respectively) of survey responses indicated that their tutor’s available technology “very effectively” or “extremely effectively” facilitated their interactions in the online tutoring environment.

In line with the numerical results of the survey, when students provided comments, they were often positive, though students also left negative feedback. Among the positive comments, students complimented their tutors: “[My tutor] was incredibly helpful”; “My tutor was FANTASTIC and incredibly helpful.” Others mentioned specific teaching strategies employed by their tutors: “She asked me to re-explain the problem to her, which helped me really understand the concept.” Within the negative comments, students cited errors made by their tutor: “Tutor didn’t do the problem right, misled me for my homework.” and a desire for longer sessions and more one-on-one tutoring (as opposed to small groups), as well as factors beyond the tutors’ control (“Internet connection was terrible.”, “…a way to have multiple people sharing screens in zoom”). Students also sometimes requested that the tutoring center hire additional tutors for specific courses.

Discussion

As Weisbord (1976) noted, helpful mechanisms should exist to monitor, identify, and inform revision of the other boxes in the Six-Box model. The attendance and survey data we collected serves as a helpful mechanism, providing feedback on the purposes and structures of the tutoring centers at U1 and U2. These data allowed us to monitor and identify what format students chose when seeking tutoring. We acknowledge that during some of these visits, particularly in-person centers, students may spend minimal time interacting with tutors and are using the space to study and work independently, knowing help is near. The data from U1 highlights that their students use the two tutoring structures (in-person and online) almost equally. In contrast, the data from U2 shows that their students are much more likely to attend in-person tutoring. Furthermore, the survey data collected suggests that students who utilized the university's online mathematics tutoring services largely found their experiences beneficial. For those students who did not find their experiences as fulfilling, the reasons they cited were mostly beyond the tutors' control.

Students at the different sites used the in-person and online environments in very different proportions. The locations of U1's and U2's mathematics tutor centers may help explain students' choices of tutoring modality. Though U1 has a larger number of residential students than U2, it witnessed a significantly higher rate of online tutoring attendance than U2. We hypothesize that this disparity may be due to U1 being split across two campuses. The tutoring center at U1 is located on Campus A, but many students live and take classes on Campus B, which sits approximately 2 miles away, making it difficult to visit the tutoring center in person for some students. For these students, online tutoring provides an option to receive help between classes on Campus B or during other times when reaching Campus A may be challenging or inconvenient. Even though the number of U2 residential students is lower than U1's, the number of students who live close to campus, either as part of the Greek system or in off campus apartments, is quite large. So, almost all undergraduate students walk or drive to campus then stay on campus for their classes.

Data from the satisfaction survey serves as a helpful mechanism by identifying components of the tutoring structures that students appreciated as well as others they would like to see changed. Some of the feedback provided by students about online tutoring is reminiscent of feedback often received about in-person tutoring; sometimes tutors are fantastic, and sometimes tutors make mistakes. Focusing on novel feedback, we were able to distinguish between aspects of the online tutoring experience that are within the control of tutors and tutoring centers and those that are beyond our control. For example, students' access to quality internet is beyond the control of tutors and tutoring centers. However, tutoring centers can place an emphasis on ensuring that tutors are provided with appropriate equipment (e.g., pen-enabled tablets) and ensuring that there are sufficient numbers of tutors for courses with particularly high demand for tutoring. In our future work, we will investigate the specific ways in which the use of appropriate equipment fosters online tutoring interactions.

The purpose of the tutoring centers at U1 and U2 is to provide access to free, high-quality tutoring. Our data suggest that students do find value in attending, as both institutions have many repeat users of their tutoring services, and those who completed the satisfaction survey largely rated their experiences positively. For students at U1, it appears that both the in-person and online tutoring options are equally valuable, as they use these services more or less equally. As mentioned above, this could be due to the difficulty in traveling between campuses at U1. We also believe U1 and U2 are offering a better online experience than 6 - 12 months ago. Although our initial implementation of online tutoring saw significantly diminished usage, we have seen higher traffic in recent months. In our future research, we will continue the use of the satisfaction survey.

In sum, we found that students valued the online option equally with the in-person option at U1, while students at U2 clearly favored the in-person option for regular tutoring but attended the in-person and online evening review sessions prior to exams in the same proportions the hours were offered. This data allows each institution to examine the modalities of their tutoring services and adjust to better suit their student populations.

Acknowledgments

This work was supported by NSF IUSE award DUE-2201747. All findings and opinions are those of the research team and do not necessarily reflect the NSF's position.

We would like to thank Tim McEldowney for his contributions to the development of training materials and the study participants for allowing us to better understand the online tutoring environment.

References

Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, *9*(1), 33-52.

Astin, A. (1993). *What Matters In College: Four critical years revisited*. Jossey-Bass.

Azevedo, R., Moos, D. C., Greene, J. A., Winters, F. I., & Cromley, J. G. (2008). Why is externally-facilitated regulated learning more effective than self-regulated learning with hypermedia? *Educational Technology Research and Development*, *56*, 45-72.

Bjorkman, K., & Nickerson, S. (2019). This is us: An analysis of the second groups within a mathematics learning center. 22nd Annual Conference on Research in Undergraduate Mathematics Education, Oklahoma City, OK.

Blackwell, L. S., Trzesniewski, K. H., & Dweck, C. S. (2007). Implicit Theories of Intelligence Predict Achievement Across an Adolescent Transition: A Longitudinal Study and an Intervention. *Child Development*, *78*(1), 246-263.

Boaler, J. (2013). Ability and Mathematics: the mindset revolution that is reshaping education. *FORUM*, *55*(1), 143-152.

Byerley, C., Campbell, T., & Rickard, B. (2018). Evaluation of Impact of Calculus Center on Student Achievement. 21st Annual Conference on Research in Undergraduate Mathematics Education, San Diego, CA.

Chetty, R., Friedman, J., Hendren, N., Stepner, M., & Team, T. O. I. (2020). *How Did COVID-19 and Stabilization Policies Affect Spending and Employment? A New Real-Time Economic Tracker Based on Private Sector Data* [Working Paper].

Cohen, G. L., & Sherman, D. K. (2014). The Psychology of Change: Self-Affirmation and Social Psychological Intervention. *The Annual Review of Psychology*, *65*, 333-371. https://doi.org/10.1146/annurev-psych-010213-115137

Dweck, C. S. (2007). Is Math a Gift? Beliefs That Put Females At Risk. In S. J. Ceci & W. Williams (Eds.), *Why aren't more women in science? Top researchers debate the evidence* (pp. 47-55). American Psychological Association. https://doi.org/https://doi.org/10.1037/11546-004

Dweck, C. S., Walton, G. M., & Cohen, G. L. (2014). *Academic Tenacity: Mindsets and Skills that Promote Long-Term Learning*. B. M. G. Foundation.

Fitzmaurice, O., Cronin, A., Ni Fhloinn, E., O'Sullivan, C., & Walsh, R. (2016). Preparing Tutors for Mathematics Learning Support. *MSOR Connections*, *14*(3), 14-21.

Good, C., Aronson, J., & Inzlicht, M. (2003). Improving Adolescents' Standardized Test Performance: An intervention to reduce the effects of stereotype threat. *Applied Developmental Psychology*, *24*, 645-662. https://doi.org/10.1016/j.appdev.2003.09.002

Johns, C., & Mills, M. (2021). Online Mathematics Tutoring During the COVID-19 Pandemic: Recommendations for Best Practices. *PRIMUS : problems, resources, and issues in mathematics undergraduate studies*, *31*(1), 99-117. https://doi.org/10.1080/10511970.2020.1818336

Kuhfeld, M., Soland, J., Tarasawa, B., Johnson, A., Ruzek, E., & Liu, J. (2020). *Projecting the Potential Impacts of COVID-19 School Closures on Academic Achievement*. Brown University.

Lepper, M. R., & Woolverton, M. (2002). The Wisdom of Practice: Lessons Learned from the Study of Highly Effective Tutors. In *Improving Academic Achievement* (pp. 135-158). Academic Press.

Moser, J. S., Schroder, H. S., Heeter, C., Moran, T. P., & Lee, Y.-H. (2011). Mind Your Errors: Evidence for a Neural Mechanism Linking Growth Mind-Set to Adaptive Posterror Adjustments. *Psychological Science*, *22*(12), 1484-1489. https://doi.org/10.1177/0956797611419520

Pascarella, E., & Terenzini, P. (1991). *How College Affects Students: Findings and Insights From Twenty Years of Research*. Jossey-Bass.

Pascarella, E., & Terenzini, P. (2005). *How College Affects Students: A Third Decade of Research* (Vol. 2). Jossey-Bass.

Rheinheimer, D. C., Grace-Odeleye, B., Francois, G. E., & Kusorgbor, C. (2010). Tutoring: A Support Strategy for At-Risk Students. *The Learning Assistance Review*, *15*(1), 23-34.

Rheinheimer, D. C., & Mann, A. (2000). Gender Matching, Floor Effects, and Other Tutoring Outcomes. *Journal of Developmental Education*, *24*(2), 10-18.

Richardson, R. C., Jr., & Skinner, E. F. (1992). Helping First-Generation Minority Students Achieve Degrees. *New Directions for Community Colleges*, *80*, 29-43.

Rickard, B., & Mills, M. (2018). The Effect of Attending Tutoring on Course Grade in Calculus I. *International Journal of Mathematical Education in Science and Technology*, *49*(3), 341-354.

Rouche, J., & Snow, J. (1977). *Overcoming Learning Problems*. Jossey-Bass.

Simão, A. M. V., Flores, M. A., Fernandes, S., & Figueira, C. (2008). Tutoring In Higher Education: Concepts and Practices. *Sísifo: Educational Sciences Journal*(7), 73-86.

Turrentine, P., & MacDonald, L. (2006). Tutoring Online: Increasing Effectiveness with Best Practices. *NADE Digest*, *2*(2), 9-18.

Weisbord, M. R. (1976). Organizational Diagnosis: Six Places To Look for Trouble with or Without a Theory. *Group & organization studies*, *1*(4), 430-447. https://doi.org/10.1177/105960117600100405

Xu, Y., Hartman, S., Uribe, G., & Mencke, R. (2001). The Effects of Peer Tutoring on Undergraduate Examination Scores in Mathematics. *Journal of College Reading and Learning*, *32*(1), 22-31.

WHICH SHOULD COME FIRST: ISOMORPHISM OR HOMOMORPHISM?

Alison Mirin
University of Arizona
amirin@math.arizona.edu

Rachel Rupnow
Northern Illinois University
rrupnow@niu.edu

Isomorphism and homomorphism are central concepts in abstract algebra, but their interrelatedness provokes questions about which should be taught first. This study investigates instructors' pedagogical preferences and rationales for their ordering of the concepts and compares them to textbooks that they use. We found that for a third of participants, there was a lack of alignment between preferences and textbooks. Additionally, the metaphors used in support of a homomorphism-first approach centered on the idea of formal definition, while those used in support of an isomorphism-first approach focused on sameness.

Keywords: Undergraduate Education, Advanced Mathematical Thinking, Curriculum.

Isomorphism and homomorphism are central concepts in abstract algebra. There are multiple ways to view isomorphism and homomorphism as being related, two of which are opposite sides of the same coin. On one side, an isomorphism can be viewed as a specific type of homomorphism (bijective). On the other side, we have the notion that homomorphism can be viewed as a generalization or abstraction of isomorphism. While the four most commonly-used (Melhuish, 2015) introductory abstract algebra textbooks in the United States (Fraleigh, 2003; Gallian, 2009; Gilbert & Gilbert, 2008; Hungerford, 2012) all introduce isomorphism first (Mirin & Rupnow, in press), there are some other, more advanced, texts that introduce homomorphism first (e.g., Dummit & Foote, 2004; Lang, 2002). Our work addresses the following questions: (1) To what extent do the participants and their chosen textbooks have aligned ordering? (2) What conceptual metaphors about isomorphism and homomorphism did the participants use to describe, justify, or explain their ordering?

Background and Literature

There is a variety of mathematics education work on isomorphism and homomorphism (Dubinsky et al., 1994; Hausberger, 2017; Larsen, 2013; Leron et al., 1995; Melhuish, 2018; Melhuish et al., 2020; Rupnow, 2021). Although not focused specifically on the ordering of isomorphism and homomorphism, some work provides insight into instructors' ordering preferences. For example, in their work on instructors' goals when assigning homework, Rupnow et al. (2021) found that some instructors would prefer to teach homomorphism first, despite being given a textbook introducing isomorphism first. In contrast, other instructors do choose to teach isomorphism first, both in following Larsen's Teaching Abstract Algebra for Understanding materials (Larsen, 2013) and in making their own decisions (Rupnow, 2021).

Gilbert and Gilbert (2008) point to a tension between generalization (isomorphism first) and specification (homomorphism first) by stating, "Isomorphism is a special case of homomorphism, while homomorphism is a generalization of isomorphism. Isomorphisms were placed first in this book with the thought that 'same structure' is the simpler idea" (p.137). It seems that, while Gilbert and Gilbert acknowledge a dilemma between generalization and specification, they eventually decide that isomorphism is an easier concept to introduce than homomorphism. This can be viewed as starting with what is conceptually simpler, even if the definition of homomorphism is more concise and with fewer conditions. We investigate the order in which instructors introduce these concepts and their rationales. Note that the notion of generalization starting with a simpler or more accessible concept is not unique to isomorphism versus homomorphism. For example, Tall (1991) explains that mathematically one must learn the notion of limit before learning the derivative. However, he points out that this instructional sequencing might result in students seeing limits as unmotivated, and that purely "logical" sequencing can be "blinkered and limiting" (p.18).

We use the notion of *conceptual metaphors* (Lakoff & Nuñez, 1997) as a framework, in which a metaphor is viewed as a relationship between a source domain and a target domain. While these domains can inform each other, the source domain is generally intended to provide insight into the target domain. For example, "An isomorphism is a special homomorphism" can be viewed as a metaphor in which *isomorphism* is the target domain, and *homomorphism* is a source domain that provides insight into the nature of isomorphism.

Methods

Participants were selected from a subset of mathematicians who were research-active in an algebra-related field and provided clear responses to a survey about sameness. Nine (labeled here as A, B, C, D, E, F, G, H, and J) who had taught abstract algebra at least once, participated in an interview. The interview questions centered on how the participants describe and characterize isomorphism and homomorphism in a variety of contexts. Here, we concern ourselves only with the data pertinent to our research questions about ordering and rationales for ordering. For eight of the nine participants, this amounts to examining their responses to the question, "do you teach isomorphism or homomorphism first, and what is your rationale?". One participant, E, addressed the question in response to an earlier question and was not explicitly asked about ordering.

We collaboratively coded to consensus using thematic analysis (Braun & Clarke, 2006). Specifically, for each participant, we catalogued and analyzed (1) the ordering preference of isomorphism and homomorphism, (2) how they used metaphors in their rationale for the ordering preference, and (3) any rationale given for an ordering that was not their stated preference. In doing so, we utilized the metaphors delineated in Rupnow and Randazzo (this volume) to see which metaphors surrounding isomorphism and homomorphism the participants used as warrants or explanations for each ordering. While Rupnow and Randazzo (this volume) note seven metaphor clusters, all metaphors in this paper relate to four clusters: sameness, sameness/mapping, sameness/formal definition, and formal definition.

Results

Six of the nine participants stated that they teach homomorphism first, and three stated that they teach isomorphism first. While the course textbooks were known for only six of the participants (B refused to state their textbook but had one in mind; C did not have a particular textbook in mind and talked about adjusting to whatever text they were given; E "hardly ever crack[s] open the textbook"), we nonetheless examine the alignment between the participants' instructional sequencing and that of their adopted textbooks. Three participants (A, F, and G) use textbooks in which isomorphism is introduced first yet teach homomorphism first. The other three participants are consistent: two use a text in which homomorphism is introduced first (Dummit & Foote, 2004) and do indeed teach homomorphism first, while one uses a text (Hungerford, 2012) in which isomorphism is introduced first and does indeed introduce isomorphism first. To summarize, of the six participants whose textbooks are known, half teach isomorphism and homomorphism in an order that aligns with their textbook, while the other half teach homomorphism first despite having textbooks that introduce isomorphism first. The ordering preferences and alignment with textbooks are summarized below (Table 1).

(Participants generally stated an author for their text; here we cite the most recent edition by the author(s).)

Table 1: Textbook Ordering and Instructional Ordering

	Isomorphism first in text	Homomorphism first in text	Unknown text
Homomorphism first teaching	A (Judson, 2019), F (Armstrong, 1997), G (Fraleigh, 2003)	D and H (both Dummit & Foote, 2004)	B
Isomorphism first teaching	J (Hungerford, 2012)		C, E

The reason for teaching homomorphism first was consistent across participants; all justified this ordering in a way that employed the metaphors capturing the notion that an isomorphism is a special type of homomorphism. This came in two distinct (targets and sources flipped) but related metaphors from the formal definition cluster: *special homomorphism* and *isomorphism without bijectivity*. For example, one participant (D) mentioned "convenience" of communication as a reason for introducing homomorphism first: "it's convenient if students already know the definition of homomorphism and are comfortable with this before defining isomorphism…I would like to be able to give the definition of isomorphism in a nice, condensed manner". Participant A, although not mentioning convenience explicitly, hinted at the idea: "we inherit the theory from homomorphism (when studying isomorphism)". Participant E even suggested that homomorphism might be "easier…from a logical point of view" due to not having to "worry" about bijectivity. Yet another participant (F) described homomorphism as "simpler". Despite these variations, we understand these notions of "simplicity", "convenience" and "easiness" as instances of the same core metaphor, stated by all participants who endorsed the homomorphism-first approach and by some who rationalized why homomorphism might be taught first (despite preferring isomorphism first): isomorphism is a *special homomorphism.*

The notion of isomorphism as an instance of homomorphism was *not* explicitly used in support of an isomorphism-first approach. Instead, participants primarily justified that approach with metaphors from sameness-based clusters. They contrasted the sameness of isomorphism with the more complicated nature of homomorphism. For example, J used sameness metaphors for both isomorphism and homomorphism to explain that isomorphism should be taught first because isomorphism sameness is more intuitive than homomorphism sameness:

> Are these objects actually the same? So that, I think, motivates the isomorphism question and the definition of isomorphism. But I feel like homomorphism needs to come a little later because homomorphism really is not talking about exact copies, it's talking about, there's kind of lots of different ways the homomorphism can manifest as I said, with collapsing or exact copies. And so there, it's a little bit fuzzier of a notion and of comparing two objects.

While there was consistency in the use of sameness-based metaphors for justifying isomorphism-first, the particular choices of metaphors varied. The most commonly used metaphor was *generic sameness* (a sameness-cluster metaphor); this was used by C and J (quoted above) to justify why they used an isomorphism-first approach, and it was additionally used by homomorphism-first participants (A, B, H) to characterize an isomorphism-first approach. Other sameness-related metaphors used to explain an isomorphism-first approach included *matching* (a sameness/mapping-cluster metaphor, used by J), *renaming/relabeling* (a sameness/mapping-cluster metaphor, used by E and C), *classification* (a sameness-cluster metaphor, used by G), *logical equivalence* (a sameness-cluster metaphor, used by G), and *structure preservation* (a sameness/formal definition-cluster metaphor, used by C). For example, C explained: "I think it's easier to think about sort of we're preserving all the structure, and isomorphism is in a sense just a relabeling, if you're in the context of group theory. Just relabeling the elements but it ends up being the same group or something".

In summary, while there was only the formal definition cluster of metaphors used to justify a homomorphism-first approach (*special homomorphism* and *isomorphism without bijectivity*), there were varied metaphors from all three sameness clusters (sameness, sameness/mapping, sameness/formal definition) justifying an isomorphism-first approach.

Discussion

Prior metaphor-focused work with isomorphism and homomorphism has separately focused on ways people think or on textbooks (Melhuish et al., 2020; Mirin & Rupnow, in press; Rupnow, 2021). Our research expands this work by using metaphors to interpret the instructional decision making of research-active algebraists. One possible direction for future research is to examine mathematics *education* researchers' views on this topic to determine whether their views align with isomorphism first (like textbooks) or homomorphism first (like some of the mathematicians) and see if there are, in fact, differences between these populations' views.

With respect to ordering, while there was some alignment between mathematicians' textbooks and descriptions of instruction, our work demonstrates that this alignment should not be assumed. These particular participants were research-active faculty with experience in algebra beyond their coursework; potentially the text could be more impactful on those who are not research-active in these areas. Nevertheless, this study raises questions about the extent to which upper-level undergraduate textbooks should be viewed as the "intended curriculum" (Fan, 2013; Zhu & Fan, 2006) as opposed to a secondary resource for reexamining material from another perspective. Furthermore, while some research has begun examining how students use dynamic textbooks in upper-level courses (Mesa & Mali, 2020), how faculty want upper-level undergraduate students to engage with textbooks remains understudied.

With respect to rationales, we saw dominant metaphor clusters for each ordering. Those who preferred to teach homomorphism first referred to the logic-based relationship between isomorphism and homomorphism encoded in their formal definitions (*special homomorphism* and *isomorphism without bijectivity* metaphors) for justification. In contrast, those who preferred to teach isomorphism first drew on sameness-based metaphors such as *generic sameness*, *matching*, *renaming/relabeling*, *classification*, *logical equivalence*, and *structure-preservation,* which together relate to all three sameness-based clusters. Thus, despite differences of opinion on which ordering is best, the participants displayed consistency in the rationale for each choice. While we acknowledge that choosing participants from respondents to a survey about sameness might lead to expectations that they would talk about sameness to justify their thinking, we note that not all participants spoke about sameness, and none used it to justify homomorphism-first. Future research should examine the extent to which these findings generalize.

Acknowledgments

This research was partially funded by a Northern Illinois University Research and Artistry Grant to Rachel Rupnow, grant number RA20-130.

References

Armstrong, M. A. (1997). *Groups and Symmetry* (2nd ed.). Springer.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, *3*(2), 77–101. https://doi.org/10.1191/1478088706qp063oa

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics*, *27*(3), 267–305. https://doi.org/10.1007/BF01273732

Dummit, D. S., & Foote, R. M. (2004). *Abstract algebra* (3rd ed). Wiley.

Fan, L. (2013). Textbook research as scientific research: Towards a common ground on issues and methods of research on mathematics textbooks. *ZDM*, *45*(5), 765–777. https://doi.org/10.1007/s11858-013-0530-6

Fraleigh, J. B. (2003). *A first course in abstract algebra* (7th ed). Addison-Wesley.

Gallian, J. (2009). *Contemporary Abstract Algebra*. Cengage Learning.

Gilbert, L., & Gilbert, J. (2008). *Elements of Modern Algebra* (7th ed.). Cengage Learning.

Hausberger, T. (2017). The (Homo)morphism Concept: Didactic Transposition, Meta-Discourse and Thematisation. *International Journal of Research in Undergraduate Mathematics Education*, *3*(3), 417–443. https://doi.org/10.1007/s40753-017-0052-7

Hungerford, T. W. (2012). *Abstract Algebra: An Introduction*. Cengage Learning. https://market.android.com/details?id=book-vRD7CAAAQBAJ

Judson, T. W. (2019). *Abstract algebra: Theory and applications.* http://abstract.ups.edu/download/aata-20190710-print.pdf.

Lakoff, G., & Nunez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–29). Erlbaum.

Lang, S. (2002). *Algebra* (Rev. 3rd ed). Springer.

Larsen, S. P. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior*, *32*(4), 712–725. https://doi.org/10.1016/j.jmathb.2013.04.006

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics*, *29*(2), 153–174. https://doi.org/10.1007/BF01274211

Melhuish, K. (2015). Determining What to Assess: A Methodology for Concept Domain Analysis As Applied to Group Theory. In M. Zandieh, T. Fukuwa-Connelly, N. Infante, & K. Keene (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 736–744). SIGMAA on RUME.

Melhuish, K. (2018). Three Conceptual Replication Studies in Group Theory. *Journal for Research in Mathematics Education*, *49*(1), 9–38. https://doi.org/10.5951/jresematheduc.49.1.0009

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *The Journal of Mathematical Behavior*, *60*, 100806. https://doi.org/10.1016/j.jmathb.2020.100806

Mesa, V., & Mali, A. (2020). Studying Student Actions with Dynamic Textbooks in University Settings: The Log as Research Instrument. *For the Learning of Mathematics*, *20*(2), 8–14.

Mirin, A., & Rupnow, R. (in press). Isomorphism and Homomorphism in Abstract Algebra Textbooks. *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education*, Boston, MA.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *The Journal of Mathematical Behavior*, *62*, 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Rupnow, R., Hegg, M., Fukawa-Connelly, T., Johnson, E., & Weber, K. (2021). How mathematicians assign homework problems in abstract algebra courses. *The Journal of Mathematical Behavior*, *64*, 100914. https://doi.org/10.1016/j.jmathb.2021.100914

Rupnow, R., & Randazzo, B. (this volume). Mathematicians' language for isomorphism and homomorphism. In *Proceedings of the 44th Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Nashville, TN.

Tall, D. (1991). The Psychology of Advanced Mathematical Thinking. In *Advanced Mathematical Thinking* (pp. 3–21). Springer Netherlands. https://doi.org/10.1007/0-306-47203-1_1

Zhu, Y., & Fan, L. (2006). Focus on the Representation of Problem Types in Intended Curriculum: A Comparison of Selected Mathematics Textbooks from Mainland China and the United States. *International Journal of Science and Mathematics Education*, *4*(4), 609–626. https://doi.org/10.1007/s10763-006-9036-9

WHAT FACTORS CONTRIBUTE TO STUDENT SUCCESS IN PRECALCULUS FLIPPED CLASSROOMS

Florence F Obielodan
University at Buffalo - SUNY
ffobielo@buffalo.edu

Dr. Ji-Won Son
University at Buffalo - SUNY
jiwonson@buffalo.edu

Flipped instruction has been flaunted as a pedagogical strategy that supports improved learning and retentive abilities of students. Nevertheless, one of the twin challenges reported in the literature, students' failing to complete preparatory work, impedes the efficacy of the model. Thus, learners' motivation and attitude to work are essential for the successful implementation of the constructivist learning-centric approach. Yet very few studies have examined the connections between students' motivation and achievement goals in flipped instruction. To address the current gap, this study investigated the relationship between students' motivation beliefs, flipped method, and achievement in precalculus using pre- and post-course surveys collected from 32 undergraduates. Both motivation beliefs and flipped instruction influenced academic achievement positively in the course and were moderated by students' efforts.

Keywords: Flipped instruction, precalculus, motivation beliefs,

The literature on undergraduate education in science, technology, engineering, and mathematics (STEM) indicated that mathematics, especially, calculus prevents many students from pursuing a career in STEM (Almeida, Queiruga-Dios, & Cáceres, 2021; Bressoud, 2015; Rasmussen et al., 2019; Sande & Reiser, 2018). Prior research faulted inadequate preparation at high school (Bressoud, Camp, & Teague, 2017; Sadler & Sonnert, 2016) and subpar teaching methods (Gilboy, Heinerichs, & Pazzaglia, 2015; Rasmussen et al., 2019). Thus, educators continue to explore active learner-centered strategies to provide individualized instructions for each student (Blumberg, 2008; Weimer, 2012). The flipped instruction method embodies this description.

The current study investigated the contributions of the flipped strategy on students' perceived motivation to learn precalculus, an extensive and rather demanding course with a high failure rate but required for success in calculus (Bressoud, 2021). Many of the students enrolled in the college precalculus have taken the course in high school, but often struggle to comprehend the course content and become unmotivated to pursue success (Bressoud et al., 2017; Sadler & Sonnert, 2016). Comparative studies have been conducted on the efficacy of flipped instruction in secondary and postsecondary courses by comparing students' performances in courses taught using flipped instruction with the same taught using the lecture method (Love, Hodge, Grandgenett, & Swift, 2014; Pattanaphanchai, 2019). However, research on the flipped strategy remained underexplored; particularly, the role of motivation on achievement goals in a flipped instruction setting. We aimed to contribute to the literature by examining the relationship between the components of motivation beliefs. We also examined how preference for flipped instruction affected students' cognitive processes and learning achievement in the course. The following research questions are used in this study:

1. What are the relationships between the components of students' motivation beliefs including expectancy beliefs, perceived value, the cost of and preference for learning precalculus in a flipped instruction?
2. To what extent do effort, motivation beliefs and preference for the flipped instructional method contribute to students' overall academic success in a flipped college precalculus course?

Research on Flipped Instruction

Flipped instruction is a constructivist learning-centric approach (Araujo, Otten, & Birisci, 2017; Gilboy et al., 2015) that provides learners the opportunities to construct knowledge of content independently and through collaborative efforts for effective learning, increased engagement, improved academic performance, and better retention (Araujo et al., 2017; Clark, 2015; Graziano & Hall, 2017; Love et al., 2014). The model involves moving all or part of content delivery outside of the classroom by having students watch video lectures and/or read assigned texts at home, while class time is used for discussions and problem-solving to promote conceptual understanding (Nielsen, Bean, & Larsen, 2018). The didactic approach promotes effective learning through the construction and reconstruction of content knowledge (Gilboy et al., 2015; Love et al., 2015). This is to say students have two attempts at learning concepts: first, independently before class, and again during class under the tutelage of the teachers and through collaborative learning with peers. However, motivation is crucial for academic success in a flipped instruction setting (Huang & Hong, 2016; Zainuddin, 2018), where the onus of learning is more student-centered. Yet, limited studies explored the direct connection between students' motivation (Zainuddin, 2018), including their ability beliefs, expectancy for success, interest and perceived value of learning, and the effort needed to succeed (Barron & Hulleman, 2015) in a flipped instruction.

Despite the enumerated benefits, a commonly reported limitation in the literature is inadequate pre-class preparations, especially failing to watch lecture videos (Araujo et al., 2017). Others are resistance to change due to increased student and teacher workload, time constraint (Araujo et al., 2017), difficulty of learning tasks (Clark, 2015; Graziano & Hall, 2017), and inability for students to receive immediate feedback during out-of-class activities (Chen, Wang, Kinhuk, & Chen, 2014). Empirical data revealed that curriculum, task difficulty, and test anxiety, correlate with students' effort and perceived beliefs of their ability to successfully complete a task (Almeida et al., 2021; Barron & Hulleman, 2015; Wigfield, 1994). Although the flipped instruction method is supportive of multiple higher-order-thinking learning strategies (Akçayır & Akçayır, 2018), the same could lead to an increased psychological state of the student and impede their academic success rather than improve it (Barron & Hulleman, 2015; Flake, Barron, Hulleman, McCoach, & Welsh, 2015). Consequently, learning achievement is dependent on the extent to which students can relate novel situations to prior learning (Dawson, Meadows, & Haffie, 2010), motivation, and the effort committed to learning content.

Several studies revealed that students have positive attitudes toward the flipped instruction approach (Clark, 2015; Graziano & Hall, 2017; Pattanaphanchai, 2019). Nonetheless, their positive perceptions about the model did not necessarily translate into learning gains. This is an indication that other factors may be influencing the achievement of desired outcomes. Thus, this study examined the role of student's motivation beliefs and commitment to learning achievement regardless of challenges experienced. While several studies have used qualitative research approaches to explore the efficacy of flipped instruction, the current study utilized quantitative method to investigate the relationship between components of undergraduates' motivation beliefs and efforts to learn content in a flipped college precalculus course.

Theoretical Framework: Expectancy-Value-Cost Theory

Achievement theorists postulated that individual's actions and inactions, including their decisions and determinations to succeed, are driven by subjective beliefs about their abilities, expectancies for success, interest, and perceived value of the desired outcome, and preferences (Wigfield & Eccles, 2000). The Expectancy-Value-Cost (EVC) theory of motivation (Barron & Hulleman, 2015) is an extension of the Expectancy-Value (EV) theory, which has been reviewed extensively by theorists in the achievement traditions (Barron & Hulleman, 2015; Getty et al., 2017). However, the cost component of the EVC model is a relatively novel field of study (Barron & Hulleman, 2015; Getty et al., 2017). The component exists because of the limitation of the EV theory at explaining why an individual with excellent expectancy and a strong perception of the utility of certain tasks, would fall short of achievement. It was inferred that other factors like the difficulty of the task, discouragement, embarrassment, related and unrelated efforts required for success were not accounted for by the EV theory (Barron & Hulleman, 2015; Flake, Barron, Hulleman, McCoach, & Welsh, 2015).

Cost, deemed as the forgotten component of the EV theory (Flake et al., 2015), was originally introduced as a subcomponent of the value component (Wigfield, 1994; Wigfield & Eccles, 2000) and was hypothesized to moderate the effect of the value component on an individual's motivation beliefs and achievement goals (Barron &Hulleman, 2015; Flake et al., 2015; Wigfield & Eccles, 2000). Cost, a barrier-related construct, is multi-dimensional and focuses on what an individual must sacrifice to achieve success (Wigfield, 1994). The construct elucidates negative appraisal of time and effort related to the task, outside of task effort, loss of valued alternatives, and emotional cost of success (Flake et al., 2015). Thus, the cost component is a negative predictor of success and mitigates the effect of both expectancy and value.

Eccles and colleagues (1983) (cited in Wigfield & Eccles, 2000) postulate the expectancy component as a two-dimension construct comprised of ability beliefs and expectancies for success (Wigfield, 1994; Wigfield & Eccles, 2000). Ability beliefs pertain to what the student perceives he or she can do now, while expectancy beliefs are apropos for the future. In this study, we focus on students' perception of their present ability to succeed at learning precalculus in a flipped classroom. Because passing the course and retention in college are utmost for first-year students who already viewed the remedial course as a setback (Kane et al., 2020).

The value component focuses on students' beliefs about the importance and interest of the task and provides answers to the questions: "Do I want to do this task?" and "Why am I doing this task?" (Barron &Hulleman, 2015). The component is described as having four subcomponents including intrinsic value, (student enjoys watching lecture videos because it is fun and interesting), utility value (actively learning precalculus because it is required for desired major), attainment value (passing the course allow students to move to Calculus I and be on track for timely graduation) and cost a negative predictor of the overall value of tasks. The same has now become the third component of the EVC theory (Barron & Hulleman, 2015; Getty et al., 2017). In this study, the EVC theory as defined by Barron and Hulleman (2015) was used to explain and interpret the data collected about the effect of students' expectancies, perceived values, and cost implications, as well as preference for the flipped instructional approach on their academic success.

Methods

The current study used observational methodology to examine students' expectancy beliefs, perceived values, and the cost of learning the content in a flipped college precalculus course. The cross-sectional study utilized data from self-reported surveys.

Participants and Data Collection

Data was collected in a flipped college Precalculus course in a four-year college in the Midwestern region of the United States during the fall 2019 semester using motivation scale. Thirty-two students (7 females and 25 males) representing 71% of the students taught that semester using the flipped method participated in the study. The participants self-registered for the course, although they are unaware of the teacher's planned teaching method. Most of the participants (82%) took Precalculus in High School and 49% reported taking Calculus. All participants indicated interest in pursuing a major in STEM-related disciplines.

Research Instrument

Students responded to two online surveys (pre-course and post-course) during the first and last week of the semester, respectively. The main component of the surveys consisted of eight motivation-related items adapted from Expectancy-Value-Cost Scale (EVC-S) (Getty et al., 2017; Kosovich, Hulleman, Barron, & Getty, 2015), and four researchers-created items to measure students' perceived liking for flipped instruction. The 12 ordinal items were theorized to measure motivation beliefs and likability for the flipped instructional strategy.

EVC-S, which was developed, field-tested, and validated is novel and had reliable psychometric properties (Getty et al., 2017; Kosovich et al., 2015). For example, a rapid version of the scale consisted of three subscales with three items measuring *Expectancy* ($\alpha = \omega = 0.82$), three measuring *Value* ($\alpha = \omega = 0.84$, and four items to measure the *cost* ($\alpha = \omega = 0.83$) subscale (Getty et al., 2017). To keep our instrument short, we dropped two of the original items, one from expectancy, and reworded a statement to combine two items from the cost subscale into one while still preserving the intended meaning of both constructs. The adapted research instrument was validated to ensure its consistency with the original EVC-scale and to determine the reliability and validity of the additional items. The following constructs were addressed.

Expectancy. This subscale focused on self-confidence, that is, the importance of believing in one's ability to succeed on a task (Barron & Hulleman, 2015; Kosovich et al., 2015; Wigfield & Eccles, 2000). Expectancy included two items (*I am sure, I can learn the material for Precalculus;* and *I am confident that I can do well in Precalculus this semester*) with a reliability coefficient of $\alpha = 0.89$.

Value. The value subscale addressed the reasons why the students would want to engage in certain actions or achieve academic success. The value subscale contained three items focused on understanding students' perceptions of the importance and personal interest for learning Precalculus, (e.g., "*Precalculus is an interesting course*," "*Precalculus is useful to me*," etc.). The value subscale had a reliability of 0.77.

Cost. The cost subscale addressed "what is invested, required, or forgone engaging in a task" (Flake et al., 2017; p.235). The construct was hypothesized to influence motivation through effort and time related and unrelated to the task, the loss of valued alternative, and negative effect (Barron & Hulleman, 2015; Flake et al., 2017; Getty et al., 2017). There were three cost items including, *I have to sacrifice too many things to do well in Precalculus, etc.* with a reliability of $\alpha = 0.88$.

Preference for Flipped Method Subscale. The remaining four were researcher-created items to measure students' preference for the flipped method. This was based on a study by

Sahin, Cavlazoglu, & Zeytuncu (2015) with calculus students. The authors investigated students' views and experience in flipped courses, and the impacts on their academic achievement, when compared with non-flipped classes. Sahin and colleagues found that students preferred watching class videos to reading their textbook and preferred courses with lecture videos. On this premise, we developed the following four questions to verify the validity of their claims. Q9-*I am sure I will learn more in a flipped instruction*; Q10-*I prefer a course with video lessons to a course without video lessons*; Q11-*I like the flexibility of choosing how I learn the course content in this course*; and Q12- *the syllabus helped me understand the goals and expectations of this course*. The last question was later dropped due to low factor loading. The subscale had a reliability of 0.92.

All responses were rated on a 5-point scale where 1 = strongly agree and 5 = strongly disagree. We recoded positively worded statements. Overall, there were 57 (pre-course = 29 and post-course = 28) usable survey data from both surveys before multiple imputations.

Data Analysis

Based on the EVC theoretical perspective as well as a review of relevant studies, the study utilized structural equation modeling (SEM) (Schumacker & Lomax, 2016) to examine the relationship between components of students' motivation and to determine predictive abilities of effort, flipped instruction method, and motivation beliefs on academic success in Precalculus.

A two-steps full-SEM (Anderson & Gerbing, 1988) in Lisrel 11 software (Jöreskog & Sörbom, 2021) is used to analyze the predictive abilities of motivation beliefs and preference for flipped instruction on academic success in a college Precalculus course. The statistical technique which combines both measurement and path analyses was selected because of its ability to explain the causal relationships between and among the latent and observed variables (Schumacker & Lomax, 2016). The measurement part was used to verify the existence of the study's latent variables (*expectancy, value, cost, preference, effort, and success*) and to determine whether the latent variables can be explained by the observed variables (using responses from the self-reported questionnaires). The structural model postulates the relationships between the variables, including the strength and direction of the causal relationships as follows. *Effort* was measured by the number of time students logged in to watch lecture videos (*Watch-Video*) is theorized to explained academic success (Dweck & Yeager, 2019; Flake et al., 2017). We hypothesized that the observed variables would measure *expectancy, value, cost*, and *preference* for the flipped method. Then *expectancy, value,* and *cost* were conjectured to measure motivation beliefs (Getty et al., 2017), and preference measured the use of the flipped method (Sahin et al., 2015). Both second-order factors would then have positive effects on *academic success* (measured by *grades*) and moderated by *effort* as shown in figure 1, the hypothetical model.

Considering our sample size, we generated values in LISREL for the first order latent variables and used them in higher-order analysis, thereby reducing the number of estimated parameters.

Several fit indices were used to assess the adequacy of the proposed model based on recommendations of Schumacker and Lomax (2016). The model was evaluated using the following indices: (a) Comparative Fit Index (CFI), (b) Goodness of Fit Index (GFI), (c) Normed Fit Index (NFI), (d) the Root Mean Square of Approximation (RMSEA), and (e) Chi-square test ($\chi 2$). The recommended cut-off that indicates a good fit for CFI, GFI, and NFI is 0.90 or higher. A non-significant chi-square is desired indicating a close fit and RMSEA values less 0.08 represents a good fit (Hu & Bentler, 1999; Schumacker & Lomax, 2016).

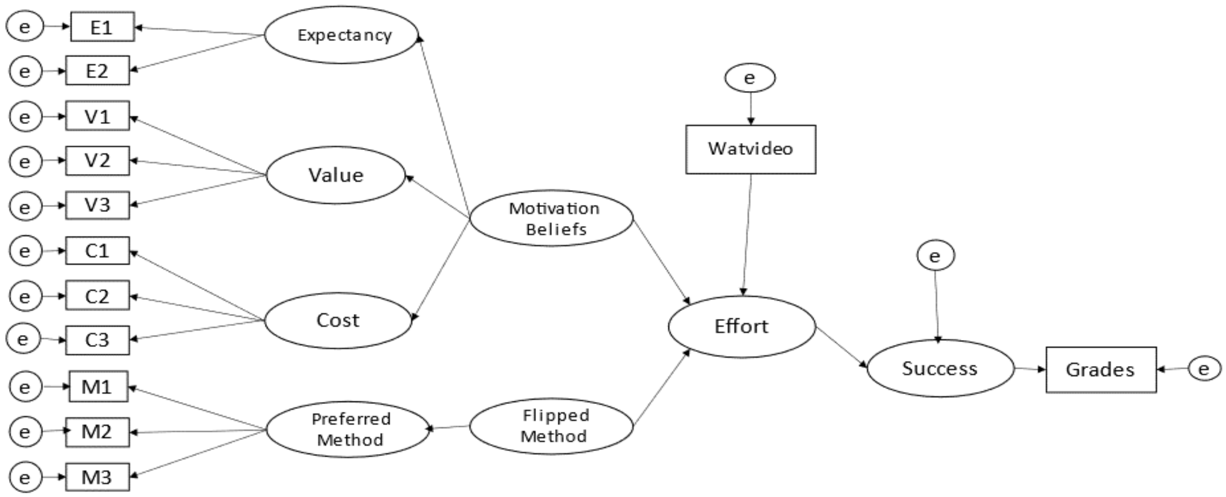


Figure 1: Research **Model**

Results

This study examined whether college students who report high expectancies and values for success would also report a higher preference for the flipped method. Using confirmatory factor analysis (CFA), we also tested whether there is a direct effect of students' motivation beliefs and flipped instruction on success moderated by students' efforts (measured by the number of logins to watch prerecorded lecture videos). Table 1 presents the direct, indirect, and total effects between motivation beliefs, preference for the flipped instructional method and students' overall academic success in a flipped college precalculus course. Figure 2 specifically shows direct effect of students' motivation beliefs (expectancy, subjective values, and cost beliefs) and flipped instruction (high preference for the method) on success (grades) moderated by students' efforts (measured by the number of logins to watch prerecorded lecture videos).

Table 1: Direct, indirect, and total effects of variables on Academic Success

Variable	Direct effect	Indirect effect	Total effect
Motivation Beliefs	0.126	-0.022	0.105
Method	**0.085**	-0.059	0.026
Effort	**-0.220**		-0.220

First, it was revealed that students who reported high *expectancy* for success also reported high *value* and are favorably disposed to learning in a flipped instruction (see Table 1). However, *effort* had a negative influence on success in the course, thereby moderating the effect of motivation and method on achievement as theorized.

Next, to determine the strengths and directions of each factor in relation to learning achievement, we further analyzed how well the hypothesized model fit our data. Figure 2 shows that the collected data supported a four-factors solution, which validates and extends the EVC theory to include preference for specific cause, in this case, flipped method.

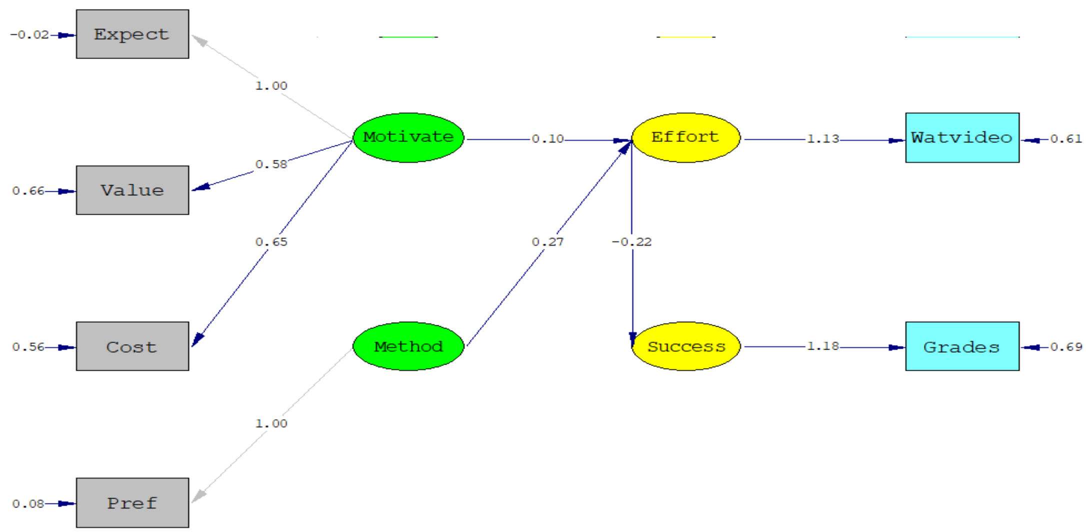


Figure 2: Analytical Model

The correlations between the factors ranges from moderate to strong positive relationship except the connection between *Grades* and *Watch-Videos* (r = -0.017, p >0.05). There is a direct effect of students' motivation beliefs (expectancy, subjective values, and cost beliefs) and flipped instruction (high preference for the method) on success (grades) moderated by students' efforts (measured by the number of logins to watch prerecorded lecture videos). As shown in Figure 2, the measurement model fits the data well showing that all indicator variables each explained only one latent variable as hypothesized. All the fit indices except RMSEA were within acceptable ranges ($\chi^2(3) = 5.29, p > 0.05$, RMSEA = 0.11, NFI = 0.94, and CFI =0.97, GFI = 0.97). Both motivation beliefs and flipped method contributed positively to success, although the significance of the paths could not be determined. The fit indices remained unchanged with the addition of the structural paths. The findings show that a combination of preferred method and motivation would lead to academic success by jointly explaining about 11% of the variations in success and 33% of students' effort. Conversely, students' commitment to learning (*Effort*) contributed negatively to success. Possible explanations may include poor learning retentions, or students may be watching the lecture just to check boxes and could become overconfident since the content is somewhat familiar. The significant error variances of *Watvideo* and *Grades* are indications that the latent variables were influenced by other confounding variables than the hypothesized indicators.

Discussion and Conclusion

We found that students who reported high *expectancy* for success also reported high *value* and are favorably disposed to learning in a flipped instruction. This finding can be supported by prior research studies (e.g., Kosovich et al., 2015; Getty et al., 2017). Yet, different from Barron and Hulleman (2015), we found positive, and significant relationships between cost and both expectancy and value. Furthermore, students' preference toward flipped method was an excellent indicator for the effective implementation of flipped instruction. We noted that a direct relationship existed between motivation and success and between flipped method and success. This may not be surprising, because motivation plays a significant role in human's achievement of desirable goals (Dweck & Yeager, 2019). Several studies found flipped instruction useful for promoting deep learning (Akçayır & Akçayır, 2018; Araujo et al., 2017; Graziano & Hall, 2017;

Huang & Hong, 2016; Zainuddin, 2018); but none considered the effect of personal preference for the model. This study shows that a combination of preferred method and motivation jointly explained about 11% of the variations in success and 33% of students' effort. Conversely, students' commitment to learn (*Effort*) contributed negatively to success needing further research.

Considering students' motivation and preference when selecting teaching methods could improve learning and achievement, the findings of this study provide teachers with tools to educate, inspire, and involve students in how they learn (e.g., freedom to choose based on personal preference). This study also addresses research gaps on motivation and effectiveness of flipped instruction. A few limitations should be noted, such as the small sample size, lack of diversity of instructors, subjects, and learner groups. More research and replications are needed to substantiate the study's findings.

References

Akçayır, G., & Akçayır, M. (2018). The flipped classroom: A review of its advantages and challenges. *Computers & Education*, *126*, 334 -345. https://doi.org/10.1016/j.compedu.2018.07.021

Almeida, M. E. B. D., Queiruga-Dios, A., & Cáceres, M. J. (2021). Differential and integral calculus in first-year engineering students: A diagnosis to understand the failure. Multidisciplinary digital publishing institute *Mathematics*, *9*(1), 61, 1 -18. Retrieved from https://doi.org/10.3390/math9010061

Anderson, J. C., & Gerbing, D. W. (1988). Structural equation modeling in practice: A review and recommended two-step approach. *Psychological Bulletin*, *103*(3), 411–423. Doi: 0033-2909/88/$00.750

Barron, K. E., & Hulleman, C. S. (2015). Expectancy-value-cost model of motivation. In J. S. Eccles & K. Salmelo-Aro (Eds.), *International Encyclopedia of Social and Behavioral Sciences*, 2nd Edition: Motivational Psychology. Elsvier. DOI: 10.1016/B978-0-08-097086-8.26099-6.

Blumberg, P. (2008). *Developing learner-centered teaching: A practical guide for faculty*. San Francisco, CA: Jossey-Bass.

Bressoud, D. (2015). Insights from the MAA national study of college calculus. *Mathematics Teacher*, *109*(3), 178-185. https://doi.org/10.5951/mathteacher.109.3.0178

Bressoud, D., (2021, July 1). Thoughts on advanced placement precalculus. *Launching, Teaching, & Learning*. *Mathematical Association of American*. Retrieved from https://www.mathvalues.org/masterblog/thoughts-on-advanced-placement-precalculus

Bressoud, D., Camp, D., & Teague, D. (2017). Background to the MAA/NCTM statement on calculus. *The Role of Calculus in the Transition from High School to College Mathematics*, 77-81. Retrieved from https://www.maa.org/sites/

Chen, Y., Wang, Y., Kinshuk, & Chen, N. S. (2014). Is FLIP enough? Or should we use the FLIPPED model instead?. *Computers & Education*, *79*, 16-27. https://doi.org/10.1016/j.compedu.2014.07.004

Clark, K. R. (2015). The effects of the flipped model of instruction on student engagement and performance in the secondary mathematics classroom. *Journal of Educators Online*, *12*(1), 91-115. Retrieved from https://files.eric.ed.gov/fulltext/EJ1051042.pdf

Dawson, D. L., Meadows, K., N., & Haffie, T. (2010). The effect of performance feedback on student help-seeking and learning strategy use: Do clickers make a difference? *The Canadian Journal for the Scholarship of Teaching and Learning, 1*(1), 1–20. doi:10.5206/cjsotl-rcacea.2010.1.6

de Araujo, Z., Otten, S., & Birisci, S. (2017). Mathematics teachers' motivations for, conceptions of, and experiences with flipped instruction. *Teaching and Teacher Education*, *62*, 60-70. https://doi.org/10.1016/j.tate.2016.11.006

Dweck, C. S., & Yeager, D. S. (2019). Mindsets: A view from two eras. *Perspectives on Psychological Science*, *14*(3), 481-496. doi: 10.1177/1745691618804166

Flake, J. K., Barron, K. E., Hulleman, C., McCoach, B. D., & Welsh, M. E. (2015). Measuring cost: The forgotten component of expectancy-value theory. *Contemporary Educational Psychology*, *41*, 232-244. https://doi.org/10.1016/j.cedpsych.2015.03.002

Getty, S.G., Hulleman, C. S., Barron, K.E., Ruzek, E. R., Flake, J. K., & Foley, K. (2017). *Using the expectancy-value-cost theory of motivation to understand student achievement and future interest in stem classrooms*

[White paper]. Retrieved from the Motivate Lab, Charlottesville: Virginia. https://www.researchgate.net/publication/326354387]

Gilboy, M. B., Heinerichs, S., & Pazzaglia, G. (2015). Enhancing student engagement using the flipped classroom. *Journal of Nutrition Education and Behavior*, *47*(1), 109-114. https://doi.org/10.1016/j.jneb.2014.08.008

Graziano, K. J., & Hall, J. D. (2017). Flipped instruction with English language learners at a newcomer high school. *Journal of Online Learning Research*, *3*(2), 175-196. Retrieved from https://www.learntechlib.org/p/172850/paper_172850.pdf

Hu, L. T., & Bentler, P. M. (1999). Cutoff criteria for fit indexes in covariance structure analysis: Conventional criteria versus new alternatives. *Structural equation modeling: A Multidisciplinary Journal*, *6*(1), 1-55.

Huang, Y. N., & Hong, Z. R. (2016). The effects of a flipped English classroom intervention on students' information and communication technology and English reading comprehension. *Educational Technology Research and Development*, *64*(2), 175-193. DOI 10.1007/s11423-015-9412-7

Jöreskog, K.G., & Sörbom, D. (2021). LISREL 11: Structural equation modeling. Chapel Hill, NC: Scientific Software International, Inc.

Kane, T., Boatman, A., Kozakowski, W., Bennett, C., Hitch, R., & Weisenfeld, D. (2020). Is college remediation a barrier or a boost? I evidence from Tennessee. *Education Next*, *20*(2), 64 - 71. https://link.gale.com/apps/doc/A618692707/AONE?u=platteville&sid=AONE&xid=fca12571

Kosovich, J. J., Hulleman, C. S., Barron, K. E., & Getty, S. (2015). A practical measure of student motivation: Establishing validity evidence for the expectancy-value-cost scale in middle school. *The Journal of Early Adolescence*, *35*(5-6), 790-816. DOI: 10.1177/0272431614556890.

Love, B., Hodge, A., Grandgenett, N., & Swift, A. W. (2014). Student learning and perceptions in a flipped linear algebra course. *International Journal of Mathematical Education in Science and Technology*, *45*(3), 317-324. DOI: 10.1080/0020739X.2013.822582

Love, B., Hodge, A., Corritore, C., & Ernst, D. C. (2015). Inquiry-based learning and the flipped classroom model. *Primus*, *25*(8), 745-762. DOI: 10.1080/10511970.2015.1046005

Nielsen, P. L., Bean, N. W., & Larsen, R. A. A. (2018). the impact of a flipped classroom model of learning on a large undergraduate statistics class. *Statistics Education Research Journal, 17(1), 121-140, http://www.stat.auckland.ac.nz/serj*

Pattanaphanchai, J. (2019). An investigation of students' learning achievement and perception using flipped classroom in an introductory programming course: A Case study of Thailand higher education. *Journal of University Teaching & Learning Practice*, *16*(5), 1-19. Retrieved from https://ro.uow.edu.au/jutlp/vol16/iss5/4

Rasmussen, C., Apkarian, N., Hagman, J. E., Johnson, E., Larsen, S., & Bressoud, D. (2019). Brief report: characteristics of precalculus through calculus 2 programs: insights from a national census survey. *Journal for Research in Mathematics Education*, *50*(1), 98-112. https://www.jstor.org/stable/10.5951/jresematheduc.50.1.0098

Sadler, P. M., & Sonnert, G. (2016). Factors influencing success in introductory college calculus. *The role of calculus in the transition from high school to college mathematics*. Retrieved from https://www.maa.org/sites/default/files/RoleOfCalc_rev_0.pdf#page=59

Sahin, A., Cavlazoglu, B., & Zeytuncu, Y. E. (2015). Flipping a college calculus course: A case study. *Journal of Educational Technology & Society*, *18*(3), 142. https://www.jstor.org/stable/10.2307/jeductechsoci.18.3.142

Schumacker, R. E., & Lomax, R. G. (2016). *A beginner's guide to structural equation modeling*. New York, NY: Routledge.

van de Sande, C. & Reiser, M. (2018). The effect of summer break on engineering student success in calculus. *International Journal of Research in Education and Science (IJRES), 4*(2), 349-357. DOI:10.21890/ijres.409264

Weimer, M. (2012). *Learner-centered teaching: Five key changes to practice*. San Francisco, CA: Jossey-Bass.

Wigfield, A. (1994). Expectancy-value theory of achievement motivation: A developmental perspective. *Educational Psychology Review*, *6*(1), 49-78. http://www.jstor.com/stable/23359359

Wigfield, A., & Eccles, J. S. (2000). Expectancy–value theory of achievement motivation. *Contemporary Educational Psychology*, *25*(1), 68–81. doi: 10.1006/ceps.1999.1015

Zainuddin, Z. (2018). Students' learning performance and perceived motivation in gamified flipped-class instruction. *Computers & Education*, *126*, 75-88. https://doi.org/10.1016/j.compedu.2018.07.003

BELIEFS OF TEACHING ASSISTANTS IN AN INTRODUCTORY PROOF COURSE

T. Royce Olarte	Micah Swartz	Sarah A. Roberts
University of California Santa Barbara	Texas State University	University of California Santa Barbara
tolarte@ucsb.edu	mas762@txstate.edu	sroberts@education.ucsb.edu

This study examined the beliefs that teaching assistants (TAs) of an introductory proof course had about their students, about how their students learned, about their teaching, and about mathematics. We sought to examine TAs' beliefs and better understand their approaches to teaching and instruction. Five mathematics TAs participated in the study and we qualitatively analyzed their responses to semi-structured interviews. We found that TAs understood that their students learned in various ways and they believed they should be able to support students' multiple ways of engaging with proofs. They also understood that introductory proof courses necessitate higher levels of cognitive demand and that many students struggle in these courses. Our preliminary findings can inform future efforts that more critically examine the beliefs of mathematics teaching assistants.

Keywords: Beliefs, Values, Teaching Assistants, Higher Education, Proof Courses

Introduction

Mathematics teachers' beliefs about their students, how their students learn, the mathematics content they are teaching, and their roles in engaging students in learning experiences are fundamental to shaping their instructional practices (e.g., Pajares, 1992; Wilkins, 2008; Goldin et al., 2009). However, there is still limited work that examines the beliefs of the individuals involved in undergraduate mathematics education - particularly, those of mathematics graduate students who serve as teaching assistants (TAs). What graduate students believe about students, learning, and the teaching of mathematics not only impacts their current roles as TAs, but are foundational to many continuing on to be mathematics faculty. In this exploratory study, we examined the beliefs of teaching assistants of an introductory proof course called Math A. Introductory proof courses mark a critical junction in the mathematical trajectories of many students. Proof-construction poses significant challenges for many students (Weber, 2001), because the problem-solving experiences and sensemaking demands associated with proof-construction are unlike those of courses such as calculus and linear algebra. Thus, TAs play important roles in supporting student learning and understanding as they transition to upper division mathematics courses that heavily rely on proof-construction competencies. It is important that we better understand what mathematics TAs believe about students, learning, and teaching at this important junction as it can inform how higher education institutions and mathematics departments can better support graduate students' pedagogies. The research question that guided this study was: How did mathematics teaching assistants reflect on their beliefs about students, learning, teaching, and mathematics in an introductory proof course?

Literature Review and Framing

Research on teachers' beliefs has evolved over the past decades. From behaviorism to cognitive science, research has investigated teachers' behaviors and the ways in which teachers think, perceive, and analyze their practices (Calderhead, 1996). While their subject matter knowledge, specialized content knowledge and pedagogical content knowledge are important aspects of teaching (Ball et al., 2008), what teachers believe about mathematics, teaching, and learning may have a greater influence on their instructional practices (Bryan, 2012). Research has illuminated that the knowledge teachers view as important and appropriate is impacted and molded by their beliefs (Pajares, 1992). Additionally, prior scholars have suggested that exploring beliefs can be a tool to understand how teachers make choices (Ernest, 1989) and how they filter new knowledge (Fives & Buehl, 2010). We take up beliefs to be the conceptions, convictions, and values that can shape and impact how one teaches and learns (Thompson, 1984; Thompson, 1992).

In this study, we adapted the work of Speer (2001) that connected teaching assistants' beliefs to their teaching practices in reform-oriented calculus courses. Speer (2001) described teaching assistants' *belief profiles* to encompass what the teaching assistants believed about their students (e.g., "Students should be independent."), what they believed about teaching (e.g., "Teachers should be guides who ask students questions."), what they believed about learning (e.g., "Learning is making something your own."), and what they believed about mathematics (e.g., "Calculus entails learning to think and acquiring skills but does not showcase the beauty of mathematics."). We situate this framework in the context of introductory proof courses to better understand what TAs believe about students and their pedagogical practices at this important mathematical junction.

Method

This study was conducted within a larger project at a Minority-Serving Institution in California about transfer students' experiences in a set of mathematics courses designed to develop proof-construction competencies and support their transition to a four-year university. Purposeful sampling (Creswell, 2013) was used to select five graduate students who were teaching assistants for an introductory proof course, Math A, that was part of this set of courses. The participants were 2nd- and 3rd- year doctoral students in the mathematics department and all had prior experiences as teaching assistants. We use the pseudonyms Federico, Nestor, Wyatt, Kaitlyn, and Lisa to refer to the five TAs interviewed for this research project. We qualitatively analyzed their responses to semi-structured interviews (Rubin & Rubin, 2011). We used the following a priori codes (Miles et al., 2020) derived from the work of Speer (2001) to categorize their beliefs: (1) beliefs about students, (2) beliefs about how students learn, (3) beliefs about how they should teach and engage students in mathematics, and (4) beliefs about mathematics. This last code encompassed their beliefs about mathematics in general and their beliefs related to introductory-proof-courses. The research team wrote memos and discussed important themes in the TAs reflections.

Findings

We present preliminary findings that highlight the four dimensions of the TAs' belief profiles. Broadly, TAs believed that students learn in various ways and that collaborative work in safe learning environments supports student learning. We observed a range of TAs' beliefs from positioning students as competent and capable to having some deficit perspectives about their engagement. They also expressed certain beliefs unique to the context of and circumstances around introductory-proof courses (e.g., it is a course mainly for mathematics majors, it is the gatekeeper to upper division courses, etc.).

TAs' Beliefs About Students

The TAs revealed a multitude of beliefs about their students broadly centering around students' capabilities and their backgrounds. For example, Wyatt noticed the diversity of students in Math A, noting: "I can tell you from my experience with teaching Math A, they're [students of color] equally capable, equally hard working." Wyatt held the belief that all students are equally capable of succeeding in Math A. This view of students contrasted Nestor's view that "if you have someone from a different culture, you know, like especially Chinese students, if you pair them up with American students, maybe it's going to be hard for them to engage." Nestor believed that students from different cultures may struggle to engage and work collaboratively with students in the dominant population, which reflects a deficit perspective of the minority students. We acknowledge that TAs may not be regularly engaging in critical reflections of their beliefs about students, but it is important to explore these beliefs, identify deficit perspectives, and then work to reframe these beliefs.

TAs' Beliefs About How Students Learn

TAs' beliefs about how students learn were related to their beliefs about pedagogical practices and departmental logistics. Firstly, TAs' expressed the belief that students are unique learners and that they engage in mathematics differently. Regarding the content students are presented with, Lisa mentioned, "student[s] won't get a deep understanding unless there's like this, you know, do the easy examples, try to do a hard interesting example." This illustrates Lisa's belief about the importance of giving students problems with varying levels of cognitive demand. She also described the value of contextualizing problems so that students can better engage with the material they are learning. Additionally, Wyatt expressed his belief about student learning when he said, "I think that the undergraduate students are better served when the class sizes are smaller." He explained that students work best in small class sizes as well as collaborative work environments. Wyatt highlighted his belief that group work was crucial in helping students learn and understand the material. We observed numerous instances of TAs indicating that students learn best in community and in safe environments, as well as the belief that students benefit from contextualized, cognitively demanding tasks.

TAs' Beliefs About Teaching

Broadly, the participants believed that their pedagogical practices should be responsive to the various ways their students learned. Wyatt mentioned, "As a TA, you have to put students first…You have to accommodate multiple ways of thinking." Here, we not only observe a student-centered approach to pedagogy but also an attention to the idea that there is not one set way to learn mathematics. We also found that TAs believed that qualities such as flexibility, personability, and approachability should guide their interactions with students. Creating a comfortable learning environment was especially important for the participants and many described efforts to create this comfortable learning environment during their sections. Kaitlyn said, "You don't want anybody to feel like they, you know, can't ask something or stupid questions or whatever. You want them to feel comfortable." Furthermore, Kaitlyn and Nestor identified that using judgmental and derogatory language to respond to students' questions is one way that can disrupt a safe and comfortable learning environment. Overall, participants' beliefs about teaching centered on attending to students' different ways of learning mathematics and ensuring that students feel comfortable and safe to take intellectual risks.

TAs' Beliefs About Mathematics

The TAs believed that the content of introductory proof courses necessitate higher levels of cognitive demand that may be unfamiliar to students and that these courses not only delineate lower division and upper division courses, but also distinguish mathematics majors from non-mathematics majors. Kaitlyn said, "There's also an intensity that comes with Math A that maybe make students feel a little bit more, I don't know, fearful about doing perfect in the class or whatever." She illustrated the frequently held belief amongst the participants that Math A is extremely challenging and intimidating for students. Federico explicitly understood introductory proof courses to be junctions that mark mathematics majors continuing to upper division courses and he mentioned enjoying opportunities to interact and engage with these students because he can provide them with advice relevant to their academic trajectories. Lastly, regarding overall beliefs about mathematics, the participants understood mathematics to be a creative discipline. Wyatt said, “If you want people to enjoy, you gotta have them recognize it's extremely creative and there are many, many ways to get to the destination.” Wyatt contended the commonly held notion of mathematics as a rigid and overly procedural discipline and instead, upholds a belief that mathematics is creative and more open-ended.

Discussion and Conclusion

Consistent with the literature (e.g., Pajares, 1992), we found connections between TAs’ beliefs about students, how students learn, and their understanding of pedagogy. TAs understood that students learn in distinct ways and that they should be responsive to these differences. Furthermore, they believed that fostering a safe learning community affords intellectual risk taking. They recognized that introductory proof courses mark a distinction in the cognitive demands of the content as well as in the demographics of students in subsequent courses. It is imperative to highlight that some TAs held deficit views of students and these may have subconsciously shaped their instructional practices. With teacher beliefs informing pedagogical practices (e.g., Ernest, 1989; Speer, 2001), we argue that TAs of introductory proof courses need to be more reflective of their beliefs as they support students at this significant mathematical junction. Limitations of this study include the limited number of participants and a more general conception of beliefs. Future research should more closely examine TAs’ beliefs through a critical lens - focusing on dimensions of their beliefs that intersect with important critical, transformative frameworks (e.g., critical race theories, antiracist education, queer theories, gender theories, etc.). Our findings shed light on the diversity of TAs’ beliefs and their approaches to pedagogy, and this can inform institutions about more equity-centered and social justice-oriented professional development opportunities for graduate students.

Acknowledgements

This project was funded by the UCSB Academic Senate Council on Research and the Instructional Resources Faculty Research Grant.

References

Bryan, L. A. (2012). Research on science teacher beliefs. In the *Second international handbook of science education* (pp. 477-495). Springer, Dordrecht.

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of Educational Psychology*. NewYork: Macmillan.

Creswell, J. W. (2013). *Qualitative inquiry and research design: choosing among five approaches*. Sage Publications.

Ernest, P. (1989). The knowledge, beliefs and attitudes of the mathematics teacher: A model. *Journal of education for teaching*, *15*(1), 13-33. https://doi.org/10.1080/0260747890150102.

Fives, H., & Buehl, M. M. (2010). Teachers’ articulation of beliefs about teaching knowledge: Conceptualizing a belief framework. In *Personal Epistemology in the Classroom* (pp. 470-515). Cambridge University Press.

Goldin, G., Rösken, B., & Törner, G. (2009). Beliefs–no longer a hidden variable in mathematical teaching and learning processes. In *Beliefs and attitudes in mathematics education* (pp. 1-18). Brill Sense.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of teacher education*, *59*(5), 389-407. https://doi.org/10.1177%2F0022487108324554.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2020). *Qualitative data analysis: A methods sourcebook.* SAGE.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research*, 62(3), 307-332. https://doi.org/10.3102%2F00346543062003307.
Rubin, H. J., & Rubin, I. S. (2011). *Qualitative interviewing: The art of hearing data*. Sage.
Speer, N. M. (2001). *Connecting beliefs and teaching practices: A study of teaching assistants in reform-oriented calculus courses*. University of California, Berkeley.
Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics 15*,105- 127. https://doi.org/10.1007/BF00305892.
Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*. New York: MacMillan.
Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational studies in mathematics*, *48*(1), 101-119. https://doi.org/10.1023/A:1015535614355.
Wilkins, J. L. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. *Journal of Mathematics Teacher Education*, *11*(2), 139-164. https://doi.org/10.1007/s10857-007-9068-2

GRADUATE TEACHING ASSISTANTS’ ENGAGEMENT OF STUDENTS’ ERRORS

Jungeun Park
University of Delaware
jungeun@udel.edu

Graduate Teaching Assistants (TAs) play an important role in undergraduate students’ learning of mathematics, especially Calculus, and spend significant time evaluating students’ work. This study examined TAs’ engagement of students’ errors by following the process of their identifying students’ errors, planning to address them, and implementing the plans in teaching. Our results showed that (a) TAs only identified a small portion of errors that students made, (b) errors TAs identified often did not include the ones made most frequently by their students, (c) many errors that TAs identify were procedural in nature and addressed with plans to provide the “correct” procedure, (d) the levels of specific details of students’ errors dropped from identifying, to planning, to teaching, and (e) even with drops, in many cases, the levels of details of students’ errors involved in TAs’ identifying was the same as the levels of their planning and teaching.

Keywords: Calculus, Undergraduate Education, Professional Development

Graduate Teaching Assistants (TAs) play crucial roles in teaching and learning of entry-level undergraduate mathematics including Calculus (American Association for the Advancement of Science (AAAS), 2019). In many U.S. universities, TAs directly impact students’ learning of Calculus (Ellis, 2014) through their class, office hours, and tutoring centers. However, our knowledge about TAs and their teaching is still limited (AAAS, 2019). Given that TAs usually spent a large amount of time grading students’ written work, we investigated if and how they engage with students’ errors in various phases of teaching. We followed the process of TAs’ identifying of students’ errors, planning how to address them, and implementing the plans in class to address the following research questions: (a) *To what extent do TAs identify students’ errors while grading students’ work on problems?* (b) *To what extent are the specifics of students’ errors used in TAs’ plans to address them and their teaching of the problems in class?*

In this study we analyzed the levels of specific details of students’ errors that TAs engaged with in their identifying, planning, and teaching. Our analysis shows the types of errors that TAs identified from students’ work, and how they used them in planning and classroom teaching.

Theoretical Background

Using students’ errors in teaching has been studied mainly at K-12 levels, but also at the college level. Those studies examined the attributes and the quality of teachers’ response to errors (Bray, 2011; Santagana & Bray, 2016); various ways of engaging with students’ errors (e.g., Ingram et al., 2015; Tulis, 2013); impact of error-handling PD on teachers’ view of using students’ errors (Brodie, 2014); teachers’ in-the-moment error handling (e.g., Ingram et al., 2015; Schleppenbach et al., 2007). In this study, we view students’ errors as sources of difficulty students experience in their learning of mathematics, that are systematic and persistent, (Brodie, 2014) which may not be quickly corrected (Ingram et al., 2015). Addressing errors in teaching is important because errors can offer students opportunities to view questions from different angles, allow teachers to reflect on their students’ thinking, and emphasize mathematical aspects related to the errors (Santagana & Bray, 2016; Van de Pol et al., 2011). This study represents beginning efforts to understand TAs’ engagement with student error throughout several phases of teaching by exposing TAs to students’ difficulties about the content *before* they teach students. Several studies suggested, in general, three steps that teachers go through while dealing with students' errors: perceiving errors (identifying), looking for reasons for errors (interpreting), and planning instructional approaches to help students "overcome the misconception" regarding errors (planning) (Heinrichs & Kaiser, 2008, p. 84). Those studies emphasized the use of specifics of students' errors and making instructional plans to address students' errors (Biza, et al., 2018; Heinrichs & Kaiser, 2008). Building on these studies, we analyzed TAs' engagement with errors regarding the levels of specificity in identifying, planning, and classroom teaching. Given that TAs often have little teaching-related background, we did not include "interpreting" as part of our design, but we added their classroom teaching to observe how they implemented their plans.

Research Design

Participants

Our study was conducted in a single-variable calculus course at a large public university in the U.S. The course is offered as coordinated lectures taught by faculty instructors for 100-150 students with discussion sections taught by TAs for 25-30 students. In the discussion sections, TAs mainly solve problems from the textbooks and review the concepts that were covered in lectures. All five TAs who taught the discussion sections voluntarily participated in this study. All TAs had one semester of experience teaching discussion sections before participating in the study, but two of them taught the course before as an instructor of record before they joined the mathematics PhD program at the institution. Prior to our study, they received general instructions about their teaching responsibility, did a 5-minute teaching demo, and received feedback from faculty, but had not received any training on how to use students' work or errors in teaching.

Problem Development and Analysis of Students' Responses

The author with extensive research experience in teaching and learning of Calculus, who has been working on PD for TAs at their institution by incorporating what they learned from the national effort to improve PD for TAs (e.g., MAA, 2022), conducted the study in collaboration with other mathematicians and graduate students. We developed 8 calculus problems from online resources (CLEAR Calculus, 2021; The Better File Cabinet, n.d.) and widely-used textbooks (Bressoud et al., 2013). We collected students' responses on the problems from the TAs' classes. We identified the errors in student responses and grouped them into error types (i.e., codes) regarding mathematical components such as functions, limit, and derivative. For each problem, we used a table to record the number of codes we identified from each TA's classroom and the number of corresponding errors for each of the codes. We also grouped the codes regarding their nature: (a) How to compute or graph (How to), (b) Meaning of mathematical objects or their relationships (Meaning or Relationship), and (c) Missing mathematical components or properties (Missing Component). For each problem, we used a table to record the number of codes from each TAs' classroom that belonged to each nature and the number of corresponding errors.

Data Collection on TAs' Identifying, Planning, and Teaching

For each problem, we ask TAs to grade their students' responses, and record students' difficulties that they identified and plans to address them in a Google Docs shared with the researchers. Although there might in principle be differences between what TAs considered as students' difficulties and what the researchers considered as error types, most difficulties that the TAs identified easily matched with the codes we developed. We also video-recorded TAs' teaching of the problems.

Analysis of TAs' identifying, Planning, Teaching

We first identified codes matched with students' difficulties that TAs submitted. For each problem we recorded the number of those codes and the number of errors corresponding to each of the codes. We then computed the percentages of those numbers in comparison to the number of codes and the number of errors that we identified from the TA's student responses. We did the same computation of the percentages in terms of the three natures of error types described above.

We then analyzed the level at which TAs used students' error types (codes) in their identifying according to the specific details of the error type that were involved as following:

- Robust: Identifying error types with specific mathematical aspects they are regarding.
- Limited: Stating general categories of error types regarding derivative components or their nature without to student work or mentioning they saw students' work in class.
- No-Data: TA does not fill out Google Docs or no mention of students' error in class

For planning, we used the following levels:

- Robust: Incorporating students' work in their plan suggesting extra representations/ problems, or alternative approaches, and revisit mathematical aspects of the errors.
- Limited: Planning to address limited aspects of student errors with extra examples/ representations or by providing correct statements/answers that student work violated.
- No-data: TA does not fill out Google Docs

For teaching, we used the same levels that we used with planning with additional subcategories for the limited level such as pointing out the part of students' solution that was incorrect, or explaining what students should or should not do in terms of solving the problem.

Results

TAs' Identifying Students' Errors

We made several important observations from the types of errors TAs identified. First, the average percentage of error types that each TA identified relative to what we, the researchers, identified ranged from 10~33%, and the average percentage of corresponding errors relative to what we identified ranged from 7~25%. A closer look at TAs' identifying shows that the error types that they identified are limited in terms of frequencies and the natures. Regarding the frequency, the error types that they identified are often made by only 1 or 2 students in their class where there were other error types made by 3 or more students (frequent error types) that they did not identify. The average percentage of each TA identifying frequent error types ranged from 8%~44% across the 8 problems. This means that 56%~92% of the frequent error types were not identified by TAs, thus not addressed in class. Moreover, among 39 TA-problem combinations (5 TAs and 8 problems except one due to missing data), only in 14 combinations, did TAs identify the most frequent error type that their students made. Among the other 25 combinations where TAs missed such error types, in 18 combinations the most frequent error types were found in 5 or more students' work, and in 14 combinations, they were found in 7 or more students' work. Regarding the natures of error types, we found that many error types that TAs identified were procedural in nature. Compared to what we identified, we found that they identified How-to error types and the corresponding errors (about 60% in both) much more frequently then Meaning-or-Relationship (about 30% in both) or Missing-Component (about 10% in both) types.

Levels TAs' Identifying, Planning, and Teaching

As a total of 58 error types that the TAs identified, 37 were at the robust level and 21 were at the limited level. When we look at the combination of identifying levels with the natures of error types, the prevalence of How-to error types become more obvious. Among 39 How-to error types that they identified, 27 (about 70%) were at the robust level, but less than half Meaning-and-Relationship error types (7 among 15) were at the robust level.

Among 56 TAs' submitted plans, 24 were at the robust level and 32 were at the limited level. The most common robust planning categories were providing extra examples/representations that address mathematical aspects of error types, and most common limited plans were providing correct answers/statements students' solutions violated. We also found that most of identifying-playing pairs have the same levels. 22 identifying-planning pairs were both at the robust level, and 18 identifying-planning pairs were both at the limited level. However, we observed a noticeable drop in the details of students' errors used from identifying to planning. There were 14 robust-identifying and limited-planning pairs. Most of those cases involved How-to error types and TAs' plan to address them with the correct procedures or mathematical statements.

Among 58 error types that TAs identified, 41 were used in their teaching. Among 41, 23 were used at the robust level, and 18 were used at the limited level. We observed that a large number of TAs' identifying-and-planning pairs were at the same level. Similarly, a large number of their identifying and teaching pairs was at the same level. However, we also noticed that there were 16 error types that TAs identified and planned to address, but did not address in their classroom teaching. Most of those error types were procedural (How-to) in nature, which the TAs planned to address by providing the correct procedures. In such cases, TAs teaching was similar to a solution that one may find in a solution manure without any evidence that they used students' errors.

Discussion and Conclusions

Our results indicate important considerations for professional development (PD) for TAs. First, we observed that the TAs' identifying of students' errors could improve; the error types they identified were a relatively small portion of what the researchers identified from their students work, and in most problems, they missed the error types that were made by most students in their class. There may be various reasons for this, such as what TAs consider important in teaching mathematics or previous student work that they encountered. However, TAs' focusing on not-frequent error types means TAs are potentially spending time on content that is not helping many students in their class, and suggests that PD could address why focusing on certain errors and possible students' ideas behind them is important in class. Also, a large portion of errors that they identified were procedural in nature. This may be associated with the belief that their primary role is to "work problems so students know how to do them" as some calculus instructors or TAs indicated in Ellis (2014a, p. 94), or with what they believe a goal of teaching calculus to be (e.g., teaching students patterns of how to solve problems) as some U.S. TAs indicated in Kim (2014). However, given that students also made a significant number of errors of other natures, such as "mathematical meanings and relationships", future PD should provide opportunities for TAs to attend to and engage with these types of errors as well.

Second, we observed that the levels of details of student errors involved in identifying, planning, and teaching in general were same although there were a significant number of drops in level. These observations suggest that their ability to identify students' errors needs more attention than their ability to plan and implement the plans to address errors that they identified, but there is room for improvement in all the aspects. Explicitly asking TAs to identify prominent errors from student work may help them to focus on those errors. PD could help them understand errors from students' point of view and engage with errors of various natures throughout several phase of teaching to reduce drops in levels. These suggestions are aligned with other suggestions that PD for TAs should emphasize. The nationally held MAA (2022) workshop for improving PD for TAs included intensive discussions and practices on understanding and interpreting student work including systematic errors. Our aim is to help TAs engage with student errors to provide responsive teaching to students' mathematical needs in their learning.

References

American Association for the Advancement of Science (2019). Levers for Change: An assessment of progress on changing STEM instruction. Retrieved from https://www.aaas.org/resources/levers-change-assessment-progress-changing-stem-instruction

Bray, W.S. (2011). A collective case study of the influence of teachers' beliefs and knowledge on error-handling practices during class discussion of mathematics. Journal for research in mathematics education, 42 (1), 2–38.

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: insights from the Mathematical Association of America national study. International Journal of Mathematical Education in Science and Technology, 44(5), 685-698.

Brodie, K. (2014). Learning about learner errors in professional learning communities. Educational studies in mathematics, 85(2), 221-239.

CLEAR Calculus (2021). CLEAR Calculus. Retrieved from https://clearcalculus.okstate.edu/

Ellis, J. (2014). Graduate students Teaching Assistants'(GTAs') beliefs, instructional practices, and student success. In Proceedings of the 17th Annual Conference on Research in Undergraduate Mathematics Education (pp. 609-616).

Heinrichs, H., & Kaiser, G. (2018). Diagnostic competence for dealing with students' errors: Fostering diagnostic competence in error situations. In T. Leuders, K. Philipp, & J. Leuders (Eds.), Diagnostic competence of mathematics teachers (pp. 79-94). Springer, Cham.

Ingram, J., Pitt, A., & Baldry, F. (2015). Handling errors as they arise in whole-class interactions. Research in Mathematics Education, 17(3), 183-197.

Mathematical Association of America. (2022, May 4) *College mathematics instructor development source (CoMInDS).* https://www.maa.org/programs-and-communities/professional-development/cominds

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and US mathematics classrooms. The elementary school journal, 108(2), 131-147.

The Better File Cabinet Home Page (n.d.). Retrieved from https://betterfilecabinet.com/

USE OF SIGNIFIERS IN MATHEMATICAL DISCOURSE IN KOREAN AND ENGLISH

Jungeun Park
University of Delaware
jungeun@udel.edu

Douglas Rizzolo
University of Delaware
drizzolo@udel.edu

We consider how the existence of different signifiers for mathematical objects in different languages manifests in discourse about those objects. Based on the observation that there is a common signifier "derivative" in English used for both the derivative at a point and the derivative function and two phonetically and semantically different signifiers for those objects in Korean, we explore the differences between one Korean teacher's discourse and one American teacher's discourse about the derivative. Our analysis uncovered differences in metarules regarding the use of signifiers, as well as differences in possible connections to colloquial discourse. Additionally, we found that, after both objects are defined, the American teacher's discourse shifts in a way that precludes the simultaneous use of the common signifier for both objects whereas, in the Korean teacher's discourse, there was no similar shift.

Keywords: commognition, derivative, calculus, mathematical object, realization tree

Recently, the impact of the language in which mathematics is practiced on mathematical discourse has been emphasized. Examining students' or teachers' uses of mathematical words is an important part in this literature (e.g., Kim et al., 2012; Han & Ginsburg, 2001). Our study examines uses of terms in mathematical discourse in different languages in an effort to understand the language-dependent nature of the discourse. We focused on the discourse about the derivative in Korean and American English (English, here after), which provides a useful context for the study. The derivative is a crucial concept in Calculus and needed for multiple disciplines. Commonly, within introductory calculus, derivatives can be separated into the derivative at a point and the derivative function. By "the derivative function" we mean a function obtained by differentiating another function. Studies have addressed challenges students face distinguishing or relating these objects (Font & Contreras, 2008; Font et al., 2007; Park, 2013).

We take a commognitive perspective (Sfard, 2008) to examine discourse on the derivative at a point, the derivative function, and the narratives connecting these objects. In the commognitive approach to mathematics education, "learning mathematics is the process in which students extends their discursive repertoire by individualizing the historically established discourse called mathematics" (Sfard, 2018, p. 222). Individualizing a discourse essentially means developing one's ability to communicate with others and oneself according to the rules of the discourse. In each language, we will refer to the historically established discourse called mathematics as the canonic discourse. As Kim et al. (2012) noted, from the commognitive perspective, one should not expect mathematical discourse in different languages to be homeomorphic. Thus, one should expect differences in the canonic discourses in different languages. From this perspective, differences in canonic discourses are differences in what students are trying to individualize.

The motivation for our study is the observation that in English, the word "derivative" is included in "the derivative at a point" and "the derivative function" and the word "derivative" alone is often used for these objects, while in Korean the phonetically and semantically different terms "mi-bun-gye-su" and "do-ham-su" are used respectively for the corresponding objects and there is no common term like "derivative" that can be used for either object. Thus, this difference between English and Korean is a difference in the canonic discourses of the two languages.

While the non-homeomorphic nature of mathematical discourse in different languages has been investigated from the student perspective, to our knowledge, our study is the first to focus on differences caused by differences within the canonic discourses. In contrast to previous studies that focus on aspects of students' discourse that one would expect students to outgrow as they attain proficiency (e.g., Favilli et al., 2013; Han & Ginsburg, 2001; Kim et al., 2012; Miller & Stigler, 1987; Paik & Mix, 2003), we consider a different aspect of language-dependency in which one language has a common term for two different but related objects and the other does not, which results in a difference in the mathematics that students in each language are trying to learn. This is an important difference to investigate because several researchers have suggested that one term/notation being used for multiple objects creates challenges for teaching and learning about those objects in different mathematical discourses (e.g., inverse in algebra or discrete mathematics, Thompson & Rubenstein, 2000; tangent line in geometry and analysis, Biza & Zachariades, 2010). Especially, multiple uses of one term in the same discourse are hard to communicate with students (e.g., limit as a number and limit as a process in (Güçler, 2013)).

To investigate the differences in the canonic discourses, we examined one Korean mathematics teacher's classroom discourse and one American teacher's classroom discourse considering those teachers as participants in the canonic discourse in Korean and English, respectively, and addressed the following research question: *What are the differences between canonic discourse in English and Korean that can be observed based on an American mathematics teacher's discourse involving a common word "derivative" and a Korean mathematics teacher's discourse involving two words "mi-bun-gye-su" and "do-ham-su"?*

Based on the comparison between two teachers' discursive characteristics, we learned about the differences between canonic discourse in each language regarding metarules about word use, in relationships between canonic discourse and colloquial discourse, and in ways mathematical objects were connected. We adopted a case study approach because it allows researchers to "closely examine the data in a specific context" (Zainal, 2007, p. 1) and many recent mathematics education studies that examine teachers' discursive characteristics through in-depth analysis adopted a case study approach (e.g., González, 2015; Kontorovich, 2021). A limitation of our approach is that it may not find all of the differences between the canonic discourses or identify which differences would be most commonly observed in a large sample of classrooms. However, in-depth engagement with one teacher's discourse involving the specific terminologies of interest in each language, which a case study allows, was needed for researchers to explore mathematical discourse in each language "in action" and compare potential differences. To make our inference about differences in the canonic discourses in each language valid we followed case study design guidelines from Check and Schutt (2017), Rubin and Rubin (1995 & 2005), and Zainal (2007). We carefully chose teachers whose discourse would allow us to learn about differences in canonic discourse in each language and collected data that would enlighten us about the connections between mathematical objects made within the canonic discourse and its connection with colloquial discourse in each language. We then produced a chain of evidence about the differences in the canonic discourses in each language based on our analysis of the data from holistic point of view by examining their discourse following commognitive research guidelines from e.g., Kontorovich (2021), Nachlieli & Tabach (2012) and Sfard (2008 & 2012).

Theoretical Background

Prior Work on Differences in Mathematical Discourses in Different Languages

The commognitive framework has been used in several settings to examine the dependence of mathematical discourse on language. This conceptual and discourse-analysis framework combines cognition and communication. It asserts that thinking is communicating with oneself and views mathematics as a type of discourse. Commognition makes the language-dependent nature of mathematics almost self-evident because languages are the medium for communicating (Kim et al., 2012), which makes it a natural framework for studying such dependence.

Language studies using the commognitive framework have focused on the discourse of students. For example, Kim et al. (2012) used this framework to study differences between students' discourses about infinity in Korean and English based on phonetic and semantic disconnections between "Korean formal mathematical discourse in infinity and its informal, colloquial predecessor" and "the cohesiveness of the infinity discourse" in colloquial and formal discourse in English (p. 95). Ní Ríordáin (2013) adopts the commognitive framework, discussing how, theoretically, the syntactical structures of Irish "lend itself to easier interpretation of mathematical meaning in comparison to English" (pp. 1581-1582), which was supported empirically by the later study (Ní Ríordáin & Flanagan, 2020).

Outside of the commognitive literature, if one views the ability to correctly solve problems as a proxy for fluency in a discourse, there is a long history of investigating phenomena similar to what Ní Ríordáin and Flanagan (2020) investigated. For example, some of these studies considered the feature of Asian language terms stating mathematical concepts clearly, such as part-whole relations in fraction words, which is not a feature in other languages (e.g., English) as an explanation for Asian-language speaking students' higher performance regarding those concepts (Han & Ginsburg, 2001; Miller & Stigler, 1987; Miura, 1987; Miura et al., 1999).

Most of these studies focus on, or give insight into, phenomena within student discourse that students are intended to outgrow as they become proficient participants in the canonic discourse. In contrast, our study considers differences within the canonic discourses, which therefore will become present and persist after students become fluent in the canonic discourse.

Canonic and Teacher Discourse

From the commognitive perspective, a discourse is a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions" (Sfard, 2008, p. 297). Canonic discourse can be difficult to identify in general because it is historically established by practice. Thus, we grounded our identification of canonic discourse in content that has broad, objectively observable acceptance. In particular, we decided to use widely accepted curricula as the basis for identifying the canonic discourse in Korea and the United States. In Korea calculus is part of the national high school mathematics curriculum set by the Korean government (Ministry of Education, 2020), which also reviews and approves high school textbooks. In the United States there is no similar nationally controlled curriculum for calculus. Rather, there is the Advanced Placement Calculus AB curriculum set by the College Board (2020), which is a private company. Although privately set, Advanced Placement (AP) Calculus courses are taught in many high schools (public and private) and high school students may receive college credit at many colleges (including highly ranked ones like Harvard University) by scoring well on the standardized AP Calculus AB Exam at the end of each year. From this, we conclude that the AP Calculus curriculum is broadly accepted in the United States. We consider certified teachers who teach these curricula to be among the participants in the canonic discourse of their country and selected our participants from among this group.

Within the commognitive framework, teaching can be defined as "the communicational activity the motive of which is to bring the learners' discourse closer to a canonic discourse" (Tabach & Nachlieli, 2016, p.303). This view is consistent with other discursive approaches. For example, as reviewed in Schutte (2018), Pimm (1987) sees learning mathematics as similar to learning a foreign language, which is both written and spoken language which has to be used extensively within the mathematics classroom, and views the teacher as similar to a “native speaker” in a new language (p. 26). Therefore, the mathematical discourse of qualified teachers teaching courses that follow broadly accepted curricula is likely to provide examples of canonic discourse, with the understanding that teachers may occasionally include non-canonic discourse through making mistakes or adopting idiosyncratic discourse in the process of trying to bring students’ discourse closer to the canonic discourse. In practice, this essentially means that, with the exception of occasional mistakes or adoptions of idiosyncratic discourse, we treat the mathematical statements endorsed as correct by qualified teachers as though they would be endorsed as correct by other qualified teachers and others who have mastered the content of the broadly accepted curriculum. We remark that the discourse of different participants in the canonic discourse may vary dramatically, see e.g. Moschkovich (2007) about the differences between academic and school discourse, but the hallmark of discourse being canonic is that it would be endorsed as correct by other participants in the canonic discourse.

Methods

This study uses teachers’ discourse in each language to learn about canonic calculus discourse in English and Korean. As noted in the introduction, we adopted a case study approach to conduct an in-depth analysis. This section details the research design with our rationale to produce valid conclusions through our selection of participants, data collection, and analysis.

Participants

To ensure the validity of our choice of teachers in terms of addressing our research question, our recruiting of teachers was guided by strategies for purposive sampling in which participants are selected for a purpose (Rubin & Rubin, 1995, as cited in Check & Schutt, 2017), such as “knowledgeable about…the situation…being studied,” and “willing to talk” with “a range of” view points” (p. 20). In selecting the American teacher, we discussed our study with faculty at a midsized U.S. university, who work at a center that organizes professional development (PD) for mathematics teachers. Of the teachers they recommended, William was recommended as the best “potential” participant given that William has taught Calculus multiple times, been engaged in PD for mathematics education, and participated in the design of the state-wide mathematics tests. William was also “willing to talk” about the subject of our study and to open their classroom being observed. We operationalized “the range of points of view” as using various communicational means in their teaching, especially including a range of representations and terminology in the mathematical context of our study. William described how they usually teach the topic of the study with the intention of using mathematical terms and concepts rigorously, as well as utilizing a variety of real-life contexts and mathematical representations (e.g. graphical and symbolic), which fit our operationalization of the third point. In selecting the Korean teacher, we considered Korean mathematics teachers from mathematics education master’s programs at highly regarded universities in Korea and also considered Korean high school teachers who attended calculus related talks at the International Congress on Mathematical Education that happened in South Korea. Kim was an experienced high school mathematics teacher, who passed a highly competitive national test to become a secondary public school teacher, taught Calculus over several years, and was also a master’s student in mathematics education. Kim also engaged in PD and taught in a mathematics program for gifted students.

Aligned with our conception of canonic discourse, the class we observed William teaching was an AP Calculus AB course in a public high school in the U.S. The Korean class we observed was a public high school course in Korea that followed the national curriculum.

Students in both classes were native speakers of the language of instruction. In general, the teachers organized their class with explanations of key words, solving examples on the board, students' individual work on problems, and then whole class discussions about the problems the they had solved. The teachers' discourse observed in their explanations and their interaction with students helped us understand how the key words of our interest were used in each language.

Data Collection

Class observations occurred at the beginning of the derivative unit before the class concentrated on computing the derivative. This part of their class was chosen to collect the relevant data for the teacher's discourse where we could potentially observe two uses of the "derivative" in English, where "mi-bun-gye-su" and "do-ham-su" were defined and connected to each other in Korean, and where other related words and real-life phenomena were discussed.

The first author video-recorded seven 50-minute lessons from Kim's class and six 90-minute lessons from William's class from the back of the classroom. There is a difference in length of the video-recorded lessons. Kim's discourse stayed related to the mathematical objects of the lesson whereas William's class included various non-mathematical discourses (e.g., information about AP exams) and reviewing of topics irrelevant to this study (e.g., computing limits), for which we did not find direct connections to our study. We included all derivative-related talk, and approximately, 400 minutes of relevant discourse were recorded in William's class.

Once the recording was completed, the first author (who is fluent in English and Korean), one Korean-speaking research assistant, and one English-speaking research assistant transcribed the videos focusing on what is said as well as what is done (non-verbal communication) following the guidelines from Sfard (2008) for transcribing the recorded data. Once the data was transcribed, the first author translated the Korean transcripts into English. A sample from the translations was checked by another mathematics education researcher who is fluent in English and Korean. Both authors reviewed the transcripts to check the match between the video data and the transcripts and added screen shots from the videos to complement "what is done".

Analysis

Our analysis of the teachers' discourse was guided by the principles of commognitive research (Kontorovich, 2021; Nachlieli& Tabach, 2012; Sfard, 2008 & 2012). In our analysis, we treated the totality of each teacher's discourse, collected over multiple days, as our unit of analysis. However, we separated lessons into several episodes to see how words were used in different contexts. Episodes were defined by each teacher's different teaching activities. Specifically, each episode was determined by completing a mathematical task, defining a new mathematical term, providing a story involving a real-life object prior to defining a mathematical term, making connections between a newly defined term and previously defined terms, or making connections between a newly defined term and a real-life object. Episodes were defined this way because changes in teaching activities correspond to changes in the context of discourse and, therefore, to potential changes in usage of words whose meaning is context-dependent (Biza & Zachariades, 2010; Thompson & Rubenstein, 2000), like "derivative" in English. Following this definition, we identified 57 episodes in Kim's class and 54 episodes in William's class.

With our interest in linguistic differences related to the signifiers "derivative" in English and "mi-bun-gye-su" and "do-ham-su" in Korean, we needed to identify the mathematical objects that these signifiers signified in each teachers' discourse. It was done based on the formalization of the derivative as a mathematical object in the commognitive framework as in (Park, 2016). After this, we examined the coded data focusing on how the derivative at a point and the derivative function were connected. Once the objects in each teacher's individualized discourse were identified, we coded each episode according to the objects and primary signifiers used to determined which objects appeared in each episode regarding how those language-specific terms were used. For example, for William's episodes, we distinguished (a) episodes in which "derivative" is used as (or as part of) a signifier for those two objects (showing the common use of the "derivative" for the two objects) from (b) episodes in which both objects signified by "derivative" appeared but in which "derivative" was only used as (or as part of) a signifier for at most one of these objects (not showing the common use of the "derivative"). Each author individually coded each of William's uses of "derivative" as described above. The coding originally matched, except for 2 episodes which were discussed until agreement was reached. The English translation of the Korean data was reviewed by another Korean and English speaking commognitive researcher, and the English-translated Korean data and English data were analyzed through discussions among the authors, one of whom is a native English speaker.

Results

Our analysis leads to three main results. First, the canonic discourse in English (AMD) contains metarules for distinguishing whether the signifier "derivative" is being used for the derivative at a point or the derivative function that has no analogue in the canonic discourse in Korean (KMD). On four occasions William provided the following rule for distinguishing the two uses of derivative: If context shows the object signified by "derivative" is a number, "derivative" is signifying the derivative at a point, and if context shows the object signified by "derivative" is a function, "derivative" is signifying the derivative function. He said, for example

> This idea of, well the derivative can be thought of as a number, which represents the slope of the tangent line at a point. But, notice that I can talk about slope of the function at every point along this curve…the slope is always changing. Now, what we get then, is a function that represents the slope of the tangent line at any point on here, as a function of x…That's called the derivative too. From the context, you understand what I'm talking about…just remember now we're talking about numbers and we're talking about functions, and we have to keep them straight. Then we get into the other half of Calc 1 and talk about integrals. There will be the same thing. There will be some integrals that are numbers, some integrals that are functions. We use the same words because the concepts are related (Day 2, 6 minutes)

One would expect that participants in AMD have some way for distinguishing between the two uses of "derivative". Thus, the existence of such metarules is not surprising, but their existence shows that the differences between AMD and KMD extend beyond simply having a different signifier structure. We concluded that this rule is part of the canonic discourse, in addition to William being a participant in the canonic discourse, because it combines two statements from the canonic discourse (that the derivative at a point is a number and that the derivative function is a function) with the generally applicable rule that one can try to use context to infer the meaning of a word that potentially has multiple meanings. We thus believe that this metarule would be endorsed by the other participants in the canonic discourse. This metarule of AMD has no counterpart in KMD because there is no signifier in KMD whose use must be distinguished.

Second, when teaching in English, one can use the fact that "derivative" is a signifier of two objects in AMD to make connections to the colloquial discourse in English for which there are no analogous connections between KMD and colloquial discourse in Korean. For example, William used "speed" in colloquial stories about reading changing numbers on a speedometer to motivate the construction of the derivative function as follows,

> As I start out on my trip, how fast am I going?...Pretty fast!…As I travel along, what is happening to the slope or my instantaneous velocity sort of thing, instantaneous rate of change? It is going down right?… I have a whole succession of changing speeds or changing rate of changes…Let's say…I have a flea, following this line and the flea has a rate of change meter on. It's the speedometer! So, he's traveling along the line, this number is changing all the time. Now, I- (Slamming hands against the board) squash the flea at one point. That number is frozen…That number I'm gonna use as the slope, and that line (drawing the tangent line at the point)…that's what we call the tangent line…that keeps visible that slope or that instantaneous rate of change at that point (Day 1, 37 minutes)

In colloquial discourse (in both Korean and English), "speed" behaves like "derivative" in AMD in the sense that it can be used as a number or as an (informal) function of time. Thus, in an English-speaking class, a teacher could explicitly recall this dual use of speed and then explain that "derivative" works in the same way students are familiar with "speed" working and, from context, one can determine if it is being used as a number or a changing quantity.

Our third result is that William's use of "derivative" shifted after he introduced the derivative function. Specifically, in episodes that only involve the derivative at a point, "derivative" was the dominant signifier used by William for this object. However, in episodes that involved both the derivative at a point and the derivative function, William dominantly used "derivative" as a signifier for the derivative function (56 out of 70 uses) and "slope of the tangent line" as a signifier for the derivative at a point. We did not observe any such shift in Kim's discourse. Rather, Kim routinely used the term "mi-bun-gye-su" to refer to the result of evaluating the "do-ham-su" at a point. Unlike our first two results, which are results about AMD and KMD, this shift depends on William's didactic choices, in particular the choice to primarily use "slope of the tangent line" to signify the result of evaluating the derivative function, and other teachers' discourse may not include such a shift. For example, one could use "derivative" assuming what it is used for would be obvious to students or always use "the derivative at a point" and "the derivative function" to avoid confusion. However, we decided that it is interesting to report because, assuming that people naturally tend to try to communicate clearly, we conjecture that not using one term for different objects in the same context is common in AMD. Similarly, we conjecture that such shifts are uncommon in KMD.

Discussion and Conclusion

In our results, we showed that the differences between the canonic mathematical discourse in English (AMD) and the canonic mathematical discourse in Korean (KMD) extend beyond the fact that AMD has a signifier "derivative" that can be used for two objects while KMD does not have common signifier for both objects. Specifically, we argued that AMD also has metarules for using context to determine which object "derivative" is signifying based on whether the object is a number or a function. Although not every instructor in AMD will necessarily include this metarule in their classroom discourse, we argued that this metarule is part of the canonic discourse, i.e. would be endorsed as correct by other qualified teachers and participants in the canonic discourse, because it combines a statement from the canonic discourse about the difference between the derivative at a point and the derivative function with the general rule that context can be used to determine the meaning of words that have multiple uses. Indeed, it has been found in other contexts that preservice mathematics teachers use context to explain the uses of other mathematical symbols that have multiple meanings (Zazkis & Kontorovich, 2016). Our result further confirms the hypothesis that Kim et al. (2012) put forward that mathematical discourse in different languages is not necessarily homeomorphic. They supported this claim by observing differences in student discourse about infinity, but we have shown differences in both the signifier structure and metarules of the canonic discourses themselves. In addition to adding to our theoretical understanding of the relationships between canonic discourses in different languages, we believe that this has some practical instructional consequences. For example, there are many students from Korea and Japan (signifiers for derivatives in Japanese and Korean are homeomorphic) who go to the U.S. for graduate school and teach calculus. These students may not be expected to know elements of AMD that do not exist in the canonic discourses of their native languages and may not be able to teach these elements to their students.

In addition to providing the metarule for distinguishing the uses of "derivative", William's first quote in Results includes a noteworthy discussion of integrals. The metarule that William provided for the use of "derivative" can be used as an analogy to explain the metarules around the use the signifier "integral" later in the course. In KMD, the terminology around integrals is similar to AMD, there is a signifier for "definite integrals" and signifier for "indefinite integrals" and a signifier like "integral" that can be used for either. However, it cannot be explained by analogy with the use of signifiers around derivatives. This further supports the conclusion of Kim et al. (2012) that "teachers need to be cognizant of those language-specific features of the discourse that may support learning and of those that may hinder successful participation" (p. 106). Because this result pertains only to the existence of certain language-specific features, whose existence we have just demonstrated, it is not limited by our case study methodology.

The discursive shift we observed in William's class, but not in Kim's leads to a theoretical hypothesis about students' learning. From commognitive theory, within higher-mathematics, which features extreme objectification and rigor that is often unfamiliar to students, new students' learning "begins with an exposure to" instructors' discursive practices (Sfard, 2014, p. 202). Then, students start "collaborating across communicational conflict" due to the differences between this new discourse and their old discourse "by observing, and then imitating, the expert's moves while also trying to figure out the reasons", which might be the only way students "come to grips with the objects" at the abstract level (Sfard, 2014, p. 202). A consequence of this is that as students start learning higher-mathematics, their discourse will include imitations of experts' discursive moves and, just as importantly, will not include moves from the canonical discourse that they have not observed (e.g, from their instructor, textbook or other resources). This leads to the hypothesis that students in courses featuring discursive shifts as in William's will initially connect evaluating the "derivative function" with obtaining "the slope of the tangent line" rather than "the derivative at a point", while students in courses like Kim's that do not feature such a shift will initially connect evaluating the "do-ham-su" with obtaining the "mi-bun-gye-su". It would be interesting to investigate whether discursive shifts as in William's class are common and to investigate the effects such shifts have on student learning.

This would be particularly interesting because the literature has shown that the connection between the derivative at a point and the derivative function can be challenging for students (Font & Contreras, 2008; Park, 2013) and implicit shifts in instructor discourse have been tied to student difficulties (Güçler, 2013). Although derivatives have been intensely studied in the literature, we have not seen shifts like the one we discuss documented before. Note, however, that the instructors in (Park, 2015) also dominantly connected "derivative function" to "slope of the of the tangent line" and this leads us to conjecture that this shift may be widespread, but not documented because previous analyses were not looking for such large-scale discursive patterns.

References

Biza, I., & Zachariades, T. (2010). First year mathematics undergraduates' settled images of tangent line. The Journal of Mathematical Behavior, 29(4), 218-229. https://doi.org/10.1016/j.jmathb.2010.11.001

Check, J., & Schutt, R. K. (2017). Qualitative Methods: Observing, Participating, Listening. In *Research methods in education*. Sage Publications. https://methods.sagepub.com/book/research-methods-in-education/i911.xml

College Board. (2020). AP Calculus AB course and exam description. Retrieved from https://apcentral.collegeboard.org/pdf/ap-calculus-ab-bc-course-and-exam-description-0.pdf

Favilli, F., Maffei, L., & Peroni, R. (2013). Teaching and learning mathematics in a non-native language: Introduction of the CLIL methodology in Italy. Online Submission, 3(6), 374-380.

Font, V., & Contreras, Á. (2008). The problem of the particular and its relation to the general in mathematics education. Educational studies in mathematics, 69(1), 33-52.

Font, V., Godino, J. D., & D'Amore, B. (2007). An onto-semiotic approach to representations in mathematics education. For the learning of mathematics, 27(2), 2-14.

González, G. (2015). The use of linguistic resources by mathematics teachers to create analogies. Linguistics and Education, 30, 81-96.

Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. Educational Studies in Mathematics, 82(3), 439-453. https://doi.org/10.1007/s10649-012-9438-2

Han, Y., & Ginsburg, H. P. (2001). Chinese and English mathematics language: The relation between linguistic clarity and mathematics performance. Mathematical Thinking and Learning, 3(2-3), 201-220.

Kim, D. J., Ferrini-Mundy, J., & Sfard, A. (2012). How does language impact the learning of mathematics? Comparison of English and Korean speaking university students' discourses on infinity. International Journal of Educational Research, 51, 86-108. https://doi.org/10.1016/j.ijer.2012.01.004

Kontorovich, I. (2021) Minding mathematicians' discourses in investigations of their feedback on students' proofs: a case study. Educational Studies in Mathematics, 107, 213–234. https://doi.org/10.1007/s10649-021-10035-2

Miller, K. F., & Stigler, J. W. (1987). Counting in Chinese: Cultural variation in a basic cognitive skill. Cognitive Development, 2, 279–305.

Ministry of Education. (2020). Mathematics curriculum. Retrieved from https://down.edunet4u.net/KEDNCM/SynapConv/skin_v2021/doc.html?fn=B3E435D1-677B-A504-E5E1-578DE8B0E05E.hwp&rs=/KEDNCM/preview2021/642249b9-1e64-42e4-8953-b0ec40fd297b/&scf=/KEDNCM/NCIC/filemng/642249b9-1e64-42e4-8953-b0ec40fd297b&lfn=3.+%EB%B3%84%EC%B1%858_%EC%88%98%ED%95%99%EA%B3%BC+%EA%B5%90%EC%9C%A1%EA%B3%BC%EC%A0%95%28%EC%A0%9C2020-225%ED%98%B8%29%EC%88%98%EC%A0%95_2020.04.14_%EC%B5%9C%EC%A2%85.hwp&ttl=%EB%B3%84%EC%B1%85+8_%EC%88%98%ED%95%99%EA%B3%BC+%EA%B5%90%EC%9C%A1%EA%B3%BC%EC%A0%95%28%EC%A0%9C2020-225%ED%98%B8%29

Miura, I. T. (1987). Mathematics achievement as a function of language. Journal of Educational Psychology,79, 79–82.

Miura, I. T., Okamoto, Y., Vlahovic-Stetic, V., Kim, C. C., & Han, J. H. (1999). Language supports for children's understanding of numerical fractions: Cross-national comparisons. Journal of Experimental Child Psychology, 74(4), 356-365.

Moschkovich, J. (2007). Examining mathematical discourse practices. For the Learning of Mathematics, 27(1), 24-30.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom–The case of function. International Journal of Educational Research, 51, 10-27. https://doi.org/10.1016/j.ijer.2011.12.007

Ní Ríordáin, M. (2013). A comparison of Irish and English language features and the potential impact on mathematical processing. In Proceedings of the Eight Congress of the European Society for Research in Mathematics Education (pp. 1576-1585). Turkey: Middle East Technical University.

Ni Ríordáin, M. & Flanagan, E. (2020). Bilingual undergraduate students' language use and meta-level developments relating to functions. Mathematics Education Research Journal 32, 475-496. https://doi.org/10.1007/s13394-019-00268-z

Paik, J. H., & Mix, K. S. (2003). US and Korean children's comprehension of fraction names: A reexamination of cross–national differences. Child Development, 74(1), 144-154.

Pimm, D. (1987). Speaking mathematically. London: Routledge.

Park, J. (2013). Is the derivative a function? If so, how do students talk about it? *International Journal of Mathematical Education in Science and Technology 44*(5), 624–640.

Park, J. (2015). Is the derivative a function? If so, how do we teach it? *Educational Studies in Mathematics 89*(2), 233-250.
Park, J. (2016). Communicational approach to study textbook discourse on the derivative, *Educational Studies in Mathematics 91*, 395-421.
Rubin, H. J., & Rubin, I. S. (1995). Qualitative interviewing: The art of hearing data. Sage Publications.
Rubin, H. J. & Rubin, I. S. (2005) Continuing the design: Making the research credible. Qualitative interviewing. Sage Publications.
Schutte, M. (2018) Subject-specific academic language versus mathematical discourse. In J. N. Moschkovich, D. Wagner, A. Bose, J. R. Mendes, & M. Schütte (Eds.), Language and communication in mathematics education (pp. 25-36). Springer, Cham.
Sfard, A. (2008). Thinking as communication. Cambridge, England: Cambridge University Press.
Sfard, A. (2012). Introduction: Developing mathematical discourse—Some insights from communicational research. International Journal of Educational Research, 51-52, 1-9. https://doi.org/10.1016/j.ijer.2011.12.013
Sfard, A. (2014). University mathematics as a discourse–why, how, and what for?. Research in Mathematics Education, 16(2), 199-203. https://doi.org/10.1080/14794802.2014.918339
Sfard, A. (2018). On the Need for Theory of Mathematics Learning and the Promise of 'Commognition'. In P. Ernest (Ed), The Philosophy of Mathematics Education Today (pp. 219-228). Springer, Cham.
Tabach, M., & Nachlieli, T. (2016). Communicational perspectives on learning and teaching mathematics: prologue. Educational Studies in Mathematics, 91(3), 299-306. https://doi.org/10.1007/s10649-015-9638-7
Thompson, D. R., & Rubenstein, R. N. (2000). Learning mathematics vocabulary: Potential pitfalls and instructional strategies. The Mathematics Teacher, 93(7), 568-574.
Zazkis, R., & Kontorovich, I. (2016). A curious case of superscript (−1): Prospective secondary mathematics teachers explain. The Journal of Mathematical Behavior, 43, 98–110. https://doi.org/10.1016/j.jmathb.2016.07.001
Zainal, Z. (2007). Case study as a research method. Jurnal Kemanusiaan, 9 (1), 1-6.

PERSONAL INFERENCES AS WARRANTS OF UNDERGRADUATE STUDENTS' ARGUMENTS IN CALCULUS CONTEXTS

Kyeong Hah Roh
Arizona State University
khroh@asu.edu

Erika David Parr
Rhodes College
parre@rhodes.edu

Derek Eckman
Arizona State University
dceckman@asu.edu

Morgan Sellers
Colorado Mesa University
msellers@coloradomesa.edu

The purpose of this paper is to highlight issues related to students' personal inferences that arise when students verbally explain their justification for calculus statements. We conducted clinical interviews with three undergraduate students who had taken first-semester calculus but had not yet been exposed to formal proof writing activities through undergraduate mathematics courses. We analyzed these students' verbal justification of four statements, each of which described a different relationship between two quantified variables in calculus contexts. In this paper, we document students' personal inferences that were evoked in their justifications and discuss how they are similar to or different from the logical inferences that have been accepted and practiced by the mathematics community for mathematical arguments.

Keywords: Calculus, reasoning and arguments, inferences, Toulmin's model of argumentation.

Reasoning and arguments are foundational aspects of mathematics. NCTM (2000) and Common Core State Standards (2010) have emphasized teaching students in all age groups to develop, evaluate, and use various types of reasoning and methods of mathematical arguments. On the other hand, research in undergraduate mathematics education has reported that students frequently make their arguments on empirical bases (Shipman & Shipman, 2013). Students often believed that examining a few cases would be sufficient as a justification in mathematics (Harel & Sowder, 1998) or stated that both example-based proofs and formal deductive proofs are equally valid (Martin & Harel, 1989). In addition, when reading an argument, students often accept it by ignoring, or without attending to, inferences inherent in the argument (Alcock & Weber, 2005). Dawkins and Zazkis (2021) reported that some students with no experience in mathematical proof construction did not envision proof as an unbroken chain of implications leading from the first line to the last line of an argument. Stylianides et al.'s (2004) study also documented that even undergraduate students who studied mathematics did not accept or use the equivalence between a conditional statement and its contrapositive. A similar phenomenon was also reported in Dawkins et al.'s (2021) study in which students did not conceive of an inference from the contrapositive equivalence in reading an argument and believed that a proof of the contrapositive of a conditional statement would not prove the original conditional statement. Instead, these students accepted an argument directly proving the converse of a conditional statement as a proof of the original conditional statement.

Likewise, we see some gap between students' personal inferences, as utilized in their proof-writing or validating others' arguments, and logical inferences, as accepted and practiced by the mathematics community. In this paper, we examine if such a gap still exists when students verbally express their arguments. Focusing on three undergraduate students who had not yet taken proof-oriented mathematics courses at university levels, we address the following research question: *What are the types of personal inferences that students naturally use, and how do they use their personal inferences as warrants to support their claims in calculus contexts*?

Theoretical Perspective

By *personal inference,* we mean an act or process that an individual draws to make a claim from things that are known to him or her. We consider students' personal inferences through the lens of radical constructivism (Glasersfeld, 1995). From this perspective, we assume that students' inferences consist of sets of action schemes which are built and refined through their experience. Students' inferences are personal, in this view, as their unique experiences have shaped the ways they support arguments. Even if students have not yet experienced reading or constructing formal mathematical proofs, they would have made inferences for their arguments through their experience. These personal inferences may or may not align with what the mathematics community considers conventional valid logical inference rules such as modus ponens, modus tollens, or various syllogisms. Additionally, from this perspective, we as researchers are limited to what we can access about students' personal inferences. At best, we can only hope to model students' personal inferences. Further, we acknowledge that the tasks used in this study may have evoked only some of the ways in which students may use personal inference to support their claims when making arguments. In other words, we do not intend to document all of a student's personal inferences. Rather, in this paper, we highlight the students' personal inferences that were *evoked* by the context and task.

We also adopt Toulmin's (2003) model of argumentation to further refine our consideration of students' personal inferences. Toulmin's (2003) model, shown in Figure 1, includes data, a claim, and warrants. In his model of argumentation, an individual perceives data, and formulates a claim, which is a conclusion made based on data. The warrant refers to the justification that the individual uses to explain how the data support the claim. In this paper, we focus on students' personal inferences as warrant types. These personal inferences (warrants) are embedded in their arguments as they justify their own data-based claims.

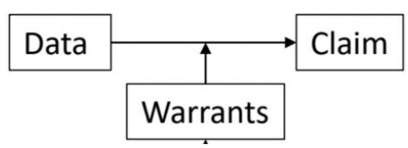


Figure 1: Toulmin's model of argumentation (2003)

Research Methodology

The data for this paper are from a more extensive study (e.g., David et al., 2019, 2020; Sellers et al., 2021). In that study, we conducted individual clinical interviews (Clement, 2000) with nine students at a large public university in the USA. This paper focuses on three students, Marie, Zack, and Hannah (pseudonyms), who had completed a first-semester calculus course but had not yet taken any proof-oriented mathematics courses. Each author of this paper served as an interviewer, witness, or researcher analyzing these students' personal inferences from their verbal justification of statements in a calculus context.

The main tasks implemented during the clinical interviews and relevant to this paper are about four statements as follows:

S1: Suppose f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then for all real numbers c in (a, b), there exists a real number N between $f(a)$ and $f(b)$ such that $f(c) = N$.

S2: Suppose f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then for all real numbers N between $f(a)$ and $f(b)$, there exists a real number c in (a, b) such that $f(c) = N$.

S3: Suppose f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then there exists a real number N between $f(a)$ and $f(b)$ such that for all real numbers c in (a, b), $f(c) = N$.

S4: Suppose f is a continuous function on the closed interval $[a, b]$, where $f(a) \neq f(b)$. Then there exists a real number c in (a, b) such that for all real numbers N between $f(a)$ and $f(b)$, $f(c) = N$.

These four statements are similar in that they are all conditional statements with the same premise, and their conclusions describe a relationship between two quantified variables, *c* and *N*. On the other hand, each of their conclusions represents different relationships between the two quantified variables. The S2 (a version of the Intermediate Value Theorem, ITV) is the only true statement among the four statements.

Without informing the students that S2 is the IVT statement, the interviewer first showed one statement at a time. Each student was then asked to interpret each statement in their own words, evaluate whether it is true or false, and explain how they could tell. Grid paper was also provided for those who wanted to sketch their example functions during the interviews. Once a student completed his/her evaluation of the four statements, the interviewer showed all four statements together and asked to compare and review their evaluations of these statements altogether.

Based on the students' utterances, gestures, and diagrams that they produced to explain their reasoning, we modeled students' arguments regarding how they would tell whether the given statements are true or false. In particular, we used Toulmin's (2003) framework to identify what students perceive (data) from the statement, their evaluation of the truth-value of the statement (claim), and how they support their claims from their data (warrants). Two types of warrants emerged from our process of analyzing students' responses to the interviewer's question about how they could determine the truth-value of the statement. One type of warrants that we found was related to students' meaning or construal of calculus concepts or their properties. Their reference to a working definition for functions and continuous functions is an example that we classified as this type of meaning-based warrants. On the other hand, the other warrants were not content-specific and were rather general rules or principles that students could apply to any content area in moving from data to claim in their arguments. In this paper, we focus on the second type of warrants, i.e., students' personal inferences that these students used as warrant types in their arguments.

Results

While some of the students changed their evaluation of the statements, all three students determined S1-S4 as true statements in their final evaluation. Thus, we identified the *claims* in these students' arguments as identical to "the statement is true" across all four statements. On the other hand, there were differences in what constituted data of each student argument as they perceived different things from the given statements. There were also differences in their personal inferences as ways in which they justified their claims about the statements' truth. In the following subsections, we present these differences in the students' *data* and detail personal inferences that the students utilized as warrants in their arguments.

Marie's Use of Biconditional Elimination and Generalization from a Particular

To support her claim that S1 is true, Marie drew a graph of a function f (see Fig. 2). She then explained that her function f is continuous on $[a, b]$ because of no holes or jumps on the graph of f. She also explained $f(a) \neq f(b)$ because $f(a) < 0$ and $f(b) > 0$. She then picked a value of c between a and b $(a < b)$ on the x-axis and then confirmed that its corresponding function value $f(c)$ is between $f(a)$ and $f(b)$ and thus N is between $f(a)$ and $f(b)$, which seemed sufficient for her to conclude S1 is true. Referring to the same graph, Marie described her reasoning about the other three statements, S2-S4, in a similar way. For instance, explaining her reasoning about S2, she said, "So N is gonna be in between here [$f(a)$] and here [$f(b)$]. And then c is gonna be in between a and b. (long pause) Okay, so N is in between $f(b)$ and $f(a)$, and because it's continuous, c is gonna be in between a and b." Here, we found what she perceived from the given statements was her graph of a continuous function f where $f(a) \neq f(b)$. Thus, we identified her graph as *data* in her arguments.

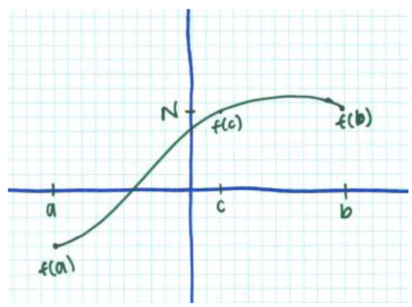


Figure 2: Marie's graph

Marie mainly used two inferences to support her claims. Her first personal inference is what is called *biconditional elimination*, $[(p \leftrightarrow p') \wedge p'] \rightarrow p$, in the conventional logic. Her use of this inference appeared when she replaced or substituted the premise and conclusion of the given statement to her own interpretation of them, respectively. In the case of S2, Marie interpreted the premise of the statement as a description of a function with no holes or jumps on its graph, and the conclusion as a description of the input of the function to be between a and b and the output of the function to be between $f(a)$ and $f(b)$. At the moment when she substituted the conclusion to these two conditions about where the input and outputs are contained, Marie neglected the quantifier words 'there exists' and 'for all.' Consequently, her interpretation of the conclusion of all four statements S1-S4 became the same. Although Marie used normatively biconditional elimination as her personal inference, her reasoning entailed invalid arguments as her evoked meaning from the given statements was not equivalent to the conclusions.

There was another inference Marie used across her arguments during the interview, which is what we refer to as a *generalization from a particular*[1], $P(x_o) \rightarrow (\forall x, P(x))$. Her function f (Fig. 2) was merely an example that satisfied her evoked meaning of the given statements. However, she drew a claim that the statements are true from this particular example. Generalization from a particular is not a valid inference in conventional mathematical logic.

1 We detailed in Sellers et al. (2021) that generalization from a particular may be connected to students' quantifications.

Zack's Use of Conjunction Elimination, Existential Instantiation, & Hypothetical Syllogism

During Zack's interpretation of S1, the interviewer recommended that he draw a graph of a function. In response, Zack drew a graph of a linear function with a positive slope (see Fig. 3) because the graph "is just simple so I can imagine myself." However, Zack described that "f will be anything, any type of a graph." In fact, he did not make his arguments based on this particular graph. His way of using a graph for illustration purposes was thus distinct from Marie's use of a graph for generalization from a particular.

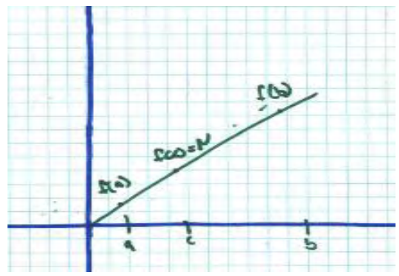


Figure 3: Zack's graph

Zack repeatedly pointed out a functional relationship between inputs and their corresponding outputs. Specifically, he started with *a*, *b*, and *c* as inputs and then described $f(a), f(b), f(c)$ as the corresponding outputs of a function, respectively. Zack's action-related explanation indicates that he mainly focused on the functional relationship between *c* and *N*. We thus identify his construal of such a functional relationship as *data* of his arguments. See the transcript below for his focus on the functional relationship.

> Zack on S1: When I put $f(a)$, we get that here (marks $f(a)$ on the graph) [...] $f(b)$ somewhere here (marks $f(b)$ on the graph). And then, [...] I would show it [*c*] over here (between *a* and *b* on the *x*-axis). I would hope it [$f(c)$] would be on the same line in between the interval from a to b ($f(a)$ and $f(b)$ in his graph, Fig. 3). That [$f(c)$] is a real number and exists right here (between $f(a)$ and $f(b)$). Now it [$f(c)$] will be like N.

We noticed that Zack leveraged the functional relationship that he was envisioning (i.e., *data*) to claim that the truth value for S1 and S2 are the same. Zack attended to the quantifier words such as "for all" and "there exists" and the order differences between quantified variables c and N in each statement. However, his attendance to such syntactic differences between the given statements was insufficient to evoke a different construal from each statement. For instance, he explained that "this one [S2] is almost a converse statement of the previous [S1]. I think the previous [S1] was asking if N existed, and this one [S2] is asking if c existed. So, I will say this statement 2 is true." His wording here also supports our model of data in his arguments that he perceived the functional relationship between *c* and *N*.

Zack used several inferences to warrant his arguments to his claims that all four statements are true, and they were often compatible with normative inferences. First, he examined only the functional relationship but excluded the other conditions from the premise of the statements, such as f being continuous and $f(a) \neq f(b)$[2]. We consider he subconsciously used what is called *conjunction elimination* ($p \wedge q \rightarrow p$) in conventional logic. His use of this inference

2 David et al. (2019) documented Zack's ways of thinking about the outputs of a function as spatial locations in grid paper, which explains why he excluded the condition $f(a) \neq f(b)$ from his interpretation of the premise because to him it is always the case for all functions.

differed from Marie's use of biconditional elimination: Marie replaced or substituted a condition to another as she treated them as equivalent, whereas Zack excluded some conditions from the given statement.

We also noted from Zack's explanation on S1 that he did not treat *N* as another label for an output $f(c)$, or use *N* to refer to the output value $f(c)$, until he checked that $f(c)$ is between $f(a)$ and $f(b)$. He said, "That [$f(c)$] is a real number and exists right here (between $f(a)$ and $f(b)$). Now it [$f(c)$] will be like N." Zack's wording here indicates that he labeled the output value $f(c)$ as N as a way to justify the existence of *N*. Once Zack showed the existence of *N* from his work on S1, he used both $f(c)$ and N to refer to the output value for the specific input value *c* in explaining S2-S4. We consider his labeling of *N* here as evidence of his use of what is called *existential instantiation* in conventional logic, i.e., $(x_0 \in X \ \wedge P(x_0)) \Longrightarrow (\exists x \in X(P(x))$. This inference is one of the conventional logical rules that are predominantly used in mathematics as a warrant while proving an existentially quantified statement. However, such an inference will unlikely be students' personal inference if they do not attend to the existential quantifier, as shown in Marie's case. Thus, it is noteworthy that Zack exhibited existential instantiation as his personal inference in his arguments.

Finally, we found that Zack used a chain of inferences when he used his evaluation of S1 to determine the other three statements S2-S4 to be true: He first compared the conclusion of S1 with the conclusion of S2 and insisted that if the conclusion of S1 is true, then the conclusion of S2 is true. Using his earlier justification regarding the conclusion of S1 to be true, he contended that the conclusion of S2 is also true. We consider it as an example of his use of a *logical syllogism* $([(p \to q) \wedge (q \to r)] \to (p \to r))$ in a normative way. On the other hand, he did not provide any further warrant to support his first inference between the conclusions of S1 and S2. He just said "there is no logical reasoning behind" in his action of choosing *N* first and then choosing *c* when thinking about the conclusion of S2, and he could do the other way "choosing *c* first and then choosing *N*" as he was thinking about the conclusion of S1. To him, the conclusions of all four statements represent the same functional relationship in which c is an input and $f(c)$, or N, is its corresponding output. Consequently, while Zack used conventional inferences normatively in his arguments, his arguments were often invalid because he construed only the functional relationship from all four statements.

Hannah's Use of Universal Generalization, Modus Ponens, & Modus Inverse

When the interviewer asked Hannah to evaluate S1, she drew a graph of a function as an example of a continuous function (Fig. 4, left). Hannah also drew two more graphs for discontinuous functions, which contained either a hole or a jump (Fig. 4, right). She used both graph types to explain her reasoning behind her evaluation. Although she drew these specific example graphs for continuous or discontinuous functions, she said that "[it] doesn't matter what it [the graph of a function] looks like." It is thus likely that what Hannah perceived from the given statements are generic functions satisfying or failing to satisfy the premise. We thus her generic examples as *data* in her arguments.

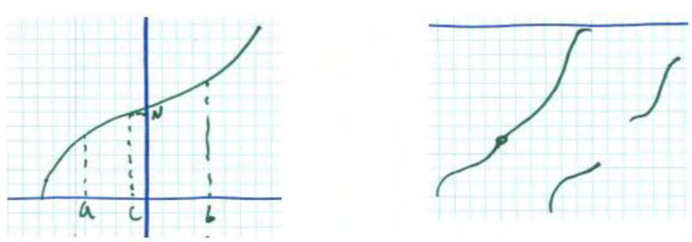


Figure 4: Hannah's graph

Hannah on S1: Because there is [are] no breaks, […] if you have this graph and there is a hole in it (Fig. 4, right), then that won't be continuous. So that would be an example of where this [the premise of S1] would be false. But since it's continuous, there is [are] no breaks [un]like this, or there is [are] no jumps [un]like that. (Fig. 4, left) So […] there has to be a $f(c)$ that equals to N, which is like, when you input any value of x between a and b, for example, c (marks c on Fig. 4, left), there has to be a y for it, a number that N equals, this roughly would be an *N* (marks *N* on Fig.4, left).

Hannah used several inferential warrants in normative ways to support her claims from her data. First of all, her use of continuous functions and discontinuous functions as any particular example indicates that she used what is called *universal generalization*, i.e., (for any fixed $x_0 \in X$, $P(x_0)) \rightarrow (\forall\, x \in X(P(x))$. We also found that Hannah used another inference that we call *modus ponens*, $(p \rightarrow q) \wedge p) \rightarrow q$, normatively in her argument for S1: She assumed the premise of S1. For her, a continuous function is equivalent to (or at least implies) that there is no break or hole in its graph. And, for her, no breaks or holes on the graph means for each input *c* between *a* and *b*, the output $f(c)$ is between $f(a)$ and $f(b)$.

Hannah attended to the syntactic differences between S1 and S2 and construed distinct meanings from each statement. Furthermore, she conceived that S1 and S2 would potentially represent different each-to-some relationships, i.e., S1 represents the each-c-to-some-N relationship, whereas S2 represents the each-N-to-some-c relationship. Her construal of different relationships from different statements characterizes her arguments distinct from Marie and Zack. However, she expressed her confusion about S3. In particular, she was not sure if the conclusion of S3 would describe a one-to-every relationship or an each-to-some relationship. Although she finally interpreted the conclusion of S3 as a description of an each-to-some relationship between *c* and *N*, she was uncertain about her interpretation of the conclusion of S3.

Hannah on S3: It sounded different [from S2]. But if you think it through, it's not saying that there is only one *N*. (reading the conclusion of S2) Okay, I am confused. Is it saying there is only one *N* or there exists a number, just any number? That's what is confusing me.

As she experienced some difficulty with S3, she focused on the sameness rather than differences between the statements, saying, "All four [statements] are equivalent because like they are just saying that there exists an input where there is a real output in this interval. So, they [S1-S4] are all saying the same thing." Here, Hannah demonstrated that some parts of the statements were distinct. However, she used her meaning of function and the input-output relationship to conclude that all four statements held the same meaning. Hannah resolved her confusion by substituting the conclusion of S3 to a conjunction of two conditions about the existence of the input and output values.

Once Hannah treated all four statements as equivalent, she also provided another way of justifying her claims about these statements, namely treating each of the given statements S1-S4 as equivalent to its inverse. She believed that she could say a statement is true by showing a statement which is what we call the inverse of a conditional statement. She thought that if a function *f* is not continuous, then the conclusion of the given statement must be false. Therefore, if *f* is continuous, then the conclusion of the statement is true.

Hannah: If you take a part of the statement, I guess, into two parts, it's saying it has to be a continuous function at first. Because this is discontinuous, in order to prove this statement true, you have to show that in a situation where it's not a continuous function, the second part [the conclusion] is not true. If you take a situation where like it doesn't meet like the basic requirements [the premise], the second part [the conclusion] can't be true for this statement to be true.

We characterize her use of such a personal inference as what we refer to in this paper as *modus inverse*, $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$. Her use of modus inverse seemed rooted in her conception of the equivalence between a conditional statement and its inverse, i.e., $(p \rightarrow q) \Leftrightarrow (\neg p \rightarrow \neg q)$. On the other hand, the modus inverse was not aligned with normative logical inferences as there are cases in that a conditional statement is not equivalent to its inverse. Thus, her use of this inference made her assertion invalid in this case.

Conclusion & Discussion

Our findings suggest that these calculus students do have intuitive personal inferences that are quite sophisticated and often in line with mathematical/logical convention. All three students spontaneously used various inferences rules normatively. Marie and Hannah's use of biconditional elimination, Zack's use of conjunctive elimination, existential instantiation, and hypothetical syllogism, and Hannah's use of universal generalization and modus ponens were examples of personal inferences that these students used normatively. It was noteworthy that these students utilized these logical inference rules normatively to support their claims even though they had not received any formal instruction for conventional logic or proof writing in mathematics.

While these students used several logical inferences normatively, their arguments often led to inaccurate conclusions. Partially, these students' invalid personal inferences (e.g., Marie's use of generalization from a particular and Hannah's use of modus inverse) explain how inaccurate conclusions were derived in their arguments. However, in some other cases, these students' use of content-specific warrants affected their argument to be invalid even if their personal inferences were valid. Marie's meanings for quantified variables influenced the invalidity in her arguments. Zack's location thinking (i.e., thinking of output values of a function as the spatial locations of points on the graph of a function) was also closely related to his invalid argument. We hope to investigate further how to leverage students to use their personal inferences with content-specific warrants for valid arguments in mathematics.

References

Alcock, L. & Weber, K. (2005). Proof validation in real analysis: Inferring and checking warrants. *Journal of Mathematical Behavior*, *24*, 125-134.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In R. A. Lesh & A. E. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 547–589). Lawrence Erlbaum.

David, E. J., Roh, K. H., & Sellers, M.E. (2019). Value-thinking and location-thinking: An empirical study of two ways students visualize points and reason about graphs in the Cartesian plane. *Journal of Mathematical Behavior*, 54, 1–18.

David, E. J., Roh, K. H., & Sellers, M.E. (2020). Teaching the representations of concepts in calculus: The case of the intermediate value theorem. *PRIMUS*, *30*(2), 191-210.

Dawkins, P. C., Roh, K. H., Eckman, D., & Cho, Y. K. (2021). Theo's reinvention of the logic of conditional statements' proofs rooted in set-based reasoning. In D. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp.518–527), Philadelphia, PA.

Dawkins, P., & Zazkis, D. (2021). Using moment-by-moment reading protocols to understand students' processes of reading mathematical proof. *Journal for Research in Mathematics Education, 52*(5), 510-538.

Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. In A. Schoenfeld, J. Kaput, & E. Dubminsky (Eds.), *Research in collegiate mathematics education*, III (pp. 234-283). Providence, RI: American Mathematical Society.

Martin, G. & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20, 41-51.

National Council of Teacher of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

National Governors Association Center for Best Practices, & Council of Chief State School Officers (2010). *Common core state standards initiative: Common core state standards for mathematics*. Washington, DC: Authors.

Sellers, M. E., Roh, K. H., & Parr, E. D. (2021). Student quantifications as meanings for quantified variables in complex mathematical statements. *Journal of Mathematical Behavior*, *61*, 1–16.

Shipman, B., & Shipman, P. (2013). Simple yet hidden counterexamples in undergraduate real analysis. *PRIMUS*, 23, 111-120.

Stylianides, A., Stylianides, G., & Phillipou, G. (2004). Undergraduate students' understanding of the contraposition equivalence rule in symbolic and verbal contexts. *Educational Studies in Mathematics*, *55*, 133-162.

Toulmin, S. (2003). *The uses of argument* (Updated Ed.). Cambridge University Press.

MATHEMATICIANS' LANGUAGE FOR ISOMORPHISM AND HOMOMORPHISM

Rachel Rupnow
Northern Illinois University
rrupnow@niu.edu

Brooke Randazzo
Northern Illinois University
bk-randazzo@niu.edu

Isomorphism and homomorphism appear throughout abstract algebra, yet how algebraists characterize these concepts, especially homomorphism, remains understudied. Based on interviews with nine research-active mathematicians, we highlight new sameness-based conceptual metaphors and three new clusters of metaphors: sameness/formal definition, changing perspectives, and generalizations beyond algebra. Implications include a way to articulate a conceptual purpose for homomorphism beyond its relationship to isomorphism: namely, as a tool for changing perspectives when problem-solving.

Keywords: Advanced Mathematical Thinking, Communication, Undergraduate Education.

Isomorphism and homomorphism are widely recognized as central to introductory abstract algebra coursework—not only do research-based curricula expressly focus on their development (Larsen, 2013) but they also serve crucial roles in fundamental theorems of algebra (e.g., the First Isomorphism Theorem (Gallian, 2009), also referred to as the Fundamental Homomorphism Theorem (Fraleigh, 2003)). Furthermore, isomorphism can be viewed as a type of sameness that expands on students' prior experiences with equality and congruence (Rupnow & Johnson, 2021) and serves an important classification role for objects like groups and rings (Randazzo & Rupnow, 2021). Nevertheless, limited research has focused explicitly on how mathematicians understand homomorphism's utility in abstract algebra.

To address this gap, we examine nine research-active mathematicians' language for isomorphism and homomorphism. In so doing, we expand on Rupnow (2021)'s characterization of instructors' language, especially with respect to homomorphism. Furthermore, we provide insight into conceptual purposes for homomorphism beyond relating it to isomorphism.

Literature Review and Theoretical Perspective

Researchers have long been interested in understanding students' views of isomorphism and enhancing students' problem-solving around isomorphism. Early research examined how students approached determining whether groups were isomorphic (Dubinsky et al., 1994; Leron et al., 1995) or approached proving theorems related to isomorphism (Weber & Alcock, 2004; Weber, 2002). More recent work has created a local instructional theory for isomorphism (Larsen, 2013), built on students' approaches to proofs using isomorphism (Melhuish, 2018), and delved into the function-nature of isomorphism (Melhuish et al., 2020).

Researchers have also begun examining students' views of homomorphism. Though many early studies examined conceptions of homomorphism in service of examining students' understanding of isomorphism (e.g., Larsen et al., 2013; Weber, 2001), more recent work has examined conceptions of homomorphism independently. Hausberger (2017) highlighted homomorphism's description as a "structure-preserving function" in textbooks, as well as what students took away from tasks intended to help students abstract the notion of ring homomorphism from their experiences with groups. Like with isomorphism, recent attention has often focused on how students coordinate the homomorphism concept and their function knowledge (Melhuish et al., 2020) as well as metaphors students use for homomorphism (Melhuish & Fagan, 2018; Rupnow, 2017).

In contrast to the many student-focused studies, limited research has been conducted on mathematicians' views of isomorphism or homomorphism. Weber and Alcock (2004) examined mathematicians' understandings in the context of proof while Ioannou and Nardi (2010) studied mathematicians' use of images in the classroom. Recent work has focused more on instructors' conceptual metaphors (described below) for isomorphism and homomorphism. Rupnow (2021) interviewed and observed instruction of two abstract algebra instructors, neither of whom did research in abstract algebra, and found four clusters of isomorphism and homomorphism metaphors: sameness, sameness/mapping, mapping, and the formal definition. Rupnow and Sassman (2021) built upon this by examining survey results from 197 mathematicians and observed examples of metaphors from each of these four clusters. To extend this work, we interviewed nine mathematicians with research specialties related to abstract algebra or category theory to see whether a larger population and a more research-oriented group of mathematicians invoked other types of conceptual metaphors for isomorphism or, especially, homomorphism.

Conceptual metaphors are a theoretical lens aimed at revealing individuals' structure of thought based on their choices of language (e.g., Lakoff & Johnson 1980; Lakoff & Núñez, 1997). Specifically, cross-domain conceptual mappings are used to connect one's cognitive structure for a target concept (e.g., isomorphism, homomorphism) to one's more developed thoughts in source domains (e.g., sameness, structure-preservation). For instance, "A homomorphism is a method for gaining information" is a conceptual metaphor that gives information about a target domain (homomorphism) by relating it to a more developed source domain with which people have had other experiences (a method for gaining information). We acknowledge that this theoretical perspective imposes the researchers' view on the mathematicians' statements; that is, we do not claim that the mathematicians intended to speak metaphorically in their responses (Steen, 2001). Nevertheless, we believe this lens does permit insight into how understandings can be clustered and the types of reasoning that can be employed when thinking about or using isomorphism and homomorphism.

Conceptual metaphors have influenced examinations of students' understandings of bases in linear algebra (Adiredja & Zandieh, 2020) as well as been used as a framework for examining college students' beliefs about mathematics (Olsen et al., 2020). More closely tied to this study, conceptual metaphors have been used to examine understandings of functions in high school and linear algebra (Zandieh et al., 2016) and in abstract algebra (Melhuish et al., 2020; Rupnow, 2017), as well as mathematicians' views of isomorphism and homomorphism (Rupnow, 2021; Rupnow & Sassman, 2021). Here we aim to extend the framework proposed in Rupnow (2021) to incorporate new clusters of metaphors based on new ideas raised by the mathematicians in this study. We thereby answer two research questions: What conceptual metaphors do research mathematicians employ to characterize (1) isomorphism and (2) homomorphism?

Methods

Data were collected from Zoom interviews conducted with nine mathematicians, given gender-neutral pseudonyms in this paper. These mathematicians had previously completed a survey about sameness in mathematics and were selected from those who had provided clear responses to the survey and had characterized themselves as research-active in abstract algebra, category theory, or a field interacting with abstract algebra. All participants had taught abstract algebra and/or category theory at least once (one once, three 2-5 times, four 6-10 times, and one 11+ times). The interviews focused on how participants characterized isomorphism and homomorphism for research, teaching, and laypeople (e.g., "How, if at all, does isomorphism play a role in your research?", "How would you describe a homomorphism to a layperson?").

Two researchers coded the interviews separately using the conceptual metaphors for isomorphism and homomorphism in Rupnow (2021) as codes (e.g., operation-preserving, journey) and then met to discuss coding and reach consensus. Each talk-turn was coded with all metaphors that appeared within that talk-turn. Although each phrase could only be associated with one metaphor, a talk-turn could contain multiple sentences and multiple metaphors. When new metaphors arose that were not clearly aligned with existing codes, these were added to the codebook, and interviews were iteratively reexamined in light of the new codes. This process aligns with codebook thematic analysis (Braun et al., 2019) in which an a priori codebook provides structure to the analysis, but space is made for emergent themes to be incorporated.

Results

We found that the original four metaphor clusters in Rupnow (2021) (sameness, sameness/mapping, mapping, and formal definition) were represented in this data as well, with some new codes being added to the sameness and mapping clusters. Additionally, three new clusters (sameness/formal definition, changing perspectives, and generalizations) were added to the framework to accommodate the new language used by participants in this study.

Sameness

This cluster contains codes referring to isomorphism or homomorphism as encoding information about the sameness of objects. This includes codes originally from Rupnow (2021), such as *generic sameness*, where participants described isomorphic objects as objects that are (essentially) the same, and *same properties*, where participants described isomorphic objects as those that have the same properties or invariants (e.g., cardinality). Several new codes were also included in this cluster and are discussed below.

Two participants were given the code *indistinguishable* when they described isomorphic objects as exact copies or not distinct from one another. For example, Avery said:

> I'm trying to teach [students] that isomorphism is measuring sameness so that they actually start to think of isomorphic objects as no longer being distinct… I might try to get them to stop distinguishing between isomorphic objects, and therefore, we can talk about *the* dihedral group of order 8 as opposed to different models of that group.

While Avery appears to be encouraging students to avoid viewing different manifestations of the dihedral group of order 8 as different, this identity-focused interpretation was not necessary to be coded as *indistinguishable*. Note that participants here were still thinking of isomorphic objects as being "the same," but used more specific language than those coded with *generic sameness*.

The code *embedding* was used when participants described homomorphisms as an embedding or a mapping into part of a larger object. This code also included sameness of homomorphic images in terms of the structure of one object appearing in another object or a copy of one object "sitting inside" another object. For instance, Indy remarked:

> [S]o I say, an isomorphism is an exact copy. These things are exactly the same. But a homomorphism,…maybe we don't have this bijection anymore. But somehow some of this structure is appearing in this other object. And the ways that that could happen—one of the ways is maybe—I have an exact copy sitting inside this larger object.

This quote was also given the code *indistinguishable* due to the comment about an isomorphism being an exact copy, which highlights the similarity between these two ideas: *indistinguishable* refers to isomorphisms as producing exact copies, whereas *embedding* refers to homomorphisms producing copies inside a larger object.

The code *logical equivalence* was given to participants who described an isomorphism or homomorphism as a transfer of knowledge in which truth values or facts about objects are preserved. When discussing different ways of writing isomorphic groups, Blair remarked:

> And so it's okay if we pick different ways of writing down what ultimately results in the same structure…At the end of the day, they're going to have the same structure because they're isomorphic in the logical sense as well. Any statement of group theory that is true about one of those structures will be true if and only if it's true in the other structure.

Others talked about isomorphism in terms of classification problems and were given the code *classification*. For instance, Hayden said:

> So I think the example of the classification of finite simple groups in the 20th century is just one of the tools of 20th century algebra. But that statement properly understood suggests that we've written sometimes in infinite families…what all the finite simple groups are up to isomorphism, not what they all are but up to isomorphism…

Notice they specify these classification problems are not listing all possible objects, just those "up to isomorphism", which relates to Avery's focus on "*the* dihedral group of order 8" above—these participants seem to view isomorphism as the "sameness" that matters in abstract algebra.

Sameness/Mapping

This cluster contains codes about isomorphism or homomorphism that showcase sameness via mappings, which are all included in Rupnow (2021). In particular, five participants spoke of isomorphism as a *renaming/relabeling*, or described isomorphic objects as the same, except for their names. Others were coded with *matching* when they talked about matching or pairing up elements between isomorphic objects in some way. The code *equivalence classes* was given to homomorphism-focused responses that explicitly mentioned equivalence classes or cosets. This code also included responses talking about homomorphisms in terms of an "orderly collapsing where things line up," or the process of "stacking [elements] into the same bins" (Indy). These latter types of responses make it clear why this code was included in this category, as they emphasize the sameness of a grouping of elements under a homomorphism. Note here that the idea of "lining things up" is what is implying the existence of *equivalence classes*; the word "collapsing" is coded with *structure loss* in the changing perspectives cluster below.

Mapping

This category includes responses that talk about isomorphisms and homomorphisms as functions or maps between mathematical objects, focusing on the map or process of mapping rather than the objects. The code *generic mapping* was given to participants who described isomorphism or homomorphism generally in terms of a function, morphism, arrow, map, or correspondence, whereas the code *journey* was given to participants who were explicit about the directionality of the map or used some sort of movement metaphor (e.g., elements being "sent to" one another). Both of these codes are included in Rupnow (2021).

The new code *invertibility* was given to four participants who highlighted the necessity of an isomorphism being reversible or comprised of maps that compose to the identity. Greer provides a clear example of the former:

> I mean, all the things I really think about when I think about the way isomorphisms would occur in the non-mathematical world maybe, really are sort of reversible processes. So now I was just thinking about cyphers and codes,…and that's a really concrete example of the way

> an isomorphism would work. It's even within the English language, but it's turning all messages in English to other messages in English in a completely reversible process.

Notice Greer's explanation of isomorphism for a non-mathematician focuses on the reversibility of isomorphism. Finley provides an example of the identity-focused version of invertibility:

> An isomorphism between groups is a homomorphism from G to H together with a homomorphism from H to G so that the composites are identities. But then, you could also say an isomorphism is a bijective homomorphism because that's a theorem that in the category of groups, the categorical isomorphisms coincide with the bijective homomorphisms. So I usually will present those two different ways.

Observe Finley distinguishes between bijective homomorphisms and homomorphisms with inverses that are also homomorphisms—although the definitions coincide in abstract algebra, they do not when generalizing to other contexts (e.g., continuous bijections need not have continuous inverses in topology).

The new code *transformation* was given to two participants who talked about isomorphism or homomorphism as a process that morphs or transforms one object into another (similar to Zandieh et al., 2016). For instance, Greer said:

> I use the word mechanism like an isomorphism really is a process to me. It's a process of turning one object into another in some sense. It's not turning the objects into each other, it's reframing your thinking from…one object to another object, I would say. Objects themselves are distinct. Completely distinct to me, and they're identified by isomorphisms.

Notice that Greer characterizes isomorphic objects as being distinct, in contrast to the examples given the *indistinguishable* code above.

Formal Definition

These are codes given to responses that utilized the formal definition to reason about isomorphism or homomorphism. All of these codes are included in Rupnow (2021). The code *literal formal definition* includes instances of describing an isomorphism as a bijection, often with special properties, or describing isomorphic objects as simply objects that have an isomorphism between them. While this code included responses using the literal string of symbols in the homomorphism property in Rupnow (2021), these types of responses did not exist in this study, likely because the mathematicians were not asked to engage in problem-solving.

Several participants also defined isomorphism based on homomorphism, and vice versa. The code *special homomorphism* was given to responses describing isomorphism in terms of homomorphism, either formally (e.g., "an isomorphism is a bijective homomorphism", Finley) or informally (e.g., "I think of isomorphism as homomorphism plus extra things", Greer). Similarly, the code *isomorphism without bijectivity* was used for responses that focused on describing homomorphism by relating to isomorphism (e.g., "[Homomorphism is] 'Sort of an isomorphism' is what comes to my mind. So we still want to preserve the structure, but maybe we don't insist on one-to-oneness anymore or one-to-one correspondence", Cameron).

Sameness/Formal Definition

Codes here include crucial parts of the formal definition (i.e., structure/operation preservation), but stated in an informal way. These two codes were originally included in the formal definition category in Rupnow (2021) because they were generally used as unexplained stand-ins for the homomorphism property/homomorphism by those participants. However, we now view them as a separate cluster because participants here seemed to use them to explain what they meant by sameness (e.g., "preserving the algebraic structure, same algebraic structure", Dallas). Cameron illustrates both *structure-preserving* and *operation-preserving*:

> And so when I cite preservation of structure, I mean that all these [mathematical sub-]fields have a notion of isomorphism in there. And you're usually referring to a bijection which preserves the structure of whatever it is you're looking at. So in topology, it's like a bijection that preserves continuity in both directions. And in abstract algebra, it's a bijection that preserves your multiplication, addition operations, whatever. In graph theory, it's a bijection that preserves adjacency.

We note here that the ideas of operation-preservation and structure-preservation are very similar, and Cameron seems to be using them interchangeably in the context of abstract algebra, though the notion of structure-preservation could carry over to other mathematical sub-fields as well.

Changing Perspectives

The metaphors in this category are all new codes involving responses that emphasize how homomorphisms force a change of perspective (*quotient group construction, structure loss*), or that explicitly mention using isomorphisms or homomorphisms to change perspective for the purpose of aiding mathematical research *(information gain*).

Two participants were given the code *quotient group construction* when they mentioned that homomorphisms arise from quotient groups or vice versa. Blair talked about viewing these two concepts interchangeably, even though students may see them as distinct initially.

> And a quotient object is exactly how you make rigorous this notion of collapsing down and the most important result… is that homomorphisms are the same things as quotients… a really important idea for students is that a group homomorphism between two groups is *the same thing* as a certain type of equivalence relation, a certain type of quotient group as well.

This quote mentions the idea of "collapsing" in relation to homomorphism, so was given the code *structure loss* as well, which involved responses talking about homomorphisms in terms of collapsing or similar ideas such as simplifying, losing, or ignoring structure. For example, Emerson describes homomorphism in the following way: "[T]aking a homomorphism is preserving some structure but losing something along the way. Hopefully, something that you are trying to ignore or that you don't care as much about as the stuff you're trying to preserve." The similar concept of getting only partial or limited information from a homomorphism was also included here. Greer observed:

> I think describing a homomorphism to a layperson… I want to say that it's about maybe collapsing and simplifying structure in mathematics, but I think it's actually a pretty foreign idea to the real world, this idea that you can take something you care about and record only partial information, and still learn something about whatever you're studying, but maybe not.

Thus, while only partial information is retained from a homomorphism, Greer believes that homomorphisms are still helpful in learning something about the relevant objects.

Four participants used metaphors related to *information gain.* This code captures the idea that homomorphisms are used to gain understanding about one or both of the objects involved. For example, Blair gave the following reason for using homomorphisms in their research but was relatively vague on the details about what sort of information is gained from this.

> And a great example of that would be like group actions. A group acts on a metric space. And even if you—a group action is a homomorphism from a group into the isometry group

> of this metric space. And even if you didn't understand that much about the metric—or you knew some things about the metric space, you knew some things about the groups, the information can go both ways. You can use things you learn about the metric space to learn things about the group, and you can use algebra that you can actually compute in the group to learn things about the metric space. So, you can gain knowledge in both directions.

Others, like Hayden, referred to this information gain as part of their mathematical tool set:

> It's our basic tool in moving around between algebraic objects [we] want to sort of exhibit. Often when you're trying to find out something about some object, you will apply some homomorphism to it to understand it, maybe in a simpler context. That's part of the grammar of doing research in algebra.

Isomorphisms were similarly mentioned as useful for shifting perspectives to gain information (e.g., "isomorphisms are used in my research, I would say, to build bridges between two different ways of thinking," Greer).

Generalizations

Here we include generalizations and analogues of isomorphism and homomorphism across different branches of mathematics. Five participants noted that isomorphisms are an example of an *equivalence relation*. Blair uses this idea to talk about isomorphisms broadly in any category.

> So *everything* I do when I talk about two things being equal or the same, there's always some explicit or implicit notion of an equivalence relation. It's up to something. And every equivalence relation gives rise to some notion of isomorphism in the right category.

Blair seems to be speaking about isomorphism in the category theoretical sense here, which includes the algebraic notion.

In a similar vein, all nine participants brought up *other branch analogues* to isomorphism in abstract algebra. Some of these were in response to being asked whether they view isomorphism in a specific context or more broadly. For example, Dallas made connections to analogous concepts in topology: "But the concept of isomorphism… extends kind of beyond just algebra.… So I think of concepts like homeomorphism, diffeomorphism, or homotopy equivalence even as being analogous to isomorphism." Finley also talked about isomorphism existing outside of algebra: "I think most mathematicians will think of [isomorphism] as a general thing across mathematics. But for me, the reason… is because it's a thing in category theory, and then you can apply it in any category." Again, we see the idea of category theory being a way to talk about these concepts in a more general way.

Some *other branch analogues* were also discussed in response to the final interview questions, which specifically asked about these analogues: "Some people answering the survey saw connections between isomorphism/homomorphism and equivalent fractions or congruence/similarity in geometry. Do you agree that there is some level of similarity between these contexts? Do you think it would be helpful to highlight these similarities with students?" Indy compared equivalent fractions to the relabeling conception of isomorphism: "I think equivalent fractions is kind of an interesting notion in the sense that they're the same number, but they're written differently. So it is kind of this idea of, we have different names for the same object." However, they didn't feel like similar triangles were a strong enough analogy to use for homomorphism: "The similar triangles… that one I don't like as much because… you're only zooming in and out. You're not even kind of like folding it… I feel like the similar triangles would give an impression of too much rigidity." This comment brings up the importance of being careful how analogues are used in the classroom, in order to avoid encouraging too narrow or loose conceptions about isomorphism or homomorphism.

Discussion

Isomorphism and homomorphism have a variety of conceptual facets to them. As previously observed, they can be interpreted formally through their definitions, through the lens of sameness, as mappings, and as sameness-focused mappings (Rupnow, 2021). One way we add to prior work is by noting new ways that sameness (e.g., *indistinguishable*) and mapping (e.g., *invertible*, *transformation*) metaphors can manifest. While these metaphors were only used by a few mathematicians, we note that they reveal opposite views of how much sameness is conveyed by isomorphism. For *indistinguishable*, objects linked by the isomorphism are not worth viewing as different—in all important ways they are the same. In contrast, *transformation* emphasizes differences still exist, even for isomorphic objects. Although these perspectives are in tension, they provide complimentary views depending on what is important for a specific context.

Furthermore, even the four original clusters do not fully capture ways in which isomorphism and homomorphism are understood. Here we see formal definitions conveying a type of sameness (formal definition/sameness cluster) through operation-preservation and structure-preservation. Though these metaphors can be used as stand-ins for the homomorphism property, the preservation aspect also highlights the sameness of elements' interactions with each other. We also see ways in which isomorphism and homomorphism are part of a broader system of interconnected ideas, permitting connections to similar concepts in other parts of mathematics (generalizations cluster). These connections can be viewed thematically (*equivalence relations*) or as specific other instantiations (*other branch analogues* like homeomorphism). Finally, the changing perspectives cluster highlights a route for viewing problems in new ways when problem solving. Specifically, these mathematicians working in or near algebra/category theory provide homomorphism a purpose of its own rather than viewing homomorphism as important only for its relationship to isomorphism or for having a tenuous connection to sameness.

Furthermore, the changing perspectives cluster highlights potential routes for future research. For instance, considering these homomorphism purposes did not arise in the prior study and were only noted by four mathematicians here, how prevalent are these notions? Similarly, would math instructors who teach but do not research algebra benefit from explicit conversations and professional development on this topic to make their teaching more relevant? Alternatively, do students find the changing perspectives cluster relevant if they are not interested in pursuing higher level math courses? Further explorations of such connections between instructors' understandings and teaching as well as teaching and students' understandings seem justified.

Finally, this examination of experts' language highlights desirable conceptions for students. Prior work has carefully examined students' use of properties and approaches to determining whether groups are isomorphic (e.g., Dubinsky et al., 1994; Leron et al., 1995) as well as focused on the function nature of isomorphism and homomorphism (e.g., Melhuish et al., 2020). Here we highlight a framework that permits and structures simultaneous examination of both while connecting to analogous topics in other mathematical subfields. Future research could examine the benefits of using particular clusters of metaphors, contexts in which different clusters are optimal, and ways to foster explicit connections among these metaphor clusters in the classroom.

Acknowledgments

This research was funded by a Northern Illinois University Research and Artistry Grant to Rachel Rupnow, grant number RA20-130.

References

Adiredja, A. P., & Zandieh, M. (2020). The lived experience of linear algebra: A counter-story about women of color in mathematics. *Educational Studies in Mathematics, 104*, 239–260. https://doi.org/10.1007/s10649-020-09954-3

Braun, V., Clarke, V., Hayfield, N., & Terry, G. (2019). Thematic analysis. In Liamputtong P. (Ed.) *Handbook of Research Methods in Health Social Sciences.* Springer: Singapore. https://doi.org/10.1007/978-981-10-5251-4_103

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267–305.

Fraleigh, J. B. (2003). *A First Course in Abstract Algebra* (7th ed). Addison-Wesley.

Gallian, J. (2009). *Contemporary Abstract Algebra.* Boston, MA: Cengage Learning.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by.* Chicago: The University of Chicago Press.

Lakoff, G., & Núñez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Larsen, S. (2013). A local instructional theory for the guided reinvention of the group and isomorphism concepts. *The Journal of Mathematical Behavior, 32*(4), 712–725.

Larsen, S., Johnson, E., & Bartlo, J. (2013). Designing and scaling up an innovation in Abstract Algebra. *The Journal of Mathematical Behavior, 32*(4), 693–711.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Melhuish, K. (2018). Three conceptual replication studies in group theory. *Journal for Research in Mathematics Education, 49*(1), 9–38.

Melhuish, K. & Fagan, J. (2018). Connecting the Group Theory Concept Assessment to core concepts at the secondary level. In Nicholas H. Wasserman (Ed.), *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers* (pp. 19–45). Cham, Switzerland: Springer Nature. https://doi.org/10.1007/978-3-319-99214-3_2

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior*, *60*, 100806.

Olsen, J., Lew, K., & Weber, K. (2020). Metaphors for learning and doing mathematics in advanced mathematics lectures. *Educational Studies in Mathematics, 105*, 1–17.

Randazzo, B., & Rupnow, R. (2021). Sameness connections across mathematics. In Karunakaran, S. S. & Higgins, A. (Eds.), *2021 Research in Undergraduate Mathematics Education Reports.* (pp. 238-246).

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education*, (pp. 259–273), San Diego, CA.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. *Journal of Mathematical Behavior*, *62*, 100867. https://doi.org/10.1016/j.jmathb.2021.100867

Rupnow, R., & Johnson, E. (2021). Mathematical sameness: An ill-defined but important concept. In Karunakaran, S. S. & Higgins, A. (Eds.), *2021 Research in Undergraduate Mathematics Education Reports.* (pp. 274-282).

Rupnow, R., & Sassman, P. (2021). Isomorphism and homomorphism as types of sameness. In Olanoff, D., Johnson, K., & Spitzer, S. (Eds.). *Proceedings of the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Philadelphia, PA (pp. 340–348).

Steen, G. (2011). From three dimensions to five steps: The value of deliberate metaphor. *Metaphorik, 21*, 83–110.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics*, *48*(1), 101–119.

Weber, K. (2002). The role of instrumental and relational understanding in proofs about group isomorphisms. In Proceedings from the 2nd International Conference for the Teaching of Mathematics, Hersonisoss, Crete: Wiley & Sons.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics*, *56*(2-3), 209–234.

RELATIONSHIPS BETWEEN DIMENSIONS OF AUTHENTICITY DURING AN INQUIRY-ORIENTED ABSTRACT ALGEBRA ACTIVITY

Holly Zolt
Texas State University
hmz10@txstate.edu

Anthony Tucci
Texas State University
aat80@txstate.edu

Kathleen Melhuish
Texas State University
melhuish@txstate.edu

One of the goals of advanced undergraduate mathematics courses is to engage students in activity that is authentic to the mathematics discipline. However, engaging students in such activity often involves managing tensions between authenticity-to-students and authenticity-to-the-discipline. In this paper, we use the Authentic Mathematical Proof Activity (AMPA) Framework to further explore potential relationships and tensions between different dimensions of authenticity. We analyzed classroom data from an inquiry-oriented abstract algebra course where instruction focused on unpacking the fundamental homomorphism theorem. Our results focus on the complexity dimension of authenticity and how this dimension relates to other dimensions of authenticity within instruction. We identify ways that instructor decisions shape authenticity even within the context of a carefully developed task.

Keywords: Undergraduate Education, Reasoning and Proof, Instructional Activities and Practices

Many mathematics educators value engaging students in “authentic” mathematical activity. The use of the term authentic often reflects ties to the work of the discipline where the goal is to engage students in activity reflective of research mathematicians (e.g., Watson, 2008). However, what constitutes authentic activity and the degree to which such activity should be the goal of school mathematics is an unsettled topic (Weber & Dawkins, 2020). Further, engendering students in authentic activity is often subject to intrinsic tensions between authenticity to students and authenticity to the discipline (e.g., Melhuish et al., 2021; Ball, 1993; Lampert, 1992; Herbst, 2002). A well-documented tension is that between authenticity-to-the-discipline (as in Weiss et al. 2009) in terms of accuracy of content and alignment with student contributions that often diverge from the norms of the discipline (e.g., Chazan & Ball, 1999). Herbst (2002) has further illustrated dilemmas in proof courses where there is a double bind on the teacher to progress a class in normative ways related to proof argumentation while also staying authentic to the contributions and activity of students in class. As noted in Dawkins et al.’s (2019) analysis of an inquiry-based instructor, this tension remains salient at the advanced undergraduate level.

In Melhuish and colleague’s (2021) recent work, they have suggested a need to better operationalize authenticity for the context of proof-based courses. Drawing on diverse design-based research projects, they built on Weiss et al.’s (2009) initial decomposition of authenticity to suggest both student/discipline dimensions and practice/content dimensions of authenticity that are at play in the advanced proof-based settings. In this paper, we build directly on this work by adapting this framework (developed in the settings of task-based interviews) to the classroom setting to better explore how different dimensions of authenticity may align or diverge in order to better understand instruction in such classes. For the scope of this paper, we forefront elements of practice, that is the nature of activities, as they have been less explored than their parallels in relation to content.

Our study is situated in the context of abstract algebra, a course taken by mathematics majors and pre-service secondary teachers. The advanced undergraduate setting is often a place where students are apprenticed into the work of research mathematicians with a focus on formal proving. It is well-documented that students often struggle to grasp the abstract concepts that are central to the course (e.g., Dubinsky et al., 1994; Melhuish et. al, 2019). As a result, there has been a growing body of research on how to improve the teaching of abstract algebra and a continued development of inquiry-oriented abstract algebra curriculum (e.g., Larsen et al., 2013) with a focus on engaging students in authentic activity. We are building on such work by elaborating on the tensions and relationships amongst competing authenticity goals observable in instruction.

Theoretical Framework

Drawing from activity theory (Engeström, 2000), we frame both students and instructors as operating within activity systems. These activity systems relate goal-driven actions to how members of a community work together toward shared goals. The assumption underlying our work is that advanced mathematics courses provide students with an opportunity to engage in activities that align with the activities of research mathematicians. Advanced mathematics courses may provide students an opportunity to use tools (e.g., examples, warranting, deformalizing) to meet objectives that can be deconstructed into motives (e.g., explore, test, construct) in reference to an object (proof, statement, concept). Evidence of participatory learning can be seen through expansions in activity within an activity system. For example, students may introduce tools, thus adding more variety and increasing their role in the division of labor.

We draw from the Authentic Mathematical Proof Activity (AMPA) framework (Melhuish et al., 2021) which was developed to capture components of student activity that reflect the work of research mathematicians. The framework includes ten tools: analyzing/refining, formalizing, deformalizing, warranting, analogizing/transferring, examples, diagrams, logic, structure/frameworks, and existent objects. These tools are used alongside three motives: constructing, exploring, and testing, in reference to three objects: proofs, statements, and concepts. Authenticity is operationalized across six dimensions to account for different, sometimes competing, notions of authenticity. Notably, the three dimensions: variety, complexity, and accuracy reflect the discipline and stem from analysis of mathematician activity while agency, authority, and alignment reflect features of authenticity to student activity and contributions. See Table 1 below for the dimensions of the Authentic Mathematical Proof Activity (AMPA) framework along with our elaboration of four levels within each dimension.

Table 1: Dimensions of Authenticity

Dimension of Authenticity	Levels
Variety: Degree of variation within tools used: formal, informal, generative, translating	Low: Only one type of tool at play Low-Mid: Two of the four types of tools at play Mid-High: Three of the four types of tools at play High: Informal, formal, translating, and generating tools at play
Complexity: Degree in which tools are used in isolation or in conjunction	Low: Single tool Low-Mid: A variety of tools are used in isolation Mid-High: Many tools used in conjunction

(and shifts from outcomes to tools)	High: Objects shift to tools
Accuracy: Degree in which the tools used would be accurate within the mathematical community	Low: Inaccurate Low-Mid: Mixed - correct tools but incorrect outcome Mid-High: Similar but imprecise High: Tools and outcomes are accurate
Alignment: Degree in which tools aligned with student contributions (whose tools are endorsed)	Low: Teacher Contributions Low-Mid: Mostly Teacher/ Some Student Contributions Mid-High: Mostly Student Contributions/ Some Teacher or Refined Student Contribution High: Tools and Outcomes are Student Contributions
Agency: Degree in which students generate tools	Low: Teacher generates/uses Tools Low-Mid: Teachers generate tools and students Use Tools Mid-High: Students Generate Tools (prompted) High: Students Generate Tools (unprompted)
Authority: Degree in which students determine how tools/ outcomes connect (validity of tools/ outcomes at play)	Low: Instructor links & explains Low-Mid: Instructor links & students explain Mid-High: Students link & explain (sometimes) High: Students link & explain (mostly)

Background: Fundamental Homomorphism Theorem and Quotient Groups

The focal task in our study involves students exploring the proof of the Fundamental Homomorphism Theorem (FHT). Both the FHT and quotient groups are key topics in an abstract algebra curriculum, but they are also two of the most difficult topics for students to understand (Melhuish et al., 2021). The FHT (see figure 1 below) involves both a homomorphism and an isomorphism to show that the quotient group is isomorphic to the image of the homomorphism.

Literature suggests that students may struggle to coordinate the homomorphism and isomorphism in the FHT (Nardi, 2000). Nardi (2000) elaborated that mathematical abstraction is particularly challenging in such proof settings. For students to productively engage with this theorem and proof, they need to coordinate a number of abstract mathematical objects including functions and quotient groups. Yet, Hazzan (1999) suggests that students often try to create less abstract environments for themselves by relying on things like the coset algorithm which in turn hides the structure of quotient groups. With the inherent challenges of abstract functions (e.g., Melhuish et al., 2021, year) and quotient groups (e.g., Dubinsky et al., 1994), we anticipate substantial opportunities to study authenticity dimensions where student contributions may often be in tension with disciplinary norms.

Theorem 1 (The First Isomorphism Theorem or the Fundamental Homomorphism Theorem).
If $\phi : G \to H$ is a group homomorphism, then

$$\frac{G}{\ker\phi} \cong \phi(G).$$

Proof. First, we note that the kernel, $K = \ker\phi$ is normal in G.

Define $\beta : \frac{G}{K} \to \phi(G)$ by $\beta(gK) = \phi(g)$. We first show that β is a well-defined map. If $g_1K = g_2K$, then for some $k \in K, g_1k = g_2$; consequently,

$$\beta(g_1K) = \phi(g_1) = \phi(g_1)\phi(k) = \phi(g_1k) = \phi(g_2) = \beta(g_2K).$$

Thus, β does not depend on the choice of coset representatives, and the map $\beta : G/K \to \phi(G)$ is well-defined.

We must also show that β is a homomorphism:

$$\begin{aligned}\beta(g_1Kg_2K) &= \beta(g_1g_2K)\\ &= \phi(g_1g_2)\\ &= \phi(g_1)\phi(g_2)\\ &= \beta(g_1K)\beta(g_2K).\end{aligned}$$

Clearly, β is onto $\phi(G)$. To show that β is one-to-one, suppose that $\beta(g_1K) = \beta(g_2K)$. Then $\phi(g_1) = \phi(g_2)$. This implies that $\phi(g_1^{-1}g_2) = e$, or $g_1^{-1}g_2$ is in the kernel of ϕ; hence, $g_1^{-1}g_2K = K$; that is, $g_1K = g_2K$. □

Figure 1: Statement and Proof of the FHT

Methods

The data for this report was collected at a large public university in the United States as a part of a design-based research project focused on orchestrating discussion around proof. More specifically, the data comes from an activity in which students were tasked with engaging with the Fundamental Homomorphism Theorem. This activity spanned one and a half class periods. On the first day, students worked in small groups on two tasks. For the first task, each group was given a different homomorphism between groups and asked to draw a function diagram where the homomorphism, kernel, cosets, and isomorphism were all labeled. This task concluded with a whole class discussion.

The second task involved unpacking the proof of the FHT. Students were first tasked with partitioning the proof into sections based on the property being proved (well-defined, homomorphism property, one-to-one, and onto). Since there were four groups, each group was assigned a section and given a set of questions about their section of the proof to facilitate a small group discussion. After each small group spent some time discussing their section of the proof, a representative from each group went to the front of their class to present their answers to each question and discuss their overall understanding of their section of the proof. The presentations occurred during the following class period.

This activity was video recorded, audio recorded, and transcribed. Using the video data from the two class sessions, we segmented the class into segments based on activity. A new segment was created when shifts in conversation such as a new topic of conversation or more major changes such as whole class to small group discussion occurred. The coding was done in two stages by two of the authors. The first stage was a trial coding of two segments, the second stage consisted of coding the rest of the segments. Using the AMPA framework, 21 segments with an average length of 6 minutes and 40 seconds were coded along six dimensions and received a code of low, low-mid, mid-high, or high. Any discrepancies between the two coders were resolved through discussion.

After we coded our data, we looked for patterns and trends throughout the coded data to find relationships and conflicts. We identified any segments that conflicted with the general trends of the data set. We discuss the findings of our analysis below.

Results

For this report, we forefront the dimension of complexity in order to better understand the relationships between complexity and other dimensions of authenticity that emerged from our analysis. Complexity captures the degree to which a set of tools are used interrelatedly to make progress towards an outcome. We use quotes and descriptions of the video data to provide context for each relationship. Note that all students have been assigned a pseudonym in these segments.

Before discussing the relationship between complexity and the other dimensions of authenticity, we present a figure (Figure 2) that summarizes the authority dimension levels across our segments. The grey boxes correspond to small group segments (focusing on the small group reported later in the results) while the white boxes are whole class segments. We note that we transition between lessons at the 80-minute mark. We can observe that authenticity dimensions are infrequently all at the high-level, with both discipline (variety, complexity, accuracy) and student (agency, authority, alignment) varying in different segments.

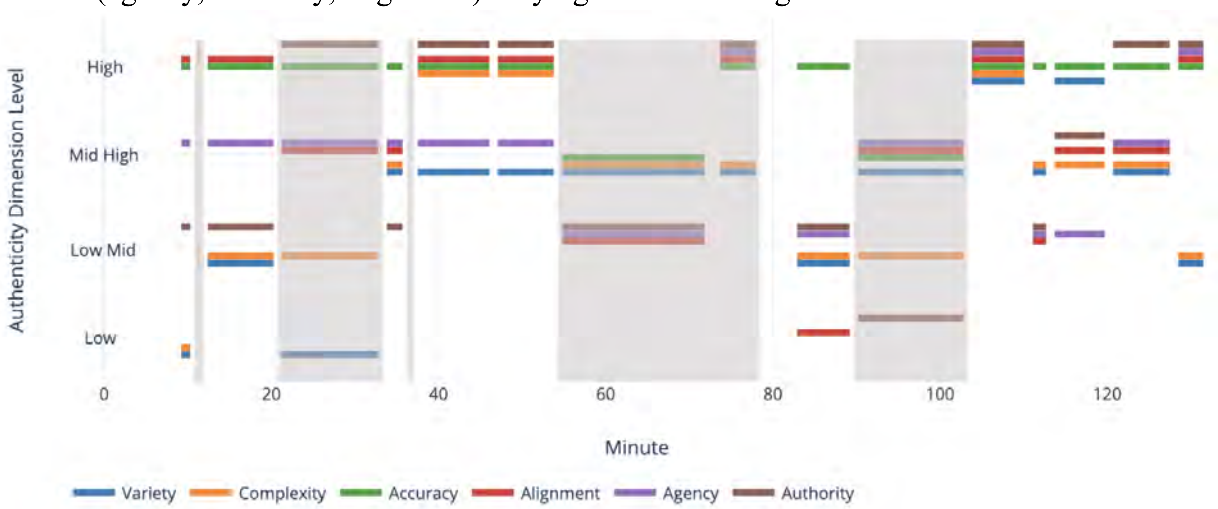


Figure 2: Coding Over Time By Whole Class And Small Group 1

Variety, a Necessary, but Not Sufficient Condition for Complexity

From a simplistic view, complexity and variety are interrelated. In order for multiple tools to be used in conjunction (high complexity), there must first be multiple tools available to use (high variety). For example, consider the following quote from the end of a segment where a student, Joe, is explaining a portion of the proof to his classmates:

> And then the last thing, which I hate. Oh, I hate this line so much is, what does it mean that beta does not depend on the choice of coset representatives, which is a really horrible way of saying that something's "well-defined". So, basically to translate what this means... I didn't know what it meant when I first saw it. A coset representative is like, if you think about our group over here, 1 and 4 would both be like coset representatives that are equal, right? Because they map to the same thing when paired with a kernel. And in our actual proof here, g_1 and g_2 are our coset representatives. And what we're trying to show in this proof is that it doesn't matter what you call them. Doesn't matter which one's being used, as long as they are equal, you will get the same result,

Joe generated several different types of tools in this segment. First, he is deformalizing when he translates the formal language in the proof to more informal language. Then, he is warranting by referring to an example, and formalizing by relating the example back to the formal proof. The variety of tools that were generated by Joe, as well as the tools being used in tandem, resulted in the segment having high levels of both variety and complexity. Across the lesson, students often generated tools with the intention of using them in conjunction with other tools. Similarly, with few tools at play (low variety), there are fewer opportunities for complex usage (low complexity.) Overall, levels of variety and complexity corresponded (both higher or both lower) in 20 of the 21 total segments.

A Deviation for the Trend of Variety and Complexity Co-Occurrence - and a Tension with Authority.

While most of the instances included variety and complexity playing out in tandem, one segment provided counter evidence to this trend. A small group of students was tasked with making sense of the onto portion of the FHT proof. Throughout this segment, students generated many different tools. For example, they referenced an example of a homomorphism between two groups, as well as deformalized the formal definition of onto by stating “everything in the codomain gets hit”. Despite generating a variety of tools, the students did not end up using any of the tools in tandem. This led to them having trouble making progress toward unpacking and understanding their portion of the proof. The instructor intervened and stated, “So, actually you have everything... So, let’s make sense of this”, and attempted to prompt students to begin using some of the tools they generated in tandem. This deviation from the trend we noticed provides evidence that students being able to successfully generate tools does not guarantee that they will use them together. The intervention by the instructor increased the level of complexity for the segment (tools began being used in tandem) but decreased authority (the instructor was primarily the one linking tools to outcomes – that is, evaluating that they had the right tools). Without the instructor’s intervention, the students appeared at an impasse in their activity. By decreasing authority (a student-related dimension), the instructor promoted increased complexity (a discipline-related dimension). Thus, this case suggests a tension between these dimensions, since linking between tools and outcomes often requires using a variety of tools in tandem.

Decreasing Agency to Increase Variety and Complexity

The final relationship we observed was that levels of agency related to levels of both variety and complexity. There were several instances where the students were not generating very many tools. If the instructor was not present, then this would necessarily lead to both low variety and complexity, because there would not be very many tools at play. However, it was often the case that if the instructor noticed that students were not generating many tools, they would step in to introduce a new tool, and sometimes even use the tool in tandem with another tool. This move by the instructor resulted in increasing both variety and complexity for the segment but decreasing agency –as the tools were no longer being generated strictly by the students. Consider the following exchange where one of the members of the group from the previous section (Nick) is presenting their explanation of the onto portion of the proof. The student started by writing out what was known: φ is onto. Then the student proceeded to attempt to explain why this necessarily meant that β was onto.

> Nick: So, now it's time for us to address β. And we were given the definition for β, what it is. And I think if I remember it correctly, it has something to do with the kernel. Do you guys remember what the definition for β is? How is β defined?
>
> Joe: Beta of G times the kernel equals □(□)?
>
> Nick: Yeah. So, it's elements in G operated with the kernel, right?
>
> Joe: Which are quotient group elements.
>
> Nick: Yeah. So, would it be too much of a jump to say that □(□)is equal to □(□) operated with K?
>
> Joe: You are allowed to do that
>
> Nick: And why am I allowed to do that? I'm actually asking you.
>
> Joe: That's the definition of beta
>
> Nick: Okay
>
> Kevin: Because why is it □(□). Right so, it is an element in the big □(□), which we just said was equal to beta of little gK or in this case, little x big K. So, you've just replaced Y with that.

Nick then notes that his classmates' contributions helped him "understand a little better" but still does not think he has the "best grasp on it." After this comment, the instructor states they have a "general insight on what is going on with the onto," but introduces a diagram (a new tool), to think through the argument. The introduction of a diagram of a homomorphism from $\mathbb{Z}_9$ to $\mathbb{Z}_6$ increased variety, since a new type of tool (informal) was being discussed. Additionally, the instructor used the tool in reference to existent objects (homomorphism, image) to warrant why the section of the proof was true. Thus, not only was variety increased, but complexity was increased as well. However, since the new tool being used was generated by the instructor, agency was decreased. This example shows that when a limited number of tools are at play and are being used in isolation, the instructor will often step in and introduce a new tool, as well as use that new tool in tandem with other tools. This results in increasing variety and complexity but decreasing agency.

Discussion

The AMPA framework was designed to document how students engage in authentic mathematical activity in a proof-based setting. Through our work, we have expanded the use of this framework to the classroom setting. We note that the course that we examined was an inquiry-oriented class which, in many ways, was naturally designed to foster an environment where students are engaging in many of these activities. Social norms within inquiry classes often include the expectation that the students are engaging actively in a disciplinary activity. Such a setting provides a robust opportunity to explore tensions in authenticity dimensions;

however, we caution that we cannot generalize these dimension relationships to a more traditional classroom.

Due to the inquiry nature of the classroom, the presence of the instructor varies from segment-to-segment. This influenced the levels of the dimensions of authenticity because when the instructor was not present in a small group, the dimensions of alignment, agency, and authority (presuming activity was occurring) all defaulted to high. The reason for this was because the instructor did not have an opportunity to prompt the students to introduce additional tools or give input on the validity of outcomes. In this paper, we shared three segments where the instructor was present throughout to better explore relationships between dimensions of authenticity.

If we turn back to the literature on authenticity, we can see evidence of the underlying tension between authenticity-to-students and authenticity-to-the-discipline found in the K-12 literature (e.g., Chazan & Ball, 1999; Lampert, 1992). The overarching goal for this abstract algebra lesson involved students engaging in authentic activities related to comprehending a theorem and a proof. In order to do this, the students needed to use a variety of tools in complex ways to make sense of a rather abstract theorem and proof. Variety was a necessity for complexity, and when variety was low, the instructor sometimes limited agency in order to introduce new tools. Further, variety did not assure complexity, and we observed the instructor lowering authority to assist students in connecting their tools to outcomes.

We also anticipate that the instructor's values and beliefs affected the authenticity profile of this class. For example, in the last exchange, the instructor introduced a diagram (which increased variety but decreased agency). We can reasonably conjecture this was for the purpose of supporting students' understanding of the onto portion of the proof. In other words, at this moment, the instructor appeared to value the students' engaging with a diagram more than ensuring that all of the tools at play were generated by the students. Further, this explanation means the increase in a disciplinary dimension was not necessarily motivated by authenticity intentions, but a consequence of other pedagogical intentions – attending to students' understanding.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. DUE-1836559. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. The Elementary School Journal, 93(4), 373–397.

Chazan, D., & Ball, D. (1999). Beyond Being Told Not to Tell. For the Learning of Mathematics, 19, 2-10.

Dawkins, P. C., Oehrtman, M., & Mahavier, W. T. (2019). Professor goals and student experiences in traditional IBL Real Analysis: A case study. International Journal of Research in Undergraduate Mathematics Education, 5(3), 315-336.

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On Learning Fundamental Concepts of Group Theory. Educational Studies in Mathematics, 27(3), 267-305.

Engeström,, Y. (2000). Activity theory as a framework for analyzing and redesigning work. Ergonomics, 43(7), 960-974.

Hazzan, O. (1999). Reducing Abstraction Level When Learning Abstract Algebra Concepts [research-article]. Educational Studies in Mathematics, 40(1), 71-90.

Herbst, P. G. (2002). Engaging students in proving: A double bind on the teacher. Journal for Research in Mathematics Education, 33(3), 176-203.

Lampert, M. (1992). Practices and problems in teaching authentic mathematics. In F. Oser, A. Dick, & J. L. Patry (Eds.), Effective and responsible teaching: The new synthesis (pp. 295–313). Jossey-Bass.

Larsen, S., Johnson, E., & Weber, K. (2013). The teaching abstract algebra for understanding project: Designing and scaling up a curriculum innovation. Journal of Mathematical Behavior, 32(4), 691-790.

Melhuish, K. (2019). The Group Theory Concept Assessment: a Tool for Measuring Conceptual Understanding in Introductory Group Theory. *International Journal of Research in Undergraduate Mathematics Education, 5*(3), 359.

Melhuish, K., Vroom, K., Lew, K., & Ellis, B. (2021). Operationalizing authentic and disciplinary activity for the undergraduate context. In D. Olanof, K. Johnson, & S. M. Spitzer (Eds.), Proceedings of the forty-third annual meeting of the North American chapter of the international group for the psychology of mathematics education (pp. 349–385). Philadelphia, Pennsylvania.

Nardi, E. (2000). Mathematics Undergraduates' Responses to Semantic Abbreviations, 'Geometric' Images and Multi-Level Abstractions in Group Theory. Educational Studies in Mathematics, 43(2), 169-189.

Watson, A. (2008). School mathematics as a special kind of mathematics. For the Learning of Mathematics, 28(3), 3-7.

Weber, K., Dawkins, P., & Mejía-Ramos, J. P. (2020). The relationship between mathematical practice and mathematics pedagogy in mathematics education research. ZDM 52, 1063-1074.

Weiss, M., Herbst, P., & Chen, C. (2009). Teachers' perspectives on "authentic mathematics" and the two-column proof form. Educational Studies in Mathematics, 70(3), 275-293.

WHAT UNDERGRADUATE STUDENTS MAKE DECISIONS ABOUT WHEN COLLABORATIVELY PROVING

Jordan Kirby
Middle Tennessee State University
jek3b@mtmail.mtsu.edu

Samuel D. Reed
Lander University
sreed1@lander.edu

Sarah K. Bleiler-Baxter
Middle Tennessee State University
sarah.bleiler@mtsu.edu

Keywords: Reasoning and Proof, Undergraduate Education

Inquiry-based classes provide an opportunity for authority (i.e., the source(s) of legitimacy within a classroom) to be shared among the members of the classroom rather than simply concentrated within the instructor or textbook (Amit & Fried, 2005; Gerson & Bateman, 2010). Inquiry-based classes focused on mathematical proof provide a context to investigate the nature of students' decision-making during the proving process (Bleiler-Baxter et al., 2021; Langer-Osuna, 2016). While mathematics educators espouse the importance of "allow[ing] students ample opportunities to be the bearers of granted authority" (Gerson & Bateman, 2010, p. 206) in mathematics classrooms, we know less about students' tendencies when granted such authority. In this study we seek to gain further insight into student decision making during collaborative group proving. We ask the following research question (Bleiler-Baxter et al., under review): To what aspects of proof do students in an undergraduate inquiry-based introduction to proof course attend when granted authority to co-construct mathematical proofs within small groups?

This study was conducted in an inquiry-based introduction to proof course in the Spring 2020 semester. Students were placed in small groups of approximately three students and were asked to co-construct group proofs for various claims over the course of the semester. During these small-group sessions, students received minimal feedback/guidance from the instructor and had to make decisions about the best ways to spend their time to develop a final proof product they would then present to the whole class. We collected nine video episodes of small-group collaborative proving before we halted data collection due to the instructional transitions caused by the COVID-19 pandemic. We adapted Stylianides' (2007) framework to define three aspects of proof to which students may attend when making decisions in their small-groups: set of accepted statements (S), modes of argumentation (A), and modes of argument representation (R). We coded the episodes by talk-turn using the S, A, R framework, and then met as a team to come to a consensus on our coding.

This poster will present findings from our research question and how we operationalized Stylianides' (2007) framework to serve as an analytic coding framework. We believe this operationalization presents a powerful tool for future research in proof that will allow researchers to dissect how students discuss and make decisions about aspects of the proving process. Further, we will use the poster as a venue to discuss some of the theoretical implications of this coding scheme. For example, there were times when single talk-turns were coded with multiple aspects of the Stylianides framework. These instances of double-coded talk-turns, such a talk-turn coded with both an "S" code and an "A" code, often indicated times when students expressed their arguments (A) with more precision due to their reliance on accepted statements (S). We believe that a poster session will serve as a valuable setting for us to engage in conversation with other mathematics education researchers who may have ideas for directions of future related work and implications for both research and practice in proof-based mathematics courses.

References

Amit, M., & Fried, M. N. (2005). Authority and authority relations in mathematics education: A view from an 8th grade classroom. *Educational Studies in Mathematics, 58*(2), 145–168. http://10.1007/s10649-005-3618-2

Bleiler-Baxter, S. K., Pair, J. D., & Reed, S. D. (2021). Let's create, not replicate: Supporting students as proof creators through the group proof activity. *Problems, Resources, and Issues in Mathematics Undergraduate Studies (PRIMUS), 31*(7), 760-774.

Bleiler-Baxter, S. K., Kirby, J. E., & Reed, S. D. (Under Review). Understanding authority in small-group co-constructions of mathematical proof.

Gerson, H., & Bateman, E. (2010). Authority in an agency-centered, inquiry-based university calculus classroom. *Journal of Mathematical Behavior, 29*(4), 195–206. http://:10.1016/j.jmathb.2010.10.003

Langer-Osuna, J. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem solving. *Mathematical Thinking and Learning, 18*(2), 107-124.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321.

Chapter 11:

Pre-Service Teacher Education

EXPLORING THE RELATIONSHIP BETWEEN PRESERVICE TEACHERS MATHEMATICAL NOTICINGS AND THEIR 360 VIDEO VIEWING

Christine K. Austin — Kent State University — causti17@kent.edu

Jennifer Heisler — Kent State University — Jheisle4@kent.edu

Karl W. Kosko — Kent State University — kkosko1@kent.edu

This study discusses the difference in a preservice teachers' (PSTs) mathematical noticings and the relationship between where they attend in a 360 video. Using a convergent mixed methods approach, our evidence provides support that PSTs who are more explicit in their written noticing of mathematical references are more focused in their viewing of a 360 video on a group of students or area of the classroom than those who look around more often.

Keywords: Teacher Noticing; Technology; Preservice Teacher Education.

Overview & Purpose

Teacher noticing is a critical skill amongst mathematics teachers (Jacobs, 2010; Sherin, 2005). Additionally, attending to students mathematical reasoning and thinking is vital to effective teaching (AMTE, 2017). This attention to a students' mathematical reasoning and the ability to interpret a students' thinking is a subset of professional teacher noticing (Jacobs, 2010; Sherin, 2005). The use of video in preservice teachers (PSTs) methods courses has been shown to support teachers' growth in professional noticing (van Es et al., 2017; Roche & Rolland, 2020). Recently, mathematics educators have begun to use and study 360 videos in teacher noticing, which record an omnidirectional view of the classroom. Evidence suggests that 360 video facilitates a PSTs' development of professional noticing (Kosko et al., 2021b; Roche & Rolland, 2020). PSTs who look around the recorded classroom more often in a 360 video attend to more generic student actions in their written noticing (Gold & Windscheid, 2020; Huang et al., 2021b). However, many of these studies have focused on attending to aspects of classroom behavior, and not to more content-specific facets. The purpose of this study is to examine the relationship between PSTs' mathematical noticing and how they view 360 videos.

Background Literature & Theoretical Perspectives

Professional noticing has been studied extensively in the context of mathematics education (Jacobs, 2010; Sherin, 2005). Professional noticing involves the three interrelated actions of attending to, interpreting of, and deciding how to respond to pedagogical events (Jacobs, 2010; Sherin & Russ 2014). Novice teachers tend to initially attend to students' generic actions (i.e., participation) and behavior, or focus on the teacher's instructional activities. By contrast, more experienced teachers attend more to students' mathematical thinking (Huang & Li, 2012; Kosko et al., 2021b). Mathematics teacher noticing can be facilitated and is the aim of many teacher education programs. However, this noticing should be structured, with an early focus of teacher educators to develop professional noticing skills (Jacobs, 2010; Star & Strickland, 2008).

In characterizing PSTs' noticing practices, and expanding upon prior conceptualizations (i.e., Jacobs et al., 2010), van Es et al., (2017) described a trajectory of noticing from generic to specific descriptions of students' mathematics. The noticing of mathematical thinking relies on specific strategies identified within students' mathematical development (Jacobs et al., 2010). In other words, the mathematics content is conveyed more precisely and defined with a higher degree of noticing. This is consistent with other findings that suggest that PSTs who are attending to other classroom happenings such as ‘student investigation’ or ‘group work’ did not describe specific mathematical content even when prompted to do so (Kosko et al. 2022).

The emerging scholarship on teachers’ noticing through 360 video suggests a relationship between teachers’ linguistic descriptions and their actual viewing behavior (Kosko et al., 2022; Weston & Amador, 2021). Much of the analysis on this phenomenon has been limited to screen recordings of PSTs’ field of view (Gold & Windscheid, 2020; Huang et al., 2021b; Kosko et al., 2021b) or interviews with PSTs re-viewing 360 videos of their own teaching (Balzeretti et al., 2019; Buchbinder et al., 2021; Weston & Amador, 2021). We conjectured that PSTs’ perceptions of where they focused attention may be an additional lens in understanding their experiences within a 360 video of a classroom. With this focus, we sought to answer:

How do PSTs’ variance in perceived focus in viewing a 360 video associate with the specificity of their professional noticing of students’ mathematics?

Method

Sample and Procedure

The sample included 47 undergraduate students who were enrolled in an educational technology course in a Midwestern University. Students enrolled in this course were required to fulfill a credit, which could be completed by participating in this study. Much of the sample identified as white (n = 43), female (n = 41), and early childhood education majors (n = 28).

PSTs were asked to record their viewing of the 360 video. The video was a 5 minute, 50 second excerpt of a fourth-grade lesson on reviewing equivalent fractions. In the video, students were provided fraction strips and tasked with finding equivalent fractions for 5/6 and then 3/8. After viewing the video initially, participants reported where they focused their attention most by marking up to 10 red Xs on a provided classroom map (see Figure 1). They then were asked to “describe any and all pivotal moments you noticed during the lesson (i.e., any moment you believe is important for teaching and/or learning of mathematics).” Next, PSTs recorded their viewing sessions and rewatched the fourth-grade lesson. After viewing, PSTs again self-reported where they focused their attention most before being asked to focus on “ONE pivotal moment” in the 360 video and describe why it was important for the teaching and/or learning of mathematics. For the purposes of this initial analysis of data, we focused on analysis of PSTs second viewing of the 360 video.

Analysis and Results

We utilized a convergent mixed methods design to examine a relationship between where PSTs reported to have attended in the 360 video on a classroom map (i.e. self-reported heat map) and their written mathematical noticing. Convergent mixed methods design involves independently analyzing both qualitative and quantitative data before merging them to better understand a given phenomenon (Creswell & Plano Clark, 2011). For this analysis, we focused on PSTs’ written attending and qualitatively analyzed writing using an adaptation of van Es et al.’s (2017) rubric for mathematics specificity. Concurrently, the self-reported heatmaps categorized along designated regions (see Figure 1). This categorical data was then used to calculate an unalikeability statistic—a non-parametric statistic of variance in nominal data. Qualitative codings were quantitized into an ordinal variable and unalikeability statistics were compared using an independent sample Kruskal-Wallis test. This allowed for examination of how variance reported aligned with the specificity of their mathematical noticing.

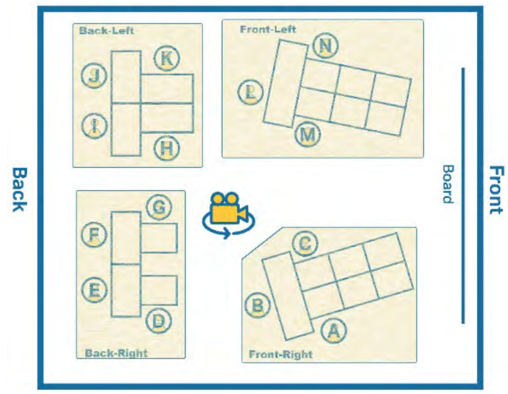


Figure 1: Classroom map for the 360 video with regions highlighted.

Qualitative analysis of PSTs' noticing. We used a coding framework for PSTs' attending to students' mathematics that was developed by van Es et al. (2017). For specificity to mathematics content, van Es et al. (2017) included three hierarchical levels: little or no attention to the mathematics in the lesson; noting the specific task or problem; and attending to / inferring from the task to the broader learning goal in the mathematics lesson. Our own adaptation included: 1) No reference to the mathematics in the lesson (i.e., equivalent fractions); 2) Generic or vague reference to the mathematics in the lesson; and 3) Reference to mathematics of the lesson. Excerpts of three participants are illustrated in Table 1 to help illustrate the difference in codes. The second and third author independently analyzed participants' written noticing using van Es et al.'s (2017) adapted framework. Findings indicated substantial agreement (K=0.714) before the authors reconciled analysis of writing. Across the sample, 14.9% of participants had no reference to the mathematics, 42.6% included generic references, and 42.6% included mathematical references in their written noticing.

Table 1: Example Excerpts of Attending to Mathematics by Classification of Focus.

No Reference	Generic Reference	Mathematical Reference
All the students are interacting and doing the activity. This allows the students to see multiple answers. It also allows them to see multiple ways of doing the question. Because the teacher is asking the students questions and asking for their answers it makes the learning experience better.	The students figured out that they could not represent the fractions because they were not on the fraction papers.	One pivotal point I noticed was when the teacher gave them a fraction in which they couldn't use their fraction strips. I think it allowed them to strengthen their multiplication skills as well as broaden their ideas of how to find similar fractions.

Quantitative analysis and results. Qualitative findings for PSTs' attending to mathematics were quantitized into ordinal data (0=No Reference; 1=Generic Reference; 2=Mathematical Reference) for use in quantitative analysis. Additionally, PSTs self-reported heat map data was coded as an unalikeability statistic (U_2). In this study, a U_2 statistic of 0 would indicate that the PST is not looking around the room and is only attending to one specific region (e.g. the front-left), whereas a 1 would indicate that a PST was looking across all regions of the room for an equal amount of time.

We used an independent sample Kruskal-Wallis test to compare U_2 statistics between PSTs who provided no reference, a generic reference, or a mathematical reference in their written noticing. The median U_2 statistic indicated a much higher degree of unalikeability (U_2 = 0.7456) for the no mathematical reference group than the generic ($U_2 = 0.6803$) or mathematical reference groups (U_2 = 0.6779). These differences were found to be statistically different ($H(2)$ = 6.291, p = 0.43). A Dunn post-hoc test indicated a statistically significant difference in the U_2 statistics between the mathematical reference group and the no mathematical reference group (p = 0.013), as well as between the general reference group and no mathematical reference group (p = 0.039). This difference may be due to the no mathematical reference group having a larger U_2 statistic, which indicates that they perceived focusing on a larger spread of the classroom than the other groups. These findings suggest that participants who perceive they focus on more of the classroom have less specificity in their professional noticing of students' mathematics.

Discussion

The reports of this study are preliminary but contribute to emerging research that the use of 360 video aides in the facilitation of professional noticing (Buchbinder et al., 2021; Walshe et al., 2021; Weston & Amador, 2021). The findings in this study support and extend observations in prior literature that PSTs have less specific noticings when they look around the 360 video frequently (Ferdig et al., 2020; Kosko et al., 2021b, Kosko et al., 2022). The results from our analysis indicates that PSTs with higher U_2 statistics tended to make no mathematical references in their written noticing. Results also suggest the possibility of a trend regarding specificity and focus, but additional data and future research are needed. These results align the work of Kosko et al. (2021b), which noted that PSTs tend to be more conceptual in their writing of mathematical reasoning when they focus on a set of students in the classroom for longer periods of time. Thus, a high variability of PST viewing may lead to PSTs attempting to focus on too many things and miss more specific pivotal moments within a classroom, such as student actions, mathematical discussions, and/or student reasoning. While the differences were not statistically significant between the mathematical reference and general reference groups, average U_2 statistics in these groups do suggest a trend.

Although analysis is ongoing, this study provides several important implications. First, mathematics teacher educators seeking to integrate 360 video may wish to scaffold PSTs' use to encourage more focus on limited number of students. Advocated by Weston and Amador (2021), this focusing may include tasking PSTs with focusing on key locations for a set period of time before looking elsewhere, and providing specific prompts associated with their viewing. A second key implication is the potential for using self-reported focusing as a possible alternative to manual analysis of PSTs' screen recordings (Zolfaghari et al., 2020). Finally, the current paper provides further evidence towards the notion of attending as a subset of perception (van Es & Sherin, 2021). Those teachers better able to be selective in what they perceive demonstrate more sophistication in how they describe what occurred in the classroom. It is possible that PSTs with lower U_2 statistics tend to be better at disregarding selected features of pedagogy than their peers with higher U_2 statistics. However, future research is needed to better understand this interplay and how it may best be used to inform mathematics teacher education for professional noticing.

Acknowledgments

Research reported here received support from the National Science Foundation (NSF) through DRK-12 Grant #1908159. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

References

Association of Mathematics Teacher Educators [AMTE]. (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Balzaretti, N., Ciani, A., Cutting, C., O'Keeffe, L., & White, B. (2019). Unpacking the potential of 360 degree video to support pre-service teacher development. *Research on Education and Media, 11*(1), 63-69. https://doi.org/10.2478/rem-2019-0009

Buchbinder, O., Brisard, S., Butler, R., & McCrone, S. (2021). Preservice secondary mathematics teachers' reflective noticing from 360-degree video recordings of their own teaching. *Journal of Technology and Teacher Education, 29*(3), 279–308. https://www.learntechlib.org/primary/p/219593/paper_219593.pdf

Creswell, J. W., & Plano Clark, V. L. (2011). Designing and conducting mixed methods research (Second edition.). SAGE Publications.

Dessus, P., Cosnefroy, O., & Luengo, V. (2016). Keep your eyes on 'em all!: A mobile eye-tracking analysis of teachers' sensitivity to students. In *European Conference on Technology Enhanced Learning* (pp. 72-84). New York: Springer. https://doi.org/10.1007/978-3-319-45153-4_6

Ferdig, R. E., & Kosko, K. W. (2020). Implementing 360 video to increase immersion, perceptual capacity, and noticing. *TechTrends, 64,*849-859. https://doi.org/10.1007/s11528-020-00522-3

Gold, B., & Windscheid, J. (2020). Observing 360-degree classroom videos–Effects of video type on presence, emotions, workload, classroom observations, and ratings of teaching quality. *Computers & Education*, *156*. https://doi.org/10.1016/j.compedu.2020.103960

Huang, R., & Li, Y. (2012). What matters most: A comparison of expert and novice teachers' noticing of mathematics classroom events. *School science and mathematics*, *112*(7), 420-432.

Huang, Y., Miller, K. F., Cortina, K. S., & Richter, D. (2021a). Teachers' professional vision in action. *Zeitschrift für Pädagogische Psychologie*. https://doi.org/10.1024/1010-0652/a000313

Huang, Y., Richter, E., Kleickmann, T., Wiepke, A., & Richter, D. (2021b). Classroom complexity affects student teachers' behavior in a VR classroom. *Computers & Education, 163*. https://doi.org/10.1016/j.compedu.2020.104100

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children's Mathematical Thinking, *Journal for Research in Mathematics Education*, 41 (2), 169-202. https://www.jstor.org/stable/20720130

Kader, G. D., & Perry, M. (2007). Variability for categorical variables. Journal of Statistics Education, 15(2). https://doi.org/10.1080/10691898.2007.11889465

Kosko, K. W., Ferdig, R. E., & Roche, L. (2021a). Editorial: Conceptualizing a shared definition and future directions for extended reality (XR) in teacher education. *Journal of Technology and Teacher Education, 29*(3), 257-278. https://www.learntechlib.org/primary/p/219894/

Kosko, K. W., Ferdig, R. E., & Zolfaghari, M. (2021b). Preservice teachers' professional noticing when viewing standard and 360 video. *Journal of Teacher Education, 72*(3), 824–841. https://doi.org/10.1177/0022487120939544

Kosko, K. W., Heisler, J., & Gandolfi, E. (2022). Using 360-degree video to explore teachers' professional noticing. *Computers & Education*, 1-13.

Roche, L., & Rolland, C. (2020). Scaffolding professional learning with 360 video for pre-service teachers. In D. Schmidt-Crawford (Ed.), *Proceedings of the Society for Information Technology & Teacher Education* (pp. 569-576), Association for the Advancement of Computing in Education (AACE). https://www.learntechlib.org/p/218203

Sherin, M. G., & Russ, R. S. (2014). Teacher noticing via video: The role of interpretive frames. In *Digital video for teacher education* (pp. 11-28). Routledge.

Sherin, M., & van Es, E. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of technology and teacher education*, *13*(3), 475-491.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education, 11*, 107–125.

van den Bogert, N., van Bruggen, J., Kostons, D., & Jochems, W. (2014). First steps into understanding teachers' visual perception of classroom events. Teaching and Teacher Education, 37, 208–216.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of technology and teacher education*, *10*(4), 571-596.

van Es, E. A., Stockero, S. L., Sherin, M. G., Van Zoest, L. R., & Dyer, E. (2015). Making the most of teacher self-captured video. *Mathematics Teacher Educator*, *4*(1), 6-19.

van Es, E. A., Cashen, M., Barnhart, T., & Auger, A. (2017). Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy. *Cognition and Instruction, 35*(3), 165-187.

https://doi.org/10.1080/07370008.2017.1317125
van Es, E. A., & Sherin, M. G. (2021). Expanding on prior conceptualizations of teacher noticing. *ZDM, 53*, 17-27. https://doi.org/10.1007/s11858-020-01211-4
Walshe, N., & Driver, P. (2019). Developing reflective trainee teacher practice with 360-degree video. *Teaching and Teacher Education, 78*, 97–105. https://doi.org/10.1016/j.tate.2018.11.009
Weston, T. L., & Amador, J. M. (2021). Investigating student teachers' noticing using 360 video of their own teaching. *Journal of Technology and Teacher Education, 29*(3), 309–338. https://www.learntechlib.org/primary/p/219535/paper_219535.pdf
Zolfaghari, M., Austin, C. K., Kosko, K., & Ferdig, R. E. (2020). Creating asynchronous virtual field experiences with 360 video. Journal of Technology and Teacher Education, 28(2), 315-320.

CULTURALLY RELEVANT FOR WHOM? REFLECTING ON CULTURE AND TASKS WITH PRESERVICE TEACHERS

John Bragelman
University of North Georgia
jabragelman@ung.edu

Kathryn M. Rupe
Western Washington University
rupek@wwu.edu

Rebecca Borowski
Western Washington University
borowsr@wwu.edu

To prepare preservice teachers (PTs) to create equitable, culturally responsive mathematics classrooms, mathematics teacher educators must support them in making connections between theory and practice. Critical examination of culture, its role in the mathematics classroom, and PTs own culture and experiences can be an entry point for this important work. In this study, preservice elementary teachers at various points in their teacher preparation programs engaged in activities that supported their understanding of culturally responsive and relevant mathematics. Using a Culturally Relevant Cognitively Demanding (CRCD) rubric (Matthews et al., 2013) to evaluate tasks and reflect upon the various issues that the tasks illuminate can be transformative experiences for PTs as they develop their identities as equitable mathematics teachers. This multiple case study examines how PTs' understanding of cultural relevance and responsiveness evolved over both content and methods courses.

Keywords: culturally responsive pedagogy; preservice teachers; teacher preparation

Equity and cultural responsiveness as a key component to teacher preparation programs has gained significant traction in recent years. There is broad support for considering both culture and equity in mathematics teaching (Gay, 2018; Ladson-Billings, 1995), and teacher preparation programs are being tasked to support preservice teachers' understanding and practice of equity, diversity, and inclusion (NCTM 2000; 2014). In this brief, we present on three teacher educators' experiences implementing a curriculum for culturally responsive teaching in elementary teacher preparation courses. Our goal of the intervention is twofold. First, we initiate critical dissonance for our PTs by asking a primarily dominant group to consider a part of their identity, one hidden by overarching systems (Leonardo, 2009). In bringing forth this dissonance, we seek to push our PTs to strive for more socially just contexts for learning and teaching mathematics by provoking a critical examination of the role of culture in math classrooms through analysis of mathematics task(s) using a culturally relevant, cognitively demanding framework (Matthews et al., 2013; Matthews et al, 2022).

Culturally Responsive Pedagogy

Culturally responsive teaching is defined as "using the cultural knowledge, prior experiences, frames of reference, and performance styles of ethnically diverse students to make learning encounters more relevant to and effective for them" (Gay, 2018, p. 36). While culturally *relevant* teaching can be considered within the broader umbrella of culturally responsive teaching, its definition and purpose are different. Culturally relevant pedagogy is a "theoretical model that not only addresses student achievement but also helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" (Ladson-Billings, 1995, p. 469). By defining both culturally *responsive* pedagogy and culturally *relevant* pedagogy, we acknowledge the differences between the two theories. We also note the role of Ladson-Billings' (1994,1995) scholarship on the fields' understanding of culturally *responsive* pedagogy and teaching for justice. So, while we position this manuscript within the theory of culturally *responsive* pedagogy, we draw on the work of scholarship in culturally *relevant* pedagogy.

Scholars have shown what it looks like to implement culturally relevant pedagogy successfully in practice through several case studies (Ladson-Billings, 1994; Matthews, 2009). However, attending to and incorporating students' cultures and lived experiences into a math classroom is difficult. For example, Sleeter (2012) discusses the potential pitfalls of mis-enacted culturally responsive teaching and tasks. Compounding this, prospective teachers have been shown to be dismissive or resistant to incorporating issues of culture and equity in their teaching (Ahlquist, 2001). However, fostering PTs' examination and reflection of their beliefs has been shown to decrease this resistance and expand how PTs might more equitably approach the teaching of mathematics (de Freitas, 2008). To provoke such critical examination and reflection, we employ the culturally relevant, cognitively demanding framework (Matthews et al., 2013).

CRCD Framework

In their framework, Matthews and colleagues' (2013) draw on research on cognitive demand (Stein et al., 2000) and culturally relevant pedagogy (Ladson-Billings, 1995) to build and evaluate mathematics tasks that make connections to students' culture and that maximize students' opportunity to learn mathematics. In Stein and colleagues' (2001) framework for cognitive demand they offer criteria for analyzing mathematics tasks by the depth of thinking that they elicit. Tasks that are more procedural without connections to broader aspects of mathematics are *low* in complexity, while tasks that draw mathematical connections and/or target conceptual understandings are *high* in complexity. Table 1 shows how Matthews and colleagues (2013) synthesize cognitive demand with culturally *relevant* pedagogy and draw upon the complexity component of cognitive demand in their analysis of task structure.

Table 1. Culturally Relevant, Cognitively Demanding Rubric

	Degree in Task Structure		
Description	**High**	**Medium**	**Low**
Mathematics task explicitly requires students to inquire (at times problematically) about themselves, their communities, and the world about them.			
May draw from connections to other subjects and issues.			
Mathematics task draws from students' community and cultural knowledge.			
Task may explicitly seek to add to this knowledge through mathematical activity.			
Task is mathematically rich and cognitively demanding, embedded in cultural activity			
Task asks students to engage discontinuity and divide between school and their own lives - home and school			
Task is real-world focused, requiring students to make sense of the world through mathematics.			
The explicit goal of the task is to critique society - that is, make empowered decisions about themselves, their communities, and the world.			

How does a curricular unit that asks PTs to analyze mathematics tasks with the Culturally Relevant Cognitively Demanding rubric shape their beliefs about culture in a math classroom?

Methods

In this multiple case study, we draw on empirical data collected from two elementary teacher preparation programs at two universities in the United States during the Spring semester of 2022. The two universities are predominantly white, and the prospective teacher programs are predominantly female. The case study examines how a culturally responsive curriculum influences preservice teachers' beliefs about culture and its place in the mathematics classroom (Yin, 2009). Qualitative methods were employed for the inquiry. Data from the curriculum implementation was collected across three early mathematics content courses and one mathematics methods and field experience course. In addition to implementation artifacts, pre- and post-surveys were given to learn more about PTs thoughts about their own culture, and more broadly, the role of culture in the mathematics classroom. Finally, as a reflexive tool, the three teacher educators journaled through the design and implementation of the curriculum. For us, this journaling presented a means at potentially avoiding misrepresentation and misinterpretation of participants (Milner, 2007), particularly when confronting systems of domination (Leonard, 2009).

Toward a Culturally Responsive Curriculum

To develop a collective understanding of culturally *responsive* teaching, preservice teachers first read and discussed an excerpt from *Advancing a Framework for Culturally Relevant, Cognitively Demanding Mathematics Tasks* (Matthews et al., 2013). The whole class analyzed and discussed the first task, *So You Think You Can Draw* with the CRCD rubric (Figure 1). Then, as small groups, the preservice teachers analyzed and discussed task 5, *Curfew Currency*, and Task 7, *Reduce, Reuse, Recycle.* Finally, the whole class analyzed and discussed a task from a prior lesson, such as the *Locker Problem* (Kimani et al., 2016). Discussions emphasized justifications for the rubric rates. At this point, we reasoned the preservice teachers had a general understanding of how to use the rubric to rate mathematics tasks with the rubric.

From here, the task was introduced to the preservice teachers. We chose a task that aligned with fifth to sixth grade standards and a task that was similar to application and style to the included tasks (Matthews et al., 2013). *Justice for Janitors* was chosen from *ReThinking Mathematics* (Gustein & Peterson, 2006). The task draws from a children's book, *¡Si, Se Puede! / Yes, we can!* (Cohn, 2002), whose context is the 2000 janitor strike in Los Angeles. For homework, the preservice teachers listened to an audio recording of the book and solved the task. During the next course meeting time, we discussed the task, analyzed the task with the CRCD rubric, and discussed our analysis.

Data Collection

To analyze the effects of the curriculum, we collected artifacts including preservice teachers' solutions and analysis of the task *Justice for Janitors*. We also conducted an open-ended pre- and post- survey which asked participants to reflect on their culture, their experiences with culture in a mathematics classroom, and the importance of culture in the teaching and learning of mathematics through the following prompts:

- What is culture? How do you define it? What do you think it is? How would you describe your culture?
- What role does students' culture play in a math classroom?

- What responsibility do math teachers have to consider and incorporate students' cultures into their teaching?
- Have you heard of culturally responsive pedagogy? What does that term mean to you? What experience do you have with it?
- What does a culturally relevant math task look like? Can you give an example from your math experiences?

In total, we collected 122 participant responses. For this brief proposal, we draw from Randy, a preservice teacher in a mathematics methods course. She is white, in her early twenties, and she is currently placed in a kindergarten classroom.

Analysis and Preliminary Results

Our analysis will orient on PTs' awareness of their own culture and the role of culture in the mathematics classroom. Given that our data collection is currently in progress, we report on pre-survey responses. PTs, in general, could articulate a definition of culture but struggled to name their own culture. For example:

> I think culture is everything that makes an individual who they are. It is their background, their experiences, their relationships, status, beliefs, all of the above. All these things affect the lens in which you view the world. I would describe my culture as my thoughts, beliefs, and attitudes in which I navigate this world. I personally have a very positive disposition and believe that people are put on this world to do good and make differences in people's lives for the better. This belief affects every decision I make in my life and is the lens in which I view the world.

Randy's description of her disposition and perspective of others speaks to the difficulty PTs have in describing their culture and beliefs. Leonardo (2009) speaks to the role of education of the dominant group and the power of normativity in perpetuating systems of domination. While Randy struggles to name her culture, she anticipates how culture shapes a math classroom:

> I definitely know that there are ways to implement culturally relevant math in your classroom, but I haven't thought much about specific tasks. I am thinking it could look something like allowing your students to connect with the content in a meaningful way. This can be done by having students collect data on their family, then represent that data using a graph and then sharing it with your family. Or it could look like looking at the current state of the world to inform a math lesson. It might be on statistics of an endangered species and comparing the percentage in which their population is dropping? This could open up conversation on why these species are endangered and how we can play a part in it. I am not entirely sure if this is what culturally relevant math looks like, but just some ideas.

Randy's responses, as one case, speaks to the need of our curricular intervention for PTs' development of their cultural understandings. Upon completion of our data collection, we will offer a comparison of responses of PTs across content and methods courses, particularly the ways PTs in different phases of their certification engage with the ideas of culturally responsive pedagogy and how these ideas shape their beliefs.

References

Cohn, D. (2002). **¡Si, se puede! /** *Yes, we can!: Janitor strike in L.A* (F. Delgado Illus.). Cinco Puntos Press.

de Freitas, E. (2008). Troubling teacher identity: preparing mathematics teachers to teach for diversity. *Teaching Education*, 19(1), 43-55.

Gay, G. (2018). *Culturally responsive teaching: Theory, research, and practice*. Teachers College Press.
Gutstein, E., & Peterson, B. (Eds.). (2006). *Rethinking mathematics*. Rethinking Schools.
Kimani, P. M., Olanoff, D., & Masingila, J. O. (2016) The locker problem: An open and shut case. *Mathematics Teaching in the Middle School*, *2*(3), 144-151.
Ladson-Billings, G. (1994). *The dreamkeepers: Successful teachers of African American children*. Jossey-Bass.
Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, 32(3), 465-491.
Leonardo, Z. (2009). *Race, whiteness, and education*. Routledge.
Matthews, L. E. (2009). "This little light of mine!" Entering voices of cultural relevancy into the mathematics teaching conversation. In D. B. Martin (Ed.), *Mathematics Teaching, Learning, and Liberation in the Lives of Black Children* (pp. 63-87). Routledge.
Matthews, L. E., Jones, S. M., & Parker, Y. A. (2022). *Engaging in culturally relevant math tasks*. Corwin.
Matthews, L. E., Jones, S. M., & Parker, Y. A. (2013). Advancing a framework for culturally relevant, cognitively demanding mathematics tasks. In J. Leonard & D. B. Martin (Eds.), *The Brilliance of Black Children in Mathematics* (pp. 123-150). Information Age Publishing.
Milner, H. R. (2007). Race, culture, and researcher positionality: Working through dangers seen, unseen, and unforeseen. *Educational Researcher*, 36(7), 388-400.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. NCTM.
National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematical success for all.* NCTM.
Sleeter, C. E. (2012). Confronting the marginalization of culturally responsive pedagogy. *Urban Education, 47*(3), 562-584.
Stein, M. K., Smith, M. S., Henningsen, M., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. Teachers College Press.
Yin, R. K. (2009). Case study research: Design and methods fourth edition. SAGE Publications.

DISSONANCE CREATED THROUGH OPEN MATHEMATICAL TASKS: PRESERVICE TEACHERS' BELIEFS ILLUMINATED

Amy Brass
Pennsylvania State University
aqb6161@psu.edu

Emily A. Mainzer
Pennsylvania State University
ead202@psu.edu

It is well-documented that elementary preservice teachers (PSTs) enter teacher education programs with beliefs about mathematics education largely formed from their own schooling experiences. These beliefs are often narrow and strongly influence how PSTs will eventually teach mathematics to their elementary students. In an effort to create dissonance regarding these beliefs, we presented PSTs with four tasks of varying levels of openness as part of their elementary math methods course. After participating in discussions surrounding these tasks, PSTs completed an open-ended questionnaire that asked them about the experience. Preliminary findings suggest the open tasks challenged PSTs' beliefs about mathematics as a subject, mathematics learning, and mathematics teaching.

Keywords: Affect, Emotion, Beliefs, and Attitude; Preservice Teacher Education

Preservice teachers' (PSTs) beliefs about the teaching and learning of mathematics are formed through their own experiences as students of mathematics (e.g., Lampert, 2002; Thompson, 1992). As PSTs' previous schooling experiences often contradict with goals of the mathematics education community, Ball (1990) called for mathematics methods courses to be a place of dissonance to interrupt the "smooth continuity from student to teacher" (p. 11). The work we report here emphasizes the breaking of such continuity and the creation of dissonance related to PSTs' experiences with mathematical tasks. We aimed to unsettle their notions of mathematical tasks through the use of tasks involving varying levels of openness (e.g., Leikin, 2018; Silver, 1995). In our analysis of PSTs' reactions to these tasks, we focused on their beliefs about mathematics education. We addressed this research question: What do elementary PSTs' perceptions of tasks of varying levels of openness illuminate regarding their beliefs about mathematics education?

Theoretical and Conceptual Framings

Through synthesizing past research, Op't Eynde et al. (2002) developed a framework of students' mathematical beliefs and defined mathematics-related beliefs as "implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematicians, and about the mathematics class context" (p. 27). Their resulting framework was comprised of a person's beliefs about mathematics education, a person's beliefs about the self, and a person's beliefs about the social context (p. 28). Here, we focus specifically on PSTs' beliefs about mathematics education, which Op't Eynde et al. viewed as the compilation of three different subsets of beliefs: (1) a person's beliefs about mathematics as a subject, (2) a person's beliefs about mathematics learning and problem solving, and (3) a person's beliefs about mathematic teaching (p. 28).

We used tasks of varying levels of openness as a platform to explore PSTs' beliefs about mathematics education. Leikin (2018) outlined three different types of openness to mathematical tasks. *Open-start tasks* allow for multiple ways to start the task and multiple strategies to solve the task. *Open-end tasks* have multiple valid answers to them. The third type of task, *open-start and open-end combined tasks*, are ones in which multiple strategies and different, valid answers are possible. Due to the various levels of openness within Leikin's classification structure, we created categorizations for the tasks we used with PSTs: (1) *open-start, closed-end* tasks allowed for different entry points and strategies but culminated in one, single answer; (2) *open-start, limited-end* tasks provided for different entry points and strategies but focused the multiple valid answers relevant to a limited number of contexts; and (3) *open-start, open-end* tasks allowed for multiple entry points, strategies, and valid answers.

Methods

This study is part of a larger exploratory study conducted for the purpose of describing PSTs' engagement in open-ended mathematical tasks. We report here on a single case – one section of an elementary math methods course – and focus specifically on what PSTs' perceptions of their own engagement in mathematical tasks of varying levels of openness illuminated about their beliefs about mathematics education.

Participants and Context

This study was conducted at a large Mid-Atlantic university. In the section of the elementary math methods course we describe here, there were 23 participants. The majority of the participants were White females; there was only one male participant.

During their math methods course, PSTs engaged in four tasks that were designed to have varying levels of openness. PSTs also participated in a discussion of each task led by their instructor. These tasks included a "Which One Doesn't Belong?" (WODB) task (Danielson, 2016), a word problem (WP) of high cognitive demand (Hallman-Thrasher & Spangler, 2020), a "How Many?" (HM) task (Danielson, 2018), and a "Notice and Wonder" (N&W) task (Fetter, 2021; Ray-Riek, 2013). Descriptions of the specific tasks are provided in Table 1.

Table 1: Task Descriptions

Task	Type	Description
Task 1: WODB	open-start, limited-end	PSTs were presented with four addition expressions, three of which had three single-digit addends and one of which had four single-digit addends. PSTs could assert any of the four options did not belong as long as they provided justification.
Task 2: WP	open start, closed-end	Given a total number of cupcakes and boxes with specific capacities for two separate flavors, PSTs were asked to find how many boxes could be made of each flavor.
Task 3: HM	open-start, open-end	PSTs were presented with an image of a large box of sidewalk chalk. The top of the box was removed and the chalk was arranged in columns according to color. Several pieces of chalk were also visible outside of the box. PSTs could choose what and how to count.
Task 4: N&W	open-start, open-end	PSTs were presented with a pictorial representation of the composition of ingredients of nine espresso-based coffee drinks. PSTs were asked to share what they noticed and wondered about the image.

Data Collection

We visited the section of elementary math methods on four separate occasions. Before each visit, we met with the instructor to choose a task that aligned with the planned content. We also collaborated with the instructor to anticipate possible PST responses and to generate potential discussion prompts. For each implementation of the task, the instructor introduced the task, gave

PSTs time to work on the task individually, then facilitated a whole-class discussion of PSTs' thinking surrounding the task. We video-recorded each task session and also took field notes as we observed. Following each discussion, PSTs completed a nine-item, open-ended questionnaire that asked them about their experience and engagement with the task. Based on PSTs' responses on the questionnaire after the first two implementations, we adapted some of the questions for the latter two implementations.

Data Analysis

In exploring our research question as to PSTs' beliefs about mathematics education, we focused our analysis on specific items from the questionnaire. Both versions of the questionnaire included a question that asked PSTs to describe their emotions and a question that asked "When you think about math, is this a task you expect? Why or why not?" PSTs' responses to these questions were included in analysis for all four task sessions. Other questions included in analysis were a question that asked about how PSTs felt about their identity as a learner of mathematics during the task and discussion (first two task implementations) and a question that asked how PSTs' experience of the task and discussion compared to their typical experiences with mathematics (third and fourth task implementations).

When jointly coding the PSTs' responses, we used a combination of inductive and deductive coding (Miles et al., 2014). We used inductive coding when examining each PSTs' responses to the above-mentioned questions for whether or not each task matched the PSTs' expectations for mathematical tasks and when analyzing PSTs' responses for the beliefs they communicated about mathematics education. In doing so, we looked for common phrases and themes within PSTs' responses. We also took note of responses that differed from these themes. We used deductive coding when we categorized our inductive codes using Op't Eynde et al.'s (2002) subsets of beliefs about mathematics education: beliefs about mathematics as a subject, beliefs about mathematics learning, and beliefs about mathematics teaching in general.

Results

For our preliminary results, we looked for predominant themes that emerged in each of these subsets of beliefs about mathematics education. These findings are briefly described below.

Beliefs About Mathematics as a Subject

When we coded PSTs' responses in the subset of beliefs about mathematics as a subject, the most commonly apparent belief involved mathematics as a content area or a set of topics. Of the 84 questionnaires completed across four tasks, 37% of responses (n=31) expressed a belief about mathematics that involved specific content. These responses most commonly involved a belief that mathematics involved numbers and operations (20 of 31 responses, or 65%). For example, when completing Task 3 (HM), one PST stated that in math, they expect "to either add, subtract, multiply, or divide something and in this case we did" (Biscuit; all names are pseudonyms). When completing Task 4 (N&W), another PST shared they were "very taken aback by the lack of numbers" (Krystal). When PSTs were coded as having expressed a belief about mathematics, that belief most often involved a content area or a set of topics.

Beliefs About Mathematics Learning

The most predominant theme in PSTs' beliefs about mathematics learning involved whether they expected one or multiple answers and/or one or multiple possible strategies. Of the 84 submitted questionnaires, 40% (n=34) provided evidence of belief in this area. Of the 34 responses involving number of answers and/or strategies, 56% (n=19) mentioned expecting a task to have one correct answer. After engaging with Task 1 (WODB), one PST explained they tend to think of math as "coming up with one particular answer… while there are many ways to solve… the answer remains the same” (Katherine). Including Katherine, 44% (n=14) of 34 PSTs mentioned the belief that math can involve multiple strategies. Maria noted of Task 3 (HM), “This problem had many ways of solving it and to me that is what I usually expect from a math task.” However, 44% (n=14) of 34 PSTs expressed a belief that math usually involves one set strategy or approach. For example, when solving Task 2 (WP), Becca shared that in her past experience as a math student, “we always were told to solve [problems] one specific way.” Importantly, across all questionnaires, no PSTs stated that they expected a task to have multiple possible answers.

Additionally, although discussions were held for all tasks so that PSTs could share their strategies and solutions, these discussions seemed out of place for PSTs in the tasks allowing for multiple valid answers. This was not the case for the for open-start, closed-end task (Task 2, WP), in which discussion seemed to be expected. For the other tasks, however, the PSTs made comments like these: “Traditionally, I think about math as less of a conversation” (Emma, Task 1, WODB), “The discussion was a bit different because I haven't really had to fully express my thought process in the past” (Oliver, Task 3, HM), and “…this task is not really similar because a lot of math problems are not open-ended and discussion based” (Biscuit, Task 4, N&W).

Beliefs about Mathematics Teaching

We also saw PSTs’ experiences as influencing their beliefs about mathematics teaching as they reacted to tasks of varying levels of openness. Although PSTs commented on the similarity or differences between the tasks and their experiences as mathematics learners, when they took a teacher’s perspective, prior experience in the math methods course influenced their beliefs. This was specifically the case with Task 1, a WODB task that was facilitated after PSTs were introduced to another WODB task earlier in the semester. When asked if this was a type of task they expect when they think about math, approximately 23% (n=5) indicated that previous experience with a WODB task in the mathematics methods course influenced their thinking. As examples, one PST responded, “Before taking this course, I would not [have] felt that this was as much math as it actually is. I had never seen this done before and I find it to be super beneficial” (Pam), and another PST commented, “Prior to this course, I would say no, as it is not something I ever experienced before. Today yes, simply because we have done it in class before” (Calleigh). Interestingly, although PSTs had also engaged in another HM task earlier in the semester, none of them mentioned that experience influencing their views on whether they expect that type of task when thinking about mathematics.

Discussion

The dissonance created by implementing mathematical tasks with varying levels of openness with PSTs illuminated their beliefs about mathematics as a subject, mathematics learning, and mathematics teaching. Preliminary findings suggest that PSTs view mathematics as a set of topics, and not as “an activity grounded in human practices, a science of patterns with problem solving at the heart of it” (Op’t Eynde et al., p. 29). Additionally, several PSTs expected a mathematics task to have one valid answer (e.g., Schoenfeld, 1992). None of the PSTs expected a task with multiple solutions, and many expressed surprise at encountering tasks that allowed for multiple answers. PSTs’ previous experiences also influenced their beliefs (e.g., Ball, 1990) for when mathematics could include discussion (e.g., open-start, closed-end tasks) and about what types of tasks they expect when they think about mathematics. In closing, we offer this question: Because moments of dissonance lead to moments of growth and learning, how do we leverage breaks in experience (Ball), such as the use of open-start, open-end tasks, with PSTs in ways that help them see the potentials, rather than the differences, of such experiences?

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics, 10*(2), 10-16.

Danielson, C. (2016). *A shapes book: Which one doesn't belong?* (teacher's guide). Stenhouse.

Danielson, C. (2018). *How many: A counting book* (teacher's guide). Stenhouse.

Fetter, A. (2021, August 10). *Making the shift towards sense making* [PowerPoint slides]. https://files.mathematicalthinking.org/annie/FetterGSDMC2021Slides.pdf

Hallman-Thrasher, A., & Spangler, D. A. (2020). Purposeful questioning with high cognitive–demand tasks. *Mathematics Teacher: Learning and Teaching PK-12, 113*(6), 446-459. https://doi.org/10.5951/MTLT.2019https://doi.org/10.5951/MTLT.2019.0297.0297

Leikin, R. (2018). Openness and constraints associated with creativity-directed activities in mathematics for all students. In N. Amado, S. Carreira, & K. Jones (Eds.), *Broadening the scope of research on mathematical problem solving* (pp. 387-397). Springer.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). Qualitative data analysis: A methods sourcebook. Sage.

Op 't Eynde, P., de Corte, E., & Verschaffel, L. (2002). Framing students' mathematics-related beliefs: A quest for conceptual clarity and a comprehensive categorization. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 13-37). Kluwer Academic Publishers.

Ray-Riek, M. (2013). *Powerful problem solving: Activities for sense making with the mathematical practices*. Heinemann.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334–370). National Council of Teachers of Mathematics.

Silver, E. A. (1995). The nature and use of open problems in mathematics education: Mathematical and pedagogical perspectives. *ZDM-International Journal of Mathematics Education, 27*(2), 67–72.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127–146). National Council of Teachers of Mathematics.

LISTENING TO TEACHERS: HARMONY AND DISSONANCE IN TAKING RESEARCH TO PRACTICE

Susan Cannon
Mercer University
Cannon_so@mercer.edu

Brittney Castanheira
Mercer University
Castnaheira_ba@mercer.edu

Ashlee Lee
Mercer University
Ashlee.lee@live.mercer.edu

Tiara Mensinger
Mercer University
Mensinger_tc@mercer.edu

In this paper, we report on two years of collaboration with inservice teachers to rethink field work in elementary mathematics education. We engaged teachers who had completed the K5 Mathematics endorsement to serve as university supervisors. Their unique position as practicing teachers and agents of the university provided insight into program improvement. The teacher mentors shared equity tools and rubrics with the preservice teachers. Inservice teachers, preservice teachers, and the researchers interrogated their practice with these tools and provided insight into each other's practices. At times there was harmony in our thinking and at other times productive dissonance between researchers, inservice teachers, and preservice teachers.

Keywords: Equity, Inclusion, and Diversity, Classroom Discourse, Preservice Teacher Education, Teacher Educators

In this paper, we investigate the collaboration between university researchers and inservice teachers to support preservice teachers (PSTs) in building equitable mathematics learning communities and attending to marginalized students or groups. We employed three tools to support our own and the PSTs' practice. In this practice, we had a trifold foci: supporting robust mathematics identities; the use of high-leverage practices to build discourse; and equitable participation in the community.

Perspectives

To shift practices in schools, teachers at all levels need practice with innovative teaching strategies or they are likely to revert to the patterns of schooling they experienced (Beyer & Zeichner, 2018; Britzman, 1991). PSTs typically begin their teaching career as part of a triad—the PST, the field supervisor from the university, and the cooperating teacher in the school. There is a critical need to explore the nature of effective supervision further, especially in light of high-stakes assessments (Donovan & Cannon, 2018). Cooperating teachers are embedded in schools but are often removed from the theory and methods taught at universities. Zeichner (2010) describes the disconnect between universities and schools as one of the "central problems" in teacher education (p. 480) and recommends the construction of hybrid spaces where university and field-based education can come together. Other studies have explored and found success in bridging the theory to practice gap by recruiting program alumni as cooperating teachers (Ragland, 2017). In this project, we have recruited inservice teachers with mathematics expertise to bridge the space between university and field by serving as field supervisors and induction mentors for PSTs.

We recruited graduates of our mathematics endorsement program (ECs) to support preservice teachers in their final internship. The focus of this support is on the use of high-leverage practices, mathematics identities, and equitable participation. We incorporated three tools to serve as pedagogical guides: the EQUIP (Equity Quantified in Participation) app (Reinholz & Shah, 2018); the Levels of Classroom Discourse (LCD) rubric (Hufferd-Ackles et al., 2004); and a culturally relevant task rubric (CRT). Boston et al. (2015) posited that tools allow teachers to focus in on aspects of instruction to "provide a concrete structure for the development of new practices by specifying criteria and identifying standards for the implementation of the intended practice" (p. 154).

Together with the inservice teachers, who had used the tools in their coursework, we considered the drawbacks and potential of each tool and how they might best be incorporated into field work. Drawing from the work of Herbel-Eisenmann and Shah (2019), our goal was to use EQUIP to help teachers and ourselves to develop critical consciousness toward a change in practice.

Elementary supervisors must possess strong mathematics content knowledge and pedagogy to support PSTs fully. By supporting PSTs in the field with ECs, we work to close the theory to practice gap for elementary mathematics teachers. At our university, 1 of the 33 existing university supervisors had a mathematics content specialty. By recruiting ECs to serve as university supervisors, student teachers had access to mentors in the field who have both a command of research-based practices and theories and were attuned to the realities of the classroom since they were currently teaching. We found that this was especially important during the 2020-2021 school year as teachers navigated various models and combinations of online and face-to-face instruction for the first time.

The questions that guided this project were:

- How do we collaborate with teachers in moving from an intention to be equitable to critical consideration of how equitable our practice is?
- What are the benefits and drawbacks of using the EQUIP, LCD, and CRT tools with inservice and preservice teachers to increase high leverage practices?
- How might the coordination of the three tools in a recursive cycle across university and schools promote the creation and maintenance of an equitable mathematics learning community?

Methods

Georgia is one of 19 states in the United States that has developed a route for endorsement of elementary mathematics specialists. This study was drawn out of a nine-semester elementary endorsement program comprised of three courses focused on equitable mathematics teaching. Each eight-week course was designed to embed content and pedagogy. The courses were designed to be fully online and asynchronous, but we offered synchronous sessions weekly, which the majority of the teachers attended. Through this exploratory qualitative inquiry, we report on three cohorts of in-service teachers (2019-2020, 2020-2021, 2021-2022) and their engagement with two cohorts of PSTs (2020-21, 2021-22).

Teaching and teacher education are complex social and political acts; therefore, we employ exploratory case study methodology (Yin, 2009) to account for context, agency, and temporality (Byrne & Callinghan, 2014). We conducted cross case analysis across the three EC cohorts and between EC and PST cohorts. Since we are building on two previously researched tools, to analyze the data we are using provisional coding (Saldaña, 2016). This means that the codes are predetermined for anticipated categories. For this study, the codes include the areas of the LCD rubric and the discourse and demographics in the EQUIP app. We also use invivo coding to "prioritize and honor the participant's voice" (Saldaña, 2016, p. 106) which is often not the case with teachers.

Results

From the data we collected with three cohorts of inservice teachers and two cohorts of preservice teachers, there is an overwhelmingly positive reaction to the use of the two tools within their practice. All the teachers expressed positive shifts in their practice and the usefulness of the tools to improve their instruction and build more equitable mathematics learning communities. We share their reflections below as they related to the areas of the LCD rubric and the EQUIP app.

In the area of *teacher role and questions*, as the teachers began using the EQUIP app and LCD rubric, several teachers first noticed the lack of time they gave their students to speak. One teacher called herself the "main talker" and another described herself as "dominating the conversation." As the teachers reflected later in their practice after using the tools several times, they were able to shift and to "take a step back and allow my students to carry through the mathematical discourse and only prompt and probe when necessary to achieve deeper thinking." One teacher reflected on this shift by stating that "allowing [students] to respectfully talk about their strategies and their answers can help them respect classmates in the long run." Another teacher reflected on this shift and identified her own understanding that "when students work together and complete mathematical tasks, they talk to each other more and rely less on me and more on each other for guidance and advice."

Though many of the teachers in our programs were in school settings that encouraged or mandated, gradual release (I do, we do, you do) instructional paradigms, the use of the LCD rubric helped them to identify and attune to key practices that would shift the mathematical authority in the classroom for teacher centered to student centered.

In the area of *Student Responsibility*, after the implementation of both the tools, several teachers identified that their students were capable of more than they had originally thought. In their reflections, the teachers demonstrated that they had begun to see their students as mathematical leaders. One teacher stated, "I have learned that if I believe my students are capable of doing what others might label as too hard for them, then they will gain the confidence in themselves and be able to complete most tasks presented to them with little to no assistance from myself." This shift is significant and difficult when teachers are being asked by their districts to show their students how to do everything, every day before they allow students to do any mathematics. Another teacher commented, "I was afraid to try higher level activities with my students." The work with the LCD rubric, as a guiding document, allowed these teachers to envision another type of mathematics classroom, a community in which students are leaders.

In the area of *Explaining Mathematical Thinking,* we posit that along with the data from EQUIP, the LCD rubric prompted teachers to listen more carefully to their students. One teacher stated, "I have been more cognizant of looking for those areas in which the students shine and finding ways to use those to help themselves and others gain a better understanding of problem solving or completing tasks."

Moving from gradual release to classrooms focused on pedagogies centering mathematical discourse changed the way that some teachers thought about the purpose of talk in their classrooms. We posit that the LCD tool provided an authoritative pedagogical model that allowed inservice teachers to push back against the mandates of their districts.

Several teachers noted that EQUIP opened their eyes to different ways of thinking about how to be equitable in their classrooms. One teacher noted, "I had never seen anything wrong with who I call on, what I ask, and how I have students respond but this had a totally different spin on it." Another teacher wrote, "Equity wasn't something that was covered in my undergraduate classes, and the importance of equity in the curriculum has never crossed my mind. …My students come from all different backgrounds and homes, so assuming that they all know what I am talking about at all times is a grave mistake."

Drawing on their experiences with the tools, ECs were able to guide PSTs to meaningfully implement the tools in their own practice. Looking across the responses of ECs and PSTs to the tools, we found that both groups became more aware of marginalized groups and individuals. However, the PSTs had fewer resources and ideas as to how to draw in the marginalized populations. Guidance from ECs was crucial in navigating both their critical consciousness and the practical moves they could make in their classrooms to respond to issues of status and marginalization.

Discussion

We found that while teachers quickly admitted that they were the main talker in the room and openly discussed their intent to provide more and different opportunities for student talk, there was less attention in the first two cohorts to directly addressing race when considering marginalized individuals and groups. Teachers discussed gender, location of students in the room, and strength in mathematics as demographic dimensions. While we do assert that the use of these two tools in concert led the teachers to foster more equitable mathematics learning communities and to increase their use of high-leverage practices, we are interested in how the teachers' consideration of their practice could be more critical.

The EQUIP app allowed flexibility for teachers to create their own discourse and demographic dimensions. We see this as a positive as the context of classrooms and schools varies widely. However, this flexibility left an opening for teachers to avoid race as a consideration in their classroom dynamics. It is too easy to be complicit in supporting or ignoring colorblindness in teacher education (Gordon, 2005).

Though EQUIP allows teachers to see patterns of participation, the ways the app is set up and the degree to which a teacher digs into the data can influence the impact of the app. When they introduced EQUIP in 2018, Rhienhalz and Shah, asked, "What are the affordances and limitations of a quantitative approach to measuring aspects of equity and inequity in classroom discourse?" (p. 141). We share this question, and we also share the acknowledgment that classrooms are incredibly complex spaces and that these and other tools should be used in concert to guide and encourage deeper reflection. However, we found, unsurprisingly, that the use of EQUIP alone cannot undo marginalization in classrooms.

In our most recent iteration of the endorsement course sequence, we have incorporated identity work with teachers, direct conversations about White Supremacist tenants in mathematics classrooms, and more guidance in setting up and reflecting on the data from the EQUIP app. We have drawn more attention to the structures and systems in the United States and in schools that disproportionately disadvantage Black and African-American students.

Acknowledgments

This project is partially funded by the Spencer Foundation grant number 202200042. Any opinions, findings, conclusions, and recommendations are those of the authors and do not necessarily reflect the views of the Spencer Foundation.

References

Beyer, L. E., & Zeichner, K. (2018). Teacher education in cultural context: Beyond reproduction. In T. Popkewitz (Ed.) *Critical studies in teacher education* (pp. 298–334). Routledge.

Boston, M., Bostic, J., Lesseig, K., & Sherman, M. (2015). A comparison of mathematics classroom observation protocols. *Mathematics Teacher Educator*, *3*(2), 154-175.

Britzman, D. P. (1991). *Practice makes practice: A critical study of learning to teach*. SUNY Press.

Bryne, D. & Callinghan, G. (2014). *Complexity theory and the social sciences.* Routledge.

Donovan, M. K., & Cannon, S. O. (2018). The university supervisor, edTPA, and the making of the new teacher. *Education Policy Analysis Archives, 26*(28), 1–26. https://epaa.asu.edu/ojs/article/view/2849

Delpit, L., & Dowdy, J. K. (Eds). (2000). The skin that we speak. The New Press

Gordon, J. (2005). Inadvertent Complicity: Colorblindness in Teacher Education. *Educational Studies*, 38(2), 135–153. https://doi.org/10.1207/s15326993es3802_5

Herbel-Eisenmann, B. Shah, N. (2019). Detecting and reducing bias in questioning patterns. *Mathematics Teaching in the Middle School, 24*(5), 282–289.

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. *Journal for Research in Mathematics Education, 35*(2), 81-116. https://doi.org/10.2307/30034933

National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematical success for all.* NCTM.

Ragland, R. (2017) Coming full circle: Using program alumni as cooperating teachers for the next generation of student teachers. *Teacher Education and Development, 4,* 1–20.

Reinholz, D. L., & Shah, N. (2018). Equity Analytics: A Methodological Approach for Quantifying Participation Patterns in Mathematics Classroom Discourse. *Journal for Research in Mathematics Education*, *49*(2), 140–177. https://doi.org/10.5951/jresematheduc.49.2.0140

Saldaña, J. (2016). *The coding manual for qualitative researchers* (3rd ed.). Thousand Oaks, CA: Sage.

Yin, R. K. (2009). *Case study research: Design and methodology* (4th ed.). Sage.

Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college-and university-based teacher education. Journal of Teacher Education, 61(1–2), 89–99.

AMPLIACIÓN DEL UNIVERSO DE ASÍNTOTA EN UNA COMUNIDAD DE PROFESORES DE MATEMÁTICAS EN FORMACIÓN INICIAL

EXTENSION OF THE ASYMPTOTE UNIVERSE IN A COMMUNITY OF MATHEMATICS TEACHERS IN INITIAL TRAINING

Henry Brian Chávez Martínez
Cinvestav-IPN
henry.chavez@cinvestav.mx

Francisco Cordero Osorio
Cinvestav-IPN
fcordero@cinvestav.mx

En este artículo se presentan los resultados de una investigación que atiende la desconexión entre la matemática escolar y la realidad, en particular se reconoce que esa desconexión ha obligado a privilegiar la representación gráfica de asíntota como recta, negando otro tipo de comportamientos asintóticos. Se construyó un diseño escolar que favoreció a modificar el tratamiento de la asíntota con 5 profesores de matemáticas en formación inicial. Se implementó en un ambiente virtual mediante zoom. El marco teórico fue la teoría socioepistemológica de la matemática educativa. Fue un estudio de caso instrumental. Como resultado se amplió el universo de asíntota en la comunidad de estudio y se conformó un marco de referencia para enseñar la asíntota.

Palabras clave: Cálculo, Conocimiento matemático para la enseñanza, Precálculo, Modelado.

Introducción

Una preocupación que hay en la matemática educativa, y en particular del profesor, es ¿cómo conectar esa matemática que se enseña con la realidad? Además, el docente es cuestionado por los estudiantes en cuanto a la "utilidad de la matemática" en contextos de la vida real. Bajo esta visión de incorporar y conectar el conocimiento que se aprende en el aula con la realidad, existen perspectivas teóricas en matemática educativa que se han preocupado por valorar las matemáticas no escolares. Por ejemplo, Rosa y Orey (2015) advierten la necesidad de valorar y promover en el currículo de matemáticas el conocimiento local desarrollado por los miembros de la comunidad que interactúan en un determinado entorno escolar. Ante esta preocupación, Codero et al. (2015) señalan que en lugar de enfocar la atención en lo que saben los estudiantes es preciso mirar cómo usan el conocimiento.

El objetivo de esta investigación fue conocer, según el modelo de socialización contemporánea de Gómez (2015), los procesos de construcción y organización del conocimiento que pasaron los profesores de matemáticas en formación al enfrentarse a situaciones específicas de la realidad en un diseño escolar y que permitieron ampliar el universo de asíntota. La aplicación del diseño se llevó a cabo mediante la plataforma de zoom. La pregunta de investigación fue ¿Cómo se dan los procesos de socialización de lo asintótico en una comunidad de profesores de matemáticas en formación al enfrentarse a un diseño de situación escolar?

Marco teórico

La teoría socioepistemológica de la matemática educativa considera la importancia de incorporar otros saberes que han estado ausentes en la escuela y favorecerían a una matemática útil para el que aprende. Cordero et al. (2015) reconocen como problemática de la enseñanza y aprendizaje de la matemática al discurso matemático escolar (dME). Este ha invisibilizado significados del cotidiano y por tanto hay una desconexión entre la realidad y la matemática escolar. Este fenómeno se reconoce como opacidad. Lo anterior, requiere trabajar en la socialización contemporánea del conocimiento matemático, de tal forma que la realidad sea un marco de referencia para enseñar matemáticas y lo que se aprende en la escuela sea útil para la vida (Gómez, 2015; Pérez-Oxte, 2021).

Gómez (2015), propone tres procesos que dan cuenta de la socialización del conocimiento matemático: proceso funcional, historial e institucional. El proceso funcional se expresa en el conocimiento útil para la comunidad: los usos. Según Cordero (2008) y Cordero, Mena y Montalto (2010) el uso del conocimiento matemático se caracteriza con referencia a dos elementos dialécticos: el funcionamiento (Fu), es decir ¿qué hace? y la forma (Fo); ¿cómo se hace? Además, Cordero y Flores (2007) afirman que las tareas pueden ser actividades, acciones y ejecuciones. El proceso historial requiere establecer relaciones entre usos de una situación específica a otra, a esto se le llama resignificar el conocimiento. Finalmente, el proceso institucional muestra la organización de las relaciones entre los usos sintetizados en categorías de conocimiento, es decir, argumentaciones basadas en lo vivencial del que aprende.

Un ejemplo de categoría es el comportamiento tendencial de funciones (CTF) (Cordero, 1998; Cordero y Solís, 2001; Buendía y Cordero, 2005). Según Cordero (1998), el CTF es un "argumento que establece relaciones entre funciones y está compuesto por una colección coordinada de conceptos y situaciones del Cálculo donde se discuten aspectos globales de variación" (p. 56). Un estudiante dará cuenta de la emergencia de la categoría CTF cuando establezca relaciones entre funciones acerca de su comportamiento en el infinito, recurriendo a procedimientos como la variación de parámetros a partir de modelos gráficos y analíticos.

El modelo de socialización es considerado como una perspectiva que provee de elementos teóricos-metodológicos para la construcción del diseño y análisis de datos.

Metodología

La investigación fue de tipo cualitativa y un estudio de casos instrumental (Skate, 2005). Se sometió a prueba que la socialización de lo asintótico propicia un entorno de relaciones recíprocas entre la asíntota y sus usos, permitiendo al docente ampliar su universo de asíntota.

La aplicación del diseño escolar en el ambiente virtual zoom, debido a las condiciones provocadas por el Covid19. La población de estudio fue conformada por 5 estudiantes del profesorado de matemáticas de la Universidad Pedagógica Nacional Francisco Morazán, en Honduras. Posterior a la aplicación del diseño se realizó una entrevista con el fin de profundizar sobre sus respuestas.

Momentos de la ruta metodológica

Momento 1. Reconocimiento de la categoría CTF. Se retomó los resultados de estudios previos (Mendoza, 2020; Soto y Vilches, 2018) sobre el CTF para estudiar la epistemología que emerge en otras comunidades como las de ingeniería y la biología. Se confrontó con lo habitual de la enseñanza de la asíntota.

Momento 2. Conformación de la epistemología. Se consideró tres situaciones específicas de tres dominios distintos, a saber: el control de temperatura de un foco (ingeniería biónica), la capacidad de soporte poblacional (biología) y la dinámica de una epidemia (epidemiología). A partir de esto se reinterpretó esta epistemología de usos para el diseño de tareas en cada situación.

Momento 3. Construcción del diseño escolar. Se construyó cada una de las situaciones y tareas con base en la epistemología de usos. Se ejecutó un pilotaje que permitió mejorarlo, validarlo y construir un guion para la entrevista no dirigida. Como resultado se estructuró en tres fases coherentes con la perspectiva de socialización: 1) evidenciar elementos de opacidad, 2) poner en juego las situaciones específicas para dar cuenta del proceso funcional e historial y 3) develar la emergencia de la categoría comportamiento tendencial de funciones, para ampliar el universo de asíntota a partir de nuevos argumentos gráficos.

Momento 4. Puesta en escena del diseño escolar. Se desarrolló a través de la plataforma Zoom, con una duración de 4 horas. Se utilizó WhatsApp como medio para que los estudiantes enviaran el desarrollo de las tareas. Posteriormente se entrevistó individualmente a cada participante, para ahondar en sus respuestas, procedimientos y argumentaciones.

Momento 5. Análisis de datos. El análisis se desarrolló en tres secciones que corresponden a las fases del diseño y la perspectiva de socialización. Por lo cual, en la sección I se analizan elementos de opacidad en la comunidad, en la sección II se analiza el proceso funcional e historial y en la sección III, se analiza el proceso institucional. Según Pérez-Oxté (2021) rendir cuentas de algún proceso significa identificar expresiones asociadas a cada proceso. De esta manera se identificó relaciones en las expresiones matemáticas de la comunidad que pudieran darse entre lo matemático y la situación específica (tabla 1).

Tabla 1: Cuestionamientos para caracterizar procesos de socialización.

Procesos	Cuestionamientos
Funcional	¿cómo usan la asíntota? ¿qué significan en cada situación? ¿cuáles son los funcionamientos de su conocimiento matemático?
Historial	¿qué debates surgen entre los usos de la asíntota en cada dominio? ¿cuáles son los procedimientos? ¿cuáles son las formas de uso en que se manifiesta su conocimiento?
Institucional	¿Cómo emerge la categoría comportamiento tendencial? ¿Qué instrumentos se presentaron en las situaciones? ¿Cuál es la relación que hay entre la asíntota y la categoría de comportamiento tendencial?

Nota: adaptado de Pérez-Oxté (2021)

En las transcripciones se utilizó P1, …, P5 para mostrar las respuestas de las participantes.

Resultados y discusión

Sección 1 del análisis

En la fase 1 del diseño se preguntó a los participantes ¿qué es la asíntota? Las respuestas mostraron que la recta tiene un papel hegemónico en las argumentaciones gráficas de la comunidad para definir la asíntota. P2 indicó que la asíntota "es una recta a la que se aproxima continuamente la gráfica de una función.", mientras P3 respondió que "Es una línea recta que prolongada de forma indefinida se acerca progresivamente a una curva sin llegar nunca a encontrarla." Los tipos de asíntota que se manifiestan son la asíntota vertical, horizontal y oblicua, evidenciando que no existen argumentos sobre comportamientos asintóticos curvos. Ante la pregunta ¿cómo enseñaría la asíntota? se evidencia la falta de marcos de referencia para enseñar la asíntota. El sentido de "aplicar del concepto" está presente a través de ejemplos gráficos, pero no hay referencias de la realidad que favorezcan al uso de la asíntota.

Sección 2 del análisis

En la fase 2 del diseño se presentan tres situaciones específicas que rescatan la epistemología de lo cotidiano en los dominios: ingeniería biónica, biología y epidemiología. Las situaciones específicas son:

- Se1: el control de temperatura de un foco,
- Se2: la capacidad de soporte poblacional y
- Se3: la dinámica de una epidemia.

Se buscó la alternancia de dominios que generan nuevos usos de la asíntota, los que están dados por establecer una relación mutua entre alcanzar y mantener una temperatura deseada, mantener la fluctuación de una población alrededor de la capacidad de soporte y determinar una tasa de recuperación favorable en una epidemia.

En la Se1 se pidió que describieran el comportamiento de la función de temperatura de un foco $T(t)$ al transcurrir el tiempo, sabiendo que debe alcanzar y mantenerse lo más cercano a la temperatura de referencia $Tr(t)$ (Figura 1).

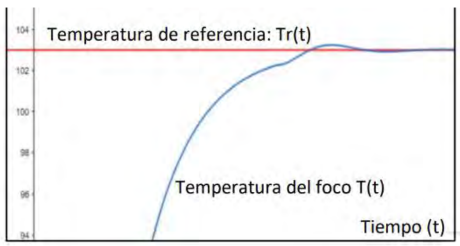


Figura 1: Temperatura $T(t)$ de un foco y temperatura de referencia $Tr(t)$.

Se sabe que un control on-off restablece la temperatura del foco $T(t)$ comparándola con la temperatura de referencia $Tr(t)$, determinando un error $\mathcal{E}(t)$ que controla el encendido y el apagado del foco y con esto su temperatura.

Se pidió escribir una expresión que relacione $T(t)$, $Tr(t)$ y $\mathcal{E}(t)$. Los participantes compararon $T(t)$ y $Tr(t)$ a través de la resta para determinar el error $\mathcal{E}(t)$. Modelando un patrón de construcción que conllevó procedimientos de variación de parámetros y organizar comportamientos asintóticos. Esto dotó de significados a la asíntota como la reproducción de la temperatura deseada considerando que se debe alcanzar y mantener. Además, se preguntó ¿Cuál debe ser el comportamiento de $\mathcal{E}(t)$ para mantener $T(t)$ lo más cercana a $Tr(t)$? Para lo que establecieron condiciones necesarias. Por ejemplo, profesora P1 describió que la condición para reproducir la temperatura deseada es que $\mathcal{E}(t)$ debe tener una tendencia hacia cero cuando el tiempo tiende al infinito (figura 2).

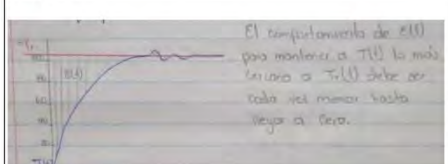
Respuesta	Transcripción de respuesta
	El comportamiento de E(t) para mantener a T(t) lo más cercano a Tr(t) debe ser cada vez menor *hasta llegar a cero.*

Figura 2: Comportamiento de E(t) en la se1.

El patrón de construcción que significó a la asíntota en la $Se1$ se resignifica en la $Se2$, donde se busca reproducir la tendencia de una dinámica de población hacia una capacidad de soporte poblacional. Aquí, la resta también es la forma de comparación que lleva a modelar el patrón $P(t) = C(t) + S(t)$. Donde $P(t)$ representa la dinámica de población, $C(t)$ la capacidad de soporte poblacional y $S(t)$ la sobresaturación.

En ambas situaciones, los profesores recurrieron a la forma de la tendencia para destacar el comportamiento global de las funciones T(t) y P(t) a partir de comparar su tendencia con Tr(t) y C(t) respectivamente. Por ejemplo P1, habla de la estabilidad entre las funciones y afirmando que P(t) tiende a la capacidad de soporte C(t). El proceso funcional describe los usos en cada situación específica y el proceso historial la resignificación entre ellos (figura 3).

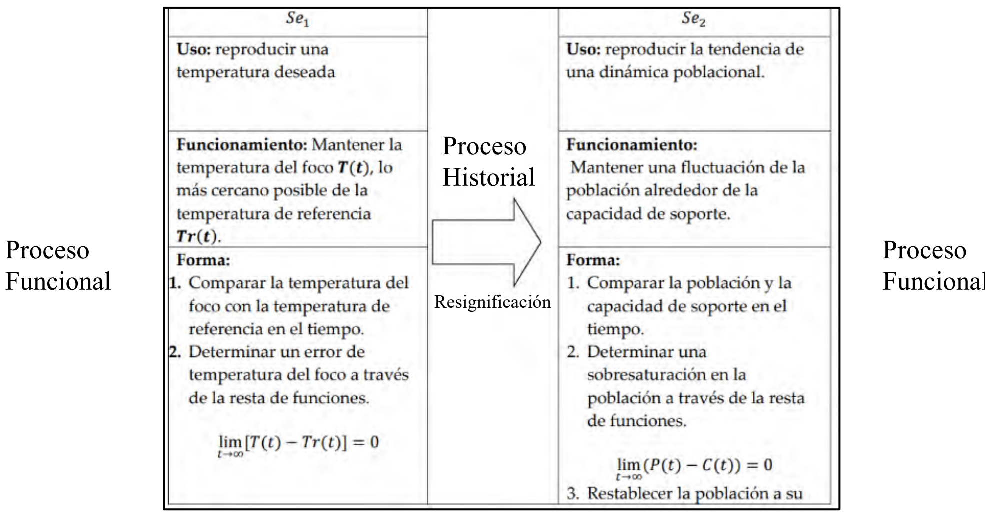


Figura 3: Resignificación de usos de la asíntota entre la Se1 y Se2.

Los profesores concluyeron que existen comportamientos asintóticos en las gráficas de temperatura y dinámica poblacional con lo que surge un primer criterio de asintoticidad: Sean $f(x)$, $g(x)$ y $h(x)$ funciones reales, donde $f(x) = g(x) + h(x)$, tal que $h(x) \to 0$ cuando $x \to \infty$. *Si* $\lim_{x\to\infty}(f(x) - g(x)) = 0$ entonces $g(x)$ es asíntota de $f(x)$. De esta manera, dada una asíntota g(x), lograron construir funciones asintóticas $f(x)$ al sumarle una función $h(x)$ tal que $h(x) \to 0$ cuando $x \to \infty$. Al variar $h(x)$ organizan infinitos comportamientos asintóticos no solo lineales sino curvos, la situación misma propicia esta ampliación.

En la Se3, dada la función $f(t) = e^{-t} + \frac{1}{t+1}$ para la tasa de infección. Se pide que elijan entre las funciones $k_1(t) = e^{-t}$ o $k_2(t) = \frac{1}{t+1}$, la tasa de recuperación que representa un mejor escenario de la epidemia. Se solicita que usen el criterio $\lim_{t\to\infty}(f(t) - k(t))$. Los participantes se dan cuenta que este criterio no es suficiente para determinar la mejor tasa de recuperación. Sin embargo, surge la expresión "aunque $k_2(t)$ decrece menos" lo que sugiere un significado para la asíntota como la rapidez de la tendencia. Es aquí donde se les presenta una noción de la epidemiología: el número básico de reproducción (R°), que permitió ampliar la situación ofreciendo tres posibilidades para decidir cuál tasa de recuperación es la mejor opción. R° representa el número de personas que un infectado puede contagiar y se obtiene a través del cociente: $R°$ = tasa de infección/tasa de recuperación. Esto llevó a una forma de comparación por cociente donde surgió un segundo criterio de asintoticidad: Sean $f(x)$, $g(x)$ y $h(x)$ funciones de x, si $\lim\limits_{x\to\infty} \left(\frac{f(x)}{g(x)}\right) = 1$ entonces $g(x)$ es asíntota de $f(x)$.

La resignificación que se mostró en las situaciones antes expuestas resultó de un debate de usos que se resumen en: reproducir comportamientos con tendencia.

Sección 3 del análisis

El proceso institucional sintetiza ese debate en una categoría de conocimiento: el comportamiento tendencial de funciones. Las tareas que provocan la construcción de lo asintótico están estrechamente relacionadas con mantener la forma y rapidez de la tendencia de una función a otra. Esto se llevó a cabo mediante la comparación del comportamiento de funciones "en el infinito" y la determinación de sus tendencias a través de la resta y/o el cociente. Lo anterior es una argumentación que modifica la enseñanza habitual de la asíntota, ampliando sus significados con base en lo cotidiano de otros dominios del conocimiento.

Se_1	Se_2	Se_3	Lo asintótico
Uso: reproducir una temperatura deseada	**Uso:** reproducir la tendencia de una dinámica poblacional.	**Uso:** reproducir una tendencia favorable en la tasa de recuperación de un virus.	**Uso:** reproducir comportamientos con tendencia
Funcionamiento: Mantener la temperatura del foco $\boldsymbol{T(t)}$, lo más cercano posible de la temperatura de referencia $\boldsymbol{Tr(t)}$.	**Funcionamiento:** Mantener una fluctuación de la población alrededor de la capacidad de soporte.	**Funcionamiento:** Distingue la tasa de recuperación que tiende con mayor rapidez a la tasa de infección.	**Funcionamiento:** Mantener la *forma* y/o rapidez de la tendencia de una función.
Forma: 1. Comparar la temperatura del foco con la temperatura de referencia en el tiempo 2. Determinar un error de temperatura del foco a través de la resta de funciones. $\lim\limits_{t\to\infty}[T(t) - Tr(t)] = 0$ 3. Restablecer la temperatura de un foco	**Forma:** 1. Comparar la población y la capacidad de soporte en el tiempo. 2. Determinar una sobresaturación en la población a través de la resta de funciones. $\lim\limits_{t\to\infty}(P(t) - C(t)) = 0$ 3. Restablecer la población a su capacidad de soporte	**Forma:** 1. Comparar las posibles tasas de recuperación con la tasa de infección a través del cociente. 2. Determinar la tasa que tiende más rápido a la tasa de infección. $\lim\limits_{t\to\infty}\left(\frac{f(t)}{k(t)}\right) = 1$	**Forma:** 1. Comparar el comportamiento de dos funciones cuando la variable independiente tiende al infinito. 2. Determinar sus tendencias a través de la resta y/o el cociente. $\lim\limits_{x\to\infty}[f(x) - g(x)] = 0$ $\lim\limits_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = 1$

Proceso Institucional

Argumento: ***comportamiento tendencial***

Ampliación del universo de asíntota

Figura 4: Proceso institucional y la ampliación del universo de asíntota

Conclusiones

Ante la pregunta ¿Cómo se dan los procesos de socialización de lo asintótico en una comunidad de profesores de matemáticas en formación al enfrentarse a un diseño de situación escolar? Se concluye que el proceso funcional describió los usos de la asíntota en cada situación específica, es decir, mostró el conocimiento que le fue útil a la comunidad:

1. Reproducir una temperatura deseada
2. Reproducir la tendencia de una dinámica poblacional
3. Reproducir una tendencia favorable entre la tasa de recuperación y tasa de infección de un virus.

El proceso historial dio cuenta del debate entre estos usos y se resumen en una función orgánica más general: reproducir comportamientos con tendencia. Donde el funcionamiento está dado por mantener la forma y/o rapidez de la tendencia de una función hacia otra; y las formas serán:

- Comparar el comportamiento de dos funciones cuando la variable independiente tiende al infinito.
- Determinar sus tendencias a través de la resta y/o el cociente.

Finalmente, el proceso institucional dilucidó la emergencia del comportamiento tendencial, lo cual dotó de significados a la asíntota a partir de patrones de comportamientos gráficos y analíticos, donde la resta y el cociente son los criterios que permiten construir funciones $\boldsymbol{f}(\boldsymbol{x})$ asintóticas a una función $\boldsymbol{g}(\boldsymbol{x})$ dada. La función asíntota se reconoce como una instrucción que organiza comportamientos con tendencia.

Se observó que los profesores establecieron relaciones entre las funciones acerca de su comportamiento global, donde los procedimientos como variar parámetros les permite construir funciones asintóticas a una función dada.

Lo anterior, amplió el universo de asíntota en los profesores de matemáticas en formación inicial que participaron en el estudio y se conformó un marco de referencia que modifica en tratamiento habitual de este objeto matemático.

Reconocimiento

La investigación tuvo apoyo de CONACYT a través de las becas de estudiantes de posgrado.

Referencias

Buendía, G. and Cordero, F. (2005). Prediction and the periodical aspect as generators of knowledge in a social practice framework. *Educational Studies in Mathematics*, 58, 299– 333.

Cordero F. y Flores, R. (2007). El uso de las gráficas en el discurso matemático escolar. Un estudio socioepistemológico en el nivel básico a través de los libros de texto. *Revista Latinoamericana de Investigación en Matemática Educativa,* 10(1), 7- 38.

Cordero, F. (1998). El entendimiento de algunas categorías del conocimiento del cálculo y análisis: el caso de comportamiento tendencial de las funciones. *Revista Latinoamericana de Investigación en Matemática Educativa,* 1(1), 56-74

Cordero, F. (2008). El uso de las gráficas en el discurso del cálculo escolar: Una visión socioepistemológica. En R. Cantoral, O. Covián, R. M. Farfán, J. Lezama and A. Romo (Eds.), Investigaciones sobre enseñanza y aprendizaje de las matemáticas: Un reporte iberoamericano (pp. 285-309). México, Díaz de Santos Comité Latinoamericano de Matemática Educativa

Cordero, F., Gómez, K., Silva-Crocci, H., y Soto, D. (2015). *El Discurso Matemático Escolar: la Adherencia, la Exclusión y la Opacidad.* Barcelona, España: Gedisa.

Cordero, F., y Solís, M. (2001). *Las gráficas de las funciones como una argumentación del cálculo.* Segunda edición. México: Grupo Editorial Iberoamérica

Cordero,F., Mena, J. y Montalto M.E. (2010) Il ruolo della giustificazione funzionale in una situazione di risignificazione dell`asintoto. *l´insegnamento della Matemática e delle Scienze Integrate.* B, 33, 457-488.

Gómez, K. (2015). *El fenómeno de opacidad y la socialización del conocimiento. Lo matemático de la Ingeniería Agrónoma.* (Tesis doctoral no publicada). Departamento de Matemática Educativa, Cinvestav-IPN.

Mendoza-Higuera, J., (2020). *El uso de la matemática en la Ingeniería Biónica. De la estabilidad a la reproducción de comportamientos en un sistema de control. Una Categoría de Modelación.* (Tesis doctoral no publicada). Departamento de Matemática Educativa, Cinvestav-IPN.

Pérez-Oxté, I. (2021). *Anticipar-periodizar: una socialización de saberes matemáticos entre la ingeniería y la docencia.* (Tesis doctoral no publicada). Departamento de Matemática Educativa, Cinvestav-IPN.

Rosa, M., & Orey, D. C. (2015). A trivium curriculum for mathematics based on literacy, matheracy, and technoracy: an ethnomathematics perspective. *ZDM*, 47(4), 587-598.

Soto, D., y Vilches, K. (2018). Resignificación de algunas nociones matemáticas en escenarios de la Biología. En Sema, Luis (Ed.), *Acta Latinoamericana de Matemática Educativa* (pp. 1324-1331).
Stake, R. E. (2005). *Investigación con estudio de casos (3ª ed.).* Madrid, Morata

EXTENSION OF THE ASYMPTOTE UNIVERSE IN A COMMUNITY OF MATHEMATICS TEACHERS IN INITIAL TRAINING

Henry Brian Chávez Martínez
Cinvestav-IPN
henry.chavez@cinvestav.mx

Francisco Cordero Osorio
Cinvestav-IPN
fcordero@cinvestav.mx

This article presents the results of a research that addresses the disconnection between school mathematics and reality, in particular, it is recognized that this disconnection has forced to privilege the graphic representation of asymptote as a straight line, denying other types of asymptotic behaviors. A school design that favored to modify the treatment of the asymptote was built with 5 mathematics teachers in initial training. It was implemented in a virtual environment using Zoom. The theoretical framework was the socioepistemological theory of educational mathematics. It was an instrumental case study. As a result, the universe of asymptotics in the study community was expanded and a framework for teaching asymptotics was formed.

Keywords: Calculus, Mathematical knowledge for teaching, Precalculus, Modeling.

Introduction

A concern in educational mathematics, and in particular for the teacher, is how to connect the mathematics being taught with reality. In addition, the teacher is questioned by students as to the "usefulness of mathematics" in real life contexts. Under this vision of incorporating and connecting the knowledge learned in the classroom with reality, there are theoretical perspectives in educational mathematics that have been concerned with valuing non-school mathematics. For example, Rosa and Orey (2015) warn of the need to value and promote in the mathematics curriculum the local knowledge developed by community members who interact in a given school environment. Given this concern, Codero et al. (2015) point out that instead of focusing attention on what students know, it is necessary to look at how they use knowledge.

The objective of this research was to know, according to the contemporary socialization model of Gómez (2015), the processes of construction and organization of knowledge that mathematics teachers in training went through when faced with specific situations of reality in a school design and that allowed the asymptote universe to be expanded. The application of the design was carried out using the zoom platform. The research question was How do the processes of socialization of the asymptotic occur in a community of mathematics teachers in training when confronted with a school situation design?

Theoretical framework

The socio-epistemological theory of educational mathematics considers the importance of incorporating other knowledge that has been absent in school and would favor a useful mathematics for the learner. Cordero et al. (2015) recognize as a problem in the teaching and learning of mathematics the school mathematical discourse (SMD). This has made everyday meanings invisible and therefore there is a disconnection between reality and school mathematics. This phenomenon is recognized as opacity. This requires working on the contemporary socialization of mathematical knowledge, so that reality is a frame of reference for teaching mathematics and what is learned in school is useful for life (Gómez, 2015; Pérez-Oxte, 2021).

Gómez (2015), proposes three processes that account for the socialization of mathematical knowledge: functional, historical and institutional process. The functional process is expressed in the knowledge useful to the community: the uses. According to Cordero (2008) and Cordero, Mena and Montalto (2010) the use of mathematical knowledge is characterized with reference to two dialectical elements: the functioning (Fu), i.e. what does it do? and the form (Fo); how is it done? Furthermore, Cordero and Flores (2007) state that tasks can be activities, actions and executions. The historical process requires establishing relationships between uses from one specific situation to another; this is called resignifying knowledge. Finally, the institutional process shows the organization of relationships between uses synthesized in knowledge categories, i.e., arguments based on the learner's experience.

An example of category is the tendential behavior of functions (TBF) (Cordero, 1998; Cordero and Solís, 2001; Buendía and Cordero, 2005). According to Cordero (1998), the TBF is an "argument that establishes relationships between functions and is composed of a coordinated collection of concepts and situations of the Calculus where global aspects of variation are discussed" (p. 56). A student will account for the emergence of the TBF category when establishing relations between functions about their behavior at infinity, resorting to procedures such as the variation of parameters from graphical and analytical models.

The socialization model is considered as a perspective that provides theoretical-methodological elements for the construction of data design and analysis.

Methodology

The research was qualitative and an instrumental case study (Skate, 2005). It was tested that the socialization of the asymptotic propitiates an environment of reciprocal relations between the asymptote and its uses, allowing the teacher to expand his universe of asymptote.

The application of the school design was through Zoom, due to the conditions caused by Covid19. The study population consisted of 5 students of the mathematics faculty of the Francisco Morazán National Pedagogical University, in Honduras. After the application of the design, an interview was carried out in order to deepen their answers.

Moments of the methodological route

Moment 1. Recognition of the TBF category. The results of previous studies (Mendoza, 2020; Soto and Vilches, 2018) on the TBF were taken up to study the epistemology that emerges in other communities such as engineering and biology. It was confronted with the usual of asymptote teaching.

Moment 2. Conformation of epistemology. Three specific situations from three different domains were considered, namely: temperature control of a focus (bionic engineering), population support capacity (biology) and the dynamics of an epidemic (epidemiology). From this, this epistemology of uses was reinterpreted for the design of tasks in each situation.

Moment 3. Construction of the school design. Each of the situations and tasks was constructed based on the epistemology of uses. A pilot test was carried out to improve it, validate it and construct a script for the non-directed interview. As a result, it was structured in three phases coherent with the socialization perspective: 1) to evidence elements of opacity, 2) to put into play the specific situations to account for the functional and historical process and 3) to unveil the emergence of the tendency behavior category of functions, to expand the universe of asymptotes based on new graphic arguments.

Moment 4. Staging of the school design. It was developed through the Zoom platform, with a duration of 4 hours. WhatsApp was used as a means for students to send the development of the tasks. Subsequently, each participant was interviewed individually, to delve into their answers, procedures and arguments.

Moment 5. Data analysis. The analysis was developed in three sections that correspond to the phases of the design and the socialization perspective. Thus, section I analyzes elements of opacity in the community, section II analyzes the functional process and history, and section III analyzes the institutional process. According to Pérez-Oxté (2021), to account for a process means to identify expressions associated with each process. In this way, relationships were identified in the mathematical expressions of the community that could occur between the mathematical and the specific situation (Table 1).

Table 1: Questions to characterize socialization processes.

Process	Questions
Functional	How do they use the asymptote? what do they mean in each situation? what are the functioning of their mathematical knowledge?
Historical	What debates arise between the uses of the asymptote in each domain? what are the procedures? what are the forms of use in which its knowledge is forms of use in which its knowledge is manifested?
Institutional	How does the category of trend behavior emerge? What instruments were present in the situations? What is the relationship between the asymptote and the category of trend behavior?

Note: Adapted from Pérez-Oxté (2021).

In the transcripts, P1, ..., P5 were used to show the participants' responses.

Results and discussion

Section 1 of the analysis

In phase 1 of the design, participants were asked what is asymptote? The responses showed that the straight line has a hegemonic role in the community's graphical arguments for defining the asymptote. P2 indicated that the asymptote "is a straight line to which the graph of a function continuously approaches.", while P3 responded that "It is a straight line that prolonged indefinitely progressively approaches a curve without ever meeting it." The types of asymptote manifested are vertical, horizontal and oblique asymptote, evidencing that there are no arguments about curved asymptotic behaviors. In response to the question "How would you teach the asymptote?", the lack of reference frameworks for teaching the asymptote is evident. The sense of "applying the concept" is present through graphic examples, but there are no references to reality that favor the use of the asymptote.

Section 2 of the analysis

In phase 2 of the design, three specific situations are presented that rescue the epistemology of everyday life in the domains: bionic engineering, biology and epidemiology. The specific situations are:

- Se1: the temperature control of a light bulb,
- Se2: the population support capacity, and.
- Se3: the dynamics of an epidemic.

The alternation of domains that generate new uses of the asymptote was sought, those given by establishing a mutual relationship between reaching and maintaining a desired temperature, maintaining the fluctuation of a population around the carrying capacity and determining a favorable recovery rate in an epidemic.

In Se1 they were asked to describe the behavior of the temperature function of a focus $T(t)$ as time elapses, knowing that it should reach and maintain as close as possible to the reference temperature $Tr(t)$ (Figure 1).

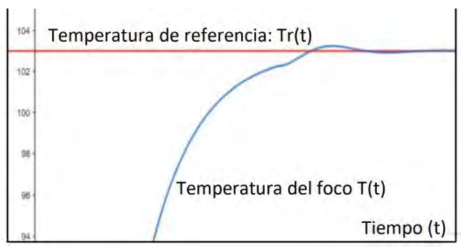


Figure 1: Temperature $T(t)$ of a focus and reference temperature $Tr(t)$.

It is known that an on-off control resets the light bulb temperature $T(t)$ by comparing it with the reference temperature $Tr(t)$, determining an error $\mathcal{E}(t)$ that controls the on-off switching of the light bulb and with this its temperature.

They were asked to write an expression relating $T(t)$, $Tr(t)$ and $\mathcal{E}(t)$. Participants compared $T(t)$ and $Tr(t)$ through subtraction to determine the error $\mathcal{E}(t)$. Modeling a construction pattern that entailed parameter variation procedures and organizing asymptotic behaviors. This endowed meanings to the asymptote as the reproduction of the desired temperature considering that it should be reached and maintained. Furthermore, it was asked what should be the behavior of $\mathcal{E}(t)$ to maintain $T(t)$ as close to $Tr(t)$ as possible? For which they established necessary conditions. For example, teacher P1 described that the condition to reproduce the desired temperature is that $\mathcal{E}(t)$ must have a trend toward zero when time tends to infinity (Figure 2).

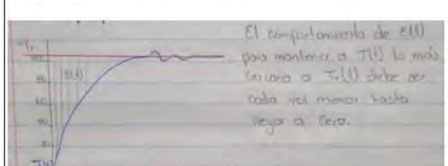
Respuesta	Transcripción de respuesta
	El comportamiento de E(t) para mantener a T(t) lo más cercano a Tr(t) debe ser cada vez menor *hasta llegar a cero.*

Figure 2: Behavior of E(t) in se1.

The construction pattern that signified the asymptote in $Se1$ is re-signified in $Se2$, where we seek to reproduce the trend of a population dynamic towards a population carrying capacity. Here, subtraction is also the form of comparison that leads to model the pattern $P(t) = C(t) + S(t)$. Where $P(t)$ represents population dynamics, $C(t)$ represents population carrying capacity, and $S(t)$ represents oversaturation.

In both situations, the teachers resorted to the trend form to highlight the overall behavior of the functions T(t) and P(t) from comparing their trend with Tr(t) and C(t) respectively. For example P1, talks about the stability between the functions and stating that P(t) tends to the bearing capacity C(t). The functional process describes the uses in each specific situation and the historical process the resignification between them (Figure 3).

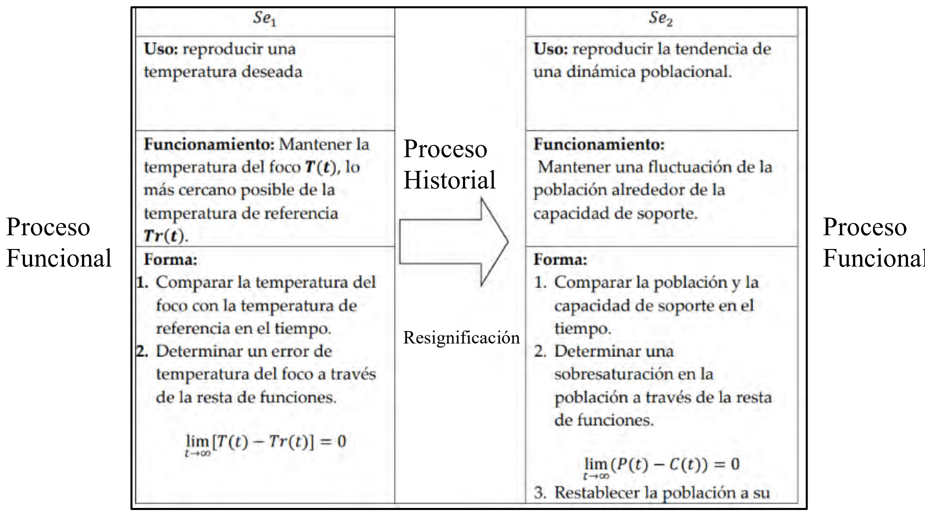


Figure 3: Resignification of uses of the asymptote between Se1 and Se2.

The professors concluded that there are asymptotic behaviors in the temperature and population dynamics plots with which a first criterion of asymptoticity emerges: Let $f(x)$, $g(x)$ and $h(x)$ be real functions, where $f(x) = g(x) + h(x)$, such that $h(x) \to 0$ when $x \to \infty$. If $\lim_{x\to\infty}(f(x) - g(x)) = 0$ then $g(x)$ is asymptote of $f(x)$. Thus, given an asymptote g(x), they succeeded in constructing asymptotic functions $f(x)$ by adding to it a function $h(x)$ such that $h(x) \to 0$ when $x \to \infty$. By varying $h(x)$ organize infinite asymptotic behaviors not only linear but curved, the situation itself is conducive to this extension.

In Se3, given the function $f(t) = e^{-t} + \frac{1}{t+1}$ for the infection rate. They are asked to choose between the functions $k_1(t) = e^{-t}$ or $k_2(t) = \frac{1}{t+1}$, the recovery rate that represents a best-case scenario of the epidemic. They are asked to use the criterion $\lim_{t\to\infty}(f(t) - k(t))$. The participants realize that this criterion is not sufficient to determine the best recovery rate. However, the expression "although $k2$ (t) decreases less" emerges which suggests a meaning for the asymptote as the rapidity of the trend. It is here that they are introduced to a notion from epidemiology: the basic reproduction number (R°), which allowed them to broaden the situation by offering three possibilities for deciding which recovery rate is the best option. R° represents the number of people an infected person can infect and is obtained through the quotient: $R°$ = infection rate/recovery rate. This led to a form of comparison by quotient where a second criterion of asymptoticity emerged: Let $f(x)$, $g(x)$ and $h(x)$ be functions of x, if $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = 1$ then $g(x)$ is asymptote of $f(x)$.

The resignification shown in the above situations resulted from a discussion of uses that can be summarized as: reproduce behaviors with tendency.

Section 3 of the analysis

The institutional process synthesizes this debate into a category of knowledge: tendential behavior of functions. The tasks that trigger the construction of the asymptotic are closely related to maintaining the shape and speed of the trend from one function to another. This was done by comparing the behavior of functions "at infinity" and determining their trends through subtraction and/or quotient. The above is an argument that modifies the usual teaching of the asymptote, expanding its meanings based on the everyday of other domains of knowledge.

Se_1	Se_2	Se_3	Lo asintótico
Uso: reproducir una temperatura deseada	**Uso:** reproducir la tendencia de una dinámica poblacional.	**Uso:** reproducir una tendencia favorable en la tasa de recuperación de un virus.	**Uso:** reproducir comportamientos con tendencia
Funcionamiento: Mantener la temperatura del foco $\boldsymbol{T(t)}$, lo más cercano posible de la temperatura de referencia $\boldsymbol{Tr(t)}$.	**Funcionamiento:** Mantener una fluctuación de la población alrededor de la capacidad de soporte.	**Funcionamiento:** Distingue la tasa de recuperación que tiende con mayor rapidez a la tasa de infección.	**Funcionamiento:** Mantener la *forma* y/o rapidez de la tendencia de una función.
Forma: 1. Comparar la temperatura del foco con la temperatura de referencia en el tiempo 2. Determinar un error de temperatura del foco a través de la resta de funciones. $\lim_{t\to\infty}[T(t) - Tr(t)] = 0$ 3. Restablecer la temperatura de un foco	**Forma:** 1. Comparar la población y la capacidad de soporte en el tiempo. 2. Determinar una sobresaturación en la población a través de la resta de funciones. $\lim_{t\to\infty}(P(t) - C(t)) = 0$ 3. Restablecer la población a su capacidad de soporte	**Forma:** 1. Comparar las posibles tasas de recuperación con la tasa de infección a través del cociente. 2. Determinar la tasa que tiende más rápido a la tasa de infección. $\lim_{t\to\infty}\left(\frac{f(t)}{k(t)}\right) = 1$	**Forma:** 1. Comparar el comportamiento de dos funciones cuando la variable independiente tiende al infinito. 2. Determinar sus tendencias a través de la resta y/o el cociente. $\lim_{x\to\infty}[f(x) - g(x)] = 0$ $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)}\right) = 1$

Proceso Institucional

Argumento: ***comportamiento tendencial***

Ampliación del universo de asíntota

Figure 4: Institutional process and the extension of the asymptote universe.

Conclusions

In response to the question How do the processes of socialization of the asymptotic occur in a community of mathematics teachers in training when facing a school situation design? It is concluded that the functional process described the uses of the asymptote in each specific situation, that is, it showed the knowledge that was useful to the community:

1. Reproduce a desired temperature
2. Reproduce the trend of a population dynamic
3. Reproduce a favorable trend between the recovery rate and infection rate of a virus.

The historical process accounted for the debate between these uses and is summarized in a more general organic function: to reproduce behaviors with trend. Where the function is given by maintaining the shape and/or speed of the trend from one function to another; and the shapes will be:

- Compare the behavior of two functions when the independent variable tends to infinity.
- Determine their tendencies through subtraction and/or quotient.

Finally, the institutional process elucidated the emergence of the trend behavior, which endowed the asymptote with meanings from graphical and analytical behavior patterns, where the subtraction and the quotient are the criteria that allow constructing functions $\boldsymbol{f}(\boldsymbol{x})$ asymptotic to a given function $\boldsymbol{g}(\boldsymbol{x})$. The asymptotic function is recognized as an instruction that organizes behaviors with trend.

It was observed that teachers established relationships between functions about their global behavior, where procedures such as varying parameters allow them to construct asymptotic functions to a given function.

This expanded the universe of asymptote in the mathematics teachers in initial training who participated in the study, and a frame of reference was formed that modifies the usual treatment of this mathematical object.

Acknowledgment

The research was supported by CONACYT through graduate student grants.

Referencias

Buendía, G. and Cordero, F. (2005). Prediction and the periodical aspect as generators of knowledge in a social practice framework. *Educational Studies in Mathematics*, 58, 299– 333.

Cordero F. y Flores, R. (2007). El uso de las gráficas en el discurso matemático escolar. Un estudio socioepistemológico en el nivel básico a través de los libros de texto. *Revista Latinoamericana de Investigación en Matemática Educativa,* 10(1), 7- 38.

Cordero, F. (1998). El entendimiento de algunas categorías del conocimiento del cálculo y análisis: el caso de comportamiento tendencial de las funciones. *Revista Latinoamericana de Investigación en Matemática Educativa,* 1(1), 56-74

Cordero, F. (2008). El uso de las gráficas en el discurso del cálculo escolar: Una visión socioepistemológica. En R. Cantoral, O. Covián, R. M. Farfán, J. Lezama and A. Romo (Eds.), Investigaciones sobre enseñanza y aprendizaje de las matemáticas: Un reporte iberoamericano (pp. 285-309). México, Díaz de Santos Comité Latinoamericano de Matemática Educativa

Cordero, F., Gómez, K., Silva-Crocci, H., y Soto, D. (2015). *El Discurso Matemático Escolar: la Adherencia, la Exclusión y la Opacidad.* Barcelona, España: Gedisa.

Cordero, F., y Solís, M. (2001). *Las gráficas de las funciones como una argumentación del cálculo.* Segunda edición. México: Grupo Editorial Iberoamérica

Cordero,F., Mena, J. y Montalto M.E. (2010) Il ruolo della giustificazione funzionale in una situazione di risignificazione dell`asintoto. *l´insegnamento della Matemática e delle Scienze Integrate.* B, 33, 457-488.

Gómez, K. (2015). *El fenómeno de opacidad y la socialización del conocimiento. Lo matemático de la Ingeniería Agrónoma.* (Tesis doctoral no publicada). Departamento de Matemática Educativa, Cinvestav-IPN.

Mendoza-Higuera, J., (2020). *El uso de la matemática en la Ingeniería Biónica. De la estabilidad a la reproducción de comportamientos en un sistema de control. Una Categoría de Modelación.* (Tesis doctoral no publicada). Departamento de Matemática Educativa, Cinvestav-IPN.

Pérez-Oxté, I. (2021). *Anticipar-periodizar: una socialización de saberes matemáticos entre la ingeniería y la docencia.* (Tesis doctoral no publicada). Departamento de Matemática Educativa, Cinvestav-IPN.

Rosa, M., & Orey, D. C. (2015). A trivium curriculum for mathematics based on literacy, matheracy, and technoracy: an ethnomathematics perspective. *ZDM*, 47(4), 587-598.

Soto, D., y Vilches, K. (2018). Resignificación de algunas nociones matemáticas en escenarios de la Biología. En Sema, Luis (Ed.), *Acta Latinoamericana de Matemática Educativa* (pp. 1324-1331).

Stake, R. E. (2005). *Investigación con estudio de casos (3ª ed.).* Madrid, Morata

SUPPORTING THE DEVELOPMENT OF POSITIVE MATHEMATICS TEACHING IDENTITIES IN PRESERVICE TEACHER EDUCATION

Lara Condon
University of Pennsylvania
condonl@upenn.edu

Janine T. Remillard
University of Pennsylvania
janiner@upenn.edu

Research suggests that preservice teachers enter teacher education with predominantly negative dispositions towards mathematics. We present a case study of an approach to supporting the development of positive mathematics teaching identities among students in a middle-level math methods course. Our findings suggest that engaging in narrative reflection in tandem with collaborative problem solving in a mathematical discourse community over time helped students to transform their relationships with mathematics and forge positive mathematics identities. Providing opportunities for students to share and work through their vulnerabilities was critical to this process. Our findings illustrate a promising approach for teacher educators to support the development of positive mathematics teaching identities in preservice teacher education.

Keywords: Preservice teacher education, problem solving, affect, emotion, beliefs, and attitudes

Introduction

Preparing preservice teachers (PSTs) to teach mathematics involves attending to the development of complex knowledge, skills, and reasoning practices. It also involves supporting the identity work that novice teachers must do to position themselves as mathematics teachers (Ntow & Adler, 2019; Lutovac & Kaasila, 2014). This identity work is made especially challenging by common approaches to mathematics instruction that permeate U.S. K-12 schools. Confronted by years of memorizing facts and procedures, in settings that emphasize being first and fastest, the majority of young people entering teaching view mathematics as a body of disconnected, procedural skills that need to be memorized, leading many to conclude that they are simply "not a math person" (Relich, 1996; Machalow et al., 2020).

This paper presents findings from a small study of an effort to support the mathematical identity work of a group of middle-level PSTs through mathematics problem solving and reflections. Building on the work of Machalow and colleagues (2020), we see PSTs' identities as teachers of mathematics as grounded in their experiences as learners of mathematics and intertwined with their orientations toward the subject (Gresalfi & Cobb, 2011; Ntow & Alder, 2019). Our research was guided by the following questions:

1. What connections between mathematics learning experiences and orientations toward the subject are surfaced through PSTs mathematics identity work?
2. How does engaging in collaborative problem solving influence PSTs' relationships with and orientations towards mathematics?

The participants in this study were students in a one-year teacher preparation program, seeking middle-level (ML) certification (grades 4-8). The context of the study was a semester-long mathematics methods course for all ML candidates, the majority of whom were specializing in subjects other than mathematics. The authors were both instructors of the course.

Conceptual Framework

The conceptual framework guiding our study builds on work by Machalow et a. (2020), which conceptualizes mathematics identity for teaching as being shaped by teachers' interpretations of their experiences and their related mathematics orientations. We understand *mathematics identity* as a learner's interpretations of themself in relation to mathematics (Darragh, 2016; Graven & Heyd-Metzuyanim, 2019). Learners develop and continuously reshape their mathematics identities through experiences learning mathematics, in which multiple social factors are implicated. When learners engage with mathematics, they are also influenced by relationships with teachers and classmates, their social identities, such as race, gender, and class, and institutional systems that often make mathematics dehumanizing (Sfard & Prusak, 2005; Gutierrez, 2002). PSTs' mathematics identities are central to their development as teachers. Teachers with negative mathematics identities are more likely to focus on rote procedures and exhibit low expectations for their students (Brady & Bowd, 2005; Relich, 1996).

Teachers' orientations toward mathematics are core to their mathematics identities. Machalow and colleagues (2020) define *mathematics orientations* as one's beliefs about the nature of mathematics knowledge. They employ Skemp's (1976) two contrasting approaches to understanding mathematics to distinguish between a view of mathematics as *instrumental,* comprised of procedural knowledge, and *relational,* characterized by understanding why rules work and the relationships between them. Often, PSTs enter teacher education having learned mathematics in highly instrumental ways and have come to think of mathematics as a series of rules and procedures. Studies suggest that when students are exposed to relationally oriented approaches to mathematics, they have more of an opportunity to develop positive relationships with mathematics as a subject (McCulloch et al., 2013; Brady & Bowd, 2005).

Literature Review

Increasingly, learning is understood as a process of identity formation or becoming a certain type of person in a particular context (Lave & Wenger, 1991). Learning to teach mathematics involves developing an identity as a mathematics teacher (Hodges & Hodge, 2017; Ntow & Adler, 2019), in which mathematics identity plays a central role. Many preservice elementary teachers, however, have had problematic relationships with mathematics and are likely to experience feelings of math anxiety or lack confidence in their own ability to effectively teach mathematics based on negative formational experiences as mathematics learners (*eg.,* Brady & Bowd, 2005, Machalow et al., 2020, McGlynn-Stewart, 2010). Research on efforts to shift PSTs' relationships to mathematics offer two distinct approaches: exploring mathematics in order to "break" from one's past experiences with the subject (Ball, 1990) and engaging them in identity work to influence their mathematics teaching identities (Ntow & Adler, 2019).

Engaging PSTs in doing mathematics in ways that emphasize relational understanding (Skemp, 1976) and are likely to differ substantially from their experience as K-12 students is a well-established approach used by teacher educators (e.g., Ball, 1990). Various studies demonstrate how such opportunities to re-learn primary mathematics in methods classes can support positive shifts in mathematics orientations and identities (Harkness et al., 2007; McGlynn-Stewart, 2010). Harkness and colleagues (2017), for example, found that by working constructively in groups to make sense of mathematics increased PSTs' self-concept and self-efficacy, as well as their understanding the role that the teacher played in facilitating this productive struggle. Researchers have also found, however, that success of these approaches is dependent upon several key components of instructional design, including emphasis on relational understanding of mathematics concepts using concrete representations, positive support from teachers and peers, and collaborative work in communities of practice (Harkness et al., 2017; Kaasila et al., 2008; Machalow, et al., 2020).

Research that examines efforts in teacher education to positively influence PSTs' mathematics identities has emerged as promising in the last decade. One widely discussed strategy is to provide PSTs with opportunities to explore their own identities through narrative accounts. In these studies, researchers (who are often teacher educators) invite students to recount their own formative mathematics learning experiences as narrative texts, hypothesizing that this type of narrative work promotes reflection and understanding of one's own mathematics learning experiences (Kaasila et al., 2008; LoPresto & Drake, 2004; McCulloch, 2013). Further, asking PSTs to reflect on how their experiences have shaped their relationship with mathematics allows them to consider how common classroom practices influence how they come to think of themselves in relation to mathematics (e.g., Machalow et al., 2020; Remillard, 1993). Zazkis (2015) further argues that personifying the object of the narrative (mathematics) can emphasize the potency of the relationship learners have with the subject. The impact of identity on teachers' future orientations towards teaching underscores the importance of identity work in preservice teacher education. When teachers are able to reframe their past learning experiences through identity work, there is an increased likelihood that they will feel a sense of agency over their future mathematics teaching (Lutovac & Kaasila, 2014) and a greater capacity to foster positive math identities to future students (Machalow et al., 2020).

Our mathematics methods course combined these two approaches to surfacing and nurturing PSTs' relationships to mathematics: we engaged them in re-learning mathematics from a relational perspective and asked them to write narratives about their relationship with a *personified* version of mathematics. These narratives provided a window into PSTs' developing orientations toward mathematics and their mathematics identities. Our research used the narratives and other artifacts and observations to examine the connections between PSTs' experiences and their orientations toward and relationships with mathematics.

Methods

The eight participants in the study comprised the 2021-22 cohort of ML teacher candidates in a one-year, K-12 program, situated in a private (PWI) university in a large east-coast city. The program aims to prepare teachers to work in poverty-impacted, urban schools. The ML certification covered all subjects for grades 4-6 and content specialization for grades 7-8. Only one of the 8 participants was seeking a specialization in mathematics; the others were specializing in ELA (n=3), social studies (n=3), and science (n=1) and enrolled in content-specific courses for grades 7-12. All were enrolled in the ML mathematics methods course, taught by the authors. The fact that the majority of students in the course would be certified to teach grades 4-6 mathematics but identified as teachers of other subjects presented an instructional challenge and a research opportunity. Our aim was to explore the potential of the approach described above for nurturing non-mathematics PSTs' identities as teachers of mathematics. Three of the participants identified as white females, one as a white male, two as black females, one as a white trans-masculine non-binary, and one as an Asian male. Both authors identify as white females.

During the semester-long mathematics methods course, students were presented with a weekly Problem of the Week, an open-strategy problem involving mathematics content appropriate for grades 5-8. In the example (Figure 1), *Squares to Stairs*, students are prompted to justify their thinking as they explain the progression of the figure, followed by subsequent questions about increasingly complex case numbers and numbers of squares.

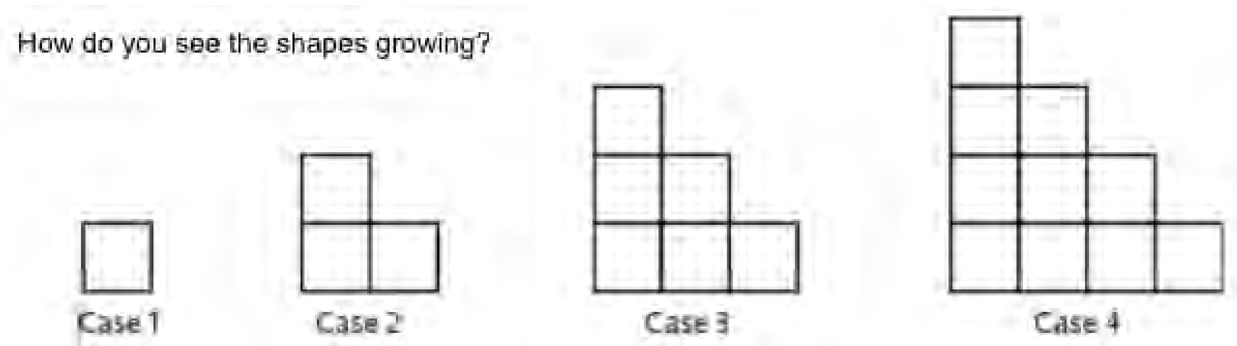


Figure 1. Squares to Stairs

Students worked on the problem outside of class and submitted a "rough draft" solution (Jansen, 2020) on an electronic document that all students had access to. The first 30 minutes of each class were devoted to paired and whole-class work on the problem, facilitated by the instructors.

The data for this study consist of two narratives written by participants, one prior to the beginning of the semester and one upon completion of the course, as well as the teacher educator/researchers' observations of students' work and interactions during in-class problem-solving sessions. The first narrative, called a "Dear Math Letter," was inspired by Bertolone-Smith and MacDonald (2020) and followed Zazkis's (2015) approach of personification. During the summer term, they were prompted to "Write a letter to Math as if Math was a person that you know." For the second narrative, students reflected on their work on the Problems of the Week in relation to the experiences and sentiments described in their initial letter.

We used inductive coding to analyze the narratives, highlighting emergent themes in the students' reflections. Themes in both narratives included salient mathematics learning experiences, shifts in views of mathematics and self-perceptions, and other influential factors such as teachers, classmates, or family members. We also applied a priori codes to indicate instrumental or relational mathematics orientations (Machalow et al., 2020). For the second narrative, we also noted specific problems mentioned by students as influential in their learning and the date during the semester when the problem was assigned. We then compared the two narratives for each participant to identify shifts and discussed themes across the trajectory of the cohort as a whole. Additionally, we used analysis of student work and observations from class sessions to supplement our understanding of the themes.

Results

Similar to previous research, we find that these PSTs entered preparation having had troubled relationships with mathematics, instrumental orientations toward the subject, and negative perceptions of themselves mathematically. PSTs described these relationships in connection to mathematics learning experiences that prioritized memorization, academic achievement, and external evaluation. These findings confirm prior accounts of the power of experiences in shaping PSTs' mathematics teaching identities (Gresalfi & Cobb, 2011; Machalow et al., 2020; Sfard & Prusak, 2005). Despite these negative entry points, students expressed a desire to improve their relationships with math for the sake of their future students.

Over the course of the semester and through participation in collaborative problem-solving sessions, we observed two parallel and related changes. As PSTs engaged with new types of mathematics learning experiences, they began to shift orientations towards mathematics and also began to form new mathematics identities, showing increased confidence and comfort with mathematics as a subject. We hypothesize that these shifts were supported by both the nature of the mathematical problem-solving tasks and the role of a supportive mathematics discourse community within the class.

The sections that follow are structured to illustrate the trajectory of PSTs' developing mathematics teaching identities as they relate to past and current mathematics experiences. We begin by describing the findings from PSTs' "Dear Math Letters", indicating the patterns in the mathematics identities and orientations surfaced through their narrative descriptions. We then outline the changes that occurred throughout the course of the semester as described in their end-of-semester reflections and corroborated by researcher observations. These changes occurred within three discernable phases: *initial apprehensions*, as PSTs began the semester deeply influenced by their prior negative associations with mathematics; *aha moments*, which occurred several weeks into the semester, as PSTs began to experience new realizations about mathematics in their problem-solving work; and, finally, *transformed relationships with mathematics*, where PSTs were adopting new strategies and approaches to problem solving and becoming more confident in their ability as learners and teachers of mathematics. We conclude by discussing how collaboration in a mathematics discourse community served as a key mediator for the transformations described by PSTs at the end of the semester.

"Dear Math, You Are Such a Heartbreaker."

The act of personifying math in their initial *Dear Math* letters brought forth the emotional nature of students' relationships with mathematics. Students revealed predominantly negative mathematics identities, which were deeply intertwined with their social identities. The letters were moving for us as instructors to read; students shared heart wrenching accounts of tears, frustration, anxiety, pressure, and feelings of inadequacy defining their past experiences with mathematics. For example, Faith began her letter telling math: "you have always frustrated me, and I won't deny that I've cried over you more than once…the academic pressure on me felt so crippling."

These accounts of negative experiences often pointed to an instrumental understanding of mathematics, in which memorization of rote procedures and a need to get the answer "right" figured significantly. Students wrote about their experiences with "drills", "missing points", the demoralization of "never getting an A" or the devastation of "getting an F." All eight students mentioned testing as contributing to a sense of inadequacy or uncertainty in relationship to mathematics. In these and other similar comments, students described math as an individual endeavor where one fails or succeeds on their own.

Along with the individualized pressure to succeed, PSTs' descriptions of mathematics learning experiences demonstrated how institutionalized presentations of math can be seen to be highly dehumanizing, perpetuating notions that mathematics ability is linked to a person's value in society. This type of comparison often occurred in relation to other dimensions of students' social identities. In many cases, the notion of being judged in relation to mathematics was highly influential in shaping students' orientations toward and relationships with mathematics. Several students noted feeling inferior to family members or classmates who were "naturally" better in math. Some of these emotions were deeply internalized: Kevin, a student who immigrated from Korea in middle school wrote to Math: "you were the stereotype I desperately tried to run away from. I think I told myself that I was 'not good' at math because I was afraid that you would take over my whole identity".

When PSTs mentioned positive associations with math, they were fleeting; students either felt positively about math when they received good grades or as a result of a supportive relationship with a teacher. Several students mentioned at least one teacher by name who helped them to improve their motivation to do math and their sense of ability to succeed in a particular mathematics course. Other positive orientations to math related to completing a problem, as Natalie described feeling like "the heavyweight champion of the world" after finally solving division problems. These positive experiences did not appear to have a lasting impact on their mathematics identities, perhaps in part because they were contingent upon temporary circumstances.

For many students, the conclusion of their letter to math expressed a desire to reconcile their own negative experiences and find ways to help their future students develop positive mathematics identities. They raised concerns about how their own problematic relationships with math would influence their identities as teachers, as Faith writes: "How can I teach my students to love and engage with you when I don't even do that myself?" As illustrated by this and other statements, PSTs may recognize their negative relationships with math, but pathways to rectify these relationships remain out of reach.

"I Didn't Know You Could Manipulate Numbers Like That!"

A primary component of the mathematics methods courses was collaborative work on the Problem of the Week. Problem solving discussions were structured around students' strategies, and instructors strategically paired students to share different approaches to solving the problem. During the whole class discussion, students compared different strategies and agreed on a reasonable solution. Based on our analysis, we saw three key stages that emerged over the course of the semester through students' work with these problem-solving activities.

Initial apprehensions. In the initial weeks of the semester, we found that PSTs continued to demonstrate behaviors that focused on finding algorithms and wanting to be sure to get the correct answer so as to not be judged by their classmates or teachers. Students also exhibited self-protective behaviors, such as adding self-deprecating notes or humorous memes to their submitted rough-draft thinking in an effort to deflect from their feelings of inferiority. This initial period of discomfort was evident in many students' reflections at the end of the semester. For example, Dylan explains:

> When this assignment first started, I felt very uneasy presenting my work to the class. I was constantly comparing myself to the other students… I feared my classmates would judge me as not being intelligent enough for the program. This was clearly reflected in my early Problem of the Week assignments…there is little evidence of work being completed and there is an emphasis on correctness.

This type of response was shared by the majority of the students in class, as they reported finding it difficult to engage with problems and practices that were counter to their prior mathematics learning environments. Students realized, however, that they needed to "unlearn" these deeply ingrained tendencies.

Aha moments. The structure of sharing work collectively in class pushed students to work through their vulnerabilities. Through discussing different strategies and visualizations for solving problems, students were exposed to new ways of thinking about mathematics, which led to a second phase that we refer to as a period of *aha moments*. Students gradually began to place more emphasis on strategies than arriving at a correct answer:

> When we worked on the problem in class I felt like something finally clicked and I realized that my visual model would have been useful if I had spent less time worrying about whether or not it was the "correct" way to solve the problem and instead just stuck it out and

recognized that my ideas do not have to be correct to be worthwhile, they have an inherent worth." (Brandon)

Two problems in particular, *Transparent Algorithms* and *Squares to Stairs*, helped students to see more mathematical meaning and develop deeper conceptual understanding. Both the nature of these problems as well as the timing mid-semester could have contributed to multiple students reaching *aha moments* at this point. Madison described her *aha moment,* stating: "On transparent algorithms, I did not know that you could manipulate numbers like that, straying from the basic, standard algorithms!" *Squares to Stairs* engaged students in recognizing patterns to help visualize figures in linear sequences. Unlike earlier in the semester when students focused on finding an algorithm to solve the problems, students used multiple ways to visualize and solve the pattern. Natalie's *aha moment* noted the importance of representing problems to find solutions: "Because of how we have been socialized, I would have thought that it was an elementary school tactic to visually represent a problem. I saw how my classmates drew out their patterns and I thought 'Why not?'". In these moments, students describe developing increased relational understandings as opposed to their initial instrumental orientations. As these orientations shifted, students also expressed more enthusiasm for problem solving. They began to position themselves as capable of breaking from the procedural rules of mathematics in favor of trying new strategies without a fear of being "wrong".

Transformed relationships to mathematics. During the final phase, students began taking on new strategies and approaches that they learned through their problem-solving discussions. As highlighted in their *aha moments*, working in collaboration with one another helped them to feel empowered to try out different strategies that focused on developing conceptual understanding rather than getting the correct answer. Once Natalie found that it was not indicative of a lack of understanding to attempt multiple strategies, she explained that she "looked for patterns, switched orders, crossed things out and used the process of elimination. I surprised myself and my confidence grew." Similarly, Faith reflects on how she began documenting "discoveries" as a result of her work with her classmates, which helped her "discover something new in every iteration" as she worked towards a solution.

As a result of engaging in problem solving discussions and reflective identity work, students made incredible shifts in their mathematics identities and orientations over the course of one semester. Several students continued to teach math in their field placements beyond what was required, and three students went from thinking they would never want to teach math to expressing interest in seeking certification to teach middle years math.

"I Had Eight Other Minds On the Job."

Looking across their reflections on their experiences engaging in the Problems of the Week, we found that the role of the mathematics learning community was a key mediator in helping students to shift their relationships with mathematics. Students trusted the class community in a way that enabled them to move through their vulnerabilities and in doing so, they allowed themselves to release the security of algorithms and accuracy. Brandon expressed how working with his peers supported this shift: "I felt like I was part of a community of inquiry dedicated to finding solutions instead of being evaluated on my ability to simultaneously learn, synthesize, and enact a new concept."

Further, the experience challenged their preexisting notions of math as individualized and dehumanizing. Students found support in one another and were able to move beyond their initial phase of vulnerability. While before this conceptual and identity work, Kevin purposefully disassociated from mathematics to avoid stereotypes, he described how working with others allowed him to grow: “I always assumed that ‘doing math’ was an individualized process. But incorporating meaningful discourse into the process of ‘inquiring’ into new mathematical concepts has evidently shown me that learning math is really a journey of growth.”

Discussion

Our study reaffirms prior findings that suggest PSTs benefit from opportunities to re-learn mathematics from a relational perspective (Ball, 1990) and from engaging in narrative identity work to reconcile their prior experiences with mathematics (e.g., Machalow et al., 2020; McCulloch et al., 2013; Remillard, 1993). Our data also help to illustrate that PSTs’ mathematics identities are complex and intertwined with other social identities that are implicated in how they learn to teach mathematics. Addressing a personified version of mathematics allowed PSTs to unpack how their prior experiences shaped their identities and helped them to problematize their instrumental understandings of math.

Our study provides a unique window into the power of engaging in this type of narrative identity work in tandem with conceptually oriented, collaborative problem solving. We believe that these two processes were mutually supportive in our PSTs’ transformation across the course of the semester, which included beginning to identify as teachers of mathematics (Ntow & Adler, 2019; Lutovac & Kaasila, 2014). The cycle of reflecting on their views of math while working with one another in community rather than individually gave students space to envision a new approach to mathematics instruction. Norms for openness, learning from mistakes, and communication throughout the problem-solving discussions helped students increase their confidence and willingness to adopt new strategies. Further, their reflections on the problem-solving experience itself helped them to see the power of community and express a desire to carry that into their own classrooms.

Importantly, we found that this process takes time and consistent, on-going work to develop a community of practice that allows students to share their vulnerabilities with one another. Although our study focuses on a small cohort of PSTs, we posit that combining opportunities for students to develop relational understandings of math with narrative identity work and community building offers opportunities for teacher educators to support the development of positive mathematics teaching identities in preservice teachers.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. For the Learning of Mathematics, 10(2), 10–16.

Bertolone-Smith, C.M., MacDonald, B.L. (2020, February). *Dear math…: Examining women’s self-identified critical experiences in mathematics* [Paper presentation]. Annual Association of Mathematics Teacher Educators, Pheonix, AZ.

Brady, P., & Bowd, A. (2005). Mathematics anxiety, prior experience and confidence to teach mathematics among pre-service education students. *Teachers and Teaching,* 11(1), p. 37-46.

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19–33.

Graven, M., & Heyd-Metzuyanim, E. (2019). Mathematics identity research: The state of the art and future directions: Review and introduction to ZDM Special Issue on Identity in Mathematics Education. *ZDM*, *51*(3), 361–377.

Gresalfi, M. S., & Cobb, P. (2011). Negotiating Identities for Mathematics Teaching in the Context of Professional Development. Journal for Research in Mathematics Education, 42(3), 270–304.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Towards a new equity research agenda. Mathematical Thinking and Learning, 4(2 & 3), 145–187.

Harkness, S. S., D’Ambrosio, B., & Morrone, A. S. (2007). Preservice elementary teachers’ voices describe how their teacher motivated them to do mathematics. *Educational Studies in Mathematics*, *65*(2), 235–254.

Hodges, T. E., & Hodge, L. L. (2017). Unpacking personal identities for teaching mathematics within the context of

prospective teacher education. *Journal of Mathematics Teacher Education*, *20*(2), 101–118.
Jansen, A. (2020) *Rough draft math: Revising to learn*. Stenhouse Publishing.
Kaasila, R., Hannula, M. S., Laine, A., & Pehkonen, E. (2008). Socio-emotional orientations and teacher change. Educational Studies in Mathematics, 67(2), 111–123.
Lave, J. & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. Cambridge: Cambridge University Press.
Lopresto, K. D., & Drake, C. (2004). What's Your (Mathematics) Story? Teaching Children Mathematics, 11(5), 266–271.
Lutovac, S., & Kaasila, R. (2014). Pre-service teachers' future-oriented mathematical identity work. *Educational Studies in Mathematics*, *85*(1), 129–142.
Machalow, R., Goldsmith-Markey, L. T., & Remillard, J. T. (2020). Critical moments: Pre-service mathematics teachers' narrative arcs and mathematical orientations over 20 years. *Journal of Mathematics Teacher Education.*
McCulloch, A. W., Marshall, P. L., DeCuir-Gunby, J. T., & Caldwell, T. S. (2013). Math autobiographies: A window into teachers' identities as mathematics learners. School Science & Mathematics, 113(8), 380–389
McGlynn-Stewart, M. (2010). Listening to Students, Listening to Myself: Addressing pre-service teachers' fears of mathematics and teaching mathematics. *Studying Teacher Education*, *6*(2), 175–186.
Ntow, F. D., & Adler, J. (2019). Identity resources and mathematics teaching identity: An exploratory study. ZDM, 51(3), 419–432.
Relich, J. (1996). Gender, Self-Concept and Teachers of Mathematics: Effects on Attitudes to Teaching and Learning. *Educational Studies in Mathematics*, *30*(2), 179–195.
Remillard, J.T. (1993, April). *Using experience to break from experience: An elementary mathematics methods course* [Paper presentation]. Annual Meeting of the American Educational Research Association, Atlanta, GA.
Skemp, R. R. (1976). Relational understanding and instrumental understanding. Mathematics Teaching, 77(1), 20–26.
Sfard, A., & Prusak, A. (2005). Telling Identities: In Search of an Analytic Tool for Investigating Learning as a Culturally Shaped Activity. *Educational Researcher*, *34*(4), 14–22.
Zazkis, D. (2015). Monsters, lovers, and former friends: Exploring relationships with mathematics via personification. *For the Learning of Mathematics, 35*(1), 33-38.

ENGAGING PROSPECTIVE SECONDARY TEACHERS IN GEOMETRY-FOCUSED ETHNOMODELING EXPLORATIONS

Siddhi Desai
University of Central Florida
siddhi.desai@knights.ucf.edu

Farshid Safi
University of Central Florida
farshid.safi@ucf.edu

Sarah B. Bush
University of Central Florida
sarah.bush@ucf.edu

Janet Andreasen
University of Central Florida
janet.andreasen@ucf.edu

Trena Wilkerson
Baylor University
trena_wilkerson@baylor.edu

Daniel Clark Orey
Federal University of Ouro Preto
oreydc@ufop.edu.br

More than 50% of PK-12 students in the United States are students of color and/or from historically underrepresented populations (NCES, 2020). Yet for so long, mathematics curriculum has not been representative of diverse student populations or inclusive of their backgrounds and their identities (AMTE, 2015; D'Ambrosio, 2017; TODOS, 2020). Through a focus on the intersections of mathematical modeling, ethnomathematics and cultural and historical practices, this ethnomodeling research study intentionally connects ways to engage and value each and every students' culture and identity while emphasizing essential geometry content. During this presentation, we share results from a study examining how a sequence of geometry-focused ethnomodeling tasks informed secondary mathematics prospective teachers' perceptions of valuing their own as well as their future students' cultures and identities.

Keywords: Ethnomathematics; Culturally Relevant Pedagogy; Equity, Inclusion, and Diversity; Geometry and Spatial Reasoning

For centuries, mathematicians and mathematics educators have debated whether mathematics is culturally bound (Bishop, 1988; D'Ambrosio, 1985) or culture free (Kline, 1980). Oftentimes, mathematics is presented as a "prized body of knowledge" stripped of its rich cultural and historical connotations, and far removed from the lives and ways of many around the world (Naresh, 2015). More than 50% of PK-12 students in the United States are students of color and/or from historically underrepresented populations (NCES, 2020). Yet for so long, mathematics curriculum has not been representative of diverse student populations or inclusive of their backgrounds, cultures, and their identities (AMTE, 2015; D'Ambrosio, 2017; TODOS, 2020). In order to confront the separation between context and content, in recent years there has been an increased interest in integrating mathematical modeling into the K-12 curriculum (Abassian et al., 2019; Niss et al., 2007; NGA & CCSSO, 2010; Pollak, 2016). In addition, a recent international survey of mathematics education researchers calls for increased research in equity, diversity, and inclusion as well as connecting mathematics to other disciplines such as integrated STEAM, familial activities, and leisure activities (Bakker et al., 2021). Mathematics should be accessible and equitable for all students by incorporating cultural diversity into mathematics lessons (AMTE, 2017; NCSM & TODOS, 2016; NCTM, 2018, 2020a, 2020b). Gutiérrez (2018) describes one way to rehumanize mathematics through valuing student's identities and their cultural traditions and practices is through an ethnomathematics perspective. For such efforts to be long lasting, it is crucial that prospective teachers (PSTs) experience this advocacy modeled in their preparation programs so they are prepared for their roles in developing students who contribute to a more equitable and just society (D'Ambrosio & D'Ambrosio, 2013).

Purpose of the Study and Research Question

The purpose of this study was to further extend and connect work by mathematics education researchers, educators, and K-12 teachers in integrating modeling with mathematics. Specifically, the ethnomodeling perspective intentionally connects students' cultural and historical identities to the teaching and learning of geometry.

As supported by NCTM (2020a, 2020b), experiencing geometry in an integrated and active manner can capitalize on the wonder, joy, and beauty of examining the world. The research goal for this study was to engage PSTs in geometry-focused ethnomodeling tasks to develop and understand new insights into honoring and sustaining cultural systems and practices through examples that are grounded in a shared commitment to equity, empowerment, and dignity. This study addressed the following research question:

In what ways can a sequence of geometry-focused ethnomodeling tasks inform secondary mathematics PSTs' perceptions of valuing

- their own cultures and identities?
- their students' diverse cultures and identities?

Conceptual and Theoretical Perspectives

There are many frameworks that inform and connect to this study including ethnomathematics (D'Ambrosio, 1960), culturally relevant pedagogy and teaching (Gay, 2002; Ladson-Billings, 1995), community funds of knowledge (Civil et al., 2002), and culturally sustaining pedagogy (Paris, 2012). Despite the critical contributions of these frameworks to the field, some argue that these frameworks can lack a focus on mathematical content. Previous studies have also created and examined frameworks that connect the mathematical modeling cycle with culturally relevant pedagogy (Anhalt et al., 2018), and with social justice issues (Aguire et al., 2019). An ethnomodeling approach attempts to apply mathematical modeling to bridge domains of ethnomathematics while preserving and valuing aspects of the local culture (Rosa & Orey, 2013). Through such engagements, mathematics teacher educators can help pre- and in-service teachers strategize and build a progression for ways of teaching mathematical modeling with the relevant cultural aspects that help shape critical consciousness for students (Anhalt et al., 2018).

Ethnomodeling

Ethnomodeling is the intersection of cultural anthropology, ethnomathematics, and mathematical modeling. Cultural anthropology refers to the aspects of identity connected to language, religion, art, and other aspects of people's daily mathematical practices. Ethnomathematics refers to the understanding of the relationship between culture and mathematics and mathematical modeling approaches that attempt to engage students in real world applications of mathematics. In much of the current literature on mathematical modeling, five perspectives are commonly discussed – realistic, educational, models and modeling, socio-critical, and epistemological (Abassian et al., 2019; Rosa & Orey, 2010). However, each of these would benefit from greater linkage to cultural components, and this is where ethnomodeling connects. One of the primary goals of ethnomodeling is to engage and empower people to mathematize their own realities, and is studied through three lenses - emic, etic or dialogic. In their previous work, Orey and Rosa worked with students and teachers in Brazil, Nepal (e.g., Orey & Rosa, 2020; Rosa & Orey, 2009), where the participants all come from very similar racial and ethnic backgrounds. In this study, the participants at a university in the United States came from diverse backgrounds and experiences.

Geometry

Another key focus particularly for this study was secondary geometry content. Previous studies have shown there is a lack of exposure to deep geometrical thinking, and many teachers need additional support to effectively teach it with depth (Clements, 2003; Driscoll, 2007; Steele, 2013). Geometry has historically been the content domain with the lowest performance on international exams such as The Trends in International Mathematics and Science Study (TIMSS, 2019). Geometry is a critical content domain that spans across our K-12 standards (NGA & CCSSO, 2010). Additionally teaching geometry is crucial in facilitating student opportunities to make connections between and within mathematical content and representations (Desai et al., 2022); and connections with the real world (Jones, 2002; Usiskin, 1980). Through this focus, the intentional connections and integrations of ethnomodeling and geometry afford participants opportunities to link mathematical domains enriched by the wonder, joy, and beauty of geometry in our lives, communities, and cultures.

Conceptual Framework

Previous studies have created and examined a framework that connects the mathematical modeling cycle with culturally relevant pedagogy (Anhalt et al. 2018), and with social justice issues (Aguire et al., 2019). For this study, the conceptual framework illustrated by Figure 1 describes the interaction between the aspects of ethnomodeling and mathematical modeling. The cyclical and iterative process of this framework informed the design and creation of the tasks as well as the process that participants engaged.

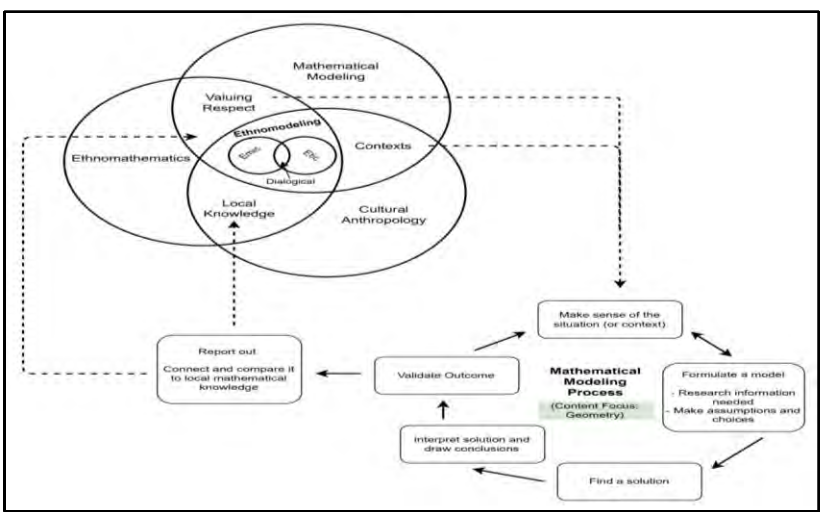


Figure 1: Ethnomodeling Conceptual Framework

Methods

Context

Previous studies have not focused explicitly on the intersection of ethnomodeling and geometry in connection to PSTs perceptions. As such, a qualitative research methodology described by Creswell (2013) was used for the purposes of this study. Within the qualitative methodology, this study was designed as a multi-case study as described by Creswell (2013), Merriam (2009), Stake (1995), and Yin (1994).

This research study took place in a mathematics education methods course in a college of education at a large suburban public university in the southeastern U.S. during the Spring 2022 semester. The participants of this study were secondary mathematics (Grade 6-12) education PSTs, as this aligns to the programmatic and course structure at the university where this study was conducted.

Participants

The total number of PSTs enrolled in this course was 20 undergraduate students from which 18 were participants for this study. Specifically, the research study took place during the first mathematics education methods course that PSTs in this program are required to take. This course was intentionally chosen because it was early in the PSTs' program and is taken prior to PSTs starting their internship in schools, and these were important criteria in helping to best inform the research question. Of the 18 participants, seven expressed interest in being selected as a case participant. In selecting the cases, multiple factors were considered in the following order of importance: (1) As much as possible keeping participants in their chosen groups to maintain natural group dynamics; (2) Heterogeneous representation amongst cases in terms of race/ethnicity; (3) Heterogeneous representation amongst coursework as well as experiences with diversity. Based on these criteria, six participants were selected- Elaine, Lucas, Helina, Diego, Tanh, and Zohar - all participant names are pseudonyms. Table 2 provides an overview of these six participants.

Table 2: Participants

Group #	Potential Cases	Race/ Ethnicity	Gender	Coursework Experience	Experience with Diversity
1	Elaine	White	F	Some	Some
1	Lucas	White	M	Some	More
2	Helina	White	F	Some	Some
2	Diego	Hispanic	M	Some	Some
3	Tanh	Asian-Vietnamese	F	Some	Some
3	Zohar	White-Jewish	M	Some	More

Data Collection and Analysis

All PSTs in the course completed the pre-survey, ethnomodeling tasks, written reflections based on these tasks, as well as the post-survey as these were regular course activities. However, only work completed by the participants chosen as the cases was analyzed as further described in the *Data Analysis* section. Additionally, the case study participants were asked to participate in three interviews. Figure 2 shares a summary of data source collection which is then described in greater detail.

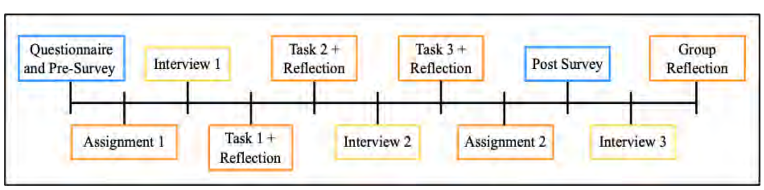


Figure 2: Data Sources Overview

Surveys. PSTs were asked to individually complete one questionnaire and one pre/post-survey. The *Academic and Demographic Background Questionnaire* contained three parts according to recommendations by Siwatu (2007) which state that for studies focusing on multicultural aspects, obtaining information about PSTs' academic and demographic background including items related to their racial and ethnic background, coursework, and experiences in multicultural settings can be helpful in gaining baseline information. Section A of the *Multicultural Efficacy Scale* (MES) (Guyton & Wesche, 2005) was used to gain insights into additional personal and professional experiences PSTs may have had in multicultural settings.

Section B and C of the *MES* was used as the pre- and post-survey. This survey contained 2 parts - Part 1: Attitude Towards Teaching Culturally Diverse Students and Part 2: Efficacy in Implementing Multicultural Teaching Strategies. As the pre-survey was used to establish a baseline for participants' perceptions of how prepared they feel to enact the teaching practices discussed. These results were then compared to the post survey through a matched pairs t-test for dependent samples to assess any changes.

Tasks, Assignments and Reflections. One of the major goals of ethnomodeling is to acknowledge, value, and respect the traditional and local knowledge and to find and develop techniques to provide a translation and contextualization of mathematical ideas. In alignment with these ideas, three tasks and two activities were developed for this study focusing on both valuing and respecting diverse cultures and histories and learning and translating geometrical ideas as well as intentionally emphasizing the iterative process of the ethnomodeling cycle. Each task began with contextualizing the task to give value and respect to the local ethnomathematics practices (*Task Launch*). In the next phase of the task, participants engaged in the mathematical modeling process to mathematize and find a solution to the given problem (*Mathematize and Find a Solution*). In the last phase, the task recontextualized the situation by connecting the task to local knowledge with a particular focus on valuing and respecting the local practices (*Reflections and Connections*).

These tasks were created using examples of ethnomodeling tasks described by Rosa and Orey (2009), and culturally relevant mathematical modeling tasks described by Anhalt and colleagues (2018), as well as built on researchers' previous work (Safi & Desai, 2017). Table 3 provides a brief overview of the ethnomodeling connections (cultural/historical and geometry connections) for each of the assignments and tasks. Following engagement in each task, PSTs submitted an individual written reflection. The assignments were completed individually outside of class and the tasks were completed in small groups of 4-5 PSTs during class time.

Interviews. The six participants chosen as the cases participated in three semi-structured interviews. Figure 2 in the Data Collection and Procedures section shares when in the process each interview took place. Each of the interviews provided participants an opportunity to extend and reflect on their group's work, as well as their own beliefs and perceptions.

Data analysis. The data collected for this study comprised both quantitative and qualitative components. The questionnaire as well as the pre-/post-surveys were analyzed quantitatively using descriptive statistics as well as a matched pair t-test. Task artifacts, task reflections, as well as interviews were analyzed qualitatively through coding and categorizing artifacts by themes based on the Ethnomodeling Conceptual Framework and finding other themes that emerged during the analysis.

Table 3: Assignments and Task Descriptions and Connections

Task	Cultural/Historical Connections	Geometry Connections
Assignment 1: Geometry Around You	Connections to local cultural and historical aspects	Geometry in the world around us
Task 1: City Explorations	Cultural and historical aspects that have influenced city and structural planning	Angles, 3D shapes, Properties of circles, Scale drawings
Task 2: Tiles and Tiling Patterns	Tiles and tiling patterns from around the world with a focus on cultural and historical significance of Mexican Talavera tiles	Geometric transformations, Ratios/ Scale
Task 3: Exploring Connections between 2D-3D Geometry	Connections to Moorish architecture and influence of cultures across time and place.	2D ←→ 3D geometry, Lines/ segments, Parallel/ perpendicular lines, Symmetry, Midpoint, Geometric modeling (HS standards)
Activity 2: Creating your own tile	Connections to own culture, history and identity.	Lines/ segments, Parallel/ perpendicular lines, Symmetry, Midpoint, Geometric transformations, Geometric modeling (HS standards)

Results

The data collection process that included both quantitative and qualitative data is summarized in this section. Specifically, these results are examined through the lens of the *Ethnomodeling Conceptual Framework* (Figure 1) which guided all aspects of this study including the task creation, data collection, and data analysis. The three main components of the ethnomodeling framework include cultural anthropology, ethnomathematics, and mathematical modeling. Following the individual summary of each, the results are synthesized to discuss PSTs' work with ethnomodeling. For the purpose of this paper, we focus the analysis on the individual and group data from Group 2.

Case Descriptions

The primary focus of this study involved aspects related to the identities and backgrounds of teachers and students and as such it would be expected that these PSTs' personal stories impacted the ways they engaged in this study. Group 2 members were Helina, Diego, and three additional PSTs. The group was diverse in terms of gender, race and ethnicity.

Helina identifies as a White female and at the time of this study was a third-year student at the university. This was her third major – prior to this she was an architecture major and then a computer science major. Outside of mathematics classrooms, she has had these discussions and experiences in her K-12 art and undergraduate architectural classes, and also enjoys doing her own research to learn about architectural styles and cultures from around the world. These experiences seemed to greatly influence her contributions to the small group and whole class discussions.

Diego identifies as a Hispanic male, and at the time of this study, Diego was a mathematics teacher at a private school. From all the PSTs in the class, he was the only one with formal K-12 teaching experience. Outside of teaching and learning mathematics Diego shared his other biggest passion is music. Diego was able to bring in his experiences as a current teacher. These perspectives were important in his interactions with the tasks as well as in the interactions and discussions he had with his peers.

Cultural Anthropology

In discussing culturally relevant and sustaining pedagogies and teaching, Ladson-Billings (1995), Gay (2000), and Paris (2012) all reinforce the importance of including students' cultural references in all aspects of learning to promote racial and ethnic equality across diverse communities. Overall, the mean of the participants' beliefs and attitudes towards culturally relevant teaching on the pre- to post-survey increased from $M = 3.37$ to $M = 3.42$, however this difference was not statistically significant at the $p<0.05$ level. For the factors relating to overall efficacy for implementing culturally relevant teaching practices, the increase in mean was statistically significant.

From the first interview with Helina, and Diego, it was evident that they individually shared some tendencies towards wanting to incorporate multicultural ideas in their classroom but not necessarily tied in with mathematical content. By the end of the study, both Helina and Diego shared that the ethnomodeling activities they engaged in opened their eyes towards the importance of incorporating students' identities and cultures in mathematics. This sentiment was also shared by other members of their group who agreed that if engaging in such tasks as PSTs had an impact on their views of their own culture and identity, such tasks were even more crucial to incorporate with K-12 students who are learning the content for the first time.

Ethnomathematics

Ethnomathematics focuses on studying practices of people in different cultures from around the world (Ascher, 2002; D'Ambrosio, 1985). From the beginning of this study, as indicated by the pre-survey data, many of the PSTs had favorable attitudes towards culturally relevant teaching. From their own experiences as K-12 students, Helina and Diego mentioned learning about the integration of multicultural ideas in their world history or world languages classes, or "we would talk about things during Black History Month and things like that but the focus was really on testing" (Helina, Interview 1). Through their engagement in this study, the PSTs were able to experience geometry focused tasks through which they saw a model for integrating such ideas within their future mathematics classrooms. As Group 2 shared in their final reflection,

> Our perceptions changed because we could see that there is so much geometry in the world around us as well as in culture that we can incorporate into our lessons. It has made us all look at a different way to go about how we can include everyone's identities and cultures that we would not have thought of before. (Group 2 Reflection)

In addition to becoming more aware of the geometry that exists in the world around them, this group also shared how the tasks from this study provided them with a model for synergistic integration of both incorporating their own as well as their future students' cultures and identities as well as teaching and learning mathematics.

Mathematical Modeling With Focus on Geometry

Students sometimes perceive school mathematics to be a narrow set of rules and algorithms that have little or no meaning to their lives. This was evident in both Diego and Helina's responses to what they think of when they hear geometry - "shapes, proofs, and the subject that so many students hate" (Interview 1). When mathematical content is devoid of relevancy to students and their lives, there is a decrease in interest and motivation for learning it. Geometric ideas and images surround students in the form of art, architecture, and patterns, and "by integrating geometry, art, and history, we can create opportunities for students to experience the excitement and beauty of mathematics that build on foundational geometric ideas and elevate student interest in mathematics and related subjects" (Desai & Safi, 2020, p. 96). An ethnomodeling approach allows for the focus on both learning about historical and cultural ideas and engaging in the mathematical modeling cycle to further learn about the mathematics of these practices.

Following each in-class task exploration, PSTs individually reflected on their experiences. During their explorations, in their interviews, as well as elsewhere in their reflections they discussed many mathematical ideas their group engaged in. However, when asked to name specific standards, they often left out key ideas. In sharing a specific example of the changes they would make for the future, for Task 2, Helina shared that her group would need to focus on connecting their plan back to the real world context. In creating a plan for tiling their wall, they used reflections which is possible on GeoGebra, however in real life where tiles are one sided this would not work. Diego added that a solution for this could be pre-designing two tiles that are reflections of one another. Such discussions point to the need for integrating real-world contexts into mathematics tasks.

Ethnomodeling

At the beginning of this study, ethnomodeling was new terminology for all of the participants. Many of them had some familiarity with engaging in mathematical modeling tasks in previous mathematics education courses, however integrating that with cultural and historical components was a new approach for them. In their final group reflections when asked how they would describe ethnomodeling tasks to a friend or colleague outside of this study, the group shared "Ethnomodeling is incorporating culture and art into mathematical concepts".

While the group was able to define key components of ethnomodeling, in sharing how these tasks will influence their own practice, the group shared they would use such tasks in introducing concepts before they go on to teaching traditional textbook content. This response calls for a greater need for continued discussions about redefining what it means to do, teach, and learn mathematics in our mathematics teacher education preparation programs.

Discussion and Concluding Remarks

While all the participants agreed that ethnomodeling tasks were necessary to incorporate in mathematics education, there was a consensus that they would need more guidance and examples of how to incorporate such tasks into other mathematical content domains. Research efforts have indicated the need for further ways to explicitly connect students' identity, culture with mathematical experiences - and ethnomodeling can be one way to achieve this goal. As educators, we have a moral responsibility to tell the story of mathematics from multiple standpoints that provide our students an immersion in mathematical experiences informed by cultural and historical perspectives (Naresh, 2015). Through engaging in ethnomodeling tasks, PSTs can learn to value and respect mathematical practices of diverse cultures and traditions as well as become more prepared to engage their students in such geometric tasks. An ethnomodeling approach creates necessary critical dissonance as it pushes on current inequitable and unjust systems, and through this integrated ethnomathematics and modeling approach we can create 'resonant harmony. Through such explorations, PSTs are able to learn mathematical content in de-siloed and connected ways, and naturally find connections between mathematics and other content domains - and explore and share their own experiences. Such experiences also empower PSTs to engage their future students to experience the wonder, joy, and beauty of mathematics in the world around them.

References

Abassian, A., Safi, F., Bush, S., & Bostic, J. (2019). Five different perspectives on mathematical modeling in mathematics education. *Investigations in Mathematics Learning, 12*(1), 53- 65.

Aguirre, J. M., Anhalt, C. O., Cortez, R., Turner, E. E., & Simic-Muller, K. (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice: The Flint water task. *Mathematics Teacher Educator, 7*(2), 7-26.

Association of Mathematics Teacher Educators. (2015). *Equity in mathematics teacher education* [Position Statement]. https://amte.net/sites/default/files/amte_equitypositionstatement_sept2015.pdf

Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for Preparing Teachers of Mathematics*. amte.net/standards.

Anhalt, C., Staats, S., Cortez, R., & Civil, M. (2018). Mathematical modeling and culturally relevant pedagogy. In Y. J. Dori, Z. Mevarech, & D. Baker (Eds.), Cognition, metacognition, and culture in STEM education (307–330). The Netherlands: Springer.

Ascher, M. (2002). *Mathematics elsewhere: An exploration of ideas across cultures*. Princeton, NJ: Princeton University Press.

Bakker, A., Cai, J. & Zenger, L (2021). Future themes of mathematics education research: an international survey before and during the pandemic. *Educational Studies in Mathematics, 107*, 1–24.

Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics, 19*(2)*,* 179-191.

Clements, D. H. (2003). Learning and teaching geometry, K-12. In G. W. Martin, D. Schifter & J. Kilpatrick (Eds.), *A research companion to principles and standards for school mathematics* (pp. 151-178). Reston, VA: National Council of Teachers of Mathematics.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches* (3rd ed.). Los Angeles, CA: Sage.

D'Ambrosio, U. (1985). Ethnomathematics and its place in the history and pedagogy of mathematics. *For the Learning of Mathematics, 5*, 44-48.

D'Ambrosio, U. (2017). Ethnomathematics and the Pursuit of Peace and Social Justice. *Digital Thematic Education, 19*(3), 653-666.

D'Ambrosio, U., & D'Ambrosio, B. S. (2013). The role of ethnomathematics in curricular leadership in mathematics education. *Journal of Mathematics Education at Teachers College, 4*, 19-25.

Desai, S., Bush, S. B., & Safi, F. (2022). Mathematical representations in the teaching and learning of geometry: A review of the literature from the United States. *Electronic Journal for Research in Science and Mathematics Education, 25*(4), 6-22*.*

Desai, S., & Safi, F. (2020). Mathematical art: Lost in translation. *NCTM Mathematics Teacher: Learning and Teaching PK-12, 113*(1), 96.

Driscoll, M. (2007). *Fostering geometric thinking: A guide for teachers, grades 5-10*. Portsmouth, NH: Heinemann.

Gay, G. (2000). Culturally responsive teaching: Theory, research, and practice. New York: Teachers College Press.

Gutiérrez, R. (2018). Why we need to rehumanize mathematics. In S. Lerman (Ed.), *Annual Perspectives in Mathematics Education: Rehumanizing mathematics for students who are Black, Latinx, and Indigenous* (pp. 1-10). Reston, VA: NCTM.

Guyton, E. M. & Wesche, M. V. (2005). The multicultural efficacy scale: Development, item selection, and reliability. *Multicultural Perspectives, 7*(4), 21-29.

Jones, K. (2002). Issues in the teaching and learning of geometry. In: L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 121-139). London: Routledge Falmer.

Kline, M. (1980). Mathematics, the loss of certainty. New York, NY: Oxford University Press.

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal, 32*(3), 465-491.

Merriam, S. B. (2009). Qualitative research: A guide to design and implementation. San Francisco, CA: Jossey-Bass.

Naresh, N. (2015). The role of a critical ethnomathematics curriculum in transforming and empowering learners. *Revista Latinoamericana de Etnomatemática, 8*(2), 450-471.

National Center of Education Statistics (NCES). (2020). *Digest of education statistics (Table 203.70.)*. U.S. Department of Education. Washington, DC: National Center for Education Statistics. Retrieved from https://nces.ed.gov/programs/digest/d20/tables/dt20_203.70.asp.

National Council of Supervisors of Mathematics and TODOS: Mathematics for ALL (NCSM and TODOS). (2016). Mathematics Education through the Lens of Social Justice: Acknowledgement, Actions, and Accountab*ility. Joint Position* Paper.
National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Reston, VA: NCTM.
National Council of Teachers of Mathematics (2020a). *Catalyzing change in early childhood and elementary mathematics: Initiating critical conversations*. Reston, VA: NCTM.
National Council of Teachers of Mathematics (2020b). *Catalyzing change in middle school mathematics: Initiating critical conversations*. Reston, VA: NCTM.
National Governors Association Center for Best Practices, Council of Chief State School. (2010). *Common core state standards mathematics.* Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.
Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P. L. Galbraith, W. Henn, & M. Niss (Eds.), *Modeling and applications in mathematics education* (pp. 3–32). New York, NY: Springer.
Orey, D. C., & Rosa, M. (2020). Positionality and creating dialouge in Nepal: Connecting ethnomathematics and modelling - The importance of place through ethnomodelling. *Social Inquiry: Journal of Social Science Research, 2*(1), 82-103.
Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational Researcher, 41*(3), 93-97.
Pollak, H. (2016). Foreword. In C. Hirsch, & A. R. McDuffie (Eds.), *Annual perspectives in mathematics education 2016: Mathematical modeling and modeling mathematics* (pp. vii-viii). Reston, VA: The National Council of Teachers of Mathematics, Inc.
Rosa, M. & Orey, D. (2009). Symmetrical freedom quilts: the ethnomathematics of ways of communication, liberation, and art. Revista Latinoamericana de Etnomatemática, 2(2). 52-75.
Rosa, M. & Orey, D. C. (2010). Ethnomodeling as a Pedagogical Tool for the Ethnomathematics Program. *Revista Latinoamericana de Etnomatemática, 3*(2). 14-23.
Rosa, M., & Orey, D. C. (2013). Ethnomodeling as a research theoretical framework on ethnomathematics and mathematical modeling. *Journal of Urban Mathematics Education, 6*(2), 62-80.
Safi, F., & Desai, S. (2017). Promoting mathematical connections using 3-D manipulatives. NCTM Mathematics Teaching in the Middle School, 22(8), 488-492.
Siwatu, K. O. (2007). Preservice teachers' culturally responsive teaching self-efficacy and outcome expectancy beliefs. *Teaching and Teacher Education, 23*, 1086–1101.
Stake, R. E. (1995). *The art of case study research.* Sage Publications, Inc.
Steele, M. D. (2013). Exploring the mathematical knowledge for teaching geometry and measurement through the design and use of rich assessment tasks. *Journal of Mathematics Teacher Education, 16*(4), 245–268.
TIMSS (2015). Mathematics for grades 4 and 8: Trends over three-time points content subscales. Retrieved from https://nces.ed.gov/timss/timss2015/timss2015_table09a.asp.
TODOS: Mathematics for All. (2020). *The mo(ve)ment to prioritize antiracist mathematics: Planning for this and every school year* [Position Statement]. https://www.todos-math.org/assets/The%20Movement%20to%20Prioritize%20Antiracist%20Mathematics%20Ed%20by%20TODOS%20June%202020.edited.pdf
Usiskin, Z. (1980). What should not be in the algebra and geometry curricula of average college-bound students? *The Mathematics Teache*r, *73*, 413-424.
Yin, R. K. (1994). Discovering the future of the case study method in evaluation research. *Evaluation Practice, 15*(3), 283-290.

PRACTICE-BASED AND SOCIAL JUSTICE APPROACHES TO MATHEMATICS TEACHER EDUCATION: MOVING FROM DISSONANCE TO HARMONY

Kristin Doherty
Michigan State University
kdoherty@msu.edu

The dissonance in the field between practice-based and justice-oriented approaches to teacher education motivated my review of the literature on the relationship between these approaches, specifically in the context of mathematics teacher education. I reviewed literature to better understand reasons for this dissonance and explore how these approaches might be harmonized within mathematics teacher education. Building on this review, I present a framework for conceptualizing practice-based approaches in justice-oriented ways. I argue social foundations can support preservice teachers in developing a critical lens to apply in their enactment of mathematical practices through practice-based approaches.

Keywords: Preservice Teacher Education, Teacher Educators, Social Justice

Problem Statement and Objectives

The dissonance in the field between practice-based teacher education (PBTE) and a teacher education oriented toward social justice (SJTE) motivated my review of the literature on PBTE and SJTE, with connections to mathematics teacher education (MTE). I reviewed the literature to better understand reasons for this dissonance and to explore if and how these approaches could be harmonized within MTE. The following research questions guided my analysis: (1) For what reasons have mathematics teacher educators adopted practice-based approaches? (2) What opportunities and challenges have mathematics teacher educators experienced when implementing PBTE with respect to issues of social justice? and (3) How can PBTE and SJTE be harmonized within MTE?

PBTE emerged as a response to a persistent problem in teacher education, the *problem of enactment* (i.e., the struggle amongst beginning teachers to bridge theory and practice) (Kennedy, 1999). This problem motivated teacher educators to conceptualize a professional knowledge base for teaching and the practices involved in the work of teaching (i.e., *high-leverage practices*), and make this learnable to PSTs through opportunities to examine and enact the practices (i.e., *pedagogies of enactment*; Grossman et al., 2009) before entering the profession (Ball & Bass, 2002; Ball & Cohen, 1999; Ball et al., 2009; Grossman et al., 2009; Holmes Group, 1986). However, as teacher educators adopted practice-based approaches, some worried these could be reductive and reinforce the status quo of whiteness (Philip et al., 2019), while others have leveraged practice-based approaches for supporting the enactment of justice-oriented pedagogy (e.g., Schiera, 2019 & 2020). This tension between different lenses of PBTE first motivated my exploration into the dissonance (Doherty, 2022; Doherty, in press). In this conference proceeding, I will build on those ideas and tie them more specifically to mathematics teacher education. My motivation for addressing the dissonance between PBTE and SJTE is to show these two approaches can be more powerful together than as stand-alone initiatives. In other words, harmonizing these approaches within MTE can better support PSTs to operationalize justice-oriented mathematics pedagogy.

Theoretical Framework

Literature Review

Differing conceptualizations of PBTE across time and across scholars seem to be one contributing factor to the dissonance between PBTE and SJTE. Forzani (2014) documents these various conceptualizations and asserts, "what teacher educators and education researchers have meant by 'practice' and 'practice-based' teacher education has varied from one era to the next" (Forzani, 2014, p. 357), and adds, "there seems to be little consensus about what it means or should mean" (Forzani, 2014, p. 358). For instance, Philip et al. (2019) describe PBTE as a reductive, prescriptive, and decontextualized approach that "amplifies oppression," whereas Schiera (2019 & 2020) leverages practice-based approaches for bridging justice in theory with practice and acknowledges these practices are contextually situated. Schiera (2019) noticed PSTs struggled to operationalize justice-oriented pedagogy, particularly in a world made by and for whiteness, and practice-based approaches can support PSTs in navigating this and acting on injustices through their teaching practice.

Studies have investigated different combinations of PBTE, SJTE, and MTE, but fewer have studied the integration of all three. For instance, studies found practice-based approaches (particularly, *pedagogies of enactment*) improved PSTs' readiness and competency to teach mathematics content (see Boerst et al., 2011; Campbell and Elliott, 2015; Ghousseini and Herbst, 2016; Lampert et al, 2013; Tyminski et al., 2014). However, these were not paired with a justice-oriented focus. Some studies harmonized PBTE and SJTE within other content areas other than mathematics, such as social studies (e.g., Schiera 2019 & 2020) or multiple subject areas (Calabrese Barton et al., 2020; Kavanagh et al., 2020). Kavanagh and colleagues (2020) found approximations of practice can improve teacher *responsiveness* to students' contexts; and Calabrese Barton et al. (2020) identified *justice-oriented high-leverage practices* they observed from teachers' instructional practices. While pedagogies of enactment (such as approximations of practice) can help operationalize pedagogy, naming practices (such as justice-oriented high-leverage practices) can contribute to a professional knowledge base of the work involved in teaching for social justice. While these studies were not specific to MTE, an empirical study by Campbell and Elliott (2015) discuss PBTE in relation to MTE, with a closer connection to SJTE. Campbell and Elliott (2015) situate their work in sociocultural theory and leverage approximations of practice for supporting PSTs in being more responsive to students, which they call part of "ambitious and equitable mathematics teaching." While the studies referenced above are notable in attempts to harmonize some combination of PBTE, SJTE, and MTE, there were no peer-reviewed published studies that I could find that wholly integrate all three. I found reductive dichotomies of theory and practice combined with different lenses of PBTE and SJTE contributes to reductive dichotomies of PBTE and SJTE within MTE.

Conceptual Framework

In this section, I present my framework for integrating PBTE and SJTE within MTE, with connections to dissonance and harmony. In music, harmony results when more than one note is played simultaneously to produce a sound arguably more powerful than the sound each stand-alone note could make. By harmonizing PBTE and SJTE within MTE, the field can realize a teacher education that is more powerful than trying to implement them as separate, stand-alone initiatives. Schiera (2019) explains integrating PBTE and SJTE entails learning to develop a critical consciousness and applying that lens to act on inequities in practice (e.g., in genuine mathematics teaching contexts). Schiera (2019) draws on Ball's (2018) notions of *discretionary* *spaces* which explains practice is never absent from belief and disposition and teachers have the power to disrupt or reinforce patterns of inequity through their teaching. Schiera explains,

> These processes of representation, decomposition and approximation could be interrogated with critical conceptual tools (Grossman et al., 2009) to investigate whether and how the enactment of a practice might reinforce or interrupt inequities. In the field, researchers and practitioners together can also investigate the ways that they enact practices in culturally responsive ways. Researchers can also look to decompose actions social justice educators take in discretionary spaces, following Ball's (2018) model, to learn further about how teachers at all stages of development respond to inequities in the moment (Schiera, 2019, p. 943).

The *SJ-PBTE Framework* (socially just practice-based teacher education framework) introduced in Doherty (in press) describes leveraging social foundations with practice-based approaches (see Figure 1). It builds on Schiera's (2019) ideas and is supported by Bowman and Gottesman (2013) who also argue for grounding PBTE in social foundations. The framework involves identifying high-leverage practices for mathematics instruction, that are made learnable to PSTs through pedagogies of enactment, situated in social foundations that support PSTs' development of a critical consciousness to apply as a lens in the enactment of the practices. The framework has connections to TeachingWorks (2022b), which acknowledges teaching is not neutral and supports grounding teacher preparation in pedagogies of enactment that support PSTs in noticing and disrupting patterns of inequity through their practice. This involves identifying the work of teaching (i.e., high-leverage practices) and how the work can advance justice, and making this learnable to PSTs through pedagogies of enactment. For instance, for *each* high-leverage practice, TeachingWorks (2022a) offers explicit statements describing how the practice can advance justice. For example, the description of the high-leverage practice, *Implementing norms and routines for discourse*, contains the subheading "How does implementing norms and routines for classroom discourse and work advance justice?" which explains content area norms and routines can help students construct and share knowledge but "often reflect dominant power and privilege." Thus, TeachingWorks (2022a) cautions that norms and routines should center student voice and in ways that do not marginalize others. Using the framework within MTE thus involves identifying high-leverage practices (1) as conceptualized within mathematics spaces and (2) specific only to mathematics, and offering similar statements that describe how the practice can reproduce or disrupt injustices in the mathematics classroom.

Another approach to using the framework might involve engaging PSTs in *pedagogies of investigation* (Grossman & McDonald, 2008) about Freire's discussion of the *banking approach* to education (as related to *traditional* mathematics), alongside *reform* mathematics (NCTM, 1989) as moving closer (but not fully) to Freire's *problem-posing* and *co-intentional education,* where students share in the co-construction of knowledge and use mathematical argumentations with justifications to decide what counts as truth rather than the teacher readily confirming answers. Pedagogies of investigation can be interwoven with *pedagogies of enactment* (Grossman et al., 2009) to provide PSTs scaffolded and mediated opportunities to carry out this work. Although not every scenario can be approximated (because teaching and learning is contextual), I argue approximating within various contexts that center student voice can help PSTs learn to be more responsive to students when later placed in other contexts. This is supported by Kavanagh et al. (2020) who found teachers' responsiveness to students improved through approximations of practice. I argue approximations and field teaching experiences should be mediated by teacher educators and mentor teachers to support PSTs in disrupting inequities that arise within the *discretionary spaces* (Ball, 2018) of mathematics instruction.

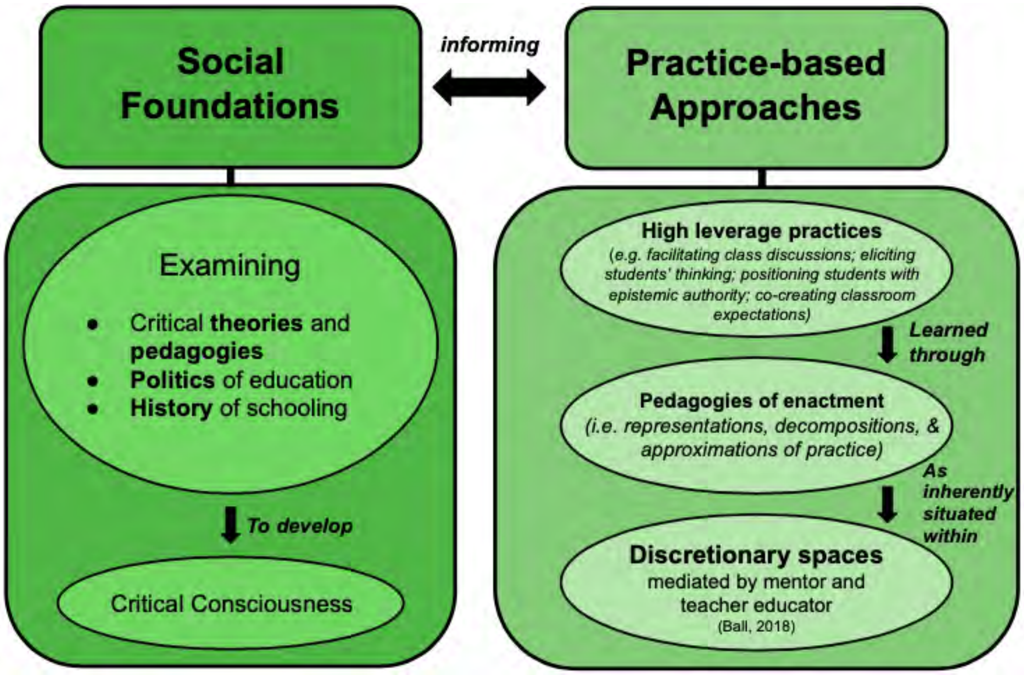


Figure 1: SJ-PBTE Framework

Discussion

Learning to teach mathematics in ways that enact ambitious and justice-oriented pedagogy is complex work, which is why I argue for supporting PSTs through a practice-based approach. Without practice-based approaches to support PSTs, many struggle with figuring out how to operationalize justice-oriented pedagogy (Kavanagh & Danielson, 2019; Schiera, 2019). Schiera explains, "SJTE critiques [PBTE] for reproducing social structures that perpetuate inequities (Philip et al., 2019). Such divergent starting points make it difficult to find common ground from which [PBTE] and SJTE begin to collaborate. This leaves novices… alone to figure out how to become 'comfortable' with practice while being 'mindful' of justice amid the complicated process of learning to teach" (Schiera, 2019, p. 2). Although broad in design, the SJ-PBTE Framework shows the potential for a socially just, practice-based teacher education situated within MTE. Because of the complex work of teaching (Ball & Forzani, 2009) and even more complex work of harmonizing these perspectives, this integration itself will be complex. The framework has implications for the field to develop more empirical studies that integrate PBTE and SJTE within MTE. It also has implications for the field to identify *justice-oriented high-leverage mathematics practices*, similar to those described in Calabrese Barton et al. (2020). Mathematics education is a highly influential field, as it serves as an academic and economic gatekeeper (Martin et al., 2010). Due to its strong influence on society, helping beginning teachers learn to enact equitable mathematics pedagogy is important work for social justice initiatives, and I believe practice-based approaches can support this.

References

Arbaugh, F., Ball, D. L., Grossman, P., Heller, D. E., Monk, D. (2015). Deans' corner: Views on the state of teacher

education in 2015. *Journal of Teacher Education, 66*(5), 435-445. doi:10.1177/0022487115602314
Ball, D. L. (2018). Just dreams and imperatives: The Power of teaching in the struggle for Public Education. American Education Research Association Annual Meeting, New York, NY. April 15, 2018. Retrieved from https://deborahloewenbergball.com/news-archive/2018/4/19/deborah-loewenberg-ball-delivers-presidential-address-at-aera-annual-meeting
Ball, D., & Bass, H. (2002). Toward a practice-based theory of mathematical knowledge for teaching. *Annual Meeting of the Canadian Mathematics Education Study Group*, 3–14. Retrieved from http://eric.ed.gov/?id=ED529557
Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In G. Skykes & L. Darling-Hammond (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 3–32). San Francisco, CA: Jossey-Bass.
Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education*, 60, 497-511. doi:10.1177/0022487109348479
Ball, D. L., Sleep, L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *Elementary School Journal*, *109*(5), 458–474. https://doi.org/10.1086/596996
Boerst, T., Sleep, L., Ball, D. L, & Hyman, B. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record*. doi:10.1177/016146811111301207
Bowman, M., & Gottesman, I. (2013). Why practice-centered teacher education programs need social foundations. *Teachers College Record*.
Calabrese Barton, A., Tan, E., & Birmingham, D. J. (2020). Rethinking high-leverage practices in justice-oriented ways. *Journal of Teacher Education, 00*(0), 1-18. doi:10.1177/0022487119900209
Campbell, M. P., Elliott, R. (2015). Designing approximations of practice and conceptualising responsive and practice-focused secondary mathematics teacher education. *Mathematics Teacher Education and Development, 17*(2), 146-164.
Doherty, K. (in press). Harmonizing practice-based and social justice approaches to teacher education: Toward a framework of SJ-PBTE. *Issues in Teacher Education.*
Doherty, K. (2022). Reconciling a false dichotomy: Seeing practice-based teacher education in harmony with social-justice teacher education. In symposium, *Tensions and Opportunities in Integrating Practice-based Teacher Education and Socially Just Teacher Education*. Session presented at the annual meeting of the *American Educational Research Association*. San Diego, CA.
Forzani, F. M. (2014). Understanding "core practices" and "practice-based" teacher education: Learning from the past. *Journal of Teacher Education, 65*(4), 357-368. doi:10.1177/0022487114533800
Gagnon, D., & Mattingly, M. J. (2012). Beginning teachers are more common in rural, high-poverty, and racially diverse schools. *The Carsey Institute at the Scholars' Repository.* Paper 173. doi:10.34051/p/2020.173
Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. *Journal of Mathematics Teacher Education, 19*(1), 29-103. doi:10.1007/s10857-014-9296-1
Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9), 2055-2100. doi:10.1177/016146810911100905
Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal*, 45, 184–205.
Holmes Group (1986). Tomorrow's teachers: A report of the Holmes Group.
Kavanagh, S. S., & Danielson, K. A., (2019). Practicing justice, justifying practice: Toward critical practice teacher education. *American Educational Research Journal*, *57*(1). doi:10.3102/0002831219848691
Kavanagh, S. S., Metz, M., Hauser, M., Fogo, B., Taylor, & M. W., Carlson, J. (2020). Practicing responsiveness: Using approximations of teaching to develop teachers' responsiveness to students' ideas. *Journal of Teacher Education, 71*(1), 94-107. doi:10.1177/002248711981884
Kennedy, M. M. (1999). The role of pre-service teacher education. In L. Darling-Hammond, & G. Sykes (Eds.), Teaching as the learning profession: Handbook of policy and practice (pp. 54–86). San Francisco, CA: Jossey Bass.
Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping It Complex: Using Rehearsals to Support Novice Teacher Learning of Ambitious Teaching. *Journal of Teacher Education*, *64*(3), 226–243. doi:10.1177/0022487112473837
Martin, D., Gholson, M., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education, 3*(2), 12-24. doi:10.21423/jume-v3i2a95

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*.
Philip, T. M., Souto-Manning, M., Anderson, L., Horn, I., J. Carter Andrews, D., Stillman, J., & Varghese, M. (2019). Making justice peripheral by constructing practice as “core”: How the increasing prominence of core practices challenges teacher education. *Journal of Teacher Education*, *70*(3), 251–264. doi:10.1177/0022487118798324
Schiera, A. J. (2019). Justice, practice and the ‘Real World’: Pre-service teachers’ critically conscious visions for teaching amid the complexities and challenges of learning to teach. *International Journal of Qualitative Studies in Education, 32*(7), 929-946, doi:10.1080/09518398.2019.1609125
Schiera, A. J. (2020). Seeking convergence and surfacing tensions between social justice and core practices: Re-presenting teacher education as a community of praxis. *Journal of Teacher Education*. Advance online publication. doi:10.1177/0022487120964959
Shulman, L. S., & Shulman, J. H. (2009). How and what teachers learn: A shifting perspective. *Journal of Education, 189*(1–2), 1–8. doi:10.1080/0022027032000148298
TeachingWorks (2022a). High leverage practices. Retrieved from https://library.teachingworks.org/curriculum-resources/high-leverage-practices/ on September 5, 2022
TeachingWorks. (2022b). Practice-based teacher education. Retrieved from https://library.teachingworks.org/our-perspective/on-justice/ on September 5, 2022.
Tyminski, A. M., Zambak, V.S., Drake, C., Land, T. J. (2014). Using representations, decomposition, and approximations of practices to support prospective elementary mathematics teachers’ practice of organizing discussions. *Journal of Mathematics Teacher Education* 17, 463–487. doi:10.1007/s10857-013-9261-4
Webel, C., & Yeo, S. (2021). Developing skills for exploring children’s thinking from extensive one-on-one work with students. *Mathematics Teacher Educator, 10*(1), 84-102. doi:10.5951/MTE.2020-0003

STUDENT VOICES AND PRESERVICE MATHEMATICS TEACHER PROGRAMMATIC IMPROVEMENT

VOCES DE LOS ESTUDIANTES Y MEJORA PROGRAMÁTICA DEL PROFESOR DE MATEMÁTICAS EN PRESERVICIO

Nicholas Fortune
Western Kentucky University
nicholas.fortune@wku.edu

Natasha Gerstenschlager
Western Kentucky University
natasha.gerstenschlager@wku.edu

Hope Marchionda
Western Kentucky University
hope.marchionda@wku.edu

Supporting our teachers means that we must continually improve our teacher preparation programs. This improvement is influenced by various sources (e.g., state policies, university culture, program environment, faculty, needs of local schools, research). An overlooked perspective on teacher preparation programs is students' voices. We conducted a multi-stage survey to inquire into students' perspectives about their mathematics teacher preparation program. We discuss mathematics teachers' beliefs about the strengths and weakness of their preparation program. Findings from both stages indicated that students experienced disconnects between what they expected to get out of certain courses and what we, as faculty, expected them to learn. Although these are not novel issues, they are pertinent issues related to improvement of teacher education and serve as the impetus for conversations around programmatic improvement.

Keywords: Systemic Change, Teacher Beliefs, Teacher Educators

Pre-service secondary mathematics teachers take a variety of mathematics, education, and/or mathematics education courses in their programs (Darling-Hammond et al., 2005). The courses and structure of the program are heavily tied to the institutional context. Local institutional contexts are always looking to improve their programs to enhance the recruitment and retention of students, as well as improve the overall teacher education experience. Student voices and perspectives are an important, yet often overlooked, component in the process of instructional changes related to improving these programs (Allen & Peach, 2007; Fielding, 2001). In this work, we share findings from student voices and perspectives on their beliefs about how their program is preparing them or has prepared them to be teachers. Our research question was: *What do preservice and inservice teachers believe about how their program is preparing them to be teachers or has prepared them to be teachers?* The goal of this research is not only to learn students' perspectives on their education, but to understand what needs they have even after leaving our programs. What role does teacher education play in professional development post-graduation? To situate this, we will also describe the specifics of our mathematics teacher preparation program. First, we summarize relevant literature. We then discuss our methods and findings.

Literature

There is much literature on teacher preparation programs, oftentimes using student achievement to discuss successful programs (e.g., Boyd et al., 2009; Koedel et al., 2015). More recently, though, focus has shifted to standards for mathematics teacher preparation that are not solely about student achievement. In 2017, the Association of Mathematics Teacher Educators (AMTE) published the *Standards for Preparing Teachers of Mathematics*. To assess candidates, AMTE recommends assessing mathematical knowledge relevant to teaching, mathematics teaching practice, and dispositions. Regarding programs, AMTE recommends assessment of stakeholder engagement, program curriculum and instruction, effective clinical experiences, and recruitment and retention. An overall assumption within these standards is that "those involved in mathematics teacher preparation must be committed to improving their effectiveness in preparing future teachers of mathematics" (AMTE, 2017, p. 2).

Thus, given these important standards and recommendations, the teacher education field needs to know how these standards are communicated to students in teacher preparation programs. Having teachers reflect on their instructional practice is a well-documented successful strategy for teacher education (e.g., Etscheidt et al., 2012); we aim to use reflection in this work at the programmatic level.

Methods

Research Design and Context

Our research question was: *What do preservice and what did inservice teachers experience in their mathematics teacher preparation program?* To answer this question, we conducted a two-phase exploratory convergent parallel design study (Creswell & Creswell, 2017; Tashakkori & Teddlie, 1998), wherein we designed a survey to collect both qualitative and quantitative data in Phase 1, and results from both data sets are used to compare or relate our findings for an overall interpretation. We then sent a follow up survey in Phase 2 to ask targeted qualitative open-ended questions from the three themes that emerged from the qualitative data in Phase 1.

In Phase 1, we sent a survey to our current undergraduate middle-grades and/or secondary mathematics education students and former graduates from these programs from the last 10 years. Participants either graduated from or are currently enrolled in a university in the southeastern United States which is a UTeach replication site. Preservice teachers seeking certification to teach middle grades and/or secondary mathematics have multiple majors. Students can either major in Mathematics or Middle Grades Mathematics. In some cases, students will choose to major in both. Mathematics certification also requires an additional major in Science and Mathematics Education (SMED). All of the mathematics courses are taught by mathematics education or mathematics faculty within the Mathematics Department. The SMED courses are taught by education faculty in the School of Education. Although there is not a formal connection between mathematics and education courses, the faculty in both units try to provide a cohesive experience. In the mathematics courses, the focus is strictly on content whereas the education classes focus on the various aspects of teaching.

Data Collection

We used alumni email lists and current student departmental email lists to ask for participation. Sixty-nine participants filled out the Phase 1 survey (15 current students and 54 former students). The survey collected demographic information, open-ended responses, and Likert-scale questions. The demographic information was gender identification, race, ethnicity, first-generation college student status, majors, minors, and classification (former student, senior, junior, sophomore, first-year). The quantitative data were 5-point Likert-scale questions (strongly agree to strongly disagree) where participants were asked how their program was preparing or has prepared them to: 1) teach mathematics in a student-centered way; 2) inquire into my students' mathematical thinking; 3) assess students' mathematical knowledge; 4) prepare mathematics lessons; 5) solve mathematical problems; 6) support diverse students and diverse student thinking; and 7) make connections between mathematics content. Example open-ended questions for former students were: 1) Please describe your teaching style and why that is your approach to teaching mathematics; 2) What about our program, in regard to your [mathematics /

education] courses, [did / did not] prepare you for your current position? Current students were asked similar questions, rephrased to be applicable to them.

In Phase 2, we asked specific questions based on the results from Phase 1. Specifically, we asked:

1. Describe how the content you learned in your courses is connected (or not connected) to the content you teach in your classes. Please discuss both mathematics and education classes, and be as specific as possible.
2. What is your definition of good mathematics pedagogy? Where have you experienced this as a student and as a teacher?
3. Concerning your response to the previous question: Describe how often and in what ways your teaching does or does not espouse (live up to) this definition of good mathematics pedagogy.
4. Classroom management is often something that teachers feel least prepared for. Please describe a specific issue you have had with classroom management during your time as a teacher. Why did you feel prepared or unprepared to deal with it?

Data Analysis

To analyze the quantitative data from Phase 1, we hoped to perform an ordinal logistic regression of the five-point scaled Likert items – comparing students' majors and their classifications. Unfortunately, the data did not meet the assumptions for this regression approach, particularly we failed the proportional odds assumption ($p < 0.001$). Additionally, in looking at the raw scores for students, many students responded similarly to one another with little deviation (i.e., most students responded with somewhat or strongly agree on all Likert items) – indicating that there was little to no difference in how students responded based on major or classification. As such, we will focus predominately on the qualitative data analysis for this project as it provides a richer description of students' voices.

To analyze the qualitative data (in both Phase 1 and Phase 2), we did open coding and thematic analysis (Braun & Clarke, 2006). As the crucial component of our study was to lift students' voices and reflections about their teacher preparation programs, we did our initial coding in vivo. Two researchers individually coded all the data. They then discussed disagreements on coding to continue to refine the codebook. Then all three researchers coded the data with the refined codebook. All three met to discuss changes that needed to occur to have an agreed upon codebook.

Results

Recall our research question was: *What do preservice and inservice teachers believe about how their program is preparing them to be teachers or has prepared them to be teachers?* We conducted a two-phase exploratory convergent parallel design study to answer this question. Three central themes emerged regarding students' experiences in our programs in the Phase 1 analysis. These were 1) connections to content they will teach, 2) what is good pedagogy, and 3) classroom management. We then sent a follow up survey to get more information specifically relating to these three themes. The results presented below combine results by theme, but clearly indicate which phase they were from.

Connections to Content They Will Teach

In the Phase 1 survey, many former students made connections between how their mathematics courses tied directly to what and how they are now teaching their students.

Specifically, there were 15 comments from former students that mentioned seeing this connection. Several comments focused on direct content connections. For example,

> My courses gave me a good background to the knowledge I already had, which allows me to more easily break down the content for students. The best example I have is the geometry course I took. We delved further into geometry proofs than normal high school curriculum goes, but that knowledge has helped me create a foundation of learning for my sophomore geometry students.

Likewise, another former student said,

> I felt like the courses I took at University X were very helpful in preparing me for teaching mathematics. I learned a lot about how connected the various parts of mathematics were which helps me to use those connections with my students currently.

Additionally, several former students discussed not only the connection between what they learned but how the classes were taught. "[I] developed a better understanding of the 'why' behind the mathematics, an understanding of how secondary concepts are used in post-secondary mathematics, and learned concepts that I now teach that I did not learn in school." And some students referenced certain classes.

> The problem solving class was THE most effective class for teaching me how to teach math, and it wasn't even a teaching class directly. It made me realize how students struggle and what makes a difficult problem fun, and I strive to recreate or use problems from that class in my own classroom. I also really think the class with [professor] helped with explicitly integrating math practices and finding research based (even multi-disciplinary) inquiry activities to do with students.

From these comments, we are encouraged that these respondents are taking their experiences in college level mathematics classes and connecting them to their practice as teachers. However, not all participants on the first survey mentioned similar experiences. Some former, as well as current students, commented on not seeing the connection between their mathematics courses and what they teach or will be teaching (i.e., Horizon Content Knowledge (Ball et al., 2008)). Specifically, there were 12 related comments. These comments often centered on the idea that the content that was taught in their undergraduate mathematics classes was beyond the content that they teach their middle or high school students. For example, one respondent said "There was not a lot of information targeted at helping us teach mathematics to students. For example, what are common misconceptions high school students have about math and how can we counteract that in the classroom." Another said, "Although I enjoyed all the mathematics courses required, the higher level classes are just not that applicable."

Additionally, several comments stated that the undergraduate mathematics classes should have discussed the aforementioned middle and high school content. "I would have appreciated more of an in-depth look at the content I'm currently teaching my students. While we covered some parts of the content, there was much that I am now teaching that I didn't see … in college."

Perhaps, the comment that summarizes these student experiences best addresses both mathematics and education classes simultaneously.

> Creation of content is one area that I wish I would have been given more of a chance to practice. [Education classes] taught me to teach lessons, and the mathematics [classes] taught me to understand the math. However, I did not have a chance to really marry the two skills. It

would have been helpful to have a course solely dedicated to creating content (test items, quizzes, activities) that checked for higher-order thinking questions and rigorous task creation.

On the Phase 2 survey, we directly asked how respondents perceived how the content they were taught connects (or does not) to the connect they teach. Nine respondents indicated that the classes they took were helpful, but not all respondents discussed the connection of content. Of those that did five said that the mathematics content they were taught was directly connected. For example, one respondent said, “the math I have been teaching is at a lower level than I was taught at the college level and has me prepared to go beyond and answer questions students may ask.” Interestingly, another respondent had a similar response but a different implication. “The math courses I took, while I found interesting, really didn’t help prepare me for teaching. The content taught is above the levels I teach.” Three respondents had similar comments that the content was not connected to what they teach. In these two comparing responses we see that both students knew that the content in college level mathematics class is beyond what they will be teaching their future middle or high school students. However, one respondent perceived this as a positive because it led them to be able to answer questions beyond the content, and the other respondent did not see this similar value.

A take-away from this finding is that students currently in our program and those who have completed it, vary in terms of what they expected certain courses to give them. The mathematics courses in our program are designed to be college-level mathematics classes that are taught in student-centered ways focusing on conceptual understanding, and then that conceptual understanding connects to the content they will teach. As this was not consistent across participants, this indicates potential programmatic clarity issues (e.g., transparently explaining to our students the purpose of the program design and specific classes).

What is Good Pedagogy?

Another theme that emerged from our data was the idea of what good pedagogy is and where students see it. Preservice teachers experience pedagogy, from a learner’s point of view, in their own classes or in their field experiences. In Phase 1, 12 participants discussed, from their point of view, how their mathematics classes were taught. For instance, one participant said, “Many of the mathematics courses within the Middle Grades mathematics degree were designed as inquiry based courses themselves. As such, they served as a good model of how to teach math using this method.” While another said, “all of the Middle Grades mathematics courses I had challenged me to think about the ‘why’ for each math skill and operation, not just the how. This deeper understanding helps me identify student misconceptions.”

There were a few instances of former students saying that courses they took were not student centered, but they often referenced specific courses (e.g., Abstract Algebra) which Secondary Mathematics majors take and are not taught by mathematics education faculty within the department. Importantly, when participants referenced good pedagogy, it most often focused on student-centered instruction. Yet, when we asked former students to describe their teaching style, 16 were coded as all or leaning student-centered, 15 were coded as all or leaning teacher-centered, and 14 were coded as a mix between the two. Notice the even split. Further, some participants who identified as leaning teacher-centered were also the same participants that mentioned the disconnects in the previous theme.

Regarding education classes, the comments on what is good pedagogy often focused on how those classes focused on inquiry-based learning but especially focused on experiences in the field. There were 18 comments from current and former students addressing the importance of field experiences. For instance, one participant said, “I believe getting to actually teach students early in [Program Name], gave me confidence in classroom management. While teaching real students, I began creating my teacher personality.”

In Phase 2, we focused on asking the participants to define good mathematics pedagogy for themselves and then state if their teaching does or does not espouse this definition of good mathematics pedagogy. Three respondents said that direct instruction works best for them and their students. One said, “a clear concise instruction of the topic that at least addresses where it can be relevant outside of the math classroom.” All respondents who said their teaching was teacher-centered said that they do achieve this style of teaching all or most of the time. Most respondents said that they teach in student-centered ways. “Good mathematics pedagogy is allowing students to construct understanding for themselves while guiding them toward that understanding.” Similarly, some respondents discussed the importance of conceptual understanding. “Good mathematics pedagogy is being able to understand math on a conceptual level, understanding how I got my answer, and why it makes sense.” Most respondents said that most of the time they can achieve this teaching style. However, two participants the hardships with this. One said,

> I take pride in my classroom and I love my students. As a result, I believe my teaching lives up to my mathematics pedagogy daily. In the current state we are in with challenges from the COVID situation, I do believe I am currently challenged more daily to live up to my expectations for my classroom experiences for my students.

Another said,

> My teaching does not live up to this definition as often as I would like due to time constraints. I cannot give students the amount of time they need to discover and construct their own understanding within the confines of my class times and the amount of content they need to see each year. I am able to do this about once every other week for an entire class, and other classes I try to build in smaller amounts of time for discussion and discovery, just not on a large scale.

This is often a very common reason that teachers do not implement active learning as often as they wish: time constraints. Similarly, the COVID-19 pandemic has impeded their ability to teach in the way they believe to be most successful for their students.

Classroom Management

Overwhelmingly, 23 participants in Phase 1 expressed a wish for more learning in their program on classroom management. One participant said, “I felt unprepared to deal with a class of 30 students when the time came.” Whereas another participant mentioned classroom management but acknowledged the importance of being in the K-12 classroom to learn about that. “I am not sure if there is a way to prepare for classroom management without having your own classroom, but that is the one skill I wish I could have gotten more experience in.”

Respondents from Phase 2 elaborated on the specifics they wished they knew about in regards to classroom management from their preparation programs. Six respondents have had issues with behavior management and disruptive students. Two respondents were not prepared to deal with students having cell phones in class and the disruption it causes. Two respondents mentioned not being prepared for students with mental health issues (exacerbated by the pandemic). Respondents also mentioned struggles with apathetic students, unsupportive parents, and a lack of administrative support. One respondent specifically discussed their role in classroom management.

> A specific issue was realizing I CAN affect and shape student behavior/misbehavior. In [education] classes, I was taught to accept all answers and let students control/lead classroom discussion. This caused me to not properly address student misbehavior and for general confusion and frustration in the classroom. For example, when doing PBI [project based instruction], students and groups would not participate in the learning, but distract students and the class.

This comment addresses a connection between teachers' desired teaching styles and learning goals and how classroom management can distract from those learning goals. These respondents overwhelmingly discussed a need for support in managing their classes; and continued support even after their teacher preparation programs.

Discussion and Conclusion

We aimed to understand what future teachers experience when they go through a mathematics teacher preparation program. Our findings are heavily tied to our institutional context as a UTeach replication site and that in particular we have several mathematics education faculty housed in the mathematics department. Thus, our findings not only inform practically how we can improve our program, but they illuminate overarching gaps between how we design our programs and what students experience when they go through them (or what they expect to get out of a program).

From our data we found three overarching themes about students' experiences in teacher preparation programs. We then followed up with additional questions regarding these three themes. First, there was a split between participants in terms of ones that experienced the connections to content they will teach when in their undergraduate mathematics classes to ones who did not see that connection. Moreover, the fact that college mathematics courses covered content beyond middle and high school content was sometimes a benefit and sometimes not for respondents. Second, what is good pedagogy was discussed in the context of both how we as mathematics education faculty teach our courses and their experiences in the field. Further, most of the time, teachers are able to live up to their definition of good pedagogy, but if they are not, it is often due to time constraints or issues related to the pandemic. Third, classroom management was something that a large majority of participants wished was a focus of their learning. Specifically, teachers have issues with behavior management, disruptive students, cell phones, unsupportive parents, and more, and these are all topics they wish were discussed in their teacher education programs.

Together, these three themes highlight an important take-away for the field of mathematics teacher education. That is, what is the purpose of a program in mathematics teacher education (and the courses within it)? Often in academia, program design is shrouded behind administrative procedures that students never need to know about (Fielding, 2001). They are at a university to get a degree. Those degrees vary by state and university. However, our findings indicate that if we want students to get the best possible experience out of their teacher education program (as they often are designed based on research or community standards (e.g., AMTE, 2017)), then an important component of that is to include students in program-level discussions and be transparent about what they should get out of an education. We acknowledge that our findings are very specific to our institutional context, however, according to Chan (2016), "student expectations and purposes for completing undergraduate education tend to be instrumental and personal, while institutional aims and purposes of undergraduate education tend to be highly ideal (i.e., life- and society-changing consequences)" (p. 19). We believe this is a larger concern in teacher preparation.

This work provides initial steps in building program reform that is rooted in students' voices and experiences of their program. Moreover, implications from this work will inform the teacher preparation research community about perspectives on their education which can be used to understanding, from a student point of view, if programs are achieving their goals. Further research is needed to ascertain if these disconnects have long lasting effects on effective teaching or on the retention of teachers in the field.

References

Allen, J. M. & Peach, D. (2007). Exploring connections between the in-field and on-campus components of a preservice teacher education program: A student perspective. *Asia-Pacific Journal of Cooperative Education*, *8*(1), 23-36.

Association of Mathematics Teacher Educators [AMTE]. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at https://amte.net/sptm.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education*, *59*(5), 389-407.

Boyd, D. J., Grossman, P. L., Lankford, H., Loeb, S., & Wyckoff J. (2009). Teacher preparation and student achievement. *Educational Evaluation and Policy Analysis*, *31*(4), 416-440.

Braun, V. & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology,* 3, 77-101.

Chan, R. Y. (2016). Understanding the purpose of higher education: An analysis of the economic and social benefits for completing a college degree. *Journal of Education Policy, Planning and Administration*, *6*(5), 1-40.

Creswell, J. W. & Creswell, J. D. (2017). *Research design: Qualitative, quantitative, and mixed methods approaches*. Sage.

Darling-Hammond, L, Hammerness, K., Grossman, P., Rust, F., & Shulman, L. (2005). The design of teacher education programs. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 390-441). Jossey-Bass.

Etscheidt, S., Curran, C. M., & Sawyer, C. M. (2012). Promoting reflection in teacher preparation programs: A multilevel model. *Teacher Education and Special Education*, *35*(1), 7-26.

Fielding, M. (2001). Students as radical agents of change. *Journal of Educational Change*, *2*, 123-141.

Koedel, C., Parsons, E., Podgursky, M., & Ehlert, M. (2015). Teacher preparation programs and teacher quality: Are there real differences across programs? *Education Finance and Policy*, *10*(4), 508-534.

Tashakkori, A., & Teddlie, C. (1998). *Mixed methodology: Combining qualitative and quantitative approaches*. Sage.

TRACKING MATHEMATICS TEACHER IDENTITY DEVELOPMENT IN ONLINE SIMULATIONS: THE CASE OF THE ONLINE PRACTICE SUITE

Dionne Cross Francis	Pavneet Kaur Bharaj	Calli Shekell
University of North Carolina	University of North Carolina	ETS
dicross@unc.edu	pkbharaj@unc.edu	cshekell@ets.org

Heather Howell	Meredith Park Rogers	Adam Maltese
ETS	Indiana University	Indiana University
hhowell@ets.org	mparkrog@indiana.edu	mparkrog@indiana.edu

We examined transitions in secondary math pre-service teachers (PSTs)' professional identities as they engaged in online practice-based, virtual simulations designed to support the development of their skills to facilitate argumentation-focused discussions. Using a single item measure, we captured a snapshot of the PSTs' identity at two time points and observed notable shifts over time. Findings indicated that identity development was related to PSTs' opportunities to practice teaching in the simulations. For some PSTs, their math teacher identity became more central during the semester, while for others it foregrounded other sub-identities.

Keywords: Preservice Teacher Education, Identity, Instructional Activities and Practices, Professional Development

Teacher identity can be defined as "teachers' understandings of themselves as teachers, which has been shaped through ongoing processes of interpretation and re-interpretation of personal and professional experiences embedded in multilayered social [historical] contexts" (Schutz et al., 2020, p. 208). Teacher identity is dynamic, develops over time, and continuously evolves (Watson, 2006). Identity is thought to be relational (Day et al., 2006); thus, it is formed and reshaped as we negotiate and internalize our interactions within our environment. This relational component undergirds the notion that no one claims an identity by themselves (Carlone & Johnson, 2007). Thus, identifying as a teacher involves how one sees themselves and how that image of self is affirmed or disaffirmed by others (Schutz et al., 2020). Additionally, identity tends to be multidimensional, meaning being a teacher is only one aspect of who teachers are. Our sub-identities (e.g., sister, Muslim) develop through experiences in different contexts which we harmonize over time (Beijaard et al., 2004).

Research on teacher identity in math education has gained traction within the last two decades (Brown & McNamara, 2011; Kaasila, 2007; Lutovac & Kaasila, 2011). This increased attention came through acknowledgement that professional identity is "a key influencing factor on teachers' sense of purpose, self-efficacy, motivation, commitment, job satisfaction" and is thus an "essential" aspect of "teacher work" (Day et al., 2006, p. 601). Not developing a robust teacher identity can manifest in actions that are destructive to career goals and choices and run contrary to teachers' desired ways of being. In more extreme cases, it can lead to early burnout and attrition (Nghia et al., 2017). As such, teacher educators have put greater emphasis on designing learning experiences for their pre-service teachers (PSTs) to support the development of positive professional identities (Izadinia, 2013; Walkington, 2005). We know that engaging in the work and lifestyles of teachers (including field experiences or internships) support identity development, so being intentional about designing enriching experiences for PSTs bode well for them developing relevant teaching competencies and a realistic, rather than idealistic, idea of the profession (Hong, 2010).

One alternative approach to developing professional identity worth exploring more closely is the use of virtual simulations. Simulations, unlike naturalistic settings such as field experiences, can eliminate the potentially limiting effect of working in one school with one mentor teacher and allow for carefully crafted experiences to practice aspects of ambitious instruction that is central to the work of teachers. This may help in supporting identity development by allowing the simulation designer to ensure engagement in the work of teaching in ways that are known to be meaningful. Magen-Nagar and Steinberger (2022) and Carrington and Pereira (2011), for example, found that PSTs had a clearer image of themselves as teachers following engagement in simulations that were integrated into their teacher education courses. In this study, we examined how participation in a suite of practice-based simulations designed to support PSTs in developing the skills to facilitate discussions focused on argumentation influenced secondary math teachers' professional identity.

Study Context

This study was a part of a larger project focused on understanding secondary PSTs' participation in a suite of three increasingly complex practice-based simulations – *Eliciting Learner Knowledge (ELK), Teacher Moments (TM),* and *Avatar-based Simulation (ABS)* - collectively called the Online Practice Suite (OPS). OPS was designed to address some of the current limitations within teacher education programs through providing PSTs support in advancing their skills around a single teaching practice, the facilitation of argumentation-focused discussions. The OPS leverages the technical aspects of online simulation platforms to decompose complex practice and provide PSTs opportunities to engage in repeated practice of different aspects of teaching. Within ELK, a pair of PSTs alternatively role play a student or a teacher using a text-based chat, with a goal that the PST playing the teacher might elicit the student's understanding of the topic. Within TM, PSTs reviewed written work from two students and planned an argumentation-based discussion based on the students' ideas. In ABS, PSTs practiced facilitating a small-group, argumentation-focused discussion about a mathematical idea with five student avatars in an online simulated classroom.

Dotger and Smith (2009) suggested teachers develop their professional identity by gradually learning the practices and expectations of a profession. Thus, participation in OPS created opportunities for PSTs to engage in three online simulations and practice their discussion facilitation skills, which created a context where professional identity could flourish. We were interested in understanding how participation in the OPS would influence pre-service mathematics teachers' professional identity development. Specifically, our research questions were: *1) How do secondary pre-service teachers identify in relation to teaching math prior to and after participation in a suite of online practice-based simulations? 2) How do these identities change over time? 3) What reasons do secondary pre-service teachers provide for their existing mathematics teacher identity?*

Participants, Data Sources and Analyses

Participants included twenty-five PSTs enrolled in secondary mathematics teacher education programs at two institutions in the U.S. To track the PSTs' identity development, we adapted McDonald et al. (2019)'s single item measure to capture a "snapshot" of the teacher's identity at a particular moment (p. 4) (Figure 1). This item was embedded at two time points. First, PSTs responded as part of an online background questionnaire (BIQ) including items about their personal characteristics and professional background, administered at the semester start. Second, they responded as part of an online survey administered after completing the final (ABS) simulation. Participants were given the following prompt, "Select the picture that best describes the current overlap of the image you have of yourself and your image of what a math teacher is." They selected among a set of seven overlapping circles varying in the degree of overlap: 1 = no overlap (do not identify as a math teacher) and 7 = near complete overlap (closely identify as a math teacher) (see Figure 1). In addition, PSTs explained why this picture best described the image of themselves as a math teacher. All PSTs completed the pre-measure; 19 completed the post-measure.

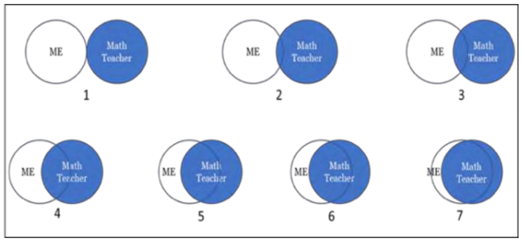


Figure 1: Single Item Measure (adapted from McDonald et al., 2019)

Participants' image selections were tabulated and the sum of the responses for each image number on the pre and post measures were included in Table 1. The reasons provided by participants who selected a particular image were grouped together by number, then read to identify common themes.

Findings and Discussion

Table 1 shows the PSTs' image selection and the total PSTs who selected each image on the pre and post measure.

Table 1: PSTs' Selection of Image from the Teacher Professional Identity Overlap Measure

	Number of PSTs Selecting Image	
Image Number	BIQ (pre-measure) (*n* =25)	Post ABS Survey (post-measure) (*n* = 19)
1 (no overlap)	0	0
2	0	2
3	7	1
4 (50% overlap)	7	3
5	10	4
6	1	7
7 (more than 90% overlap)	0	1

Results showed that prior to OPS, a majority of the PSTs (24/25) selected images with medium overlap (Images 3-5) between their current image of self (indicated as ME in the overlapping circles) and their image of a mathematics teacher. However, there was one PST who indicated strong overlap (Image 6) showing that this PST strongly identified as a mathematics teacher. We noticed how PSTs' professional identity shifted towards the end of participation in the OPS. Of the 19 participants who completed the post measure, eight selected images with maximum overlap (Images 6-7) indicating that they strongly identified as mathematics teachers suggesting that PSTs' experiences in the OPS may have supported their identity development as mathematics teachers.

We also closely examined the shifts in PSTs' image selections from pre- to post-administration and made some interesting observations. We only used data from the 19 participants who completed the post-measure in this analysis. First, the range of image numbers changed from 3 to 5 showing greater variability in the degree to which PSTs identified as mathematics teachers. For the pre-measure, PSTs selected images between 3 and 6, however, on post-measure they selected images between 2 and 7. Second, PSTs shifts were multi-directional; 12 PSTs identified more strongly as math teachers (moving from a lower image number to a higher image number), three showed no change (same image number), and four responses moved from a higher to a lower image number.

We examined PSTs' reasons for their image selections to better understand influences on the PSTs' identity development. Two factors stood out on responses at the mid-level (Images 3 and 4) for either pre or post response or both. One, responses indicated an awareness that their math teacher identity was one of multiple identities they held. As Garry noted, *"While math teacher is a part of me, it does not define me as a person"*. The second factor was related to how strongly they connected being a math teacher with having actual teaching experience; thus, for those who had none prior to OPS, they did not see themselves as math teachers. Hannah captured this well "*Since I* *have not had a lot of in-person experience teaching math, I do not fully picture myself as a 'real' math teacher yet, but it is still something that I have been studying a lot."* For those PSTs who selected Images 6 and 7 during post-administration, they indicated that their experiences within the OPS activities provided them an opportunity to practice teaching, which made them feel more confident to engage in the work of a math teacher. This underscores the connection between identity and efficacy in that individuals tend to identify more closely with a role for which they feel more confident performing (Schutz et al., 2020; Kelchtermans, 2009). We noted a subtle distinction between the responses associated with Image 6 and Image 7 with respect to feelings of preparedness and levels of comfort. More specifically, Ellie who selected Image 7, commented, *"I feel super prepared after everything this semester to be a teacher!".* Responses associated with Image 6 also reflected a sense of being better prepared, but coupled this with noting they still had more to learn. This is reflected in the response by Nancy who stated, *"I feel really comfortable teaching in my placement, but there is still so much to learn!"* PSTs who selected Images 6 and 7 in the post explicitly stated that participation in the OPS helped them to perform aspects of a challenging teaching practice, which would support them in their role as a math teacher.

We were also interested in understanding how identity could change through participation in OPS, so we took a closer look at Sharon who selected Image 6 on pre and Image 4 on the post. Sharon described the reason Image 6 reflected their profession identity stating,

> I have often been told that I am the embodiment of a math teacher. People say that I am logical…and have expertise in math. I have a love for math that exists outside of my classes and teaching experiences... However, there are dimensions of me that are separate from my identity as a math teacher, such as my hobbies and personality.

They identified strongly as a math teacher because they were recognized as such by others, had a love for the discipline and had a connection with math beyond their academic life. We note here that they mentioned that there were still aspects of self (i.e., hobbies and personality) that are distinct from being a mathematics teacher. Interestingly, at the end of participation in the OPS, Sharon selected Image 4 and stated the reason, "*I have many interests separate from math education, but my career embodies who I am in terms of my love for children, math, and learning*". By the end of their participation in the OPS, it appeared that other interests were more central to their identity widening the gap between their current self and their mathematics teacher identity.

Conclusion

In the OPS, mathematics teacher identity development took multiple pathways. Participation appeared to strengthen and make teacher identity more central for some PSTs, while for others it foregrounded other sub-identities. These findings showed the dynamic nature of identity and that how identity changes and develops is related to individuals' personal interpretations of these experiences. Aligned with existing literature, we also observed interconnectedness of mathematics teacher identity and teacher efficacy (c.f. Marschall, 2021), that recognition by others (e.g., Sharon) (Carlone & Johnson, 2007) and opportunities to engage in the work of teaching (e.g., Ellie, Nancy) (Hong, 2010) engendered a stronger mathematics teacher identity (Images 6 and 7). In this regard, online practice-based simulations are a valuable resource as they provide opportunities for PSTs to practice teaching without some of the drawbacks of field experiences, which bodes well for developing a robust mathematics teacher identity. We also noted that the PSTs who indicated having stronger professional identities on the post attributed it to participation in OPS. Given that OPS comprise researcher-designed simulations, these experiences may not be considered akin to the realistic experiences advocated for PSTs (Hong, 2010). However, this may be related to the degree of authenticity the PSTs themselves ascribe to the simulation. This is an area for further exploration.

Acknowledgments

This grant was funded by the National Science Foundation (grant #2037983). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Beijaard, D., Meijer, P. C., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and Teacher Education, 20*, 107–128.

Brown, T., & McNamara, O. (Eds.). (2011*). Becoming a mathematics teacher: Identity and identifications*. Springer.

Bruner, J. S. (2003). The narrative creation of self. In L. E. Angus & J. McLeod (Eds.), *The handbook of narrative and psychotherapy: Practice, theory and research* (pp. 3–14). Thousand Oaks: Sage.

Carlone, H. B., & Johnson, A. (2007). Understanding the science experiences of successful women of color: Science identity as an analytic lens. *Journal of Research in Science Teaching: The Official Journal of the National Association for Research in Science Teaching, 44*(8), 1187-1218.

Carrington, K., & Pereira, M. (2011). Assessing the social impacts of the resources boom on rural communities. *Rural Society*, *21*(1), 2-20.

Day, C., Kington, A., Stobart, G., & Sammons, P. (2006). The personal and professional selves of teachers: Stable and unstable identities. *British Educational Research Journal, 32*(4), 601–616. doi:10.1080/01411920600775316

Dotger, B. H., & Smith, M. J. (2009). "Where's the Line?"—Negotiating Simulated Experiences to Define Teacher Identity. *The New Educator*, *5*(2), 161-180.

Furlong, C. (2013). The teacher I wish to be: Exploring the influence of life histories on student teacher idealised identities. *European Journal of Teacher Education, 36*(1), 68-83. http://dx.doi.org/10.1080/02619768.2012.678486

Hong, J. Y. (2010). Pre-service and beginning teachers' professional identity and its relation to dropping out of the profession. *Teaching and Teacher Education, 26*(8), 1530- 1543. http://dx.doi.org/10.1016/j.tate.2010.06.003

Izadinia, M. (2013). A review of research on student teachers' professional identity. *British Educational Research Journal, 39*(4), 1–20.

Kaasila, R. (2007). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM, International Journal of Mathematics Education, 39*, 205–213.

Kelchtermans, G. (2009). Teachers' emotions in educational reforms: Self-understanding, vulnerable commitment and micropolitical literacy. *Teaching and Teacher Education, 21*, 995–1006.

Lutovac, S., & Kaasila, R. (2011). Beginning a pre-service teacher's mathematical identity work through narrative rehabilitation and bibliotherapy. *Teaching in Higher Education, 16*(2), 225–236.

Magen-Nagar, N., & Steinberger, P. (2020). Developing teachers' professional identity through conflict simulations. *Teaching Education*, 1-21.

Marschall, G. (2021). The role of teacher identity in teacher self-efficacy development: the case of Katie. *Journal of Mathematics Teacher Education*, 1-23. https://doi.org/10.1007/s10857-021-09515-2

McDonald, M. M., Zeigler-Hill, V., Vrabel, J. K., & Escobar, M. (2019). A single-item measure for assessing STEM identity. In Frontiers in Education. Frontiers, 4. https://doi.org/10.3389/feduc.2019. 00078

Nghia, T. L. H., & Tai, H. N. (2017). Preservice teachers' identity development during the teaching internship. *Australian Journal of Teacher Education (Online)*, *42*(8), 1-15.

Pillen, M., Beijaard, D. & den Brok, P. (2013). Tensions in beginning teachers' professional identity development, accompanying feelings and coping strategies. *European Journal of Teacher Education, 36*(3), 240-260. http://dx.doi.org/10.1080/02619768.2012.696192

Schutz, P., Hong, J. & Cross Francis, D. (2020). *Teachers' Goals, Beliefs, Emotions, and Identity Development: Investigating Complexities in the Profession*. Taylor and Francis

Walkington, J. (2005). Becoming a teacher: Encouraging development of teacher identity through reflective practice. *Asia-Pacific Journal of Teacher Education, 33*(1), 53–64. doi:10.1080/ 1359866052000341124

Watson, C. (2006). Narratives of practice and the construction of identity in teaching. *Teachers and Teaching: theory and practice, 12*(5), 509-526.

Zembylas, M. (2005). Discursive practices, genealogies, and emotional rules: A poststructuralist view on emotion and identity in teaching. *Teaching and Teacher Education, 21*(8), 935-948. http://dx.doi.org/10.1016/j.tate.2005.06.005

DESIGNING SIMULATIONS OF STUDENT THINKING FOR PRESERVICE SECONDARY TEACHER EDUCATION

Duane Graysay
Syracuse University
dtgraysa@syr.edu

Hillary Bermudez
Syracuse University
hbermude@syr.edu

Mathematics teacher educators need pedagogical strategies to orchestrate opportunities for prospective secondary mathematics teachers (PSMTs) to develop their capacities to elicit and build on student thinking. We share preliminary results of an ongoing project to design protocols for clinical simulations that will create such opportunities and that will press PSMTs to act on those opportunities in ways that elicit and build on student thinking.

Keywords: Preservice Teacher Education

Over the past few decades, researcher have documented the positive impact of teaching that elicits and builds on student thinking on students' achievement in mathematics across K-16 (e.g., Carpenter & Fennema, 1992; Laursen et al., 2014). Scholars focused on issues of equity in mathematics education suggest that eliciting and using evidence of student thinking is a necessary component of equity-oriented teaching that positions students as active agents in constructing mathematical knowledge (Chao, Murray, & Gutiérrez, 2014; Seda & Brown, 2021). For these reasons, among others, the Association of Mathematics Teacher Educators (AMTE, 2014) and the National Council of Teachers of Mathematics (NCTM, 2014) describe the practice of eliciting and using evidence of student thinking as a core practice for mathematics teachers. Because the practice is a complex activity (Van Zoest et al., 2017), prospective secondary mathematics teachers (PSMTs) need support in their professional education for learning how to engage in eliciting and building on evidence of student thinking.

However, though research and recommendations support teaching that incorporates tasks that promote student thinking and that elicits and builds on that thinking, teachers face a number of barriers related to socioeconomic status of communities, school or district policies that constrain their authority, demands related to high-stakes testing, and other factors, each of which can limit the extent to which teachers are able to act on those recommendations (Serrano Corkin et al., 2019). In the face of those barriers, many mathematics teachers choose (or are constrained) to teach in ways that emphasize memorization, drill, and practice with limited opportunities for students to think independently or creatively (Banilower et al. 2006; Weiss & Pasley, 2004). So, while PSMTs need opportunities to engage in eliciting and responding to student thinking as part of their professional education (Didis et al., 2016), we cannot rely solely on clinical field experiences to provide those opportunities. We face a fundamental challenge in mathematics teacher education to complement clinical experiences with other experiences designed to create opportunities for PSMTs to engage in eliciting and building on student thinking.

In this brief report, we describe initial stages of our work to investigate the design of clinical simulations of student thinking as a pedagogy for orchestrating opportunities for PSMTs to observe, elicit, and build on student thinking.

Designing Clinical Simulations

We ground the project in the *pedagogies of practice* model (Grossman et al., 2009), positing that effective teacher education requires close connections between content that teachers learn in professional coursework and practical skills developed in clinical experience. The perspective has been influential for designing coursework for teacher education (e.g., Core Practices Consortium, 2017). Our work bridges the development of knowledge and skill by orchestrating "iterative and interactive relationship[s] between teachers' development of principles for teaching and practical tools [for teaching]" (Grossman et al., 2009, p. 278). Further, according to principles of situated cognition (Brown, Collins, Duguid, 1989), learning to teach requires participation in the kinds of environments in which that teaching will occur.

Our design builds on others' uses of clinical simulations to engage prospective teachers with problems of practice (e.g. Dotger & Smith, 2009; Dotger et al., 2017). Our design aligns with the use of approximations of practice to engage prospective mathematics teachers in developing high-leverage instructional practices (Arbaugh et al., 2019, 2020; Campbell & Elliott, 2015; Graysay & Freeburn, 2018; Graysay & Masingila, 2021; Lampert et al., 2010; McDonald, Kazemi, & Kavanagh, 2013).

We based designs on two guiding principles. First, simulation must create opportunities for a PSMT to observe realistic depictions of behaviors that students exhibit while engaged in mathematical thinking. Second, to engage PSMTs in eliciting and building on student thinking, simulation must allow aspects of student thinking to remain opaque unless elicited by the PSMT. We designed three protocols to guide actors in taking actions that represent realistic depictions of student thinking and that serve as *triggers* (Dotger & Smith, 2009) to communicate information about the student's approach. We designed simulations around a task from Rivera (2007). Rivera reported, based on extensive study of ninth-grade Algebra students' responses to this task, two types of strategies that students use: *figural* strategies focus on spatial relationships among Patterns and *numerical* strategies focus on quantities derived from each pattern. Rivera reported that many students were able to extend the pattern iteratively yet struggled to express an explicit generalized expression for the number of tiles needed for an arbitrary figure. Our designs are based on simulating student responses that represent those findings.

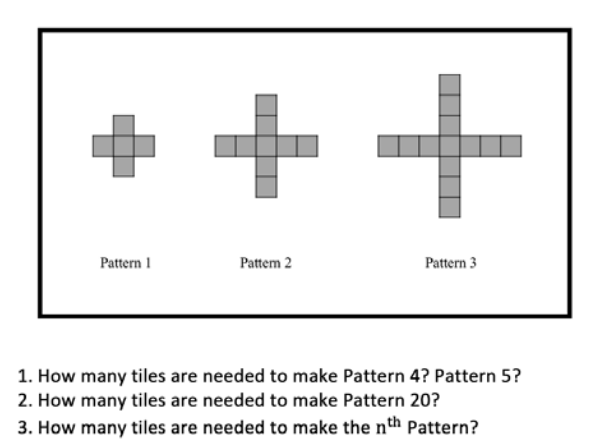


Figure 1. The Tiling Task (Rivera, 2007)

Each protocol includes an initiating trigger in which the student shows a table that she has constructed showing the number of tiles in the first three Patterns and asks for an opportunity to talk through the task with a teacher. The protocols then diverge, each presenting a different simulated figural or numerical strategy to extend the table and an initial, albeit imperfect, expression of her generalization. In the Nell McNally simulation, Nell skip-counts by four to find the number of tiles for the 20^{th} figure and proposes $n+4$ as her generalization. In the Suzy Greenberg simulation, Suzy perceives Pattern 2 as Pattern 1 plus a group of four additional tiles, Pattern 3 as Pattern 1 plus two groups of four and proposes *5+n*4* as a generalization. In the Michelle Gordon simulation, Michelle perceived each pattern as two overlapping rows of tiles with a shared tile at the center and proposes *n*2-1* as a generalization. We instructed actors to respond to questions from the teacher and to allow the teacher to help them revise their generalization. We trained actors in one-hour sessions outlining each role and the triggers that they should present during the simulations. To address proof of concept we chose actors who self-reported ambivalence about their mathematical ability, no prior training as actors, and no teaching experience. In this report we refer to protocols with female names, but in practice we asked actors to self-select a gendered name consistent with their identity.

Our goal in this stage of the project was to determine the effectiveness of the designs for creating conditions in which PSMTs would have opportunities to engage in eliciting and responding to student thinking. We sought to answer the following:

1. Are the protocols and training sufficient for guiding the actors in portraying their role?
2. In what ways, if any, do the simulations provoke PSMTs to elicit and build on student thinking?

Data Collection and Analysis

We collected data in two events separated by several days. Prior to the first event we provided four participating PSMTs (Emily, Fiona, Gabriella, and Heather (pseudonyms)) in their final year of an undergraduate teacher education program with copies of the task. We asked them to generate multiple possible approaches to the task. We explained the nature of the simulation experience. We provided no information about approaches that Nell, Suzy, or Michelle would represent. In the first data collection event each PSMT met individually with an actor portraying Michelle. In the second event each PSMT participated in one conversation with an actor portraying Nell and one conversation with an actor portraying Suzy. We recorded conversations to capture the presentation of triggers and the responses from PSMTs.

We coded PSMTs' responses using the Teacher Response Coding Scheme (TRC) (Van Zoest et al., 2021). The TRC is a multidimensional framework based on the idea that for each mathematical idea that a student expresses, a teacher's response invites some Actor (*teacher* or *student*) to make some Move to advance the conversation. Responses vary in the extent to which the student would recognize their own idea in the response (*explicitly, implicitly*, or *not at all*), and in terms of how the student's idea anchors the Move (*core, peripheral, other*, or *not at all*). Space precludes including the fifteen possible Moves included in the TRC; examples include *Dismissing, Clarifying*, or *Justifying* the idea, *Collecting* a new idea, and *Checking In* for understanding. We added Move codes for responses that we could not clearly identify with an existing code from the TRC. For student statements we created a separate code set: An *Intended Trigger* is a statement that is explicitly provided in training, an *Unplanned Trigger* is consistent with the role but not explicitly provided in training, and an *Off-Protocol* statement is an improvisation by the actor in response to a PSMT question that the training did not anticipate. We divided transcripts into two sets of six, each coded the transcripts in one set, then reviewed the other's coding and resolved differences of opinion to arrive at consensus coding for the twelve transcripts. As we coded we noted patterns in response within each simulation, within each PSMTs' set of simulations, and across the simulations for each role (Nell, Suzy, or Michelle).

Preliminary Findings

Effectiveness of Protocols

We found the protocols moderately effective at guiding actors' portrayals of Nell, Suzy, or Michelle. In each simulation, actors presented intended triggers, though the timing of triggers varied depending on how each PSMT responded. For example, in response to Nell's extension of her table of values by skip counting to find the number of tiles for Pattern 4 and Pattern 5, Emily quickly interjected to guide Nell away from writing out the table to get to the 20th pattern and began introducing and developing a different approach. In contrast, after Nell presented the same skip-counting trigger, Gabriella encouraged Nell to follow her approach, allowing Nell to write out the table up to Pattern 20.

Our goal was to create conditions that would press the PSMTs to elicit and build on student thinking. We found the protocols not very effective at helping the actor deflect the introduction of alternative approaches by PSMTs. We had asked actors to portray Nell, Suzy, and Michelle as insistent on following their own approach by including a deflection trigger, a paraphrasing of "I want to do it my way first." However, we noted that in multiple simulations PSMTs were able to introduce and pursue an alternative approach that the protocol did not prepare the actor for, suggesting that the deflection triggers and actor training were not sufficient to ensure that the simulation followed our intended student approach.

Opportunities to Elicit and Build on Student Thinking

Given that we found evidence in each simulation of actors presenting the intended triggers, we find that simulations were effective at creating opportunities for PSMTs to elicit and build on student thinking. However, simulations were not universally effective at pressing PSMTs to act on those opportunities. In each simulation, PSMTs made some inquiries into how the student had arrived at their initial extension of the table, which allowed the actor to present a trigger representing their numerical approach (Nell) or their way of conceptualizing the structure of the Patterns (Suzy and Michelle). Some PSMTs responded, as Gabriella did, by asking the student to develop the student's idea, eventually helping the student revise her generalization in a way that was consistent with her approach. However, as described in the preceding illustrative anecdote, others responded -- as Emily did -- by replacing the student's idea with a different approach and then working to develop the new approach instead of building on the student's thinking.

Discussion

We provide early, preliminary results of an ongoing project to design simulations that create conditions in which PSMTs are pressed to engage in eliciting and building on student thinking. Our findings suggest, first, that we can create protocols that will effectively guide actors to make statements and take actions that simulate authentic ways that students respond to a given task, regardless of the actor's self-reported mathematical confidence, prior training as an actor, or experience as a teacher or teacher educator. However, the current designs were not effective at creating conditions in which PSMTs are pressed to respect student thinking and build on that thinking rather than replacing students' approaches with the teacher's preferred alternative. Our next step in this ongoing project is to incorporate triggers that actors can present to deflect attempts by PSMTs to guide them away from one approach in favor of the PSMTs' preferred alternative.

References

Arbaugh, F., Graysay, D., Freeburn, B., & Konuk, N. (2020). Investigating secondary preservice teachers' instructional vision: Learning to teach through pedagogies of practice. Journal of Teacher Education. Online

first. https://doi.org/10.1177/0022487120965938
Arbaugh, F., Graysay, D., Konuk, N., & Freeburn, B. (2019). The Three-Minute-Rehearsal Cycle of Enactment and Investigation: Preservice secondary mathematics teachers learning to elicit and use evidence of student thinking. Mathematics Teacher Educator, 8(1), 22–48.
Association of Mathematics Teacher Educators [AMTE]. (2017). Standards for preparing teachers of mathematics. Available online at amte.net/standards
Banilower, E. R., Boyd, S. E., Pasley, J. D., & Weiss, I. R. (2006). Lessons from a decade of mathematics and science reform. Horizon Research.
Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. Educational Research, 18(1), 32–42. https://doi.org/10.3102/0013189X018001032
Campbell, M. P., & Elliott, R. (2015). Designing approximations of practice and conceptualising responsive and practice-focused secondary mathematics teacher education. Mathematics Teacher Education and Development, 17(2), 146–164.
Carpenter, T. P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. International Journal of Educational Research, 17(5), 457–470. https://doi.org/10.1016/S0883-0355(05)80005-9
Chao, T., Murray, E., & Gutiérrez, R. (2014). What Are Classroom Practices That Support Equity-Based Mathematics Teaching? A Research Brief. NCTM. https://www.nctm.org/Research-and-Advocacy/Research-Brief-and-Clips/Classroom-Practices-That-Support-Equity-Based-Mathematics-Teaching/
Core Practice Consortium. (2017). The Problem We Are Trying to Solve. http://corepracticeconsortium.com/
Didis, M. G., Erbas, A. K., Cetinkaya, B., Cakiroglu, E., & Alacaci, C. (2016). Exploring prospective secondary mathematics teachers’ interpretation of student thinking through analyzing students’ work in modelling. Mathematics Education Research Journal, 28, 349–378. https://doi.org/ 10.1007/s13394-016-0170-6
Dotger, B., Dotger, S., Masingila, J., Rozelle, J., Bearkland, M., & Binnert, A. (2017). The right ‘fit’: Exploring science teacher candidates’ approaches to natural selection in a clinical simulation. Research in Science Education. https://doi.org/10.1007/s11165-016-9582-2. (Published online April 11, 2017).
Dotger, B. H., & Smith, M. J. (2009). Where's the line? Negotiating simulated experiences to define teacher identity. The New Educator, 5(2), 161–180. https://doi.org/10.1080/1547688X.2009.10399570
Graysay, D., & Freeburn, B. (2018). Preservice teachers’ responsiveness to student mathematical thinking in rehearsals of instructional practices. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 825). Greenville, SC: University of South Carolina & Clemson University.
Graysay, D. T., & Masingila, J. O. (2022). Simulations in secondary mathematics teacher education: The role of debriefings in providing opportunities to develop knowledge for teaching. In B. Dotger & K. Chandler-Olcott (Eds.), Clinical simulations as a signature pedagogy: Educator preparation across the disciplines. Harvard Education Press.
Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching, 15(2), 273–289. http://dx.doi.org/10.1080/13540600902875340
Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In M. K. Stein & L. Kucan (Eds.), Instructional Explanations in the Disciplines (pp. 129–141). Dordrecht, the Netherlands: Springer.
Laursen, S. L., Hassi, M. L., Kogan, M., & Weston.T. J. (2014). Benefits for women and men of inquiry-based learning in college mathematics: A multi-institution study. Journal for Research in Mathematics Education, 45(4), 406–418.
McDonald, M., Kazemi, E., & Kavanagh, S. S. (2013). Core practices and pedagogies of teacher education: a call for a common language and collective activity. Journal of Teacher Education, 64(5), 378–386. https://doi.org/10.1177/0022487113493807
National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: NCTM.
Rivera, F. (2007). Visualizing as a mathematical way of knowing: Understanding figural generalization. Mathematics Teacher, 101(1), 69–75.
Seda, P., & Brown, K. (2021). Choosing to See: A Framework for Equity in the Math Classroom. Dave Burgess Consulting.
Serrano Corkin, D., Coleman, S. L., & Ekmekci, A. (2019). Navigating the challenges of student-centered mathematics teaching in an urban context. The Urban Review, 51, 370–403. https://doi.org/10.1007/s11256-018-0485-6

Weiss, I. R., & Pasley, J. D. (2004). What is high-quality instruction? Educational Leadership, 61(5), 24–28.
Van Zoest, L. R., Peterson, B. E., Rougée, A. O. T., Stockero, S. L., & Leatham, K. O., & Freeburn, B. (2021). Conceptualizing important facets of teacher responses to student mathematical thinking. International Journal of Mathematical Education in Science and Technology, Online First. https://doi.org/10.1080/0020739X.2021.1895341
Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. Mathematical Thinking and Learning, 19(1), 33–54. http://dx.doi.org/10.1080/10986065.2017.1259786

MATHEMATICAL MAKING IN TEACHER PREPARATION: RESEARCH AT THE INTERSECTIONS OF KNOWLEDGE, IDENTITY, PEDAGOGY, AND DESIGN

Denish Akuom	Steven Greenstein	Eileen Fernández
Montclair State University	Montclair State University	Montclair State University
akuomd1@montclair.edu	greensteins@montclair.edu	fernandeze@montclair.edu

In this proposal, we share research that explores the potential benefits of a novel Making experience within mathematics teacher preparation that we hypothesized would inform the pedagogical and curricular thinking of prospective teachers of elementary mathematics (PMTs). That experience had PMTs exploring at the intersection of content, pedagogy, and design to digitally design, 3D print, and share an original manipulative with a child to promote their mathematical thinking. We share several vignettes of our research that aim to discern some of the potential benefits the experience might offer PMTs. These take a variety of theoretical and methodological approaches at the intersections of teacher knowledge, identity, pedagogy, and design. Implications of our findings for teacher preparation and professional learning are provided throughout the paper and in its conclusion.

Keywords: Teacher Knowledge, Technology, Problem-Based Learning, Integrated STEM/STEAM

Prospective elementary teachers (PMTs) have been characterized as coming to teacher preparation with limited conceptions of mathematics (AMTE, 2013) and a model of mathematics teaching that appeals mostly to rules and procedures (Ball, 1990; Ma, 1999). Consequently, teacher preparation must offer opportunities that challenge this model, and provide pathways to meaningful interactions and deepened understanding of both content and pedagogy. Connecting with a body of research that conceives of Making in education as the creative practice of designing, building, and innovating with analog and digital tools and materials (Halverson & Sheridan, 2014), we present one such opportunity that we centered in a novel Making experience within mathematics teacher preparation. That experience tasks PMTs with digitally designing, 3D printing, and sharing an original manipulative with a child to engage and advance their mathematical understanding. In seeking to determine what this experience might offer PMTs as they prepare to teach mathematics, our work has taken a number of theoretical and methodological approaches to address research questions at the intersections of teacher knowledge, identity, pedagogy, and design. These questions address the project's broader agenda, which is framed by the following question: *What are the potential benefits of a Making experience within mathematics teacher preparation*?

In this proposal we share some of the findings of our research along with their implications for teacher preparation and professional learning. Because the theoretical framings and methodological approaches we've taken are specific to each of these projects, this proposal is not organized in the conventional manner. Instead, we begin with the broader rationale for this project and then we present three vignettes of our research, each situated within their own literature, framings, and approaches. We conclude by looking across these and other projects to offer some reflections on the potential value of STEAM-integrated curricular experiences in teacher preparation.

Rationale

Schad and Jones's (2019) review of the research on the Maker movement in K12 education finds that students' learning through Making dominates that literature, with foci that include the improvement of STEM learning outcomes, increasing motivation and interest in STEM, and increasing equity by broadening notions of what counts as Making in STEM education. The extent to which this review mentions research on what *teachers* learn through Making is through studies of how they learn to design makerspaces and integrate maker-centered learning strategies (Clapp et al., 2016) into existing curriculum. Thus, there is almost no research on supporting teacher learning through Making. Our work (Akuom & Greenstein, 2021; Greenstein et al., 2017, 2018, 2019, 2020, 2021, 2021, under review) is situated within that gap in the research.

This work connects with a body of literature that frames *teachers as designers* (e.g., Brown, 2009; Maher, 1987) of learning experiences and of the material resources that mediate them. We conceive of design quite broadly to include the "intentional activity of transforming ideas and knowledge" (Carvalho et al., 2019, p. 79) into "tangible, meaningful artifacts" (Koehler & Mishra, 2005, p. 135). Our purpose in doing so is to introduce a pedagogically genuine, open-ended, and iterative design experience into mathematics teacher preparation that is centered on the Making of an original physical manipulative for mathematics teaching and learning. We hypothesized that the experience would afford unique pathways of diversified engagement that could promote epistemic and pedagogical shifts toward inquiry-oriented creative and participatory practices that support teaching and learning mathematics with joy and understanding. Accordingly, we view this Making experience as a *Learning by Design* (Koehler & Mishra, 2005) approach that honors the proposition that it is productive to develop teacher knowledge within a context that recognizes the interactions and connections among its constituent domains of knowledge. We also view the experience from a constructionist perspective (Harel & Papert, 1991), which argues that meaningful learning happens through the designing and sharing of digital or physical artifacts "that learners care about and have some degree of agency over" (Schad & Jones, 2019, p. 2). Indeed, when teachers take agency over the design of their own curriculum materials, they may challenge the *dissonance* that often arises at being tasked with implementing curriculum they neither chose nor endorsed. A *harmony* occurs, as a result, as these teachers come to see themselves as agents of curricular and pedagogical reform (Leander & Osborne, 2008; Priestley et al., 2012).

The Curricular Context and Experience

The study took place over three semesters in a specialized mathematics content course for PMTs at a mid-sized public university in the northeastern United States. Situated in an instructional context in which the teacher educators of those courses modeled an inquiry-oriented pedagogy, the course engaged students in a Making experience defined by the following task: "The purpose of this project is for you to 3D design and print an original physical tool (or 'manipulative') that can be used to teach a mathematical idea, along with corresponding tasks to be completed by a learner using the tool." The data corpus was comprised of video recordings of the in-class design sessions, the design of the tool, and these four written project components: 1) a "Math Autobiography," 2) an "Initial Idea Assignment," 3) a "Project Rationale," and 4) a "Final Paper/Reflection" that presents the findings from problem-solving interviews conducted by the PMTs with their tool and an elementary-age target learner.

The PMTs learned to use the Tinkercad (Autodesk, Inc., 2020; see Figure 1, left) digital modeling platform to design their manipulatives. They worked on their designs in in-class design sessions during three or four of the weekly class meetings. These sessions were held in a design lab (Figure 1, right), that we deliberately chose as we imagined that the PMTs' design activity would be more inspired in an environment intentionally configured to accommodate the kind of immersive, collaborative social space that nourishes it.

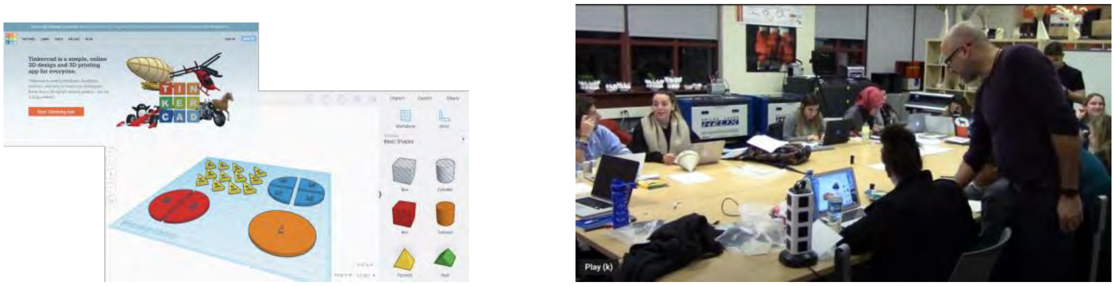


Figure 1: The Tinkercad design environment (left) and the design setting (right).

The first implementation of the Making experience occurred in the context of a pilot study (Greenstein, et al., 2019). Findings from exploratory and revelatory case studies (Yin, 2009) revealed that as the PMTs designed their manipulatives, they leveraged an appreciably rich and mature repertoire of teacher knowledge domains that we are not typically afforded opportunities to see. This finding betrays essentializing characterizations of elementary teachers as lacking in knowledge for teaching mathematics (AMTE, 2013) and suggests the promise of the Making experience. Accordingly, our pilot work became the launching point on a trajectory of further research.

Three Vignettes

In the findings that follow, we present three vignettes of research we've undertaken on that trajectory. Should this proposal be accepted, we will share the findings of these and other projects in our presentation. We propose that this body of work offers evidence of the broadly formative value of a making-oriented, learning by design experience in mathematics teacher preparation.

The Interplay of Discourses of Identity, Mathematics, Pedagogy, and Design in Mathematical Making

Here we took a commognitive perspective on learning (Sfard, 2007, 2008) in order to explore the premise that learning to teach mathematics can be seen as changes in discursive activities that include narratives about mathematics and identity (Heyd-Metzuyanim & Sfard, 2012). We adopted this perspective by foregrounding the identities (Sfard & Prusak, 2005) of teachers as learners in order to recognize what affective, interpersonal, and social matters can bring to this conversation. The following question guided the inquiry: *As prospective teachers of elementary mathematics Make new manipulatives to support the teaching and learning of mathematics, what might their discourses reveal about the epistemology of learning to teach mathematics?*

Methods. We addressed the question through a revelatory case study (Yin, 2009) of a prospective elementary teacher named "Moira" and by framing mathematics learning as the interplay between discourses about mathematical objects (*mathematizing*), participants of the discourse (*identifying*), teaching and learning (*pedagogy*), and design activity (*designing*). This framework provided us with a lens through which to study how the process of making a manipulative can provoke the four discourse activities and make visible the intertwined nature of a teacher's learning. We chose Moira because her initial design was a tool intended to simulate the "keep-change-flip" algorithm for fraction division. However, when the course's teacher educator pushed back on the idea because it did not meet the expectations for a tool that would support a students' conceptual learning, she tried to make sense of the algorithm but could not. Eventually she abandoned the idea altogether. We sought to understand this change through the lenses of the four discourses.

Findings. In this section, we present just one of the central results, which came from a follow-up interview we conducted with Moira in order to understand her rationale for the change in her design idea. It concerns our analysis of this change through the discourses of Mathematizing [*M*], Pedagogy [*P*], Designing [*D*], and Identifying [*I*]. Moira explained, "I wanted to make something that could be interpreted in many different ways [*M/P/D*] ..." As she considered her initial "keep-change-flip" tool, she explained how she realized that, "flipping the fraction upside down in my initial tool... it was just not useful [*M/P/D*] ... Then I came up with this [fraction comparison tool]" [*M/D/I*].

These reflections revealed how Moira's decision to abandon her initial design in light of the *dissonance* she wrestled with as she contemplated its purpose was not just about mathematizing, it was also about identifying. As a teacher, it was important to her that her students have the opportunity to develop their own ways of thinking about fractions with a tool that can be used in a variety of personally meaningful ways. Moira acknowledged that the pedagogy promoted in the course was also part of her decision to change her design: "Well, [the change of design] was because we were talking and you [the teacher educator] said, 'you're just teaching them how to – you're just giving them a way to solve the problem.' And I realized, you're right ..." [*M/P/D/I*]. By switching to a design for comparing fractions, Moira found *harmony* in the realization that she could participate in the discourse endorsed in the course and honor the teacher she wanted to be [*P/I*].

Implications. As in a woven tapestry, learning to teach mathematics weaves together four threads – or discourses – that are unique to a PMT's discursive experiences and particular to a learning community where an inquiry pedagogy is promoted. In this sense, to characterize Moira's learning to teach mathematics as a complex structure of discursive activities interwoven in dialectical unity is to illuminate the brilliance of a tapestry threaded by what she wants to teach (mathematizing), how she wants to teach it (pedagogy), decisions about what resources to make available (designing), and the kind of teacher she wants to be (identifying). This finding of the intertwined nature of the four discursive activities establishes that identity is as central to learning to teach mathematics as is the learning of mathematics, pedagogy, and design. And its implications speak to the formative potential of interdisciplinary project-based experiences as venues for the cultivation of prospective teachers' identities as *teachers becoming* (Greenstein et al., under review).

Designed Manipulatives as Anchors for Teacher Knowledge

This next vignette presents research that explores the 'life' of teachers as designers (e.g., Brown, 2009) of their own curricular materials, tracing their design activities from *tool-design* to *tool-use*. The aim is to explore how a PMT's designed manipulative can mediate the perennial gap (Ünver, 2014) that exists between teacher preparation and practice. The project proceeded in two phases. We began by examining the conceptual, pedagogical, social, and material resources that PMTs bring to their design decisions, their rationales for their uses of those resources, and how these intersect to mediate those design decisions. This phase is depicted in the inner ring of Figure 2. We then extended our inquiry to explore whether the designed tool could possibly be some sort of anchor for their conceptual/pedagogical visions in practice (see the outer ring of Figure 2). The following questions guided the inquiry: *As prospective teachers Make new* *manipulatives for mathematics teaching and learning, what are the resources and rationales they bring to their design decisions? Can connections be made between the resources for their design decisions and how their designs mediate the pedagogical moves they make in practice?*

Methods. To convey just one of many "images of the possible" (Shulman, 2004, p. 147), we present some of the findings of a revelatory case study (Yin, 2009) with "Anyango," a PMT whose written work explicitly expressed a wealth of design decisions and whose problem-solving interview demonstrated how her embeddings of pedagogical and conceptual knowledge in her designs served as anchors for that knowledge in practice.

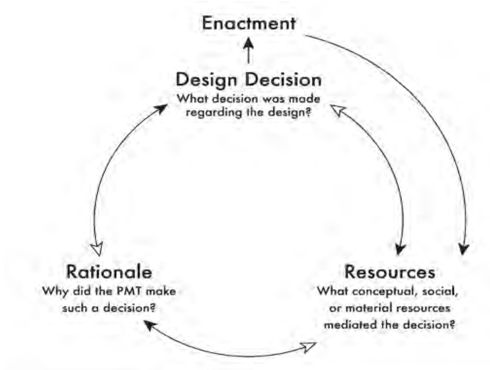


Figure 2: The 3 elements of a design decision (inner ring); Mediating resources may also be evoked in enactment (outer ring).

As the PMTs designed their manipulatives, it was their intention (Malafouris, 2013) to provide their designs with particular affordances (Gibson, 1977) for utilization schemes (Verillon & Rabardel, 1995) that they hypothesized would enable the child to abstract, through their sensorimotor manipulations, the perceptual elements that are the basis of the target concepts. Schön's (1992) notion of "knowing in action" (p. 2) directed our attention as our *Learning by Design* approach (Koehler & Mishra, 2005) enabled us to characterize the interplay between the PMTs' knowledge, experiences, intentions, and other resources that mediated their design decisions as they were invoked and made visible during the iterative design of their tool. In addition, we used the analytic concept of an *embedding* to connote a design element that embeds a PMT's pedagogical and/or conceptual knowledge in their tool. We referred to instances in which the tool served the PMT as a resource for (i.e., a reminder of) knowledge they embedded in the tool as an *anchoring* phenomenon.

Results. Anyango conceived of her design idea in response to the needs of a child she had worked with during problem-solving interviews earlier in the course. She wanted "to help… students visualize and deepen their understanding as they explore fraction relationships." Her design is "a 3D version of fraction strips (Figure 3). Each strip was made to be a rectangular/square piece that slides into individual pegs… [the] blocks stack vertically... to indicate height as value and amount." With several fractions mounted on a single "platform with the 1 (whole) always being visible… the student could begin to grasp how all the smaller parts can equate and compare to the whole." Technological knowledge, the mathematics of fractions, and a responsive pedagogy (e.g., Smith et al., 2016) mediated these and other design decisions that embed fraction values and concepts into the tool.

Anyango posed the following task in her problem-solving interview with the child: *Jack and his two friends each had the same size pizzas for lunch. Jack ate 5/8 of his pizza. Judy ate 2/3 of* *her pizza. And Sam ate 3/6 of his pizza. Who ate the most pizza? Who ate the least?* The child responded by stacking five one-eighth pieces, two one-third pieces, and three one-sixth pieces,

Figure 3: Anyango’s fraction tool

each on their own pedestal with their labels facing her (Figure 3, right). Anyango’s intention was for the child to compare “heights as amount” and identify the tallest as the one “who ate the most,” and shortest as the one “who ate the least.” When she asked the child, “Who ate the most?” the child attended exclusively to the symbolic representations engraved on each of the pegs and concluded that “It’s Jack” (represented by the 5/8 piece). She went on to say that, “5 out of 8 is the biggest of all of them… 2 out of 3 is smaller and 3 out of 6 is… kind of small.” Then, when Anyango asked the child to justify her answer, the child explained, “The top is two and the bottom is three.” We inferred from this response that the child was basing her comparisons on interpretations of fractions as two separate whole numbers. According to this way of thinking, 5/8 is greater than 2/3.

We interpret Anyango’s next move as a noticing one (Sherin et al., 2011) as she leveraged her pedagogical knowledge about the efficacy of interpreting and attending to students’ thinking:

> Anyango: If I turn this [pedestal] around [Figure 2, left, such that the child’s gaze can no longer be restricted to the fraction labels on the pieces], who ate the most?
> Child: <*Pointing to Judy’s stack of two one-third pieces:*> This one.
> Anyango: Who has the least?
> Child: <*Pointing to Sam’s stack of three sixth-pieces:*> This one.

In this excerpt, an unintentional design affordance enabled Anyango’s “flipping” move and served as an anchor for her pedagogical knowledge in action. In a similar anchoring move soon thereafter, Anyango returned the tool to its initial, label-facing orientation so that the child could connect the physical representation of the amount to the symbolic one.

Implications. The diversity of design decisions made by Anyango and other PMTs, as well as the breadth of resources they leveraged and embedded within their designs, speaks to the generative power of the Making experience in terms of the agency PMTs enacted through their design activity and the wealth of conceptual resources that mediated it. In addition, the identification of anchoring phenomena in practice suggests that the Making experience yielded material epistemic scaffolding (in physical manipulative form) that supported PMTs and their commitments to the models of knowing and learning they construct in teacher preparation.

Dare to Care: A Case Study of a Caring Pedagogy on Mathematical Making, Teaching, and Learning

The mathematics and Maker cultures can be interpreted as exclusionary (Stinson, 2004; Gutiérrez, 2017; Barton et al., 2017), thereby suggesting opportunities for broadening learning opportunities to these spaces through *caring* and *Maker* pedagogies. We selected three participants for a revelatory case study (Yin, 2009), each with accompanying “outsider” traits to the project. The teacher educator (TE) and PMT (“David”) brought caring pedagogies to their work but viewed themselves as interlopers to the Making culture. David’s kindergarten student, “Vincent,” is a student with disabilities (SWD) on the autism spectrum whose embodied acts of learning are not typically embraced in traditional mathematics classrooms. By focusing on *caring-centered relationships*, we illustrate how *together*, the participants redefined values associated with Making, traditional mathematics, and what can get celebrated as learning.

Hackenberg's (2010) *mathematical caring relations* (MCR) honor the mathematical and affective dimensions of learning. To navigate an MCR, a teacher must *decenter* "from his or her own perspectives… to help students realize and expand their ideas and worlds" (p. 239). In our project, we honor the open-ended nature of designing and Making a mathematics manipulative, the sometimes uneasy navigation through emergent mathematical "unknowns," the child's unique needs and experiences, and the tensions that are negotiated by carers (Noddings, 2012) in balancing these considerations. We therefore asked: *How does enacting a caring pedagogy during a Making-centered experience impact and broaden opportunities for meaningful mathematics learning? How does this challenge traditional notions of who can Make or participate in mathematics, and who cannot?*

Methods. We utilized purposeful sampling (Creswell, 2007) to focus this case study on our three participants. MCRs were analyzed and revealed through participants' verbal utterances and intonations, body language, actions, and mutual positionings (Simmt, 2000). The possibility of intersecting caring and Making theories called for a grounded theory approach (Glaser & Strauss, 1967) to analyzing and cross-referencing our data sources.

Results. David's Making experience began with an easy "answer" to the project task by designing an already-existing manipulative with a classmate. However, when the TE noticed the warm interactions between David and Vincent in a video recording of an earlier interview, she invited David to design a manipulative that was responsive to these interactions. When David articulated his trepidation in undertaking this more open-ended task, the TE promised to support him. We recognized this moment as the TE accepting responsibility for supporting David in caring for Vincent, and in navigating the discomfort and tensions (Noddings, 2012) that accompany this pedagogical decision. David, in turn, accepts responsibility for Vincent's care by sharing and utilizing Vincent's knowledge and love of shapes. After a few sessions with Vincent, David opts to design triangular, square, and hexagonal prisms with holes and corresponding inserts intended to create a one-to-one matching task.

During a design session, David noticed that some printed inserts did not fit into their intended holes. The TE utilized this moment of struggle to support David through his technological anxieties, and recommended including the "mis-shapes" in the matching task. David reflected on this being a "teachable moment" because his "mis-shapes" became usable for his and Vincent's learning. In another teachable moment, Vincent showed David how every shape and insert need not "match" to fill the holes (e.g., Vincent drops hexagonal inserts into the square hole). These uninhibited moments of insight suggested a transition in Vincent's attention from a shape's sides to its genus—a driving force underlying the concept of topological equivalence. These explorations culminated in Vincent aligning the hexagonal and square prisms with unlike holes to peer through them, and David receptively arranging the pieces between them to form a telescope (see Figure 4)! Together, they locked eyes and exchanged laughter and words of affirmation in an MCR where David decentered from the intended activity to *literally see his child's point of view* (Hackenberg, 2005).

Implications. Our focus on Making something *for* and *with* a specific student enabled a TE and PMT to leverage caring-centered pedagogies, and speaks to the inclusivity that caring brings to learning. As a member of the SWD community, Vincent's inclusive participation enabled him to learn with his characteristic, embodied enthusiasm in ways that defy the exclusionary notion that SWDs are not expected to participate in problem solving. The TE and David's caring-

Figure 4: Vincent sees similarities in different-shaped holes.

centered pedagogies embraced David's "mistakes" as an important part of his learning in addition to Vincent's inclination to know and learn with his body. By providing a platform "to demonstrate care for individual students and for the subject matter itself" (Bartell, 2011, p. 54), this case demonstrates how Making can create a novel opportunity to honor and invite the participation of supposed outsiders to the mathematics and Maker cultures and embrace the mathematical struggle, surprise, and discovery of all learners.

Conclusion

In addressing the question, *What are the potential benefits of a Making experience within mathematics teacher preparation?*, our research has revealed a number of positive outcomes. We shared these findings and their implications for teacher learning at the conclusion of each of the vignettes we presented above. With the conference theme of critical dissonance and resonant harmony in mind, we now provide a summary overview of the findings reported here and of other projects associated with this research.

Over and over, our findings demonstrate that immersing prospective teachers in a communal design environment and tasking them with a pedagogically genuine design experience generates opportunities for them to *harmonize emerging dissonances*. Moira's shift to a design for a conceptually promising learning tool harmonized the dissonance she faced as she confronted the rote understandings associated with her initial design idea. Anyango reconciled the dissonance that arose in her interaction with a learner by enacting a harmonizing pedagogy for the child's sound learning using a novel use of her tool. And the dissonances David felt in his interactions with a teacher and learner were harmonized as he navigated a caring pedagogy that gave rise to unforeseen and innovative perspectives on his tool for that child's learning. In addition, in projects not reported here, we observed two learners harmonize the dissonance they experienced in making sense of fraction division through embodied actions (e.g., Abrahamson, & Lindgren, 2014) with a physical "Fraction Orange" manipulative (Greenstein et al., 2021). We also observed the dissonance one PMT felt as she hesitated to leverage her cultural funds of knowledge (Moll et al., 1992) being harmonized as another PMT encouraged her to leverage it in their design of a tool for counting (Akuom & Greenstein, 2021).

These findings contribute to bodies of research on a "teacher's becoming" (Vågan, 2011; Greenstein et al., under review). They also generate new opportunities for research that moves the field forward regarding the potential value of constructionist, STEAM-integrated curricular experiences in teacher preparation. Future research could more closely explore the design of these environments in teacher preparation, the teacher educator's role in designing and facilitating these experiences, and the subsequent in-service instruction of teachers who participated in these experiences during teacher preparation.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Award No. 1812887.

References

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In R. K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (2 ed., pp. 358-376). Cambridge University Press.

Akuom, D. & Greenstein, S. (2021). *Prospective Mathematics Teachers' Designed Manipulatives As Anchors for Their Pedagogical and Conceptual Knowledge*. Proceedings of the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Philadelphia.

Association of Mathematics Teacher Educators (AMTE). (2013). *Standards for elementary mathematics specialists: A reference for teacher credentialing and degree programs*. Retrieved from San Diego, CA: http://amte.net/sites/all/themes/amte/resources/EMS_Standards_AMTE2013.pdf

Autodesk Inc. (2020). Tinkercad [Computer software]. Retrieved from https://www.tinkercad.com/

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4), 449-466. doi:10.2307/1001941

Bartell, T. G. (2011). Learning to Teach Mathematics for Social Justice: Negotiating Social Justice and Mathematical Goals. *Journal for Research in Mathematics Education, 41*(0).

Barton, A. C., Tan, E., & Greenberg, D. (2017). The makerspace movement: Sites of possibilities for equitable opportunities to engage underrepresented youth in STEM. *Teachers College Record, 119*(6), 11-44.

Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17–36). Routledge.

Carvalho, L., Martinez-Maldonado, R., & Goodyear, P. (2019). Instrumental genesis in the design studio. *International Journal of Computer-Supported Collaborative Learning, 14*(1), 77-107.

Clapp, E. P., Ross, J., Ryan, J. O., & Tishman, S. (2016). *Maker-Centered Learning: Empowering Young People to Shape Their Worlds*. Wiley.

Creswell, J. W. (2007). *Qualitative inquiry and research design: Choosing among five approaches*, 2nd ed. Sage Publications, Inc.

Gibson, J. J. (1977). The Theory of Affordances. In R. Shaw & J. Bransford (Eds.), *Perceiving, Acting, and Knowing: Toward an Ecological Psychology.* (pp. 67-82).

Glaser, B. G., & Strauss, A. L. (1967). *The Discovery of Grounded Theory: Strategies for Qualitative Research*: Aldine.

Greenstein, S., Fernández, E. & Davidson, J. (2019). *Revealing Teacher Knowledge Through Making: A Case Study of Two Prospective Mathematics Teachers*. Proceedings of the 41st Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education. St. Louis, MO.

Greenstein, S., Jeannotte, D., Fernández, E., Davidson, J., Pomponio, E., & Akuom, D. (2020). *Exploring the Interwoven Discourses Associated with Learning to Teach Mathematics in a Making Context*. In A.I. Sacristán, J.C. Cortés-Zavala & P.M. Ruiz-Arias, (Eds.). Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico (pp. 810-816). Cinvestav/AMIUTEM/PME-NA.

Greenstein, S., Jeannotte, D., & Pomponio, E. (under review) *Making as a Window into the Process of Becoming a Teacher: The Case of Moira.*

Greenstein, S. & Olmanson, J. (2018). Reconceptualizing Pedagogical and Curricular Knowledge Development Through Making. *Emerging Learning Design Journal, 4*(1)*,* 1-6.

Greenstein, S., Pomponio, E., & Akuom, D. (2021). *Harmony and Dissonance: An Enactivist Analysis of the Struggle for Sense Making in Problem Solving*. Proceedings of the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Philadelphia.

Greenstein, S. & Seventko, J. (2017). *Mathematical Making in Teacher Preparation: What Knowledge is Brought to Bear?* In E. Galindo & J. Newton, (Eds.). Proceedings of the 39th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 821-828). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Gutiérrez, R. (2017). Political conocimiento for teaching mathematics. In A. Lischka, A. Tyminski, S. Kastberg, & W. Sanchez (Eds.), *Building support for scholarly practices in mathematics methods* (pp. 11-37): Information Age Publishing.

Hackenberg, A. (2005). A model of mathematical learning and caring relations. *For the Learning of Mathematics, 25*(1), 45-51.
Hackenberg, A. (2010). Mathematical caring relations in action. *Journal for Research in Mathematics Education, 41*(3), 236-273.
Halverson, E. R., & Sheridan, K. M. (2014). The maker movement in education. *Harvard Educational Review, 84*(4), 495-504, 563, 565.
Harel, I., & Papert, S. (1991). *Constructionism*. Ablex.
Heyd-Metzuyanim, E., & Sfard, A. (2012). Identity struggles in the mathematics classroom: On learning mathematics as an interplay of mathematizing and identifying. *International Journal of Educational Research, 51*, 128-145.
Koehler, M., & Mishra, P. (2005). Teachers learning technology by design. *Journal of Computing in Teacher Education, 21*(3), 94-102.
Leander, K. M., & Osborne, M. D. (2008). Complex positioning: Teachers as agents of curricular and pedagogical reform. *Journal of Curriculum Studies, 40*(1), 23-46.
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum Associates.
Malafouris, L. (2013). *How things shape the mind*. MIT press.
Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of Knowledge for Teaching: Using a Qualitative Approach to Connect Homes and Classrooms. *Theory into Practice, 31*(2), 132-141.
Noddings, N. (2012). The caring relation in teaching. *Oxford Review of Education, 38*(6), 771-781.
National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics.* Reston, VA, National Council of Teachers of Mathematics.
Priestley, M., Edwards, R., Priestley, A., & Miller, K. (2012). Teacher agency in curriculum making: Agents of change and spaces for manoeuvre. *Curriculum Inquiry*, *42*(2), 191-214.
Schad, M., & Jones, W. M. (2020). The Maker movement and education: A systematic review of the literature. *Journal of Research on Technology in Education, 52*(1), 65-78.
Schön, D. A. (1992). Designing as reflective conversation with the materials of a design situation. *Knowledge-based systems, 5*(1), 3-14.
Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *The Journal of the Learning Sciences, 16*(4), 565-613.
Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge University Press.
Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher, 34*(4), 14-22.
Sherin, M., Jacobs, V., & Philipp, R. (Eds.). (2011). *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes* (1st ed.). Routledge. https://doi.org/10.4324/9780203832714
Shulman, L. S. (1986, February). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.
Shulman, L. (2004). *The wisdom of practice: Essays on teaching, learning, and learning to teach*. Jossey-Bass.
Simmt, E. (2000). *Mathematics knowing in action: A fully embodied interpretation*. Paper presented at the Proceedings of the 2000 Annual Meeting of the Canadian Mathematics Education Study Group.
Smith, K., Gamlem, S. M., Sandal, A. K., & Engelsen, K. S. (2016, 2016/12/31). Educating for the future: A conceptual framework of responsive pedagogy. *Cogent Education, 3*(1), 1227021.
Stinson, D. W. (2004). Mathematics as "gate-keeper" (?): Three theoretical perspectives that aim toward empowering all children with a key to the gate. *The Mathematics Educator, 14*(1), 8-18.
Ünver, G. (2014). Connecting Theory and Practice in Teacher Education: A Case Study. Educational Sciences: *Theory and Practice, 14*(4), 1402-1407.
Vågan, A. (2011). Towards a sociocultural perspective on identity formation in education. *Mind, Culture, and Activity, 18*(1), 43-57.
Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77-101.
Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes*. Harvard University Press.
Yin, R. K. (2009). *Case study research: Design and methods* (4 ed.). Sage.

A PROSPECTIVE TEACHER'S SHIFTING VIEWS IN HOW MATHEMATICAL AUTHORITY OPERATES IN THE CLASSROOM

Michael Hamilton
University of Georgia
mwhamilton@uga.edu

This case study investigated how one prospective teacher's views of how mathematical authority operates in the classroom were influenced by her student teaching practicum by comparing mathematical authority diagrams produced prior to and after student teaching. I found there were similarities across the diagrams, but many differences in the diagrams resulted directly from her student teaching experience(s). The most noticeable difference was in her views of students as mathematical authorities. Practical implications, particularly for mathematics teacher educators, are discussed.

Keywords: Preservice Teacher Education, Teachers' Perspectives

Introduction

Many discussions of effective instructional strategies in mathematics classrooms have implications for how authority should operate in the classroom. For instance, many strategies and recommendations made in *Principles to Actions* (NCTM, 2014) require teachers to leverage and build upon *students'* ideas, *students'* solution strategies, and *students'* experiences as means for students to develop their own conceptions of mathematics. Following these recommendations, teachers who enact productive instructional strategies consistently position students as authorities in the classroom particularly when it comes to the mathematics that is reasoned with, discussed, and presented. However, consistently positioning students as authorities in the mathematics classroom can be challenging for mathematics teachers (Amit & Fried, 2005; Hamm & Perry, 2002; Wilson & Lloyd, 2000). One reason it can be challenging for teachers to frequently and consistently positions students as authorities is that multiple sources of authority, other than the teacher and students, may be operating within the mathematics classroom (Wagner & Herbel-Eisenmann, 2014). Because enacting instructional strategies that consistently position students as mathematical authorities is challenging and complex, the purpose of this study is to investigate a prospective teacher's views of how mathematical authority operates in classrooms and how those views were influenced by their student teaching experience(s).

Literature Review and Theoretical Framing

Mathematical authority in classrooms is a relatively new line of inquiry in mathematics education research. Consequently, few conceptions of mathematical authority exist and many scholars provide descriptions of mathematical authority rather than defining the construct (see Gresalfi & Cobb, 2006; Stein et al., 2008; Wilson & Lloyd, 2000). Moreover, many descriptions of mathematical authority describe only the teacher and students as potential authorities and leave out other individuals or objects outside of the immediate classroom content. Gresalfi and Cobb, for instance, described mathematical authority as "the degree to which students are given opportunities to be involved in decision making and whether they have a say in establishing priorities in task completion, method, or pace of learning" (p. 51). However, Wagner and Herbel-Eisenmann's (2014) study suggests individuals and objects, in addition to teachers and students, can be positioned as mathematical authorities in the classroom. Hence, I contend that a new definition of mathematical authority that allows for a variety of potential sources of mathematical authority is needed to further our understanding of mathematical authority relations in classrooms.

To develop my conception of mathematical authority, I draw from tenets of Gresalfi and Cobb's (2006) description of mathematical authority along with Weber's (1947) conception of authority. Weber perceived authority as a relationship in which one individual chooses to follow the orders of another. Furthermore, Weber claimed that when an authority relation exists, the individual chooses to follow the orders of the other when the other is seen to have a valid claim to legitimacy by the individual. Amit and Fried (2005) summarized Weber's conception of an authority relation well when they wrote an authority relation "has as much to do with those who obey it as it does with those who command it: a relationship of authority is a quasi-reciprocal one" (p. 147). The primary tenets of Weber's conception of authority that I draw on are authority being described as a *relationship* and one that hinges of the *legitimacy* of the individual who is viewed as an authority. Tying those tenets of authority together with Gresalfi and Cobb's description of mathematical authority, I define mathematical authority as a relationship in which at least one individual views another subject (i.e. person, community, object) as a legitimate source of mathematical knowledge or mathematical reasoning and, thus, able to make meaningful mathematical contributions.

Methods

This study was a particularistic and descriptive case study (Merriam, 1998) of one prospective teacher's views of mathematical authority in the classroom and how those views were influenced by their student teaching practicum. As part of a larger study, Grace was a student in the secondary mathematics education program at a large public university in the southeastern United States and completed her student teaching practicum in a high school classroom during the Fall 2021 semester. Grace took part in three interviews prior to student teaching, one interview during student teaching, and three interviews after student teaching. Grace also developed two diagrams representing how she thought mathematical authority operated in the classroom (see Figures 1 and 2). The purpose of the diagrams was to provide Grace a medium in which she could explore and explicate aspects of her experiences and thoughts related to mathematical authority in the classroom (Black & Halliwell, 2000; Wagner & Herbel-Eisenmann, 2014). Grace developed her initial diagram during the second interview. Prior to the sixth interview, Grace developed the second diagram and recorded a video explanation of her diagram. During the sixth interview, Grace was asked to further describe aspects of her second diagram and compare her second diagram with her first diagram side-by-side. Grace was chosen for this case study because of the influence student teaching had on her views of mathematical authority in the classroom, particularly when it came to students positioned as mathematical authorities.

The data from the interviews and the video describing the second diagram were transcribed. These transcripts were then coded with broad codes such as *teacher as authority*, *students as authority*, and *objects as authority*. Within each code I then determined emergent themes and created subcodes describing those themes (Strauss & Corbin, 1998). With the new subcodes I re-coded the transcripts again and asked if new codes needed to be included. This process continued until no new codes emerged. Both diagrams were analyzed in tandem with the coded transcripts. Specifically, aspects of the diagrams that were not discussed in the interviews were analyzed based on Grace's descriptions of the diagram, particularly how she described relationships between sources of authority in the diagram (e.g. via arrows between sources).

Results

Prior to student teaching Grace viewed all students as mathematical authorities and some students were viewed as more of a mathematical authority in relation to their peers. Moreover, the legitimacy by which students were positioned as mathematical authorities was primarily rooted in their mathematical conceptions. To represent mathematical authority operating among students and the teacher in her first diagram, Grace drew bidirectional arrows between all the students, represented by squares, and between the teacher and the students (see Figure 1). Grace explained that students viewed their peers as mathematical authorities due to their peers' knowledge of a certain mathematical topic or because they can "find value in how they're thinking about the math." As she described the teacher positioning students as mathematical authorities, Grace claimed, "there are always ways to either look at problems or solve problems that I would never think of that my students would think of." Furthermore, a student could be viewed as more of a mathematical authority by both their peers and the teacher if they consistently contributed productive mathematical insights and/or ideas in class discussions. Consistent in Grace's descriptions of what made a student a mathematical authority was their mathematical thinking or insights. In addition to the teacher and students, Grace included other sources of mathematical authority in her first diagram that were divided into those positioned by the teacher as authorities (such as Department Heads) and those positioned by students as authorities (such as Tutors).

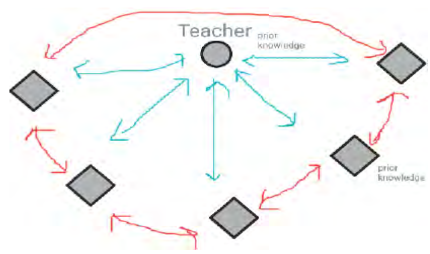


Figure 1: Students and teacher represented as mathematical authorities in first diagram

When asked to compare diagrams, Grace noted there was much more detail in her second diagram. Grace's second diagram did not include many new sources of mathematical authority, but some sources that were once viewed as sources for the teacher were now viewed as sources for both students and teachers. For instance, in the first diagram Grace included *fellow teachers* as a source that the teacher might view as a mathematical authority when it comes to increasing content knowledge and how to best teach certain mathematical concepts. In the second diagram Grace espoused a similar view of how teachers might position other teachers as mathematical authorities, but she also described how students might view other teachers as mathematical authorities. What is notable in her description of this shift is it is grounded in her experiences as a student teacher. During student teaching, Grace saw students go to their previous teachers for help on math homework which she indicated influenced differences in how she developed her second diagram. While describing her second diagram, and as she compared the two diagrams, Grace consistently attributed changes in her views of how mathematical authority operates in the classroom to her student teaching experience(s).

The most salient change in Grace's views about mathematical authority was related to students and their position as mathematical authorities. From the teacher's perspective, Grace continued to view all students as mathematical authorities in the classroom, however her view of students positioning their peers as mathematical authorities shifted. When discussing her second diagram, Grace claimed, "after finishing my student teaching there's no denying that some students have more mathematical authority than others". In isolation this claim may seem no different than Grace's claim prior to student teaching that some students could be viewed as more of a mathematical authority in relation to their peers. Yet, the primary reasons Grace held this view after student teaching were the differences she noticed in students' personalities and/or dispositions. For instance, Grace viewed some students being positioned as more of a mathematical authority than others due to differences in their confidence, willingness to participate, and abilities to clearly communicate their ideas. These views are shown in her second diagram (see Figure 2) by representing students with various sized rectangles and with unidirectional arrows going between students. When elaborating on why she represented students this way, Grace said, "there could also be students that don't have any arrows going to them at all," meaning some students may not be viewed as a mathematical authority by their peers. Influenced by her student teaching experience(s), Grace no longer held the view of students positioning all their peers as mathematical authorities. Moreover, what is notable in this shift is that students' personalities and/or dispositions were the primary factors that led to them being positioned as mathematical authorities by their peers, not necessarily the validity of their mathematical ideas or conceptions. In other words, the legitimacy by which students were primarily positioned as mathematical authorities shifted from their mathematical ideas and insights to their personalities and/or dispositions.

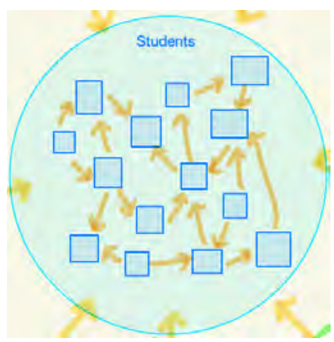


Figure 2: Students as mathematical authorities to other students in second diagram

Discussion and Implications

Although this study focuses on just one student teacher's shifting views of how mathematical authority operates in the mathematics classroom, Grace's story is notable due to the change in her view of students as mathematical authorities as well as what influenced those changes. Britzman (2003) claimed learning to teach is an active and interactive endeavor in which "one's past, present, and future are set in dynamic tension" (p. 31). Because student teachers experience this dynamic tension it would be reasonable to assume their student teaching experience(s) lead to changes in their beliefs, conceptions, and perspectives, as was the case with Grace. Given the complex nature of student teaching, student teachers like Grace may need support from mathematics teacher educators (MTEs) navigating and reflecting upon their shifting views and the implications of those shifts on their instructional practice. Additionally, and echoing a suggestion made by Wagner & Herbel-Eisenmann (2014), mathematical authority diagrams may be a powerful tool to engage teachers in such reflection on their instructional practices. Through this support and reflection, novice teachers may be better prepared to enact instructional strategies that consistently position students as mathematical authorities and, as a result, support students in developing their own meaningful conceptions of mathematics.

References

Amit, M., & Fried, M. N. (2005). Authority and authority relations in mathematics education : A view from an 8th grade classroom. *Educational Studies in Mathematics*, *58*(2), 145–168.

Britzman, D. (2003). Practice makes practice: A critical study of learning to teach (revised edition). New York: SUNY.

Black, A., & Halliwell, G. (2000). Accessing practical knowledge: How? why? *Teaching and Teacher Education, 16*, 103-115.

Gresalfi, M. S., & Cobb, P. (2006). Cultivating students' discipline-specific dispositions as a critical goal for pedagogy and equity. *Pedagogies: An International Journal*, *1*(1), 49–57. https://doi.org/10.1207/s15544818ped0101

Hamm, J. V., & Perry, M. (2002). Learning mathematics in first-grade classrooms: On whose authority? *Journal of Educational Psychology*, *94*(1), 126–137. https://doi.org/10.1037/0022-0663.94.1.126

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. Josey-Bass Publishers.

NCTM. (2014). *Principles to Actions: Ensuring Mathematical Success for All*. Reston, VA: Author.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning*, *10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techiniques and procedures for developing grounded theory.* SAGE Publications.

Wagner, D., & Herbel-Eisenmann, B. (2014). Mathematics teachers' representations of authority. *Journal of Mathematics Teacher Education*, *17*(3), 201–225. https://doi.org/10.1007/s10857-013-9252-5

Weber, M. (1947). *The Theory of Social and Economic Organization* (A. R. Henderson & T. Parsons (eds.)). Free Press.

Wilson, M. S., & Lloyd, G. M. (2000). Sharing mathematical authority with students: The challenge for high school teachers. *Journal of Curriculum and Supervision*, *15*(2), 146–169.

(RE)DESIGNING A TOOL TO SUPPORT MENTOR–NOVICE CO-LEARNING EQUITY-ORIENTED TEACHING

Ruth Heaton
Teachers Development Group
ruth.heaton@teachersdg.org

Kara Jackson
University of Washington
karajack@uw.edu

Taylor Stafford
University of Washington
taylores@uw.edu

Torrey Kulow
Portland State University
kulow@pdx.edu

Mary Alice Carlson
Montana State University
mary.carlson5@montana.edu

Imani Goffney
University of Maryland
igoffney@umd.edu

Melinda Knapp
Oregon State University-Cascades
melinda.knapp@osucascades.edu

Typically, mentor teachers (MTs) help teacher candidates (TCs) learn by supporting them before and after lessons and observing TCs' practice. This design research study reimagines the MT-TC relationship to be one of co-learning and tests a provisional theory and design of a Collaborative Learning Structure (CLS) tool to support MT-TC co-learning equity-oriented teaching. We draw from a case in which a pair of educators used an early iteration of the CLS. Our analysis demonstrates that rather than asking teachers to plan to confer in 'curious or uncertain' moments, it may be more beneficial to confer in 'decision-making moments' while teaching. We discuss implications of our theory of co-learning equity-oriented teaching.

Keywords: Preservice Teacher Education, Equity, Inclusion, and Diversity, Mentor Teachers,

Teachers engage in learning while teaching as they navigate in relation to students and subject matter, a process Lampert (2009) calls "learning teaching." Shulman (2005), like Lampert characterizes teaching as necessarily entailing a "pedagogy of uncertainty." The relational and improvisational nature of the work constantly presents a necessity for teachers of all levels to make new decisions about "how to act under conditions where knowledge is limited but actions must be taken" (Shulman 2005, p.20).

Teacher education programs routinely place teacher candidates (TCs) in field experiences, primarily designed to give them opportunities to practice teaching and learn from mentor teachers (MTs). Mentors typically offer support before and after teaching and position themselves as observers, rarely engaging with novices "inside the action of teaching" (Schwille, 2008). This arrangement assumes teaching is an individual act and leaves teachers alone to "grapple independently with the complexities that may arise" (Ghousseini, 2021, p.1). Drawing on concepts from theater studies, Ghousseini et al. (2021) liken these norms of noninterference to the "fourth wall" that separates actors from the audience. They argue that this "entrenched wall" separates "teachers who gather to improve their practice and the practice itself" (Ghousseini, 2021, p.1). That mentors and novices are typically positioned as spectators of one another's teaching has implications for the learning opportunities available in field experiences. Our study seeks to design learning opportunities for *both* MTs and TCs supported by tools which break through the fourth wall, thereby opening up practice to collaboration, experimentation, inquiry.

Purposes and Research Questions

We are in early stages of a four-year NSF funded project to reimagine the MT-TC as *co-learners* (i.e., Thompson et al., 2015) of *equity-oriented teaching* (e.g., Aguirre, et al., 2013). Using design research methodology (English, 2003), we are creating tools that we term Collaborative Learning Structures (CLSs) which create opportunities for mentors and novices to co-learn in the context of field experiences. The CLS, in focus here, works to make more explicit the implicit work of teaching, which occurs during instruction, in an effort to support teachers in learning about equity-oriented teaching together. Striving for equitable teaching and learning in a mathematics classroom is an ongoing process (NCTM/TODOS, 2016) characterized by an orientation to teaching centered on the needs and assets of traditionally marginalized students and is, therefore deeply relevant and urgent for all educators including MTs and TCs alike (Gutierrez, 2018; Shah & Crespo, 2018). Teachers must be on a daily quest for understanding the dissonance felt by traditionally underserved students in their math classrooms and act in ways that elicit, draw on, value, and validate students' perspectives, strengths, knowledge, and skills.

Methods

We started our project with a provisional theory of co-learning between MTs and TCs and, over the next several years, intend to revise it as we iteratively study MT-TC dyads' use of CLSs, and revise the CLSs accordingly (Cobb et al., 2003; English 2003). Our current CLS tool prompts mentor-novice pairs to focus on equity in students' participation patterns during a lesson. Before the lesson, the MT-TC dyad are asked to confer about the upcoming lesson (either person may be teaching) and to anticipate moments to confer *during* the lesson when the person teaching might "feel curious or uncertain" regarding their noticings of students' mathematical understandings and/or participation patterns. The intent is for the MT-TC dyad during the lesson to "co-notice" students' participation, confer at least one time to discuss an uncertain moment, and deliberate about what to do next with students. After the lesson, the MT and TC are asked to reflect on noticing either made during the lesson, any instructional decisions jointly made, and identify new goals for co-learning. Due to logistical reasons related to the global pandemic, we were unable to collect data using a CLS with MTs and TCs. However, we were able to study a coaching cycle between a teacher and a coach as they engage using parts of our initial CLS tool. While the case focuses on co-learning between in-service professionals (rather than MT-TC dyads), our analysis supported a revision of a key aspect of our theory of co-learning and the CLS. The case below exemplifies how our team is relating a teacher's in-the-moment decision-making to opportunities for MT-TC co-learning. Our analysis highlights how we are thinking about the design of tools to support MT-TC dyads co-learning equity-oriented teaching.

Participants

The focal participant in this study is Sally, a white middle school math teacher with 19 years of teaching experience, with the last 10 years as a 6th grade math teacher at this school. Sally's school serves approximately 1000 students, nearly 40% of whom are students of color, the majority of whom are Latinx. Nearly half of the students at the school qualify for free/reduced lunch. Four years ago, Sally and a colleague successfully initiated efforts to de-track mathematics in their school, which is now in its second year of being de-tracked.

Context, Data Collection, and Analysis

We gathered data on use of some of the CLS tool prompts in a coaching cycle, including planning, teaching, and debriefing, between the district math coach and Sally. A project researcher audio recorded, observed, and took field notes throughout the cycle, including during the 70-minute math class. This coaching cycle occurred after a professional development session featuring "worked examples" (Renki, 2017). Sally and the coach planned a lesson using worked examples that Sally intended to use during three class periods the next day. The goal for Sally's lesson was for students to identify the origin and represent five teams participating in a race on a straight line divided into equal intervals with this information: Team A is in the lead; Team B is 4 miles behind Team C; Team C is 2 miles behind Team A and 8 miles ahead of Team D; and Team E is 5 miles ahead of Team B. Students were to note the position of each team relative to others and be able to explain how they knew. Sally expected students to use positive and negative numbers to express spatial relationships. She and the coach discussed advantages of using worked examples to explore and compare multiple solutions. During the lesson the math coach and project researcher joined in co-noticing and conferred with Sally about what they noticed. Sally made a number of changes to the original lesson plan in-the-moment but her pedagogical reasoning remained invisible to them.

We analyzed excerpts from the lesson debriefing in service of further refinement of the CLS tool and the theory of co-learning underlying its design. Our analytic questions included: What was uncertain for Sally while teaching? What did she notice during the lesson about students who were currently experiencing low status? What decisions did she make that related to elevating those students and on what basis did she make the decisions? How might what we learn from analyzing Sally's reflections on her in-the-moment decisions and the reasoning behind them inform refinement of the CLS tool for both novices and experienced teachers?

Results

The following two excerpts from a transcript during the lesson debriefing session make visible a sample of many decisions Sally made while teaching and her reasons for making these decisions (Ball, 2018). The numbers in Table 1 and Table 2 indicate Sally's reasons verbatim from the transcript. Our analysis references Sally's reasoning with these numbers. Sally changed her plans from using worked examples and, instead, selected and sequenced her students' work (Smith & Sherin, 2019) The coach asked her to explain her reasoning around this decision.

Table 1: Excerpt 1 from Sally's Lesson Debrief with Coach and Researcher

Sally's Reasoning from the Debrief *indicates reasons known ahead of the lesson ** indicates reasons unknown ahead of the lesson	**Analysis**
1. One of the reasons I made the shift is I could see the waning attention span of this group.** **2.** And I was looking at the time, and I needed to make sure that they could understand the relationship of pieces together. So, I decided to table this [worked examples] for now.** **3.** That, and the fact that we had 3 kids who pulled up numbers to show the distance between. That seemed more authentic than doing this [worked examples]. ** **4.** The last thing as to why I did make the shift is…which of the kids who responded today were my IEP kids? Which of my kids, whose journals went up on the overhead, were my IEP kids? And which four voices came out that haven't contributed anything all year? So, I took it as a win.*	Sally entered class knowing that some students have shorter attention spans than others and because of this she watches how much time she spends on activities (**1**, **2**). She also noticed several students' work and voices she wanted to elevate (**4**) do work on the math problem in ways that offered her ideas to share that would benefit other students (**3**).

Table 1 offers insight in Sally's deep knowledge of students that prompted in-the-moment decision changes to her lesson plans. Table 2 represents Sally's pedagogical reasoning when she was asked by project researcher to elaborate on her decisions who she called on and why.

Table 2: Excerpt 2 from Sally's Lesson Debrief with Coach and Researcher

Sally's Reasoning from the Debrief	Analysis
5. And the two IEP kids, one of them had the numbering system I wanted.** **6.** Not Will, but this was the kid that showed the numbers…** **7.** James needs to have his status raised with other students. James knows what he's doing. He knows why he's doing it. He's not comfortable giving his ideas out to kids who claim to have math status.* So, I wanted his status to be raised for those people to see Jonathon as a mathematician.** **8.** Mike doesn't participate a lot. *And, so, when I saw the numbers on his paper and that we could connect to Will…I wanted his ideas to come up because he never shares. And because this was a deep thought to share about the exact relationship between things …** **9.** Brandon just needs all the status he can get. He's a copier. He's very good at it.* And, so, the fact that he had something different from his group, and they were arguing about it, because his was in a separate spot than everybody else's, and the two girls [in his group] have really high math status in the classroom.**	James and Brandon are students with IEPs who Sally knows are seen by others and themselves as having lower status and she noticed they both had ideas that she knew could be useful to other students (5). She could see that James knew how to solve the problem (5, 7). Brandon usually copies others but this time he had not copied and his group was arguing about his ideas and the ideas of two girls with high status in his group (5, 9). Will was not someone who needed his status elevated (6) but he had math ideas that Sally thought would allow her to connect Mike's (8) deep thinking with Will's idea (8).

What Sally learned about how to attend to students' participation patterns and status *while* teaching are things she could not have known earlier. These improvised moments represent Sally's efforts to be responsive to who her students are outside and inside the math class, what they know, and what they need to feel engaged, seen, and successful. However, despite the math coach and project researcher's attempts to engage with Sally while she was teaching, most of the rich pedagogical reasoning which she engaged in during the lesson remained invisible to them. This analysis leads us to refine the CLS tool to shift from conferring during the lesson in moments that feel 'curious or uncertain,' to "decision-making moments" in teaching. We aim to position MTs and TCs as co-learners in shared enactments of equity-oriented teaching and create routines supportive of co-learners in predicting when these moments are about to occur and pausing for collaborative sharing of co-noticing and decision-making. We will test a revised theory of co-learning that centers moments of decision-making, specifically regarding how to center traditionally marginalized students' mathematical sensemaking.

Acknowledgments

This paper is based in part on work supported by the National Science Foundation under grant No.2010634. Any opinions, findings, conclusions, or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Aguirre, J., Mayfield-Ingram, K. & Martin, D. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. Reston, VA: NCTM.

Ball, D.L. (2018). Just dreams and imperatives: The power of teaching in the struggle for public education. Presidential Address at 2018 American Educational Research Association Annual Meeting, New York, NY, April 15, 2018

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32 (1), 9-13.

English, L. (2003). Reconciling theory, research, and practice: A models and modeling perspective. Educational Studies in Mathematics, 54, 225-248.

Feiman-Nemser, S. (2012). Teachers as learners. Cambridge, MA: Harvard University Press.

Ghousseini, H., Kavanagh, S. S., Dutro, E., & Kazemi, E. (2021). The Fourth Wall of Professional Learning and Cultures of Collaboration. Educational Researcher, 0013189X211058751.

Gutierrez, R. (2018). The need to rehumanize mathematics. In I. Goffney, R. Gutierrez, & M. Boston, Rehumanizing Mathematics for Black, Indigenous, and Latinx Students (pp.1-8). Reston, VA: NCTM

Lampert, M. (2009). Learning teaching in, from, and for practice: What do we mean? Journal of Teacher Education, 61(1-2), 21-34.

National Council of Supervisors of Mathematics (NCSM) & TODOS: Mathematics for ALL. (2016). Mathematics education through the lens of social justice: Acknowledgement, actions, and accountability.

Renkl, A. (2017). Learning from worked examples in mathematics: students relate procedures to principles. ZDM, 49(4), 571-584.

Schulman, L. (2005). Pedagogies of uncertainty. Liberal Education,18-25.

Schwille, S. A. (2008). The Professional Practice of Mentoring. American Journal of Education, 115(1), 139–167. https://doi.org/10.1086/590678

Shah, N. & Crespo, S. (2018). Cultural narratives and status hierarchies: Tools for identifying and disrupting inequities in mathematics classroom interactions. In R. Hunter, M. Civil, B. Herbel-Eisenmann, N. Planas, and D. Wagner, Mathematical Discourse that Breaks Barriers and Creates Space for Marginalized Learners, pp. 23-37. Boston, MA: Brill.

Shah, N. & Lewis, N. (2019). Amplifying and attenuating inequity in collaborative learning: Toward an analytical framework. Cognition and Instruction, 37(4) 423-452.

Smith, M. & Sherin, M., (2019). The five practices in practice [middle school]: Successfully orchestrating mathematics discussion in your middle school classroom 1st edition. Thousand Oaks, CA: Corwin Press.

Thompson, J., Hagenah, S., Lohwasser, K., & Laxton, K. (2015). Problems without ceilings: How mentors and novices frame and work on problems-of-practice. Journal of Teacher Education, 66(4), 363-381.

EXPLORING THE USE OF ONLINE MODULES FOR SUPPORTING MATHEMATICS FOR TEACHING

Jennifer Holm
Wilfrid Laurier University
jholm@wlu.ca

This research examines the use of online modules in supporting the development of mathematics for teaching in pre-service teachers. This paper focuses on a single participant in the research study to provide some overarching themes that were presented in the work. Although the results were mixed on even improving mathematics content knowledge, the case provides evidence for considering how to improve an online platform in supporting future teachers.

Keywords: Mathematical Knowledge for Teaching; Preservice Teacher Education; Online and Distance Education

This research is based in Ontario in a Bachelor of Education program, where individuals would attend the program after a four-year degree. Since the program itself is a two-year program, there are limitations in the number of hours provided to mathematics. In the studied university, there are two required mathematics methods courses (one in each year of the program), but these courses explore pedagogical methods for teaching mathematics in combination with mathematics for teaching. Since previous research has pointed to concerns about the level of mathematics for teaching in pre-service teachers (e.g., Kajander, 2010), the creation of an online portal provided extra time for pre-service teachers to focus on mathematics knowledge for teaching. In this research, a single participant from the piloted study is examined in detail to provide some preliminary questions and starting points for expanding the online platform.

Literature and Theoretical Framework

The research is framed by the discussion of mathematics for teaching presented in the research literature. Mathematics for teaching has been identified as an understanding of mathematics content which "bridges content knowledge and the practice of teaching" (Ball et al., 2008, p. 389). In creating the modules, the focus was on an understanding of models, manipulatives, and alternative algorithms that would be needed by a teacher, but not necessarily someone studying mathematics (Baumert et al., 2010; Hill, 2010; Ma, 1999). The mathematics for teaching knowledge needed by teachers also differs from the knowledge a mathematics student would need or gain during a classroom learning experience (Chamberlin et al., 2008; Davis & Simmt, 2006; Kajander, 2010). In particular, the lens used in analyzing the participant responses focused on the notion that merely stating a procedure or procedural steps would not be enough to show mathematical reasoning and support a claim when teaching (Ball & Bass, 2000). In evaluating the responses, the responses were coded to examine whether they followed a procedure or used an alternative method for solving the question.

More research, especially during the pandemic, has been conducted on using online technology as a platform to teach without specifically considering video-interventions. For example, Mulenga and Marbán (2020) examined pre-service teachers and their engagement during an online mathematics course and noted that the ability to personalize the program supported development, although attitude related to mathematics and technology impacted performance. Although there is much that can be learned from the research about online teaching in mathematics, since this research focused on the use of video-interventions, the literature discussed focuses on this area. The majority of research on video-based interventions with pre-service teachers focuses on the use of classroom observation videos for prompting discussion of teaching and learning practices (e.g., Borko et al., 2008). In these interventions, teachers are often shown short videos of active classrooms and asked to observe teacher practices and student behaviours. Barth-Cohen et al. (2018) used video analysis with the intent of increasing the reflective practices of the pre-service teacher. This study also points to how video is an underused and understudied form of professional development. Johnson et al. (2019) examined the use of virtual teaching playgrounds with a video component in supporting pre-service and in-service teachers and determined no difference between the in person and virtual learning outcomes. The study presented in this paper adds to this existing literature by focusing on an intervention to support mathematics knowledge for teaching through video-based interventions.

Methods

The participants in the research study were pre-service teachers enrolled in the Bachelor of Education program. Since the online modules were an addition to the program, pre-service teachers self-selected if they wished to participate. The platform for the program was housed within the university system limiting the possible participants to those who had enrolled in the program. The modules in the platform were completed and adjusted in an iterative process, with participants being enrolled and allowed to work, then additional information or modules were created. At the time of the research, three modules were completed: measurement, algebra, and fractions. Each of the modules contains sub-modules related to different areas of the content strand based on complexity determined by the Ontario curriculum up to and including grade 8 (Ontario Ministry of Education, 2020). For example, the algebra module contains sub-modules related to repeating patterns, linear patterns, and non-linear patterns. The number of sub-modules varied between the different content modules.

Each of the sub-modules followed an identical pattern to allow familiarity with the course content, as well the initial platform had a linear layout, so this prevented overly complicating the platform. Each sub-module began with an initial mathematics problem that was then submitted to the site. Upon submission of their work, participants would be able to view videos of different ways to solve the problem, as well as see an answer to the initial mathematics problem. Next, participants would be invited to complete and submit a solution to a second, similar problem. A survey was also provided to ask participants about their experience in the sub-module. Figure 1 shows the first and second question in the measurement sub-module on area. Data collected for the entire project consists of the solutions to both sets of problems and the information provided in the survey.

A gardener has a 1.5 m^2 garden where she will plant flowers. She decides to plant bluebells on an area that is 0.6 of the total garden. One how many total square meters can she plant bluebells?

A gardener has a 2.2 m^2 garden where she will plant vegetables. She decides to plant squash on an area that is one-quarter of the total garden. On how many total square meters will she plant squash?

Figure 1. Questions from area sub-module

Note. Question 1 is on the top. This question would appear before the video intervention. Question 2 is on the bottom.

The videos and questions in the program were designed to be completed on the participants' own time and are elective, so participants chose which problems to do and to what extent they wished to complete them. At time of writing, 19 participants volunteered to be part of the project; however, only 9 had completed any work within the platform. The data collection is ongoing, and the number of questions that the participants have completed varies between the participants. Data was collected through the online learning management system of the university, and ethics was obtained through the university Research Ethics Board. The participant in this summary was one of two participants to complete all sub-modules in the measurement section, and the only individual to complete any sub-modules in the algebra module. For the data analysis, descriptive statistics were first collected on the number of correct and incorrect solutions both before the video intervention and after it. Solution methods were then analyzed to see if there was evidence of mathematics for teaching and if the solution method differed from the start to the finish.

Results

For the purposes of this discussion, I will use the pseudonym of Terry. Since gender was not a question in the data collection, the pronoun "they/them" will be used throughout when discussing Terry's solutions. Table 1 provides a summary of the questions answered correctly and incorrectly before the intervention and after watching the videos. The number of questions that were answered correctly increased for the second question that was attempted.

Table 1: Summary of Data

Question	Correct	Incorrect
Question #1 (pre-intervention)	9 (36%)	16 (64%)
Question #2 (post-intervention)	16 (64%)	9 (36%)

In exploring the survey results, Terry rated the majority of the questions as "easy" both before and after the intervention. There were questions that Terry got correct before the intervention that they got incorrect after, so there was no consistency in the application of correct procedures after intervention. As well, there were questions that Terry got incorrect both before and after intervention, such as the volume of rectangular prisms sub-module.

In the second part of the analysis, the individual solutions of the individual were examined. This data led to some interesting results where only one solution method changed from the pre-intervention question to the post-intervention question. Even with the change in method, Terry still gave an incorrect answer. The solutions Terry submitted can be found in Figure 2. The question was found in the irregular shapes sub-module, where participants were asked to find the area and perimeter of an irregular shape in question 1, and then a different irregular shape in question 2. In one sub-module, Terry did not show a solution method in the second question, but the answer was incorrect both before and after the video.

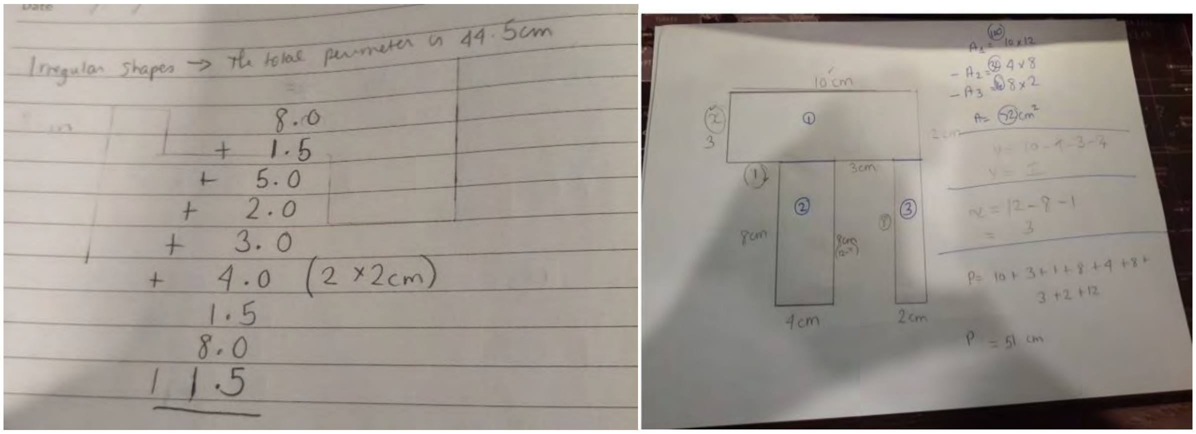


Figure 1. Solutions to a question related to finding the area and perimeter of an irregular shape. The solution on the left was the first question.

Discussion and Conclusion

This single participant provided some interesting initial results related to the use of the online modules. A limitation of the study was the fact that Terry mainly completed the modules within a three-day period right at the beginning of their participation in the study. At the time of the data collection, the province of Ontario had instituted a mandatory mathematics exam for licensing (Ontario College of Teachers, 2019), so this likely also led to some of the focus on simply solving questions instead of focusing on mathematics for teaching since the goal of the Test was to solve mathematics questions correctly and in a set time frame (Education Quality and Accountability Office, 2020). Terry did show gains in mathematics knowledge between the two questions; however, there was no evidence that the interventions had an impact on their mathematics knowledge for teaching.

In conclusion, the project has added to the discussion of video-based interventions for learning mathematics for teaching but has also raised some suggestions about how to create a better platform for implementation. In examining the results for strictly increasing mathematics knowledge, there is evidence that it does help. In considering mathematics for teaching, the results were not as promising. With the implementation of the exam, it could be that Terry was simply focused on increasing mathematics content knowledge and did not believe that there was a need for anything different. In reflecting on the platform used in the study, clearly more needs to be done to point participants towards considering the importance of mathematics for teaching, which may change future results. At present, this was only a pilot study exploring the platform to learn more about changes that could be made to support learners. Creating a new platform while considering the need to highlight mathematics for teaching would open possibilities for a revised project that could support adding extra mathematics-focused hours to a teacher education program.

Acknowledgements

This research was funded by the Social Sciences and Humanities Research Council.

References

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. C. Phillips (Ed.), *Constructivism in education: Opinions and second opinions on controversial issues: Ninety-ninth yearbook of the National Society for the Study of Education* (pp. 193–224). National Society for the Study of Education.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407.
Barth-Cohen, L. A., Little, A. J., & Abrahamson, D. (2018). Building reflective practices in a pre-service math and science teacher education course that focuses on qualitative video analysis. *Journal of Science Teacher Education*, *29*(2), 83–101.
Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A. Klusmann, U., Krauss, S., Neubrand, M., & Tsai, Y. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*(1), 133–180.
Borko, H., Jacobs, J., Eiteljorg, E., & Pittman, M.E. (2008). Video as a tool for fostering productive discussions in mathematics professional development. *Teaching and Teacher Education*, *24*(2), 417–436.
Chamberlin,M. T., Farmer, J. D., & Novak, J. D. (2008). Teachers' perceptions of assessments of their mathematical knowledge in a professional development course. *Journal of Mathematics Teacher Education*, *11*, 435–457.
Davis, B., & Simmt, E. (2006). Mathematics-for-teaching: An ongoing investigation of the mathematics that teachers (need to) know. *Educational Studies in Mathematics*, *61*, 293–319.
Education Quality and Accountability Office. (2020). *The Ontario Mathematics Proficiency Test (MPT): Assessment blueprint*. https://www.mathproficiencytest.ca/docs/blueprint-20200426/MPT_Assessment_Blueprint-EN.pdf
Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education*, *41*(5), 513–545.
Johnson, H. L., Dunlap, J. C., Verma, G., McClintock, E., DeBay, D. J., & Bourdeaux, B. (2019). Video-based teaching playgrounds: Designing online learning opportunities to foster professional noticing of teaching practices. *TechTrends*, *63*(2), 160–169.
Kajander. A. (2010). Mathematics teacher preparation in an era of curriculum change: The development of mathematics for teaching. *Canadian Journal of Education*, *33*(1), 228–255.
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Erlbaum.
Mulenga, E. M., & Marbán, J. M. (2020). Prospective teachers' online learning mathematics activities in the age of COVID-19: A cluster analysis approach. *EURASIA Journal of Mathematics, Science and Technology Education*, *16*(9), 1–9.
Ontario College of Teachers. (2019). *New certification requirement: Math Proficiency Test*. https://www.oct.ca/public/media/announcements/new-certification-requirement-math-proficiency-test

A PRELIMINARY LOOK AT PROSPECTIVE MATHEMATICS TEACHERS' PRODUCTIVE STRUGGLE

Nitchada Kamlue
Western Michigan University
nitchada.kamlue@wmich.edu

Laura R. Van Zoest
Western Michigan University
laura.vanzoest@wmich.edu

This exploratory study aims to describe what productive struggle looks like when prospective mathematics teachers in a middle school mathematics methods course engage with a challenging mathematics task. We hypothesize that a productive struggle consists of different types of struggles—goal struggle, strategy struggle, and sub-strategy struggle—that can coexist simultaneously. We provide insight into the complexity of prospective teacher productive struggle and how it differs from that of middle school students. This information is useful for teacher educators who want to capitalize on the opportunity productive struggle offers for prospective teacher learning.

Keywords: Instructional Activities and Practices, Preservice Teacher Education

In general, people perceive struggles as things they should avoid. However, the National Council of Teachers of Mathematics (NCTM; 2014) views students' mathematical struggles as "opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions" (p. 48). Several researchers have worked to better understand those opportunities by studying school students' (e.g., Warshauer, 2015) and prospective teachers' (e.g., Zeybek, 2016) efforts to make sense of mathematics when engaging with challenging mathematical activities. Despite these gains, we lack research that reveals how and when learning occurs during the time that learners are engaged in a productive struggle. This information would help mathematics teacher educators to recognize and leverage their PTs' productive struggle and thus better support those PTs to be able to *support productive struggle in learning mathematics* (NCTM, 2014) in their future K-12 classes. Thus, this study explores the inner workings of PTs' productive struggle to investigate how and when those learning moments arise.

Literature Review

Productive struggle is a growing area of research that has encompassed three major themes: (a) types of struggles (e.g., DuCloux et al., 2018; Warshauer, 2015), (b) factors that foster productive struggle in classes (e.g., Roble, 2017; Russo et al., 2021), and (c) ways to help PTs to support productive struggle in their future classes (e.g., El-ahwal & Shahin, 2020; Warshauer et al., 2021). Since our work focuses on PTs' productive struggle, we elaborate on the first theme. Warshauer (2015) investigated 327 middle school students' struggles and developed a productive struggle framework that included four dimensions: (a) Task, (b) Student Struggle, (c) Teacher Response, and (d) Outcome. Within the Student Struggle dimension, she identified four kinds of student struggles: (a) *get started*, (b) *carry out a process*, (c) *give mathematical explanation*, and (d) *express misconception and errors* (Warshauer, 2015). These identified kinds of struggles have served as the foundation for other research in this theme. Sayster and Mhakure (2020) used Warshauer's kinds of student struggles to discuss 28 high school students' struggles with simplification of rational algebraic expressions. Their results aligned with Warshauer's kinds of student struggles. Other researchers have investigated PTs' struggles. Zeybek (2016) used Warshauer's kinds of student struggles to document the struggles of 48 pre-service middle grade teachers when engaging in non-routine tasks in a geometry course. Zeybek's findings aligned well with the results of Warshauer's study, thus she concluded that "struggle types are similar in all classrooms regardless of grade level or participants as long as participants are engaged in a high level task" (p. 411). DuCloux et al. (2018) used Warshauer's kinds of student struggles as a guide to analyze the perceptions of 32 prospective elementary, middle, and secondary teachers about their struggles. Their participants identified perseverance and struggling together with their peers as important aspects of their productive struggles. Although these studies provide us some information about PTs' productive struggles, they stop short of investigating the inner workings of productive struggle. Hence, our study aims to build on the existing research by exploring this research question: *What do PTs' productive struggles look like when they are engaged in a challenging mathematics task in a middle school mathematics methods course?*

Theoretical Framework

We ground our work in Warshauer's (2015) productive struggle framework mentioned above, with our adaptations shown in Figure 1. Rather than looking at tasks with a variety of cognitive demands, we used a *doing math* task (Box 1; Stein et al., 1996). Similarly, rather than looking at all possible outcomes, we looked only at interactions that led to a productive outcome (Box 4). To operationalize a productive outcome, we defined a *productive struggle* as an interaction that begins with some expression of struggle, verbal (e.g., "I'm confused", "I don't know", or long pause), or physical (e.g., deep sigh, putting hands in head) that provides the opportunity to better understand a well-specified statement of a mathematical truth—what we call a mathematical point (MP; Van Zoest et al., 2016)—and ends when some sense has been made of the MP and the focus of the interaction changes. Our work is situated in Boxes 2 and 3.

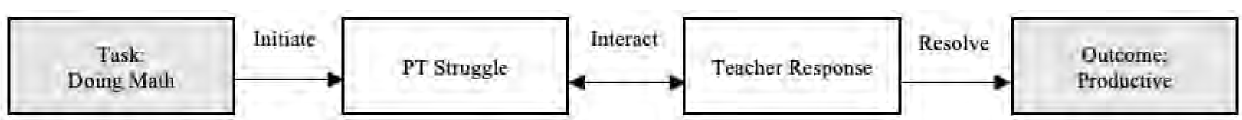


Figure 1: PT Productive Struggle Framework (Developed from Warshauer, 2015)

Methodology

This exploratory study is part of a larger ongoing project that investigates the teaching and learning that occurs in a middle school mathematics methods course (e.g., Ochieng, 2018; Van Zoest & Levin, 2021). The current study took place with the 18 PTs in the Fall 2021 course during the first two 90-minute class sessions. The PTs were engaged in the Frog Problem (see Figure 2), which met the criteria for a *doing math* task (Stein et al., 1996). (For more details about the Frog Problem, see Andrews (2000) and Dixon and Watkinson (1998).)

Figure 2: The Frog Problem Prompt and Representation of Two Frog Teams of Size Three

Data collection included videos and audio recordings of both class sessions, which were primarily whole-class discussion on the first day and small-group discussions on the second-day. We also gathered electronic copies of all PT written work and reflections from both days to help us better understand what a productive struggle looks like in detail. We first identified interactions that met our definition of productive struggle (see above). These interactions—

referred to as "productive struggles"—served as our primary data set for the study, with the other data sources referred to as needed to help us make sense of each productive struggle. Our analysis of these productive struggles was supported by coding the answers to these questions: (a) Who is struggling?, (b) What are they struggling with?, and (c) Why is it a productive struggle? In our broader analysis we looked at the role of the teacher response, but here we narrow the discussion of our results to what we found out about the Student Struggle dimension of Warshauer's (2015) framework, identified as PT Struggle in Box 2 of Figure 1.

Results & Discussion

Based on our preliminary analyses, we hypothesize that a productive struggle consists of different types of struggles that can coexist simultaneously. We identified three types of struggles: *goal struggle*, *strategy struggle*, and *sub-strategy struggle*. We use an illustrative productive struggle (IPS) to elaborate on these types and to provide insight into the inner workings of PTs' productive struggles and how they may differ from school students' struggles (Warshauer, 2015). This particular productive struggle took place during a small-group discussion among four PTs (PT1, PT2, PT3, and PT4) that occurred on the second day of the classes' engagement with the Frog Problem. At this point in the discussion, the group knew the actual numbers for frog teams of size 1, 2, 3, and 4. PT3 suggested an arithmetic sequence [$a_n = a_1 + (n - 1)d$] as a template for finding an equation that represents the fewest number of moves for any size team of frogs and the other PTs agreed. (Note that the difference of the fewest moves from each consecutive case is not constant, thus an arithmetic sequence does not apply to this situation.) The IPS began with the following exchange:

PT1: So wait, what would *d* be then?
PT3: That's what I'm trying to figure out because the difference increases so [*d*]'s not consistent. So I don't know that it's technically arithmetic because the, an arithmetic sequence is consistent.
PT1: That's true.
PT3: So I don't know if I can change the *d* to be something like that.

This exchange initiated a productive struggle because it provided the PTs the opportunity to better understand this MP: *d is a constant that represents a common difference between each consecutive term in an arithmetic sequence.* The IPS ended approximately ten minutes later when the group made sense of that MP and agreed on a plan for a different approach (discussed in more detail below).

After the transcript excerpt above, the other PTs joined PT3 in her struggle by attempting to find *d* even though they recognized that the *d* in an arithmetic sequence needs to be constant. We noticed that their struggle was related to *express misconception and errors*, one kind of student struggle from Warshauer's (2015) study. However, the result from our study is slightly different, since the PTs realized the misconception of using *d* as a common difference in arithmetic sequence when the differences in their data are not constant. We hypothesize that the PTs additional experience in learning mathematics may be responsible for their ability to recognize their misconception, leading to a more sophisticated version of this kind of struggle than that seen in Warshauer's study with middle school students.

The PTs were struggling with making sense of *d* in an arithmetic sequence. However, while struggling with this specific concept, they also had been struggling with how to use an arithmetic sequence as a template to find the equation that represents their solution to the task. This is an example of what we call a *strategy struggle*. Hence, struggling with finding *d* became a *sub-*

strategy struggle since it represents small steps or elements in the proposed strategy. While engaged in both of these types of struggles, they also had been struggling with how to find an equation to represent the fewest number of moves for any size team of frogs. This struggle is defined as a *goal struggle* since the goal of this task is to generalize their obtained data to any size frog team. Since struggling with this sub-strategy struggle—making sense of *d* in an arithmetic sequence—brought an opportunity for the PTs to better understand the stated MP (see above), we hypothesize that with *doing math* tasks, resolving sub-struggle strategies may be where PTs' learning—their coming to better understand an MP—occurs. To describe more about how that learning occurs, we need to take into consideration context outside the IPS. From our broader analysis, the challenging mathematical task—the Frog Problem—initiated the PTs' goal struggle. Then, the PTs' goal struggle initiated the PTs' strategy struggle, followed by the PTs' sub-strategy struggle that led them to further their learning of the MP of this IPS. Thus although in this case there were three struggles going on simultaneously that were moving the PTs towards the goal of the lesson, it was the sub-strategy struggle that made this particular interaction productive in the sense of learning mathematics.

Towards the end of the 10-minute IPS, making a decision to change to a different focus after making sense of the *sub-strategy struggle* provided evidence that learning had occurred. Below is when the PTs realized they could not use the current approach and proposed a next strategy for the group to consider.

PT2: So instead of making it *d*, could almost like put a variable that was like the last variable I'm gonna add two then in place of d like that's what it would represent there?

PT1: So what do you mean?

PT2: Like, Ok so [PT3: I know what you're saying] Yeah, Like if you have like when you [PT3: But then we will have two variables] use the last one. Yeah that's the only thing is that there'd be a lot of variables.

PT4: Like and now we can do some substitution or elimination to figure out the other variable

PT1: So I feel like the total moves for the first four groups we can plug that in and maybe solve for a variable.

PT2: Yeah

Making these decisions as a group provided evidence that they were all engaged in making sense of the MP and had benefited from the learning opportunity.

Conclusion

The results of this exploratory study revealed the complexity of the inner workings of PTs' productive struggle. We hypothesize that a productive struggle can consist of different types of struggles—*goal struggle*, *strategy struggle*, and *sub-strategy struggle*—that may coexist simultaneously. These types of struggles expand our knowledge about struggle by adding detail about the nature of struggles to the already existing information about kinds of struggles (e.g., Warshauer, 2015; Zeybek, 2016). These findings highlight the importance of mathematics teacher educators recognizing the mathematical opportunities that may be present in their PTs' struggles and provide some insight into how mathematics teacher educators might leverage their PTs' productive struggles to better prepare those PTs to support their future students' productive struggles.

References

Andrews, P. (2000). Investigating the structure of Frogs. *Mathematics in School*, *29*(2), 7-9.

Dixon, J., & Watkinson, R. (1998). The investigating experience: Frogs. *Mathematics in School*, *27*(5), 42-43.

DuCloux, K., Gerstenschlager, N., Marchionda, H., & Tassell, J. (2018). Characterizing prospective mathematics teachers' productive struggle. In Venenciano, L. & Redmond-Sanogo, A. (Eds.), *Proceedings for the 45th Annual Meeting of the Research Council on Mathematics Learning* (pp. 9-16). Baton Rouge, LA.

El-ahwal, M., & Shahin, A. (2020). Using video-based on tasks for improving mathematical practice and supporting the productive struggle in learning math among student teachers in the faculty of education. *International Journal of Instructional Technology and Educational Studies*, *1*(1), 25-30.

National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematical success for all*. Author.

Ochieng, M. A. (2018). *The bellringer sequence: Investigating what and how preservice mathematics teachers learn through pedagogies of enactment* [Unpublished doctoral dissertation]. Western Michigan University.

Roble, D.B. (2017). Communicating and valuing students' productive struggle and creativity in calculus. *Turkish Online Journal of Design Art and Communication*, *7*(2), 255-263.

Russo, J., Bobis, J., Downton, A., Livy, S., & Sullivan, P. (2021). Primary teacher attitudes towards productive struggle in mathematics in remote learning versus classroom-based settings. *Education Sciences*, *11*(2), 35. https://doi.org/10.3390/educsci11020035

Sayster, A., & Mhakure, D. (2020). Students' productive struggles in mathematics learning. In K. Tirri, & A. Toom (Eds), *Pedagogy in basic and higher education - Current developments and challenges*. IntechOpen.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455–488. https://doi.org/10.3102/00028312033002455

Van Zoest, L. R., & Levin, M. (2021). Artifact-enhanced collegial inquiry: Making mathematics teacher educator practice visible. In M. Goos & K. Beswick (Eds), *The learning and development of mathematics teacher educators.* Research in Mathematics Education. Springer, Cham. https://doi.org/10.1007/978-3-030-62408-8_9

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., & Peterson, B. E. (2016). Theorizing the mathematical point of building on student mathematical thinking. In C. Csíkos, A. Rausch, & J. Szitányi (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 323-330). Szeged, Hungary: PME.

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375-400.

Warshauer, H. K., Starkey, C., Herrera, C. A., & Smith, S. (2021). Developing prospective teachers' noticing and notions of productive struggle with video analysis in a mathematics content course. *Journal of Mathematics Teacher Education*, 24(1), 89-121. https://doi.org/10.1007/s10857-019-09451-2

Zeybek, Z. (2016). Productive struggle in a geometry class. *International Journal of Research in Education and Science*, *2*(2), 396-415.

RECRUITING AND RETAINING MINORITIZED PRESERVICE TEACHERS: CRITICAL RESISTORS

Nick Kim
The University of Tennessee
nkim2@utk.edu

Lynn Hodge
The University of Tennessee
lhodge4@utk.edu

Shande King
Trevecca Nazarene University
nking2@trevecca.edu

The primary purpose of this research was to provide insight into the narratives and experiences of minoritized (based on race) preservice teachers (PST) interested in teaching mathematics and how those experiences impact recruitment and retention of those minoritized teachers. Narrative inquiry framed through equity and identity lenses were chosen as the methodology for gathering and analyzing data from one-on-one, structured interviews as well as a focus group interview. A finding from the data analysis resulted in one specific resonant thread, Critical Individuals: Resistors. In summary, the PSTs' minoritized identities clearly impacted their decisions to pursue and remain in mathematics teacher education. All participants could recall "aha moments" when dealing with important people in their lives who either supported or attempted to resist their journeys to pursue teacher education.

Keywords: Preservice Teacher Education, Teacher Educators, High School Education

Teacher shortages across the United States in elementary, intermediate, and secondary schools are a recognized concern and can be credited to the difficulties in recruiting and retaining highly qualified teachers. The American Association for Employment in Education (AAEE, 2016) found data signifying a substantial teacher shortage in the Southeast United States. More specifically in Tennessee, forecasts designate that half of the more than 65,000 teachers in the state will leave or retire in the next decade (Aldrich, 2017). Teacher preparation programs, policymakers, and administrators are particularly familiar with said difficulties in that they are continuously tasked with the challenge of finding highly qualified teachers in response to a reduction in enrollment for teacher educator programs (TEP) (Flannery, 2016; Westervelt, 2015), a high rate of teachers leaving the profession only after a few years, and a high proportion of veteran teachers retiring.

Shortages of specifically minoritized teachers coupled with the growth of minoritized students attending schools raise additional challenges. According to McFarland et al. (2018) and National Center for Educational Statistics (NCES, 2019), minoritized students made up approximately 52% of public-school students in 2015-2016. By contrast, however, minoritized teachers made up approximately 20% of U.S. public school teachers during the same timeframe – constituting a mere 4% growth since 1999-2000 (McFarland et al., 2018; NCES, 2019). These statistics evidence the unlikelihood of many minoritized students seeing a proportional number of teachers representing their own communities. This is significant in that minoritized teachers are more likely to honor and support the cultures and interests of minoritized students, and according to research, such exemplification has been found to be a source of motivation for some minoritized students (Guarino et al., 2006).

Issues and Problem Statement

Research has addressed issues of teacher shortages and recruitment in general as well as more specific issues of teacher shortages among disciplines including science, technology, engineering, and mathematics (STEM). Fisher and Royster (2016) found that secondary level core subject teachers such as mathematics and science teachers as well as special education teachers are most likely to leave the teaching profession. Fisher and Royster (2016) take a look at multiple studies that offer different accounts as to why mathematics teachers leave the profession including: not enough preparation in the mathematics content, stressful situations from poor student behavior and lack of administrative support, and pressure from high-stakes testing. Fisher and Royster (2016) also offer resolutions to reduce teacher stress levels and ease the teacher retention problem by offering recommendations for stronger professional development, more effective mentoring for new teachers, and more productive peer collaboration.

In spite of these studies, we still know little about how to recruit and retain mathematics educators in practical ways. Therefore, we are forced to look for new and innovative ways to recruit more teachers, especially mathematics teachers, in practical ways that do not require extensive budget reforms that are unlikely to be implemented. This study addressed the broader issues of diversifying the teacher workforce and teacher shortages, particularly in the area of mathematics education. In doing so, the study shined light on the effective recruitment and retention strategies by identifying common stories amongst minoritized participants. The primary research question addressed in this study: How do minoritized preservice teachers' experiences (identity) contribute to their decision to pursue mathematics education? The sub question addressed in this study: How does an identity as a minoritized individual based on race and ethnicity affects one's pursuit of mathematics education? In this study, experiences are defined as life events that impacted the individuals' identity as a minoritized individual.

Theoretical Framework

The notion of identity informs the design of this study. An individual's identity is created as one participates in the social and cultural practices within a community of practice (Lave & Wenger, 1991). In this instance, concentrating on the involvement in social experiences also indicates a clear emphasis on the individual as a participant of the sociocultural community (Lave & Wenger, 1991). This situated perspective of identity comprises in what ways individuals see themselves, how society views them, and how they interpret these views in relation to their positions within communities of practice (Matheny, 2016). In addition, Hodges and Hodge (2017), drawing on Wenger (1998), described identity as a "constant becoming" where identities evolve based on their experiences within communities of practice.

More specifically and making identity more concrete, Gresalfi and Cobb (2011) and Hodges and Hodge (2017) identify one's "personal identity" as formed when one comes to associate with certain practices valued within a community. "The idea of personal identity provides structure to a narrative approach by focusing our attention on the normative ways of acting and valuations of these norms from a participant's perspective (Hodges & Hodge, 2017, p. 104)." In this way, the notion of personal identity informs the design of this study by focusing attention on whether or not and to what extent PSTs experiences as minoritized individuals influenced their decisions to pursue mathematics teacher education. In addition, this identity lens emphasizes the resources, including other individuals and interactions, that contribute to or delimit the participants' identities.

Narrative Inquiry

Clandinin and Connelly (2000) refined the narrative research approach to a narrative inquiry methodology that commences and concludes "in the midst of living and telling, reliving and retelling, the stories of the experiences that made up people's lives, both individual and social" (p. 20). Narrative inquiry is mutual as the researcher and the participants "reach a joint intersubjective understanding of the narratives that occur during the research process" (Moen, 2006). The cooperative association between participants and researcher is also expressed in the re-storying of their experiences. Human beings give meaning to experience by situating it in time, place, and relationship to others (Connelly & Clandinin, 2006). This is the reason why the study is situated in Tennessee as the participants and researcher have collaborated in various learning environments prior to the study.

Data Analysis

This study consisted of seven participants of varying races, genders, and status as a university undergraduate student. During this study, preliminary and primary data analysis occurred concerning both the individual narrative interviews as well as the focus group interview. Preliminary data analysis during both interview types assisted in providing structure and quality-checking of participant responses. In addition, this process provided an opportunity to start initial analysis for possible implications and overall recurrent themes. The researcher continued to use the research journal during this preliminary analysis process to keep note of themes, irregularities, and noteworthy findings. Patterns and findings from the preliminary analysis portion of the individual narrative interviews influenced modifications in the questions and process of conducting the focus group interview. The analysis of the data collected in the research journal was then transferred to a spreadsheet with each individual participant's quotes and initial codes. This was the beginning process of the primary data analysis.

Primary data analysis of the transcriptions from the audio from both interviews served as the primary source of data in this study. In addition, video recordings of the Zoom Video Conferencing interviews allowed the researcher to focus not only on the words chosen by the participant but also their body language, facial expressions, changes in tone, and verbal emphases while answering personal questions about their experiences. The experiences and narratives collected were then coded *in vivo* through continuous coding cycles with constant comparisons (Saldaña, 2013). In the opening round of coding, each individual interview was evaluated separately through multiple readings of the transcript while examining the notes on body language and facial expressions to be for significant findings to address the research questions. These codes were created from concrete experiences contributing to participants' journey to become a minoritized mathematics teacher and recorded in the spreadsheets mentioned above.

Results

Resonant Narrative Threads

Clandinin defined resonant threads as "threads that echoed and reverberated across the accounts" (Clandinin et. al., 2012, p.14). Therefore, similar themes and significant differences were identified across the narratives. The next section will describe how the participants' interactions with critical individuals in their lives resisted their journey to become a mathematics teacher.

Critical Individuals: The Resistors

Critical individuals resisted the participants' intentions to pursue mathematics teaching. The major reason for resisting becoming a mathematics teacher came from their families concern about the adequacy of teacher salaries. Sydney, Tiana, Peter, and Il-seong's families all wanted them to "earn higher than average salaries" since they were in lucrative fields of mathematics or engineering in Peter's case. The rationale was due to financial stability. All participants questioned and revisited their decisions to pursue mathematics teacher education because of the concerns expressed by their families, but ultimately the participants decided that a high salary was not the major factor in their decisions to become a teacher. This theme is made clear in Sydney's (a Black, female) comments about her decision to continue pursuing becoming a mathematics teacher. As she notes:

> My grandparents, they were like, Sydney, if you're going to school for math, don't be a teacher. And they kind of skewed my viewpoint a bit. And I was like, should I do something else? Because they made me feel like I was settling... They were like, you're smart, so why are you going to pursue a math degree, then just going settle on being a teacher? And I was like, I guess, yeah, you're right. I mean, why am I just going to settle on being a teacher, but then I realized that I wasn't settling on being a teacher. They wanted me to go and work and get money, which is fine. Money is great, don't get me wrong, but I was like, I was not going to be happy with that. Man, of course, I'd be happy, but was I going to be genuinely happy? And I realized I wasn't going to be genuinely happy with that, so I went back to teaching.

For most of the participants, becoming a teacher serves more than just earning a salary; it is about making a difference in students' lives, similarly how other teachers have changed their lives.

Reflecting one significant difference, Gabriel's (a Latino, male) resistor came in the form of an eighth-grade mathematics teacher. His teacher was a White, female who recommended Gabriel take below grade-level mathematics coursework during his freshman year of high school. Gabriel has always been confused about why he was placed into such a low performing class. Instead of harboring negative feelings, Gabriel used the actions of the teacher as motivation to pursue high- level mathematics in high school. As the participants' journeys make clear, they decided to carve their own paths in spite of resistance from some critical individuals in their personal lives. It seemed that these tough life decisions created critical decisions in which the participants chose to listen to their own voice rather than the voice of their resistors. However, the resistors did play significant roles in motivating participants in some cases, such as Gabriel's. These critical individuals, these resistors, often play a role in what I call aha moments when individuals decide to pursue teaching.

Conclusion

Minoritized PSTs have experiences with critical individuals in their journey to pursue a mathematics teacher license. These individuals can often be resistors not just supporters. This research examined the critical individuals who resisted PSTs decision to pursue mathematics education through various ways. The resistance came from family members as well as teachers. The first recommendation based on this research is to inform PSTs with all of the financial issues with becoming a teacher. Starting with low salary- presenting the statistics that show teacher salaries combined with benefits such as retirement and step raises actually are competitive with many other professions. In addition, identifying the teach grants or federal teacher loan forgiveness programs that can ease some of the financial burden.

Another recommendation based on the research is to allow current secondary teachers, as well as community college or university instructors, to actively search for potential mathematics teachers by explicitly providing opportunities for minoritized students to teach or tutor or by explicitly recommending them to consider mathematics teaching. In contrast, current teachers should be made aware of their decisions on how they track their students and the long-term impact those decisions have on their potential students' potential career choices.

References

Aldrich, M. (2017). Common core is out. Tennessee academic standards are in. Here's how teachers are prepping for the change. Retrieved from http://www.chalkbeat.org/posts/tn/2017/06/26/common-core-is-out-tennessee-academicstandards-are-in-heres-how-teachers-are-prepping-for-the-change/

American Association for Employment in Education. (2016) Educator supply and demand report 2015-16: Executive summary

Clandinin, D., & Connelly, F. (2000). Narrative inquiry: Experience and story in

qualitative research. San Francisco, CA: Jossey-Bass.

Clandinin, D., Lessard, S., & Caine, V. (2012). Reverberations of narrative inquiry: How resonant echoes of an inquiry with early school leavers shaped further inquiries. *Educacao, Socidedade & Culturas, 36,* 7-24.

Connelly, F. M., & Clandinin, D. J. (2006). *Narrative inquiry. In J. Green, G. Camilli, & P.*

Elmore (Eds.), Handbook of complementary methods in education research (pp. 375–385). Mahwah, NJ: Lawrence Erlbaum.

Fisher, M. H., & Royster, D. (2016). Mathematics teachers' support and retention: Using Maslow's hierarchy to understand teachers' needs. *International Journal of Mathematical Education in Science and Technology,47*(7), 993-1008. DOI:10.1080/0020739X.2016.1162333.

Flannery, M. (2016). Survey: Number of future teachers reaches all-time low. *NEA Today.* Retrieved from http://neatoday.org/2016/03/15/future-teachers-at-all-time-low/

Guarino, C., Santibanez, L., & Daley, G. (2006). Teacher recruitment and retention: A review of the recent empirical literature. *Review of Educational Research*, 76(2), 173-208.

Hodges, T., & Hodge, L. (2017). Unpacking personal identities for teaching mathematics within the context of prospective teacher education. *Journal of Mathematics Teacher Education, 20*, 101-118.

Lave, J., & Wenger, E. (1991). Situated learning: Legitimate peripheral participation.

Cambridge University Press.

Matheny, E. (2016). *Becoming a mathematics teacher: Identity, narratives, and resources.* [Doctoral dissertation, University of Tennessee]. Trace: Tennessee Research and Creative Exchange.

McFarland, J., Hussar, B., Wang, X., Zhang, J., Wang, K., Rathbun, A., & Mann, F. (2018). *The Condition of Education 2018* (NCES 2018-144). Washington, DC: U.S. Department of Education, National Center for Education Statistics. Retrieved from https://nces.ed.gov/pubs2018/2018144.pdf

Moen, T. (2006). Reflections on the narrative research approach. *International Journal of Qualitative Methods, 5*(4), 56-69. DOI: 10.1177/160940690600500405

National Center for Education Statistics. (2019).

Saldaña, J. (2013). *The coding manual for qualitative researchers*. Los Angeles: Sage.

Westervelt, E. (2015). *Where have all the teachers gone?* Retrieved from https://www.npr.org/sections/ed/2015/03/03/389282733/where-have-all-the-teachers-gone

MAESTROS, ESTUDIANTES, LENGUAJE Y MATEMÁTICAS: PRÁCTICAS PEDAGÓGICAS PARA UNA FORMACIÓN CON VISIÓN GLOBAL

TEACHERS, STUDENTS, LANGUAGE, AND MATHEMATICS: PEDAGOGICAL PRACTICES FOR PRE-SERVICE TEACHERS WITH A GLOBAL VISION

Gladys Krause	Adriana Ávila	Elgar Gualdrón
William & Mary	Universidad Autónoma de Bucaramanga	Universidad de Pamplona
ghkrause@wm.edu	aavila2@unab.edu.co	egualdron@unipamplona.edu.co

Describimos la segunda fase de una investigación en curso que se pregunta sobre la manera en que los maestros en formación asumen sus prácticas pedagógicas en un contexto global donde el lenguaje y las matemáticas se conjugan. Tal intención obedece al propósito de formar maestros que respondan a los retos educativos actuales. Uno de estos retos es el uso del inglés como lengua global. Se presenta una experiencia de trabajo conjunto entre estudiantes y docentes de dos cursos de enseñanza de métodos y didáctica de las matemáticas, uno en Estados Unidos y otro en Colombia. Esto implicó el diseño de actividades para la enseñanza de las matemáticas usando como referencia cuentos infantiles. Dentro de los resultados se identifica que las actividades fueron influenciadas más por el uso del lenguaje que por el contenido matemático que se quería enseñar.

Palabras clave: Teacher Educators, Preservice Teacher Educators, Elementary School Education

Este artículo presenta resultados preliminares de una investigación en curso que tiene como centro la formación de docentes de la escuela primaria y preescolar en el área de las matemáticas. Además, nuestra investigación intenta identificar cómo esta formación contribuye a el desarrollo de docentes globales (Krause et al., 2021).

El uso de la palabra "globalización" explotó en los años noventa porque capturó la naturaleza cada vez más interdependiente de la vida social en nuestro planeta (Steger, 2013). Esta interdependencia ha llevado a grandes cambios sociales, políticos, y económicos a nivel mundial, que han influenciado los sistemas educativos locales. Esta interdependencia ha establecido procesos de estandarización en el campo educativo enfatizando en exámenes que miden y posicionan a los estudiantes, a los maestros y a los sistemas educativos de acuerdo con un ranking establecido por agentes económicos (por ejemplo, la Organización para la Cooperación y el Desarrollo Económico - OCDE), que al mismo tiempo se encargan de patrocinar dichos exámenes (por ejemplo, las pruebas PISA) (Meyer & Benavot, 2013). Como consecuencia, el aula escolar en el mundo está cambiando y este cambio, necesariamente, nos obliga a hablar sobre la formación de maestros. Como formadores de maestros estamos obligados a preparar maestros que puedan asumir los retos que traen estos cambios en los sistemas educativos. Los tres autores de este artículo nos dimos en esta tarea hace dos años. En 2020 iniciamos nuestro trabajo colaborativo enfocándonos en entender cómo se están formando los maestros desde nuestros contextos locales, específicamente desde los cursos relacionados con los métodos y didáctica de las matemáticas. Resultados preliminares de la primera fase fueron publicados y presentados en esta misma conferencia el año anterior (Krause, et al., 2021). En dicha publicación documentamos la planeación, desarrollo e implementación inicial de retos matemáticos en dos clases: una de métodos de enseñanza de las matemáticas en Estados Unidos y la otra de didáctica de las matemáticas en Colombia. En este estudio encontramos cuatro aspectos en común en los dos contextos; los maestros en formación (MEF): (1) usaron diferentes estrategias para solucionar los problemas matemáticos, (2) reflexionaron sobre los posibles efectos negativos de la memorización de fórmulas sin entender conceptos, (3) se dieron cuenta de que entienden más matemáticas que lo que realmente creían, (4) reportaron el disfrutar el trabajo de los retos matemáticos (Krause, et al., 2021). Otros aspectos que identificamos se refieren a que los MEF en el contexto colombiano mostraron características de trabajo colaborativo, mientras que en el contexto de Estados Unidos los MEF expresaron tendencias más individualistas. Adicionalmente, en el contexto de Estados Unidos los MEF estuvieron más atentos al uso del tiempo mientras que en Colombia consideraciones de tiempo para la solución de problemas no fueron notorias. A partir de estos resultados pasamos a una segunda fase de nuestro estudio en la que, informados por los resultados anteriores, decidimos integrar los dos contextos en una actividad conjunta.

En esta fase le pedimos a los MEF que diseñaran y enseñaran actividades matemáticas para los MEF en el otro país. Esta integración tomó varios meses de planeación principalmente porque debíamos establecer puentes de comunicación entre dos contextos monolingües, uno en inglés y otro en español. En este artículo documentamos la experiencia de diseño, integración y resultados preliminares que han emergido al analizar este trabajo. En este artículo nos centramos en contestar la siguiente pregunta de investigación: *¿de qué manera asumen los maestros en formación sus prácticas pedagógicas en un contexto donde el lenguaje y las matemáticas se conjugan*?

En las siguientes secciones describimos la actividad pedagógica que desarrollamos para integrar las dos clases, cómo se implementó, y las decisiones de instrucción que tomamos informados en la literatura. Por último, presentamos un resumen de lo que aprendimos y los pasos a seguir en el futuro.

Antecedentes y Marco Teórico

Nuestro estudio parte del principio que las matemáticas deben estar en el centro de las discusiones de equidad en la sociedad y por ende en el salón de clase (Moses & Cobb, 2002; Gutierrez, 2013). Este principio rige el marco teórico de nuestras prácticas pedagógicas. En el contexto estadounidense la clase de métodos de enseñanza de las matemáticas se enfoca en proveer oportunidades que fomentan el extender y expresar ideas matemáticas. Un aspecto para resaltar en esta clase es el enfoque que tiene en fomentar la participación de estudiantes multilingües. En particular un pilar importante de esta clase es el promover las discusiones matemáticas especialmente aquellas que mantienen el lenguaje y la identidad cultural de los estudiantes (Krause, et al., 2021; Krause & Colegrove, 2020). En el contexto colombiano, la clase Didáctica de las Matemáticas centra su interés en proveer a los futuros licenciados de estrategias y recursos que les permitan integrar la formación matemática en la educación infantil (Murcia & Henao, 2015).

Cuando estudiamos la globalización en el contexto educativo es muy común encontrar artículos y estudios que hacen referencia a los “ciudadanos globales” (por citar algunos (Spring, 1998; Steger, 2013; Subedi, 2013; Triandis, 2001)). En muchos casos, estos ciudadanos globales son aquellos ejecutivos que se mueven fluidamente entre continentes y estudiantes internacionales que en muchas ocasiones gozan de los mismos privilegios económicos. Sin embargo, muy poco se dice sobre otros ciudadanos globales que también se mueven entre países y culturas, pero con menos ventajas económicas. Este contraste lo describe claramente Doerr (2020), quien hace un paralelo mostrando cómo los diferentes estatus que proveen la raza, el nivel socioeconómico y el nivel educativo de algunos ciudadanos globales contrastan con otros ciudadanos globales que cruzan fronteras en busca de seguridad, educación y empleo. En las escuelas públicas de los Estados Unidos, estos ciudadanos globales son un número bastante grande. La pobreza es una de las razones más comunes descrita por ellos como el motivo principal para dejar su país (Bigelow, 2006). Particularmente en Centro y Latino América, los Estados Unidos ha jugado un papel importante en contribuir a este estado de pobreza. El acuerdo *North American Free Trade Agreement* (NAFTA) es quizá una de las razones más mencionadas y estudiadas en las aulas de clase como una consecuencia directa del incremento en las migraciones masivas de Centro América a los Estados Unidos. Sólo en México, 10 años después de la firma de este acuerdo, el salario mínimo declinó 50%, las exportaciones de maíz de Estados Unidos a México incrementaron 14% y el precio del maíz mexicano bajo 45% entre los años 2001 y 2005. Según reportes económicos en esa misma década 3 millones de empleos que dependían de la agricultura mexicana desaparecieron (Bigelow, 2006).

Otro aspecto que se menciona comúnmente al hablar de globalización es el uso del inglés como lengua global (Melitz, 2016). Al hablar de la globalización como un proceso de interconexión, el lenguaje se convierte en un medio casi obligatorio para esta interconexión, sirviendo como herramienta en los procesos de estandarización que mencionábamos anteriormente. En el contexto colombiano, bajo el argumento de la necesidad de un país más competitivo en tiempos de la globalización, el gobierno central aprobó una serie de reformas educativas en el año 2005 en las cuales el aprendizaje y enseñanza del inglés adquiere un papel prioritario en el sistema escolar nacional (Usma, 2009).

Identificados estos dos grandes aspectos de la globalización, influenciando cada uno de los contextos donde trabajamos, nos dimos a la tarea de diseñar la actividad en común.

Descripción y justificación de la actividad

La orientación del ejercicio propuesto para nuestros MEF (tanto estadounidenses como colombianos) fue diseñar una actividad para enseñar un concepto o desarrollar un pensamiento matemático a partir de cuentos para niños. La actividad debía además ser enseñada a sus pares del otro país. Esta decisión la tomamos informados por la literatura. Sabemos que frecuentemente los educadores de matemáticas usan los ensayos pedagógicos como medio para modificar sus prácticas de instrucción (Webb & Wilson, 2022). Estos ensayos son herramientas que permiten a los MEF imaginar nuevas posibilidades de enseñanza y aprendizaje. Además, estos ensayos proveen la oportunidad al MEF de reflexionar en *cómo y por qué* actuamos de la manera que lo hacemos desde el punto de vista del maestro y del estudiante (Webb & Wilson, 2022). De esta manera, los ensayos pedagógicos abren la puerta para reflexionar en nuevas formas de aprender y en por qué respondemos de la manera que lo hacemos en el mismo momento de la instrucción. Además, la reflexión colectiva contribuye a alinear nuestras prácticas de instrucción a través de la observación y discusión (Webb & Wilson, 2022).

En preparación de la actividad cada pareja de Estados Unidos y de Colombia tuvo la posibilidad de escoger un cuento y tener 6 semanas para diseñar la actividad de enseñanza. La actividad debía trabajar algún concepto o pensamiento matemático. Los MEF tenían la libertad de escoger el concepto matemático que ellos consideraran adecuado. Luego, dispusimos diferentes espacios en línea para que los MEF de los dos países se pusieran en contacto y se enseñaran la actividad.

La idea de usar los cuentos surge con la intención de proveer a los estudiantes con un punto de partida en común. Tal intención obedece a la diferencia de idiomas que se manejan en los dos contextos, por tanto, de cada cuento se tenía su versión en inglés y en español. El uso de cuentos para enseñar matemáticas ha sido empleado por los investigadores en sus clases. Por ejemplo, en la clase de Estados Unidos el primer autor ha diseñado una actividad donde los MEF crean una figura geométrica usando las letras de su nombre. Como preámbulo de esta actividad el primer autor trae varios cuentos que hacen referencia a los nombres. Uno de estos cuentos es Crisantemo (y su versión en inglés Chrysanthemum) (Henkes, 2008), *The Name Jar* (Yangsook, 2003), *Marisol McDonald Does not Match* (Brown, 2011), entre otros. Durante la clase hay un espacio para que los estudiantes lean los cuentos, vean las imágenes, y compartan sus apreciaciones del libro con sus compañeros. Luego, los discutimos como clase. La discusión del cuento se centra en varios aspectos de pertenencia e identidad y la relevancia que estos tienen en el aprendizaje de las matemáticas. Al mismo tiempo, esta actividad abre un espacio en la clase para hablar sobre cómo hacer conexiones en clase de matemáticas con la lectura. Además del uso de libros que incluyen diferentes representaciones culturales y lingüísticas en el salón de clase. Luego, los MEF pasan a realizar una actividad matemática que requiere reemplazar y despejar variables usando valores asignados a cada letra del nombre de cada uno. Así se generan pares ordenados que luego se ubican en el plano cartesiano. De esta manera se generan figuras geométricas usando sus nombres. En esta actividad el cuento es una herramienta para ayudarnos a discutir el tema de pertenencia e identidad, es el preámbulo de la clase. La actividad matemática se conecta con este preámbulo y el cuento nos ayuda a unir los nombres con la matemática. En preparación a nuestra actividad colaborativa, Colombia adoptó esta actividad también. Teniendo en cuenta que el monolingüismo en cada uno de nuestros contextos locales podría generar una barrera de comunicación, decidimos ubicar cuentos que estuvieran disponibles en inglés y español, o que no tuvieran palabras, solo imágenes. También necesitábamos que los MEF pudieran tener acceso a los libros de forma digital. Aquí nos apoyamos en los recursos de las bibliotecas en cada una de nuestras instituciones. Siguiendo estos parámetros tomamos como base cinco cuentos infantiles: *Un Charco Azul* (Amaco, 2007); *Cómo Atrapar Una Estrella* (Jeffers, 2010); *Crisantemo* (Henkes, 2008); *Mi León* (Mandana, 2006); estos cuatro dispuestos en inglés y en español y *Journey* (Becker, 2013) que contiene sólo imágenes.

Las pautas para la actividad fueron las siguientes: (1) trabajar en grupos de dos. Cada grupo fue asignado a otro par en el contexto de la otra clase y fueron presentados con 6 semanas de anterioridad a la ejecución de la actividad. Basados en nuestras experiencias pedagógicas sabemos que el trabajo en parejas proveería un poco más de seguridad para que los estudiantes en cada contexto se sintieran apoyados en el diseño y ejecución de la actividad. Esto era importante para nosotros, dado que era la primera vez que estábamos realizando la actividad y además sabíamos que los estudiantes en los dos contextos se sentían un poco atemorizados de afrontar este reto, dado las diferencias en el lenguaje. (2) Diseñar una actividad matemática a partir de uno de los cinco cuentos seleccionados. (3) Cada grupo debía enseñar la actividad al otro grupo asignado de la otra clase. Es decir, un grupo de Colombia debía diseñar una actividad y luego enseñarla a un grupo asignado en Estados Unidos y, de la misma manera, un grupo de Estados Unidos debía diseñar una actividad y luego enseñarla a un grupo de Colombia. De esta manera cada grupo asumiría el papel de maestros y de estudiantes. Estas dos parejas tenían en común el mismo cuento. (4) El "intercambio" académico se hizo a través de la plataforma Zoom y pedimos que se grabara esta interacción. Esto lo hicimos con el fin de tener acceso a cada momento de instrucción de los grupos y así poder estudiar el desarrollo de la actividad. El tiempo que se destinó para que cada pareja ejecutara su actividad fueron 15 minutos. Así, en total, cada grupo debía tomar 30 minutos para el intercambio. (5) Finalmente, cada estudiante debió escribir una reflexión sobre la experiencia y sus aprendizajes. La reflexión la usamos para conocer las apreciaciones individuales de cada estudiante. Al mismo tiempo, la usamos para aprender sobre el impacto que tuvo en los dos grupos. La reflexión tenía 7 puntos específicos donde pedíamos a los MEF describir desde el diseño de la actividad hasta su experiencia enseñándola y participando como estudiantes de la actividad del otro grupo. También, preguntamos por sus sugerencias sobre cómo hacer este intercambio más significativo para su formación como maestros y una descripción de lo que aprendieron sobre cómo enseñar matemáticas, el lenguaje y, en general, de toda la actividad.

Métodos

Datos

Los datos que analizamos para nuestro estudio se conforman de las actividades diseñadas por cada grupo, las reflexiones individuales, y los videos donde los maestros en formación participaron del intercambio de actividades en los dos roles: maestros y estudiantes. En total tuvimos 13 parejas en cada contexto, es decir 26 grupos en total. Este trabajo produjo 26 actividades de matemáticas, 52 reflexiones, y 13 videos. El análisis de los datos se hizo siguiendo el modelo hermenéutico dialéctico (Martínez, 2013), con el cual revisamos los datos desde diferentes dimensiones como la enseñanza, el aprendizaje, la comunicación y la didáctica misma de la experiencia, a fin de descubrir significados y aprender de ellos.

Cada uno de los instructores leyó las reflexiones e identificó aspectos diferenciadores en cada reflexión y aspectos que fueron comunes. Al mismo tiempo, cada instructor vio cada uno de los videos y marcó episodios donde aspectos de uso del lenguaje sobresalieran en referencia a cómo se comunicaban conceptos matemáticos, tanto en el papel de maestro o de estudiante. Luego, a través de diferentes sesiones de trabajo, los investigadores discutieron cada uno de estos episodios y aspectos emergentes de las reflexiones y seleccionaron aquellos que encontraron en común. De este análisis salieron los resultados que describimos a continuación.

Resultados

En este primer análisis, cuatro ideas sobresalieron: (1) los cuentos infantiles no fueron ese punto de partida de lenguaje común que consideramos. El cuento fue usado por algunos MEF como el medio de instrucción. Es decir, no tuvieron en cuenta que los compañeros del otro país ya conocían el cuento, en cambio optaron por leerlo y usar esta lectura como medio para la enseñanza de las matemáticas. Por ejemplo, el grupo 1 usó el cuento *Un Charco Azul* (Amaco, 2007). En este grupo la pareja de Estados Unidos leyó el libro en español a su pareja colombiana. A medida que leían el libro pausaban y hacían preguntas sobre las imágenes del libro, por ejemplo, preguntaron "¿cuántas golondrinas estaban tomando agua?" "¿Cuántos animales han visto en total hasta este punto?" Aún sabiendo que su pareja colombiana tenía el libro en español. Algo similar sucedió en Colombia, aun sabiendo que la pareja en Estados Unidos tenía el libro en inglés, los estudiantes decidieron leerlo en inglés al grupo de Estados Unidos. (2) Los temas matemáticos cubiertos se centraron en su mayoría en contar e identificar figuras geométricas y patrones porque, según la explicación de algunos estudiantes, son temas que no requieren un mayor uso del lenguaje. Un estudiante escribió en su reflexión:

> "… we chose our topic because we felt that shapes and patterns required little knowledge of English/Spanish words in order to comprehend or explain." [… escogimos este tema porque sentimos que las figuras y los patrones requieren poco conocimiento de palabras de inglés/español para entender o explicar las ideas."].

(3) La experiencia fue diferente para cada estudiante dependiendo del papel que asumieron. Este resultado es quizá el que muestra con mayor claridad indicaciones del impacto que tuvo la

actividad en los MEF y que sirve como punto de partida para expandir esta investigación. Cuando los MEF reflexionaron sobre su papel como maestros, ellos hablaron sobre el número de veces que practicaron, los recursos que usaron para asegurarse de traducir las palabras correctas, los materiales que decidieron usar, y de cómo sus habilidades docentes fueron puestas a prueba. Sin embargo, cuando asumieron el otro papel, el de estudiantes, la experiencia fue un poco diferente. En ese momento ellos tenían que responder sin aviso previo o preparación para lo que debían contestar. Por ejemplo, un estudiante de Estados Unidos escribió en su reflexión:

> I was expecting to be more nervous as an instructor than the student, but the opposite
>
> happened for me. I knew exactly what I was going to say and teach for my own lesson, but I was nervous about doing the activities for the other group correctly. I got nervous when I did not understand the directions because I wasn't sure how to ask, but we eventually got it and it was really fun!
>
> [Esperaba estar más nervioso como instructor que como estudiante, pero al contrario sucedió para mí. Sabía exactamente lo que iba a decir y enseñar para mi propia lección, pero estaba nervioso por hacer correctamente las actividades para el otro grupo. ¡Me puse nervioso cuando no entendía las instrucciones porque no estaba seguro de cómo preguntar, pero finalmente lo conseguí y fue muy divertido!]

En Colombia una estudiante escribió:

> Me sentí muy bien, como volviendo a ser niña de nuevo, aunque un poco ansiosa ya que nos estaban observando y esperando nuestra respuesta, pero aun así fue satisfactorio ya que se pudo realizar la actividad y nos entendieron al momento de dar nuestras respuestas.

Acá cabe anotar también que, en las reflexiones, fue claro que cada MEF trabajó y dedicó tiempo para diseñar las actividades, sin embargo, este esfuerzo no incluyó tiempo para conocerse un poco más entre ellos o comunicarse con alguna frecuencia antes de la actividad, a pesar de que los canales de comunicación estuvieron dispuestos con bastantes semanas de anterioridad.

Adicionalmente otras ideas emergieron del análisis. Por ejemplo, los MEF concluyeron que pueden usar diferentes métodos de comunicación y que el no ser bilingües en realidad no fue una barrera tan grande como la que inicialmente temían al momento de presentarles la asignación. Esto es algo que queremos seguir estudiando y analizando, ya que pueden surgir alternativas pedagógicas importantes para la formación de maestros. También encontramos que el uso de recursos (por ejemplo, herramientas de traducción, materiales, imágenes, etc.) fue bastante usado por los diferentes grupos. Este es un tema bastante relevante para nuestra investigación. Por ejemplo, preguntas como ¿cuáles recursos son los más eficientes? ¿por qué se usan o por qué no? ¿quién tiene acceso a estos recursos? Están entre algunas de las ideas que aún seguimos explorando.

Discusión y Conclusiones

Iniciamos este trabajo con la intención de identificar de qué manera los MEF asumían los retos que pudieran presentarse al diseñar e implementar una actividad de matemáticas en un contexto donde dos idiomas y la matemática convergían. Los resultados preliminares de nuestro estudio nos dirigieron a dos puntos esenciales que informan y contribuyen al trabajo de formadores de maestros. Uno es la diferencia entre la preparación y el tiempo que dedicaron los futuros maestros en nuestro estudio para asumir los papeles de maestros y de alumnos. En las reflexiones de los MEF encontramos que ellos dedicaron bastante tiempo a preparar la actividad y a ensayar cómo presentar la actividad. Sin embargo, este esfuerzo se enfocó sólo en eso, en su preparación. No hubo ningún registro de comunicaciones previas para conocer un poco más de quienes asumirían el papel del estudiante. No hubo registro de comunicaciones para conocerse un poco más. Ninguno de los grupos intentó reunirse con anterioridad para familiarizarse un poco más y establecer estrategias para comunicarse. En las indicaciones de la actividad intencionalmente dejamos por fuera el "requerir" que se comunicaran con anterioridad, sin embargo, los presentamos con 6 semanas antes porque queríamos ver si ellos sentían la motivación o necesidad de hacerlo en preparación a la actividad. El encontrar que esta parte estuvo ausente en todos los grupos, nos lleva a concluir que es necesario enfatizar prácticas pedagógicas que guíen al maestro en formación para que trabaje y aprenda a conocer más de cerca a sus estudiantes. En el contexto de los Estados Unidos, particularmente este resultado fue bastante sorprendente. Uno de los pilares de la clase de métodos de la enseñanza de las matemáticas del primer autor (en Estados Unidos) es el formar a los futuros maestros para que aprendan a conocer más a fondo a cada uno de sus estudiantes y usar este conocimiento para informar sus prácticas pedagógicas. Específicamente, los MEF en esta clase desarrollan una tarea donde cada uno de ellos debe entrevistar a un estudiante focal en sus salones de práctica. Esta tarea tiene varios propósitos. Una es averiguar más sobre el estudiante, incluyendo sus intereses y las actividades que realiza fuera de la escuela. Otro propósito es averiguar más acerca de las ideas del estudiante, actitudes y disposiciones hacia las matemáticas. De esta manera, los maestros en formación tienen una base para empezar a trabajar en el diseño de problemas de matemáticas y otras actividades que se desarrollan en la clase (Krause & Colegrove, 2020). Esta tarea es una las primeras que realizan los MEF en el semestre. Por lo tanto, ya para el momento que realizaron este intercambio entre Colombia y Estados Unidos, los MEF en la clase de Estados Unidos habían realizado esta entrevista. Este resultado es bastante diciente ya que los MEF particularmente en el contexto de Estados Unidos van a estar trabajando en escuelas con una diversidad cultural y lingüística amplia. El conocer los fondos de conocimiento que traen los estudiantes al salón de clase es esencial en cómo los futuros maestros asumen sus prácticas pedagógicas (Civil et al., 2005). Al mismo tiempo, este resultado aclara un poco por qué los MEF no usaron los cuentos como lenguaje común. Su enfoque se centró tanto en ellos mismos que no permitió ver los recursos que ya aquellos, asumiendo el papel de estudiantes, tenían como presaberes. Como mencionamos anteriormente, el ignorar este recurso hizo que cada actividad consumiera más tiempo del que en realidad necesitaba. Lo que puede ser una razón para justificar la superficialidad del contenido matemático que se enseñó. Este sólo punto de enfocarse en ellos mismos prácticamente desencadenó un efecto dominó al momento de enseñar la actividad. De igual manera para la clase de Didáctica de las Matemáticas en el contexto colombiano, sorprende que los MEF no se interesaran por conocer a sus estudiantes de manera previa. Esto riñe con lo que de manera permanente se trabaja en su formación para ser maestros en educación infantil. En este nivel es particularmente importante reconocer quiénes son los alumnos, cuál es su contexto, qué les gusta, qué saben. Las respuestas a tales preguntas fundamentan las bases del constructivismo y el aprendizaje significativo (Ministerio de Educación Nacional, 2014) que se espera alcancen los niños y las niñas.

El otro punto esencial que nos muestran los resultados es el contenido matemático escogido para las actividades. Todavía es necesario estudiar este punto con detenimiento, pero por los resultados preliminares conjeturamos que el lenguaje sí jugó un papel importante en la decisión del contenido matemático que se enseñó. En varias de las reflexiones los maestros en formación explícitamente dijeron que usaron patrones y figuras geométricas porque así no tendrían que usar tantas palabras. El debate existente en la enseñanza de las matemáticas en contextos bilingües es común en los Estados Unidos. Existen grupos avocando por educación bilingüe que incluya instrucción de matemáticas en la lengua materna de los estudiantes y al mismo tiempo existen grupos avocando por instrucción bilingüe, pero excluyendo la matemática (Jacobs, 2016). Independientemente de este debate, los resultados de esta investigación revelan un aspecto del uso del lenguaje que poco se menciona en estos debates. La calidad de la instrucción de las matemáticas (Jacobs, 2016). En este trabajo los resultados mostraron que los MEF, en los dos contextos, se prepararon para usar el lenguaje de sus estudiantes y que además el uso del lenguaje fue lo que condicionó la decisión de qué enseñar. Algo que esperábamos que surgiera de la interacción previa (que no ocurrió) era que entre los grupos hablaran y compartieran un poco sobre sus experiencias en las prácticas escolares que son parte de su formación como maestros y que tomaran de ahí las ideas sobre el contenido matemático. Este aspecto particular del uso del lenguaje nos hace pensar en la idea de la estandarización de prácticas educativas como consecuencia de la globalización y el uso del inglés como lengua de la globalización. Particularmente estos dos aspectos del lenguaje se conectan con la creciente población de estudiantes inmigrantes en las escuelas de Estado Unidos. ¿Qué pasa con la calidad de la educación que reciben estos estudiantes mientras adquieren un nivel de inglés adecuado para comunicarse con sus maestros que, en la gran mayoría, son monolingües? Esto es una pregunta que todavía no se contesta en nuestro campo de educación de matemáticas bilingüe y que tampoco hemos visto en la literatura sobre la globalización y la educación.

El intercambio pedagógico que nos atrevimos a diseñar resultó ser una herramienta eficaz para apoyar la formación de los maestros en formación. Al mismo tiempo, abrió la puerta para continuar caminos de investigación que claramente se necesitan para fortalecer la formación de maestros para que enseñen matemáticas.

Referencias

Amaco, A. (2007). *Un charco azul*. Sieteleguas Ediciones.

Becker, A. (2013). *Journey*. Candlewick.

Bigelow, B. (2006). *The line between us*. Rethinking Schools.

Brown, M. (2011). *Marisol McDonald doesn't match*. Children's Book Press.

Civil, M., Planas, N., & Quintos Alonso, B. (2005). Immigrant parents' perspectives on their children's mathematics education. *Zentralblatt für Didaktik der Mathematik*, *37*(2), 81-89. https://doi.org/10.1007/BF02655717

Doerr, N. (Ed.). (2020). *The Global Education Effect and Japan: Constructing New Borders and Identification Practices*. Routledge.

Gutierrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in* Mathematics Education, 44(1), 37-68. https://doi.org/https://doi.org/10.5951/jresematheduc.44.1.0037

Henkes, K. (2008). Chrysanthemum. Mulberry Books.

Jacobs, J. (2016). Learning English: Accountability, Common Core and the College-for-All movement are transforming instruction. Education Next, 16(1), 38-45. https://www.educationnext.org/learning-english-accountability-common-core-college-instruction/

Jeffers, O. (2010). Cómo atrapar una estrella. Fondo de Cultura Económica.

Krause, G., & Colegrove, K. S. S. (2020). Preparing bilingual pre-service teachers to foster equitable and open communication with Latinx immigrant parents en la enseñanza de matemáticas. Teaching for Excellence and Equity in Mathematics, 11(3), 41-49.

Krause, G., Ávila, A., & Gualdrón, E. (2021). Retos matemáticos en la formación de maestros: Dos contextos, un escenario global. In D. Olanoff, K. Johnson, & S. M. Spitzer(pp. 941-946).

Mandana, S. (2006). Mi León. Fondo de Cultura Económica.

Martínez, M. (2013). Ciencia y arte en la metodología cualitativa. Trillas.

Melitz, J. (2016). English as a global language. In V. Ginsburgh & S. Weber (Eds.), The Palgrave handbook of economics and language (pp. 583-616). https://doi.org/https://doi.org/10.1007/978-1-137-32505-1_21

Meyer, H., & Benavot, A. (Eds.). (2013). Pisa, power, and policy. Symposium Books.
Ministerio de Educación Nacional. (2014). Sentido de la Educación Inicial. https://www.mineducacion.gov.co/1780/articles-341810_archivo_pdf_sentido_de_la_educacion.pdf
Moses, R. P., & Cobb, C. E. (2002). Radical equations: Civil rights from Mississippi to the algebra project. Beacon Press.
Murcia, M. E., & Henao, J. C. (2015). Educación matemática en Colombia, una perspectiva evolucionaria. Entre Ciencia e Ingeniería, 18, 23-30.
Spring, J. (1998). Education and the rise of the global economy. Lawrence Erlbaum.
Steger, M. (2013). Globalization. A very short introduction. Oxford University Press.
Subedi, B. (2013). Decolonizing the curriculum for global perspectives. Educational Theory, 63(6), 621-638. https://doi.org/https://doi.org/10.1111/edth.12045
Triandis, H. C. (2001). Individualism and collectivism: Past, present, and future. In D. Matsumoto (Ed.), The handbook of culture and psychology (pp. 35-50). Oxford University Press.
Usma, J. (2009). Globalization and language and education reform in Colombia: A critical outlook. Íkala, Revista de Lenguaje y Cultura, 14(22), 19-42.
Webb, J., & Wilson, P. H. (2022). Designing rehearsals for secondary mathematics teachers to refine practice. Mathematics Teacher Educator, 10(2), 129-142. https://doi.org/10.5951/MTE.2020.0073
Yangsook, C. (2003). The name jar. Dragonfly Books.

TEACHERS, STUDENTS, LANGUAGE, AND MATHEMATICS: PEDAGOGICAL PRACTICES FOR PRE-SERVICE TEACHERS WITH A GLOBAL VISION

We describe the second phase of an ongoing investigation that asks about the way in which pre-service teachers assume their pedagogical practices in a global context where language and mathematics are combined. Such a focus supports the goal of preparing teachers who can respond to current educational challenges. One of these challenges is the use of English as a global language. We present an experience from collaborative work between students and professors of two courses teaching methods and didactics of mathematics, one in the United States and the other in Colombia. This included the design of activities for the teaching of mathematics using children's books as a reference. Among the results we noted that the activities were influenced more by the use of language than by the mathematical content to be taught.

PRESERVICE AND BEGINNING TEACHERS' PERSPECTIVES ON EQUITY

Rebecca McGraw
University of Arizona
rmcgraw@math.arizona.edu

Anthony Fernandes
University of North Carolina – Charlotte
Anthony.Fernandes@uncc.edu

Becca Jarnutowski
University of Arizona
jarnutowski@math.arizona.edu

Jennifer A. Wolfe
University of Arizona
jwolfe@math.arizona.edu

Abstract: This report is part of a larger, longitudinal study focusing on the development of equity-related knowledge, beliefs, and practice across 68 individuals and five teacher preparation programs. In this brief report, we seek to unpack the ways five preservice and beginning mathematics teachers think about equity, especially as it relates to their current and future teacher practice. Analysis of interview data from these participants suggest as many as twelve different aspects of equity reflected in their thinking, as well as multiple actions teachers could take to promote equity including raising expectations, rejecting deficit views, and using complex instruction.

This material is based upon work supported by the National Science Foundation under Grant No. 1758401.

Keywords: Preservice Teacher Education, Equity, Inclusion, and Diversity

In this brief report, we describe preliminary analysis of data from an ongoing, longitudinal study of middle/high school mathematics teacher education and beginning teacher practice in the U.S. as it relates to equity. Our goal is to understand how teachers develop and use their knowledge of equity - how they think, what they think, and how they apply their knowledge. We are guided in this work by ideas from culturally responsive teaching (Gay, 2002; 2010; 2015) and equity literacy (Gorski 2014, Gorski & Swalwell, 2015), as well as from work by experts in the field of educational equity in mathematics (e.g., Aguirre, Mayfield-Ingram, & Martin, 2013; Bartell et al., 2017; Gutierrez, 2009; Moschkovich, 2013; Ramirez & Celedon-Pattichis, 2012). We adopt Aguirre, Mayfield-Ingram, and Martin's (2013, p. 45-48) categories of access and advancement, the use of cultural and linguistic resources, and the development of identity and agency to structure our thinking about teacher knowledge and its development. Further, we incorporate self-determination theory (Ryan & Deci, 2000) into our conceptual framework as a lens through which to consider teacher agency and perceived constraints to increasing equity.

Using a combination of data sources including interviews, work samples, surveys, written reflections, and videos of instruction, our research group is constructing data sets for each of 67 preservice or beginning teachers (PT/BT) across five teacher education programs in the U.S. Each PT/BT participates in the research project for four years, encompassing the final two years in their education program and the first two years of their post-program classroom teaching. Most study participants are/were enrolled in undergraduate programs at their institutions, but some are/were enrolled in a graduate certification program while teaching concurrently. For this brief report, we examine the knowledge and beliefs of five PT/BTs, which was a convenience sample but also spanned four of the five programs. We present our preliminary analysis of the end-of-program, structured interviews, and focus on the following research question: *How do*

PT/BTs think about equity in relation to their teacher education programs and their emerging teaching practice?

The interview protocol for each PT/BT included the following questions: (1) Since you began your teacher education program at [institution], what experiences have particularly impacted how you think about teaching mathematics? (2) What are some of the most important things you have learned about teaching and about students in your teacher education program? (3) What, if any, instruction or experiences related to equity have you had in your teacher education program? (Note: a definition of equity was <u>not</u> provided) (4) What have you learned about equity through your teacher education program? (5) How do you feel personally about equity? (6) What actions should math teachers take to promote equity in opportunities to learn, participation in learning, and learning outcomes? (7) What do you wonder about, or what questions do you have, about equity and mathematics teaching? We used an open-coding process (Strauss & Corbin, 1990) involving four interviews and two coders which resulted in six codes. Next, three researchers coded one common interview independently using the six codes and met to discuss differences, refine coding categories, and add new codes as appropriate. Finally, three researchers independently coded three interviews each using the refined coding scheme and met again. The final coding scheme we established accounted for the sections of data we interpreted as equity-related, with some sections of text receiving two or more codes.

Using the process described above, we achieved kappas of 0.9^+ across coders for these six codes: **personal meanings** - personal meanings, experiences or definitions of equity; **understanding/sense-making** - productive struggle, mathematical reasoning, cognitive demand, creativity, learning from mistakes, critical thinking and problem solving; **building relationships with students** - includes relationships with individual students or groups, attending to social/emotional needs, listening to students and sharing with them beyond the mathematics at hand, knowing about students' homes, knowing who needs what to help them learn; **connecting the math/classroom to the real world** - building math instruction upon students' interests and lives, bringing current events/issues into the classroom, attending to social (in)justices, connecting to lived experiences outside the classroom; **inequities in education and actions to take** - includes expressions of awareness of educational issues such as bias, deficit views, and privilege (from a personal or systemic perspective), descriptions of inequities in general or specific terms, as well as strategies for addressing inequity; and **constraints to increasing equity** - focus on barriers identified by PT/BTs that they believe will materially limit their ability to create more equitable classrooms, or barriers for the world to be more equitable, also includes expressions of lack of knowledge about how to achieve a vision of equity. For this paper, we focus on initial analysis of the "personal meanings" code and responses to the questions "How do you feel personally about equity?" and "What actions should math teachers take to promote equity in opportunities to learn, participation in learning, and learning outcomes?"

A constant-comparative analysis of the five PT/BT interview transcripts resulted in 12 "equity associations" demonstrating the many ways these five PT/BTs connected the notion of equity to themselves and to their emerging teacher practice.

Equity as Accommodation - "equality is everyone gets the same resources, but equity is where you, um, accommodate for those who would need the additional resources to be able to perform at the same level as everyone else around them"

Equity as Productive Struggle - "I know that sounds evil, and almost inequitable, but um, I mean, there's this idea of a constructive, I think, constructive struggle that students need to go through so that they've reached an understanding themselves"

Equity as Asset-Based Teaching - “it's very, very easy to prioritize and . . . spend a lot of time on boost up the students who are also good students . . . I think that it's so easy to just fall into like this compare and contrast. I think you like you at first do it in your own head. . . . you start thinking all these negative things and then like it starts coming out in your actions and your words . . . it’s just something you have to be really conscious of.”

Equity as Accessibility and Choice - “it'll make math more accessible to all students because you, you know, as long as you follow like some certain basic rules, like you can solve, like coming, coming up with a new way to solve a problem is, is actually a better thing.”

Equity as Assigning Competence - “giving them that public specific praise in class to raise their status of not only of themselves, but also how other students view them.”

Equity as Differentiation - “supporting those students who might need extra support . . . challenging all of your students to just go above and beyond and push past your expectations.”

Equity as Personal Experience - “and just being like a female in a math class was you could tell, as soon as I walked in the room, they didn't even look at me twice.”

Equity as Culturally Responsive Teaching - “so for me, it's like the culturally responsive teaching. . . . you have to know who your students are to be able to meet their needs best.”

Equity as Social Justice - “it's sort of the responsibility of an educator to address these systems of oppression and inequity that are looming around us and making our students aware”

Equity as Empowerment - “maybe have a focus on empowering students . . . explaining to them the reality of the world and then working with them to break down the barriers.”

Equity as Personal Transformation - “a lot of the time, I thought, well, why don't people just work harder if they're not doing as well?. . . . [the program] made me realize that there are barriers that other people have had to fight through systemically that I've never faced.”

Equity as a Systemic Issue - “issues with the system is how we should be viewing it rather than how that idea kind of portrays it as issues with the person. . . . dismantling hierarchies of status in class trying to almost like level the playing field.”

Although we can imagine other associations that PT/BTs might make that we did not find in the interviews of these five individuals, for example, equity as racial justice or collective liberation, initial analysis suggests that PT/BTs are able to connect the idea of equity to their own lives and their emerging teacher practice in many different and potentially powerful ways. Going forward, we are interested in determining whether/how different ways of thinking about equity cluster together within and across individuals and teacher preparation programs, and whether/how particular associations are expressed in the classroom.

Regarding actions that mathematics teachers could take to promote equity in opportunities to learn, participation in learning, and learning outcomes, all five PT/BTs noted that teachers should engage students in problem solving that draws on contexts that are familiar or interesting to students and related to their experiences. One PT/BT, Frank, believed the experience would help students understand that mathematics could be useful in navigating their world.

> I think math teachers have to take the opportunity to use real world issues in the class. Not just like word problems, but actually bringing about, in the right way, in the right context, issues that face their students in a very real way […]. Saying, okay, we can use math to help solve this issue or we can use math to at least analyze this issue that many of you are facing or dealing with. And that can be one way for maybe students who haven't dealt with that issue before in my class to kind of open their eyes to it. (Frank, PT/BT)

Three PT/BTs suggested that teachers should differentiate their instruction to meet students where they are and build on their existing knowledge. The PT/BTs noted that engaging students in problem solving could be challenging. Two PT/BTs suggested that it was important that teachers have non-deficit views about the students and their capabilities. Teachers should believe their students can engage in challenging mathematics with scaffolding. One of the PT/BTs discussed the importance of allowing students to engage with productive struggle for learning.

> In regard to overall instruction, I think learning to allow your students to struggle. And I know that sounds evil, and almost inequitable. I mean, there's this idea of a constructive struggle that students need to go through so that they've reached an understanding themselves, rather than the instructor providing the answer for them. (Colin, PT/BT)

In addition to believing that students are capable of handling challenging work, three PT/BTs stated that the teachers need to develop a supportive classroom environment to encourage student discourse. They pointed out the importance of classroom discussions among the students, and the students and the teacher, and they highlighted the work that the teachers need to do to make the classroom into a discussion-rich space.

> I really think teachers ought to use this idea of complex instruction. Whether it be exactly that or some sort of form of it, it's just what I've been able to see in the classroom. There have been students who have been so quiet at the beginning of the year, but to see their growth using this, this kind of teaching, this group work building on students' strengths and giving them that public specific praise in class to raise their status of not only of themselves, but also how other students view them in class. I think, is it just, it does a lot for when it comes to equitable participation in class, you see students grow, you see them finally feeling like they have a voice. I would say using that strategy is definitely a big teacher move (Frank PT/BT)

Discussion

In this paper, we focus on ways of thinking about equity as expressed by five PT/BTs who were interviewed near the completion of their respective teacher education programs. As we move forward in our research, we are interested in considering how frameworks for thinking about equity, such as that proposed by Gutierrez (2009), could be useful in characterizing this data. For example, comparing the 12 “equity associations” made by the five PT/BTs with the axes of access, identity, power, and achievement (Gutierrez, 2009), we notice that they attended collectively to all four axes, but with differing emphases. With regard to the categories and framework developed by Aguirre, Mayfield-Ingram and Martin (2013), these PTs expressed ways of thinking about equity falling into all three categories - access and advancement, the use of cultural and linguistic resources, and the development of identity and agency. One component of equity literacy (Gorski 2014, Gorski & Swalwell, 2015) we did not find in the data for these five PT/BTs is empowering students to collectively take action to create meaningful changes inside or outside their schools. This is not to suggest that the PT/BTs do not think in this way, but rather that this idea was not evoked during the interview.

With respect to influences on PT/BTs’ knowledge, both Colin and Frank pointed to university courses and their instructors as they described the importance of productive struggle and complex instruction, respectively. In fact, we found an emerging theme in the data thus far as the PT/BTs repeatedly return to the importance of maintaining non-deficit views of students and allowing them to engage in problem-solving and productive struggle. Going forward, we will investigate how these ways of thinking about equity may serve as “launch pads” for designing increasingly equitable classrooms, and how they may constitute “comfort zones” allowing teachers to maintain the status quo.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Towards a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education, 48*(1), 7-21.

Gay, G. (2002). Preparing for culturally responsive teaching. *Journal of Teacher Education, 53*(2), 106-116.

Gay, G. (2010). *Culturally responsive teaching: Theory, research, and practice* (2nd ed.). New York, NY: Teachers College Press.

Gay, G. (2015). Teachers' beliefs about cultural diversity: Problems and possibilities. In H. Fives & M. G. Gill (Eds.), *International handbook of research on teachers' beliefs* (pp. 436-452). New York: Routledge.

Gorski, P. C. (2014). *Reaching and teaching students in poverty: Strategies for erasing the opportunity gap*. Teachers College Press.

Gorski, P. C., & Swalwell, K. (2015). Equity literacy for all. *Educational Leadership, 7*(6), 34-40.

Gutierrez, R. (2009). Framing equity: Helping students ''play the game'' and ''change the game''. *Teaching for Excellence and Equity, 1*(1), 4–9.

Moschkovich, J. (2013). Principles and guidelines for equitable mathematics teaching practices and materials for English language learners. *Journal of Urban Mathematics Education, 6*(1), 45-57.

Ramirez, N. G., & Celedón-Pattichis, S. (2012). Second language development and implications for the mathematics classroom. In S. Celedón-Pattichis & N. G. Ramirez (Eds.), *Beyond good teaching: Advancing mathematics education for ELLs* (pp. 19-37). Reston, VA: National Council of Teachers of Mathematics.

Ryan, R. M., & Deci, E. L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist, 55*(1), 68-78.

Strauss, A. L., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Sage.

ANALYSIS OF PRE-SERVICE TEACHERS' VIRTUAL NUMBER TALKS PRACTICE: IMPLICATIONS FOR THE PREPARATION OF FACILITATING MATHEMATICS DISCUSSIONS

Audrey Meador	Nicole Fletcher
West Texas A&M University	Fairfield University
ameador@wtamu.edu	nfletcher@fairfield.edu

Efforts within mathematics teacher education have recognized and attended to the importance of developing pre-service teachers' skills for leading mathematical discussions. Number Talks have been utilized in teacher preparation programs to create opportunities to practice discussion facilitation moves. A virtual Number Talk field experience was enacted to provide pre-service teachers the experience of facilitating Number Talks while navigating technology. Two purposefully selected pre-service teachers' experiences conducting virtual Number Talks were analyzed using a formative assessment that decomposed the discussion practice. As a mode of using research to inform teacher educator practice, results of the analysis provide implications for assisting pre-service teachers in orchestrating productive mathematics discussions using technology in order to move from critical dissonance to resonant harmony in praxis.

Keywords: pre-service teacher education, technology, classroom discourse, teacher knowledge

Facilitating meaningful mathematics discourse is recognized as an essential element of effective mathematics teaching practice (NCTM, 2014). To orchestrate productive mathematical discussions, teachers must attend to discussion components in the moment and possess an understanding of mathematics content, student thinking about content, and pedagogical practices to develop students' mathematical understanding (Smith & Stein, 2018). Recent efforts within mathematics teacher preparation programs have attended to pre-service teachers' (PSTs) skills in leading discussions that support students' development of conceptual understanding (Shaughnessy et al., 2021). Number Talks have been utilized in this regard to build fundamental knowledge and skills for leading productive mathematical discussions (Woods, 2021).

In Fall 2020, we implemented a Number Talks-based, virtual field experience to allow PSTs to practice and refine skills in conducting a productive mathematics discussion (Fletcher & Meador, 2022). The virtual Number Talk teacher learning cycle (VNT TLC) was our response to creating field experiences for students when access to schools was unavailable (Joswick et al., 2021). While this modification served a purpose, it also created a critical dissonance in PST practice. Not only were PSTs required to juggle the demands of conducting a mathematical discussion, but they had to do so in a virtual space. Conducting a productive mathematical discussion requires attention to the areas of *framing, orchestrating,* and *recording/representing content* (Shaughnessy et al., 2021) in real time. When enacting a mathematical discussion in a virtual classroom, consideration must also be given to the use of technology in these areas. We analyzed PSTs' skills in facilitating mathematical discussions, demonstrated in their virtual Number Talks videos, to move our own practice as teacher educators from critical dissonance to resonant harmony. The following research questions guided our analysis: (1) Which areas of work in a mathematics discussion (*framing, orchestrating, recording/ representing content*) are revised through participation in a teacher learning cycle focused on virtual Number Talks? (2) Which areas of work in a mathematics discussion do PSTs attend to, or not attend to, when developing a virtual Number Talk practice? As we respond to these research questions, we focus on the voice of the PST as a formative assessment of our own teaching practice. We use our analysis of PSTs' experiences implementing virtual Number Talks to inform our teaching on facilitating meaningful mathematics discussions, using technology to achieve pedagogical goals, and attending to equitable opportunities for student voice in the mathematics classroom.

Theoretical Framework

Facilitating a meaningful mathematics discussion requires the building of shared understanding "of mathematical ideas by analyzing and comparing student approaches and arguments" (NCTM, 2014, p. 35). By implementing mathematics discussions, teachers also engage in several effective mathematics teaching practices (Smith & Stein, 2018), such as using and connecting mathematical representations. Teachers must also identify a mathematical concept to guide the discussion so that the flow of the conversation is directed to this point (Shaughnessy et al., 2021; Sleep, 2012) and ensures that the sharing of ideas involves discourse and not merely a "show and tell" of ideas (Kazemi & Stipek, 2001; Smith & Stein, 2018). We utilized a decomposition of the facilitation of mathematics discussions in a formative assessment framework developed by Shaughnessy et al. (2021) as our theoretical frame. Shaughnessy and colleagues (2021) created a checklist of skills for leading mathematics discussions centered on the decomposition of three main areas of work, *framing, orchestrating,* and *recording and representing content*, along with associated teacher moves (see Table 1).

Table 1: Mathematics Discussion Skills Assessment Checklist (Shaughnessy et al., 2021)

Areas of Work	**Teacher Moves**	
Area of Work: Framing	Launching	Efficiently engages students in the mathematical work
		Directs attention towards the mathematics intended for the discussion
	Concluding	Makes a closing statement that indicates the conclusion of the discussion.
		Supports students in remembering or making sense of at least one key mathematical topic or practice
Area of Work: Orchestrating	Eliciting contributions	Elicits multiple solutions, strategies, or ideas
		Elicits a range of student understanding or methods
		Engages several students in sharing their thinking
	Probing students' thinking in relation to the mathematical goals	Poses questions to get students to explain their thinking about processes
		Poses questions to get students to explain their understanding of key mathematical ideas
		Follows up on responses to make more student thinking about the mathematics available
		Provides support for students to complete their contribution or clarify their thinking
	Orienting students to the contributions of peers	Ensures that the class can hear others' ideas
		Poses questions to students about others' ideas and contributions, including asking students to comment on, add to, or restate another student's idea
		Supports the listening of the class through the use of moves that require all students to respond to others' work
		Encourages students to attend, listen, and respond to peers' contributions in order to maintain productive and focused interaction
	Making contributions	Uses moves such as redirecting, revoicing, and highlighting to keep the discussion on track
		Ensures that substantive and relevant analysis is part of the discussion
		Makes mathematical contributions which enrich the core ideas of the discussion and keep the discussion focused on the learning targets

Area of Work: Recording and Representing Content	Keeping accurate records	Records student ideas in ways that are true to the students' contributions
		Attends to the accuracy of records and representations
	Choosing and using appropriate representations to convey key mathematical ideas.	Records in ways that are clear, organized, and visible to the class
		Uses recordings which support student understanding and participation

This frame was also selected for analysis to capture the technology component of the virtual Number Talks. After study of the framework and initial assessment of PSTs' virtual Number Talk videos, we found that technology could be addressed in the *orchestrating* and *recording/representing content* areas of the assessment. We also considered the use of the teacher learning cycle and the ability of the framework to highlight areas of work PSTs enacted throughout the cycle. Additionally, the assessment checklist provided exemplar decomposition move descriptions (Shaughnessy et al., 2021) that provided guidance and clarity of recognition when evaluating the virtual Number Talks videos for evidence of these moves.

Methods

As part of a larger study, we considered a purposeful sample of two teachers' Number Talk videos for qualitative analysis using the assessment framework. These teachers were deliberately chosen for two main reasons. The first selection criteria for analysis included PSTs who demonstrated development of a variety moves in each of the three main areas of the checklist as they progressed through our learning cycle. The final selection criteria included PSTs with a range of prior teaching experience, while also taking into consideration their respective field placements. As a result, two teachers were chosen as cases for this study. Institutional review board approval was obtained and for anonymity of participants, pseudonyms are being used.

Setting and Participants

A virtual field experience based on Number Talks was implemented in Fall 2020 in an elementary mathematics methods course required for an elementary education master's program at a university in the northeastern United States. 29 students were enrolled in the course, with 23 consenting to having their course assignments analyzed for research purposes.

Dakota was in their second year of the elementary education master's program and was enrolled in the course in the fall semester prior to student teaching. Dakota was working for a third semester as an intern at a suburban public K-4 school and had no other prior teaching experience. Dakota's school was implementing a hybrid model but did not permit videotaping, so Dakota completed the VNT TLC field experience (Fletcher & Meador, 2022) with three third grade students recruited with the help of the instructor. The students did not know one another or Dakota prior. Dakota had not heard of the phrase "Number Talks" before the project but had seen the practice while observing classes during their internship.

Emerson was in their graduate year of a five-year integrated Bachelor's/Master's program in elementary education and was enrolled in the course prior to student teaching. Emerson was in their first semester of working as an intern in a suburban public school for grades 3-5. Emerson had prior experience working as a paraprofessional in a summer school program for grades 9-12. Emerson's school implemented a hybrid cohort model at the start of the semester and later shifted to primarily in-person instruction with some students learning remotely. Emerson completed their Virtual Number Talks field experience with three third grade students from their internship. Emerson was unfamiliar with Number Talks prior to the project.

Data Sources

Sources of data consisted of two complete virtual Number Talks videos submitted by each participant as part of the VNT TLC (Fletcher & Meador, 2022). At the start of the cycle, the PSTs learned about Number Talks through readings, videos, discussions, and demonstrations. Next, the PSTs planned their first Number Talk by identifying the mathematical focus of the Number Talk, selecting a set of problems from a provided list [most of these problem sets were taken from Parrish's (2010) *Number Talks* book], anticipating student responses and strategies along with follow-up questions (Smith & Stein, 2018), selecting technological tools, and considering participation norms. The instructor provided feedback on the plans, and then PSTs rehearsed their Number Talk with peers from the course, reflected on the experience, and made any necessary modifications to their plan based on feedback. The PSTs then implemented and video recorded their Number Talk with K-6 students and reflected on the experience. Following the first Number Talk, PSTs engaged in further learning and planning for a second Number Talk, using the knowledge and skills developed during their first to inform their planning for the second. They facilitated and video recorded a second Number Talk and then completed a final reflection as part of the learning cycle. Figure 1 details an iteration of the VNT-TLC cycle.

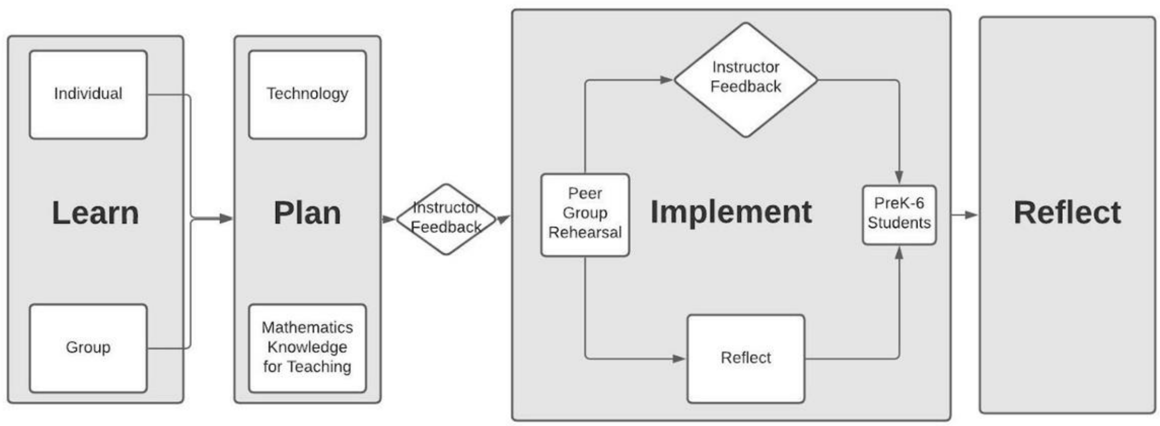


Figure 1: One Iteration of the VNT-TLC Cycle (Fletcher & Meador, 2022)

Data Analysis

The analysis of the data sources was conducted in three phases. The first phase began by individually evaluating the first and then the second virtual Number Talk videos for each of the purposefully selected participants using the assessment checklist. Each video was broken into one-minute segments for unit analysis and each minute was evaluated for evidence of the decomposed teacher move areas of the checklist. A code of 1 was recorded if the participant demonstrated proficient use of the teacher move, a 0 if the participant demonstrated developing use of the teacher move, and an X if the move was not applicable or not observed. Through rounds of watching and coding participants' Number Talks videos, a consensus of proficient and developing practice was achieved inductively to develop a common definition and description for analysis using the skills checklist. Notes were also recorded to provide rationale for assignment of codes. The second phase of data analysis consisted of the comparison of codes assigned to each Number Talk video by going through each unit of analysis in an open coding format (Strauss & Corbin, 1990; Vollstedt & Rezat, 2019). Discrepancies of code assignment were discussed, notes were compared, and videos were revisited at the unit under consideration before a consensus on the code was determined. The final phase of data analysis consisted of constant comparison analysis (Glaser & Strauss, 1967; Leech & Onwuegbuzie, 2007) of the *framing*, *orchestrating*, and *representing/recording data* decomposition areas coded for each participant between the first and second Number Talk.

Results

The results are presented by participant and are organized by the three areas of work in leading a mathematics discussion developed by Shaughnessy et al. (2021): *framing*, *orchestrating*, and *recording and representing content*. Because the checklist was intended to be used with videos of mathematical discussions rather than transcripts of this data source (Shaughnessy et al., 2021), we describe the occurrence of moves from the checklist observed in the Number Talk videos. The number of occurrences of each move was documented during data analysis, but because the number of moves is affected by the number and complexity of the problems presented, we instead report whether or not the moves were attempted.

Dakota's Areas of Work

Dakota's *framing* of the mathematical discussions through launching of both Number Talks included prompts that "efficiently engaged the class in the mathematical work." Dakota did not "direct attention towards the mathematics intended for the discussion" in the launch of their first Number Talk, but this was observed in their second. In concluding both discussions, Dakota "made statements to indicate the conclusion of the discussion." Dakota also attempted to "support students in remembering or making sense of at least one key mathematical topic or practice" in both Number Talks, though they executed this move more effectively in the second.

During the *orchestration* of the mathematical discussions in both Number Talks, Dakota elicited contributions by "eliciting multiple solutions, strategies, or ideas" and "engaging several students in sharing their thinking." Dakota was not observed "eliciting a range of student understanding or methods" during their first Number Talk, but this did occur during their second. However, this could be due to the task, as Dakota was facilitating dot arrangement Number Talks, and less of a range of understanding would be expected with this task. Dakota probed student thinking frequently by "posing questions to get students to explain their thinking about processes" during both Number Talks, a key component of the routine structure. "Posing questions to get students to explain their understanding of key mathematical ideas" was observed in both of Dakota's Number Talks, but the use of this question type was far more limited than questions to get students to explain their thinking. Dakota did not "follow up on responses to make more student thinking about the mathematics available" in either Number Talk. "Providing support for students to complete their contribution or clarify their thinking" was not observed in their first Number Talk but was observed in their second.

The orienting students to the contributions of peers' component of *orchestrating* a mathematical discussion was used less frequently than other moves during Dakota's Number Talks. Dakota did not "pose questions to students about others' ideas and contributions, including asking students to comment on, add to, or restate another student's ideas" during their first Number Talk but did use this move during their second Number Talk. Similarly, "encouraging students to attend to, listen, and respond to peers' contributions in order to maintain productive and focused interaction" was observed only in Dakota's second Number Talk. Dakota did not "support the listening of the class through the use of moves that require all students to respond to others' work" during either Number Talk. Dakota also did not use any moves to "ensure that the class can hear others' ideas," but this was not needed during either of Dakota's small group virtual Number Talks as all students could be heard clearly.

Dakota showed significant improvement in the making contributions component of *orchestrating* a mathematical discussion. In their first Number Talk, Dakota did not use moves such as “redirecting, revoicing, and highlighting to keep the discussion on track” or “make mathematical contributions which enrich the core ideas of the discussion and keep the discussion focused on the learning targets.” Yet they did use the move “ensure that substantive and relevant analysis is part of the discussion” once during the first Number Talk. However, in their second Number Talk, Dakota employed all three of these types of moves throughout the discussion. Dakota’s second Number Talk was double the length of their first Number Talk, even though the problem type (dot arrangements), number of problems, and the number of students all remained the same from the first to the second Number Talk. The increased length was due to Dakota’s use of the making contributions moves frequently throughout, which extended and deepened the discussion of each problem.

Dakota’s *recording and representing content* also improved significantly between their first and second Number Talk. Dakota did not record student thinking during their first Number Talk. When a teacher facilitates a dot arrangement Number Talk in-person, they may record student thinking, or they may use gestures to demonstrate student thinking about groupings and number decomposition. Because Dakota’s Number Talk was virtual, they were not able to gesture towards the dot arrangement image on the screen as a teacher could gesture towards an image on the board or a large hand-held card (Dakota did not use the Zoom function of using a slide as a virtual background). In the virtual setting, recording would be necessary to make student thinking available to others, but this move was not used. In their second Number Talk, Dakota “recorded student ideas in ways that are true to the students’ contributions,” “attended to the accuracy of records and representations,” “recorded in ways that are clear, organized, and visible to the class,” and “used recordings which support student understanding and participation.” Throughout the Number Talk, Dakota recorded students’ thinking by circling dot groupings and recording equations along with branches to demonstrate number decomposition.

Emerson’s Areas of Work

Emerson *framed* the mathematical discussions by launching with moves that “efficiently engaged the class in the mathematical work” and “directed attention towards the mathematics intended for the discussion” in both Number Talks. Emerson concluded the discussions by “making statements to indicate the conclusion of the discussion” in both Number Talks and by “supporting students in remembering or making sense of at least one key mathematical topic or practice” in their second Number Talk. Emerson elicited contributions during their *orchestration* of the mathematical discussions by “eliciting multiple solutions, strategies, or ideas” and “engaging several students in sharing their thinking” in both Number Talks. “Eliciting a range of student understanding or methods” was observed in Emerson’s first Number Talk but not in their second. This is potentially explained by Emerson’s choice of tasks–Emerson used dot arrangements for their first Number Talk and a sequence of double-digit addition problems for the second. Facilitating a Number Talk focused on double digit addition may require further developed Mathematical Knowledge for Teaching (Ball et al., 2008) than is necessary for a discussion of dot arrangements, making “eliciting a range of student understanding or methods” more challenging with this type of task.

To probe student thinking as part of the *orchestrating* area of work, Emerson “posed questions to get students to explain their thinking about processes” in both Number Talks. Emerson “posed questions to get students to explain their understanding of key mathematical ideas” in their first but not second Number Talk, which again could be explained by task complexity. “Follow up on responses to make more student thinking about the mathematics available” was not observed in either Number Talk. Emerson did not use moves to “provide support for students to complete their contribution or clarify their thinking” in their first Number Talk but did so in their second. Emerson employed some moves to orient students to the contributions of peers more than others. Emerson “encouraged students to attend to, listen, and respond to peers’ contributions in order to maintain productive and focused interaction” in both Number Talks, and their frequency in the use of this move increased from the first to the second. “Posing questions to students about others’ ideas and contributions, including asking students to comment on, add to, or restate another student’s ideas” was not observed in Emerson’s first Number Talk and was observed once in their second. Emerson did not “support the listening of the class through the use of moves that require all students to respond to others’ work” in either Number Talk. Due to a student’s internet issues during the first Number Talk, Emerson “ensured that the class can hear others’ ideas,” but this move was not needed in the second Number Talk.

Emerson demonstrated some variation in the making contributions component of *orchestrating* a mathematical discussion. Emerson “used moves such as redirecting, revoicing, and highlighting to keep the discussion on track” during both Number Talks. “Making mathematical contributions which enrich the core ideas of the discussion and keep the discussion focused on the learning targets” was not observed in Emerson’s first Number Talk but was observed in their second. Emerson used moves to “ensure that substantive and relevant analysis is part of the discussion” in their first Number Talk but did not use this type of move in their second. Again, the increase in task complexity may have been a contributing factor.

Emerson showed a marked improvement in their *recording and representing content*. Interestingly, this improvement occurred midway through their second Number Talk. During the first Number Talk and first half of the second, Emerson recorded student thinking by typing students’ contributions verbatim. Emerson “recorded student ideas in ways that are true to the students’ contributions” but did not “record in ways that are clear, organized, and visible to the class” due to font size and use of small virtual sticky notes on a blank whiteboard. This method of recording did not allow Emerson to “attend to the accuracy of records and representations” or to “use recordings which support student understanding and participation” as a student’s explanation for adding a two-digit number that demonstrated understanding of place value was recorded verbatim. Midway through the second Number Talk, Emerson switched from typing to recording equations using the pen tool. Emerson’s clear and accurate recording with equations highlighted the place value understanding inherent in the student’s partial sums strategy, supporting other student’s comprehension of the strategy.

Discussion and Conclusion

Analyzing Dakota and Emerson’s virtual Number Talks using the formative assessment checklist for prospective teachers’ skills in leading mathematics discussions (Shaughnessy et al., 2021) revealed both proficiencies and areas in need of further development–critical dissonance and resonant harmonies. Both Dakota and Emerson demonstrated skills in *framing* their Number Talks during both the launch and conclusion. Both PSTs elicited multiple solutions and strategies throughout both number talks and “accepted, respected, and considered” all answers, a key component of Number Talks (Parrish, 2010, p. 19). Dakota and Emerson also made efforts to engage all students during both Number Talks, although tendencies to call on certain students were first observed. An implication for future practice is moving beyond involving multiple students in the discussion, to ensuring more equitable opportunities for student engagement.

Most PSTs conducted their Number Talks with small groups of students. Though small group size was used for flexibility of implementation during COVID, a benefit of this was that it allowed for a focus on mathematical content, technological tools, and discussion facilitation rather than classroom management. Although the small group size shifted the focus to the important work of discussion facilitation, our analysis revealed that positioning students to make sense of or attend to the contributions of others during discussion *orchestration* was observed less frequently. Prior research has identified positioning students to interact with the ideas of their peers as a challenge for teachers due to the cognitive demand of productively utilizing student solutions and strategies (Smith & Stein, 2018). Even teachers aware of the benefits of incorporating student thinking into their instruction can be apprehensive about responding to student thinking “on the fly” (Smith & Stein, 2018, p. 9). The move of orienting students to the contributions of peers presented an area of critical dissonance in need of resolution. When discussion facilitation does not encourage students' to consider the contributions of peers, it brings about the question of whose voice in particular is being centered. While both PSTs were developing in terms of this move, lack of effective moves to orient students to strategies suggested by their peers minimizes the opportunity for discourse as the conversation proceeds in a “show and tell” format that does not highlight or illustrate important mathematical ideas (Smith & Stein, 2018, p. 6). To achieve resonant harmony, we need to further emphasize the importance of this practice and support PSTs in learning to implement this move.

Recording and representing content was an area of dissonance that shifted to harmony for both PSTs. Representing and recording students' thinking is a challenging but vital way to support students' development of conceptual understanding by connecting language, notation, and visual representations during mathematics discussions (Garcia et al., 2021). Within the cycle, teacher participants were provided with various technologies for *recording and representing content* when facilitating a Number Talk virtually (Dimas, 2020). Both PSTs demonstrated growth from their first to their second Number Talk in using technology for pedagogical goals. Representations corresponded more accurately to students’ thinking and were more effective in supporting other students’ understanding of shared ideas as teachers became more familiar with the virtual space. We hypothesize that the effectiveness of PSTs' recording of student thinking may vary based on task selection, depending on PSTs' depth of knowledge and comfort level with the operations, relevant concepts, strategies, and representations relevant to selected tasks. Still, facilitating Number Talks virtually afforded PSTs the opportunity to use technology tools as they developed skills needed to effectively record and represent mathematics for instruction.

Analyzing pre-service teachers' work in the areas of *framing, orchestrating*, and *recording and representing content* during virtual number talks provided a formative assessment of the strengths PSTs bring to classroom discussions as well as the knowledge and skills needing further attention in future iterations of the VNT TLC and course. Research of this type allows for an awareness of the areas where teacher education experiences may create a critical dissonance in praxis and provides an opportunity for dissension resolution to achieve resonant harmony. Moreover, and in agreement with Shaughnessy et al. (2021), we acknowledge that assessment of mathematical discussion practice with this framework is a dynamic process. We are also cognizant that the framework can be tailored to serve various purposes, such as learning about supports for noticing and areas that are important but not addressed in the checklist framework (Shaughnessy et al., 2021). Regardless, engagement with decompositions of mathematical discussions and analyses of experiences created by teacher educators to develop this practice elaborate the field’s understanding of this instructional component for teaching and learning.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389-407.

Dimas, C. (2020, August 18). Virtual Math Talks: Exploring Ways to Stay Rooted in Meaningful Collaboration and Discourse [Live online workshop/video]. TODOS: Mathematics for ALL. https://vimeo.com/449287286

Fletcher, N., & Meador, A. (2022, February 3). *Developing preservice teachers' TPACK through a virtual number talks field experience: A case study* [Refereed Proceedings]. The 12th Congress of the European Society for Research in Mathematics Education (CERME12), Bolzano, Italy.

Garcia, N., Shaughnessy, M., & Pynes, D. (2021). Recording student thinking in a mathematics discussion. *Mathematics Teacher: Learning and Teaching PK-12 114*(12), 926-932. https://doi.org/10.5951/MTLT.2021.0117

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Aldine.

Joswick, C., Meador, A., & Fletcher, N. (2021). Virtual number talks: A case for transforming K-12 mathematics classroom teacher pedagogy through a virtualization of routines. In M. Niess & H. Gillow-Wiles (Eds.), *Transforming Teacher's Online Pedagogical Reasoning for Teaching K-12 Students in Virtual Learning Environments*. IGI Global.

Kazemi, E., & Hintz, A. (2014). *Intentional talk: How to structure and lead producing mathematical discussions*. Stenhouse Publishers.

Leech, N. L., & Onwuegbuzie, A. J. (2007). An array of qualitative data analysis tools: A call for data analysis triangulation. *School Psychology Quarterly*, *22*(4), 557-584. https://www.doi.org/10.1037/1045-3830.22.4.557

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Author.

Parrish. S. (2010). *Number talks: Helping children build mental math and computation strategies, grades K-5*. Math Solutions.

Shaughnessy, M., Garcia, N. M., O'Neil, M. K., Selling, S. K., Willis, A. T., Wiles, C. E., Salazar, S. B., & Ball, D. L. (2021) Formatively assessing prospective teachers' skills in leading mathematics discussions. *Educational Studies in Mathematics*, *108*, 451-472.

Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Educational Research Journal*, *49*(5), 935-970.

Smith, M. S., & Stein, M. K. (2018). *5 practices for orchestrating productive mathematics discussions* (2nd ed.). NCTM.

Strauss, A. L., & Corbin, J. M. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Sage.

Volstedt, M., & Rezat, S. (2019). An introduction to grounded theory with a special focus on axial coding and the coding paradigm. In G. Kaiser & N. Presmeg (Eds.), *Compendium for early career researchers in mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15636-7_4

Woods, D. M. (2021). Enacting number talks in a simulated classroom environment: What do preservice teachers notice about students? *International Journal of Technology in Education*, *4*(4), 772-795. https://doi.org/10.46328/ijte.148

SUBVERTING DOMINANT SCRIPTS OF MATHEMATICS TEACHING: EXPLORING PROSPECTIVE ELEMENTARY TEACHERS' (RE)IMAGININGS OF A CLASS DISCUSSION

Sheila Orr
Michigan State University
smorr11@msu.edu

Sandra Crespo
Michigan State University
crespo@msu.edu

Recent calls to (re)humanize mathematics education require a clearer articulation of how equity-oriented pedagogy can be taught to and learned by teacher candidates. In this article we propose and theorize "flipped scripts" as a research and pedagogical construct that can help identify prospective teachers' (PTs) potential to (re)imagine mathematics classrooms as inclusive and equitable spaces for learning. We use examples of PTs' imagined representations of a mathematics class discussion to examine explicit and hidden messages they communicate about students as capable or struggling learners of mathematics. We conclude by offering implications and affordances of studying PTs' generated "flipped scripts" as indicators of their potential to take up and enact humanizing mathematics pedagogical approaches.

Keywords: Classroom Discourse, Elementary School Education, Preservice Teacher Education, Culturally Relevant Pedagogy

Efforts to conceptualize equity-oriented pedagogies in our field have been ongoing and have taken multiple forms such as teaching mathematics for social justice (Gutstein, 2006), culturally relevant mathematics teaching (Tate, 1995), and most recently the call to (re)humanized mathematics classrooms (Gutiérrez, 2017). Despite past and present efforts to conceptualize equity-oriented pedagogies, teachers of all experience levels find it challenging to stray away from dominant forms of mathematics instruction which set up classrooms as hierarchical and exclusionary spaces for learning (Louie, 2017). This challenge is compounded for those at the initial stages of teacher preparation and for mathematics teacher educators (MTEs) interested in supporting prospective teachers (PTs) to reimagine mathematics classrooms as loving and nurturing spaces for learning. In this study we seek to understand PTs' potential for subverting dominant scripts for mathematics teaching. We conceptualize dominant instructional scripts as representations of practice that center intellectual authority solely on the teacher and that set up classrooms as exclusionary spaces for learning. We explore PTs' capacity for generating "*flipped scripts*" which we conceptualize as representations of instructional practice that counter exclusionary and dominant forms of mathematics teaching practice.

Theoretical Framework

We enter the conversation of reimagining mathematics classrooms as humanizing spaces for learning by building on previous studies focusing on culturally relevant mathematics teaching (Leonard et al., 2010; Parker et al., 2017), teaching mathematics for social justice (Kokka, 2015; Gutstein, 2006) and (re)humanizing mathematics (Goffney & Gutiérrez, 2018). These previous studies have highlighted the instructional practices of teachers who have challenged dominant forms of mathematics instruction and align with the conceptualization of humanizing pedagogy as described by critical scholars outside of mathematics education (del Carmen Salazar, 2013; hooks, 1994; Freire, 1993). To be clear, humanizing pedagogical practices are not new to mathematics education but rather have been a topic of continued interest and scholarship. Bartell and colleagues (2019) share their exploration into working with prospective teachers to draw on students' funds of knowledge (Moll et al., 1992) to build connections between their lived experiences and the content in the classroom. Additionally, Yeh and colleagues (2020) explore how (re)humanizing classrooms become a place where students are free of deficit narratives and challenge the way students with dis/abilities are excluded from traditional mathematics classrooms. Humanizing pedagogies disrupt familiar dominant scripts of mathematics classrooms that position teachers in control of the content and form of the classroom discourse.

Louie (2017) describes the dominant script of mathematics classrooms as embedding a culture of exclusion and describes it as “the restrictive and hierarchical culture that has historically dominated American mathematics education” (p. 489). A culture of exclusion sorts students into those who know and do not know while creating only narrow forms of participation in the classroom. Cultures of exclusion are characterized by dominant instructional scripts in the classroom that set the stage for competitive student behaviors that are unproductive and potentially harmful. Running counter to this culture of exclusion is an inclusive mathematics classroom. Inclusive classrooms encourage multiple solution paths and forms of participation, and it also encourages students to engage in collaborative learning practices. An inclusive classroom, as described by Louie, is a classroom working to (re)humanize mathematics.

Our study also draws from studies of *lesson plays* and of *lesson scripts* of mathematics classrooms which are used as artifacts that make visible the imagined instructional practice of practicing and prospective teachers (Zazkis et al., 2009; Zazkis & Herbst, 2017). We use a collection of classroom dialogues generated by prospective teachers to explore the extent to which they reproduced dominant (exclusionary) instructional scripts and/or produced what we are calling *flipped* (inclusive) instructional scripts. We use Louie’s framework of exclusionary and inclusive mathematics classroom culture to identify and unpack salient characteristics of *dominant* and *flipped* classroom scripts that the PTs in this study generated. We argue that flipped classroom scripts may be good indicators of PTs’ readiness and potential for enacting humanizing pedagogy in their future mathematics classrooms.

Research Methods

This is an exploratory study that uses a small portion of data collected for a larger study of PTs’ generated representations of mathematics teaching practice (Crespo, 2006). This data consisted of 17 imagined classroom dialogues generated by PTs who were at the end of their senior year elementary mathematics methods courses, prior to their yearlong internship. This study uses one of the project’s teaching scenario prompts that features *The Coins Problem* as the mathematical object for a potential class discussion with third graders (Ball & Bass, 2003; Stylianides, 2007). The teaching scenario prompt (Figure 1) was designed to elicit PTs’ representations of their emerging mathematics teaching practice by asking them to write teacher questions they might use with the given task and by generating a classroom dialogue that illustrated a possible class discussion a teacher and their students could have around this mathematics problem.

Imagine you have given your students the following problem to work on.

The Coins Problem

You have a bunch of coins in your pocket, and you know that there are some of each—Pennies, Nickels, and Dimes. Suppose you take out two coins. How much money could you pull out?

. Write 3 questions you want to make sure you ask the students about this problem. Say a little bit about what your goal is for asking these questions.

a. Imagine students have spent some time working on the coins problem, you call everyone’s attention to begin the class discussion about the problem. What can you imagine students will say and do in return? Write out as much of your imaginary class discussion as you can in the following format:

Class Exchange	Explanation
T: *Let's have a discussion about the coin problem, is everyone ready?* *Ss: Yes* *T:*	*Getting everyone's attention and make sure everyone is ready to share and listen*

Figure 1: Generate a Classroom Dialogue (Two-Coins Problem) Task

We drew on Louie's (2017) exclusion/inclusion framework to characterize the PTs' generated classroom dialogues as following or not following dominant scripts of math class discussions. As seen in Table 1, scripts classified as flipped (inclusive) set the stage for a collaborative classroom culture and include teacher questions that create openings for diverse solutions and answers communicating the message that all answers (including mistakes) and forms of participation are valued. In contrast, scripts classified as dominant (exclusionary) set the stage for a competitive classroom culture and include teacher questions that have a narrow set of possible solutions and answers creating a sorting mechanism for those who do and do not know the answer. Through the data analysis a third category emerged where the dialogues were a mixture of both flipped and dominant scripts.

Table 1: Examples of Flipped and Dominant Scripts

Script Type	Class Exchange	Explanation
Flipped Script	T: Can 3 students come up and share their solutions and how they got them Ss with different solutions and methods come up and put their work on the board	
	T: Why are these so different? Are they all correct? S: They are all correct but people picked different coins. The coins have different amounts so you will get different totals.	To show a problem can have many solutions
	T: What happens when you add a 3rd coin? Ss: we get even more different answers	Compare different situations
	T: Why? Ss: Because we had more choices	Justification
	T: What does that mean? S: The more choices there are the less chance there is of getting the same thing.	Finding a solution/rule
Dominant Script	T: What are the values of the coins? S: one cent, 5 cents, 10 cents T: if there are 2 of each coin in his pocket, how many different combinations could you make? S: 6 T: how did you get that? S: 3x2 Ss: No there has to be more than 6 T: Why did you say that?	Speak up to get class attention, once all eyes are on you and all talking has stopped you can start your discussion with an intriguing question.

To unpack the distinctions between dominant and flipped scripts we examined: (1) PTs' questions in both parts (a) and (b) of the task and the extent to which these questions set the stage for exclusionary and inclusive classroom discussion, (2) how students are represented in terms of being active/passive participants and competent/not competent, (3) PTs' explanations for their teaching moves. Each of these domains highlighted a different aspect of how the prospective teachers were imagining their class discussions. We then re-read the scripts with a focus on coding each of these three areas as either adhering to or subverting the dominant scripts of mathematics instruction.

Findings

Of the 17 responses to the coins problem task, we categorized 5 of them (29%) as meeting the criteria of dominant scripts, 3 of them (18%) met our criteria of "flipped script", and the remaining 9 (53%) were a mixture of both, dominant and flipped scripts. Although it may seem discouraging that only 3 (18%) of the dialogues were classified as flipped scripts, we see the latter types of mixed scripts as also suggestive of PTs' potential for learning and taking up (re)humanizing mathematics pedagogy.

Our analysis of the questions the PTs generated in the flipped and mixed scripts found an important difference between the questions the PTs generated in part (a) and (b) of the teaching scenario prompt that we interpreted as PTs' willingness to go "off-script" as another feature of their potential to subvert dominant scripts. The questions PTs had generated in the part (a) of the task focused exclusively on the mathematics of the coins problem task, whereas when they then attempted to enact those questions within the class discussion those initial questions were left out and instead the questions used were more dialogical and had a dual focus on the mathematics and the students' participation. In doing so, they represented the classroom as a more humanizing space where students could be more active (rather than passive) participants. Instead of representing students as answer givers to factual questions asked by the teacher, the students in the flipped scripts were represented as generating and constructing ideas collaboratively. Flipped scripts also (re)imagined teachers as inviting students into the space by asking questions to generate discussion and pressing students to explain their reasoning once an answer was shared.

Reading the scripts with a focus on teacher-student interaction and student-peer interaction we also found PTs' flipped scripts represented class discussions that break off from the dominant Initiation-Response-Evaluation (IRE) pattern of classroom discourse and instead represented the teacher pressing for explanations with multiple back and forth between the teacher and student(s). However, it is important to note that the student responses in the flipped scripts were not as clearly specified as the answers represented in the dominant scripts. There was typically a sentence or two included about students sharing answers or collaboratively generating a list. We interpret this difference as suggesting that PTs may have a general vision of what inclusive mathematics class discussions might look like but are not yet able to produce specific teaching moves that align with their vision.

Discussion & Implications

We offer the construct of "flipped scripts" as an indicator of PTs' potential to take up and enact humanizing mathematics pedagogical approaches. We offer it as a useful analytical lens to analyze PTs' generated representations of mathematics teaching at different stages of development of (re)imagining mathematics instruction as equitable and humanizing. PTs generated classroom dialogues with mixed scripts can provide MTEs with the opportunity to see the initial attempts of PTs to create humanizing pedagogical approaches. In identifying ways PTs attempt to flip the script from exclusionary to inclusive forms of mathematics instruction, MTEs can build on the assets PTs bring to the teacher education classroom.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 0546164. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 27–44). National Council of Teachers of Mathematics.

Bartell, T. G., Drake, C., McDuffie, A.-R., Aguirre, J. M., Turner, E. E., & Foote, M. Q. (Eds.). (2019). *Transforming mathematics teacher education: An equity-based approach*. Springer International Publishing. https://doi.org/10.1007/978-3-030-21017-5

Crespo, S. (2006). Exploring prospective teachers' learning the practices of posing, interpreting, and responding during and beyond teacher preparation. National Science Foundation, NSFCAREER award (Grant No. 0546164).

del Carmen Salazar, M. (2013). A humanizing pedagogy: Reinventing the principles and practice of education as a journey toward liberation. *Review of Research in Education*, *37*, 121–148.

Freire, P. (2018). *Pedagogy of the oppressed: 50th anniversary edition*. Bloomsbury Publishing USA.

Gutiérrez, R. (2017, October). Living mathematx: Towards a vision for the future. *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. https://eric.ed.gov/?id=ED581384

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. Taylor & Francis.

hooks, bell. (1994). *Teaching to transgress: Education as the practice of freedom*. Routledge.

Kokka, K. (2015). Addressing dilemmas of social justice mathematics instruction through collaboration of students, educators, and researchers. *Educational Considerations*, *43*(1), 13–21.

Leonard, J., Brooks, W., Barnes-Johnson, J., & Berry, R. Q. (2010). The nuances and complexities of teaching mathematics for cultural relevance and social justice. *Journal of Teacher Education*, *61*(3), 261–270. https://doi.org/10.1177/0022487109359927

Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, *48*(5), 488–519. JSTOR. https://doi.org/10.5951/jresematheduc.48.5.0488

Moll, L. C., Amanti, C., Neff, D., & Gonzalez, N. (1992). Funds of knowledge for teaching: Using a qualitative approach to connect homes and classrooms. *Theory Into Practice*, *31*(2), 132–141. https://doi.org/10.1080/00405849209543534

Parker, F., Bartell, T. G., & Novak, J. D. (2017). Developing culturally responsive mathematics teachers: Secondary teachers' evolving conceptions of knowing students. *Journal of Mathematics Teacher Education*, *20*(4), 385–407. https://doi.org/10.1007/s10857-015-9328-5

Stylianides, A. L. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321. https://doi.org/10.2307/30034869

Tate, W. F. (1995). Returning to the root: A culturally relevant approach to mathematics pedagogy. *Theory Into Practice*, *34*(3), 166–173. https://doi.org/10.1080/00405849509543676

Yeh, C., Ellis, M., & Mahmood, D. (2020). From the margin to the center: A framework for rehumanizing mathematics education for students with dis/abilities. *The Journal of Mathematical Behavior*, *58*, 100758. https://doi.org/10.1016/j.jmathb.2020.100758

Zazkis, R., & Herbst, P. (Eds.). (2017). *Scripting approaches in mathematics education: Mathematical dialogues in research and practice*. Springer.

Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson plays: Planning teaching versus teaching planning. *For the learning of mathematics*, *29*(1), 40-47.

PRESERVICE MATHEMATICS TEACHER BELIEFS REGARDING PROCEDURAL VERSUS CONCEPTUAL TEACHING BEFORE AND AFTER METHODS COURSES

Derek Pope
Stony Brook University
derek.pope@stonybrook.edu

Angela Kelly
Stony Brook University
angela.kelly@stonybrook.edu

After a two-semester graduate mathematics pedagogy sequence, nine preservice mathematics teachers showed significant changes in their beliefs away from mathematics as a set of rules and procedures and towards mathematics as a process of enquiry. Interviews corroborated quantitative findings and pointed towards practical conceptually-focused strategies in the courses as a driving factor in affecting students' beliefs. Implications for preservice mathematics teacher education are discussed.

Keywords: Teacher Beliefs; Preservice Teacher Education

Teaching mathematics with a focus on developing conceptual understanding, and not merely procedural fluency, is called for in the *Common Core Mathematics Standards* (National Governors Association, 2010). The link between teachers' conceptual views of mathematics and the achievement of their students is well documented (e.g., Staub & Stern, 2002; Tchoshanov, 2011). However, in a recent study of U.S. mathematics teachers, only 61% of high school teachers and 49% of middle school teachers considered themselves very well prepared to develop students' conceptual understanding (Banilower et al., 2018). In a study of preservice mathematics teachers, only 24.1% of teachers proposed instructional strategies that promoted conceptual understanding and qualitative reasoning (Cankoy, 2010).

Why might there be dissonance between the benefit of teaching students conceptually and teachers' beliefs that mathematics should be taught this way? This study sought to understand preservice teachers' beliefs about how mathematics should be taught before and after two content pedagogy (methods) courses, and to identify programmatic or other factors that contributed to changes in beliefs. The following research questions guided the collection and analysis of data: (1) What is the nature of secondary mathematics preservice teachers' beliefs about the nature of mathematics (as a more conceptual or procedural discipline) before and after their methods courses? (2) Which aspects of students' preservice teacher preparation program impacted their beliefs about how mathematics should be taught?

The conceptual framework for the present study is derived from research that suggests that teachers' pedagogical beliefs are precursors to their enacted pedagogy (Pajares, 1992). This study also drew inspiration from Conney et al.'s (1998) and Gomez and Conner's (2020) belief structures to help explain why teachers changed their beliefs.

Methods

This was a convergent parallel study that explored the beliefs of a group of nine preservice mathematics teachers. Although a preliminary analysis, this study was significant in its examination of secondary preservice teachers specifically, its mixed methods approach, and its emphasis on understanding not just how beliefs changed but why (Gomez & Conner, 2020).

The research sample consisted of a subset of a cohort of preservice teachers in a secondary mathematics teacher preparation program at a large public research university. Of the 19 students enrolled in *Introduction to Methods of Teaching Mathematics* (Methods 1) and *Advanced Methods of Teaching Mathematics* (Methods 2) in Fall 2020 and Spring 2021, nine students volunteered to participate in the study. Two were undergraduates, five were graduates in the Master of Arts in Teaching program, and two were in a five-year combined program. All but one student were women. The primary researcher was also the course instructor.

The Intervention: Methods 1 and Methods 2

Methods 1 and 2 were both three-credit courses which met for 160 minutes per week for 14 weeks. Both courses were taught synchronously online due to the COVID-19 pandemic. Course activities in Methods 1 centered around exploring how students learn mathematics and creating mathematical tasks that foster conceptual understanding. The instructor shared many examples of learning activities from his own experience as a high school mathematics teacher. Two important components of Methods 1 were observations of secondary mathematics classrooms and practice planning, implementing, and reflecting on two lessons which students taught to their peers. Methods 2 built upon this foundation and focused on sequencing lessons into coherent unit plans as well as assessment of student learning. Students also had their third opportunity to plan, implement, and reflect upon a lesson to their peers.

Data Sources

Data sources included: (1) a survey taken in the beginning of Methods 1 and again after Methods 2; (2) a VoiceThread assignment completed in the beginning of Methods 1; and (3) a 45-minute semi-structured interview over Zoom after Methods 2.

The validated survey consisted of 33 six-point Likert-scale items that measured participants' beliefs about mathematics and how it should be taught (Tatto et al., 2012). The first six survey statements related to beliefs about mathematics as a set of rules and procedures and the next six to beliefs about mathematics as a process of enquiry. Composite scores were calculated for the two categories, indicating participants' preferences toward statements about mathematics as a process of enquiry as opposed to mathematics as a set of rules and procedures.

At the same two points during participants' teacher preparation program, interviews were conducted to provide further detail on survey choices and to attempt to understand factors that had influenced any belief changes. The first interview was conducted virtually using VoiceThread. After the completion of the two-semester methods sequence, participants sat for a 45-minute semi-structured interview over Zoom. Interview guides were developed based on prior research to provide different ways for participants to demonstrate their beliefs about mathematics and how it should be taught. To triangulate results, multiple types of questions were used, including open-ended questions, reactions to student work, and reactions to a video of a very procedural approach towards teaching the fact that $x^0 = 1$ for $x \neq 0$.

Data Analysis

Quantitative analysis involved performing a paired-samples *t*-test to identify significant differences in participants' composite scores before and after the Methods sequence. A power analysis performed using SPSS v28 indicated that the sample size of nine was adequate for >80% power and a large effect size.

Qualitative analysis involved an iterative coding process. The project team coded interview transcripts line-by-line using a priori provisional coding (Miles & Huberman, 1994), grouped thematically into axial codes, with elements of grounded theory (Saldaña, 2009).

Results

Quantitative Results

Figure 1 shows how students' preference toward mathematics as a process of enquiry over mathematics as a set of rules and procedures changed from before Methods 1 (first bar) to after Methods 2 (second bar).

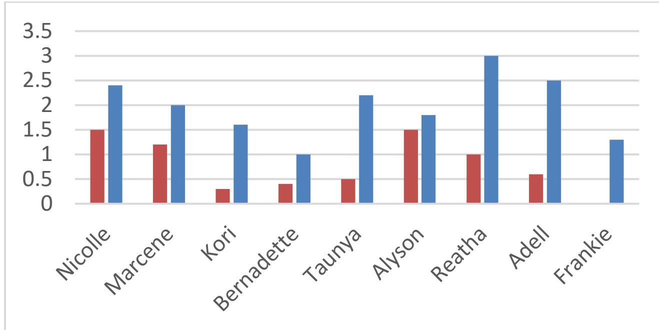


Figure 1: Pre-Methods 1 and Post-Methods 2 Survey Responses

The fact that all values were nonnegative for the pre-Methods survey indicated that all students came into their Methods courses preferring statements towards mathematics as a process of enquiry, with one student showing no preference. All values increased from the beginning of Methods 1 (M = 0.78, SD = 0.54) to after Methods 2 (M = 1.98, SD = 0.63), indicating that students developed significantly stronger preferences towards mathematics as a process of enquiry, $t(8) = 6.063$, $p < .001$, with a large effect size, Cohen's $d = 2.02$.

Qualitative Results: What Changed?

Interview transcripts corroborated students' positive shifts towards mathematics as a process of enquiry over mathematics as a set of rules and procedures, with implications for how the preservice teachers planned to teach mathematics in the future. When asked if her views had shifted on how to teach students rules such as $x^0 = 1$, Adell said:

> Absolutely. And I think it's really just because of the Methods courses. I think those were definitely the most powerful courses that I was able to take. I don't ever want to just spit out facts to my students, or rules… I would still make everything as conceptual as possible, use different methods, use manipulatives, use technology like Desmos, GeoGebra. It would feel wrong if I just gave them out as rules.

Changes in beliefs were particularly notable given that many students affirmed that they were taught mathematics in very procedural, teacher-centered ways, and that they had planned to teach this way until their participation in the teacher preparation program. For example, in reaction to a procedural video about $x^0 = 1$, Bernadette stated that "before [Methods] I would probably teach it like that… I don't think I would have much of a problem with just giving students a rule like $x^0 = 1$. But after the program definitely I would teach them to understand why."

Qualitative Results: Why Were There Changes?

Interview transcripts also gave insight into why changes occurred after the Methods sequence. Adell, who showed one of the greatest changes in her preference toward mathematics as a process of enquiry, expressed her desire to have a classroom full of technology, manipulatives, and group work, saying that the Methods classes gave her pedagogical tools that she had not previously possessed which she could use to develop students' conceptual understanding. Adell expressed how her changed beliefs influenced her actions as a tutor and as a mathematics student:

> I think the Methods classes even just changed the way I thought, like I could see even how I was tutoring changed completely, just based on the Methods classes. The way that I felt about my other math classes definitely changed. I was so happy in my algebra class

> when I saw a pattern myself, in the textbook, and then I noticed that it was just given to me as a rule… It was just so nice to finally see a pattern myself.

In reference to teaching a rule such as $x^0 = 1$, Adell said that she did not understand why it worked until it was discussed in her Methods class. She recalled writing the number two to various descending powers and noticing a pattern and planned to teach it the same way in the future. Adell was an example of a reflective connectionist who was able to formulate a coherent philosophy of teaching that connected new information with her experience (Cooney et al., 1998). Another student, Allyson, also said that although she always wanted a collaborative mathematics classroom (as evidenced by her relatively high score on the pre-Methods 1 survey), during Methods she learned many strategies for actualizing the type of classroom she wanted.

Reatha, another student who showed a large increase in her score pre- to post-Methods, said that in Methods she was introduced to the importance of groupwork, self-exploration, and developing mathematical ideas with teacher assistance, rather than lecturing. She also mentioned her belief that students should be presented problems before learning the procedures, as opposed to learning a procedure first and then simply repeating it. Reatha even said that after the class discussion on why $x^0 = 1$, she contacted her friend to teach her why the rule worked, giving evidence that an activity in the Methods class had a direct and immediate impact on her pedagogical practice.

Based on interview transcripts, course elements that appeared to have positive effects on students' productive beliefs about mathematics and its teaching included: (1) use of interactive technology (e.g., Desmos and GeoGebra); (2) use of physical and virtual manipulatives (e.g., Algebra Tiles); (3) use of collaborative groups; (4) practice writing exams that assessed both conceptual understanding and procedural fluency; (5) learning about why mathematical rules work; (6) reading the book *Mathematical Mindsets* (Boaler, 2016), with its emphasis on teaching mathematics creatively instead of through memorization; (7) use of topic-specific examples of teaching for conceptual understanding; (8) realizing through discussion and readings that many students are not successful learning mathematics in procedural ways; (9) observing secondary lessons taught conceptually; and (10) reflecting on their personal experiences learning mathematics procedurally in light of alternative methods.

Conclusion

A sequence of Methods courses purposefully designed to convince preservice teachers of the merits of teaching mathematics for conceptual understanding, and the detriments of teaching mathematics for merely procedural competency, significantly affected their beliefs about the nature of mathematics towards mathematics as a process of enquiry. The findings presented here imply that beliefs can change when preservice teachers are provided opportunities to learn various strategies for teaching mathematics conceptually, practice using these strategies with their peers, and reflect on their experiences learning and teaching mathematics this way. Preservice teacher programs should be structured in ways that support these changes in beliefs and practice through a variety of strategies, so that students have both the motivation to change their beliefs and the pedagogical toolkit to modify their practice accordingly. Future research could explore how other aspects of secondary mathematics preservice programs, such as clinical experience, work in tandem with Methods courses to influence preservice teachers' beliefs and practices.

References

Banilower, E. R., Smith, P. S., Malzahn, K. A., Plumley, C. L., Gordon, E. M., & Hayes, M. L. (2018). *Report of the 2018 NSSME+*. Horizon Research, Inc.

Boaler, J. (2016*). Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. Jossey-Bass.

Cankoy, O. (2010). Mathematics teachers' topic-specific pedagogical content knowledge in the context of teaching a^0, 0! and a / 0. *Educational Sciences: Theory and Practice, 10(2),* 749-769.

Cooney, T.J., Shealy, B.E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education*, *29(3)*, 306-333. https://doi.org/10.5951/jresematheduc.29.3.0306

Gomez, C. N., & Conner, A. (2020). Impact of Cooney, Shealy, and Arvold's (1998) belief structures: A literature review and citation analysis. *Journal for Research in Mathematics Education, 51(4)*, 468-503. https://doi.org/10.5951/jresematheduc-2020-0046

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Sage Publications, Inc.

National Governors Association. (2010). *Common Core State Standards (Mathematics).* National Governor's Association. http://www.corestandards.org/Math/

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. Review of Educational Research, 62(3), 307-332. https://doi.org/10.2307/1170741

Saldaña, J. (2009). *The coding manual for qualitative researchers*. Sage Publications Ltd.

Staub, F. C., & Stern, E. (2002). The nature of teachers' pedagogical content beliefs matters for students' achievement gains: Quasi-experimental evidence from elementary mathematics. *Journal of Educational Psychology, 94(2),* 344-355. https://doi.org/10.1037/0022-0663.94.2.344

Tatto, M. T., Peck, R., Schwille, J., Bankov, K., Senk, S. L., Rodriguez, M., Ingvarson, L., Reckase, M., & Rowley, G. (2012). *Policy, practice, and readiness to teach primary and secondary mathematics in 17 countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-MM).* International Association for the Evaluation of Educational Achievement.

Tchoshanov, M., Lesser, L. M., & Salazar, J. (2008). Teacher knowledge and student achievement: Revealing patterns. *NCSM Journal of Mathematics Education Leadership, 10(2),* 38-48.

PTS' EXPERIENCE DEVELOPING CULTURALLY RESPONSIVE MATHEMATICS LESSONS IN A MIDDLE SCHOOL MATHEMATICS METHODS COURSE

Zareen G. Rahman
James Madison University
rahmanzx@jmu.edu

The proposed presentation describes 20 prospective teachers' (PTs) experience developing culturally responsive mathematics lessons in their middle school mathematics methods course. One mathematics teacher educator guided the PTs in developing and implementing lessons as they engaged in micro-teaching episodes. During Spring 2020, Fall 2020 and Spring 2021 semesters, the PTs engaged in micro-teaching guided by the lesson analysis tool developed by Aguirre and Zavalla (2013) to make culturally responsive mathematics teaching (CRMT) explicit. Engagement in the activity allowed the PTs to learn about developing and implementing lessons that are mathematically rigorous as well as sensitive to social justice issues. The proposed presentation aims to deconstruct how micro-teaching guided by the CRMT tool was employed and to describe the opportunities for PTs' learning.

Keywords: Culturally Relevant Pedagogy, Equity, Inclusion, and Diversity, Social Justice and Preservice Teacher Education.

Incorporating culturally responsive pedagogy (CRP) in mathematical context is challenging (Nasir, Hand, & Taylor, 2008). PTs in their methods courses must experience ways to weave CRP with the content they teach (Cochran-Smith, Davis, & Fries, 2004). For example, using students' funds of knowledge (FoK) to teach mathematics (Moll & Gonzalez, 2004). The research on FoK encourages teachers to incorporate students' wider experiences into teaching mathematics. However, this doesn't necessarily include issues of equity and social justice (Aguirre & Zavala, 2013). CRMT includes knowledge, beliefs, and teaching practices that highlight mathematical thinking, culture, language, as well as issues of social justice (Aguirre & Zavalla, 2013; Aguirre, 2009; Gay, 2009; Gutiérrez, 2009; Kitchen, 2005; Leonard, Napp, & Adeleke, 2009; Turner et al., 2012). It includes 8 dimensions that incorporate elements of pedagogical content knowledge (PCK) and CRP to guide equitable mathematics teaching.

Mathematics education research with a focus on social justice views mathematics as a tool to analyze social justice issues locally or globally (Aguirre, 2009; Christiansen, 2008; Gutiérrez, 2009; Gutstein, 2006; Mukhopadhyay & Greer, 2008; Tate, 1994; Turner & Strawhun, 2007; Gutstein, 2006; Mukhopadhyay & Greer, 2008; Peterson, 2005; Varley-Gutierrez, 2011; Kitchen & Lear, 2000). When teaching mathematics for social justice, teachers provide opportunities for students to use mathematics to challenge and understand existing structures of power.

According to Aguirre and Zavala (2013) developing culturally responsive mathematics teachers requires that they become aware of the larger context within which they teach mathematics. They must conceptualize their role as a teacher to prepare their students to become engaged members of a larger social, cultural or political context. Teachers must recognize that teaching mathematics is a political activity, it is not the neutral subject it's often perceived to be. This means becoming aware of inequities in mathematics education and the role of mathematics as a gatekeeper to advance courses and careers. To this end, the specific research question guiding this study is: *How did engagement in a micro-teaching activity guided by the CRMT tool provide opportunities for PTs' learning?*

Methodology

Action research methodology was employed to investigate how micro-teaching guided by the CRMT tool provided opportunities for PTs' learning over three semesters (Johnson, 2005; Kemmis & McTaggart, 2000; Mills, 2003; O'Brien, 2001; Willis & Edwards, 2014). Micro-teaching activity included PTs developing lesson plans and video recording a short 5-10 minute teaching episode from that lesson. The Mathematics teacher educator (MTE) and peers provided feedback on the video followed by the PTs reflecting on their experience.

The study took place at a public university in the mid-Atlantic region of United States (US). Data sources include student lesson plans, video lessons, student reflections, and peer feedback provided by the PTs during Spring 2020, Fall 2020 and Spring 2021 semesters. Qualitative research methods, specifically thematic analysis (Braun & Clarke, 2019) was employed to systematically code the data and look for emergent themes. Data was organized into 8 CRMT dimensions when analyzing the lessons. Dimensions 1 through 5 guided the analysis on how the lessons attended to students' mathematical thinking. (1-intellectual support, 2-depth of student knowledge and understanding, 3-mathematical analysis, 4-mathematical discourse & 5-communication and student engagement). Dimensions 6a and 6b provided information about students' language (6a-academic language support for ELL & 6b-use of ESL scaffolding strategies). While dimensions 7 and 8 guided the lesson analysis on culture and social justice components (7-funds of knowledge/culture/community Support & 8-social justice).

Findings

Analysis of data collected over three semesters unearthed PTs' perceptions about CRMT and their challenges. During Spring 2020 semester PTs' understanding of CRMT ranged from 'using mathematics to solve real-life problems' to 'developing a math lesson superficially using context from another (non-US) culture'. Most PTs sprinkled some context over mathematics. Such lessons pointed towards the need to improve my PTs' perceptions about CRMT. I planned to help them see connections between CRP, social justice and teaching mathematics. During the next two semesters I devoted more time to unpacking CRMT and introduced it earlier on in the semester. Using existing resources (Berry, Conway, Lawler, & Staley, 2020) I highlighted how CRMT dimensions could form the foundation of a math lesson. PTs examined a social justice based mathematics lesson in class guided by the CRMT tool and developed their own ideas about CRMT. PTs appreciated seeing a math lesson where math and social justice were given equal importance. I also provided a list of social justice standards (Teaching Tolerance, 2016) to be used in the lesson plans along with mathematics standards. The resulting student work from Fall 2020 and Spring 2021 semesters showed marked improvement in students' lesson plans.

Description of two Lessons from Fall 2020 and Spring 2021

During Fall 2020 one PT developed a lesson on linear equations and gender wage gap. The guiding question for the lesson was, How do functions represent the relationship between two variables? The PT stated the following mathematics standard and social justice goals: "The student will investigate and analyze linear and quadratic function families and their characteristics both algebraically and graphically, including connections between and among multiple representations of functions using verbal descriptions, tables, equations, and graphs.", "Students will be able to recognize inequality in the world around them and discuss the issues critically." and "Students will be able to do their own research on inequalities and voice their biggest concerns in that area." During this semester the MTE did not ask the PTs to include social justice goals into their lesson however this PT included them. Among other activities in the lesson, the PT planned to have the students engage in a Desmos activity comparing graphs of two linear functions depicting the wages of two different people. Later in the lesson, the PT planned for the students to react to an image depicting gender wage gap data through Jamboard. The lesson presented a balance between a focus on mathematics and the social justice issue being considered. The lesson did not touch upon the needs of students whose first language is not English. A deep connection to students funds of knowledge and their culture was also missing.

In another example of a lesson from Spring 2021, a PT aimed to engage students in discussions about digital activism while learning about histograms, stem-and-leaf plots and dot plots. The PT stated the following mathematics and social justice goals: "The student, given data in a practical situation, will (a) represent data in a histogram; (b) make observations and inferences about data represented in a histogram; and (c) compare histograms with the same data represented in stem-and-leaf plots, and line (dot) plots" and "I take responsibility for standing up to exclusion, prejudice and injustice" The PT planned to collect data on the number of letters in student names and make a histogram from this data. The PT also planned to ask the students questions like, "What could happen to our histogram if we asked this same question in another country? A different era in history? Another culture?" Such questions could potentially lead to a discussion about similarities and differences in naming conventions of different cultures. While this lesson also included social justice standards, the inclusion of social justice standards did not lead to a deep connection between mathematics and social justice. This semester the students were explicitly asked by the MTE to include social justice standards so inclusion may have been a response to that requirement. The lesson did have the potential to connect to students' home cultures but did not attend to learning needs of students whose first language was not English.

These lessons showed a shift away from superficial use of context. The inclusion of social justice standards may have encouraged the PTs to choose contexts that justified their selection of social justice standards. Their intentional goal selection for both mathematics and social justice may have allowed the PTs to think of both these factors as important in their planning. Overall the analysis of PTs' lessons over three semesters showed a decrease in the use of contexts such as, "ice cream activity" or "graphing flowers" and an increase in contexts such as, "planning a community garden", "using American sign language" or " learning about affordable housing".

Challenges and Growth during Spring 2020, Fall 2020 and Spring 2021

Engagement in the micro-teaching activity itself provided the PTs with the opportunity to learn about CRMT, become aware of their challenges, and experience excitement in connecting issues they feel passionate about to mathematics. The PTs shared similar concerns and patterns of growth over the three semesters. I describe these findings below.

Finding a context was challenging. Some of the PTs' concerns were general in nature because of the novelty of CRMT, for example, "When I saw that we had to come up with a culturally relevant lesson, I got nervous. I had never been forced to come up with a lesson that wasn't all about math. This assignment pushed me out of my comfort zone and forced me to consider other people and their perspectives on the world." (Spring, 2020) PTs shared specific concerns as well, and their biggest challenge was deciding on a context. As one PT shared, "It was also more difficult coming up with an idea to do my lesson on and incorporating the culture I wanted to." (Spring, 2020) The specific challenge was to first find a connection between mathematics and a real world situation that was also culturally relevant.

CRMT was perceived as beneficial for students. Despite their challenges, the PTs found that connecting mathematics to a context was beneficial for their students. Some were motivated to overcome this challenge. One PT shared, "I learned that CRMT is not always going to be the easiest method of teaching, but it is something I am going to have to work on getting used to and trying to implement in my classroom whenever possible." (Spring, 2021) Another PT explained their thinking by saying, "Prior to this course I hadn't had much experience thinking about students' funds of knowledge can be implemented into a math lesson. I have learned that it is extremely important to make content relevant for students so that they feel connected to it and thus more interested in it." (Spring, 2021) The PTs found the idea of CRMT useful for their students. In addition, PTs' initial concerns were allayed with support from MTE as well as their peers. Through peer and professor feedback, cycles of revision and many discussions, the PTs became less apprehensive about CRMT. One PT shared, "Developing lessons based on culturally relevant pedagogy is extremely important, because it allows all students to feel valued and represented." (Spring, 2020)

CRMT was perceived as meaningful for the PTs. In addition to being beneficial for their students many PTs shared finding the process of incorporating context into mathematics lessons exciting. For example one PT shared, "I really liked doing this lesson. It was cool to incorporate something that is a big part of my own homelife culture into a lesson … I got feedback saying that they thought the sign language was a cool touch."(Spring, 2020) Even though the goal was to learn about their students and find contexts relevant to their fictional students, the PTs found it rewarding to connect their own passion to mathematics. A PT commented, "This topic is very close to me and I wish I could have focused more on mental health in teenagers." (Fall, 2020) The PTs started thinking about what contexts were important to them as well as their students.

Discussion and Implications

The experience with the micro-teaching activity allowed the PTs to broaden their perception of teaching and learning of mathematics. It was the first time each PT had heard about culturally responsive mathematics teaching. The experience of learning ways to develop culturally responsive mathematics lesson plans allowed them to unpack their own beliefs about being a mathematics teacher. Many of the PTs were uncomfortable with the idea of social justice in a mathematics classrooms and needed support to unpack how the two were connected. Using the CRMT tool to examine existing lesson plans in class, class discussions, peer feedback on lesson plans and having access to social justice standards all proved to be sources of support.

Through this ongoing action research project I am learning about the actionable steps MTEs can take to develop supports for their PTs. In particular MTEs can help their PTs consider ways of incorporating social justice contexts into teaching and learning mathematics. Allowing PTs to express their concerns about discussing difficult issues in a mathematics classroom can be a first step in allaying those concerns. MTEs must provide tools and examples to show how PTs can develop CRMT but more importantly MTEs must support their PTs. It's a challenging task to help PTs change their perception about mathematics as more than just a stand-alone subject that is devoid of context. This work requires the MTEs to have patience with PTs in order to support their growth.

Conclusion

Findings indicate that MTEs need to be intentional in their course design to highlight the connections between mathematics and social justice. With guidance, future teachers can begin to see how to design classroom experiences to support mathematical knowledge that underpins social issues. This work addresses the need to develop effective mathematics teachers for diverse learners (Civil, 2007; Leonard, 2008; Civil, 2002; González et al. 2001).

References

Aguirre, J. (2009). Privileging mathematics and equity in teacher education: Framework, counter- resistance strategies and reflections from a Latina mathematics educator. In B. Greer, S. Mukhopadhyay, S. Nelson-Barber, & A. Powell (Eds.), *Culturally responsive mathematics education* (pp. 295–319). New York, NY: Routledge.

Aguirre, J. M., & del Rosario Zavala, M. (2013). Making culturally responsive mathematics teaching explicit: A lesson analysis tool. *Pedagogies: An international journal, 8*(2), 163-190. DOI: 10.1080/1554480X.2013.768518

Berry III, R. Q., Conway IV, B. M., Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice.* Corwin Press.

Braun, V., & Clarke, V. (2019). Reflecting on reflexive thematic analysis. *Qualitative Research in Sport, Exercise and Health, 11*(4), 589-597.

Christiansen, I. (2008). Some tensions in mathematics education for democracy. In B. Sriraman (Ed.), *International perspectives on social justice in mathematics education* (pp. 69–86). Charlotte, NC: Information Age Publishing.

Civil, M. (2002). Culture and mathematics: A community approach. *Journal of Intercultural Studies*, *23*(2), 133-148.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nasir and P. Cobb (Eds.) *Improving access to mathematics: Diversity and equity in the classroom*. (pp. 105-117). New York, NY: Teachers College Press.

Cochran-Smith, M., Davis, D., & Fries, K. (2004). Multicultural teacher education: Research, prac- tice, and policy. In J. A. Banks & C. A. M. Banks (Eds.), *Handbook of research on multicultural education* (2nd ed., pp. 931–978). San Francisco, CA: Jossey-Bass.

Gay, G. (2009). Preparing culturally responsive mathematics teachers. In B. Greer, S. Mukhopadhyay, S. Nelson-Barber, & A. Powell (Eds.), *Culturally responsive mathematics education* (pp. 189–206). New York, NY: Routledge.

González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for students placed at risk*, *6*(1-2), 115-132.

Gutiérrez, R. (2009). Helping students to play the game and change the game. *Teaching for Equity and Excellence in Mathematics, 1*(1), 4–8.

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice.* New York, NY: Routledge.

Johnson, A. P. (2005). *A short guide to action research* . Pearson Education.

Kemmis, S. , & McTaggart, R. (2000). Participatory action research. In N.Denzin, & Y.Lincoln (Eds.), *Handbook of qualitative research* (pp. 567–605). Sage.

Leonard, J. (2008). *Culturally Specific Pedagogy in the Mathematics Classroom: Strategies for Teachers and Students*. New York: Routledge.

Mills, G. E. (2003). *Action research: A guide for the teacher researcher.* Merrill/PrenticeHall.

Kitchen, R. S. (2005). Making equity and multiculturalism explicit to transform mathematics edu- cation. In A. J. Rodriguez & R. S. Kitchen (Eds.), *Preparing mathematics and science teachers for diverse classrooms: Promising strategies for transformative pedagogy* (pp. 33–60). Mahwah, NJ: Lawrence Erlbaum Associates.

Kitchen, R. S., & Lear, J. M. (2000). Mathematizing Barbie: Using measurement as a means for girls to analyze their sense of body image. In W. Secada (Ed.), *Changing the faces of mathematics: Perspectives on multiculturalism and gender equity.* Reston, VA: National Council of Teachers of Mathematics.

Leonard, J., Napp, C., & Adeleke, S. (2009). The complexities of culturally relevant pedagogy: A case of two secondary mathematics teachers and their ESOL students. *The High School Journal, 93*(1), 3–21.

Moll, L., & González, N. (2004). Engaging life: A funds-of-knowledge approach to multicultural education. In J. A. Banks & C. A. Banks (Eds.), *Handbook of research on multicultural education* (2nd ed., pp. 699–715). San Francisco, CA: Jossey-Bass.

Mukhopadhyay, S., & Greer, B. (2008). How many deaths? Education for statistical empathy. In B. Sriraman (Ed.), *International perspectives on social justice in mathematics education*. Charlotte, NC: Information Age Publishing.

Nasir, N. S., Hand, V., & Taylor, E. V. (2008). Culture and mathematics in school: Boundaries between "cultural" and "domain" knowledge in the mathematics classroom and beyond. *Review of Research in Education, 32*(1), 187.

O'Brien, R. (2001). An overview of methodological approach of action research. In R.Richardson (Ed.), *Theory and* *practice of action research* . João Pessoa, Brazil: Universidade Federal da Paraíba (English version). Available online. Accessed on September 09, 2020.

Peterson, B. (2005). Understanding large numbers. In E. Gutstein & B. Peterson (Eds.), *Rethinking mathematics: Teaching social justice by the numbers* (pp. 76–79). Milwaukee, WI: Rethinking Schools.

Teaching Tolerance. (2016). *Social justice standards: The teaching tolerance anti-bias framework.* Retrieved from https://www.learningforjustice.org/frameworks/social-justice-standards.

Turner, E. E., Drake, C., Roth McDuffie, A., Aguirre, J. M., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematics knowledge bases. *Journal of Mathematics Teacher Education*, *15*(1), 67–82. doi: 10.1007/s10857-011-9196-6

Varley-Gutierrez, M. (2011). "I thought this U.S. place was supposed to be about freedom": Young Latinas engage in mathematics and social change to save their school. *Rethinking Schools, 24*(2), 36–39.

Willis, J. W. , & Edwards, C. (eds.). (2014). *Action research: Models, methods, and examples*. Information Age Publising.

THE SELF AS NOTICER: SUPPORTING PRESERVICE TEACHERS' DEVELOPING AWARENESS

Ethan Rubin
ebrubin@uci.edu

Elizabeth A. van Es
evanes@uci.edu

Gaining insight into how one's noticing shapes decision making can enable a teacher to reflect on how they frame, interpret, and respond to classroom activity and disrupt the influence of dominant ideologies. Working in the context of teacher education, we conjectured that systematically analyzing and reflecting on their own noticing can enable preservice teachers (PSTs) in mathematics to develop more equitable practices. Using data from summative assignments in a course on advancing equitable teaching, we investigate how PSTs use lenses of equitable teaching to make sense of their noticing and develop conceptions of equity. Analysis reveals that PSTs engaged in meaningful reflection and adopted terms from the course but avoided discussing the sociopolitical dimensions of instruction. These findings have implications for course design and facilitation in the context of developing PSTs' noticing for equity.

Keywords: Preservice teacher education, instructional vision, social justice

Objectives of the study

Teacher noticing, which encompasses how teachers attend to and interpret details of classroom activity, has been recognized as an important construct for equitable teaching practice (Schack et al, 2017; Scheiner, 2021). How teachers notice interactions is situated within dominant narratives from the history of mathematics education as a field (Sherin et al, 2011). Especially in Western education, mathematics has been characterized as an objective subject without social or political implications, despite a complex legacy of racialized, gendered, and otherwise biased structures and institutions (Gutierrez, 2018). Through that framing, math education continues to be a site for reproducing dominant narratives about what counts as "smart," who is capable of mathematics, and related ideologies that undermine efforts toward equitable education (Louie, 2017). One approach to disrupting those ideologies is for teachers to develop an awareness of how their noticing is shaped by broader sociopolitical systems and their personal histories within those systems. As the influence of dominant narratives becomes more visible, teachers are able to recognize biases and blind spots in their noticing, empowering them to reframe their conceptions of classroom activity and cultivate instructional practices that support equity (Mendoza et al, 2021). Gaining insight into oneself as a noticer and understanding how noticing shapes moment-by-moment decision making can therefore enable a teacher to reflect on how they frame, interpret, and respond to classroom activity and disrupt the influence of dominant ideologies (Louie et al., 2021; Patterson Williams et al, 2020).

Working in the context of teacher education, we conjectured that developing insight into oneself as a noticer can enable preservice teachers (PSTs) to develop more equitable practices. We enter this conversation by investigating pedagogies for supporting preservice teachers in the process of gaining insight into their noticing practices. In response to the constraints imposed on education by the COVID-19 pandemic, we adapted a course for PSTs focused on learning to systematically notice and analyze teaching practice (see Sherin et al, 2009). The course centered on developing "lenses" for noticing and advancing equitable teaching, including attending to student thinking, discourse, positioning, and identity. We structured the course around a series of noticing tasks that entailed using those lenses to analyze how noticing, instruction, and commitments come to life in moment-to-moment interactions. As a summative assignment, the PSTs reflected on and analyzed those noticing tasks in order to represent their noticing as a system informed by the lenses the course had introduced. In this study, we ask: how do preservice teachers use lenses of equitable teaching to make sense of their noticing practices? What do their self-analyses reveal about their emerging conceptions of equitable instruction?

Theoretical framework

Noticing is especially central to teaching for equity, as noticing is shaped and constrained by ideologies that frame what teachers deem worthy of attention and how they interpret those details (Louie, 2017). Noticing occurs continuously as an active and subjective process, whether or not the teacher in question is conscious of attending to certain phenomena over others or interpreting details in particular ways. Becoming aware of one's noticing, however, enables a teacher to understand, question, broaden, and disrupt their noticing practices (Erickson, 2011; Mason, 2009). Similarly, ideologies are shaping noticing whether a teacher is aware of them or not; awareness helps one see what ideologies are shaping noticing and how, i.e. are they supports or barriers to advancing equity in teaching.

The concept of frames is of particular importance for unpacking the relationship between ideologies and noticing. Hand (2012) describes frames as structures that establish "expectations for how the emerging activity should unfold and for the roles that different individuals will take within it" (p. 251). Louie (2021) builds on this understanding, characterizing frames as narratives that shape both *what* and *how* we notice, and which "take on authority as they are told and retold [to] influence their tellers' and others' subsequent framing" (p. 3). Frames are both internal structures that teachers rely on to make sense of classroom activity *and* modes of representation and communication with students about the content and context of learning in the discipline. Frames that are based in dominant ideologies, e.g., white racial knowledge or race denialism ("colorblindness"), can perpetuate those ideologies (Reisman et al, 2020; Bonilla-Silva, 2006). On the other hand, frames based in challenging inequitable systems and rehumanizing learning can disrupt dominant ideologies (Gutierrez, 2018; Louie, 2017; McKinney de Royston et al, 2021).

The relationships between ideologies, frames, and noticing have prompted attention to how teachers' self-awareness in noticing develops. For instance, Patterson Williams et al (2020) offer a model of teacher noticing that centers on cultivating an "inner witness" by attending to how equity unfolds in "micro-moments within fleeting classroom discourse" (p. 505). Mason (2009) parses the development of teacher awareness into phases of preparation (e.g. structuring and planning), paration (the enactment of teaching), and postparation (e.g., reflection and interpretation). Postparation practices then provide the basis for preparation in continuing instruction. Philip (2019) describes practices for gaining insight into one's teaching through narrating, re-narrating, and re-envisioning episodes of teaching. Across these frameworks, authors highlight the importance of reflecting on past practice to improve future practice in iterative cycles. In the context of teacher education, this insight indicates pedagogies for supporting the development of self-awareness with pre-service teachers. In the course design that led to this study, we drew on these frameworks to engage PSTs in representing, reviewing, and reinterpreting their noticing as the manifestation of commitments, beliefs, and ideologies they consciously and unconsciously hold.

Research methods

Study context

This study was situated in a required course for a combined Master of Arts in Teaching and teaching credential program that took place in the fall of 2020. The course, titled "Learning to Learn from Teaching," engaged preservice teachers in analyzing representations of teaching practice (e.g., videos and transcripts) to develop their noticing practices and their *awareness* of their noticing. Through course readings, discussions, and assignments, we provided PSTs with multiple lenses that together constitute a model of responsive and equitable teaching. PSTs were directed to use the lenses to analyze and reflect on representations of teaching in a series of noticing tasks assigned throughout the course. Their observations and reflections were recorded in a noticing journal, which documented PSTs' noticing in a form they could review and reference. Examples of lenses and associated readings are listed in Table 1.

Table 1: Lenses and readings from the course

Lens	Summary	Readings
Understanding students' identities	Knowing, doing, and being are intertwined aspects of learning. Students' identities, practices, and knowledge are all continuously expressed and developing in the classroom.	Herrenkohl & Mertl, 2010
Student thinking (strengths-based perspective)	"Smartness" in math has been narrowly defined in terms of speed, accuracy, and "right" answers. Teachers can challenge this ideology by focusing on students' strengths instead of deficits and moving beyond the right/wrong binary to understand and appreciate student thinking.	Louie, 2017; Jilk, 2016
Discourse and accountable talk	Classroom discussions can be understood through three forms of accountability: to knowledge, to standards of reasoning, and to the learning community. Rich classroom discourse involves students collaborating to navigate the process of sensemaking and guide discussion through questioning, reasoning, and mutual support.	Hufferd-Ackles et al, 2004; Michaels et al, 2008

Prior iterations of the course culminated in PSTs planning, enacting, recording, and analyzing a lesson in their student teaching placements. Due to the COVID-19 pandemic, however, PSTs were not in classrooms and therefore could not produce and analyze representations of their own teaching. As an alternative, we engaged PSTs in a method of self-analysis using their interpretations of videos and transcripts in the noticing journals as data. In prior work, we developed an analytic method to characterize the relationship between teachers' commitments, noticing practices, and instructional practices (see Authors, 2022), which we refer to as a system of noticing. We conjectured that the process of constructing a personal system of noticing through reflection and analysis of noticing tasks would support PSTs developing the kind of awareness that would support equitable practice in teaching.

Developing a system of noticing first required PSTs to review their responses to noticing tasks and make distinctions between their observations and their inferences. PSTs identified patterns in what they tend to notice (observations) and how they interpreted those details (inferences). They then grouped their patterns of observations and the associated inferences into clusters that shared a noticing lens. Figure 1 provides an example of a cluster of observations and inferences around a noticing lens.

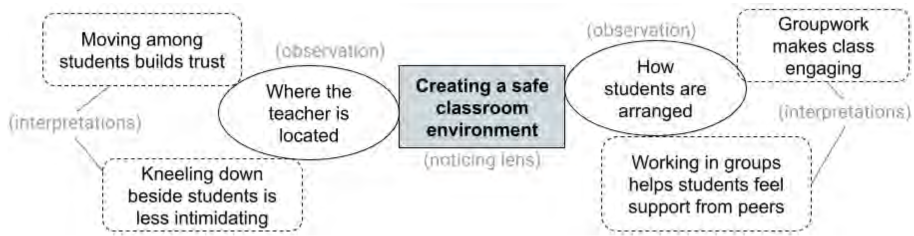


Figure 1. Example of a system of noticing

Through this process, we conjectured that PSTs would develop the ability to distinguish between an objective observation and an interpretation, thereby revealing the influence of commitments, ideologies, and dominant narratives on how they interpret what they see. Grouping the observations and inferences and naming that cluster as a lens then helps preservice teachers identify the specific narratives, ideologies, and commitments at play. By making these aspects of noticing visible, we intended to position preservice teachers to consider how their noticing may or may not support equity and determine aspects of their noticing they want to refine, expand, or disrupt. In addition to creating a visual representation of their system of noticing, the preservice teachers wrote a paper explaining how they developed and interpreted their system of noticing using their noticing task responses and the literature from the class. The paper consisted of three parts: 1, explain the noticing lenses using evidence from the noticing tasks; 2, analyze the lenses using the course readings; and 3, identify areas of noticing to expand or disrupt and describe a plan for doing so.

Data analysis

Of the 11 mathematics candidates enrolled in the course, five consented to participate in this study. Data consists of their summative system of noticing assignments, which include a visual representation of the system of noticing and a reflection paper explaining the data from their noticing tasks and the concepts from course readings that they used to develop that representation. We employed iterative qualitative methods (Miles & Huberman, 1994) to examine the ways that PSTs took up lenses of equitable and responsive teaching and the conceptions of equitable teaching their work revealed. First, both authors read through, made margin comments, and wrote memos about each paper. The authors then discussed their comments and memos, identifying categories and themes that stood out across their impressions. These themes concerned which lenses were raised by the PSTs, what kinds of explanations they provided to justify prioritizing those lenses, and how their explanations related to the course materials and goals. Following these discussions, the authors returned to the data to conduct rounds of focused coding (Saldaña, 2011) around three areas: how terms were used (e.g. definitions and examples), critical statements (e.g. “students often feel vulnerable”, “math is seen as irrelevant,”) and normative or values statements (e.g. “teachers should do X”, “I want to create a classroom that is Y”). Finally, the authors reviewed the readings that PSTs cited most to consider how student definitions aligned with the definitions of concepts in the course materials and the extent to which their critical and normative stances were consistent with the aims of the course.

Results

Analysis revealed two main findings. First, PSTs appear to have taken up the fundamental aims and practices of the course by trying on lenses to re-envision and re-imagine schooling in light of their commitments. Second, PSTs applied language from the course readings and discussions, but often defined those terms in partial or adapted forms that were more compatible with their pre-existing frames. These redefinitions showed a tendency to avoid the broader systemic issues and ideologies that influence classroom activity.

Finding 1: Interpreting noticing through beliefs and commitments

Although this portion of the assignment asked PSTs to describe what they noticed and inferred in the noticing tasks, all five PSTs in this study consistently made normative or aspirational statements about teaching. Student 1, for instance, made the arguments that "students should develop ideas, have authority in the classroom, and be responsible for their knowledge" and "it is crucial to create a safe, supporting, and inclusive place…students should have the opportunity to practice their identity, individuality, and whom they are." Student 2 stated that "we need to value students for who they are and acknowledge their differences" and Student 5 made claims like "students should lead the direction of the discussion" and "teachers need to learn how to really listen to their students." In a similar vein, Student 4 linked their understandings of key concepts to value statements: "Tools are shared through communicating with one another, thus, communication is a practice that must be developed to promote learning"; "Students develop their identity through interactions, thus participation is key in providing students with an opportunity to develop their identity"; "Knowing, doing, and being are all intertwined, hence it is important to study all aspects." These claims about the purposes of teaching indicate that these PSTs were articulating their beliefs and commitments about how teaching *should* be and understanding those commitments as the basis for their noticing.

In several cases, PSTs expressed normative or aspirational claims by pairing critiques of schooling with alternatives. Student 1 articulated these dyads in each section of their paper, starting with: "in many cases, the teacher is considered as the only authority in the classroom…I believe in a student centered classroom." In their next argument, they state, "students cannot see the connection between what they learn as mathematics and how it is used…students must become aware of…the applications of the subject." Finally, they point out that "all students come from different backgrounds and stories and we should not expect them to adjust their identities to the dominant culture. Students should be appreciated as individuals with unique thoughts, beliefs, values, and identities." Student 4 used the same format, at one point unpacking a specific observation by explaining that "this student might have felt left out…teachers must not turn away different representations of participation because it may hinder students' opportunities to learn. Thus, students must not only be encouraged to participate but more importantly must be offered assistance to participate." Later in their paper, Student 4 presses on assumptions about the nature of doing mathematics, noting that "students have been conditioned to believe that if they are fast and accurate, then they are smart. However, my noticing lens disrupts this narrative because my… noticing lenses value participation and use participation to redefine smartness."

Students 2, 3, and 5 reverse these dyads, with Student 2 saying, "I look at the positive interactions or ideas…teachers may focus too much on what students are doing wrong," and Student 3 expressing that they pay more attention to student-led discussions because they have observed classrooms in which "there is not a lot of room for discussion, just lectures and note taking." Student 3 also provides a more personal example, saying, "I remember being in the classroom and not wanting to provide my opinions because I was afraid of being wrong and other students judging me for it. So, when I observe students, I look to make sure that everyone feels comfortable being wrong and learning from it." Student 5 makes a broader claim about their beliefs on the nature of learning: "students should be given the opportunity to construct knowledge for themselves. If students are constantly spoon-fed knowledge without a chance to build it for themselves, we position them as incompetent learners that need someone to build it for them." Across these examples, we see PSTs using their normative commitments as frames to interpret classroom activity. Instructional practices they observed, interpersonal dynamics, and messages implied by teacher actions were interpreted in relation to beliefs about what is valuable in teaching and learning, which in turn pointed PSTs to alternatives they conceptualized as solutions to persistent issues in schooling.

Finding 2: Redefining terms from the course to fit with pre-existing frames

PSTs used terms from the course, but often adapted them to fit with their pre-existing frames. This was most visible when PSTs explicitly provided definitions and illustrations that were partial or selective versions of the concepts introduced in class. Student 2, for instance, begins their paper by defining "noticing" as "a critical and analytical tool that…pertains to student attainment, cognition, and thinking, as well as the educational environments where such facets take place." The framing emphasized in the course, however, sought to define noticing as a typically unconscious manifestation of a teacher's identity, their personal and social history, and the systems in which they work, rather than as an analytical tool that teachers can choose to aim at students. Student 2's focus on attainment and cognition appears again when they use the term "accountability" to mean "where students are held accountable in understanding certain content material…[and] held accountable in finding answers, persevering if struggling." This conception of accountability indicates a partial understanding of "accountable talk" from the course readings, as it omits Michaels et al's (2008) emphasis on the social dimension of accountability to a learning community. The idea of "perseverance through struggle", on the other hand, does *not* appear in the course materials. Student 3 applied a similar partial understanding to interpret a video shown during class, writing that "during the discussion, some students were right while others were wrong, but there was never any judgment in the room." In the course, however, the rationale for showing that video was to demonstrate a teacher rejecting the frame of the "right/wrong binary", valuing student sensemaking *instead* of thinking in terms of students being right or wrong.

The tendency toward partial or selective understanding is further illustrated by what PSTs did *not* talk about when explaining or analyzing a given concept. For instance, four of the five PSTs cited Philip et al (2016) as evidence that they attended to power and positioning. The reading, whose title refers to "becoming racially literate" and "racial-ideological micro-contestations" (p. 361), unpacks a classroom interaction in terms of the teacher's avoidance of talk about race, which tacitly enables racial antagonism between students and excludes a Black student by delegitimizing his insights about content that involves Black communities. Student 5 argued that the problems in that situation arose because the teacher "positioned himself as the authority figure in the room by acting as the 'gatekeeper of knowledge'", in accord with a claim that asking the question "was the teacher acting as the 'gatekeeper of knowledge'...allows us to analyze who has the power in the classroom." While questions of authority and gatekeeping are relevant to this example, the absence of race, identity, and power dynamics beyond the interpersonal scale in this account is conspicuous. Likewise, Student 3 diagnosed the problems in the Philip et al reading as "[the student] being disregarded so much and constantly defending his opinion to the point where he just stopped." Again, this analysis brings up relevant points, but there is scant mention of race despite the authors' insistence that race, and particularly the teacher's refusal or inability to *discuss* race, is crucial to understanding the example. Student 1, in contrast, did mention race in the context of this example, but they interpreted it in colorblind terms: "we ostracize an individual due to simply the color of their skin, but at the end of the day, we are all human, so why do these discrepancies exist?" This response conflicts with the authors' argument that a student's race entails legitimate differences in experience and perspective, which can afford insights that should be recognized.

With regard to Philip et al (2016) and several other examples, PSTs showed a tendency to omit the sociopolitical aspects of a concept or situation by choosing to focus on smaller scale interpersonal perspectives. All five PSTs prioritized some version of students feeling safe, welcome, and included, and four made claims about attending to positioning. In their explanations and examples, they focused primarily on how teachers' actions affected students. Student 4, for example, stated that "teachers can create an inclusive classroom by asking students questions and inviting students to participate." Student 5 claims, "I very much focus on how students are positioned in the classroom. For example…where the knowledge is coming from contributes to the power dynamic between teacher and student." Their understanding of positioning related to how students were positioned *within* the classroom through interactions, without engaging with how students were already positioned *coming into* the classroom based on their identities and histories. Likewise, PSTs' accounts of students feeling safe/unsafe or welcome/unwelcome focused on how teachers' actions could *cause* a student to feel that way, rather than considering how certain students may not feel safe or welcome within school institutions even before a teacher acts.

Discussion

Findings suggest that PSTs took up the *practices* of developing self-awareness that were facilitated by the course design, although their responses did not necessarily capture the details of the content. Even when PSTs brought up different ideas than were discussed in the course, their practices of surfacing assumptions and interpretations, critiquing the status quo, and reimagining instruction were consistent with the course aims. The prevalence of normative and aspirational statements indicates that PSTs were examining their noticing in the context of their beliefs and commitments, which complements the course focus on revealing the power of ideologies and frames. Through reflection and analysis of representations of their own practices, PSTs were beginning to see the connections between a teacher's ideologies and commitments, what they notice, and the actions they can take to embody a vision of responsive and equitable teaching. This is consistent with other research on developing PSTs awareness (e.g., Philip, 2019). While the conceptions they applied within those practices were at times partial or misaligned with the course content, it is important to recall that these were *emerging* conceptions; many of the ideas and frameworks in the course were entirely new for the PSTs. Research on developing frames for equitable teaching speaks to the difficulty of changing frames, even with experienced educators who participate in extended professional development (Louie, 2017; Reisman et al, 2020). For PSTs, this process is additionally complicated because they lack the frame of reference on which experienced teachers rely. Without their own experiences of the classroom context to draw on, pre-service teachers often struggle with what details are noteworthy in a case and how to interpret those details (Darling-Hammond & Hammerness, 2002; van Es et al, 2017). Given the novelty of the course material, the fact that PSTs were applying relevant practices in efforts to make sense of their noticing in terms of their commitments indicates they began to adopt the perspectives intended by the course.

It is noteworthy, however, that the areas in which PSTs' responses departed most from the course aims revolved around sociopolitical awareness. Despite efforts throughout the course to center and unpack the effects of social histories and systems on instruction, the PSTs rarely mentioned the sociopolitical dimension. As a counterpoint, McKinney de Royston et al (2021) consider how Black educators enact frames of care and protection towards Black students and find that their frames are characterized by "clarity about the historical and political landscape" that illuminates "racialized disparities and experiences in schools…as by-products of systemic racism" (p. 73). This clarity is expressed in "politicized caring", which motivates and empowers the teachers and school leaders to disrupt harmful narratives and structures in the name of making students feel safe and cared for. By contrast, the PSTs here bring in a frame of *apolitical* caring. In these reflections, PSTs emphasize creating a sense of safety and students feeling welcome or at home, but without addressing the broader social and systemic issues that affect students' experiences of schooling. This pattern was visible even in PSTs' responses to representations of explicitly racialized situations, as in the Philip et al (2016) example. This fits with a more generalized tendency to view the classroom as a neutral space until it is shaped by teacher actions. Treating the classroom as a neutral space (rather than one inherently pre-shaped by sociopolitical factors) frames creating a sense of student safety as a question of *not* doing things to make them feel *unsafe* as individuals within classroom interactions, rather than disrupting or counteracting broader sociopolitical narratives. The absence of a sociopolitical frame, even under the guise of neutrality or valuing interpersonal relationships, ultimately reproduces a colorblind ideology that perpetuates harmful narratives and delegitimizes the experiences of students from historically non-dominant communities (Bonilla-Silva, 2016). These findings therefore raise questions regarding what kinds of pedagogies, artifacts, and supports are needed for PSTs to cultivate systemic *and* interpersonal perspectives as compatible and complementary frames (Weis & Fine, 2012).

Limitations and future research

The systems of noticing that PSTs developed and the reflections they analyzed to do so were produced by engaging with a set of artifacts, representations, and readings that were selected by the authors. As teacher educators, we were not neutral or objective in the selection of these materials. Rather, we attempted to curate a set of materials that support a vision of equitable noticing and instruction that we hoped to cultivate with this group of PSTs. It is likely that engaging with different representations or being informed by different literature would reveal other tendencies in PSTs' noticing, ideologies, and commitments. The study is also limited by its small size, both in terms of the number of PSTs involved and by virtue of being a single instance of the course. This study represents only one iteration of guiding this process of development, and findings reveal both areas of success and possibilities for improvement. Further research is needed to consider the implications of these findings for course design and facilitation in the context of developing noticing for equity and social justice. Moreover, a single course is clearly insufficient for deep and lasting change given the complexity of noticing and the persistent nature of dominant ideologies and narratives. Future research will also address how this process can be supported and sustained in other PST courses and in professional development for early career teachers.

References

Bonilla-Silva, E., (2006). *Racism without racists: Color-blind racism and the persistence of racial inequality in the United States*. Lanham, Maryland: Rowman & Littlefield Publishers, inc.

Darling-Hammond, L. & Hammerness, K. (2002) Toward a pedagogy of cases in teacher education. *Teaching Education, 13*(2), 125-135. doi: 10.1080/1047621022000007549

Erickson, F. (2011). On noticing teacher noticing. In M. G. Sherin, R. Phillip, & V. R. Jacobs (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 17-34). Routledge.

Gutiérrez, R. (2018). The need to rehumanize mathematics. In Goffney, I., Gutiérrez, R. & Boston, M. (Eds). *Rehumanizing mathematics for Black, Indigenous, and Latinx Students: Annual perspectives in mathematics education* (pp. 1-10). National Council of Teachers of Mathematics: Reston, VA.

Hand, V. (2012). Seeing culture and power in mathematical learning: Toward a model of equitable instruction. *Educational Studies in Mathematics, 80*, 233-247

Herrenkohl, L. R., & Mertl, V. (2010). *How students come to be, know, and do: A case for a broad view of learning*. New York: Cambridge University Press. Introduction, pp. 1-11, 15-17.

Hufferd-Ackles, K., Fuson, K.C., & Sherin, M.G. (2004). Describing levels of components of a math talk learning community. Journal of Research in Mathematics Education, 35(2), 81-116.

Jilk, L. (2016). Supporting teacher noticing of students' mathematical strengths. *Mathematics Teacher Educator, 4*(2), 188-199.

Louie, N. (2017). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics, 97*, 55–69. doi: 10.1007/s10649-017-9775-2

Louie, N., Adiredja, A.P. & Jessup, N. (2021). Teacher noticing from a sociopolitical perspective: the FAIR framework for anti-deficit noticing. *ZDM Mathematics Education* 53, 95–107. doi: 10.1007/s11858-021-01229-2

Mason, J. (2009). Teaching as disciplined enquiry. *Teachers and Teaching: Theory and Practice, 15*(2), 205–223. doi: 10.1080/13540600902875308

McKinney de Royston, M., Madkins, T.C., Givens, J.R., & Nasir, N.S. (2021). "I'm a teacher, I'm gonna always protect you": Understanding Black educators' protection of Black children. *American Educational Research Journal. 58*(1), 68-106. doi: 10.3102/0002831220921119

Mendoza, E., Hand, V., van Es, E., Hoos, S., & Frierson, M. (2021). 'The ability to lay yourself bare': centering rupture, inherited conversations, and vulnerability in professional development. *Professional Development in Education, 47*(2-3), 243-256. doi: 10.1080/19415257.2021.1891955

Michaels, S., O'Connor, C. & Resnick, L.B. (2008). Deliberative discourse idealized and realized: Accountable talk in the classroom and in civic life. *Studies in Philosophy and Education 27*, 283–297. doi: 10.1007/s11217-007-9071-1

Miles, M. and Huberman, A.M., 1994. *Qualitative data analysis*. 2nd ed. Thousand Oaks, CA: Sage.

Patterson Williams, A., Athanases, S., Higgs, J., & Martinez, D. (2020). Developing an inner witness to notice for equity in the fleeting moments of talk for content learning. *Equity & Excellence in Education, 53*(4), 504-517. doi: 10.1080/10665684.2020.1791282

Philip, T. M., Olivares-Pasillas, M. C., and Rocha, J. (2016). Becoming racially literate about data and data-literate about race: Data visualizations in the classroom as a site of racial-ideological micro-contestations. *Cognition & Instruction, 34*(4), 370-379

Philip, T. (2019). Principled improvisation to support novice teacher learning. *Teachers College Record, 121*(4), 1-32.

Reisman, A., Enumah, L., & Jay, L. (2020) Interpretive frames for responding to racially stressful moments in history discussions. *Theory & Research in Social Education, 48*(3), 321-345. doi: 10.1080/00933104.2020.1718569

Saldaña, J. (2011). Fundamentals of qualitative research: Understanding qualitative research. Oxford University Press.

Schack, E., Fisher, M., Wilhelm, J. (Eds.) (2017). *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks*. Springer.

Scheiner, T. (2021). Towards a more comprehensive model of teacher noticing. *ZDM Mathematics Education, 53*, 85–94. doi: 10.1007/s11858-020-01202-5.

Sherin, M.G., Jacobs, V.R., and Philipp, R.A., Eds. (2011). *Mathematics teacher noticing: seeing through teachers' eyes*. New York, NY: Routledge.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting Video Clips to Promote Mathematics Teachers' Discussion of Student Thinking. *Journal of Teacher Education, 60*(3), 213–230. doi: 10.1177/0022487109336967 van Es, E., Hand, V., Agarwal, P., & Sandoval, C. (2022). Multidimensional noticing for equity: Theorizing mathematics teachers' systems of noticing to disrupt inequities. *Journal for Research in Mathematics Education*, *53*(2), 114-132. doi: 10.5951/jresematheduc-2019-0018

van Es, E., Cashen, M., Barnhart, T., & Auger, A. (2017) Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy. *Cognition and Instruction, 35*(3), 165-187. doi: 10.1080/07370008.2017.1317125

Weis, L., & Fine, M. (2012). Critically bifocality and circuits of privilege: Expanding critical ethnographic theory and design. *Harvard Educational Review, 82*(2), 173–201. doi: 10.17763/haer.82.2.v1jx34n441532242

FORMATIVELY ASSESSING NOVICES' CAPABILITIES WITH MODELING CONTENT

Meghan Shaughnessy
Boston University
mshaugh@bu.edu

Nicole Garcia
University of Michigan
nmgarcia@umich.edu

This paper examines the possibilities of designing a formative assessment that gathers information about novice elementary teachers' skills with modeling content and makes sense of such information. A decomposition of the practice of modeling content was developed and used to design the assessment. Participants included ten first-year teachers who graduated from a range of different teacher education programs. The findings reveal that our formative assessment works to gather information about teachers' capabilities with modeling content and that the associated tools support making sense of the information gathered.

Keywords: Assessment; Instructional Activities and Practices; Preservice Teacher Education

Ms. Hazard is a third-grade teacher who is currently teaching her students how to name shaded parts of areas as fractions using the definition of a fraction in the Common Core State Standards (CCSSO, 2010). She knows that it is crucial that students grapple with the importance of making equal parts, naming one of the equal-sized parts, and then counting the number of shaded parts to name the fractions. In her initial lesson, she presents students with the task shown in Figure 1. There is disagreement about the fraction of the rectangle that is shaded, with students suggesting 1/3, ¼, and 1/2. She holds a rich discussion where children share their thinking and consider the thinking of others, including asking questions of their classmates. The class talks about how some students arrived at ½ because they used part of the rectangle as the whole and other students, who arrived at 1/3 or ¼, used the whole rectangle. They further talk about how some students noticed that one of the parts was bigger and that they added a line to make the parts the same sizes and that this is how some people arrived at 1/3 and others arrived at ¼. At the end of the discussion, it seems that a convincing argument for ¼ has been shared and that the class is tentatively in agreement that the shaded area is ¼ of the large rectangle. Ms. Hazard recognizes that the class needs to consolidate the core ideas shared.

What fraction of the rectangle is shaded gray?

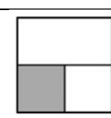


Figure 1: Third Grade Fractions Task

Ms. Hazard concludes by saying to the class, "We had such an important discussion today and I want you to listen carefully as I share how I think about naming this fraction because it connects to lots of ideas that were shared today and can help us think about other fraction problems." Ms. Hazard goes on, "The first thing I ask myself is, 'What is the whole?', and I outline the whole. Our whole is the big rectangle." As she says this, Ms. Hazard outlines the big rectangle with a whiteboard marker. "Then, I ask myself, 'Is the whole divided into equal parts?' Let's look, here the parts in the whole are different sizes. See this part [pointing to the rectangle on top] is bigger than these parts [pointing to the parts on the bottom]. So, I need to make the parts the same size, so I can divide this part [pointing to the rectangle on top] in half to make the parts the same size." Ms. Hazard uses her red marker to split the part in half. Ms. Hazard goes on, "Now I ask myself, 'how many equal parts does it take to make the whole?' I can count one, two, three, four." As Ms. Hazard counts, she points to each part and labels them one, two, three, four. Ms. Hazard goes on, "Then, I ask myself, 'What do you call one of the parts?' We call one of the parts, one out of the number of equal parts. We have four equal parts so we can call one of our equal parts, one-fourth. We have one equal part shaded in our rectangle, so one-fourth of the rectangle is shaded." Ms. Hazard labels the shaded part as one-fourth on the board. Ms. Hazard concludes, "We will continue to work with fractions and it is really important that we ask ourselves four questions. What is the whole? Trace around the whole [while tracing]. Is the whole divided into equal parts? If not, make the parts the same size [traces line with finger]. How many equal parts does it take to make the whole? What do we call one of the equal parts? We can call it one-fourth." Ms. Hazard records these questions on the board as she says them.

Ms. Hazard first engaged students in grappling with mathematical ideas related to naming a shaded part of area as a fraction and concluding that ¼ of the rectangle is shaded. She then provided access to thinking by naming, highlighting, and scaffolding important ideas surfaced by students in ways that support the whole class in doing complex mathematics. Ms. Hazard named the core ideas and provided scaffolding questions that support students in learning what to ask themselves as they approach such mathematical work.

In recent decades, there has been increased attention to several instructional practices that teachers can use to support students in constructing understanding of mathematics, including selecting rich mathematical tasks (i.e., Smith & Stein, 1998) and facilitating discussions of students' work on such tasks, either in small group or whole class (Chapin, O'Connor, & Anderson, 2013; Smith & Stein, 2011; Kazemi & Hintz, 2014). But we argue that these instructional practices are, by themselves, insufficient to enable every student to be successful with complex mathematical work and to experience of joy of mathematics as they can often leave core mathematical ideas "in the ether" rather than supporting students to synthesize and generalize the ideas they have developed individually and collectively.

Fundamentally, modeling content is about intentionally and thoughtfully providing access to content that may otherwise remain hidden to some students. Modeling, which requires the teacher to think aloud while demonstrating a skill, makes visible those practices and processes that happen internally and often remain invisible to learners if not explicitly named, explained, and shared. As we saw in the vignette from Ms. Hazard's classroom, when modeling, a teacher names, highlights and scaffolds the topics and practices in ways that support students in doing complex mathematics without taking over the work for them or compromising students' agency and their opportunity to engage in complex mathematical work.

The concept of "explicitness" in modeling is distinct from "direct instruction," which is a teacher-centered approach for delivery of content instruction (Archer & Hughes, 2011). Explicitness in modeling is not meant to reinforce or recreate the patterns of telling that are often seen in U.S. mathematics classrooms (Stein, Smith, Silver, & Henningsen, 2000), but rather it is aimed at making "access" a reality rather than a goal of collaborative development of mathematical ideas. But this sort of explicitness has often had an uncomfortable place in mathematics education. We argue both that this sort of explicitness is important and that the consideration of when to make content explicit through modeling is crucial for ensuring that all students have opportunities to engage with complex mathematical ideas. The key here is what is made explicit and what is left for students to figure out, and what "making explicit" means for the role of the teachers and students. Yet, partly due to concerns that focusing on modeling content might result in direct instruction, mathematics teacher educators have often backgrounded the work of making complex mathematics accessible through modeling. Further, this practice is complex and difficult and there is much to learn about how to support new teachers in learning to model content in productive ways (Charalambous et al., 2011).

We believe that formative assessment could make a substantial contribution to preparing new teachers to model content to support children's mathematical learning. By formative assessment, we mean assessments used to formulate subsequent learning opportunities (Cizek, 2010). Formative assessment enhances learning by revealing the current state of learners' knowledge, skills, and dispositions and ensuing action that facilitates growth (Black & Wiliam, 1998; Hattie & Timperley, 2007; Shute, 2008). We use the term skills to describe teacher candidates' (TCs) abilities to carry out specific aspects of the work of teaching at a particular moment in time, fully recognizing that TCs' capabilities will grow and change over time. Studies of the development of expertise have found that practice opportunities alone do not sufficiently support TCs to improve. Practice opportunities need to be coupled with structured directive coaching (Ericsson & Pool, 2016), which formative assessment can provide. Thus, formative assessment is a critical component in teacher preparation (AMTE, 2017; Darling-Hammond et al., 2005) and we need additional tools to assess TCs' skills with particular practices and components of teaching (Boerst et al., 2020; Shaughnessy & Boerst, 2018; Shaughnessy, Boerst, & Farmer, 2019; Shaughnessy et al., 2021). We argue that if we develop and refine formative assessments of teaching practice and tools that support teacher educators in providing timely feedback to TCs on their skills, then TCs will develop more robust skills with these teaching practices, impacting student learning. Thus, we sought to investigate whether it would be feasible to design a formative assessment focused on modeling content for use with elementary TCs. Specifically, we investigated whether such an assessment could elicit and reveal detailed aspects of TCs' skills with modeling content.

The Teaching Practice: Modeling Content

Modeling Content

Although the field of mathematics education does not yet have a shared conceptualization or common language for decomposing modeling content, a number of scholars have worked to specify and define instruction related to modeling content in ways that support our conceptualization of this instructional practice, which is distinct from behavioral modeling (demonstrating how to complete a task or behavior). The work of Collins, Brown, and Newman (1989) and Collins, Brown, and Holum (1991) in their development of the cognitive apprenticeship as a way to make thinking visible to students, name modeling as the work of performing a task while making internal processes external. Leinhardt's seminal research on instructional explanations (e.g., Leinhardt, 2010; Leinhardt et al., 1991; Leinhardt & Steele, 2005) has illustrated the complexities of explanations that are accountable to both the discipline of mathematics and pedagogical purposes. They also suggest criteria for such explanations.

Other scholars have worked to decompose the work of modeling content by examining specific types of teacher moves related to explicitness. Selling (2016) offered a decomposition of eight types of teacher moves that made different aspects of mathematical practices explicit in middle and high school mathematics discussions. This included moves like highlighting and naming when students were engaged in particular practices. Furthermore, Selling highlighted how the work of making aspects of mathematics explicit often happens at the end or in the middle of lessons, rather than at the beginning, which is also supported by other empirical research (e.g., Schwartz & Bransford, 1998; Schwartz & Martin, 2004). Further, teachers' choice and use of particular mathematical examples can support, constrain, or even obscure the mathematics that is available for students to learn within a particular domain (Ball et al., 2005; Rowland, 2008) and the choices teachers make about particular mathematical representations influence the nature of students' learning opportunities (Ball, 1993).

Overall, the mathematics education research base supports our conceptualization of modeling content and suggests that this practice is related to student learning (Cohen, 2018). The research also supports our conception of modeling as being appropriate and effective after students have had an opportunity to grapple with the mathematical ideas themselves (Schwartz & Martin, 2004). We recognize that the way that this practice is often enacted does not align with our view of this practice. When Cohen summarized her study of explicit instructional strategies (modeling & strategy instruction) in elementary ELA and mathematics lessons, she reflected, "explicitness need not be directive or prescriptive, though the instruction in this study suggests that it often is" (Cohen, 2018, p. 322). This suggests that, given the tendency for directiveness and prescriptiveness in classrooms, it is crucial that we support teachers in learning how to make context explicit in productive ways that are not directive or prescriptive.

Decomposing the Teaching Practice

To assess teachers' capabilities in modeling content, we drew upon Grossman et al.'s (2009) notion of parsing teaching practice into specific areas of work to create a "decomposition" of the practice. Because we were interested in using decomposition in teacher preparation, we attended to the importance of decomposing in ways that the practice can be taught and learned by TCs (Boerst et al., 2011). We drew upon prior research on the importance of modeling content for student learning and research on moves that teachers make when demonstrating, explaining, and modeling content to develop an unpacking of the work of modeling content, a "decomposition" of the teaching practice. We identified different aspects of what teachers have to do to model content. We acknowledge that a decomposition is a living document (Jacobs & Spangler, 2017; Shaughnessy, Ghousseini, et al., 2019; Shaughnessy et al., 2021). One challenge of decompositions is that while they are meant to provide details about the work of carrying out the practice, they cannot name the complete set of knowledge and skills required to carry out the practice or they will be unwieldy. While we recognize that teaching practice is dependent on mathematical knowledge for teaching and teachers' views of mathematics and children's ideas, the decomposition is focused squarely on teachers' enactment of the teaching practice.

Our decomposition deliberately foregrounds certain aspects of the work of modeling content with students—specifically those that we consider crucial for entry-level teaching. We further considered the aspects of modeling content that are most likely to be accessible to and learnable by novices. We began by identifying specific aspects of leading modeling content and what teachers of any level of experience and expertise do to try to accomplish those. For example, in modeling content, one important goal is to make thinking visible that might otherwise be invisible. This entails specific techniques such as annotating, which refers to adding in ideas necessary to support students' understanding, including clearly articulating what you are doing and why you are doing it. These moves are also foundational for the more complex work experienced teachers do in modeling content. We organize our decomposition for modeling content into six areas of work: (1) planning to model, (3) framing, (3) doing content area work, (5) highlighting core ideas, and (6) making thinking visible by emphasizing thinking and key elements, and (6) using language and representations carefully (see TeachingWorks, 2019 for a list of moves associated with each area of work).

Assessing Teacher Candidates' Skills with Modeling and Explaining Core Content

The design of the assessment was influenced by our view of teaching as interactive and situated in contexts (Cohen et al., 2003; Lampert, 2001). We aimed to design an assessment focused squarely on TCs' ability to model content in real time. Three considerations led us to design a standardized assessment in which TCs video recorded themselves modeling content in response to a prompt without children present. First, skillful modeling is responsive to the ideas and resources that students bring and learning goals for students. A standardized assessment enabled us to design a scenario in which we could specify the prior work that children had done and the learning goals for students. Second, modeling content can be done by the teacher, students, or co-constructed by teachers and students. These approaches differ in how much can be seen about a teacher's skills with modeling. Our standardized assessment created an instance in which we could see TCs doing all of the work rather than handing some of it off to students. Third, because the selection of content matters and we wanted a design that would enable us to see patterns in performances across TCs, a standardized assessment allowed us to standardize the content being modeled and the example and representations used. We purposefully selected content that is core to the upper elementary curriculum (comparing fractions) and focused on the use of a particular representation (the number line) to see how all TCs could model content in this situation. Specifically, we asked TCs to model comparing the fractions 5/4 and 5/6 using the number line. Figure 2 contains the "class background" and instructions. Aware that TCs might have different ideas about the practice of modeling, we included a definition of the work of modeling in the full instructions and student learning goals.

Class Background
Your fourth grade students have been working on identifying and ordering fractions. They are familiar with number lines and understand the importance of creating segments of equal length when partitioning the whole into segments of equal length. Students have worked with common fractions such as halves, thirds, fourths, sixths and eighths. They are comfortable with the relationship between halves, fourths, and eighths. Students in general know that if you divide a unit interval into n equal segments that n/n is one whole.
Instructions
You will model how to use a number line, or two number lines, to accurately place and compare two fractions. You can choose any number line representation(s) that you think best communicates these ideas to fourth grade students. • Begin with unmarked line(s) and model placing the necessary whole numbers. It is recommended that you place the whole numbers far enough apart that you can easily partition the intervals into fourths and sixths. • Show how you determine, with precision, where each of the fractions falls on the line. Specifically, remind students how to partition the unit interval(s) and use that information to locate the fractions on the line. • Explain how to determine which fraction is greater using the number line. You should model the process of using this technique as you would for a group of fourth graders. The goal is to make explicit the process for placing and comparing the fractions 5/4 and 5/6 on the number line.

Figure 2: Assessment Task

We used the decomposition to develop a checklist that would support us in noticing the skills with modeling content demonstrated by the TCs. We focused on five areas of work: *framing*, *doing the content area work*, *highlighting core ideas, making thinking visible by emphasizing thinking and key elements,* and *using language and representations carefully*. We excluded *planning* because we identified the content to be modeled and the example and representations for the TCs. Our checklist focuses on the teacher's use of specific techniques within our decomposition. In other words, techniques are what we look for to see if the practice happens. We could look directly for most of the techniques. However, in some cases, we had to further unpack ideas. For example, within *highlighting core ideas*, we had to identify the core ideas.

Present Study

We used the tool we had developed to investigate whether an assessment could elicit and reveal details in teachers' skills in modeling content. While the assessment was designed to be used with TCs enrolled in a teacher education program, we studied the practice of first-year teachers from a range of teacher education programs because this enabled us to study the use of the assessment in ways that were not constrained to one teacher preparation program.

Methods

The ten participants, all in their first year of teaching, spanned grades 1–5. All of the teachers were teaching in the midwestern USA. Contexts varied across urban, suburban, and rural schools. We recruited a diverse sample of teachers with respect to grade level, school district, and teacher preparation program. This sample was not intended to be representative of all first-year teachers. All teacher names used in subsequent sections are pseudonyms.

Data sources include a video record of each participant modeling in response to the provided prompt. We met with each teacher individually in their classroom after school hours or during a prep period. We gave teachers the assessment instructions and allowed 20 minutes of preparation time. Teachers were aware that we were studying their skills in modeling content. Teachers were told that they had up to 10 minutes to complete the modeling.

Data analysis proceeded in two phases. First, members of the research team independently watched each video and used the checklist to capture attributes of the performance. Second, the team discussed cases where there were differences in observations on the checklist, referencing and refining a codebook as needed to reach consensus.

Findings

Framing

Framing included both launching and closing the modeling. For launching, we found that eight of the 10 teachers made an opening statement that *named what was about to be modeled*. Two teachers *explained the purpose* of what was being modeled. Three teachers *connected the modeling to students' prior learning*, future learning, or background and/or experience. None of the teachers *closed the modeling* by summarizing or revisiting core ideas and results.

Doing Content Area Work

The content area work included four components. Six of the 10 teachers marked whole units (intervals) on the number line. To place the first fraction on the number line, six teachers marked the number line and labeled the fraction. To place the second fraction, eight teachers marked the number line and labeled the fraction. Seven teachers stated which fraction is greater.

Highlighting Core Ideas

With respect to *backgrounding* ideas, we found that only two of the 10 teachers avoided highlighting aspects of the content or task that are distracting or may lead to misconceptions.

This means that eight teachers highlighted aspects of the content or task that were confusing or might lead to misconceptions. For example, Mr. Houston highlighted that he was using 24 cm as the unit length because it was divisible by both 6 and 4.

Three of the teachers *foregrounded* mathematical ideas by using explicit verbal and non-verbal markers to draw students' attention to important aspects of the content. Ms. Wheeler used gestures to emphasize that the value of a fraction on a number line represents a distance from zero by motioning from the zero on the number line to the point representing the fractional value.

Teachers varied in *marking ideas*. We focused on five core mathematical ideas. Two of the 10 teachers elaborated the whole unit before partitioning, five teachers highlighted the meaning of the denominator (with respect to one or both of the fractions), three teachers explained the partitioning using the language of equal parts (or same-sized parts), nine teachers showed how to use the numerator and the parts that had been marked to locate the fractions (with respect to one or both fractions), and five teachers stated how to use the representation to determine the comparison. There was variation in how many ideas each teacher marked. Further, we examined the *sequencing of teachers' marking of the core ideas* and found that only three teachers compared the fractions on a number line in a logical manner.

Making Thinking Visible by Emphasizing Thinking and Key Elements

We examined whether teachers made use of three different sorts of techniques for making thinking visible by emphasizing thinking and key elements: annotating, marking metacognition, and thinking aloud. Two of the ten teachers engaged in *annotating*, adding ideas necessary to support students' understanding, including clearly articulating what you are doing and why you are doing it. For example, Ms. Wheeler emphasized why she was drawing two (stacked) number lines and the importance of lining up the number lines to later support the comparison. One teacher, again Ms. Wheeler, engaged in *thinking aloud* as a means to make thinking visible for students. After Ms. Wheeler marked both fractions on the number line, she engaged in a think aloud about how to compare the fractions. None of the teachers *marked metacognition* by using verbal, tone, or visual markers to indicate to students when thinking was being made visible.

Using Language and Representations Carefully

When looking at teachers' use of language, we found that nine of the 10 teachers consistently *used language economically*. All 10 teachers consistently *used language which was grade level appropriate*. However, only two of the teachers consistently *used language which was mathematically accurate and precise*. The mathematical language used by teachers that was not accurate and/or precise included referring to numbers as higher and/or lower, and saying fours and sixes rather than fourths and sixths. When we looked at teachers' use of representations, we found that only four teachers *drew number lines in accurate ways* (e.g., having arrows on both ends of the number line, marking 0, and making equal parts). Nine of the teachers *produced writing and representations that were legible and visible*. Only four teachers *organized the board in a way that supported understanding* how to use a number line to compare two fractions.

Discussion

Having access to detailed information about TCs' developing proficiencies with teaching practices is crucial for quality teacher education. Moreover, teacher educators must be able to use the data gathered from such assessments to focus efforts to support TCs' development. This study sought to examine the utility of using an assessment to assess teachers' skills with modeling content for formative purposes. The findings reveal details about skills with modeling content displayed by each of the teachers in the study.

We identify four key limitations of the design and use of the assessment and checklist. One limitation concerns the use of a checklist, yielding information primarily about the presence or absence of each technique rather than the use of a rubric, distinguishing the quality or quantity of teachers' enactments above the threshold. However, checklists allow for in-the-moment scoring, whereas a rubric requires that a teacher educator watch the entire performance and often rewatch it before making a judgment. Further, the checklist addressed our goal of capturing whether a TC could enact a particular technique. A second challenge is the potential interaction of mathematical knowledge for teaching (MKT) and modeling content. Even with the focus on content that matters for teaching mathematics, and supports built into the materials (e.g., student learning goals), there were instances in which MKT appeared to be a factor in the enactment. This is unsurprising as MKT is clearly intertwined throughout the work. In these cases, it was unclear whether we were accurately capturing a teacher's skill with modeling content or whether their use of MKT prevented them from demonstrating their modeling skill. This is an inherent problem in examining teaching practice. A third limitation is that we do not know whether the teachers in this study would perform differently if they modeled content in a different context. The context could be interpreted along multiple dimensions, including, but not limited to, the mathematics task used, the grade level of the students, or even engaging in the modeling work with actual children rather than through a simulation. This is an inherent problem; however, because we are not making claims about teachers' skills more broadly, we argue the assessment reveals important information about teachers' skills with modeling that can be used for formative purposes. A fourth limitation is that our study does not provide evidence of whether these teachers could decide what and when to model in their own teaching. We call out this point because, as we stated in the introduction, the goal of a teacher modeling in math class is to expand access to opportunities to engage in cognitively-demanding mathematical work. The aim is not to increase access by lowering the cognitive demand and spoon-feeding content to students. Instead, modeling is a means to consolidate ideas that students have explored. Thus, it is crucial that other sorts of information be gathered to determine (and, if necessary, intervene on) TCs' beliefs about what, when, and why to model.

Despite these limitations, our findings suggest that a standardized assessment of modeling content can reveal important information about TCs' skills. As we did not intend to make claims about this sample of teachers or a larger population of TCs, the small sample was appropriate for our goals. However, the variations in performance that are used to illustrate the capabilities of the assessment and tool cannot be interpreted as representing the skills of a larger population of TCs. This standardized assessment accomplished many of our design goals. Its ability to capture a range of skills could make the assessment and scoring tool useful in teacher education. Teacher educators and programs could use such assessments to track TCs' growth over time and to identify areas of strength and weakness with respect to the practice, which would allow for targeted support and program-level curriculum design.

We close by noting two questions that arose from this study. First, we might explore how this assessment could fit into a trajectory of assessments for assessing TCs' skills with modeling content. Second, in light of the questions that arose in this study about the role of content knowledge, we need to investigate further how we can reliably assess teaching practice in ways that account for the role of content knowledge for teaching.

Acknowledgments

This work was supported by the Bill and Melinda Gates Foundation under Grant No. OPP1053743. The authors wish to acknowledge and thank Deborah Loewenberg Ball, Michaela

Krug O'Neill, Charles Wilkes II, Amber Willis, Sabrina Bobsin Salazar, and Sarah Kate Selling for their contributions to this project. Any opinions, findings, and recommendations expressed are those of the authors.

References

Archer, A. L., & Hughes, C. A. (2011). *Explicit instruction: Effective and efficient teaching*. Guilford.

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Available online at amte.net/standards.

Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*(4), 373–397.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide?. *American Educator*, (Fall), 14–22.

Black, P., & Wiliam, D. (1998). Inside the black box raising standards through classroom assessment. *Phi Delta Kappan, 80*, 139–148.

Boerst, T., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record, 113*(12), 2844–2877.

Boerst, T. A., Shaughnessy, M., DeFino, R., Blunk, M., Farmer, S. O., Pfaff, E., & Pynes, D. (2020). Preparing teachers to formatively assess: Connecting the initial capabilities of preservice teachers with visions of teaching practice. In C. Martin, D. Polly, & R. Lambert (Eds.), *Handbook of Research on Formative Assessment in Pre-K through Elementary Classrooms* (pp. 89–116). IGI Global.

Chapin, S. H., O'Connor, C., & Anderson, N. C. (2013). *Talk moves: A teacher's guide for using classroom discussions in math, grades K–6*. Math Solutions.

Charalambous, C. Y., Hill, H. C., & Ball, D. L. (2011). Prospective teachers' learning to provide instructional explanations: how does it look and what might it take? *Journal of Mathematics Teacher Education, 14*(6), 441–463.

Council of Chief State School Officers (CCSSO) (2010). *Common Core State Standards*. National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington D.C.

Cizek, G. J. (2010). An introduction to formative assessment: History, characteristics, and challenges. In. H. L. Andrade & G. J. Cizek (Eds.), *Handbook of formative assessment* (pp. 3–17). Routledge.

Cohen, D. K., Raudenbush, S., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119–142.

Cohen, J. (2018). Practices that cross disciplines?: Revisiting explicit instruction in elementary mathematics and language arts. *Teaching and Teacher Education, 69*(3), 324–335.

Collins, A., Brown, J.S., & Holum, A. (1991). Cognitive apprenticeship: Making thinking visible. *American Educator, 15*(3), 6–11, 36–46.

Collins, A., Brown, J. S., & Newman, S. E. (1989). Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics. In L. B. Resnick (Ed.), *Knowing, learning, and instruction: Essays in honor of Robert Glaser* (pp. 453–494). Erlbaum.

Darling-Hammond, L., Pacheco, A., Michelli, N., LePage, P., & Hamerness, K. (2005). Implementing curriculum renew in teacher education: Managing organizational and policy change. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing worlds* (pp. 442–479). Wiley.

Ericsson, A., & Pool, R. (2016). *Peak: Secrets from the new science of expertise*. Houghton Mifflin Harcourt.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *The Teachers College Record, 111*(9), 2055–2100.

Hattie, J., & Timperley, H. (2007). The power of feedback. *Review of Educational Research, 77*(1), 81–112.

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in k-12 mathematics teaching. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 766–792). NCTM.

Kazemi, E., & Hintz, A. (2014). *Intentional task: How to structure and lead productive mathematical discussions.* Stenhouse Publishers.

Lampert, M. (2001). *Teaching problems and the problems of teaching*. Yale University Press.

Leinhardt, G. (2010). Introduction: Explaining instructional explanations. In M. K. Stein & L. Kucan (Eds.), *Instructional explanations in the disciplines* (pp. 1–5). Springer.

Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 87–113). JAI Press Inc.

Leinhardt, G., & Steele, M. D. (2005). Seeing the complexity of standing to the side: Instructional dialogues. *Cognition and Instruction, 23*, 87–163.
Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics*, *69*(2), 149–163.
Schwartz, D. L., & Bransford, J. D. (1998). A time for telling. *Cognition and Instruction, 16*(4), 367–398.
Schwartz, D. L. & Martin, T. (2004). Inventing to prepare for future learning: The hidden efficiency of encouraging original student production in statistics instruction. *Cognition and Instruction, 22*(2), 129–184.
Selling, S. K. (2016). Making mathematical practices explicit in urban middle and high school mathematics classrooms. *Journal for Research in Mathematics Education, 47*(5), 505–551.
Shaughnessy, M., & Boerst, T. (2018). Uncovering the skills that preservice teachers bring to teacher education: The practice of eliciting a student's thinking. *Journal of Teacher Education, 69*(1), 40–55.
Shaughnessy, M., Boerst, T., & Farmer, S. O. (2019). Complementary assessments of preservice teachers' skill with eliciting student thinking. *Journal of Mathematics Teacher Education, 22*(6), 607–638.
Shaughnessy, M., Garcia, N. M., O'Neill, M. K., Selling, S. K., Willis, A. T., Wilkes II, C. E., Salazar, S. B., & Ball, D. L. (2021). Formatively assessing prospective teachers' skills in leading mathematics discussions. *Educational Studies in Mathematics*, *108,* 451–472.
Shaughnessy, M., Ghousseini, H., Kazemi, E., Franke, M., Kelley-Petersen, M., & Hartmann, E. (2019). An investigation of supporting teacher learning in the context of a common decomposition for leading mathematics discussions. *Teaching and Teacher Education, 80*, 167–179.
Shute, V. (2008). Focus on formative feedback. *Review of Educational Research, 78*(1), 153–189.
Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Teaching Mathematics in the Middle School, 3*(5), 344–350.
Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions.* NCTM.
Stein, M., Smith, M., Henningsen, M., & Silver, E. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development.* Teachers College.
TeachingWorks. (2019). Explaining and modeling content, practices, and strategies. Retrieved from https://library.teachingworks.org/wp-content/uploads/ExplainingandModeling_Decomposition.pdf.

MATHEMATICAL ARGUMENTATION: ALIGNMENT OF TASK GOALS WITH PRESERVICE TEACHERS' ENACTMENTS IN A SIMULATED CLASSROOM

Calli Shekell	Jamie Mikeska	Heather Howell
ETS	ETS	ETS
cshekell@ets.org	jmikeska@ets.org	hhowell@ets.org

Dionne Cross Francis	Pavneet Kaur Bharaj	Jennifer Dix
University of North Carolina at Chapel Hill	University of North Carolina at Chapel Hill	Towson University
dicross@unc.edu	pkbharaj@unc.edu	Jdix4@students.towson.edu

Mathematical argumentation is a practice in which K-12 students should be engaging but supporting preservice teachers (PSTs) in learning to facilitate argumentation among students is challenging. The tasks used to support PSTs' learning in this area should be as carefully selected as the tasks that K-12 classroom teachers choose for their students. In this study, we examine a performance task that was designed for PSTs to practice facilitating an argumentation-focused mathematics discussion in a simulated classroom environment. Findings showed that most PSTs both articulated and enacted at least one feature of argumentation that was embedded in the performance task's student learning goal. However, evidence of PSTs' attention to critique in both the learning goal and the enacted discussion was limited. This suggests that careful consideration of alignment between a specific discursive practice and the design features of a performance task could yield better learning opportunities for PSTs when using online simulations.

Keywords: Classroom discourse; preservice teacher education; teacher educators; task design; mathematical argumentation

Introduction

Mathematical discussions have been a focus in mathematics teacher education for decades with a great deal of attention paid to supporting teacher learning (Boerst et al., 2011; Ghousseini, 2008; Stein, et al., 2008). Related research has been widely covered in mathematics education literature and has included examining cognitively-oriented math tasks that can support K-12 students' engagement in discussions (Stein & Lane, 1996), exploring how to effectively launch and implement those tasks to support students in having productive discussions (Jackson et al., 2012), and developing frameworks for orchestrating productive discussions (Stein et al., 2008). While justification and proof have always been critical components of mathematics, the advent of the Standards for Mathematical Practice (National Governors Association, 2010) made the term argumentation prominent in the mathematics education world. Since then, several scholars have published on how to support PSTs in learning how to facilitate mathematical argumentation with their students (Conner & Singletary, 2021; Rumsey & Langrall, 2016; Stylianou & Blanton, 2018). Learning how to engage students in productive mathematical argumentation requires targeted learning opportunities, especially for PSTs who have limited experience with this practice. While some research has examined the tasks used by teachers to engage K-12 students in mathematical argumentation, more limited research has been conducted to examine the performance tasks used by teacher educators to support PSTs as they learn how to engage in this aspect of the work of teaching mathematics. This study is directly designed to address this gap by examining how PSTs perceived and used one performance task that was designed to support them in learning how to facilitate argumentation-focused discussions.

Theoretical Framework

One promising approach for helping PSTs learn how to engage in novel teaching practices, such as facilitating argumentation-focused discussions, is the use of approximations of practice where PSTs can try out and refine an instructional skill in settings of reduced complexity (Grossman, Compton et al., 2009). Approximations of practice are one type of pedagogy of practice advocated for in a theory of practice-based teacher education. This theory hypothesizes that teacher learning is supported by opportunities for them to learn in and from their practice. This stance suggests that teachers, especially PSTs, need to have opportunities to practice discrete components of the work of teaching mathematics, ideally in settings of reduced complexity and with opportunities for feedback, reflection, and repeated practice.

Peer teaching has been one kind of approximation used in practice-based teacher education and involves a PST playing the role of teacher while their peers act as students (Kazemi et al., 2016). A newer technology that is being used to engage PSTs in approximations of practice to support their learning is simulated classrooms (Howell & Mikeska, 2021). Simulated classrooms readily allow for repeated practice, interactions with students without having to find classroom placements and reflection through video recorded practice sessions (Gibson et al., 2011). From a research perspective, these practice spaces support analyses of PST learning opportunities. Because they are designed spaces, researchers can also pay close attention to the functioning of the practice spaces themselves and optimize them over multiple iterations of study.

Within both simulated and real-life learning environments, the kinds of tasks teachers use in their classrooms to engage students can impact their learning opportunities (Hiebert et al., 1997; National Research Council, 2001; Stein et al., 2000). While much work has focused on cognitively-oriented task choice in the K-12 setting, an area less explored is performance-based task choice for PST learning. As educators we hypothesized that similar logic applies here, i.e., the nature of the tasks used to support PST learning likely relates to the quality of learning opportunities they experience. How those tasks are interpreted can also impact what students and/or teachers learn from and interact with them (Watson & Ohtani, 2015). Part of interpreting a task is deciding what the learning goal is for a given instructional sequence and then making choices about how to engage students around the task to reach the learning goal (Hiebert & Grouws, 2007; Sleep, 2012).

Study Purpose

In this study, we examined the features of a performance task designed to engage PSTs in an approximation of practice where they practiced one ambitious mathematical teaching practice: facilitating discussions that engage students in mathematical argumentation. We explored how the PSTs interpreted the task's goal and examined how they subsequently engaged in the simulated discussion. We set out to answer the following research questions: *RQ1. How do PSTs articulate the goal of a discussion task focused on argumentation?; RQ2. To what extent do PSTs engage student avatars in mathematical argumentation that targets the task's goal in the simulated classroom?*

We made design choices in the performance task development phase that led us to hypothesize that this task provided opportunities for PSTs to practice engaging the student avatars in three key features of mathematical argumentation: (a) explaining or justifying their work, (b) comparing different approaches to solving the problem, and (c) critiquing one another's work. Specifically, we anticipated the PSTs would do so, respectively, by providing the students with opportunities to discuss the choice and utility of rates used in the solution (pounds per day versus days per pound) and/or justify the approaches used; note similarities and differences in the approaches used; and consider the efficiency and utility of different solution approaches.

Methods

Study Context

This study was part of a larger research project examining teacher educators' use of and PSTs' learning within an online suite of simulated teaching experiences, one of which is an Avatar-Based Simulation (ABS) (Mikeska et al., 2022). This study reports on the task design phase when we used the task with PSTs to gauge its success and inform subsequent updates. In this study, we were piloting a mathematics performance task that we developed to see how PSTs interpreted the task materials and how they engaged in facilitating the argumentation-focused discussion in the simulated classroom. This task was developed for the middle school level but was designed along the same structure as used in our prior work for the elementary level (Howell & Mikeska, 2021).

Mathematical Task

The content focus of the mathematical task, *Hungry Hungry Huskies,* is proportional reasoning, and the specific problem the student avatars (hereafter referred to as students) engage with is a story problem in which two dogs consume dog food at different rates. The students are asked to use the rates of consumption to predict how long it would take to consume a given quantity. We designed the story line of the task as if the five middle school students worked on the problem in two groups, with one group using a pictorial representation and the second using a table. Similarities and differences across the two solution approaches were deliberately constructed to provide opportunities to compare different ways of solving the same math task, with the intention that these will also open avenues to support argumentation. For example, both groups reached a correct answer, a design feature intended to help the PST focus on methods over answers. While both approaches are successful in this case, neither is optimal in general. This aspect of the task design was intentional so instead of the PST favoring one approach over the other, they would be encouraged to identify affordances in each. Each approach also relied on a form of visual inspection to draw conclusions, leaving the possibility that the approaches might not extend effortlessly with less friendly numbers (i.e., if the amount of dog food they were looking for did not appear in the table). This was an intentional task feature to provide PSTs a space to press on the limitations of the visual approach. One group uses unit rates and the other does not, providing a point of contrast and a place the PST might press to find out why unit rates were used and what they afford. In addition, one group represents the ratio as pounds per day where the other represents it as days per pound, creating an opportunity for the PST to think about the significance of 'per unit' while working on proportional reasoning tasks.

Study Design

Seven PSTs enrolled in secondary mathematics teacher education programs across the U.S. participated in this study. For data collection, each PST completed: (a) an online background questionnaire about their personal characteristics and professional background; (b) a 20-minute discussion in the simulated classroom using the *Hungry Hungry Huskies* mathematics task; (c) an online task survey to provide information about their discussion preparation and their perceptions on the task components and their discussion performance; and (d) a semi-structured interview where they answered questions about the discussion goal, the task materials and their discussion performance. Each discussion was video recorded and then later transcribed for analysis purposes.

Before the PSTs engaged in their ABS session, they were given a task packet to prepare to facilitate a 20-minute argumentation-focused discussion among five students around the students' work on the mathematics problem. The task packet included the students' work, the learning goal, the mathematics problem that they solved, tips about engaging the students in argumentation, and other information about what to expect in the simulated classroom. PSTs facilitated this discussion with the students from their computers.

Analysis

Analysis included multiple approaches. First, we conducted qualitative content analysis to identify patterns in the PSTs' open-ended responses from the interview and survey about the goal of the discussion. We looked for alignment between PSTs' articulations of the goal of the discussion and these three features of an argumentation-focused discussion: students explaining/justifying their work; students comparing their approaches; and students evaluating and/or critiquing each other's work. Using hypotheses described above that emerged from the task design phase of development, we looked for PST alignment with these features in their responses to questions about the goal of the task. Second, we reviewed video records from each ABS session to determine whether there was evidence that each PST provided the students with opportunities to engage in the specific features of mathematical argumentation embedded in the task design as articulated above.

Findings

In terms of PSTs' articulations of the task's goal (RQ 1), one of the PSTs reported all three of the features of argumentation-focused discussions that we were looking for (justification, comparison, critiquing). Three of the PSTs articulated two of the three (justification and comparison). Two PSTs each reported on one feature of the discussion: one each for justification and critique. There was one PST who did not articulate any of the features that we expected when asked what the goal of the task was. Instead, she stated that she "wanted students to explore each other's methods and think about their own method." While relevant, her response was not detailed enough to determine whether she was addressing any of the key features that we anticipated. Table 1 summarizes findings across the 7 PSTs.

Table 1
PSTs' Articulations of Features of Argumentation in the Task Goal

PST (pseudonyms)	Justification/ Explanation	Comparing Approaches	Critiquing Each Other's Work
Taylor	X	X	X
Addy			X
Mo	X	X	
Saylor	X		
Molly			
Lynn	X	X	
Caleb	X	X	

One of the PSTs, Taylor, included all three features, stating:

> "(1) All students will present and explain their ideas; (2) All students will listen to the justifications of others and respond appropriately to further understanding; (3) All students will interact with each other to have a meaningful math discussion; (4) Students will analyze others' work in order to compare and contrast ways to solve the problem; (5) Students will be engaged and feel involved throughout the entire discussion."

In Taylor's response, she noted that there would be justifications and that students would compare and contrast the ways of solving the problem. Taylor was alluding to critique in stating that the students would "respond appropriately to further understanding" and in her comment about contrasting the approaches to the problem. Another PST, Lynn, said

> "The overall goal of the discussion I wanted them to be able to compare and contrast the strategies, that was a big one, …to justify their work and say how their procedure connected to their answer."

Here, Lynn was highlighting two of the features: justification and comparing approaches to the problem. She went a step further to state that the students should connect their work to their answer. Critique, however, was not evident in her response.

The fact that a majority of PSTs (n = 6) provided evidence of understanding at least one component of the goal that we considered to be a key feature underscores that most PSTs in our study were pedagogically aware of one or more of the key features of mathematical argumentation they should aim for when facilitating their discussion. However, the lack of PSTs' attention to critique in the learning goal suggests it did not stand out as part of the goal.

Our analyses addressing the extent to which the PSTs engaged the students in argumentation (RQ 2) suggest that most PSTs did support the students to justify their work in the discussion (n=7) and to compare their approaches (n=6). This evidence points to alignment between how PSTs articulated the goal and engaged the students in the discussion. All PSTs supported the students to justify their work by typically asking the students to share their solutions or to talk about how they came to their answer. Caleb did so by stating, "You were asked to come up with an answer that the group agreed on, along with evidence to support your answer. So what group would like to [share] first?" Here, Caleb was reminding the students that they were to justify their solution with some sort of evidence. A typical way that PSTs engaged the students in comparison was by asking them to identify similarities and differences in their solutions. In her discussion, Taylor said,

> "So now we're going to do a little bit more of an advanced skill. Now that you've both solved the problem, I'm happy to tell you that both groups got the answer right. So we're just going to compare and contrast. So that means we're going to point out similarities and differences between the two ways to solve the problem."

Taylor let the students know, now that they shared their work, it was time to compare approaches. One of the students, Ethan, indicated that a difference was that his group used a drawing and the other group used a table. Another student, Dev, said that his group used unit rates in the solution but the other group did not. The groups were considering each other's work and comparing it to their own.

While the PSTs engaged the students in justification/explanation and comparing approaches quite a bit, critique was less evident. Only 3 of the 7 PSTs engaged the students in critique. When evident, critique was usually around the limited utility of a given approach. For example, when using a table to solve rate problems, very large numbers could become difficult to represent in a table. Molly encouraged her students to think in this direction by stating,

> "Do you guys want to talk about maybe what do you like best about this method and then maybe where Savannah identified maybe an area of concern, like if this would work for other problems...identify something you liked about this method and then something you didn't like about this method."

Molly encouraged a critical lens by asking the students to identify a part of the solution that they disliked, but also clued them in on thinking about other problems. Addy, who both described critique as part of the learning goal and enacted her discussion with evidence of critique, encouraged a critical lens in a different way. To set the students up to critique, she had them complete a "See, Think and Wonder" chart while observing each group's written work. More explicitly, she also told the students "We are mathematicians and we're going to act like great mathematicians, evaluating and critiquing each other." A little later on, one of the students, Savannah, asked a group if they thought their picture would work to solve other problems.

Discussion

To date we have analyzed how PSTs articulated the goal of the discussion they facilitated and found that most PSTs (n=6) articulated and enacted at least one of the key features of argumentation that we intended in the task design phase. This evidence suggests that PSTs were interpreting the performance task's associated learning goal as intended. Just as the PSTs articulated more justification and comparison in the task's goal, they also engaged in more of that dialogue in their simulated discussions. While this finding suggests that the task's learning goal needs to better support PSTs to engage their students in critique, it also demonstrates that the PSTs' enactments, at least in part, aligned with their understanding of the goal in the simulated environment. An open question, then, is whether including ideas around critique in the learning goal will support PSTs to engage the students in more of it.

In this study, we were able to examine the alignment between the performance task's goal and the PSTs' enacted practices in the simulated discussion, but we did not consider the quality of the argumentation that occurred. That is, we identified where it occurred, but not how or what the mathematical content of the argumentation was. Future work will need to analyze the quality and content of argumentation in the simulated discussions.

The literature supports the conclusion that facilitation of argumentation is difficult, and novice teachers often struggle with this practice, an issue the larger study seeks to address. The affordance of this analysis is in allowing us to draw conclusions about where the designed task might be improved in ways that increase PST opportunity to learn to engage students in critique, justification and comparison. With the understanding that tasks influence opportunities to learn, these findings prompted our team to re-design the task and its affiliated learning goal around which PSTs facilitated the simulated discussion to better support them with learning opportunities to practice engaging the students in the full set of argumentation features when facilitating the mathematics discussion. Future work will examine how PSTs make use of the revised task in the simulated discussions, including addressing the mathematical point of the argumentation that is evident (Sleep, 2012).

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Baker, B., Cooley, L., & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education*, 557–578.

Ball, D.L. (2001). Teaching with respect to mathematics and students. In T. Wood, B. S. Nelson & J. Warfield (Eds.), Beyond classical pedagogy: Teaching elementary school mathematics, 11-22. Mahwah, NJ: Erlbaum.

Boaler, J., & Brodie, K. (2004, October). The importance, nature and impact of teacher questions. *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 774-782).

Boerst, T., Sleep, L., Ball, D., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record*, *113*(12), 2844-2877.

Bos, B. (2011). Professional development for elementary teachers using TPACK. *Contemporary Issues in Technology and Teacher Education, 11*(2). Retrieved from http://www.citejournal.org/vol11/iss2/mathematics/article1.cfm

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics.* Retrieved from [Math Standards.indb (corestandards.org)]

Conner, A., & Singletary, L. M. (2021). Teacher support for argumentation: An examination of beliefs and practice. *Journal for Research in Mathematics Education*, 52(2), 213-247.

Crespo, S. (2003). Learning to pose mathematical problems: Exploring changes in preservice teachers' practices. *Educational Studies in Mathematics*, *52*(3), 243–270.

Gibson, D., Christensen, R., Tyler-Wood, T. & Knezek, G. (2011). SimSchool: Enhancing Teacher Preparation through Simulated Classrooms. In M. Koehler & P. Mishra (Eds.), *Proceedings of SITE 2011--Society for Information Technology & Teacher Education International Conference* (pp. 1504-1510). Nashville, Tennessee, USA: Association for the Advancement of Computing in Education (AACE). Retrieved February 1, 2022 from https://www.learntechlib.org/primary/p/36508/

Ghousseini, H. N. (2008). *Learning with routines: Preservice teachers learning to lead classroom mathematics discussions* (Doctoral dissertation).

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. Teachers College Record, 111(9), 2055-2100.

Herbst, P., & Chazan, D. (2006). Producing a viable story of geometry instruction: What kind of representation calls forth teachers' practical rationality? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez. (Eds.), *Proceedings of the 28th North American Group for the Psychology of Mathematics Education Conference* (Vol. 2, pp. 213–220). Mérida, México: UPN.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H. & Wearne, D. (1997). Making mathematics problematic: A rejoinder to Prawat and Smith. *Educational Researcher, 26*(2), 24-26.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. *Second handbook of research on mathematics teaching and learning*, *1*(1), 371-404.

Howell, H., & Mikeska, J. N. (2021). Approximations of practice as a framework for understanding authenticity in simulations of teaching. Journal of Research on Technology in Education, 53(1), 8-20.

Howell, H., Mikeska, J. N., Tierney, J., Baehr, B., & Lehman, P. (2021). M1 Ordering Fractions Mathematics Elementary Task. Qualitative Data Repository. https://doi.org/10.5064/F6W3BY0S

Jackson, K. J., Shahan, E. C., Gibbons, L. K., & Cobb, P. A. (2012). Launching complex tasks. *Mathematics Teaching in the Middle School, 18*(1), 24-29.

Kazemi, E., Ghousseini, H., Cunard, A., & Turrou, A. C. (2016). Getting inside rehearsals: Insights from teacher educators to support work on complex practice. Journal of Teacher Education, 67(1), 18-31. https://doi.org/10.1177/0022487115615191

Mikeska, J. N., Shekell, C., Maltese, A. V., Reich, J., Thompson, M., Howell, H., ... & Rogers, M. P. (2022, April). Exploring the Potential of an Online Suite of Practice-Based Activities for Supporting Preservice Elementary Teachers in Learning How to Facilitate Argumentation-Focused Discussions in Mathematics and Science. In *Society for Information Technology & Teacher Education International Conference* (pp. 1793-1803). Association for the Advancement of Computing in Education (AACE).

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors.

National Research Council. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.
Rumsey, C., & Langrall, C. W. (2016). Promoting mathematical argumentation. *Teaching Children Mathematics*, 22(7), 412-419.
Sleep, L. (2012). The work of steering instruction toward the mathematical point: A decomposition of teaching practice. *American Educational Research Journal*, *49*(5), 935-970.
Smith, J. B., Lee, V. E., & Newman, F. M. (2001). *Instruction and achievement in Chicago elementary schools*. Chicago, IL: Consortium on Chicago School Research.
Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and Instruction*, 25(2-3), 161-217.
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*(4), 313-340.
Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*(1), 50-80.
Stein, M.K., Smith, M.S., Henningsen, M.A., & Silver, E.A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.
Stylianou, D. A., & Blanton, M. L. (2018). *Teaching with Mathematical Argument: Strategies for Supporting Everyday Instruction*. Heinemann.Rasmussen, C., & Ellis, J. (2015). Calculus coordination at PhD-granting universities: More than just using the same syllabus, textbook, and final exam. In D. Bressoud, V. Mesa, & C. Rasmussen (Eds.). *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 107–115). Washington, DC: The Mathematical Association of America.
Watson, A., & Ohtani, M. (2015). Themes and issues in mathematics education concerning task design: Editorial introduction. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI study* 22 (pp. 3–15). Cham: Springer International Publishing

HOW PRESERVICE ELEMENTARY TEACHERS VIEW LESSON COHERENCE

Ji-Won Son
University at Buffalo - SUNY
jiwonson@buffalo.edu

Seungjung Jo
SUNY Geneseo
Seungjung.jo@gmail.com

Florence F Obielodan
University at Buffalo - SUNY
ffobielo@buffalo.edu

Samantha Wallace
University at Buffalo - SUNY
Sw232@buffalo.edu

Adam Heck
University at Buffalo - SUNY
adamheck@buffalo.edu

The purpose of this study is to investigate how preservice elementary teachers view mathematics lesson coherence. It is often reported that how lessons are less coherent in U.S. mathematics classrooms compared to high-achieving countries. Stigler and Hiebert (2009) urged the importance of teacher' conception and their learning to achieve coherent lessons. In this study we examined 18 preservice teachers' views of lesson coherence. We developed theoretical framework by extensively reviewing previous studies. By crosschecking preservice teachers' responses across our survey questions using the framework, we illustrate what knowledge of lesson coherence the preservice teachers have and have not, and thus suggest how teacher educators help preservice teachers to develop and analyze lessons to be more coherent.

Keywords: Lesson coherence, Preservice teachers, Teacher beliefs

Lesson coherence is important predictors that contribute to teachers' high quality of mathematic teaching and classroom practice (Hiebert et al., 2003; Stigler & Perry, 1990; Stigler & Stevenson, 2005). When lesson coherence is achieved, students have more chances to conceptually understand mathematics better. Lesson coherence is in close relation not only to teachers' effective mathematics instruction but also to students' learning. Accordingly, lesson coherence has gained significant attention to address how U.S. teachers did in their everyday classroom (Jacobs & Morita, 2002; Stigler & Perry, 1990; Stigler & Stevenson, 2005; Wang & Murphy, 2004). It has been reported that lesson coherence can be established when topics, activities, and teachers' discourse moves are structurally and contextually connected each other (Sekiguchi, 2006; Stigler & Stevenson, 2005). The purpose of this study is to investigate how preservice elementary teachers (PSTs) view mathematics lesson coherence. Despite the important relationship between teacher view and instructional practices, there is relatively a few research studies on teacher view of mathematical lesson coherence (e.g., Cai, Ding, Wang, 2014; Sekiguchi, 2006; Stigler & Perry, 1990; Stigler & Stevenson, 2005; Wang & Murphy, 2004). Considerable amounts of research have dealt with teacher lesson coherence, but far less research has dealt with in-service and preservice teachers' view of this topic in the US context. The research questions that guided the study are: How do preservice elementary teachers view mathematics lesson coherence?

Theoretical framework

Figure 1 shows our conceptualization of lesson coherence drawn from prior research studies (e.g., Hiebert et al., 2003a; 2003b; Newmann et al., 2001; Wang, Cai, & Wang, 2015; Wang & Murphy, 2004). Coherence often relates to connectedness and interrelations to mathematics content and instructional activities (Wang, Cai, & Wang, 2015; Wang & Murphy, 2004). In particular, lesson coherence refers to 1) connectedness of lesson structures and events within and across lessons (Cai et al., 2014), 2) goal-oriented and content-focused classroom activities to achieve learning outcomes (Liang, 2013), 3) “how well lessons follow a logical and structured sequence of events and how well lessons focus on one or related topics and proceed from simple to complex situations in the process of concept development” (Chen & Li, 2009, p.713), 4) relations among themes or topics in the lesson (Sekiguchi, 2006), and 5) “coherence was defined by the group as the (implicit and explicit) interrelation of all mathematical components of the lesson… a lesson with a central theme that progressed saliently through the whole lesson” (Hiebert et al., 2003a, p. 196). Hiebert et al. (2005) argue that it is not enough to simply include all components but individual components work together to achieve lesson coherence.

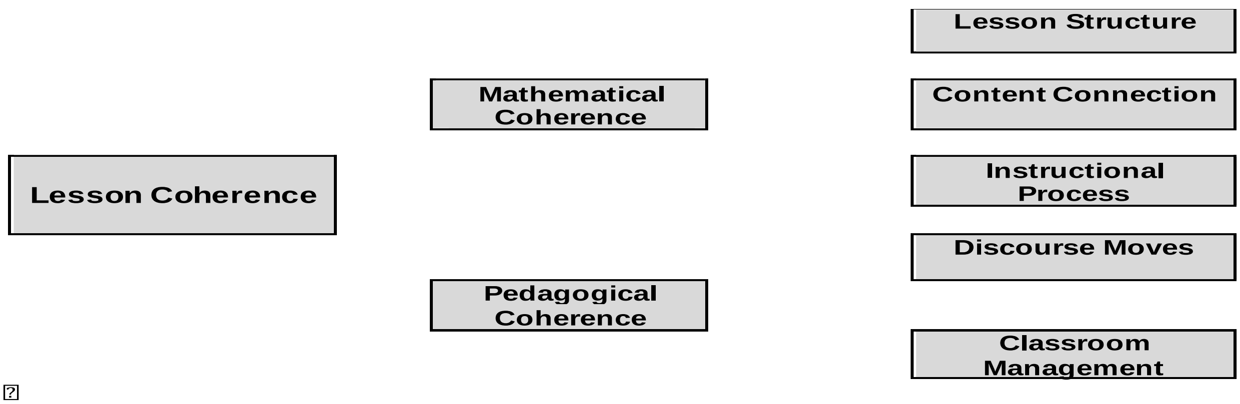


Figure 1. Theoretical framework for our study

Methods

Participants A total of 18 PSTs participated in the study from two elementary mathematics methods classes at a large Northeastern university of teacher preparation in the United States that offers a five-year teacher preparation program. As part of the undergraduate major in elementary education, students complete a core sequence of educational psychology courses and two mathematics content courses before taking a three credit-hour methods course in their senior year. Participants were recruited from their second methods course during their internship year. Their ages ranged from 22 to 35 years, with 54 females and 4 males.

Instruments Since the purpose of this study is to explore PSTs’ views and perceptions, we employed open-ended survey questions. Open-ended survey method allows respondents to include more information, attitudes, and understanding of a particular subject by responding to the questions with their own wording directly (Reja, Manfreda, Hlebec, & Vehovor, 2003). At the beginning of the semester, the PSTs were asked to answer three questions that were adapted from Cai et al. (2014). The questions include: 1) When people say a lesson is very coherent, what does the word “coherent” mean to you? What are the characteristics of a coherent lesson?; 2) If you mentor a new teacher, how would you guide the new teacher to achieve coherence in her or his teaching?; 3) Some people say that a coherent lesson can foster students’ learning. Do you agree with this statement? Why?

Data analysis We used an inductive content analysis approach (Grbich, 2007) for the written response. We initially organized raw data into a spreadsheet, read all of the responses, and created codes based on the raw data. More specifically, data analysis involved these four processes: (a) an initial reading of each PST’s response, (b) identifying the subcategories under each analytical aspect according to the framework shown in Table 1, (c) coding the categories and subcategories, and (d) interpreting the data quantitatively and qualitatively (Creswell, 1988).

In order to answer the research question examining PSTs' view of lesson cohere, PSTs' responses were categorized based on the themes that emerged as researchers read multiple cases.

Summary of Results

Figure 2 presents our finalized categories that describe PSTs' conception of lesson cohere. When PSTs were asked to describe their conception on lesson coherence, two additional categories emerged in addition to two categories drawn from previous studies: student learning opportunities and what teachers need to know and do. Of the 18 PSTs, 17 defined mathematics lesson coherence with respect to mathematical coherence, and 3 PSTs viewed it from pedagogical coherence. Ten PSTs additionally stated student learning opportunities resulted from coherent lessons, and five PSTs additionally mentioned what teachers need to know and do in order to create a coherent lesson.

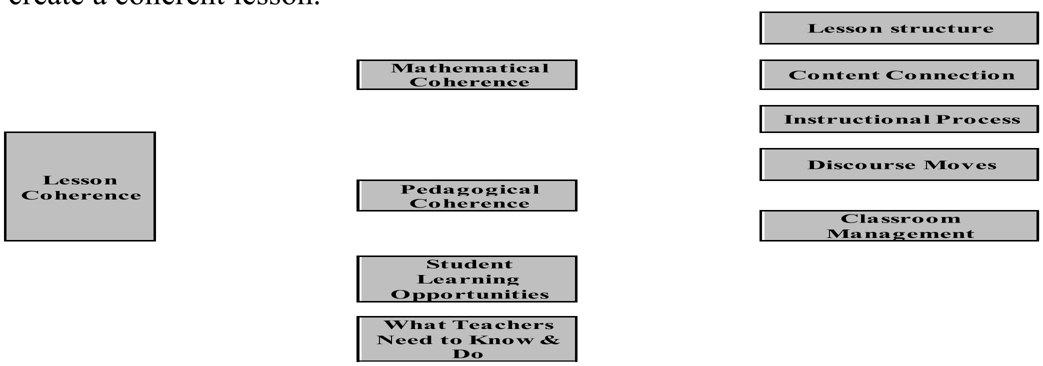


Figure 2. A framework that describes PSTs' conception of lesson coherence.

Mathematical coherence

When the PSTs defined lesson coherence from mathematical coherence, they displayed similar distributions among three subcategories: the lesson structure (10 PSTS), content connection (9 PSTs), and instructional process (11 PSTs) (**Figure 3**). Nine PSTs discussed across two characteristics, either lesson structure and content connection, lesson structure and instructional process, or content connection and instructional process. Our PSTs showed clear attention to mathematical coherence with reference to lesson coherence. However, the responses of PSTs were mostly broad and did not include narrowed topics and a several activities that need to be arranged for understanding of the topics the previous studies emphasized (e.g., Sekiguchi, 2006; Stigler & Perry, 1990). The PSTs displayed a lack of attention to pedagogical coherence. As pedagogical coherence, particularly teachers' intentional discourse moves help students to obtain mathematical knowledge (e.g., Chen & Li, 2009), PSTs need support to develop content rich discourse throughout their lesson.

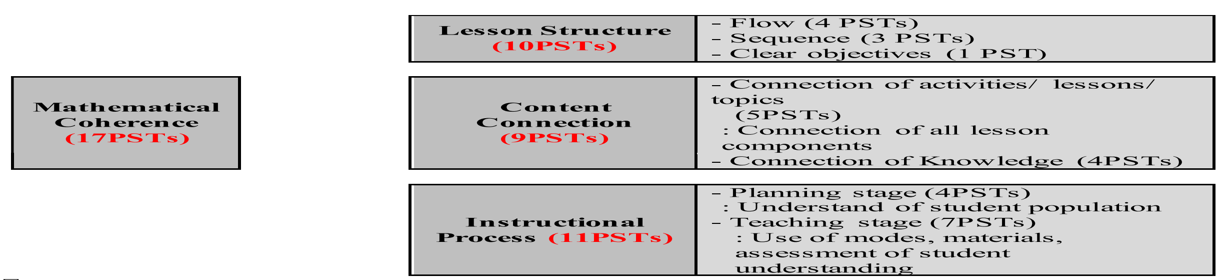


Figure 3. Distribution of PSTs' view on mathematical lesson coherence

Specifically, ten PSTs described lesson structure as an important characteristic of a coherent lesson. Among them, four PSTs emphasized flow, stating that “A coherent lesson is broken up into parts, [so] it flows nicely” (PST #4). Three PSTs mentioned the sequence of the lessons, stating that “A coherent lesson should be structured as beginning, middle, and end”. Among them, PST #17 stated, “A coherent lesson includes a general introduction of the topic, some problems that include that topic and methods in going about solving that problem.” Only one PST (PST #4), emphasized clear and focused lesson objectives in the lesson structure.

Interestingly, none of the PSTs mentioned mathematic problem complexity to create a coherent lesson, such as organizing mathematical problems to proceed from simple to complex. In general, our PSTs tended to focus on the lesson structure, using a broad concept of ‘flow’ and ‘sequence’ rather than how activities are systematically and logically organized in a lesson.

Discussion and Conclusions

This study contributes to the current literature on teacher view on lesson coherence and the knowledge base of teacher education. This study has implication for teacher educators working to design mathematics education courses for PSTs, as well as for researchers interested in furthering understanding of teachers’ knowledge and beliefs on lesson coherence. Very few PSTs had such complex understanding regarding lesson coherence. The findings of this study suggest that teacher educators need to find a better way to help PSTs perceive lesson coherence differently. Future studies need to be done with different research tools and in multiple contexts, possibly using interviews or observations to provide more detailed explanations for preservice teachers’ responses.

References

Cai, J., Ding, M., & Wang, T. (2014). How do exemplary Chinese and U.S. mathematics teachers view instructional coherence? Educational Studies in Mathematics, 85 (2), p. 265-280.

Chen, X, & Li, Y. (2009). Instructional coherence in Chinese mathematics classroom: A case study of lessons on fraction division. International Journal of Science and Mathematics Education, 8, 711–735.

Fortus, D., Adams, L. M. S., Krajcik, J., & Reiser, B. (2015). Assessing the Role of Curriculum Coherence in Student Learning About Energy. Journal of Research in Science Teaching, 52 (10), p.1408–1425

Grow-Maienza, J., Hahn, D., & Joo, C. (2001). Mathematics instruction in Korean primary schools: Structures, processes, and a linguistic analysis of questioning. Journal of Educational Psychology, 93 (2), 363-376.

Hiebert, J., Gallimore, R., Garnier, H., Giwin, K., Hollingsworth, H., Jacob, J., et al. (2003). Teaching mathematics in seven countries: Results from the TIMMS 1999 video study. Washington, DC: USA Department of Education National Center for Educational Statistics.

Hiebert, J., Stigler, J. W., Jacobs, J, K., Givvin, K., Garnier, H., Smith, M., Hollingsworth, H., Manaster, A., Wearne, D., & Gallimore, R., (2005). Teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. Educational Evolution and Policy Analysis, 27 (2), 111-132.

Jacobs, J. K., & Morita, E. (2002). Japanese and American teachers' evaluations of videotaped mathematics. Journal for Research in Mathematics Education, 33 (3), p. 154-175.

Liang, S. (2013). An example of coherent mathematics lesson. Universal Journal of Educational Research, 1 (2), 57-64.

Reja, U., Manfreda, K. L., Hlebec, V., & Vehovor, V. (2003). Open-ended vs. close-ended questions in web questionaires. Developments in Applied Linguistics, 19, 159-177.

Sekiguchi, Y. (2006). Coherence of mathematics lessons in Japanese-eight grade classroom. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), Proceedings of the 30th conference of the international group for the psychology of mathematics education, 5,

AMBITIOUS MATHEMATICS INSTRUCTION FROM TEACHER PREPARATION TO ELEMENTARY CLASSROOMS: AN EXAMINATION ACROSS CASE STUDIES

Casedy Ann Thomas
The University of Virginia
cat6hu@virginia.edu

Faith Peddie
Cambium Learning Group
Faith.Peddie@gmail.com

This study cross analyzes the results of two multiple case studies that focus on ambitious mathematics instruction within two institutions. The first study examines how three elementary mathematics methods instructors in the same teacher education program (TEP) provide their teacher candidates with learning opportunities. The second study examines how two elementary teachers who graduated from that TEP enact mathematics instruction. Findings suggest that teacher beliefs influence both the content that students have the opportunities to learn (OTL) and the nature of the learning opportunities. The results describe how instructors and teachers perceive the purpose of elementary mathematics instruction and how their beliefs align with the enactment of ambitious mathematics instruction with learners.

Keywords: Elementary School Education, Preservice Teacher Education, Teacher Beliefs, and Instructional Activities and Practices

Purpose

Mathematics education research has yet to reach a consensus on the construct of ambitious mathematics instruction (Kazemi et al., 2009; Newmann & Associates, 1996; Windschitl et al., 2018). This has implications for TEPs that aim to support teachers in enacting such instruction and for novice teachers (i.e., those with zero to three years of experience) who attempt to draw upon their pre-service teacher experiences as practicing teachers. In this study, the researchers conceptualize ambitious instruction as comprised of a set of teaching practices that foster students' deep, conceptual understanding of standards-based mathematics concepts (Newmann & Associates, 1996). The National Research Council (NRC; 2001) defines conceptual understanding as "an integrated and functional grasp of mathematical ideas" (p. 18).

In an effort to further operationalize ambitious instruction, the researchers examine the complexity of enacting such instruction within three methods courses at one TEP and in two novice teachers' classrooms who graduated from that program. By focusing on the enactment of ambitious instruction at various stages, this study acknowledges the complexities that novices face in enacting such instruction. By doing so, we argue that a shared vision of ambitious mathematics instruction within a TEP could have positive implications for the ways that novices transfer these opportunities and experiences into their own teaching practices.

The purpose of the study is to examine what perceptions of ambitious instruction from a TEP carry over into novice teachers' beliefs while remaining conscious of the many factors that influence their beliefs and instruction. Various factors impact novice teachers' instruction, yet in order for us to develop a shared vision of what ambitious instruction means and looks like in the classroom, we seek to examine the trajectory of these beliefs from a TEP into elementary schools. To do so, this study cross-analyzes the results of two multiple case studies that were nested within a larger longitudinal study; one study was bounded within a University's TEP and focused on three instructors who taught elementary mathematics methods courses (Thomas, 2021); the other study was bounded within a school and focused on two graduates of the University's TEP.

Theoretical & Conceptual Frameworks

Theoretical Framework: Activity Theory

Both studies drew upon activity theory in order to examine teachers' interactions with the socio-material context in which they develop mathematics teaching practices and create learning opportunities for their students. Activity theory emerged from the Vygotskian theory of cognition, examining cognitive activity from a socio-cultural lens (Peddie, 2019; Chahine, 2013). As Pai, Adler, and Shadiow (2006) explain, "There is no escaping the fact that education is a sociocultural process," or an artifact-mediated activity (p. 6). Therefore, activity theory is useful for understanding how teachers and educators may be influenced, and how their teaching practices provide opportunities to learn for students. Within the framework for activity theory, a subject is influenced by mediating artifacts both conceptual and material to reach, develop, or create some object. In the first study, elementary mathematics methods instructors (subjects) engaged with artifacts to create learning opportunities for teacher candidates (objects). In the second study, novice teachers (subjects) were influenced by numerous factors (artifacts) as they planned for and enacted instruction with elementary students (objects). Although tools were conceptualized differently in each study, both researchers examined tools as mediated artifacts as they relate to pedagogies of practice (Grossman et al., 2000).

Activity theory was instrumental in both studies and informs the comparative analysis across studies as it relates to OTL and beliefs about and the enactment of ambitious elementary mathematics instruction. Activity theory provides a lens for examining how and why methods instructors provide particular OTL for teacher candidates and how such OTL seem to translate into novice teachers' beliefs and instructional activities with elementary students within an artifact-mediated activity system.

Conceptual Frameworks

Opportunity to learn. The first study (Thomas, 2021) draws upon Schmidt and colleagues' (2011) framework for OTL, defining it as "the content to which future teachers are exposed as part of their teacher preparation programs" (p. 140). The second study defines OTL as the "the set of experiences that schools organize to help students acquire the knowledge, skills, and abilities specified in those official standards (Schmidt & McKnight, 2012, p.13). Although there is variability within these definitions of OTL, as it relates to the contexts and learners, both studies focus on OTL ambitious elementary mathematics in pedagogy and practice.

Influence of beliefs on practice. We take the position that *beliefs* are more cognitively ingrained and harder to change than *perceptions* (Philipp, 2007). Likewise, teachers may believe that it is important for students to grapple with cognitively demanding tasks, but if the psychological strength of this belief is not strong, then they may not follow through to create such opportunities; further, just because a belief has strength in one context does not mean that it will have as much strength in another (Cooney et al., 1998; Thomas, 2021). When examining how teacher educators' and novice teachers' beliefs impact mathematics instruction, both studies take into consideration the teachers' beliefs and their actions in the classroom; reporting upon beliefs that were considered psychologically strong, supported with evidence, and demonstrated consistently within the actions of the participants. Hence, we take the position that, "Teachers' beliefs influence the decisions that they make about the manner in which they teach mathematics" (NCTM, 2014, p. 10).

Research Questions

1) How do methods instructors and novice teachers perceive the purpose of elementary mathematics instruction?
2) How do the methods instructors' and novice teachers' beliefs of teaching mathematics align with the enactment of ambitious mathematics instruction?

Methods

Sample

Two researchers implemented multiple case studies to examine teachers’ perceptions of elementary mathematics instruction. Both studies were nested within the same longitudinal mixed-methods study that focused on the ways that five TEPs support novices' development of ambitious instruction. The first researcher's study took place at one TEP in a southeastern state, called Robin University (pseudonym). During that time, there were three total elementary mathematics methods instructors at Robin University and each participated in the study. The second researcher's study took place the following school year within two elementary classrooms in the same southeastern state at McCaskey Elementary School (MES). The teachers taught at the same school and both attended Robin University.

Data Collection

In the first study, the researcher observed each instructor two to three times for a duration of two and a half hours each, for a total of 20 hours of observation. The researcher took fieldnotes alongside each observation. Additionally, each instructor participated in two rounds of interviews that lasted between 45 and 60 minutes each. The first interview was structured to better understand the methods instructors’ perceptions of their instructional strategies (prior to observations) and the second interview was semi-structured and took place after observations to further examine emerging findings. In the second study, the researcher observed and recorded three one-hour mathematics lessons for each teacher for a total of nine hours of observation. In addition, the teachers participated in one structured interview that focused on topics related to classroom expectations, curricular resources, and beliefs of teaching mathematics. The study also featured three semi-structured interviews with each teacher that took place after each lesson and focused on the ways that teachers planned and enacted their mathematics instruction. The interviews lasted approximately 30 minutes each. Finally, each teacher provided supporting artifacts such as lesson plans and lesson handouts.

Analytical Strategies

Each researcher drew upon an instrument called the Mathematics Scan (M-Scan; Berry et al., 2017) to analyze their observation data. M-Scan represents a schema of instruction that can be used to measure teachers’ implementation of standards-based teaching practices. The instrument is useful in research that focuses on ambitious mathematics instruction because it captures differences between teaching for conceptual understanding versus teaching for acquisition of procedural knowledge (Berry et al., 2017). M-Scan has nine dimensions that measure teaching practices in four domains: task selection and enactment, the use of representations, the use of mathematical discourse, and lesson coherence.

The researchers captured fieldnotes from each lesson observation and transcribed their own interviews. Analytic logs were developed alongside each set of field notes and transcriptions. The researchers independently analyzed each of their data sets. Additionally, each used inductive and deductive coding to develop emerging themes that incorporated Erickson's (1986) model of analytic induction and constant comparison (Miles, Huberman, & Saldaña, 2014) respectively. The themes were triangulated across each of the data sources to support assertions in each research study. Both researchers compared confirming evidence and disconfirming evidence for each of their emerging assertions and continued to adjust the assertions until all evidence was accounted for.

For the purposes of this study, the researchers cross-analyzed the archival interview transcriptions from both studies, examining the similarities and differences in perceptions of ambitious instruction, and they simultaneously engaged in deductive coding. Based upon emerging themes from this data analysis, the researchers triangulated video recordings and field notes from observed lessons and analytic memos to further explore OTL provided to students and how these compared to each individual's stated perceptions and beliefs.

Results

Across studies, assertions suggested that both instructors' and teachers' perceptions of teaching elementary mathematics seemed to be associated with their enactment of ambitious mathematics instruction. The results also feature examples of OTL that instructors provided to students at Robin University in addition to experiences that the novice teachers seemed to draw upon from their pre-service experience to enact such instruction. For example, a teacher at MES described the importance of encouraging students to use various strategies when solving mathematics tasks. In turn, it was common for the teacher to provide students the opportunity to represent content using words, pictures, and symbols. This teacher described how the experiences at Robin University allowed her to practice doing mathematics as a child would, which helped her conceptualize how the use of various strategies could strengthen students' understanding of mathematical concepts. The findings from the first study confirm that the methods instructors at Robin were providing OTL that emphasized the importance of providing multiple mathematical representations inclusive of mathematical tools and engaging in problem solving that incorporated multiple strategies.

Furthermore, the results describe counter-examples that focus on various mediating artifacts that seemed to be associated with novices' enactment of ambitious mathematics instruction. This was witnessed in the first study when novices felt pressured within the context of their field placements to conform to cooperating teacher norms. Later, teachers complied with the school-wide expectations at MES in order to develop their mathematics lessons. The teachers also had relationships with their grade-level colleagues that seemed to influence the ways that they selected and enacted mathematics tasks. Grade-level colleagues with more teaching experience provided materials and lesson implementation ideas to teachers that they then enacted in their own classrooms. There was also a relationship between Robin University and MES such that the teachers were provided professional development opportunities by the University.

Discussion & Significance

Math education is making strides in studying the development of ambitious mathematics instruction at the pre-service level (Cavanna et al., 2017; Kazemi et al., 2009). However, more work needs to be done focusing on the capabilities of novice teachers in enacting ambitious mathematics instruction (Jong, 2016). This study is significant in that it focuses on OTL across methods instructors and classrooms. We also focus on the various perceptions of instructors and novices as there is still more to be understood about the ways that beliefs seem to influence the enactment of ambitious mathematics instruction. This study highlights significant ideas that seemed to have transferred from one university to novice teachers' classroom settings while using activity theory as a lens to examine the complexities of defining ambitious mathematics instruction. Future research could benefit from examining the impact of a shared vision at different levels as well as the continued study of various mediating artifacts that seem to be associated with the enactment of ambitious mathematics instruction.

References

Berry III, R. Q., Rimm-Kaufman, S. E., Ottmar, E. M., Walkowiak, T., & Merritt, E. G. (2017). The mathematics scan (M-Scan): A measure of mathematics instructional quality. *Unpublished measure.*

Cavanna, J. M., Drake, C., & Pak, B. (2017). Exploring elementary mathematics teachers' opportunities to learn to teach, *Psychology of Mathematics Education-North American Chapter 2017 Conference Proceedings*, Research Report Session.

Chahine, I. (2013). Delineating the epistemological trajectory of learning theories: Implications for mathematics teaching and learning. *Mathitudes,1*(1), 1-18.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education*, 29(3), 306-333.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. Wittrock (Ed.), *Handbook of research on teaching*, (3rd Ed.), pp. 119-161. New York: Macmillan.

Grossman, P. L., Valencia, S.W., Evans, K., Thompson, C., Martin, S. (2000). Transitions into teaching: Learning to teach writing in teacher education and beyond. *Journal of Literacy Research, 32*(4), 631-632.

Jong, C. (2016). Linking reform-oriented experiences to teacher identity: The case of an elementary mathematics teacher. *The Journal of Educational Research*, *109*(3), 296-310.

Kazemi, E., Franke, M., & Lampert, M. (2009). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 12-30). Adelaide, SA: MERGA.

Miles, M. B., Huberman, A. M., & Saldana, J. (2014). *Qualitative data analysis: A methods sourcebook (3rd Ed.).* Thousand Oaks, CA: SAGE Publications.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: National Council of Teachers of Mathematics.

Newmann & Associates. (1996). *Authentic achievement: Restructuring schools for intellectual quality.* San Francisco, CA: Jossey Bass.

National Research Council (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Pai, Y., Adler, S. A., & Shadiow, L. K. (2006). *Cultural foundations of education (4th ed.)*. Upper Saddle River, NJ: Merrill/Prentice Hall.

Peddie, F. (2019). *An examination of novice teachers' planning and enactment of ambitious instruction.* Unpublished doctoral dissertation. University of Virginia.

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte, NC: Information Age Publishing.

Schmidt, W. H., Cogan, L., & Houang, R. (2011). The role of opportunity to learn in teacher preparation: An international context. *Journal of Teacher Education*, *62*(2), 138-153.

Schmidt, W. H., & McKnight, C. C. (2012). *Inequality for all: The challenge of unequal opportunity in American schools*. New York, NY: Teachers College Press.

Thomas, C. A. (2021). One university's story on teacher preparation in elementary mathematics: Examining opportunities to learn. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-021-09487-3

Windschitl, M., Thompson, J., & Braaten, M. (2018). *Ambitious science teaching*. Cambridge, MA: Harvard Education Press.

DECIDING QUALITY: LENSES, CHALLENGES, AND OPPORTUNITIES

LA CALIDAD DE DECIDIENDO: LENTES, LIMITACIONES, Y DESAFÍOS

Jonathan Thomas
University of Kentucky
Jonathan.thomas1@uky.edu

Cindy Jong
University of Kentucky
Cindy.jong@uky.edu

Molly Fisher
University of Kentucky
Molly.fisher@uky.edu

Edna Schack
Morehead State University
e.schack@gmail.com

Walker Mask
University of Kentucky
Walker.mask@uky.edu

This study is focused on different lenses that may be used to make quality determinations with respect to prospective elementary teachers' instructional decisions. Using a professional noticing framework, we analyze individuals' deciding responses within a video-anchored case to explore the varying ways that quality of such decisions might be determined. Specifically, we examine how quality may be ascribed to decisions via holistic and analytic lenses. Our findings suggest key differences and tensions in quality determinations when using analytic and holistic lenses to make quality determinations with respect to instructional decisions.

Keywords: Teacher Noticing; Instructional Activities and Practices; Preservice Teacher Education

Purpose

The purpose of this preliminary study is to explore the variations of quality with respect to instructional decisions via a professional noticing framework. Specifically, we focus on *how different rating lenses may be used to evaluate the quality of prospective elementary teachers' (PSETs') decision making skills within their professional noticing of children's mathematical thinking.* This study was conducted as part of a larger funded venture focused on development of PSETs' professional noticing capabilities with respect to dimensions of equity.

Professional Noticing and Deciding

Focus on teacher noticing and variants thereof has been sustained for some time (Schack, Fisher, & Wilhelm, 2017); Sherin, Jacobs & Philipp, 2011a) and, as such, is increasingly shaping practitioner guidance in a variety of ways including noticing along mathematical progressions (Thomas, et al., 2014) and noticing via equity lenses (Jackson, Taylor, & Buchheister, 2018). While some portrayals of noticing focus primarily on attending and interpreting (i.e., sense-making) (Sherin, Jacobs & Philipp, 2011b), Jacobs, Lamb & Philipp (2010), describe a third, interrelated component process, *deciding*, which connotes intended activity resulting from the interpretation of a mathematical moment stemming from one's attending to certain features of that moment. As such, the component process of deciding represents the bridge between an individual's cognitive processing of a mathematical moment (attending, interpreting) and intended action in response to that moment.

We focus this investigation on the deciding component and the different ways in which the quality of such decisions may be examined. Historically, decision quality has been considered through a lens of routine. Different types of decisions were categorized with little attention to the perceptive and interpretive activity that guided such actions (Leinhardt, Weidman, & Hammond, 1987). However, professional noticing frameworks recast decisions as resultant of perception and interpretation allowing for more expansive applications of lenses with respect to quality. For example, quality determinations might be a function of alignment with a particular mathematical learning progression (Schack, et al., 2013). From this perspective, for example, a task aimed at helping a student move from constructing perceptual unit items to figural unit items was determined as higher in quality in that it matches well with typical mathematical development along a learning progression. This lens for evaluating decisions is predicated on detecting subtle details of mathematical activity, interpreting them through a developmental lens, and responding to create productive space for mathematical construction. Another lens to determine decision quality might foreground the extent to which a particular decision addresses multiple and varied critical incidences or perspectives within a mathematical moment (Fisher, Thomas, Jong, Schack, & Dueber, 2018). For example, in a whole class discussion, a decision that connects with and furthers the mathematical reasoning of multiple, varied student perspectives might be deemed more productive than a decision that merely focuses on a single individual. Stockero et al. (2017) describe the distinction between these two lenses as *noticing within* and *noticing among.* Lastly, researchers may evaluate decisions more holistically via considerations of context to determine the extent to which a proposed decision is centered on student reasoning and has the potential to bend that moment toward conceptual growth and/or productive activity. From this perspective scoring processes such as rubrics or flowcharts (Fisher et al, 2018; Schack et al., 2013) may recede or be used with increased flexibility to enhance interpretive space for raters. Further, more holistic, contextualized and interpretive lenses render distinctions between noticing within and noticing among less salient as they relate to quality. Rather, quality determinations are guided by some central, desirable element or outcome (e.g., decisions that centralize student thinking or activity) as it may apply to a given context. For this study, we compare two approaches to evaluate the quality of decision-making within a professional noticing framework. One rating approach was heavily structured via the use of flow-process charts and privileged decisions that reflected the detection of nuanced details as well as address of multiple students. We refer to this lens as analytic scoring. The other rating approach focused adopted a more holistic approach to evaluating decisions and quality determinations were made according to the presence or absence of broadly desirable elements. We refer to this lens as holistic scoring. *Our aim was to compare these two lenses (analytic scoring, holistic scoring) in the context of evaluating PSET decisions to better understand how differing perspectives and processes influence quality determinations.*

Methods

Context and Participants

Eight modules to integrate equity and professional noticing of children's mathematical thinking were developed by the authors. To facilitate embedding in a class session, each module had a duration of approximately 20 minutes focused on mathematical topics typical of a methods class as well as a particular dimension of equity (Gutiérrez, 2009**).** Some participating faculty members elected to engage in full implementation (minimum of seven modules). Other faculty elected to engage in partial implementation (use of four, five, or six modules). One module, which introduced the concepts of equity and professional noticing, was required as the initial module for all implementation sites, full or partial. This paper reports on one semester of data collected at three institutions, one research and two regionals, in two southeastern states. Two of the settings, one rural, one urban/suburban, completed full implementation and one rural completed a partial implementation.

Data Collection

The Video Assessment on Noticing and Equity (VANE) consisted of prompts for professional noticing and equitable practices observed in an approximately two-minute clip of a 2nd grade class exploring the concept of equality. A problem, 10 + 10 = __ + 5 was presented to the class. The students were encouraged to share their solution and strategy. The clip captures three students sharing, two of whom respond with an incorrect answer but some reasoning and the third with the correct answer with different reasoning. After watching this video, PSETs were tasked with responding to items corresponding to professional noticing component processes (attending, interpreting, deciding). Germane to this study are the PSET responses to the deciding prompt – Imagine that you are the classroom teacher. What might you do next? Provide a rationale for your decision. For this preliminary investigation of varying lenses, we randomly selected 23 deciding matched pre/post responses from full implementation sites and 13 matched pre/post responses from partial implementation sites. We elected to engage in this comparative investigation with relatively small sample sizes as an exploratory exercise that will inform rating processes for the larger study focused on PSETs' development of equitable noticing practices.

Analytic Processes

The selected responses were each scored by two raters using both the holistic scoring lens and the analytic scoring lens (see Figure 1).

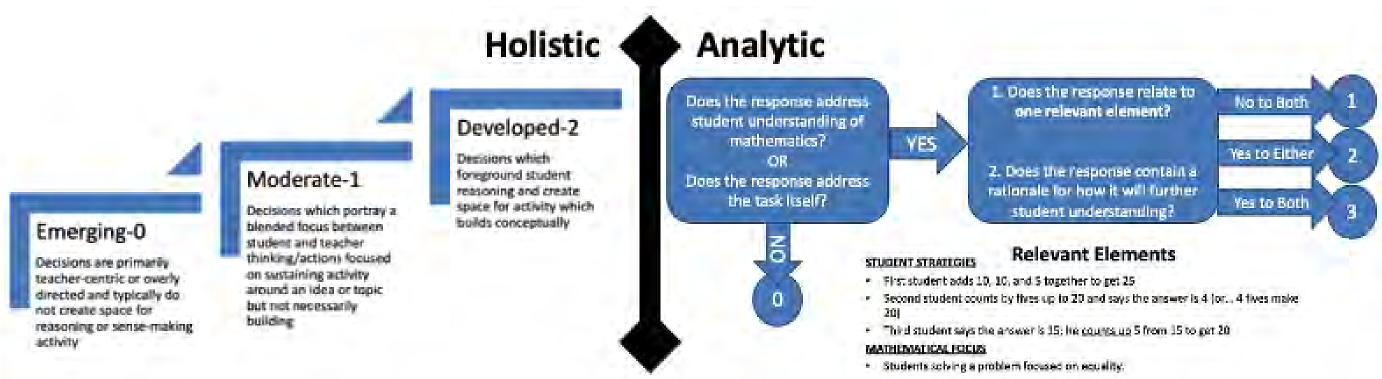


Figure 1: Holistic and Analytic Scoring Lenses

Note, the holistic scoring lens provides descriptions of quality at three levels and creates considerable interpretive space for raters, while the analytic lens is far more structured and provides discrete pathways to guide the rater. The raters had previously engaged in iterative, independent scoring activity with each lens and associated calibration conversations. Thus, there was considerable shared familiarity. For instances of disagreement (for both lenses), raters engaged in negotiated discussion to arrive at a consensus score.

Findings and Discussion

Again, we note that this investigation represents a preliminary examination of different lenses as they apply to PSET deciding. We first examined the outcomes from each lens in terms of rudimentary counts (see Tables 1 and 2). We also conducted 2-tailed paired t-tests to determine statistical significance of rating outcomes (see Table 3).

Table 1: Holistic Scoring

	Score 0	Score 1	Score 2
Full Implementation Pre	20	3	0
Full Implementation Post	15	7	1
Partial Implementation Pre	8	5	0
Partial Implementation Post	10	3	0

Table 2: Analytic Scoring

	Score 0	Score 1	Score 2	Score 3
Full Implementation Pre	2	1	11	9
Full Implementation Post	2	4	13	4
Partial Implementation Pre	1	3	6	3
Partial Implementation Post	1	1	6	5

Table 3: 2-Tailed Paired t-Tests:

Holistic Scoring p-value	Analytic Scoring p-value	
0.324174	0.151834	All paired data
0.436265	1	Partial implementation paired data
0.055668	0.076272	Full implementation paired data

Comparing changes in response counts, pre/post, across the two lenses, PSET decisions tended to receive a low rating (0) when scored holistically and higher ratings (2, 3) when scored analytically. Given the more interpretive nature of the holistic lens, this finding suggests tension between structured ratings processes (Fisher et al., 2018) and the raters more intuitive sense of quality with respect to PSET decisions when ratings processes are more loosely guided. Further, p-values for both partial and full implementations more closely approach statistical significance when scored via a holistic lens vs. an analytic lens. This finding strikes us as somewhat counterintuitive given the notion that more regulated and structured processes tend to minimize chance outcomes.

The deciding component of professional noticing has consistently presented challenges in terms of measurement. (Schack et al., 2013; Thomas, 2017). Determining the quality of decisions requires negotiating the space between specific and highly contextualized moments and broader principles of teaching. In these complex arenas, researchers tend to emphasize structure and guided processes to make the terrain more negotiable. However, our findings suggest that reducing structure and process guidance and creating space for more holistic interpretation in the context of instructional decisions may result outcomes more consistent with researchers' understanding of quality and less likely to be attributable to chance. We intend to further investigate this intriguing hypothesis in subsequent study.

Acknowledgments

This research was supported by a grant from the NSF IUSE Program - Award #s 1431133, 1431631, 1431098. The opinions expressed herein are those of the authors and do not necessarily reflect views of the NSF.

References

Fisher, M.H., Thomas, J., Jong, C., Schack, E.O., & Dueber, D. (2018). Professional noticing in complex mathematical contexts: Examining preservice teachers' changes in performance. Psychology of Mathematics Education – North America Annual Conference. Greenville, SC.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game.". Teaching for Excellence and Equity in Mathematics, 1(1), 4-8.

Jackson, C., Taylor, C. E., & Buchheister, K. (2018). Seeing mathematics through different eyes: An equitable approach to use with prospective teachers. In T. G. Bartell (Ed.), Toward equity and social justice in mathematics education (pp. 263–285). Dordrecht, The Netherlands: Springer.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169–202.

Leinhardt, G., Weidman, C., & Hammond, K. M. (1987). Introduction and integration of classroom routines by expert teachers. Curriculum Inquiry, 17(2), 135-176.

National Council of Teachers of Mathematics (NCTM). (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: NCTM.

Schack, E., Fisher, M., Thomas, J., Eisenhardt, S., Tassell, J., & Yoder, M., (2013). Preservice teachers professional noticing of children's early numeracy. Journal of Mathematics Teacher Education, 16(5), 379-397.

Schack, E.O, Fisher, M.H., Wilhelm, J (Eds.) (2017). Teacher Noticing – Bridging and Broadening Perspectives, Contexts, and Frameworks. In, Research in Mathematics Education Series. Springer. New York: NY.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.). (2011a). Mathematics teacher noticing: Seeing through teachers' eyes. New York: Routledge.

Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011b). Situating the study of teacher noticing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.). Mathematics teacher noticing: Seeing through teachers' eyes. (pp. 3-13). New York: Routledge.

Stockero, S. L., Leatham, K. R., Van Zoest, L. R., & Peterson, B. E. (2017). Noticing distinctions among and within instances of student mathematical thinking. In Schack, E.O., Fisher, M.H., & Wilhelm, J. (eds.) Research Trends in Mathematics Teacher Education. (pp.467-480). New York: Springer.

Thomas, J., Fisher, M., Eisenhardt, S., Schack, E., Tassell, J., & Yoder, M. (2014). Professional Noticing: Developing Responsive Mathematics Teaching. Teaching Children Mathematics, 21(5), 295-303.

Thomas, J. (2017). The Ascendance of Noticing: Connections, Challenges, and Questions. In Schack, E.O., Fisher, M.H., & Wilhelm, J. (eds.) Research Trends in Mathematics Teacher Education. (pp.507-514). New York: Springer.

USING ASSESSMENT TOOLS FOR TEACHER LEARNING PURPOSES

Merav Weingarden
University of New-Hampshire
Merav.weingarden@unh.edu

This paper offers a socio-cultural perspective on the mechanism by which research-based assessment tools designed to evaluate mathematics teaching, are used as pedagogical tools for teacher learning. This process is exemplified by the Realization Tree Assessment tool (RTA), showing how pre-service teachers refer to the pedagogical messages valued and devalued by the assessment tool, and which conflicts between these messages they attend to.

Keywords: Preservice Teacher Education, Teacher Educators, Assessment.

Introduction

In recent years there has been a growing interest in reform-oriented pedagogies, such as *student-centered*, *cognitively demanding* and *ambitious* teaching (Brown, 2003; Kazemi et al., 2009; Stein et al., 2000), which aim to support mathematics learning of *all* students at the highest possible level (NCTM, 2014). Accordingly, many efforts have been made to develop tools, such as the IQA – Instructional Quality Assessment (Boston & Smith, 2009) and the MQI – The Mathematical Quality of Instruction (Hill et al., 2008), for evaluating how these pedagogies are being implemented in classrooms. A small but growing strand of research has recently examined the use of these assessment tools also as pedagogical tools, aiming to help teachers implement ambitious teaching practices in their classrooms (e.g., Boston & Candela, 2018). These studies focus on implementing a particular assessment tool and describing teachers' gain from engaging with this specific tool. From a socio-cultural perspective on teacher learning, this paper aims to explore the mechanism by which research-based tools designed to assess teaching are used as pedagogical tools for pre-service teachers (PSTs) who learn how to teach mathematics.

Theoretical Framework

Teacher learning from a socio-cultural point of view draws upon the idea that teacher learning is not just a matter of mastering specific skills or acquiring certain bits of pedagogical knowledge (Horn & Kane, 2015; Nachlieli & Heyd-Metzuyanim, 2021). It is a more holistic process that often requires teachers, especially PSTs, to shift between *Contextual Discourses (CDs)*, which are different perceptions of messages about teaching and learning communicated in various social, cultural, and institutional situations (Thompson et al., 2013). The *pedagogical messages*, which are messages about teaching and learning, include how students learn (e.g., memorizing or exploring), who is the main authority of mathematical ideas (teacher or students), and what are the teacher's roles (e.g., delivering knowledge or supports student thinking). Thompson et al. (2013) differentiate two CDs. The first, *ambitious CD*, values deep understanding of ideas and student engagement in solving complex problems rather than procedural tasks. The second, *traditional CD*, values traditional teaching practices, where the teacher presents a procedure using a model problem and students practice on similar problems.

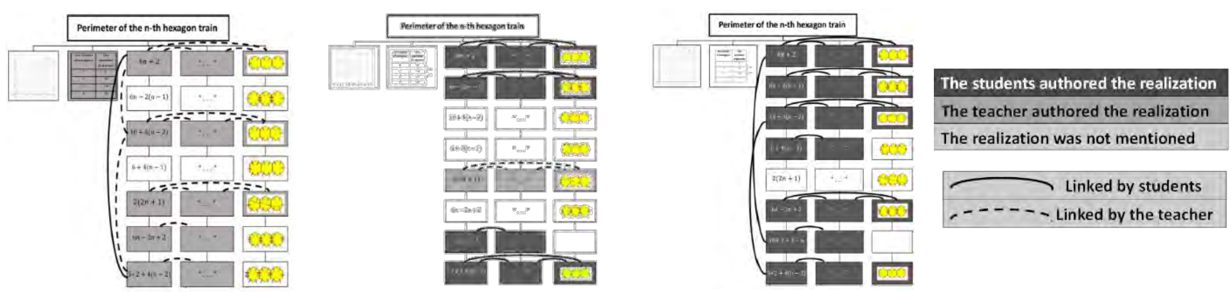
Initiating PSTs into ambitious CD stands at the core of multiple teacher education programs (e.g., Van Es et al., 2017) that often use various pedagogical artifacts or teaching representations such as students' work or video excerpt to promote PST learning processes (Herbst & Kosko, 2014; Kazemi & Franke, 2004). Recently, studies have examined the use of assessment tools as a pedagogical means for teacher learning (Boston & Candela, 2018; Kraft & Hill, 2020). The

Realizations Tree Assessment tool (RTA), for example, was designed to evaluate classroom discussions and was found to be helpful for PSTs to attend to the mathematical objects at the core of a lesson (Weingarden & Heyd-Metzuyanim, 2020). Boston and Candela (2018), which examine the use of the IQA as a reflection tool for teachers, concluded this process: "what is assessed becomes what is valued, hence classroom observation tools have the power to shape and support instructional reforms by highlighting or foregrounding specific features of instructional practice" (p. 427). In what follow I examine the process of "what is assessed" and "what is valued" by relying on the traditional and ambitious CDs. Eventually, allowing to examine how PSTs, while engaging with such tools, refer to various pedagogical messages from different CDs and the tension between them. I exemplify this process on the RTA tool.

The Realization Tree Assessment Tool (RTA)

The Realizations Tree Assessment tool (RTA) (Weingarden et al., 2019) was designed to evaluate the extent to which class discussions provide opportunities for students to attend the various realizations (representations) of the mathematical object and the links between them. The RTA visually depicts the mathematical object at the core of a class discussion, its various realizations, and the extent to which students, rather than the teacher, author narratives about the object. Figure 1 exemplifies three different class discussions assessed by the RTA (Weingarden et al., 2019; Weingarden & Heyd-Metzuyanim, 2021). These class discussions focused on the Hexagon Task in which students were asked to find the perimeter of n hexagon concatenation (the mathematical object). The cells in the RTAs describe the realizations of the mathematical object (e.g., algebraic realizations such as 4n+2, visual realizations of the hexagons pattern) according to who articulated them, the students, or the teacher.

Lesson A Lesson B Lesson C



RTA score: 0.39 RTA score: 0.47 RTA score: 0.87

Figure 1: Three RTA Images and Measures Represent Different Learning Opportunities

Each lesson was depicted by an RTA image quantified into a number between 0 to 1 – the RTA score, based on the ratio of student-authored and teacher-authored narratives and the number of realizations and links mentioned. The presented images and measures show the different extent and levels of opportunities offered during the lesson. Lesson A describes a teacher-centered lesson with almost all realizations and links authored by the teacher. Lesson C, in contrast, shows the students as the main authority in the class discussion. Thus, Lesson A received a lower score (0.39) than Lesson C (0.87). In Lesson B, although students authored most of the realizations and links, the solutions were not linked to each other (by vertical links) as in Lesson C. Thus, Lesson B received a lower score (0.47) than lesson C (0.87). These RTA images and scores (Figure 1) exemplify different pedagogical messages aligned with traditional and ambitious CDs. The pedagogical messages of student-centered teaching (Lesson B, C) and linking between realizations (Lesson C) align with ambitious CD, whereas the pedagogical messages of teacher-centered (Lesson A) and show-and-tell (Lesson B) align with traditional CD.

This paper aims to examine how the RTA can be used as a pedagogical tool for PST learning to teach mathematics by attending to messages from different CDs. Thus, the research questions are: *How do PSTs, by using the RTA, attend to pedagogical messages aligned with traditional and ambitious CDs? And what conflicts between the CDs were illustrated by the PSTs?*

Method

The current study took place within a course for pre-service secondary mathematics teachers. During the course, 33 PSTs were introduced to the RTA tool and its theoretical aspects, then compared and discussed different RTAs (similar to those in Figure 1). Subsequently, they were administered the Calling-Plans task (Institute For Learning, 2015) and an empty RTA of this task (Figure 2). They were asked to shade three RTAs in different ways describing three types of learning opportunities and discuss the similarities and differences between the shaded RTAs.

Long Distance Telecommunication Company A charges a base rate of 45 NIS (equivalent of dollars) per month for phone calls plus 5 agoras (equivalent of cents) per minute of call. Long Distance Telecommunication Company B charges a base rate of only 20 NIS per month, but charges 30 agoras per minute of call. Which service would you choose, and why?

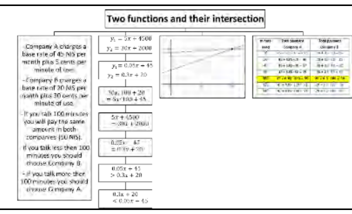


Figure 2: The Calling-Plans Task and the RTA for the Task

The PSTs worked in 11 groups (3 PSTs in each group), and thus 11 worksheets, including 33 RTAs, were analyzed. The analysis was performed in two stages and included analyzing the PSTs' written descriptions about the RTAs they shaded. First, coding the PSTs' utterances according to the pedagogical messages they referred to (e.g., student-centered, rich discussion, teacher directs too much, teacher-centered, student struggle). The coding categories were adopted from studies that examined teacher learning as transitioning between CDs (e.g., Heyd-Metzuyanim, 2019). Second, identifying conflicts between the two CDs illustrated when the PSTs described their shaded RTAs.

Findings

Attending Traditional and Ambitious Contextual Discourses

The most common pedagogical message referred to by the PSTs (9 groups) was the extent to which the teacher and students authored narratives during the lesson. Figure 3 presents an example of two RTAs designed by two groups.

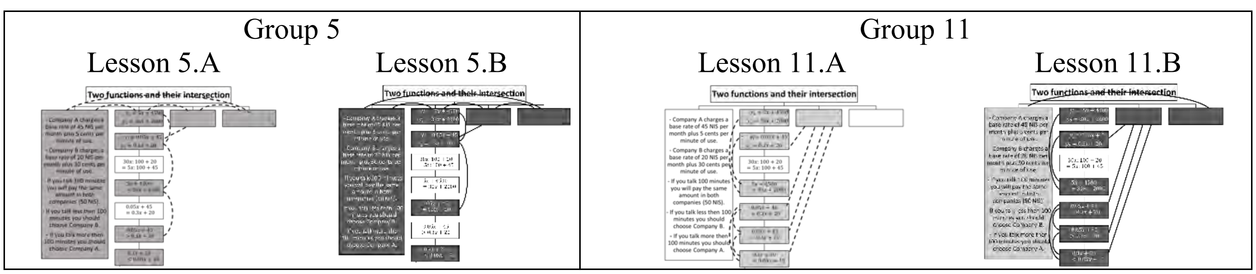


Figure 3: The Extent of Student Authority in the RTAs

Lessons 5.A and 11.A are described by the PSTs as teacher-centered lessons in which the teacher "suggests all of the ideas by herself" and "takes over the discussion and simply shows the students a complete solution to the problem". Lessons 5.B and 11.B are described by the PSTs as student-centered lessons in which "students participated and tried to solve the question on their own" and "discovered many links between realizations". The PSTs, by the RTAs they designed, attended to a pedagogical message assessed by the RTA tool – the extent to which students and the teacher authored mathematical narratives. They attended to this pedagogical message both as aligned with traditional CD (light-shaded RTAs: Lesson 5.A and 11.A), and as aligned with ambitious CD (dark-shaded RTAs: Lesson 5.B and 11.B).

Conflicts Between Contextual Discourses

Although most of the PSTs described the teacher-centered RTAs as less valuable (e.g., "failed attempts") compared to the student-centered lessons (e.g., "ideal lesson"), some of the PSTs pointed out some valued pedagogical messages in these teacher-centered lessons aligned with traditional CD. For example, the PSTs in group 11, while describing Lesson 11.A (Figure 3), indicated that although "the students did not participate in the lesson", this lesson has some advantages. They wrote: "since the teacher explains the solutions by herself, it is possible to go further and deeper with the ideas… students may ask questions that they may not have asked if the teacher had not explained". The PSTs referred to two pedagogical messages aligned with different CDs. One, teacher authority ("explain by herself") aligned with traditional CD and the second, engagement with important mathematics ("go deeper with the ideas") aligned with ambitious CD. They tacitly attend to a conflict between the two discourses by referring to a teacher-centered lesson (traditional CD) as supporting and promoting engagement with important mathematics (ambitious CD). Another example of such a conflict appears in group 6's RTAs. They showed how student-centered lessons are not always valuable as teacher-centered lessons. They shaded an RTA with many student-authored realizations but no links and indicated that "in this situation, the teacher needs to direct students to make these links". The PSTs referred to the conflict between the importance of linking between realizations, aligned with ambitious CD and the teacher as "directs the students too much", aligned with traditional CD.

Discussion

This paper examines how the RTA, designed to evaluate teaching mathematics, is used as a learning tool for PST learning to teach. This was done by relying on the idea that teacher learning occurs when teachers initiate into the ambitious CDs and adopt the pedagogical messages aligned with this CD. Initial findings show that the PSTs, by the RTAs they shaded, attended to different types of pedagogical messages aligned with traditional and ambitious CDs. This allowed them to attend to conflicts and tensions that could occur while two different pedagogical messages from different CDs are embedded in the same lesson.

PSTs attending to different, often competing CDs and the conflicts between them can contribute to teacher educators' efforts to engage PSTs with ambitious teaching practices (Ma & Singer-Gabella, 2011; Thompson et al., 2013). Examining the use of assessment tools as pedagogical tools by focusing on the valued and devalued pedagogical messages aligned with ambitious and traditional CDs, can theoretically contribute to the growing strand of research examining teacher learning using assessment tools (e.g., Boston & Candela, 2018).

Further studies are required to examine how different assessment tools attend to different sets of pedagogical messages and which conflicts can be addressed by using assessment tools as a pedagogical means. This study is the first step towards a better understanding of the mutual relationship between teaching assessment and teacher learning.

References

Boston, M. D., & Candela, A. G. (2018). The Instructional Quality Assessment as a tool for reflecting on instructional practice. *ZDM - Mathematics Education*, *50*(3), 427–444. https://doi.org/10.1007/s11858-018-0916-6

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education*, *40*(2), 119–156. https://doi.org/10.2307/40539329

Brown, K. L. (2003). From teacher-centered to learner-centered curriculum: improving learning in diverse classrooms. *Education*, *124*(1), 49–54. https://eds.b.ebscohost.com/eds/pdfviewer/pdfviewer?vid=1&sid=cf1bfdb1-c959-4486-bb9a-3d70716cdaba%40sessionmgr104

Herbst, P., & Kosko, K. W. (2014). Using representations of practice to elicit mathematics teachers' tacit knowledge of practice: A comparison of responses to animations and videos. *Journal of Mathematics Teacher Education*, *17*(6), 515–537. https://doi.org/10.1007/s10857-013-9267-y

Heyd-Metzuyanim, E. (2019). Changing teaching practices towards explorative mathematics instruction – the interweaving of teacher identity and pedagogical discourse. *Teaching and Teacher Education*, *86*, *102862*. https://doi.org/10.1016/j.tate.2019.06.016

Hill, H. C., Blunk, M. L., Charalambos, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., & All, D. L. (2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An Exploratory Study. *Cognition and Instruction*, *26*(4), 430–511.

Horn, I. S., & Kane, B. D. (2015). Opportunities for Professional Learning in Mathematics Teacher Workgroup Conversations: Relationships to Instructional Expertise. *Journal of the Learning Sciences*, *24*(3), 373–418. https://doi.org/10.1080/10508406.2015.1034865

Institute For Learning. (2015). *Mathematics, Grade 8, Calling Plans (Part 2)*. University of Pittsburgh. file:///C:/Users/mw1415/Downloads/9-LessonGuide-CallingPlans2-MS-Brovey.pdf

Kazemi, E., & Franke, M. L. (2004). Teacher Learning in Mathematics: Using Student Work to Promote Collective Inqiry. *Journal of Mathematics Teacher Education*, *7*, 203–235. https://doi.org/10.1023/B:JMTE.0000033084.26326.19

Kazemi, E., Franke, M., & Lampert, M. (2009). Developing Pedagogies in Teacher Education to Support Novice Teachers ' Ability to Enact Ambitious Instruction Developing Pedagogies in Teacher Education to Support Novice. *Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia*, *1*(May), 12–30.

Kraft, M. A., & Hill, H. C. (2020). Developing Ambitious Mathematics Instruction Through Web-Based Coaching: A Randomized Field Trial. In *American Educational Research Journal* (Vol. 57, Issue 6). https://doi.org/10.3102/0002831220916840

Ma, J. Y., & Singer-Gabella, M. (2011). Learning to teach in the figured world of reform mathematics: Negotiating new models of identity. *Journal of Teacher Education*, *62*(1), 8–22. https://doi.org/10.1177%2F0022487110378851

Nachlieli, T., & Heyd-Metzuyanim, E. (2021). Commognitive conflicts as a learning mechanism towards explorative pedagogical discourse. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-021-09495-3

National Council of Teachers of Mathematics (2014). Principles to Actions: Executive Summary. Reston, VA: NCTM.

Schoenfeld, A. H. (2018). Video analyses for research and professional development: the teaching for robust understanding (TRU) framework. *ZDM - Mathematics Education*, *50*(3), 491–506. https://doi.org/10.1007/s11858-017-0908-y

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Standards-Based Mathematics Instruction: A Casebook for Professional Development*. Teachers College Press.

Thompson, J., Windschitl, M., & Braaten, M. (2013). Developing a Theory of Ambitious Early-Career Teacher Practice. *American Educational Research Journal*, *50*(3), 574–615. https://doi.org/10.3102/0002831213476334

Van Es, E. A., Cashen, M., Barnhart, T., & Auger, A. (2017). Learning to Notice Mathematics Instruction: Using Video to Develop Pre-service Teachers ' Vision of Ambitious Pedagogy. *Cognition and Instruction*, *35*(3). https://doi.org/10.1080/07370008.2017.1317125

Weingarden, M., & Heyd-Metzuyanim, E. (2021). Opportunities for Explorative Participation in Different Achievement Groups. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education, Vol. 4* (pp. 264–272).

Khon Kaen, Thailand: PME.
Weingarden, M., & Heyd-Metzuyanim, E. (2020). The Realization Tree Assessment (RTA) tool as a representation of explorative teaching. In H. Borko & D. Potari (Eds.), *Proceedings of the 25th conference of the International Commission on Mathematical Instruction – ICMI study 25. Teachers of Mathematics Working and Learning in Collaborative Groups.* (pp. 708–715).
Weingarden, M., Heyd-Metzuyanim, E., & Nachlieli, T. (2019). The realization tree assessment tool – examining explorative participation in mathematics lessons. *Journal of Mathematical Behavior*, *56*, *100717*. https://doi.org/10.1016/j.jmathb.2019.100717

USING MINDFULNESS TO SUPPORT THE WELL-BEING OF PRESERVICE K-8 TEACHERS DURING A PANDEMIC

Rachael M. Welder
University of Nevada, Reno
rwelder@unr.edu

Megan Burton
Auburn University
meb0042@auburn.edu

Heidi Eisenreich
Georgia Southern University
heisenreich@georgiasouthern.edu

This paper reports initial findings from the implementation of a Mindfulness-Based Intervention (MBI) during the COVID-19 pandemic in an effort to mitigate the effects of stress and anxiety on preservice K-8 teachers (PTs) as learners of mathematics for teaching. Three teacher educators, at three U.S. institutions, implemented the MBI in their mathematics content and methods courses for PTs, as a way to connect with and holistically support students during a semester of pandemic-induced online instruction. Initial analysis of survey data (n=136) shows that PTs reported mindfulness helping them to become self-aware and learn strategies for regulating their behaviors. PTs saw value in applying mindfulness outside of the classroom and with their future students. By introducing coping strategies, we were able to show PTs the importance of caring for their mental and emotional well-being and the well-being of their future students.

Keywords: Preservice Teacher Education, Affect, Emotion, Beliefs, and Attitudes, Teacher Educators, Sustainability (of Teachers)

Teaching is a particularly demanding profession (Roeser et al., 2012) and teachers have long endured some of the highest rates of work-related stress among all occupations (Felver, 2016; Johnson et al., 2005). Higher levels of stress have been linked to higher rates of teacher burnout (McCormick & Barnett, 2011). In fact, prior to the COVID-19 pandemic, more than 44% of new teachers were leaving the profession within five years (Ingersoll et al., 2018) and the full effects of the pandemic on teacher retention is yet to be seen. Thus, we argue that supporting preservice K-8 teachers (PTs) in developing strategies to mitigate the negative effects of stress and anxiety should be a consideration of teacher preparation programs. "Mindfulness is an aspect of social-emotional competence that may protect them [teachers] from experiencing burnout and its negative consequences" (Abenavoli et al., 2013, p. 57). Furthermore, research has shown that a significantly larger percentage of elementary PTs experience high levels of mathematics anxiety when compared to other undergraduate students (Harper & Daane, 1998). Therefore, mathematics courses for PTs may provide a fruitful context for introducing such strategies.

The COVID-19 pandemic illuminated these long standing mental-health challenges related to the teaching and learning of mathematics and exacerbated these issues as classes moved online, instructional expectations and situations underwent frequent changes, and teachers and students felt isolated. Thus, three mathematics teacher educators, at three U.S. universities, designed and implemented a Mindfulness-Based Intervention (MBI) for PTs in mathematics content and methods courses, as a way to connect with and provide holistic support for their students during the fall of 2020, a semester of pandemic-induced online instruction.

Background

Mindfulness

Kabat-Zinn (2003) describes mindfulness as "the awareness that emerges through paying attention on purpose, in the present moment, and nonjudgmentally to the unfolding of experience moment by moment" (p. 144). Mindfulness is often associated with the practice of traditional meditation, which is rooted in Buddhism; however, over the past three decades secular mindfulness as a state, trait, process, and intervention has been adapted in settings such as schools, businesses, and medical facilities (Brazier, 2016; Gupta, 2019). The practice of mindfulness involves three key features of attention-training, "paying attention to the present moment, recognizing and classifying emotions, and experiencing more refined self-awareness in the present" (Ahmed et al., 2017, p. 26). By combining self-awareness, self-regulation, and self-compassion (Roeser et al., 2012), mindfulness can reduce stress (Bamber & Schneider, 2016; Poulin et al., 2008), increase quality of attention (Good et al., 2015), improve resiliency (Salmon et al., 2004), and provide people with tools that allow them "to 'respond' to a potentially anxiety-producing situation with greater effectiveness rather than to 'react' with escalation, panic or fear" (Miller et al., 1995, p. 197).

Mindfulness and Education

Recent and rapid growth in research addressing mindfulness-based interventions (MBIs) for both students and teachers (Felver & Jennings, 2016; Jennings, 2015) has included dozens of studies assessing the effects of mindfulness on college students (Bamber & Schneider, 2016, Bellinger et al., 2015, Brunyé et al., 2013). Research suggests that MBIs, ranging from brief informal activities to formal, sustained programs, may be particularly well-suited for supporting teachers, PTs, and college students in mitigating the negative effect of stress and anxiety (Burrows, 2018; Meiklejohn et al., 2012; Ramsburg & Youmans, 2013). Research regarding teachers indicates that mindfulness training can reduce reactivity, stress, and burnout while increasing teachers' sense of self-efficacy, acceptance of self and others, self-regulation, well-being, classroom management skills, creativity, and ability to cultivate positive student relationships (e.g., Bliss, 2017; Burrows, 2015; Jennings & Greenberg, 2009; Meiklejohn et al., 2012). Less research has examined the effects of mindfulness on PTs (Brown, 2017), although the Mindfulness-Based Wellness Education program at the University of Toronto has found some promising results in increased self-efficacy and social and emotional competence for PTs (Poulin et al., 2008; Poulin, 2009; Soloway, 2011; Soloway et al., 2011).

In addition to reducing negative states such as stress and anxiety (e.g., Bamber & Schneider, 2016), mindfulness has been linked to improving factors that can affect learning such as sustained attention, working memory, and concentration (Chambers et al., 2008; Jha et al., 2007), cognition (Rodgers & Raider-Roth, 2006; Zeidan et al., 2010), knowledge retention (e.g., Ramsburg & Youmans, 2013), and academic achievement (Bakosh et al., 2015; Bellinger et al, 2015; Brunyé et al., 2013; Weger et al., 2012). These findings inspired our design and implementation of an MBI to support our K-8 PTs during the pandemic in an effort to mitigate the effects of stress and anxiety while learning mathematics for teaching. Our work utilizes the theoretical framework of self-regulation learning theory, which postulates that stress can be alleviated and learning can be more effective when learners are able to self-regulate (Roeser et al., 2013; Zimmerman, 1989a, 1989b).

Methodology

During the fall of 2020, we implemented a semester-long MBI in our mathematics content and methods courses for PTs. Integrating mindfulness was seen as an important step in supporting our PTs as well as recognizing the balance needed in the teaching profession and the importance of humanizing education beyond the content. Our intervention consisted of introducing and engaging with students in two short (5-10 minutes) mindfulness practices per week, one completed at the beginning of each class meeting. The MBI was designed to introduce a new form of mindfulness each week, e.g., mindful breathing, mindful listening, mindful movement, body scan meditations, and gratitude journaling, for 15 weeks. Each week, the PTs engaged in two different, yet related, practices for each form. Most practices were implemented by sharing publicly-accessible videos or images made by leaders in the mindfulness community, e.g., Dan Harris and Sharon Salzburg (Happify, 2015).

As part of our data collection, PTs were asked to complete two surveys regarding mindfulness, one at the midterm and another at the end of the semester. Each survey consisted of 10 Likert scale items and 4 open-ended response questions. This report shares the qualitative analysis of responses from 136 PTs to the open prompt, "Mindfulness has helped me…".

Preliminary Qualitative Data Analysis

Initially the researchers read the data and one took the lead to identify major codes that emerged. Then the three researchers met and discussed the emergent codes and reorganized them based on mindfulness literature. This conversation led to code revisions and a new code list with definitions for each code. Next the data were recoded using this finalized list. To establish interrater reliability, the codes were checked by a second researcher who then coded the data a second time. The second coder produced codes that were matched by the original coder by 100%. However, the original coder had some additional codes, which were discussed to reach mutual agreement. Below, we will report on the three strongest themes that were revealed by the three most prevalent codes: self-awareness, self-regulation, and applications (listed in the order in which we will discuss them, see Table 1).

Table 1: Codes/Themes, Definitions, and Count

Code/Theme	Definition	Mid	Post	Total
Self-awareness	Awareness of bodily sensations (e.g., tension), emotions, attention, or automatic reactions	28	20	48
Self-regulation	Behavioral changes made as PTs learned strategies for responding to or managing stress, and/or anxiety	119	78	197
Applications	PTs identified a specific audience or situation where they found or could foresee benefits of using mindfulness practices (such as when focusing in class or for use with future students)	26	27	53

Results

Self-awareness

Self-awareness was a major theme identified by many participants. This included awareness of physical sensations in one's body (such as the breath or muscular tension), emotions, attention/focus, and typical responses to stressors. For example, one PT wrote, "Mindfulness has helped me become more aware of my emotions and feelings." Similarly, a PT reported, "Mindfulness helped me become more aware of how I am feeling and know how to handle the tensions and stressors of the day." Another PT shared, "[I am] more aware of my emotions so I am able to reevaluate my situation." One PT noted, "[I have] become more aware of how I have previously handled stress and anxiety and find new and healthier ways to handle those feelings."

Self-regulation

Self-regulation was the strongest theme identified by far and can already be noted in the sample PT comments above. This included being able to modify one's reactions to stress and anxiety, such as slowing down the body or breathe, being able to relax and calm oneself, and having more control over one’s emotions. The ability to regulate one’s feelings of stress and anxiety was mentioned more than any other code. PTs noted that mindfulness helped them “deal with the stress of school work better,” “control my anxiety,” and “learn how to take time to breathe and take my mind off of what is stressing me out.” One PT shared that the mindfulness practices helped them “realize that in times of stress, I can't JUST push through. I have to take a deep breath and mentally remind myself that I CAN DO HARD THINGS.” Another stated, “It is easy to get caught up in the stress and anxiety of the day, but doing these [mindful] exercises help me when I get worked up.” One PT reported “learn(ing) how to regulate my emotions outside of class. Another prominent code within this theme was a sense of empowerment that seem to arise through the development of strategies to apply in moments of dysregulation. One PT noted, that they can now “identify stressors and develop ways to cope and overcome them.”

Applications

The third code that emerged is that PTs acknowledged specific ways in which they used or plan to use the mindfulness strategies they learned through this MBI. PTs noted that mindfulness helped them concentrate in class. For example, one shared that the MBI helped them, “calm down a bit before class starts.” Another said the strategies helped them to “begin each class with a clean slate, by helping me forget all the stressors and become more focused for class.” One PT even noted that the practices helped them “wake up” so they could pay more attention in early morning classes. Specific to the needs of PTs enrolled in mathematics courses, one noted that mindfulness helped them “handle my math-related anxiety.” Some PTs who were further in their programs and/or involved in field experiences acknowledged potential benefits of using mindfulness practices with their future students. One PT said that it helped them, “think about what to do for some students facing stress,” while another said participating in the activities helped them “find strategies that I will be able to use with my future students to manage stress.”

Discussion

The results of analyzing this single open-response question already shows evidence that the MBI helped PTs develop meta-awareness in ways that empowered them to effectively modify their behaviors, i.e., self-regulate (Vago & Silbersweig, 2012), through the development and application of new strategies. It is interesting that students reported their abilities to regulate their behaviors or mindsets at a much higher rate than their awareness of such things. However, intentional self-regulation requires awareness of the experience (i.e., thought pattern, emotion, reaction, or bodily sensation) one wishes to regulate (Bandura, 1991). Thus, even though the self-awareness code appeared fewer times, the prevalent of the self-regulation code suggests that there was more self-awareness occurring than the PTs reported or were consciously aware of.

This study also found that PTs acknowledged mindfulness as being beneficial to them within the context of the course the MBI was implemented. Thus, future research might explore how MBIs impact PTs’ mathematical achievement and/or self-efficacy in their teacher preparation programs. However, we hope that the benefits PTs experience will support their development of resilience by helping them mitigate the negative effects of stress and anxiety, especially within the first few years of teaching. Therefore, we see value in extending this work with longitudinal research studying the impact of MBIs on teachers as they enter the profession and exploring if they stay in the profession long-term. This study also found that PTs foresaw longer-term applications of mindfulness for supporting their future students. This was particularly interesting as the PTs were not directly asked about their intentions to use mindfulness in their future classrooms. However, longitudinal research could explore the effects of the MBI on PTs’ future application of mindfulness with their future students.

References

Abenavoli, R. M., Jennings, P. A., Greenberg, M. T., Harris, A. R., & Katz, D. A. (2013). The protective effects of mindfulness against burnout among educators. *The Psychology of Education Review*, *37*(2), 57–69.

Ahmed, K., Trager, B., Rodwell, M., Foinding, L., & Lopez, C. (2017) A review of mindfulness research related to alleviating math and science anxiety. *Journal for Leadership and Instruction*, v16 n2 p26-30.

Bakosh, L. S., Snow, R. M., Tobias, J. M., Houlihan, J. L., & Barbosa-Leiker, C. (2016). Maximizing mindful learning: Mindful awareness intervention improves elementary school students' quarterly grades. *Mindfulness*, *7*(1), 59-67.

Bamber, M. D., & Schneider, J. K. (2016). Mindfulness-based meditation to decrease stress and anxiety in college students: A narrative synthesis of the research. *Educational Research Review*, *18*, 1–32.

Bandura, A. (1991). Social cognitive theory of self-regulation. *Organizational Behavior and Human Decision Processes*, *50*(2), 248-287.

Bellinger, D. B., DeCaro, M. S., & Ralston, P. A. S. (2015). Mindfulness, anxiety, and high-stakes mathematics performance in the laboratory and classroom. *Consciousness and Cognition*, *37*, 123–132.

Bliss, S. A. (2017) Exploring Shunyata (emptiness) and the cultivation of mindfulness practices: Educators finding their zero-point balance, *Childhood Education*, *93*(2), 114-118, DOI: 10.1080/00094056.2017.1300489

Brazier, D. (2016). Mindfulness: Traditional and utilitarian. In R. E. Purser & D. F. Adam (Eds.), Handbook of mindfulness: Culture. *Context, and Social Engagement*. Springer.

Brown, R. (2017). The perceived impact of mindfulness instruction on pre-service elementary teachers, *Childhood Education*, *93*(2), 136-146. DOI: 10.1080/00094056.2017.1300492

Brunyé, T. T., Mahoney, C. R., Giles, G. E., Rapp, D. N., Taylor, H. A., & Kanarek, R. B. (2013). Learning to relax: Evaluating four brief interventions for overcoming the negative emotions accompanying math anxiety. *Learning and Individual Differences*, *27*, 1–7. https://doi.org/10.1016/j.lindif.2013.06.008

Burrows, L. (2015). Inner Alchemy. Transforming dilemmas in education through mindfulness. *Journal of Transformative Education*, *13*(2), 127–139.

Burrows, L. (2018). Safeguarding mindfulness in schools and higher education: A holistic and inclusive approach. New York, NY: Routlege.

Chambers, R., Lo, B. C. Y., Allen, N. B. (2008). The impact of intensive mindfulness training on attentional control, cognitive style, and affect. *Cognitive Therapy and Research*, *32*, 303-322. DOI 10.1007/s10608-007-9119-0

Felver, J. C. (2016). Book review: Patricia A. Jennings: Mindfulness for teachers: Simple skills for peace and productivity in the classroom. W. W. Norton, New York, NY, 2015, 288 pp., *Mindfulness*, *7*, 286-288.

Felver, J. C., & Jennings, P. A. (2016). Applications of mindfulness-based interventions in school settings: An introduction. *Mindfulness*, *7*, 1-4. https://doi.org/10.1007/s12671-015- 0478-4

Good, D. J., Lyddy, C. J., Glomb, T. M., Bono, J. E., Brown, K. W., Duffy, M. K., Baer, R. A., Brewer, J. A., & Lazar, S. W. Lazar. (2015). Contemplating mindfulness at work: An integrative review. *Journal of Management*, *42*(1), 114-142.

Gupta, K. (2019). The sacred to the secular: Using mindfulness and meditation as instructional methods in academia. *Meditation and Mindfulness in Adult Education*, *161,* 21-32. DOI: 10.1002/ace.20308

Happify (2015, March 27). *Meditation for Beginners - Featuring Dan Harris and Sharon Salzberg* [Video]. YouTube. https://www.youtube.com/watch?v=mtsdz_jhB7c&t=1s

Harper, N. W., & Daane, C. J. (1998) Causes and reduction of math anxiety in preservice elementary teachers, *Action in Teacher Education*, *19*(4), 29-38. DOI: 10.1080/01626620.1998.10462889

Ingersoll, R., Merrill, E., Stuckey, D., & Collins, G. (2018). Seven trends: The transformation of the teaching force, updated October 2018. Research Report (#RR 2018–2). Consortium for Policy Research in Education, University of Pennsylvania.

Jennings, P. A. (2015): Mindfulness for teachers: Simple skills for peace and productivity in the classroom. W. W. Norton, New York, NY.

Jennings, P. A., & Greenberg, M. T. (2009). The prosocial classroom: Teacher social and emotional competence in relation to student and classroom outcomes. *Review of Educational Research*, *79*(1), 491–525.

Jha, A. P., Krompinger, J., & Baime, M. J. (2007). Mindfulness training modifies subsystems of attention. *Cognitive, Affective, & Behavioral Neuroscience*, *7*(2), 109-119.

Johnson, S., Cooper, C., Cartwright, S., Donald, I., Taylor, P. and Millet, C. (2005). The experience of work-related stress across occupations, *Journal of Managerial Psychology*, *20*(2), 178-187.

Kabat-Zinn, J. (2003). Mindfulness-based interventions in context: Past, present, and future. *Clinical Psychology: Science & Practice*, *10*(2), 144-156.

McCormick, J. & Barnett, K. (2011). Teachers' attributions for stress and their relationships with burnout. *International Journal of Educational Management*, *25*, 278–293.

Meiklejohn, J., Phillips, C., Freedman, M. L., Griffin, M. L., Biegel, G., Roach, A., ... Saltzman, A. (2012). Integrating mindfulness training into K-12 education: Fostering the resilience of teachers and students. *Mindfulness*, *3*(4), 291-307.

Miller, J. J., Fletcher, K., & Kabat-Zinn, J. (1995). Three-year follow-up and clinical implications of a mindfulness meditation-based stress reduction intervention in the treatment of anxiety disorders. *General Hospital Psychiatry*, *17*, 192-200.

Poulin, P. A., Mackenzie, C. S., Soloway, G., & Karayolas, E. (2008). Mindfulness training as an evidenced-based approach to reducing stress and promoting well-being among human services professionals. *International Journal of Health Promotion and Education*, *46*(2), 72-80.

Poulin, P. A. (2009). Mindfulness-based wellness education: A longitudinal evaluation with students in initial teacher education (Unpublished doctoral dissertation). University of Toronto, Canada.

Ramsburg, J. T., & Youmans, R. J. (2013). Meditation in the higher-education classroom: Meditation training improves student knowledge retention during lectures. *Mindfulness* *5*(4), 431–441. DOI: 10.1007/s12671-013-0199-5

Roeser, R. W., Schonert-Reichl, K. A., Jha, A., Cullen, M., Wallace, L., Wilensky, R., & Harrison, J. (2013). Mindfulness training and reductions in teacher stress and burnout: Results from two randomized, waitlist-control field trials. *Journal of Educational Psychology*, *105*(3), 787–804.

Roeser, R. W., Skinner, E., Beers, J. & Jennings, P. A. (2012). Mindfulness training and teachers' professional development: An emerging area of research and practice. *Child Development Perspectives*, *6*, 167–173.

Rodgers, C. R., & Raider-Roth, M. B. (2006). Presence in teaching. *Teachers and teaching: Theory and Practice*, *12*(3), 265-287. DOI: 10.1080/13450600500467548

Salmon, P., Sephton, S., Weissbecke, I, Hoover, K., Ulmer, C. & Studts, J. I. (2004). Mindfulness meditation in clinical practice. *Cognitive and Behavioural Practice, 11*, 434-46.

Soloway, G. B. (2011). Preparing teacher candidates for the present: Exploring the praxis of mindfulness training in teacher education (Unpublished doctoral dissertation). University of Toronto, Canada.

Soloway, G. B., Poulin, A., & Mackenzie, C. S. (2011). Preparing new teachers for the full catastrophe of the 21st century classroom: Integrating mindfulness training into initial teacher education. In A. Cohan & A. Honigsfeld (Eds.), *Breaking the mold of preservice and in-service teacher education* (pp. 221–227). Lanham: R and L Education.

Vago, D. R. & Silbersweig, D. A. (2012, October 25). Self-awareness, self-regulation, and self-transcendence (S-ART): A framework for understanding the neurobiological mechanisms of mindfulness. *Frontiers in Human Neuroscience*, *6*, 1-30. https://doi.org/10.3389/fnhum.2012.00296

Weger, U. W., Hooper, N., Meier, B. P., Hopthrow, T. (2012). Mindful maths: Reducing the impact of stereotype threat through a mindfulness exercise, *Consciousness and Cognition*, *21*(1), 471-475.

Zeidan, F., Johnson, S. K., Diamond, B. J., David, Z., & Goolkasian, P. (2010). Mindfulness meditation improves cognition: evidence of brief mental training. *Conscious Cognition*, *19*(2):597-605. DOI: 10.1016/j.concog.2010.03.014

Zimmerman, B. J. (1989a). A social cognitive view of self-regulated academic learning. *Journal of Educational Psychology*, *81*, 329-339.

Zimmerman, B. J. (1989b). Models of self-regulated learning and academic achievement. In B. J. Zimmerman & D. H. Schunk (Eds.), *Self-regulated learning and academic achievement: Theory, research, and practice* (pp. 1-25). New York, NY: Springer.

EARLY MATHEMATICS TEACHER PREPARATION EVALUATION RUBRICS FOR THE CONTEXT OF LIVE DISCUSSION FORUMS

Jeremy Zelkowski
The University of Alabama
jzelkowski@ua.edu

Tye Campbell
Crandall University
tye.campbell@crandallu.ca

Teacher candidates bring many beliefs and interpretations of mathematics teaching and learning at the start of their teacher preparation coursework (e.g. methods courses, field experiences, assessment). Well-prepared beginning teachers in many instances requires programs and designed experience to breakdown unproductive beliefs and/or improve dispositions to align to best practices and equitable dispositions for the teaching and learning of mathematics. Our study focused on developing and validating two rubrics to evaluate teacher candidates' talk during live discussion forums in the first month of their initial teacher preparation program coursework with the intent to inform to varying degrees, where teachers candidate talk is situated (or not) in alignment to foundational readings and productive beliefs. Early validity evidence for rubric use is presented with suggestions for informative use and practice.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Preservice Teacher Education; Teacher Beliefs; Measurement

Many secondary mathematics teacher candidates (TCs) enter their preservice preparation programs with traditional (e.g. cultural norms, conventional) beliefs about education, believing, for example, that all or most student learn best in teacher-centered classroom instructional methods and/or performing rote practice with learn-by-example replication/practice (Cady et al., 2006; Conner et al., 2011; Cooney, 1999). Combating such beliefs is laborious since literature suggests TCs are likely to replicate practices most prevalent in their own educational experiences in K-12 mathematics classrooms (Borko at al., 1992; Cross, 2009). This idea is still generally supported by the more recent publication of the National Council of Teachers of Mathematics' (NCTM) *Catalyzing Change in High School Mathematics* (NCTM, 2017), where it is noted that only pockets of excellence exist within the high school for classrooms modeling the eight effective mathematics teaching practices (MTPs) (NCTM, 2014) developed from research over the last few decades. More troublesome in such conventional and culturally/historically normed mathematics teaching methods is the rooting of inequitable opportunities for non-dominant cultures creating a group of students becoming historically marginalized as an outcome (Aguirre et al., 2013). Our work is designed to fit within the challenges of preservice mathematics teacher education that presses to create critical dissonance within the status quo of teacher preparation programs with the designed intent of a resonant harmony of well-prepared, equitable teaching by newly minted mathematics teachers. We recognize this challenging work to rock the teacher preparation magnate model that has generally continued the cyclic pattern of student to TC to beginning teacher that does not significantly alter on a grander scale the teaching and learning of mathematics embedded in the eight MTPs.

Over the course of the last two decades, our preparation program has used and/or developed program measures as a means and intent to be predictive, yet deeply informative, with respect to desired outcomes for interns' (full-time student teachers) mathematics teaching practices and ultimately first-year teaching. We view the importance of internal program measures as needing validity evidence for their effective use to inform and shape the development of all teacher candidates over time that helps shifts beliefs and practices when needed to desired TC outcomes. While we recognize the limitations of the data collection and rubrics presented in this paper, we believe this work presses preparation programs to push boundaries and challenge barriers that have not moved quickly enough to eradicate conventional and culturally/historically normed practices in mathematics teacher preparation to ultimately produce systematic change in K-12 schools in the teaching of mathematics to be equitable for all students. Our work firmly fits within most guiding questions for PME-NA 2022 program, as well as adding to the psychology of how the field views mathematics teacher preparation program components as a learning to teach experience that prepares and aligns to best practices as a systematic requirement to become a licensed teacher of mathematics. Our research questions are as follows:

1. What is the predictive nature of two rubrics scoring first semester teacher candidate discussion forums in relation to entering the teaching profession as validity evidence?
2. To what degree does the alignment of teacher candidates' first semester talk with assigned readings relate to completion of their preparation program?
3. To what degree do teacher candidates publicly stated first semester beliefs (productive or unproductive) during discussions relate to completion of their preparation program?
4. Do males and females (as identified on self-report) show any differences within the analyses given TC imbalance of 2:1 female to male within the preparation program?

Theoretical Framework

The American Educational Research Association, American Psychological Association, and the National Council on Measurement in Education *Standards for Educational and Psychological Measurement in Education* (AERA, APA, & NCME, 2014) introduces the need for validity evidence for measurement within educational contexts. The validation framework for the development, use, and interpretation of rubrics is the central tenet of our study.

Rubric development for affective domains in teacher education is an arduous task. Constructing a case for rubric validity is an ongoing process involving multiple facets of validity evidence, including the notion of the construct(s) in which are sought to be determined. Instruments in social sciences vary in precision and quality in the measurement of a particular construct, whereas the validation process should be centered on rubric interpretation and use (AERA et al., 2014; Bostic et al., 2019; Lavery et al., 2019). Kane (2013) denotes the use and interpretations of instruments [rubrics] require more validity evidence than is the case for less ambitious applications of instruments [rubrics]. Kane (2016) later indicated validation research is not easy but that it is generally sensible to grow solid evidence with manageable efforts.

Situational Perspectives of Rubric Use in Teacher Preparation

Rubrics are used extensively in teacher education preparation programs for assessment and for accreditation purposes, but there has been a lack of validity evidence (published) with arguments for specific rubrics and their appropriate interpretation and use (Hill & Shih, 2009; Howell et al, 2019; Lavery et al, 2019). Well-constructed, systematically developed instruments with validity evidence in mind for their intended purposes have strong potential to provide teacher educators valuable and predictive data in the development trajectory of TCs. Traditionally, rubrics used programmatically are grounded in the documentation for accreditation purposes. That is to say for example, that for the Council for the Accreditation of Educator Preparation (CAEP) or jurisdictional level accreditation, rubrics are used to report aggregate level performance of program completers on an annual basis at different points in time within TCs' program coursework and field experiences. Yet, most of the rubric data is not generate with a TC-level predictive nature in mind but rather for program accountability and documentation to become/remain accredited (CAEP, 2013, 2022). We challenge this status quo in teacher preparation in the use of rubrics beyond that of accountability, but significantly more so, with predictive validity in mind as a means to inform program faculty on their effectiveness of shaping TCs over time to align practices and beliefs to the mathematics education literature that ultimately removes barriers for marginalized students' opportunities to engage with and learning mathematics. Should such internal program rubrics exist, ultimately program faculty can modify program curricular experiences and provide TCs feedback identified earlier. We recognize that some TCs have only a single semester of preparation before the full-time (or extensive) student teaching internship. This fact is presented in our discussions and limitations later.

Teacher Candidate Beliefs in Teacher Preparation

Of particular importance to this study is NCTM's (2014) *Principles to Actions*, designed to describe "the conditions, structures, and policies that must exist for all students to learn" (p. vii). *Principles to Actions* (PtA) advances and redefines NCTM's (2000) six guiding principles for school mathematics: teaching and learning, access and equity, curriculum, tools and technology, assessment, and professionalism. Within each of these domains, NCTM (2014) outlined productive and unproductive beliefs that support or hinder mathematical learning for all students. NCTM's delineation between productive and unproductive beliefs in *PtA* serves as a standard by which TCs begin to think about their beliefs and how potential practices as a result of such beliefs may not be in the best interest of teaching and learning for all students.

Research suggests whole class discussions can positively influence TC beliefs within a whole class (Stohlmann, 2015; White et al., 2016) though less is known about the variation of influence and change on an individual TC level. The field would benefit from empirical research investigating the nature of individual contributions during whole class discussion and how those contributions shift beliefs toward or align to productive visions of teaching and learning mathematics within a teacher preparation programs. For example, if individual contributions during whole class discussion significantly influence TCs' teaching vision and beliefs, program faculty may utilize whole class discussions to learn about TCs' situated beliefs early enough to create situational interventions and even potential remediation activities. Stohlmann (2015) and White et al. (2016) both found engaging TCs in whole class discussions with purposefully chosen questions/tasks/prompts aided in TCs developing productive beliefs. Utilizing whole class discussion allows TCs to construct their own knowledge through discussing relevant material, but it further reinforces positive dispositions by enculturating TCs into a desirable learning environment. Our study is grounded in these ideas about using whole class discussions, which we refer to as TC live discussion forums, as the setting for the purposeful development of two rubrics to record and monitor TCs contributions over three live discussion forums in the first month of teacher preparation [education] coursework.

Methodology

This paper reports on three distinct phases of the study in which rubric development was sought to create a tool to more reliably assess what teacher candidates put forward during live discussion forums. First, the development of two rubrics was needed to align to our research questions. Second, our team then looked to start the validity and revision process of testing the rubrics to move forward to the third phase. The final phase was to analyze the data in which the rubrics development sought to examine TCs alignment to the readings and productive beliefs (or not). Consider that in all mathematics teacher preparation programs, TCs who begin coursework all do not finish the program as certified mathematics teachers. Therefore, our study methodology was to develop rubrics, test the rubrics on existing data to generate validity evidence, and then ultimately have validated rubrics situated to have a predictive ability to improve TC outcomes and program completion aligned to the desired characteristics in the field of mathematics teachers who exemplify productive visions and beliefs about teaching and learning mathematics (i.e. implementing the MTPs).

Context of Participant Generated Discussion Forum Data

Within the teacher preparation program at The University of Alabama, three live discussion forums have existed in the first month of the program for more than a decade. The notes about TC talk have been analyzed qualitatively in the assessment of progress in the program, but never has such analyses resulted in a mechanism to understand which TCs might program faculty provided deeper feedback on their development visions and beliefs about teaching and learning mathematics at such an early stage of the program. A seven-year period of data collection was used in which all three live discussion forums were implemented without deviation, holding the format, readings, and prompts to be consistent, as well as the mechanism for how data was noted and recorded in a spreadsheet. A small subset (~10%) via random selection of the existing data across all seven years was used for the development, test, revision, and initial validation work to finalize the rubrics for the full analysis. It is important to paint a vivid picture of how the data is collected during the live discussion forums.

The data used in this study was collected in a first semester TPP course (fall semester, junior year) which is embedded in a sequence of three consecutive mathematics teaching methods courses prior to the student teaching internship (see Zelkowski et al, 2018, 2021) for more details of program structure and sequence. The three live discussion forums took place at the start of the mathematics technology methods course during the first four weeks where each class was held twice a week for 1-hour 15-minutes (class meeting 2, 4, and 7). Lessons in the methods course between each of the three discussion forums included technology driven activities (see Zelkowski (2013), Bismarck et al., (2014) for examples) with TCs exposed as engaged learners to the eight Standards for Mathematical Practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Data Collection

Over seven years, 110 PSMTs (77-female, 33-male) were enrolled in the methods course.

The readings were the first three chapters of *Teaching Secondary and Middle School Mathematics* (Brahier, 2013) titled: (a) Mathematics as a Process, (b) Principles of Mathematics Education, and (c) Learning Theories and Psychology in Mathematics Education. During each live discussion forum, the methods course professor projected a PowerPoint presentation with discussion prompts based on the focus questions of each chapter. The professor remained silent concentrating on accurate notetaking, allowing TCs to engage in an open, student-led forum related to each discussion prompt based upon the focus questions of each chapter. There were no restraints regarding how much individual TCs could talk during each prompt or discussion forum, though the professor encouraged equitable opportunities for all and respect among peers.

For this particular method of data collection (i.e. researcher as complete observer), Creswell (2003) discusses many advantages to this design such as it is useful in exploring topics that may be uncomfortable for participants to discuss, information can be discovered as it is revealed, and unusual aspects can be noticed with the researcher's firsthand experience with the participants. Creswell points to some limitations such as the researcher being intrusive or lacking good attention and observation skills. To mitigate such concerns, the methods professor had already spent five years observing and recording data in such forums. Meetings with students allowed for the methods professor to check and verify the notes taken in years prior to the data used in this study. Given the five years of data collection and checking with students, we are confident that most comments captured reflect the TCs spoken words during the seven-year period of this study's data collection.

The professor took notes by constructing a grid with each TCs name as one column with additional columns for each prompt. TCs had name-tents on display for complete accuracy of comments being recorded. The quality of discourse and level of contribution during each of the three live discussion forums constituted about 3% of the final course grade (about 9% total) as an encouragement to participate early with peers at the start of the methods course sequence.

Rubric Development Process

The development of two rubrics and the validation work for their use to evaluate TCs' discussion contribution were rating two constructs of interest. The first rubric was designed to capture the alignment and interpretation of TCs' responses with the assigned chapter readings. The second rubric was designed to capture the alignment and interpretation of the responses to the unproductive and productive "beliefs about teaching and learning mathematics" within *Principles to Actions (PtA)* (NCTM, 2014, p.11) prior to TCs reading *PtA*.

We initially began with a four-level rubric (0-3) and two raters. We randomly scored about 3% of the forum data using the first rubric iteration. We discovered that an additional rubric-level would be needed and that a revision of the language was necessary as there were too many examples of implicit scoring decisions (e.g. higher or lower with less agreement). The rubrics were revised and retested with a new random sample of responses. After a third small revision of language in a few cells of the rubric, the third iteration was finalized.

In terms of each rubric, our revision process noted above went through three cycles in scoring a small sample until we were confident of removing as much subjectivity as possible between each level of the rubric. Ultimately, having enough levels in Rubric-1 was needed to consider the follow-up talk of a TC in terms of whether they were giving opinion supported by the interpretation of the reading or just giving opinion not supported by the reading. Whereas, Rubric-2, the sole purpose was to identify statements that do not align or do align to the productive beliefs tables in *PtA*. The rubric final iterations are shown in Figure 1.

Rubric-1. Alignment of Response with Reading

Level 0	Level 1	Level 2	Level 3	Level 4
Student did not engage in the discussion forum	Responses across prompts are mostly unrelated to the reading OR reveals misinterpretation of content of chapter	Responses across prompts are mostly related to the reading but is based on opinion rather than the content of the chapter	Responses across prompts mix personal opinion and correct interpretation of content of the chapter	Responses across prompts rely on content of the chapter but might also include some personal interpretation

Rubric-2. Alignment of Response with *PtA* Beliefs about Teaching & Learning Mathematics

Level 0	Level 1	Level 2	Level 3	Level 4
Student did not engage in the	Response across prompts reveals statements which align with	Response across prompts partially reveals statements which align with at	Response across prompts reveals more statements which	Response across the prompts reveals little evidence of statements that align

discussion forum	unproductive beliefs	least one productive belief	align with productive beliefs than not	with unproductive beliefs

Figure 1: Finalized Rubrics for Scoring TCs Alignments within the Discussion Forums

Results

Rater Reliability of Rubrics

One rater scored all three discussion forums for all N=110 TCs with both rubrics which generated 660 spreadsheet cells. We randomly selected 10% of the cells for each rubric for a second rater to score independently. The rater agreement was 78.8%. We discussed our scores that were not in alignment in reference to the rubrics and the TCs discussion forum contributions. Rater one revisited all 660 cells and rescored. Independently, rater two randomly selected another 10% of cells to score for each rubric. There were 6 cells of overlap with the initial 10% selected for a total of 126 cells (19.1% of all). Rater two rescored the initial 66 cells and scored the additional 60 cells. The rater agreement was 88.1%. The results of each rubric scoring are presented in Table 1. We proceeded then to compute the associated Cohen's Kappa, Cronbach Alpha, and Intra-class Correlation Coefficient (ICC) presented in the note of Table 1.

Table 1: Rater Agreement for Rubric-1 and Rubric-2

Rubric-1	Rater 2					Rubric-2	Rater 2				
Rater 1	0	1	2	3	4	Rater 1	0	1	2	3	4
0	6	0	0	0	0	0	6	0	0	0	0
1	0	1	0	0	0	1	0	9	3	0	0
2	0	0	29	4	0	2	0	0	19	2	0
3	0	0	1	19	0	3	0	0	1	20	1
4	0	0	0	2	1	4	0	0	0	1	1

Note. Rubric-1 Statistics: Kappa=0.819, ICC=0.934, Cronbach Alpha=0.966. Rubric-2 Statistics: Kappa=0.822, ICC=0.939, Cronbach Alpha=0.969. All statistics sig. @ p<0.001 level.

Relationship of Rubric Scores to Program Completion

Rubric-1 and Rubric-2 produce two scores for each of the three live discussion forums and total scores (sums of each forum). Of the N=110 TCs, 64 were program completers (~58%), 39 left the program (~35%) for academic and/or personal decisions not to enter teaching, and seven (~6%) eventually entered teaching through an alternative non-certified route. With this knowledge, our interest is whether Rubric-1 and/or Rubric-2 have any predictiveness in classifying TCs as a likely program completer or not from semester-1 discussions. More importantly, rubric scores should provide methods faculty early insights to improve outcomes related to core program readings, beliefs, and ultimately practices.

Two quantitative analysis methods were employed to address the research questions. Zelkowski (2011) stated, "the basic idea underlying this statistical method was to determine whether these groups differed significantly with respect to the mean of" (p.34) rubric levels individually and collectively. Zelkowski further described the use of Discriminant Analysis as a method that indicates the nature of the predictor variables (i.e. both rubrics) in contributing to group separation (i.e. outcome group) through a linear modeling process. We used Chi-square for each rubric across each of the three live discussion forums (six total analyses) considering our TCs three possible outcomes. Because research suggests males and females contribute differently to whole class discussions (e.g. Guzzetti & Williams, 1996), we further examined if there were any differences across TCs gender.

Statistically Significant Findings

Rubric-1 was found to be least predictive of program outcome for all three forums. However, the Chi-Square analyses found statistically significant results for Rubric-1 with the third discussion forum chapter (Learning Theories and Psychology in Mathematics Education) for females but not males. Given the ratio of 77:33 TCs' gender imbalance, this is not surprising. This is further examined in the discussion.

The strongest results were found with the Discriminant Analyses. We found correctly classify males was significant in predicting a final outcome, but not females alone and not all N=110 TCs together in the analyses. Based on each Rubric-2, as well as the sum of both rubrics, males were correctly classified in the analyses into their outcome between 60% and 76%. Rubric-2 scores for chapter 1 forum predicted the outcome 60.6% correctly, chapter 2 forum predicted the outcome 69.7% correctly, and the sum of all three forums predicted the outcome 75.8% correctly. When considering both rubric scores on all three forums (six total scores summed), male outcomes in the program were predicted 69.7% correctly.

Discussion

The purposes of the discussion forums have always been driven by introducing TCs to three foundational chapters setting the stage for mathematics teaching and learning. More importantly, the design engages TCs with peers to hear, listen, and discuss with each other's interpretations of the readings and injecting their own beliefs. Because these three forums immediately provide about 9% of the course grade, there is an early responsibility of TCs to read, be ready to discuss, and openly discuss critical foundations of mathematics education with their peers. We report here that less than 5% of TCs did not engage at least in at least one of three discussion forums.

Program Use of Rubrics

Our intent in the development of these rubrics and working through validation processes for their use and interpretation with desired outcomes, was and still is, to understand TCs potential cognitive conflicts with aspects of mathematics teaching and learning best practices. We further wanted to understand if early semester-1 discussions could be assessed to improve attrition and begin building improved belief structures. Our use of these rubrics in the program serves as informative early indicators of cognitive conflict with readings and/or productive beliefs about teaching and learning mathematics. In our previous program analyses (Zelkowski et al., 2018, 2021), we focused on the impact of key program assessments, coursework, and structural sequence on content knowledge and pedagogical content knowledge for program completers. These two rubrics across three live TC discussion forums provides an early indicator in which personal emails, one-on-one conversations, and high-quality feedback can be provided to TCs to increase their likelihood of program completion if teaching is truly their career desire. We do not see these rubrics as valuable in saving or rescuing those who decide to change majors out of teaching, though we do see their use as indicative and valid to improve outcomes and reduce attrition.

Implications for Mathematics Teacher Education

Rubrics are widely used in teacher education as we previously discussed, including mathematics teacher education. Rubrics serve excellent purposes regarding accreditation of programs, admission to teacher education programs, assignments and grades in coursework, and to provide feedback to TCs. However, the literature rarely points to the issues with the validity of using self-created rubrics for assessment without rigorous validity work having ensued (Hill & Shih, 2009; Howell et al., 2019; Lavery et al., 2019). That is to say, such rubrics generally lack one or more of the categories of validity evidence, or have none at all, for establishing arguments for rubric use in making important interpretations and decisions in mathematics teacher preparation. More precisely, what constructs do rubrics measure and to what degree are the use and interpretation valid? These are difficult questions to face without some process in the validation of the rubrics that methods course faculty may use. We provide our results and findings as a way to encourage rubric use but by generating validity evidence for use and their interpretation. Rubrics as any sort of measure in mathematics teacher education programs, ultimately should be providing faculty and TCs some predictive value about their development towards first-year teaching. Further, we demonstrate a methodology that aligns to the AERA, APA, and NCME (2014) for using rubrics for purposes of decision making, assessment, and programmatic outcomes with aims to reduce attrition and provide objective scoring for TCs.

Conclusion

As researchers, we are interested in examining the 'next step' of remediation when TCs do not exhibit movement towards productive beliefs regarding teaching and learning and/or share personal experiences as 'proof' over well-structured readings (i.e. Brahier). That is, how can methods faculty support learners whose beliefs regarding teaching and learning are not influenced by readings and discussions? We made several revisions to our program design over the last decade to account for these considerations (Zelkowski et al., 2018; 2021; Zelkowski & Gleason, 2018), but there is much to learn regarding effectiveness of program components, course designs, and the validation of key assessments.

We hope this work provides stimulus for discussion, research, and practice towards aiding TCs in developing productive beliefs about teaching and learning mathematics within their preparation program. Developing and validating rubric use and interpretation is a critical piece to not only improving mathematics teacher education, but also measuring the effectiveness of interventions aimed at stronger TC development. This work may serve as an example of the utility of rubrics in mathematics teacher education and provide methods faculty with a process for the validation of rubric use in critical junctions of teacher preparation programs.

Acknowledgments

This work was supported in part by the National Science Foundation (NSF) Grants #1849948, 1340069. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the author(s) and do not necessarily reflect the views of the NSF.

References

Bismarck, S., Zelkowski, J., & Gleason, J. (2014). Predicting future gas prices using the standards for mathematical practice. *Mathematics Teacher, 107*(9), 694-700.

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics: Rethinking Equity-based Practices*. Reston, VA: National Council of Teachers of Mathematics.

Brahier, D. J. (2013). *Teaching secondary and middle school mathematics* (4th edition). Pearson.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194–222.

Bostic, J., Krupa, E., & Shih, J. (2019). Introductions: Aims and scope for assessments in mathematics education contexts: Theoretical frameworks and new directions. In J. Bostic, E. Krupa, & J. Shih (Eds.), *Assessment in mathematics education contexts: Theoretical frameworks and new directions* (pp. 1-11). Routledge.

Cady, J., Meier, S. L., & Lubinski, C. A. (2006). The mathematical tale of two teachers: A longitudinal study relating

mathematics instructional practices to level of intellectual development. *Mathematics Education Research Journal, 18*(1), 3–26.
Conner, A., Edenfield, K.W., Gleason, B.W. & Ersoz, F.A. (2011). Impact of a content and methods course sequence on prospective secondary mathematics teachers' beliefs. *Journal of Mathematics Teacher Education, 14*(6), 483–504.
Cooney, T. J. (1999). Conceptualizing teachers' ways of knowing. *Educational Studies in Mathematics, 38*, 163–187.
Council for the Accreditation of Educator Preparation (CAEP). (2013). CAEP Initial Level Standards. Author.
Council for the Accreditation of Educator Preparation (CAEP). (2022). CAEP Initial Level Standards. Author.
Creswell, J. W. (2003). *Research design: Qualitative, quantitative, and mixed methods approach* (2nd edition). Sage.
Cross, D. (2009). Alignment, cohesion, and change: Examining mathematics teachers' belief structures and their influence on instructional practices. *Journal of Mathematics Teacher Education, 12*, 325–346.
Guzzetti, B. J., & Williams, W. O. (1996). Gender, text, and discussion: Examining intellectual safety in the science classroom. *Journal of Research in Science Teaching, 33*(1), 5-20.
Hill, H., & Shih, J. (2009). Examining the quality of statistical mathematics education research. *Journal of Research in Mathematics Education, 40*(3), 241-250.
Howell, H., Stone, E., & Kane, M. (2019). Future directions in the measurement of mathematics teachers' competencies. In J. Bostic, E. Krupa, & J. Shih (Eds.), *Quantitative measures of mathematical knowledge: Researching instruments and perspectives* (pp. 230-252). Routledge.
Kane, M. (2013). Validation the interpretation and uses of test scores. *Journal of Educational Measurement, 50*(1), 1-73.
Kane, M. (2016). Validation strategies: Delineating and validating proposed interpretations and uses of test scores. In S. Lane, M.R. Raymond, & T.M. Haladyna (Eds.), *Handbook of test development* (2nd ed., pp. 64 80). Routledge & Taylor & Francis Group.
Krupa, E., Bostic, J. & Shih, J. (2019). Validation in mathematics education: An introduction to quantitative measures of mathematical knowledge: Researching instruments and perspectives. In J. Bostic, E. Krupa, & J. Shih (Eds.), *Quantitative measures of mathematical knowledge: Researching instruments and perspectives* (pp. 1-13). Routledge.
Lavery, M., Jong, C., Krupa, E. & Bostic, J. (2019). Developing an instrument with validity in mind. In J. Bostic, E. Krupa, & J. Shih (Eds.), *Assessment in mathematics education contexts: Theoretical frameworks and new directions* (pp. 12-39). Routledge.
National Council of Teachers of Mathematics. (2014). P*rinciples to actions: Ensuring mathematical success for all.* Author.
National Council of Teachers of Mathematics. (2017). *Catalyzing change in high school mathematics: Initiating critical conversations.* Author.
National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards Mathematics.* Author.
Stohlmann, M., Moore, T., Cramer, K., & Maiorca, C. (2015). Changing pre-service elementary teachers' beliefs about mathematical knowledge. *Mathematics Teacher Education and Development*, *16*(2), 4-24.
The American Educational Research Association, American Psychological Association, & the National Council on Measurement in Education. (2014). *Standards for Educational and Psychological Measurement in Education.* American Educational Research Association.
White, D. Y., DuCloux, K. K., Carreras-Jusino, Á. M., González, D. A., & Keels, K. (2016). Preparing preservice teachers for diverse mathematics classrooms through a cultural awareness unit. *Mathematics Teacher Educator*, *4*(2), 164-187.
Zelkowski, J. (2011). Defining the intensity of high school mathematics: Distinguishing the difference between college-ready and college-eligible students. *American Secondary Education, 39*(2), 27-49.
Zelkowski, J. (2013). Making sense of extraneous solutions. *Mathematics Teacher, 106*(6), 452-458.
Zelkowski, J., Campbell, T.G., & Gleason, J. (2018). Programmatic effects of capstone math content and math methods courses on teacher licensure exams. In Smith, W.M., Lawler, B.R., Strayer, J.F. & Augustyn, L. (Eds.). *Proceedings of the 7th Annual Mathematics Teacher Education – Partnership Conference* (pp. 91-96). Association of Public Land-grant Universities.
Zelkowski, J. & Gleason, J. (2018). Programmatic effects on high stakes measures in secondary math teacher preparation. In Venenciano, L. and Redmond-Sanogo, A. (Eds.). *Proceedings of the 45th Annual Meeting of the Research Council on Mathematics Learning.* Baton Rouge, LA.
Zelkowski, J., Campbell, T.G., Moldavan, A.M., & Gleason, J. (2021). Maximizing teacher candidate performances

based on internal program measures: Program design considerations. In Smith, W.M. & Augustyn, L. (Eds.). *Proceedings of the 10th Annual Mathematics Teacher Education – Partnership Conference* (pp. 39-42). Association of Public Land-grant Universities.

EXPLORING THE AFFECTIVE EXPERIENCES OF PRE-SERVICE ELEMENTARY TEACHERS DURING A FRACTIONS UNIT

Domonique Caro-Rora
Florida State University
dr11e@fsu.edu

Jessica Smith
Vanderbilt University
jlsmith503@gmail.com

Lama Jaber
Florida State University
ljaber@fsu.edu

Ian Whitacre
Florida State University
iwhitacre@fsu.edu

Keywords: Affect, Emotions, Beliefs, and Attitudes, Learning Trajectories and Progressions, Pre-service Teacher Education

Learning trajectories focus on expectations related to how learners' ideas develop during an instructional sequence in relation to a learning goal (Simon, 1995). Thus, research on learning trajectories has given little attention to affective aspects of learners' experiences in understanding and exploring mathematical ideas. However, findings from a wide range of studies continue to document the central role of affect in learning (e.g., Cobb et al., 1989; deBellis & Goldin, 2006; Grootenboer & Marshman, 2015). The goal of our study is to foreground the affective experiences of preservice elementary teachers (PSTs) during a fractions unit and to consider how these relate to the conceptual aspects of their learning process.

The affect-focused literature regarding elementary PSTs is sparse and mostly limited to math anxiety (Kelly & Tomhave, 1985; Stoehr, 2017). We seek to both broaden and deepen our understanding of PSTs' affective experiences. Understanding patterns and variabilities in PSTs' affect during their learning is of value to teacher educators. The conceptual challenges PSTs face in coming to better understand fractions and fraction operations are coupled with emotional challenges. It is important to anticipate and plan for these challenges in designing instruction. We draw on the construct of *epistemic affect* (Jaber & Hammer, 2016) which refers to the various feelings, emotions, and motivations that are experienced within work centered on the production, communication, and critique of ideas to explore phenomena and solve problems. This affect might include feelings of vexation when faced with inconsistencies, excitement when figuring things out, and motivation to pursue a puzzling question. These affective experiences are not separable from but rather entangled with the work of sensemaking in mathematics.

The data used in this study consist of discussion boards, written reflections, recordings of class discussions, and interviews with one cohort of PSTs. We focus on weekly reflective assignments that students completed at the end of each class, designed to capture their emotions and feelings experienced during and after each meeting. Early analysis indicates students moved through a variety of affective experiences as the unit progressed. Rather than moving from anxiety to pleasant emotions, PSTs are moving between a variety of emotions that are influenced by external factors including but not limited to new or challenging content, overcoming difficult concepts, and interactions with classmates. Attending to affect could be beneficial to teacher educators as they support PSTs as they move through learning trajectories, especially those charged with emotions such as anxiety.

References

Cobb P., Yackel E., & Wood T. (1989). *Young children's emotional acts while engaged in mathematical problem solving*. In McLeod, D.B., & Adams, V. M. (Eds.) Affect and Mathematical Problem Solving (pp. 117–148). Springer.

DeBellis, V.A., & Goldin, G.A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies Mathematics 63*, 131–147 (2006). https://doi.org/10.1007/s10649-006-9026-4

Grootenboer, P., & Marshman, M. (2015). *Mathematics, affect and learning: Middle school students' beliefs and attitudes about mathematics education*. Springer.

Jaber, L. Z., & Hammer, D. (2016). Learning to feel like a scientist. Science Education, 100(2), 189–220.

Kelly, W. P., & Tomhave, W. K. (1985). A study of math anxiety/math avoidance in preservice elementary teachers. *The Arithmetic Teacher, 32*(5), 51–53.

Stoer, K. J. (2017). Mathematic anxiety. *School of Teacher Education, 68*(1), 69–84.

TEACHING FOR JUSTICE: REFINING A PRACTICUM COURSE FOR PRE-SERVICE TEACHERS IN HIGH-NEEDS SCHOOLS

Monica Grillo
William & Mary, Noyce Scholars Program
mgrillo@wm.edu

Meredith Kier
William & Mary, School of Education
mwkier@wm.edu

Keywords: Culturally Relevant Pedagogy, Social Justice, Preservice Teacher Education

It is more important than ever to prepare teachers who can advocate for minoritized students and social justice. It has been found that preservice mathematics teachers who have opportunities to critically reflect upon the opportunity gap in education and interrogate their own positionality as equity-minded teachers are better equipped to advance social justice in classrooms. In addition to direct cross-cultural experiences, teacher preparation programs can support the preparation of preservice teachers by broadening their perspectives of family and communities.

This poster will share experiences from a course designed to: (1) develop an understanding of pedagogical practices and educational strategies for successful teaching in the high-need setting, especially in the math classroom; and (2) cultivate both cultural self-awareness and cross-cultural consciousness in one's ability to adapt to the high-need environment in a culturally responsive way. The poster depicts how key assignments have been adapted over time to encourage scholars to interrogate their positionalities toward working with diverse others and advocate for marginalized youth as preservice teachers. The assignments we highlight are: Reflections on the utility of key texts; Early field experiences to diverse high-poverty schools; Making positive parent phone calls; and, Engaging in Critical Reflection practices.

Methodology

Each year of implementation, an external evaluation team from outside the scholarship program and university conducted focus-group discussions with scholars about the course, the program, and their teacher-preparation experience. We analyzed the anonymized focus-group data in addition to course assignment submissions, weekly and final reflections on the course, work samples, and anonymous course evaluations.

Findings

The predominant themes that emerged from course and assignment reflections were scholars' increased awareness of how they might "fit" and be successful within a HNS, a more thorough understanding of the inequities in schools and classrooms, and a clearer vision for how they can advocate for students and a more socially-just school system as an early career teacher. The poster highlights our scholars' voices to show how specific assignments contributed to their future thoughts and actions in HNSs. Overall, the course was influential to scholars' understanding of equity issues in education, as evidenced by their reflections on observations, readings, and experiences with prior alumni. The assignments in the course predominately promoted scholars' introspection and reflection of how they might position themselves to address inequities in HNSs. The voices of our scholars iteratively guided our revisions to the course, alongside our personal growth and professional development as social justice teacher educators.

Acknowledgments

This material is based upon work supported by The National Science Foundation under Grant No. DUE1758419. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the NSF.

PRE-SERVICE TEACHERS' CLASSIFICATION AND MODIFICATION OF TASKS

Ji-Eun Lee
Oakland University
lee2345@oakland.edu

Keywords: pre-service teacher education, task modification, cognitive demand

Theoretical Perspectives and Research Question

Effective mathematics teaching calls for engaging students in solving and discussing cognitively demanding tasks that promote mathematical reasoning and problem solving (NCTM, 2014). The cognitive demand of tasks has been categorized by their functional and qualitative characteristics, such as *memorization, procedures without connections, procedures with connections, and doing mathematics* (Smith & Stein, 1998). As the opportunity to learn to develop skills of selecting and modifying appropriate tasks is critical, this study investigates the following question: What types of content and pedagogical knowledge do pre-service teachers (PSTs) demonstrate while engaging in the task classification and modification activities?

Data Collection and Analysis

Four fraction-related tasks were provided to 54 elementary PSTs in a Midwestern University in the US. They were asked to complete the following activities: (a) match them with the levels of cognitive demand, and (b) modify two tasks identified as lower-level to higher-level tasks along with justifications and expected students' responses. For the first activity, frequencies were identified. PSTs' written and drawn work in the second activity were analyzed following the inductive content analysis approach (Grbich, 2013).

Summary of Findings and Implications

PSTs were more successful in identifying lower-level than higher-level tasks — *procedures without connection* (87%), *memorization* (85%), *procedures with connection* (74%), *doing mathematics* (72%). Major modification strategies focused on changing the forms and requirements of the tasks (e.g., incorporating story problems and drawings, requiring to explain, asking to solve in different ways). However, some modifications did not raise the level of cognitive demand (e.g., routine story problems focusing on computations only). This tendency is similar to a prior study, reporting teachers' tendency of selecting tasks according to "surface-level features" (Boston & Smith, 2009, p. 123). Some PSTs' anticipated student responses included false representations or excessively detailed work, which were not included in the task directions. This study suggests that PSTs need to be exposed to various tasks and opportunities to analyze and develop tasks beyond the surface level work. Also, giving proper directions aligned with the purpose of the task is another area PSTs need to learn and refine.

References

Boston, M. D., & Smith, M. S. (2009). Transforming secondary mathematics teaching: Increasing the cognitive demands of instructional tasks used in teachers' classrooms. *Journal for Research in Mathematics Education, 40*(2), 119–156.

Grbich, C. (2013). *Qualitative data analysis: An introduction (2nd ed.)*. Thousand Oaks, CA: Sage.

National Council of Teachers of Mathematics (2014). *Principles to action: Ensuring mathematical success for all*. Reston, VA: NCTM

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School, 3*(5), 344–349.

'CONVINCING ARGUMENT' IN THE PROBLEM-SOLVING CONTEXT: PROSPECTIVE 1-8 TEACHERS' INTERPRETATIONS

Marta T. Magiera
Marquette University
marta.magiera@marquette.edu

Keywords: Problem Solving, Reasoning and Proof, Teacher Knowledge, Pre-service Teacher Education

Problem-solving constitutes an important aspect of school mathematics. Little, however, is known about how prospective teachers (PSTs) think about mathematical argumentation in context-rich situations and ill-defined problems that lack a clearly defined solution path or single correct answer. Francisco and Maher (2005) argued that in the context of problem-solving, students could learn the practice of explaining and convincing others of the validity of their ideas and develop skills of critically examining and evaluating arguments and ideas shared by their peers. This research adds to the current efforts to support PSTs in meeting the challenges of fostering argumentation in school mathematics by examining PSTs' interpretations of the term convincing mathematical argument. Prior to negotiating classroom norms for argumentation, we explore (a) *How do PSTs characterize convincing arguments in inquiry-oriented problem-solving contexts*, (b) *How do they judge if an argument is convincing when listening to arguments shared by their peers*, and (c) *What do PSTs perceive as a challenging aspect of writing convincing mathematical arguments in a problem-solving context.*

In this research, a mathematical argument in the problem-solving context is conceptualized as a case for a mathematical claim—a solution to a given problem situation. An argument is interpreted as a product or a discursive practice that leads to discovering new mathematical ideas and convincing oneself or others that the given claim (result) is true (Leitão, 2000). Analyzed were written journal responses completed by 33 grades 1-8 teacher candidates enrolled in a Problem-Solving Course (99 journal entries analyzed). In their definitions of convincing arguments, PSTs addressed the validity of claims, logical flow, clarity of assumptions, and accuracy of mathematical representations. When asked to describe convincing arguments in the context of class discussions and presentations, PSTs characterized convincing arguments by their informative power (i.e., how well they could understand the argument, argument clarity, precision, overall organization, level of details). Some PSTs considered the presenter's confidence as a criterion for assessing convincing arguments. Most of the PSTs reported that deciding on the necessary level of details to make an argument comprehensive is a challenging aspect of generating convincing arguments. The results provide a direction for helping PSTs develop a robust knowledge of mathematical argumentation in teacher preparation programs by revealing PSTs' initial views about convincing arguments in problem-solving contexts.

References

Leitão, S, (2000). The potential of argument in knowledge building. *Human Development*, *43*(6), 332-360.

Francisco, J. M., & Maher, C.A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, *24*(3-4), 361–37

"I AM GOOD AT MATH WHEN…": PRE-SERVICE TEACHERS' MATHEMATICS EFFICACY BELIEFS

Christine M. Phelps-Gregory	Gabrielle E. Mynatt	Martha Frank
Central Michigan University	Central Michigan University	Central Michigan University
phelp1cm@cmich.edu	mynat1ge@cmich.edu	frank1m@cmich.edu

Keywords: Affect, emotions, beliefs and attitudes, Teacher beliefs, Preservice teacher education

Mathematics self-efficacy is defined as a student's belief about their ability to be successful in mathematics (Bandura, 1986; Zimmerman & Cleary, 2006). Previous research has shown the benefits of high mathematics self-efficacy (believing you are good at mathematics). In particular, self-efficacy affects future performance, persistence, effort, self-regulation, resilience, and success on difficult tasks (Cassidy, 2012; Ruch et al., 2014; Sawtelle et al., 2012; Usher et al., 2018; Holenstein et al., 2021). However, most self-efficacy work has been quantitative, often using Likert scales to assess self-efficacy. This quantitative focus often means we lack understanding of what students mean when they say, "I am good at math." In addition, the work has rarely focused on pre-service teachers (PTs). Do PTs define "good" in ways that align with researchers and their instructors?

Research Question, Design, and Analysis

This poster presents preliminary results from a qualitative study that used journaling to examine PTs' definitions of being "good" at mathematics. This work adds to existing work on PTs' mathematics self-efficacy (e.g., Phelps, 2010; Zuya et al., 2016). Participants in this study were 23 elementary PTs enrolled in a mathematics class for PTs taken before the mathematics methods course. Participants completed two written journal entries on their beliefs, which were analyzed with codes developed inductively; two authors worked together to ensure reliability of the codes following Campbell and colleagues (2013). Future work will examine additional beliefs, such as PTs' beliefs about teaching.

Findings and Implications

Results suggest PTs had a variety of definitions of being "good" at mathematics. Some participants (n = 17, 74%) defined being good at math as applying their skills to new contexts and problems; Olivia said, "A successful learner can apply the concepts to different situations and can solve problems using a bank of different strategies." Other common definitions included being able to explain math to others (n = 16, 70%), getting good grades (n = 13, 57%), being fluid (n = 12, 52%) and working hard (n = 6, 26%). However, there was distinct disagreement among some PTs' definitions; for example, some PTs said being "good" at math meant always getting the right answer but an almost equal number of PTs (n = 11, 48%) expressed doubt that right answers or high grades alone meant you were good at math. Mia said, "The grade that I receive is not always a reflection of how successful I was. I have completed many math courses and received an A without actually mastering the material." Overall, these results, and other data we will share in the poster, suggest that PTs may not define being "good" at math in the same way as researchers, their classmates, or their instructors. This has implications for understanding PTs' mathematics self-efficacy beliefs, in particular, what they may mean when they say, "I am good at math." In addition, this work can help teacher educators understand PTs' future teaching and the goals they will strive for their students to achieve.

References

Bandura, A. (1986). Self-Efficacy. In *Social foundation of thought and action: A social-cognitive view* (pp. 290-453). *Englewood Cliffs*.

Campbell, J. L., Quincy, C., Osserman, J., and Pedersen, O. K. (2013). Coding in-depth semistructured interviews: Problems of unitization and intercoder reliability and agreement. *Sociological Methods & Research*. *42*(*3*): 294 – 320.

Cassidy, S. (2012). Exploring individual differences as determining factors in student academic achievement in higher education. *Studies in Higher Education*, *37*(7), 793 – 810.

Holenstein, M., Bruckmaier, G., & Grob, A. (2021). How do self-efficacy and self-concept impact mathematical achievement? The case of mathematical modelling. British Journal of Educational Psychology, e12443.

Phelps, C. M. (2010). Factors that pre-service elementary teachers perceive as affecting their motivational profiles in mathematics. *Educational Studies in Mathematics*, *75*(3), 293 – 309.

Ruch, W., Weber, M., Park, N., & Peterson, C. (2014). Character strengths in children and adolescents: Reliability and initial validity of the German values in action inventory of strengths for youth (German VIA-Youth). European journal of psychological assessment, 30(1), 57 – 64.

Sawtelle, V., Brewe, E., & Kramer, L. H. (2012). Exploring the relationship between self‐efficacy and retention in introductory physics. *Journal of Research in Science Teaching*, *49*(9), 1096 – 1121.

Usher, E. L., Li, C. R., Butz, A. R., & Rojas, J. P. (2018). Perseverant grit and self-efficacy: Are both essential for children’s academic success? Journal of Educational Psychology, 111(5), 877 – 902.

Zimmerman, B. J., & Cleary, T. J. (2006). Adolescents’ development of personal agency: The role of self-efficacy beliefs and self-regulatory skill. In *Self-Efficacy Beliefs of Adolescents* (pp. 45 – 69).

Zuya, H. E., Kwalat, S. K., & Attah, B. G. (2016). Pre-service teachers' mathematics self-efficacy and mathematics teaching self-efficacy. *Journal of Education and Practice*, *7*(14), 93 – 98.

A NARRATIVE INQUIRY OF A BEGINNING MATHEMATICS CONTENT INSTRUCTOR

Kelsey Quaisley
University of Nebraska-Lincoln
kquigley@huskers.unl.edu

Keywords: Teacher Educators, Instructional Activities and Practices, Preservice Teacher Education

Learning to teach mathematics for prospective elementary teachers (PTs) is challenging. Mathematics content instructors must be prepared to not only teach mathematics content, but engage PTs in understanding children's mathematical thinking strategies, learning trajectories, and misconceptions (Carpenter et al., 1996; Carpenter & Moser, 1982; I et al., 2020). When considering how instructors might best be prepared and supported however, one must recognize that there is significant variation in the backgrounds and expertise of mathematics content instructors (Masingila et al., 2012; Yow et al., 2016) and little is known about the preparation, knowledge, and experiences of mathematics content instructors (Even, 2008; Goos, 2009; Masingila et al., 2012; Oesterle, 2011; Zaslavsky & Leikin, 2004). Furthermore, most instructors who are newer to teaching PTs do not feel prepared and additionally report an absence of training, resources, and support at their institutions (Goodwin et al., 2014; Masingila et al., 2012; Yow et al., 2016). Mathematics content instructors, especially those newer to teaching PTs, need preparation and support that account for their background and teaching context.

In this poster, I describe a beginning mathematics content instructor's experiences learning to teach elementary PTs in their respective teaching context. Specifically, I aim to draw connections from the instructor's background and preparation to their challenges and successes around teaching and learning to teach mathematics content for elementary PTs.

I view instructor learning as occurring through *interactions* with others, while also occurring through *reflection* and inquiry into one's own practices and the practices of others (Cobb & Bauersfeld, 1995). I further recognize that an instructor's interactions and reflections are inseparable from *context, time, and place*, and are deeply rooted in socio-cultural norms (Gutiérrez, 2008; Lave & Wenger, 1991; Rubel & Nicol, 2020; Vygotsky, 1978). Challenges and successes may offer substantial insight into instructor learning. Since they are embedded in experiences, understanding experience is essential to understanding learning. Hence, I engage in narrative inquiry, drawing on Clandinin and Connelly's (2000) *three-dimensional inquiry space* to make sense of one beginning instructor's experiences learning to teach mathematics for elementary PTs. For one semester, I observed all of the instructor's classes and meetings with other instructors, conducted three interviews with the instructor, collected the instructor's teaching and learning autobiographies, and collected the instructor's weekly reflection journals.

Narrative themes connecting the instructor's contexts to their learning to teach mathematics for elementary PTs will be presented. For example, transitioning from evaluating students' work in introductory-level mathematics courses to students' work in mathematics content courses may present challenges for instructors in mathematics departments. I will discuss stories that resonate or create dissonance with the literature on mathematics content instructor preparation and support, and theories for harmonizing instructors' backgrounds with their current experiences.

References

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20. https://doi.org/10.1086/461846

Carpenter, T. P. & Moser, J. M. (1982). The development of addition and subtraction problem-solving skills. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 9-24). Routledge. https://doi.org/10.4324/9781003046585

Clandinin, J. & Connelly, M. (2000). *Narrative inquiry: Experience and story in qualitative research*. Jossey-Bass Publishers.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures*. Psychology Press.

Even, R. (2008). Facing the challenge of educating educators to work with practicing mathematics teachers. In B. Jaworski & T. Woods (Eds.), *The international handbook on mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 57-73). Rotterdam, Netherlands: Sense Publishers.

Goodwin, A. L., Smith, L., Souto-Manning, M., Cheruvu, R., Tan, M. Y., Reed, R., & Taveras, L. (2014). What should teacher educators know and be able to do? Perspectives from practicing teacher educators. *Journal of Teacher Education, 65*(4), 284-302.

Goos, M. (2009). Investigating the professional learning and development of mathematics teacher educators: A theoretical discussion and research agenda. Crossing Divides: Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia (Vol. 1), 209-216.

Gutiérrez, K. D. (2008). Developing a sociocritical literacy in the third space. *Reading Research Quarterly, 43*(2), 148-164. https://doi.org/10.1598/RRQ.43.2.3

I, J., Martinez, R., & Dougherty, B. (2020). Misconceptions on part-part-whole proportional relationships using proportional division problems. *Investigations in Mathematics Learning, 12*(2), 67-81. https://doi.org/10.1080/19477503.2018.1548222

Lave, J. & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Masingila, J., Olanoff, D., & Kwaka, D. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education, 15*(5), 347-358.

Oesterle, S. M. (2011). *Diverse perspectives on teaching Math For Teachers: Living the tensions* (Doctoral dissertation, Education: Faculty of Education).

Rubel, L. H., & Nicol, C. (2020). The power of place: spatializing critical mathematics education. *Mathematical Thinking and Learning, 22*(3), 173-194. https://doi.org/10.1080/10986065.2020.1709938

Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds. & Trans.). Harvard University Press.

Yow, J. A., Eli, J. A., Beisiegel, M., McCloskey, A., & Welder, R. M. (2016). Challenging Transitions and Crossing Borders: Preparing Novice Mathematics Teacher Educators to Support Novice K-12 Mathematics Teachers. *Mathematics Teacher Education and Development, 18*(1), 52-69.

Zaslavsky, O., & Leikin, R. (2004). Professional development of mathematics teacher educators: Growth in practice. *Journal of Mathematics Teacher Education, 7*(1), 5–23.

SUPPORTING A TEACHER RESIDENCY PROGRAM EMPHASIZING STEM EDUCATION IN RURAL COMMUNITIES

George J. Roy
University of South Carolina
roygj@mailbox.sc.edu

Kristin E. Harbour
University of South Carolina
kharbour@mailbox.sc.edu

Thomas E. Hodges
University of South Carolina
hodgeste@mailbox.sc.edu

Keywords: Equity, Inclusion, and Diversity; Elementary School Education; Systemic Change; Preservice Teacher Education

We present the development and implementation of Carolina Transition to Teaching, a cohort-based, 18-month teacher residency program focused on the recruitment, preparation, and retention of diverse teachers in rural communities. The significance of this work is two-fold: (a) the dearth of teaching candidates in STEM fields at all levels (Cowan et al., 2016); and (b) the ongoing lack of racial diversity found in teacher preparation programs (King et al., 2016). These factors are situated at a time when states are investing less in higher education (Mitchell et al., 2019), leading to the need for innovative approaches alongside more traditional pathways. In this poster we highlight the deliberate interplay of recruitment, preparation, clinical practice, and attention to certification requirements that lead a master's degree and state teaching licensure.

Through a U.S. Department of Education Teacher Quality Partnership (TQP) grant, a flagship university in the Southeastern part of the U.S. developed a teacher residency program with two partner districts located in rural communities. A Network Improvement Community, Carolina Transition to Teaching, including university-based faculty and staff from the College of Education, external evaluators, and district representatives, used systematic methods of inquiry within an improvement science approach (Bryk et al., 2017) to explore and inform implementation as well as to analyze ongoing and retrospective findings as a mechanism for programmatic refinements.

A goal of the Carolina Transition to Teaching residency program is to diversify the professional pipeline to include individuals with a bachelor's degree who are currently working within our partner districts as paraprofessionals or long-term substitute teachers as well as career changers from rural communities. Built upon research-informed components of teacher residencies (e.g., Guha et al, 2017), the structural components of the Carolina Transition to Teaching residency program include: (a) providing a minimum living wage stipend; (b) yearlong practicum in classrooms within partner schools, (c) multiple layers of support through coaching and ongoing professional development; (d) graduate coursework; (e) completing a series micro-credentials to satisfy teacher certification requirements, and (f) induction support during the first three years of classroom teaching.

The initial findings are promising for our recruitment and preparation efforts. For instance, one goal of the Carolina Transition to Teaching residency is to support and prepare teachers while diversifying the workforce within rural districts; through our two rounds of recruitment and preparation, 87% of our participating residents (N=23) self identified as African American or Black, and 17% self identified as male. These data stand out as only 21% of the new teachers in our state identify as non-white and 19% identify as male. Further, 100% of the residents range from 30-70 years old, bringing with them a wealth of knowledge to their residency experiences. As we continue to analyze our research efforts to recruit, prepare, and support the residents in the program, we will share additional findings related to supporting our two cohorts who are or will be practicing teachers in classrooms within our partner school districts.

References

Bryk, A. S., Gomez, L. M., Grunow, A. & LeMahieu, P. G. (2017). *Learning to improve: How America's schools can get better at getting better*. Harvard Education Press.

Cowan, J., Goldhaber, D., Hayes, K., & Theobald, R. (2016). Missing elements in the discussion of teacher shortages. *Educational Researcher, 45*(8)*,* 460–462. https://doi.org/10.3102/0013189X16679145

Guha, R., Hyler, M. E., & Darling-Hammond, L. (2017). The teacher residency: A practical path to recruitment and retention. *American Educator, 41*(1), 31-34, 44.

King, J. B., McIntosh, A., & Bell-Ellwanger, J. (2016). *The state of racial diversity in the educator workforce.* Washington, DC: US Department of Education.

Mitchell, M., Leachman, M., & Saenz, M. (2019). *State higher education funding cuts have pushed costs to students, worsened inequality*. Washington, DC: Center on Budget and Policy Priorities.

USING REHEARSALS TO SUPPORT PROSPECTIVE TEACHERS IN TEACHING WITH TECHNOLOGY

Alyson E. Lischka
Middle Tennessee St Univ
Alyson.Lischka@mtsu.edu

Barbara A. Burns
Canisius College
burns1@canisius.edu

WenYen (Jason) Huang
SUNY - New Paltz
huangj18@newpaltz.edu

Dexter Lim
Tusculum University
dlim@tusculum.edu

Christopher W. Parrish
University of South Alabama
parrish@southalabama.edu

Karoline Smucker
Eastern Oregon University
ksmucker@eou.edu

Keywords: Technology, Preservice Teacher Education

Teaching with *mathematical action technology* (MAT) is a complex practice (e.g., Lagrange & Monaghan, 2009; Lee & Hollebrands, 2008) that requires attention to learners' thinking and to the role of MAT in supporting productive discussions. McCulloch et al. (2021) found that few teacher preparation programs provide opportunities for prospective teachers (PTs) to engage in implementation of MAT tasks. One way to support PTs in developing complex teaching practices is through the use of teaching rehearsals (Kazemi et al., 2016). To provide PTs an opportunity to engage in the implementation of MAT tasks, we implemented rehearsals focused on orchestrating whole-group discussion of MAT tasks in secondary mathematics methods courses.

Rehearsal is a pedagogy that provides opportunities for PTs to approximate teaching practice (Grossman et al., 2009) with peers in the role of learners, in a space that allows for reflection and growth (Kazemi et al., 2016). In rehearsals, PTs learn to implement pedagogical practices and tools to support learning. Mathematical action technologies are technology tools that allow learners to engage with mathematical objects in a dynamic way as they explore mathematical concepts and make sense of mathematical relationships (Dick & Hollebrands, 2011).

In three secondary mathematics methods courses at three different institutions, we implemented a rehearsal cycle with 20 PTs in Fall 2021. Prior to the rehearsal cycle at all three sites, PTs read about MATs (Dick & Hollebrands, 2011), considered ways in which MATs could serve as amplifiers or reorganizers (Sherman & Cayton, 2015), and discussed ways in which MATs might be enacted (Drijvers et al., 2010). PTs were then required to select, create, or adapt an existing MAT-based lesson using Desmos or Geogebra for use in a rehearsal in which the focus was class discussion of a mathematical concept. The rehearsal cycles included video-recorded enactment of the discussion and a debrief following a viewing of the videos to support reflection. Common data collection included a pre-rehearsal planning survey, videos of the enacted rehearsals, peer rehearsal feedback forms, and a post-rehearsal reflective survey.

Preliminary analysis of the data shows that combining the pedagogies of rehearsals with the use of MATs is a plausible way to fill the gap in mathematics teacher preparation noted by McCulloch et al. (2021). Rehearsals can be used to engage PTs in thinking about decisions they make while integrating MATs in their lesson design and teaching. Findings show PTs felt that this was an authentic experience that helped to grow their thinking about teaching with technology. One PT commented on the experience saying: "it was very helpful to be able to rewatch the video, pause, reflect, and then continue. This allowed for thoughtful, extended critical thinking for our responses." Analysis will continue so as to determine the specific ways in which this pedagogy informs teacher preparation. In this poster, we seek to share findings from our ongoing data analysis.

References

Dick, T. P. & Hollebrands, K. F.. (2011). *Focus in High School Mathematics: Technology to Support Reasoning and Sense Making*. National Council of Teachers of Mathematics.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213-234.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, *111*(9), 2055-2100.

Kazemi, E., Ghousseini, H., Cunard, A., & Turrou, A. C. (2016). Getting inside rehearsals: Insights from teacher educators to support work on complex practice. *Journal of Teacher Education, 67*(1), 18-31.

Lagrange, J.-B., & Monaghan, J. (2009, January 28-February 1). *On the adoption of a model to interpret teachers' use of technology in mathematics lessons* [Paper presentation]. CERME6 Conference, Lyon, France.

Lee, H., & Hollebrands, K. (2008). Preparing to teach mathematics with technology: An integrated approach to developing technological pedagogical content knowledge. *Contemporary Issues in Technology and Teacher Education, 8*(4), 326-341.

McCulloch, A. W., Leatham, K. R., Lovett, J. N., Bailey, N. G., & Reed, S. D. (2021). How we are preparing secondary mathematics teachers to teach with technology: Findings from a nationwide survey. *Journal for Research in Mathematics Education*, *52*(1), 94-107.

Sherman, M., & Cayton, C. (2015). Using appropriate tools strategically for instruction. *The Mathematics Teacher*, *109*(4), 306-310.

A CASE STUDY EXAMINATION OF THE RELATIONSHIP BETWEEN PRE-SERVICE TEACHERS' NOTICING AND QUESTIONING

Christine Phelps-Gregory
Central Michigan University
phelp1cm@cmich.edu

Sandy Spitzer
Towson University
sspitzer@towson.edu

Keywords: Preservice Teacher Education; Teacher Noticing; Technology

Mathematics teacher educators and researchers have proposed that a critical knowledge domain for prospective teachers (PTs) is *professional teacher noticing of student mathematical thinking*, often referred to simply as "teacher noticing." Emerging research suggests that teacher noticing supports both high-quality mathematics teaching in the short term (e.g., Jacobs et al., 2010) as well as long-term growth in teacher expertise (Santagata, 2011). Further, research suggests that teacher noticing might also be associated with high quality teacher questioning (Weiland et al., 2014). However, more research is needed to better understand the relationships between teacher noticing and questioning and how these skills support each other.

Research Question, Design, and Analysis

In this poster, we present results from a larger study in which PTs (n = 18) conducted a simulated interview, focused on geometric thinking, with an AI chatbot named "Matilda" (which had been programmed based on children's responses). Previously, we analyzed PTs' questioning patterns in the interview (Spitzer & Phelps-Gregory, 2021). After the interview, PTs wrote an essay describing Matilda's knowledge (including misconceptions) about quadrilaterals, interpreting the evidence and identifying her van Hiele level, and describing what Matilda's teacher should do next. PTs' essays were analyzed in terms of the three domains of teacher noticing, *Attending*, *Interpreting*, and *Deciding* (Jacobs et al., 2010). We then categorized PTs as having either high, medium, or low noticing skills and qualitatively analyzed for differences in their questioning patterns.

Findings and Implications

Our emerging findings suggest that teacher questioning is critical to eliciting the kinds of student mathematical responses that teachers can use as evidence in their noticing. During their interviews, PTs with stronger noticing skills asked more open-ended questions (e.g., "What is shape 2?") and less leading questions (e.g., "Is shape 2 a rectangle?"). In contrast, PTs with weaker noticing skills appeared to struggle to find the right questions to diagnose Matilda's thinking, asking more questions overall and more restatements of previous questions.

Several cases from our data help illustrate these findings. For example, Abigail used a productive questioning pattern, starting with asking "What is shape 2?" When Matilda called it a diamond, Abigail asked multiple follow-up questions, such as "What is a diamond?" In her essay, Abigail focused on the answers to these questions to provide strong evidence for her appropriate diagnosis of student thinking. In contrast, Brianna used many leading questions, such as "So is shape 2 a rectangle?" Brianna asked less follow-up questions than Abigail and, when Matilda named the shape a diamond, responded with "If it is a square, doesn't that also mean it is a rectangle?" In Brianna's essay, she inappropriately cited this exchange as evidence that Matilda had learned. In this poster presentation, we will present these and other cases that exemplify the relationships between questioning and noticing. This has implications for teacher educators in helping PTs develop both the critical skills of noticing and questioning.

References

Jacobs, V. R., Lamb, L. L. C. & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 41(2), 169 – 202.

Santagata, R. (2011). From teacher noticing to a framework for analyzing and improving classroom lessons. In Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 152 – 168). New York, NY: Routledge

Spitzer, S., & Phelps-Gregory, C. M. (2021). Prospective Teachers' Questioning Patterns in a Simulated Student Interview. In *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Philadelphia, PA.

Weiland, I. S., Hudson, R. A., & Amador, J. M. (2014). Preservice formative assessment interviews: The development of competent questioning. *International Journal of Science and Mathematics Education*, *12*(2), 329-352.

THE EFFECTS ON PRESERVICE TEACHERS' SELF-EFFICACY FROM ENGAGING IN A MATHEMATICAL MODELING TASK

Zachary Stepp
University of Florida
zsguitar@ufl.edu

Hyunyi Jung
University of Florida
hyunyi.jung@coe.ufl.edu

Megan Wickstrom
Montana State University
megan.wickstrom@montana.edu

Sheunghyun Yeo
The University of Alabama
syeo@ua.edu

Keywords: Self-Efficacy, Design Based Research, Mathematics Education, Pre-Service Teachers

Mathematical modeling is the process of interpreting real world situations with the use of mathematics. The benefits to students when they engage in mathematical modeling include developing reasoning abilities, entrepreneurial thinking, and conceptual understanding of mathematics and other relevant contents (Blum & Niss, 1991; Lesh et. al, 2000; Zbiek & Conner, 2006). It is recommended that preservice teachers' (PSTs) learn mathematical modeling to engage their future students with authentic problem-solving experiences (Association of Mathematics Teacher Educators, 2017). However, research is needed in the area of such learning opportunities and in the area of learners' perspectives while responding to working through mathematical modeling (Asempapa & Sastry, 2021).

This study investigates the effects on PSTs self-efficacy from engaging in a mathematical modeling task. Self-efficacy is defined as "judgements of their capabilities to organize and execute courses of action required to attain designated types of performances" (Bandura, 1986, p. 391). According to Bandura (1986), motivation and behavior influence each other, and through interventions and strengthening these beliefs can increase academic achievement. While much research has involved learning opportunities in mathematical modeling, the importance of teacher's self-efficacy in continuing the practice needs to be explored.

Using design-based research, the three authors worked with PSTs on a modeling task involving two parts: (1) the design of a classroom and (2) the development of a rating or ranking system of the classroom design. The first part of the modeling aimed to use both two- and three-dimensional concepts in geometry for elementary grades, while the second part was designed to compare the classroom designs, as well as learn how involved a grading system is implemented. This modeling task was employed over a course of four classrooms: three undergraduate classes, and one graduate class of students all enrolled in an elementary mathematical content and methods course. With the third author enacting the first iteration of the task, we then revised the lesson, and used the task again in three other classrooms: two undergraduate classrooms with the first author, and one graduate class with the second author.

Coding reflective journals from the PSTs, we utilized Bandura's (1994) four main sources of influence: mastery experiences, vicarious experiences, social (or verbal) persuasion, and somatic and emotional states (pp. 2-3) to guide our analysis. Preliminary results indicate an overall increase in PSTs self-efficacy from this modeling task, in part due to the nature of the task, collaborative efforts of their classmates, with the instructors as a facilitator in the modeling task. In our poster session, we will illustrate the sources of influence on self-efficacy and its connection to specific aspects of the mathematical modeling task. We will also look at the differences between the three authors enacting the mathematical modeling task and highlight general themes from the reflective journals relating to the PSTs self-efficacy.

References

Asempapa, R. S., & Sastry, A. M. (2021). Examining Preservice Teachers' Familiarity and Experiences with Mathematical Modeling Practices. Investigations in Mathematics Learning, 13(3), 214-229.

Bandura, A., & Wessels, S. (1994). Self-efficacy (Vol. 4, pp. 71-81). na.

Bandura, A. (1986). Prentice-Hall series in social learning theory. Social foundations of thought and action: A social cognitive theory. Englewood Cliffs, NJ.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – State, trends and issues in mathematics instruction. Educational Studies in Mathematics, 22(1), 37-68.

Lesh, R., Hoover, M., Hole, B., Kelly, A. & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In R. A. Lesh, & A. Kelly (Eds.), Handbook of research design in mathematics and science education (pp. 591–646). Mahwah, NJ: Lawrence Erlbaum Associates.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. Educational Studies in Mathematics, 63(1), 89-112.

EXPLORING METHODOLOGIES USED IN THE STUDY OF (MATHEMATICS) TEACHER IDENTITY

Molly Sutter	Will Hall	Ashley Whitehead
Washington State University	Washington State University	Appalachian State University
molly.sutter@wsu.edu	w.hall@wsu.edu	whiteheadan@appstate.edu

Keywords: Research Methods; Teacher Beliefs; Preservice Teacher Education

Purpose

This poster summarizes findings from a review of the literature on teacher identity formation. Through this summary, we aim to explore the different methods and theoretical frameworks used by researchers in their teacher identity work. We will discuss strengths and weaknesses of these methodologies, how different methods and frameworks may fit together, and ultimately how they fit with mathematics teacher identity research.

Summary of Methodologies and Theoretical Frameworks Reviewed

Identity is complex with socially formed multiple identities that continue to grow and develop over time (Akkerman & Meijer, 2011; Beauchamp & Thomas, 2009; Chung-Parsons & Bailey, 2019; Ruohotie-Lyhty, 2013). A majority of the studies reviewed employed qualitative methodologies including interviews, narratives, and reflections. Through these qualitative methods, researchers focused on how preservice teachers perceive the teaching profession and how their lived experiences affect their beliefs of teaching (Cross & Hong, 2012; Losano et al., 2018; Pillen et al., 2013). In addition, implementing interviews, narratives, and reflections within the practice of preservice teachers afford students opportunities to reflect and reevaluate the influence their own lived experiences have on their mathematics teacher identities. Along with narratives and reflections, autoethnography has been utilized as a self-reflective cultural narrative, which requires the author of the narrative to reflect and analyze experiences to understand culture and the impact it has on identity, both a process and a product (Anderson, 2006; Austin & Hicky, 2007; Duarte, 2007; Ellis et al., 2011; Hains-Wesson & Young, 2017; Hamilton, 2021; Pinner, 2018; Yung, 2020). While most studies utilized qualitative methods, there were a couple that used quantitative methods including surveys or questionnaires (Harvath et al., 2018; Hong, 2010). Each of these methods bring different strengths and weaknesses into the study of teacher identity in answering different research questions.

Studies examined have also utilized different theoretical frameworks to analyze identity formation within teacher education, for this poster we will highlight three. The first being Gee's four views of identity: Nature, Institution, Discursive, and Affinity (Carrier et al., 2017; Tsybulsky & Muchnik-Rozanov, 2019), another is the two-dimensional Cognitive Process model (Sutherland et al., 2010), and finally, Bronfenbrenner's Ecological System (Cross & Hong, 2012). Each of these frameworks were used to analyze the different experiences, feelings, and/or environments impact on teacher identity formation and were chosen as highlights for this poster given their perceived potential to incorporate autoethnographic activities within each framework. We layout the main structure and ideas in each of these frameworks in order to highlight strengths and weaknesses within teacher identity research as well as their potential to specifically focus on *mathematics* teacher identity.

References

Akkerman, S. F., & Meijer, P. C. (2011). A dialogical approach to conceptualizing teacher identity. *Teaching and Teacher Education*, *27*(2), 308–319. https://doi.org/10.1016/j.tate.2010.08.013

Anderson, L. (2006). Analytic Autoethnography. *Journal of Contemporary Ethnography*, *35*(4), 373–395. https://doi.org/10.1177/0891241605280449

Austin, J. & Hicky, A. (2007). Autoethnography and Teacher Development. *The International Journal of Interdisciplinary Social Sciences: Annual Review*, *2*(2), 369–378. https://doi.org/10.18848/1833-1882/CGP/v02i02/52189

Beauchamp, C., & Thomas, L. (2009). Understanding teacher identity: An overview of issues in the literature and implications for teacher education. *Cambridge Journal of Education*, *39*(2), 175–189. https://doi.org/10.1080/03057640902902252

Carrier, S. J., Whitehead, A. N., Walkowiak, T. A., Luginbuhl, S. C., & Thomson, M. M. (2017). The development of elementary teacher identities as teachers of science. *International Journal of Science Education*, *39*(13), 1733–1754. https://doi.org/10.1080/09500693.2017.1351648

Chung-Parsons, R., & Bailey, J. M. (2019). The hierarchical (not fluid) nature of preservice secondary science teachers' perceptions of their science teacher identity. *Teaching and Teacher Education*, *78*, 39–48. https://doi.org/10.1016/j.tate.2018.11.007

Cross, D. I., & Hong, J. Y. (2012). An ecological examination of teachers' emotions in the school context. *Teaching and Teacher Education*, *28*(7), 957–967. https://doi.org/10.1016/j.tate.2012.05.001

Duarte, F. P. (2007). Using Autoethnography in the Scholarship of Teaching and Learning: Reflective Practice from 'the Other Side of the Mirror'. *International Journal for the Scholarship of Teaching and Learning*, *1*(2). https://doi.org/10.20429/ijsotl.2007.010221

Ellis, C., Adams, T. E., & Bochner, A. P. (2011). Autoethnography: An Overview. *Forum: Qualitative Social Research*, *12*(1), 1–13.

Hains-Wesson, R., & Young, K. (2017). A collaborative autoethnography study to inform the teaching of reflective practice in STEM. *Higher Education Research & Development*, *36*(2), 297–310. https://doi.org/10.1080/07294360.2016.1196653

Hamilton, M. (2021). An Autoethnographic Self-Study Navigating the Transition to Becoming a STEM Teacher Educator. *The Qualitative Report*. https://doi.org/10.46743/2160-3715/2021.4462

Hodges, T. E., & Hodge, L. L. (2017). Unpacking personal identities for teaching mathematics within the context of prospective teacher education. *Journal of Mathematics Teacher Education*, *20*(2), 101–118. https://doi.org/10.1007/s10857-015-9339-2

Losano, L., Fiorentini, D., & Villarreal, M. (2018). The development of a mathematics teacher's professional identity during her first year teaching. *Journal of Mathematics Teacher Education*, *21*(3), 287–315. https://doi.org/10.1007/s10857-017-9364-4

Maye, D., & Day, B. (2012). Teacher Identities: The Fingerprint of Culturally Relevant Pedagogy for Students At Risk. *International Journal for Professional Educators: The Delta Kappa Gamma Bulletin*, *78*(2), 19–25.

Pillen, M., Beijaard, D., & den Brok, P. (2013). Professional identity tensions of beginning teachers. *Teachers and Teaching*, *19*(6), 660–678. https://doi.org/10.1080/13540602.2013.827455

Pinner, R. S. (2018). Re-learning from experience: Using autoethnography for teacher development. *Educational Action Research*, *26*(1), 91–105. https://doi.org/10.1080/09650792.2017.1310665

Ruohotie-Lyhty, M. (2013). Struggling for a professional identity: Two newly qualified language teachers' identity narratives during the first years at work. *Teaching and Teacher Education*, *30*, 120–129.

Sutherland, L., Howard, S., & Markauskaite, L. (2010). Professional identity creation: Examining the development of beginning preservice teachers' understanding of their work as teachers. *Teaching and Teacher Education*, *26*(3), 455–465. https://doi.org/10.1016/j.tate.2009.06.006

Tsybulsky, D., & Muchnik-Rozanov, Y. (2019). The development of student-teachers' professional identity while team-teaching science classes using a project-based learning approach: A multi-level analysis. *Teaching and Teacher Education*, *79*, 48–59. https://doi.org/10.1016/j.tate.2018.12.006

Yung, K. W.-H. (2020). Becoming a teacher educator through being a student teacher: An autoethnography in the practicum. *Journal of Education for Teaching*, *46*(2), 248–250. https://doi.org/10.1080/02607476.2020.1724655

EXAMINING VIDEO SELECTIONS THAT FOCUS PROSPECTIVE TEACHERS' NOTICING OF STUDENT RESOURCES TO SUPPORT PRODUCTIVE STRUGGLE

Christina Starkey
Kentucky Wesleyan College
christina.starkey@kwc.edu

Christine Herrera
California State University, Chico
caherrera@csuchico.edu

Shawnda Smith
Texas Woman's University
ssmith115@twu.edu

Hiroko Warshauer
Texas State University
hw02@txstate.edu

Keywords: Culturally Relevant Pedagogy; Equity, Inclusion, and Diversity; Preservice Teacher Education; Teacher Noticing; Productive Struggle

Preparing prospective elementary and middle school teachers for the diverse classrooms in which they will teach is of critical importance (AMTE, 2017). Learning to notice children's mathematical thinking and their cultural funds of knowledge provides teachers with opportunities to leverage these student resources to support students equitably and inclusively in their classrooms (Turner & Drake, 2016). The use of videos to support teachers' professional teacher noticing skills and practices has been well documented (Jacobs et al., 2010; van Es, Cashen, Barnhart & Auger, 2017). As part of a larger study, we focus here on salient features of the video selection as considerations in supporting prospective teachers' (PSTs) development of noticing skills specific to student thinking, and the use of cultural funds of knowledge to support productive struggle.

Purpose

While research suggests that productive struggle is an important component of learning mathematics (Hiebert & Wearne, 2003; Hiebert & Grouws, 2007), supporting productive struggle in the classroom remains challenging for teachers to implement (NCTM, 2014; Warshauer, 2015). Thus, we explore specific features of video selections designed to support PSTs understanding of noticing and productive struggle while also drawing their attention to equitable teaching practices that support children's Multiple Mathematical Knowledge Base [MMKB] (Roth McDuffie et al., 2014; van Es, E., Hand, V., & Mercado, J. 2017).

Methods

The study included 40 PSTs enrolled in their final mathematics content course for elementary teachers. They completed three writing assignments with prompts incorporating frameworks on Noticing, Productive Struggle, and MMKB (Jacobs, et al., 2010; Author, 2015; Roth McDuffie et al., 2014) to reflect on and analyze video episodes of productive struggle and the teachers' use of MMKB in the classroom interactions. The video episode selection criteria were based on the presence of: Interaction type, Mathematical content, Productive Struggle, and MMKB.

Results and Implications

We report the specific dimensions considered including the type of interpersonal interaction present (e.g. 1-1 student and teacher interaction or whole class discussion lead by teacher), the difficulty of mathematical content (level of cognitive demand related to PSTs class learning), level of productive struggle, and the complexity of noticing required (inclusion and emphasis on

MMKB). Selection, progression, and use of videos with these dimensions can add an equity lens towards PSTs viewing a more socially just context for learning and teaching mathematics.

References

Association of Mathematics Teacher Educators (AMTE) (2017). *Standards for Preparing Teachers of Mathematics*. Available online at amte.net/standards

Hiebert, J. & Grouws, F. (2007). The effects of classroom mathematics teaching on student learning. In F. Lester (Ed.). *Second Handbook of Research on Mathematics Teaching and learning* (pp. 371-404). Charlotte: Information Age Publishing.

Hiebert, J., & Wearne, D. (2003). Developing understanding through problem solving. In H. L. Schoen & R. I. Charles (Eds.), *Teaching mathematics through problem solving: grades 6-12* (pp. 3-13). Reston: NCTM.

Jacobs, V., Lamb, L., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202.

National Council of Teachers of Mathematics. (2014). *Principles to Actions: ensuring mathematical success for all*. Reston: NCTM.

Roth McDuffie, A., Foote, M., Bolson, C., Turner, E., Aguirre, J., Bartell, T., & Land, T. (2014). Using video analysis to support prospective K-8 teachers' noticing of students; multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education, 17*(3), 245-270.

Turner, E. & Drake, C. (2016). A review of research on prospective teachers' learning about children's mathematical thinking and cultural funds of knowledge. *Journal of Teacher Education, 67*(1), 32-46. DOI:10.1177/0022487115597476

van Es, E., Hand, V., & Mercado, J. (2017). Making visible the relationship between teachers' noticing for equity and equitable teaching practice. *In teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 251-270). Cham, Switzerland: Springer.

van Es, E., Cashen, M., Barnhart, T. & Auger, A. (2017). Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy. *Cognition and Instruction, 35*(3), 165-187. DOI:10.1080/0370008.2017.131725

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education, 18*(4), 375-400.

INVESTIGATING SECONDARY PRESERVICE TEACHERS' MATHEMATICAL CREATIVITY DURING PROBLEM SOLVING

Anne N. Waswa
University of Georgia
Anne.Waswa@uga.edu

Kevin C. Moore
University of Georgia
kvcmoore@uga.edu

Understanding and supporting preservice teachers' (PSTs') mathematical creativity (MC) is important to encourage their teaching with and for mathematical creativity. Research on MC with a focus on teachers has mostly been concerned with in-service and pre-service teachers' conceptions of MC (e.g., Andrade & Pasia, 2020; Beghetto, 2007; Bolden et al., 2010; Lev-Zamir & Leikin, 2011;2013, Shriki, 2010; Author 1, 2019). We extend this line of work by investigating PSTs' MC *as they engage* in problem solving in the context of quadratic and linear growth. The purpose of the study was to investigate the components of MC that are evident when secondary PSTs engage in problem solving including how these components relate to their mathematical meanings for quadratic and linear growth. Participants included three secondary mathematics PSTs at a university in the Southern part of the US. Data were collected using teaching experiment (Steffe & Thompson, 2000) and analyzed using ongoing analysis and retrospective analysis (Steffe & Thompson, 2000).

We adopted Lithner's (2008) framework for creative and imitative reasoning. The framework's criterion for MC includes *novelty* (developing a new sequence of reasoning or re-creating a forgotten one), *flexibility* (using multiple approaches fluently and adapting to new situations), *plausibility* (arguing for ones' choice and/or implementation of a strategy), and *mathematical foundation* (anchoring arguments in the intrinsic mathematical properties of the components entailed in reasoning). We leveraged literature on quantitative and covariational reasoning, (e.g. Carlson et al., 2002; Ellis et al., 2020; Author 2, 2013; Author 2 & Thompson, 2015; Paoletti & Author 2, 2017; Saldanha & Thompson,1998; Thompson, 2011; Thompson & Carlson, 2017) which means attending to the manner in which quantities change together, and specifically coordinating first- and second-differences in quantities. We used the fundamental principles of radical constructivism (von Glasersfeld, 1995) to drive and form the kinds of descriptions that we provide regarding PSTs' mathematical creativity.

PSTs' meanings for linear growth included: constant first differences in the dependent variable with uniform changes in the independent variable and constant rate of growth. Their meanings for quadratic growth included: constant second differences in the dependent variable with uniform changes in the independent variable, increasing rate of growth (constant amount in which the addition increases each time), and conceptualizing quadratic growth using first and second derivatives to describe rate of change and the difference in rate of change respectively. *Novelty* and *flexibility* were evident in PSTs' conceptualizations of linear and quadratic growth through their use of multiple approaches and representations such as tables, graphs, and equations to elaborate these meanings. However, one PST experienced perturbations with the *plausibility* and *mathematical foundation* in terms of her quantitative organization in conceptualizing linear and quadratic growth. Initial findings of this emphasizes the development of conceptual meanings beyond multiple representations to show linear and quadratic growth and leveraging quantitative and covariational reasoning through task design and questioning to develop PSTs' meanings for quadratic and linear growth and support PSTs' MC.

References

Andrade, R. R., & Pasia, A. E. (2020). Mathematical Creativity of Pre-service Teachers in Solving Non-routine Problems in State University in Laguna. Universal *Journal of Educational Research, 8*(10), 4555-4567.

Author, (2019)

Beghetto, R. A. (2007). Does creativity have a place in classroom discussions? Prospective teachers' response preferences. *Thinking Skills and Creativity, 2*(1), 1-9.

Bolden, D. S., Harries, T. V., & Newton, D. P. (2010). Pre-service primary teachers' conceptions of creativity in mathematics. *Educational Studies in Mathematics, 73*(2), 143-157.

Lev-Zamir, H., & Leikin, R. (2011). Creative mathematics teaching in the eye of the beholder: focusing on teachers' conceptions. *Research in Mathematics Education, 13*(1), 17-32.

Lev-Zamir, H., & Leikin, R. (2013). Saying versus doing: Teachers' conceptions of creativity in elementary mathematics teaching. *ZDM - International Journal on Mathematics Education, 45*(2), 295-308.

Lithner, J. (2008). A research framework for creative and imitative reasoning. *Educational Studies in Mathematics*, *67*(3), 255-276. .

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen et al. (Eds.), *Proceedings of the 20th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 298–303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Shriki, A. (2010). Working like real mathematicians: Developing prospective teachers' awareness of mathematical creativity through generating new concepts. *Educational Studies in Mathematics, 73*(2), 159-179.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of research design in mathematics and science education*, 267-306.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L.L. Hatfield, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e, University of Wyoming, Laramie, WY (2011), pp. 33-57*

Thompson, P. W., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). Reston, VA: National Council of Teachers Mathematics.

von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Washington, D.C.: Falmer Press.

CHARTING A VIABLE LEARNING TRAJECTORY FOR THE DEVELOPMENT OF PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF HIERARCHICAL GEOMETRIC RELATIONSHIPS

Ian Whitacre
Florida State University
iwhitacre@fsu.edu

Domonique Caro-Rora
Florida State University
dcarorora@fsu.edu

Keywords: Communication, Geometry and Spatial Reasoning, Learning Trajectories and Progressions, Preservice Teacher Education

Scarce literature focuses on PSTs' reasoning related to hierarchical geometric relationships (Browning et al., 2014). Some articles address ways in which PSTs' reasoning has not measured up to researchers' expectations: Researchers find that PSTs have difficulty defining shapes inclusively, exhibit "misconceptions" (Pickreign, 2007), and rely on "personal figural concepts" (Fujita & Jones, 2007). Such findings provide baseline data but do not illuminate a path forward.

In a report of an intervention study, Yi et al. (2020) offer evidence that PSTs' knowledge of geometry can improve significantly over a three-week period. Their description mentions instruction on hierarchical relationships but does not elaborate. The assessment included items involving properties and relationships, focusing on quadrilaterals, but did not address particularly challenging aspects (e.g., categorization of trapezoids). Brunheira and da Ponte (2019) report on a teaching experiment focused on classification of quadrilaterals and then of prisms. An example that they provide of a student's work (their Figure 11) involves what seems to us to be two very different interpretations of Venn diagrams. This distinction is left unexplored. The authors report progress in PSTs' reasoning but also lingering difficulties, and they suggest the need to explore "other factors" beyond those identified in the framework of Fujita (2012), which is based on Van Hiele levels. They refer to such factors as "language interpretation and logical reasoning" (p. 80).

Overall, the literature leaves us with unanswered questions. It does not provide insights into PSTs' geometric reasoning, nor does it document viable learning trajectories for the development of that reasoning (Browning et al., 2014). We believe a key to understanding PSTs' reasoning and supporting their learning related to hierarchical geometric relationships lies in those aforementioned "other factors" that we have not seen explored in the literature to date.

Focusing on PSTs' reasoning about relationships among quadrilaterals, we are charting a viable learning trajectory (Simon, 1995) consisting of two intertwined strands: (a) interpretation and use of language and representations concerning hierarchical relationships and (b) reasoning about properties and relationships. In the first strand, progress is afforded by the development of shared language, most notably to distinguish two ways of using and interpreting Venn diagrams ("set–subset" and "similarities/differences"). Related to the second strand, PSTs explore and discuss relationships in sequence (rectangle–square, parallelogram–rhombus, etc.) with the help of dynamic geometry software. These activities, together with analogies to other contexts (e.g., relating the relationship between rectangles, rhombi, and squares to that between mint ice cream, chocolate chip ice cream, and mint chocolate chip ice cream), appear to support progress in PSTs' distinguishing between inclusive and exclusive definitions, reasoning inclusively about hierarchical relationships, and identifying overlapping sets, leading to the ability to construct "one big diagram" representing a hierarchy of quadrilaterals. Given the contrasting ways of using and interpreting language and diagrams at the beginning of the unit, this is a nontrivial feat. Our ongoing work documents the development of PSTs' reasoning about hierarchical relationships by working from their prior knowledge with an emphasis on communication. It has yielded insights that clarify for us PSTs' reasoning and enable us to articulate a learning trajectory.

References

Browning, C., Edson, A. J., Kimani, P. M., & Aslan-Tutak, F. (2014). Mathematical content knowledge for teaching elementary mathematics: A focus on geometry and measurement. *The Mathematics Enthusiast, 11*(2), 333–384.

Brunheira, L., & da Ponte, J. P. (2019). From the classification of quadrilaterals to the classification of prisms: An experiment with prospective teachers. *Journal of Mathematical Behavior, 53*, 65–80.

Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education, 9(1–2)*, 3–20.

Fujita, T. (2012). Learners' level of understanding of the inclusion relations of quadrilaterals and prototype phenomenon. *Journal of Mathematical Behavior, 31*(1), 60–72.

Pickreign, J. (2007). Rectangles and rhombi: How well do preservice teachers know them? *Issues in the Undergraduate Mathematics Preparation of School Teachers, 1.*

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics and Science Education, 26*(2), 114–145.

Yi, M., Flores, R., & Wang, J. (2020). Examining the influence of van Hiele theory-based instructional activities on elementary preservice teachers' geometry knowledge for teaching 2-D shapes. *Teaching and Teacher Education, 91*, 103038.

Chapter 12:

Professional Development and In-Service Teacher Education

USING VIDEO TO IDENTIFY WHAT IS NOT KNOWN IN STUDENTS' MATHEMATICAL THINKING

Madhuvanti Anantharajan
Stanford University
madhuvan@stanford.edu

Michael Jarry-Shore
Boise State University
mjarryshore@boisestate.edu

Remaining continually curious about students' mathematical thinking is challenging, yet worthwhile, in teaching practice. This paper describes and analyzes two video-based professional learning (PL) activities designed to help teachers go beyond their initial perceptions of what students understand and to identify what else they might learn about students' thinking. The findings suggest the potential of the activities to evoke different types of curiosity about student-thinking and the conditions that may support such questioning.

Keywords: Student-thinking, teacher-curiosity, professional development, early mathematics

Uncovering students' mathematical thinking is accepted as an important part of ambitious mathematics teaching (Lampert et al., 2013). However, the emphasis on immediacy in the classroom often results in teachers lacking the time to be curious about students' thinking and to understand all that is there. Teachers may also not know what else to look for or be curious about when a student seems to understand certain mathematical ideas well. In other words, teachers may not attend to what they themselves do not know about students' thinking. Both these cases may result in teachers paying less attention to student-thinking than needed and responding before having given adequate attention to such thinking. In these challenging situations, it is important for teachers to slow down their own thinking and be curious about the details of students' thinking.

This paper describes two video activities designed to help teachers slow down their own thinking regarding students' mathematical ideas, and support teacher-curiosity about students' mathematical thinking when counting. The first activity helped teachers identify what they could learn about a student's thinking when the student's understanding of early number concepts appeared unclear, and the second helped teachers identify how they might extend what they knew about a student who appeared to have a strong understanding of the concepts involved in the activity. The activities are based on the construct of teacher-curiosity (Anantharajan, 2020) and were part of a professional learning (PL) module designed to support teachers' curiosity about student-thinking.

Literature review and conceptual framework

This study is theoretically based on the approach of Cognitively Guided Instruction (CGI) (Carpenter et al., 1996) and the framework of teacher-noticing of student thinking (Jacobs et al., 2010), which prioritize attending to student-thinking, and using it to determine how teachers can respond to students. The steps involved in these approaches are translated to questions that guide teachers to notice, interpret and respond to student thinking. However, skilled noticing does not automatically help teachers figure out how to respond (van Es & Sherin, 2008). In this context, the framework of teacher-curiosity suggests one way for teachers to respond by introducing specific intermediate steps to help teachers identify what they wish to learn about student thinking, and how they can find out about that aspect of their students' thinking, thus providing a concrete way to move from noticing to responding.

Teacher-curiosity is defined as follows: An instance of teacher-curiosity is one where teachers recognize something as unknown, unfamiliar, puzzling, uncertain, or new in the context of teaching and learning, and feel motivated to initiate inquiry into that instance (Anantharajan, 2020). This conceptualization of teacher-curiosity is based on literature in teacher professional learning, and mathematics education, and understandings of curiosity in philosophy and psychology. Teacher-curiosity is comprised of cognitive, motivational, and active aspects (Audi, 1995, 2017). The cognitive aspect of teacher-curiosity involves an experience of dissonance, not knowing, or confusion and can include surprise, ambiguity, puzzlement, or novelty (Berlyne, 1966; Kashdan, 2004; Lowenstein, 1994) about student-thinking. The motivational aspect addresses the desire to learn more about those aspects of student-thinking that the teacher finds surprising, ambiguous, puzzling, or new, that the teacher wants to learn more about. The active element of teacher-curiosity refers to how teachers go about trying to learn about aspects of student-thinking they are curious about.

Teacher-curiosity can be directed towards many aspects of instruction, including students' thinking. When teachers' curiosity is directed at something they observed or experienced, it may be regarded as 'specific'. For example, if a student counts and says a total number that is one more than the actual total, the teacher may wonder "Did this student count an object twice by mistake or do they not know their number sequence?" While it may relate to understanding a particular instance, specific curiosity may not always be satisfied by information from that instance itself. For instance, in the preceding example, a teacher observing the student in real time would be unable to re-watch the student count. However, they may design other activities or questions to answer the question. When what teachers observe or experience evokes curiosity about something that was not immediately observed or experienced, their curiosity may be regarded as 'diversive.' For example, a teacher may see a student group objects by five and wonder what other numbers the student can group by (Grossnickle, 2016).

The PL at the heart of this study focused on teachers' curiosity about children's mathematical thinking in the domain of counting. Counting involves *counting principles* like one-to-one-correspondence, number-sequence, and cardinality, as well as an understanding of counting *strategies* like grouping (Carpenter et al., 2017).

The current paper focuses on the cognitive aspect of teacher-curiosity. The PL activities were designed to surface the range of potential ideas in student-thinking that teachers could be curious about. Prior work on noticing has focused on identifying what students know and are able to do (Jacobs et al., 2010). The goal of the video activities and corresponding instruments analyzed here was to shift the focus away from determining what students know and can do, to what the teacher-participants *did not* know about what students knew. Approaching children's thinking in this way was intended to provide a starting point to take a stance of curiosity, rather than one of certainty, about student-thinking. The research question addressed here is: How do two video-based PL activities help participants identify what they do not know about students' mathematical thinking?

Methods

The PL focused on Counting Collections. In this activity children count collections of objects. Teachers observe students and ask them to explain their strategies and try to understand students' thinking. The study had six participants. They taught grades TK[1] to 1st grade in three

1 Transitional kindergarten (TK) in the California public school system includes 4-year-olds who turn five after September 2 of the school year, and will enter kindergarten the following year.

schools in California, and four of these participants taught combined K-1 classrooms. They had worked as teachers for between 1 and 20 years, and all of them implemented Counting Collections prior to the PL as well. The PL was informed by research on teacher-learning (Borko et al., 2014; van Es & Sherin, 2008) and supporting curiosity (Kashdan & Fincham, 2004), and was facilitated by the first author. The entire PL comprised of six, weekly after-school sessions of 90-minutes each. Sessions included video-based discussions that were guided by the teacher-curiosity framework. Participants were individually interviewed before and after the PL using a protocol that operationalized the teacher-curiosity framework. The participants were paid a nominal honorarium for their time.

The current paper analyzes participant responses to two video-based activities in the second session of the PL which focused on identifying what is unknown, and therefore possible to be curious about, with respect to student-thinking. Guidelines on planning video-based professional development for teachers (Borko et. al 2014), a framework of criteria for choosing videos portraying students' mathematical thinking (Sherin et al., 2009) and empirical guidelines for interventions to support curiosity (Kashdan & Fincham, 2004) informed the selection of video-clips. Each video contained elements that could potentially push teachers to examine what is and is not known about students' thinking. Each brief video was of a student engaged in a counting activity and who was of a similar age as those the teachers taught. Participants responded individually to instruments developed for these specific video activities. The data analyzed in this paper consist of the responses to these two instruments (Table 1). For this activity the participants were given a research-based list of counting principles and strategies they could refer to as they watched and commented on the student videos.

Video 1: Christian Counts Bears

The purpose of this activity was to draw participants' attention to the unpredictable and the unexpected in students' thinking. The activity was structured in four steps: Provide partial information about student-thinking; invite participants' prediction based on the partial information; provide more information that might be surprising to participants and have participants assess their prediction; and finally, have participants identify what they now do not know about the student's thinking. The conjecture was that participants would want to know what might explain any gaps between their prediction and what the student actually did.

Table 1: Christian Counts Bears Instrument

Questions
[Video paused]
1. Based on what you have seen, what do you expect Christian might do or say next?
2. Why would you expect that?
[Video resumes]
3. Did you see what you expected? [If not, what was different?]
4. Based on this video, what do you not know, or what could you try to find out about Christian's thinking?

In this video, the student, Christian, is seated at a table and a researcher gives him several plastic bears to count. The researcher asks Christian to count all the objects together, count objects by color, and compare the size of different color groups. Christian's counting initially suggests that if objects were to be added to his collection, he would count the entire collection again to find out the total number, rather than remembering how many he previously counted and counting on from there. At one point in the video, the researcher adds more objects to Christian's collection and asks how many he has in total. In the PL, the video was paused just prior to this point and participants were asked to predict in their response instruments (Table 1) what Christian might do next, based on what they had seen so far. The participants then saw the rest of the video, where he seems to count on mentally and says the new total number. Asked how he knew, he explains that he held the previous total in his mind, started to count mentally from the next number and arrived at the new total. This ability was not apparent in the preceding part of the video. Participants were then asked to note whether he did what they had predicted. Finally, participants were asked to reflect and identify what they felt they did not know about Christian's mathematical thinking.

Mohammad Counts Cookies

The purpose of this activity was to explore what else there may be to learn about a student whose understanding is both strong and visible. Unlike the previous activity, here the participants were shown the whole video. They were then invited to propose informed conjectures of what remained unknown but relevant based on the information in the video.

In this video, the student, Mohammad, is seated at a table with a researcher. The researcher asks him to imagine that he has three boxes with five cookies in each. He is then asked how many cookies he has in all. The researcher provides Mohammad a set of blocks which he uses to help him count. He first places three yellow blocks side by side. These may represent the boxes although he does not say so, because he does not count these yellow blocks going forward. On each yellow block he stacks five blocks of other colors. He then counts the first two stacks as 5 and 10. Then he counts on the remaining five by ones. He says that he has fifteen cookies in all. In the video he devises a clear counting strategy and explains his thinking without apparent difficulty. This video gave participants the opportunity to reflect on what they might learn about a student who did not indicate any confusion or struggle. The instrument for this activity had only one question that participants responded to after watching the whole video: Based on what you see in the video, what could you try to find out about Mohammed's thinking?

Coding and Analysis

The data was coded for the type of curiosity that participants expressed, and the mathematical principles and strategies that the participants mentioned in their responses. The responses were also coded for whether participants perceived a strong or partial understanding of the principle or strategy (Table 2). Additionally, for the first video, the analysis also looked at participants' statements about whether or not Christian did what they had predicted.

Inter-rater agreement was calculated with Kappa statistics at the parent and child code levels, and discussion and consensus at the grandchild code level. The Kappa value for the parent and child codes ranged from 0.7-1.0. Agreement values were calculated for the pre- and post-interview transcripts[2]. This instrument and data set operationalized all the elements of the teacher-curiosity framework and was a reliable indicator of the types of responses in more activity-specific instruments like the video response tool that operationalized only the cognitive aspect of teacher-curiosity. After reaching agreement on the interview data the two coders discussed the data from all other instruments, including the instruments analyzed in the current paper to confirm that the

2 In these interviews participants watched a video and responded to questions based on the teacher-curiosity framework, as well as broader questions on the role of curiosity in their own work. The codebook used in the present study corresponds to the questions in the interviews and the elements of the framework.

codebook could be applied to these data as well. The kappa calculations mentioned above were thus for data sources (i.e., interview transcripts) not the focus of this study, but mentioned here because coding those was necessary to developing the codebook used here. Disagreements were discussed to reach consensus.

Table 2: Codes Applied

Type of curiosity	
Specific curiosity	Participant is curious about something that they observe a student do or say.
Diversive curiosity	Participant is curious about something a student might do or say, which they cannot observe at that time.
Mathematical ideas that participants perceive in students' thinking	
a. Counting principles	Principles teachers perceive as constituting child's understanding of counting. (Carpenter et. al. 2017, NRC 2001) Includes: • One-to-one correspondence • Cardinality • Number sequence • Abstraction principle • Order irrelevance • Other principles or concepts (e.g., number conservation, operations, base-ten system, place value).
ai. strong understanding	Teacher implies the child has a comfortable/ fluent/ consistent understanding.
aii. partial understanding	Teacher implies the child is working on understanding, or that the child's understanding is not always consistent.
b. Counting strategies	Strategies children use to count objects. Includes: • Grouping by number • Strategies to keep track • Sorting based on non-numerical qualities • Visual arrangements • Associating number with object • Other strategies
bi. strong understanding	Teacher implies the child has a comfortable/ fluent/ consistent understanding.
bii. partial understanding	Teacher implies the child is working on understanding, or that the child's understanding is not always consistent.

Table 3. Sample of Coded Responses

Question	Bella	Codes	Beth	Codes
2. Why would you expect that?	*He understands that he can touch each one & count it out loud.* *He does not seem to count on.*	ai. One-to-one correspondence; strong understanding bii. Counting on; partial understanding	*He doesn't seem to be keeping track or counting on as he is answering the questions.*	bii. Keeping track and counting on; partial understanding

The responses for each instrument were organized into a single table where each row represented a question in the instrument and each column represented a participant. Thus, all the participants' responses were visible side-by-side. The first author then coded the responses for the mathematical ideas that the participants refer to. Table 3 indicates a sample of two participants' responses to question 2 and the codes applied to Christian Counts Bears.

Results

The coded data indicated multiple patterns. The results for each video are discussed below.

Christian Counts Bears

After watching the first portion of the video, five of the six participants predicted in their written response that Christian would recount the entire collection. This was consistent with what they had observed until that point, where Christian tended to count all the objects each time objects were added. One participant wrote that he would count on from the previous total he counted, if he could remember the total. Four of the six participants also wrote that they expected that Christian would not arrive at the correct total number of objects after counting the final time.

Five participants said he did not do what they expected – in other words, they said their prediction was not correct. One said she had expected that if he counted on he would begin at 14 but he began at 16, and in this sense he did not do what she expected. The aspect of Christian's counting that surprised the participants was his ability in the second portion of the video to count on from a number he held in his mind. This aspect of counting on is what all participants said they could learn more about. For example, Lillian stated in her response, "I would ask him to show me how he counted them all together by adding on" and Hannah stated, "We know he can do the basic idea of counting on. We don't know if he can do it accurately."

Three participants also identified an additional mathematical idea that they wanted to look for and learn about in the student's thinking, namely addition. Bella wondered "Can he add bigger numbers together? What can he add together in his head?"

The participants all expressed two types of curiosity about Christian's thinking. On the one hand, they wished to know more about things that would help explain why he had counted on, and why their own predictions were not accurate. For example, Beth stated, "I would like to find out how high he can count with one-to-one correspondence. (When he counted the 12 bears, he did not have one to one)." The other type of questions were related to what else Christian knew or could do as a kindergartener, or related to the general body of ideas related to number and counting. For example, Stacey wondered, "Would he be able to add together 2 smaller groups of bears?" In summary, the participants used what they had seen upto the point the video was paused to predict Christian's future moves and discovered that their prediction was not accurate. Rather, participants found that Christian's understanding and skills were more complex and advanced than they anticipated. Consequently, participants perceived that there were aspects of his thinking that they did not know about and identified what they could learn about his thinking.

Mohammad Counts Cookies

In their responses after watching the entire video of Mohammad, all the participants went beyond Mohammad's 'correct answer' and identified further aspects of his thinking they could try to learn about. All the participants attended to the details of what they saw Mohammad do and posed questions about what he understood. Lillian wondered if he could subitize the quantity of five blocks. Bella said she would ask him to explain his strategy for "figuring out 15."

The participants all wondered if Mohammed understood how to count by groups of 5 or 10. This ability to skip count or count by grouping can be seen as a precursor to multiplication and the participants may have wanted to determine this before exploring multiplication with him. For example, Hannah wondered, “Can he count by 5s? if there were 25 would he count on starting at 11 still? Or would he make groups of 10? Count by 5s?”

Without a particular element of surprise or dissonance evoked by the video, the types of questions that participants posed were of two types. The first was to clarify aspects of what they saw him do. The questions posed by the participants also make clear that despite completing the task with apparent confidence and giving the right answer, there were aspects of what the student did in the video that were not clear and thus possible to clarify. For example, Beth stated, “He was good at counting on. I would like to know more about how he keeps track.” The other type of questions that participants posed had to do with related mathematical ideas that they could explore with the student, to find out what else he understood and to what extent. For example, Rita wrote, “I could try to find out what he knows about 10s. [Can he] use a 10s frame?” In summary, the participants identified aspects of Mohammad’s thinking that were related to what he was able to do but were not apparent in the video.

Discussion

This paper looks at how two video-based activities helped participants in a PL identify what they could further learn about students’ mathematical thinking. The experience of the participants in the two activities was quite different. For the Christian Counts Bears video, participants were likely curious about what was unknown because of the nature of the video the facilitator chose, the way the viewing of the video was structured (i.e., pausing at an opportune time), and how inviting predictions required participants to process the information they had up to that point. These steps appeared to seed dissonance and give rise to participants’ puzzlement. The dissonance or disequilibrium that resulted from participants predicting what the student could do then finding out that the prediction was not accurate likely evoked participants’ questions about what more they could learn. It was also striking to note that the participants readily stated that the student did not do what they predicted, which indicates a certain humility and openness on the part of the participants to revise their thinking. It also potentially suggests that the structure of the activity may have helped them revise their thinking by focusing on their surprise rather than evaluating whether or not their prediction was correct.

For the Mohammad Counts Cookies video, participants had to draw on their own knowledge and experience to assess the information from the video and imagine what other information they would like to have. The response of the participants affirms Kashdan and Fincham’s (2004) recommendations for interventions to facilitate curiosity, which informed the design of the task. These include developing “tasks that capitalize on novelty, complexity, ambiguity, variety, and surprise,” “enjoyable, group-based activities,” and “tasks that are personally meaningful” (p.490). The video-based PL activities may also provide participants cues for practice, to attend to the unexpected and unknown aspects of their own students’ thinking.

With the Christian Counts Bears video, making an inaccurate prediction and seeing some surprising aspects of the student’s thinking may have provided a starting point to learn more about aspects of the student’s thinking, such as counting on or keeping track of counting, as well as consider other mathematical understandings that may explain what he did, such as addition. The teachers’ questions might have been motivated by a desire to resolve the dissonance they experienced, which can be a strong motivating factor in learning (Chinn & Brewer, 1993).

For the Mohammad Counts Cookies video, participants considered a wide range of ideas they could learn about in Mohammad’s thinking, despite his apparent confidence and correct answer. The structure of the video activity may have helped participants identify what else they could learn about Mohammad's thinking. This includes the sequencing of the video after Christian Counts Bears, which may have primed the participants to reflect on what is known or unknown about the mathematical thinking of the student in a video. This mental priming, as well as the explicit invitation to consider what else the student may know, may have helped participants identify the wide range of mathematical ideas they could explore with this student.

Although participants did not explicitly refer to Mohammad's understanding of multiplication in Mohammad Counts Cookies, they did want to learn about his understanding of grouping and skip counting, which can be a step towards multiplication. Teachers often tend to associate multiplication with older students (Carpenter et al., 2017). However, the participants' responses suggest that activities like these may provide opportunities to slow down, look beyond the correct answer, and uncover the seeds of such "advanced" ideas in younger students as well.

Both types of curiosity that participants displayed towards student-thinking – specific and diversive – implicitly acknowledge that there is more to learn about students' understanding than is initially apparent. The findings however indicate that specific curiosity may be further unpacked into questions that seek two types of answers: information that will help resolve a feeling of dissonance as the result of a surprising occurrence or an incorrect prediction, and information to explain what is observed. The former can help teachers correct misconceptions or assumptions about students, and the latter can help teachers make sense of the knowledge and understanding that a student demonstrates in the moment.

In the context of video-based PL activities, it is possible that some specific questions can be resolved by re-watching the video. For example, Rita asked of Christian's counting, "Where does his 1-to-1 [correspondence] stop?" Watching the video again with this question in mind may help her answer this question. But not all questions can be answered by re-watching the video because the information to answer some of the specific questions that participants asked is not present in the video. They would need to interact with the student and give him more tasks to find out, like asking him to count all bears by ones, or asking him to explain how he keeps track, asking him to count the collection again using the strategy that he just used – all ideas proposed by the participants. Thus, these questions indicate specific curiosity to the extent that what the student does in the video triggers participants' questions though the answers may not be in the video.

On the other hand, diversive questions can help teachers use their observations as a starting point to wonder what else the student may know or understand. In this sense, diversive questions may help teachers plan their response to and next steps even with students who seem to have strong mathematical understandings. In the context of the PL itself, diversive or "what else" questions may be an effective way to close the video activities, so as to invite teachers to continue thinking about what they might learn about a student's thinking, including their own. Developing the capacity to pose these questions can also help teachers approach student-thinking with the attitude that there is always more to learn. PL activities can thus be designed to elicit all these different types of questions about student-thinking.

Limitations and future work

As this is a small dataset, further research is needed to determine to what extent these findings are applicable to other PL contexts. Further, the participants in the study already implemented Counting Collections and were interested in teaching that elicits and responds to student-thinking. It would require further research to understand whether and in what ways teachers with other pedagogical approaches respond to the activities described in this study.

References

Anantharajan, M., (2020). *"What Do You Wonder?" Developing and Exploring the Construct of Teacher-Curiosity About* Young *Children's Mathematical Thinking.* (Publication Number 28241775). [Doctoral dissertation, Stanford University]. ProQuest Dissertation Publishing.

Audi, R. (1995). Acting from virtue. *Mind*, 104(415), 449-471.

Audi, R. (2017). Role Modeling and Reasons. *Journal of Moral Philosophy*, 14(6), 646-668.

Berlyne, D. E. (1966). Curiosity and exploration. *Science*, 153(3731), 25-33.

Borko, H., Jacobs, J., Seago, N., & Mangram, C. (2014). Facilitating video-based professional development: Planning and orchestrating productive discussions. In Y. Li, E. Silver, and S. Li (Eds) *Transforming mathematics instruction* (pp. 259-281). Springer, Cham.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, *97*(1), 3-20.

Carpenter, T.P., Franke, M.L., Johnson, N.C., Turrou, A.C., & Wager, A.A. (2017). *Young Children's Mathematics: Cognitively Guided Instruction in Early Childhood Education*. Heinemann.

Chinn, C. A., & Brewer, W. F. (1993). The role of anomalous data in knowledge acquisition: A theoretical framework and implications for science instruction. *Review of Educational Research*, 63(1), 1-49.

Grossnickle, E. M. (2016). Disentangling curiosity: Dimensionality, definitions, and distinctions from interest in educational contexts. *Educational Psychology Review*, *28*(1), 23-60.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education*, 169-202.

Kashdan, T. (2009). Curiosity and Exploration Inventory (CEI-II). Retrieved from http://psychfaculty.gmu.edu/kashdan/CEI-II.pdf

Kashdan, T. B., & Fincham, F. D. (2004). Facilitating curiosity: A social and self-regulatory perspective for scientifically based interventions. *Positive psychology in practice*, 482-503

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., ... & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, 64(3), 226-243.

Loewenstein, G. (1994). The psychology of curiosity. *Psychological Bulletin*, 116, 75–98.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student-thinking. *Journal of Teacher Education*, 60(3), 213-230.

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and teacher education*, 24(2), 244-276.

A CATALYST FOR CHANGE: A TEACHER'S EXPERIENCES WITH SUPPLEMENTARY CURRICULAR MATERIALS ENRICHED WITH INTERACTIVE SIMULATIONS

Sebnem Atabas
University of Southern California
atabas@usc.edu

Ian Whitacre
Florida State University
iwhitacre@fsu.edu

This study aims to understand a middle school mathematics teacher's instruction and reflections on her experiences with supplementary curricular materials enriched with interactive simulations—PhET interactive simulations (sims) and sim-based materials—over two years. We conceptualize Linda's (pseudonym) teaching in terms of both thinking and doing. Regarding Linda as doer, results show significant differences in Linda's instruction, favoring sim lessons compared to non-sim lessons. Regarding Linda as thinker, results show increased attention to problems of practice related to teaching, as well as more productive framing of problems of practice. When these two findings are taken together, shifts in what Linda did and thought illustrate the potential for high-quality supplementary materials to function as a catalyst for change as we map the flow of shifts in what Linda thought and did across two years.

Keywords: teacher learning, interactive simulations, problems of practice, framing

Purpose

In the light of research suggestions, teachers have been tasked to empower students to experience mathematics as a dynamic process of exploration (Boaler & Greeno, 2000; Langer-Osuna, 2017) rather than recalling a static body of knowledge. This task demands teachers to create a fundamentally different learning environment in which teachers "draw information out of students" (Boaler, 2003, p. 4) rather than transferring the information to students. Given the challenges of "moving away from a language of skills ('students will calculate slopes') to the language of understanding ('students will identify common features of linear growth')" (p. 62), to transition from one teaching approach to the other is not an easy task (Horn, 2012).

Even if many teachers engage in informal reflection on their teaching on a daily basis, it might be challenging for teachers to be critical about their instructional decisions and know what to change about their teaching (Hart et al., 1992). Then, the question becomes how to create opportunities for teachers to be critical about what they do in classrooms and "*rethink* their teaching, rather than merely extend their existing practice" (Horn et al., 2017, p. 51). Related to this question, there are some promising research findings on the use of technology (Goldenberg, 2000) and supplementary curricular materials (Matewos et al., 2019) in challenging routine instructional practices and the existing interplay between students, teachers, content, and activity (Zbiek et al., 2007). As a special technological tool, we focus on PhET sims (phet.colorado.edu) and supplementary curricular materials designed around them—sim-based materials.

Teachers often use online supplementary curricular materials with the intention of improving student engagement or learning (Polikoff et al., 2018). However, it is well-documented in the literature that teachers may end up being the ones engaging in learning. For example, in their effort to create a learning environment for students to engage in mathematics in more meaningful ways, Wood et al. (1991) realized how "the classroom had simultaneously and unintentionally become a learning environment for the teacher as well" (p. 588). Building on research on learning through teaching, teachers often develop their understanding of students, content, and teaching as they use unfamiliar tasks (Leikin, 2005).

Building on the research findings on the potential of supplementary curricular materials and instructional technologies to be catalysts for change (Kaufman et al., 2018; Matewos et al., 2019), this study explores the potential of sims and sim-based materials to motivate shifts in a middle school mathematics teacher's approach to teaching mathematics. For this purpose, we recorded what Linda *did* in her sim and non-sim lessons and what she *thought* about her experiences with these materials over two years. Accordingly, we asked the following questions:

1. How did Linda's instructional practices in the sim and non-sim lessons compare in terms of instructional quality over two years?
2. What problems of practice did Linda identify as she used sims and sim-based materials over two years? How did she frame these problems of practice?
3. What, if any, alignments were there in Linda's identification and framing of problems of practice and her instructional practices as evidence of professional growth?

Conceptual Framework

Given the complexity of teaching, identifying shifts and changes is not an easy task. To capture a complete picture of teacher change, we conceptualize teachers as both *doers* and *thinkers* (Horn et al., 2017) (Table 1). The field of mathematics education made important progress in developing measures for instructional quality based on instructional practices aligned with a reform approach in which the instruction builds on student thinking and ideas (Boston, 2012; Marder et al., 2010; Thompson & Davis, 2014). Some observable indicators of what teachers *do* in the classroom include the rigor of tasks as planned and implemented, teacher questioning (e.g., exploring mathematical relations), accountability of student–teacher and student–student interactions (e.g., linking mathematical ideas), teacher's press for knowledge or thinking, and students providing knowledge or thinking in response (Boston, 2012).

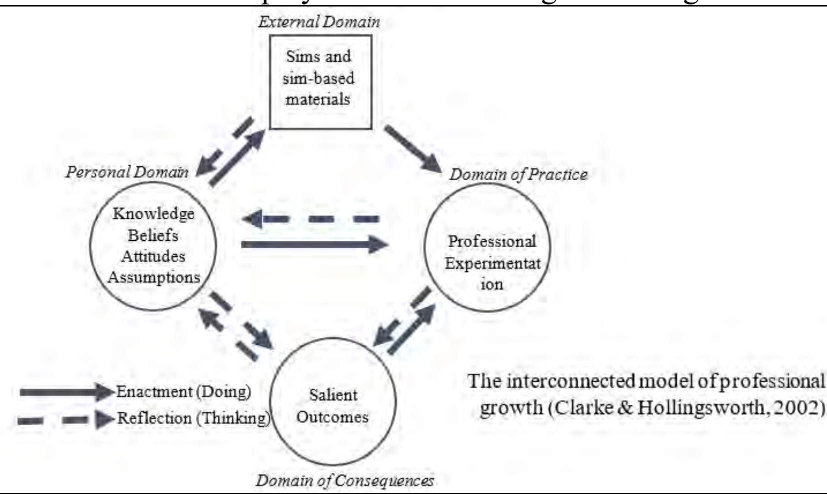
One important resource to get access to teachers' *thinking* is problems of practice that they identify in their talk (Horn & Little, 2010; Vedder-Weiss et al., 2018; Windschitl et al., 2011). For example, 'students' difficulties in mathematics' can be a problem of practice that teachers may identify. In addition to identifying this problem of practice, how teachers frame the problem of practice (Dyer, 2020; Vedder-Weiss et al., 2018) adds an interpretive stance of teachers' *thinking.* Building on the same example, the teacher may frame the problem as an inherent student characteristic (e.g., some students are born with a "math gene" and some are not) or as a lack of learning opportunities available in the classroom (e.g., the context of the problem did not afford students to make sense of the content). These two framings have important consequences for what teachers might do in the classroom.

Aligned with the conceptualization of teachers as doers and thinkers, Clarke and Hollingsworth (2002) presented an interconnected professional growth framework including teacher reflection and enactment, showing the interplay between personal domain (e.g., knowledge), external domain (e.g., curricular materials), domain of practice (e.g., teacher's instructional practices), and domain of consequences (e.g., outcomes of teacher questioning) (Table 1). Our conceptual framework enabled us to capture what a teacher did and thought and the interplay between the two across the domains of interconnected professional growth model.

Table 1: Conceptual Framework

Teacher as a *Thinker*	Teacher as a *Doer*

• What *problems of practice* teachers identify (Horn & Little, 2010; Vedder-Weiss et al., 2018; Windschitl et al., 2011) in their talk. • The way(s) teachers *frame* the problems of practice (Bannister, 2015; Dyer, 2020; Jackson et al., 2017; Snow & Benford, 1988; Vedder-Weiss et al., 2018) they identify in their talk.	• Instructional quality assessment (IQA) (Boston, 2012) comprised of eight rubrics below: - Potential of the Task - Implementation of the Task - Student Discussion Following the Task - Rigor of Teachers' Questions - Teacher's Linking Contribution - Student's Linking Contribution - Teachers' Press for Knowledge or Thinking - Students Providing Knowledge or Thinking • Each rubric is scored on a scale from 0 to 4
Teacher as doer and thinker (Horn et al., 2017)	
The Interplay between Thinking and Doing	



Methods

Participant

Linda was a middle school mathematics teacher with 11 years of teaching experience at the beginning of the study. During her participation in the study, she was teaching in a Title 1 school in the southeastern United States. Her students were predominantly white (64%), with students who were identified as African American (15%), Hispanic (18%), and others (3%). Overall, Linda shared many characteristics not uncommon with other middle school mathematics teachers. She identified her teaching as teacher-centered. She was knowledgeable about student-centered pedagogy based on her professional learning experiences (e.g., undergraduate program). She was struggling with student engagement and looking for some resources to increase student engagement. At the start of this research, Linda participated in a two-day workshop where she was familiarized with PhET simulations and provided with sample sim-based lessons. Besides the workshop, she received no structured support (e.g., professional development). The instructional materials designed by the PhET research team and other teachers were accessible to her through the website. She decided which sims and corresponding sim-based materials to use.

Our goal with our case study selection was to understand a middle school mathematics teacher's use of supplementary curricular materials enriched with interactive simulations. Yin (2018) highlighted the goal of doing case study research is to develop "analytic generalizations"

(p. 23), which are defined as "The logic whereby case study findings can apply to situations beyond the original case study, based on the relevance of similar theoretical concepts or principles" (p. 349). We believe this case study can inform other teachers in situations beyond Linda's case and make connections to the theory of teacher learning (LeCompte et al., 1993).

Data Collection

We observed and recorded lessons in two of Linda's 7th grade math periods during the 2017–2018 and 2018–2019 school years. In the first year of the study, Linda taught three sim-based lesson modules, each comprised of 3–4 consecutive days of teaching with the use of sim-based materials, in one of her class periods in which data were collected. She taught the corresponding non-sim modules in another class period in which data were collected. The data set included video recordings of seven sim lessons and six non-sim lessons in Year 1 and 10 sim lessons in Year 2. Linda used the same sim-based modules in the first year and second year of the study. We also conducted interviews with Linda at three time points: the beginning of Year 1, the end of Year 1, and the end of Year 2, as well as collected written reflections after she taught each sim-based module (Figure 1).

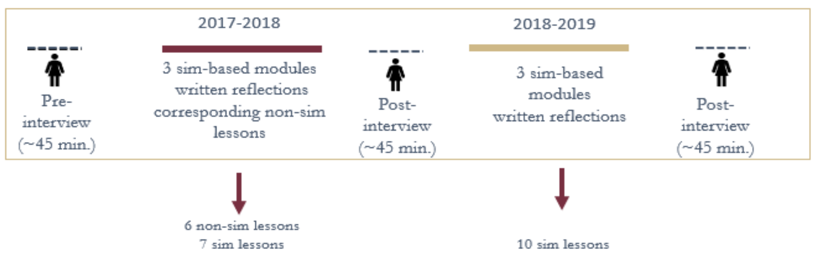


Figure 1: Data Collection

Data Analysis

To answer the first research question on what Linda *did,* we used video recordings of Linda's teaching in sim lessons and non-sim lessons. We analyzed the instructional quality of each lesson by using IQA (Boston, 2012; Matsumura et al., 2002) and assigned a rubric-specific IQA score for each type of lesson—sim lessons versus non-sim lessons. To examine whether sim lessons resulted in statistically higher levels of instructional quality, we ran seven permutation tests (one for each of our seven IQA rubrics) (Good, 2013).

We used Linda's written reflections and interviews to answer the second research question on Linda's thinking. We separated them into idea units (Jacobs et al., 1997; Tekkumru-Kisa & Stein, 2015). In each idea unit, we identified what problem of practice (Horn & Kane, 2015) Linda discussed and how she framed the problem of practice (Dyer, 2020; Snow & Benford, 1988). We organized our coding chronologically to identify shifts across two years.

To answer the last research question, we examined the interplay between what Linda did and thought as she started using sims and sim-based materials over two years by using the interconnected model of professional growth framework (Clarke & Hollingsworth, 2002). This framework enabled us to map the flow of change between four domains: external domain (e.g., sims and sim-based materials), personal domain (e.g., knowledge, assumptions), domain of practice (e.g., sim-based activities used by the teacher, questions the teacher pose to students), and domain of consequences (e.g., student engagement, student learning).

Results

Differences in What Linda Did in Class with a Focus on Instructional Quality

Regarding Linda as *doer*, results showed that IQA scores were significantly higher in sim vs. non-sim lessons on four rubrics: implementation of the task, student discussion following the task, rigor of teacher's questions, and teacher's linking (Table 2). More specifically, in sim lessons, Linda encouraged students to elaborate their thinking and explore mathematical meanings with some effort to make connections between their contributions. Thus, students had more opportunities to develop their own strategies and present their work and their thinking to the whole class. Please note that we excluded the rubric on the potential of the task because sim-based materials were provided to Linda. We were interested in what Linda was doing with sim-based materials when she used them in her lessons rather than the potential of these tasks.

Table 2: Mean of IQA Rubric Scores in Non-sim and Sim lessons

	Potential of the task	Implementation of the task	Student Discussion Following Task	Rigor of Teachers' Questions	Teacher Linking	Student Linking	Teacher Press	Students Providing
Non-sim Lessons (Year 1)	2.50	2.00*	1.50*	1.50*	1.33*	1.17	2.17	2.00
Sim Lessons (Year 1 & Year 2)	3.80	3.20	2.80	2.87	2.07	1.80	2.93	2.67

*$p < .05$.

To give the reader a sense of what Linda's sim and non-sim lessons looked like, we briefly describe the lessons focusing on the same topic—scale factor, part of the unit on proportional relationships. In the non-sim lesson, Linda went through a worked example in the book, which comprised of three steps as follows: (1) write the scale as a fraction, (2) convert the fraction to a unit rate, (3) multiply the unit rate by actual length to find the missing value. In contrast, the sim lesson started with an open-play time for students to explore how the sim works (click to open the sim). After exploring the sim, Linda asked students to create a figure on the sim and scale it by a scale factor of 2 and 3, as well as to predict how the perimeter and area of the figures would change as the original figures were scaled. After students compare their predictions with the actual values (the sim can show the actual values of the perimeter and area of the figures with a click on a button), Linda asked students:

> What do you think happened to the area? Why was not as simple as it is, just multiplying it [the area of the original figure] by 3?... Any ideas of why you think that is? It [the scale factor] definitely affected the area differently? (Sim Lesson on Scale, Year 2)

Based on these qualitative descriptions, in the non-sim lesson, finding the missing values asked in the problems was the end of the discussion. Whereas, in the sim lesson, the scaled figures served as the data for students to develop a rule that explains how scale factor affects the perimeter and area of a shape—if they affect the perimeter and area differently, what was the reason? While these contrasts between sim and non-sim lessons were important, Linda's takeaway from this experience was worthwhile, as discussed below.

What Problems of Practice Linda Identified and How She Framed Them?

Regarding Linda as *a* thinker, there was a shift in Linda's attention from problems of practice related to students to problems of practice related to teaching. At the beginning of the study, Linda heavily focused on problems of practice related to students (e.g., students not engaged)

(89% of the problems of practice). In Year 2, by contrast, Linda focused on problems of practice related to *her* teaching (e.g., ask good questions) (62.5% of the problems of practice) (Table 3).

Table 3: Number of Problems of Practice Related to Students and Teaching

	Number of Problems of Practice related to Students (%)	Number of Problems of Practice related to Teaching (%)	Total Number of Problems of Practice (%)
Pre-Interview	8 (89%)	1 (11%)	9 (100%)
Year 1	13 (52%)	12 (48%)	25 (100%)
Year 2	9 (37.5%)	15 (62.5%)	24 (100%)

Aligned with Linda's increasing attention to her teaching, in the post-interviews, Linda talked about her struggle in asking good questions, limitations in her content knowledge, and how much to tell students to scaffold their thinking (Table 4a and 4b). In addition to these new problems of practice emerged, Linda's description of these problems got more sophisticated. For example, in Year 1, Linda framed the importance of teacher questioning especially for students who might struggle, whereas, in Year 2, Linda reframed teacher questioning as a tool to support student learning, not only for students who struggle but also for expanding student thinking as she said:

> Coming up with the right questions to ask them. Like both ends of the spectrum … when they are struggling but also … when they think they got it … there is always more. I mean, aside from the generic "explain your thinking," … how to encourage them… and also like "can you find another way," … for ones who are struggling but also when they do get it. And how to push them but not just give them more work to do. (Post-interview, Year 2)

Although these percentages show a shift in Linda's attention, it is important to note that an increase in the number of problems of practice related to teaching does not necessarily indicate a positive change over time. It is equally important to examine Linda's framing of these

Table 4a: Problems of Practice Related to Students Over Two Years

	Students not engaged	Students not motivated	Students have a mindset of "I am not good in math"	Some students struggle more	Some students play schooling	Do not notice patterns Linda expects	Struggle with working on open-ended tasks	Struggle with understanding math concepts (e.g., coming up with a rule)
Pre-interview	3	1	2	2	0	0	0	0
Year 1/Sim	0	1	1	2	1	2	6	0
Year 2/Sim	0	0	1	0	1	0	4	3

Table 4b: Problems of Practice Related to Teaching Across Two Years

	Attend student experience	Avoid mis-conceptions	Know your goal	Anticipate student struggle and responses	Teacher's limited content knowledge	Ask good questions	Decide how much to tell	Anticipate what students know	Step back
Pre-interview	1	0	0	0	0	0	0	0	0
Year 1/Sim	1	2	1	1	2	3	2	0	0

Year 2/Sim	0	0	1	1	1	9	1	1	1

problems of practice related to students and teaching. Our analysis of Linda's framing of these problems of practice showed evidence that Linda's conception of what it meant to be a good student was challenged. She was surprised about students who were typically on track and following instructions because these students struggled with new expectations in sim lessons and expected Linda to tell them what to do. In a post-interview, Linda said,

> Often in the sim activities, they [students] are asked a question that they do not know how to answer right off their head. So, there is an immediate pushback; well, I do not wanna do this, I do not wanna do this… but when I just stop giving them how to, there has been a difference. (Post-interview, Year 2)

Linda was also surprised by students who were typically unengaged because in sim lessons, these students made important contributions to the discussions. In her written reflections in Year 1, Linda wrote, "The students that surprised me the most were the students that are usually the 'unmotivated' ones." Thus, Linda's prior framing of students who were *on track* or *unengaged* was challenged. Another reframing was about student engagement. In the pre-interview, Linda expected sims and sim-based materials to increase engagement; however, she was not clear about what aspects of sims would make that difference. Starting in Year 1, Linda framed student engagement in connection to the curricular materials she used. In a post-interview, Linda said,

> I realized that me going over problems at the board and asking the students questions required little from the students… That is what I am trying to say in a sim lesson versus a non-sim lesson. There is not much opportunity for them to contribute their ideas, whereas, in sim lessons, there is much more opportunity, which is also a part of more engagement. (Post-interview, Year 2)

Students were contributing their ideas and figuring things out by themselves rather than following Linda going over the problems on the board, and that was the reason for increased student engagement in sim lessons.

The Interplay Between What Linda Did and Thought as Evidence of Professional Growth

Regarding the last research question, we identified three flows of change evident in the interplay between what Linda did and thought over two years. The first flow of change was about Linda's seeing students as more capable and problematizing who were motivated and engaged. As Linda started using sims and sim-based materials—external domain—she practiced not telling students what to do and encouraging them to explore—domain of practice. In response, she observed students figuring things out by themselves, especially the ones who were considered unmotivated and unengaged—domain of consequences. This consequence created an opportunity for Linda to reflect on her assumptions about who was engaged and motivated to learn mathematics and what students were capable of doing—personal domain. Overall, induced from her observations in her classroom, Linda started seeing students more capable and student engagement in connection to the curricular materials she used.

The second flow of change started with Linda stepping back and listening to students—domain of practice. Consequently, students brought different ideas and thoughts, often different than what Linda expected to hear—domain of consequences. Linda repeated her questions hoping that students would bring what she expected to hear; however, this did not happen, and Linda ended up limiting student thinking by asking close-ended questions through the end of the lessons—domain of practice. This flow was reflected in Linda's personal domain as an expressed struggle in asking good questions to leverage student thinking. Importantly, not telling students what to do created a need for some practices that Linda was not readily comfortable with, and she showed ownership of the problems of practice (e.g., asking good questions).

The final flow of change was about teaching mathematics with a conceptual orientation. As Linda started using sims and sim-based materials, the nature of mathematics they were working on became more about the concepts, relations, and connections rather than procedures and computations. Thus, in sim lessons, she more frequently asked "why" and "how" questions to uncover student ideas—domain of practice. In addition to Linda, students also asked questions to make sense of the mathematical ideas and concepts (e.g., "What does random mean?") part of their effort to make sense of the mathematics. However, Linda and her students struggled in coming up with those explanations—domain of consequences—and this struggle was resolved with Linda's shift to procedural focus through the end of the lessons. Again, Linda identified this as a problem of practice related to her personal domain by identifying limitations in her content knowledge in connection to her struggle in maintaining the mathematical focus on conceptual ideas—personal domain. Although Linda was not readily capable of maintaining the conceptual focus, she developed an awareness and, more importantly, a need to improve her content knowledge.

Discussion and Conclusion

These findings support the recent findings on the potential of supplementary curricular materials in creating shifts in teachers' instructional roles and doubt induced by their own teaching (Matewos et al., 2019). Linda started using sims and sim-based materials with the intention of increasing student engagement; however, "the classroom had simultaneously and unintentionally become a learning environment" (Wood et al., p. 588) for her as well. She showed some shifts in her instructional practices, in the problems of practice she identified, and in her framing of these problems of practice. Using sims and sim-based materials created a rich experience where she problematized her previous conceptualization of student engagement as stable and connected it to the nature of work students are tasked to do. She started seeing students as more capable, became critical of her own instructional practices and content knowledge, and identified areas for improvement. These shifts, especially seeing students as more capable and student engagement in connection to what students are tasked to do, are important steps to engage in efforts to improve teachers' instruction (Jackson et al., 2017).

As discussed before, creating learning environments aligned with reform recommendations—more equitable learning opportunities for students—is not an easy task for teachers (Horn, 2012). Teachers' experimentation with student-centered pedagogy as they use well-designed supplementary curricular materials can create opportunities for teachers "to *rethink* their teaching" (Horn et al., 2017, p. 51), identify what particular practices need to be improved in their own teaching, and a desire to improve (Horn & Kane, 2015). Given the teachers' more frequent use of online resources and social media with the pandemic (Aguilar et al., 2021), teachers are becoming more aware of the available resources on online platforms. Thus, there is a need to understand the influence of these materials on what teachers *do* in their classrooms and what teachers *think* about their experiences with these materials. Doubt induced by teachers' own teaching and ownership of problems of practices (e.g., asking good questions) has the potential to further teachers' efforts to improve their instructional practices. In future work, we aim to explore the role of interactive simulations, tasks, and student contributions in affording the shifts we identified in what Linda did and thought over two years.

References

Aguilar, S. J., Rosenberg, J. M., Greenhalgh, S. P., Fütterer, T., Lishinski, A., & Fischer, C. (2021). A different experience in a different moment? Teachers' social media use before and during the COVID-19 pandemic. *AERA Open*, *7*, 23328584211063898.

Bannister, N. A. (2015). Reframing practice: Teacher learning through interactions in a collaborative group. *Journal of the Learning Sciences*, *24*(3), 347-372.

Boaler, J. (2003). Studying and capturing the complexity of practice-The case of the 'Dance of agency.' *Proceedings of the International Group for the Psychology of Mathematics Education*, Honolulu, Hawaii 1,3–16.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 171–200). Westport, CT: Ablex.

Boston, M. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal*, *113*(1), 76-104.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*(8), 947-967.

Dyer, E. (2020). The Dynamic Nature of Teachers' Framing of the Problem of Student Mathematical Difficult, *Proceedings of the International Conference of the Learning Sciences, Nashville, TN.*

Goldenberg, E. P. (2000). *Thinking (and talking) about technology in math classrooms: K-12 mathematics curriculum center issues paper series.* Newton, MA: Education Development Center, Inc.

Good, P. (2013). *Permutation tests: A practical guide to resampling methods for testing hypotheses.* New York: Springer Science & Business Media.

Hart, L. C., Schultz, K., Najee-ullah, D., & Nash, L. (1992). Implementing the professional standards for teaching mathematics: The role of reflection in teaching. *Arithmetic Teacher 40*, 40–42.

Horn, I. S. (2012). *Strength in numbers: Collaborative learning in secondary mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Horn, I. S., Garner, B., Kane, B. D., & Brasel, J. (2017). A taxonomy of instructional learning opportunities in teachers' workgroup conversations. *Journal of Teacher Education*, *68*(1), 41-54.

Horn, I. S., & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: Relationships to instructional expertise. *Journal of the Learning Sciences*, *24*(3), 373-418.

Horn, I. S., & Little, J. W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers' workplace interactions. *American Educational Research Journal*, *47*(1), 181-217.

Jackson, K., Gibbons, L., & Sharpe, C. (2017). Teachers' views of students' mathematical capabilities: Challenges and possibilities for ambitious reform. *Teachers College Record*, *119*(7), 1-43.

Jacobs, J. K., Yoshida, M., Stigler, J., & Fernandez, C. (1997). Japanese and American teachers' evaluations of mathematics lessons: A new technique for exploring beliefs. Journal of Mathematical Behavior, 16(1), 7–24.

Kaufman, J. H., Opfer, V. D., Bongar, M., & Pane, J. (2018). *Changes in what teachers know and do in the Common Core era*. Santa Monica, CA: RAND.

Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education, 48*(3), 237–247.

LeCompte, M. D., Preissle, J., & Tesch, R. (1993). *Ethnography and qualitative design in educational research*. Academic Press.

Leikin, R. (2005). Qualities of professional dialogue: Connecting graduate research on teaching and the undergraduate teachers' program. *International Journal of Mathematical Education in Science and Technology*, *36*(2-3), 235-254.

Marder, M., Walkington, C., Abraham, L., Allen, K., Arora, P., Daniels, M., Dickinson, G., Ekberg, D., Gordon, J., Ihorn, S., & Walker, M. (2010). The UTeach Observation Protocol (UTOP) Training Guide. UTeach Natural Sciences, University of Texas Austin.

Matewos, A. M., Marsh, J. A., McKibben, S., Sinatra, G. M., Le, Q. T., & Polikoff, M. S. (2019). Teacher learning from supplementary curricular materials: Shifting instructional roles. *Teaching and Teacher Education*, *83*, 212-224.

Matsumura, L. C., Garnier, H., Pascal, J., & Valdés, R. (2002). Measuring instructional quality in accountability systems: Classroom assignments and student achievement. *Educational Assessment*, *8*(3), 207-229.

Polikoff, M., Le, Q. T., Danielson, R. W., Sinatra, G. M., & Marsh, J. A. (2018). The impact of speedometry on student knowledge, interest, and emotions. *Journal of Research on Educational Effectiveness, 11*(2), 217-239.

Snow, D.A., & Benford, B. (1988). Ideology, Frame Resonance, and Participant Mobilization. *International Social Movement Research*, *1,* 197-217.

Tekkumru Kisa, M., & Stein, M. K. (2015). Learning to see teaching in new ways: A foundation for maintaining

cognitive demand. *American Educational Research Journal*, *52*(1), 105-136.
Thompson, C. J., & Davis, S. B. (2014). Classroom observation data and instruction in primary mathematics education: Improving design and rigour. *Mathematics Education Research Journal*, *26*(2), 301-323.
Vedder-Weiss, D., Ehrenfeld, N., Ram-Menashe, M., & Pollak, I. (2018). Productive framing of pedagogical failure: How teacher framings can facilitate or impede learning from problems of practice. *Thinking Skills and Creativity*, *30*, 31-41.
Windschitl, M., Thompson, J., & Braaten, M. (2011). Ambitious pedagogy by novice teachers: Who benefits from tool-supported collaborative inquiry into practice and why? *Teachers College Record*, *113*(7), 1311-1360.
Wood, T., Cobb, P., & Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal*, *28*(3), 587-616.
Yin, R. K. (2018). *Case study research and applications: Design and methods*. Thousand Oaks, CA: Sage.
Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: The perspective of constructs. In F. Lester (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 1169–1207). Reston, VA: NCTM.

INVESTIGATING MATHEMATICS DEPARTMENT LEADERS' EXPERIENCES AND UNDERSTANDINGS OF EQUITY

David Fifty
Oregon State University
fiftyd@oregonstate.edu

Stacey Zimmerman
Western Carolina University
szimmerman@wcu.edu

Mary Beisiegel
Oregon State University
mary.beisiegel@oregonstate.edu

ELITE PD Research Group[1]
EHR #2013590, 2013563, and 2013422

In the wake of national movements calling for attention to equity, colleges, departments, and instructors are now faced with various responsibilities to implement practices and structures to support diversity, equity, and inclusion (DEI). This project aims to design a professional development program for mathematics graduate teaching assistants (MGTAs) by helping them learn evidence-based teaching practices to support diverse groups of learners in engaging mathematics activities. Part of these efforts included investigating department leaders' understanding of equity. This paper focuses on interviews with department leaders at a large, public, research university. Our analysis shows that despite the implementation of university structures focused on improving equity, department leaders had very different understandings of equitable teaching, and reported differences in how equity factors into their roles.

Keywords: Professional Development, Equity, Inclusion, and Diversity

Introduction

The majority of mathematics graduate teaching assistants (MGTAs) spend between 12 to 20 hours each week working for mathematics departments as instructors or teaching assistants (Selinski & Milbourne, 2015). In some departments, MGTAs have the opportunity to teach their own courses, while in other departments, MGTAs do not teach their own courses but instead lead recitations or lab sessions that support large enrollment courses taught by faculty members. During a two-year master's degree program, MGTAs might have the opportunity to teach as many as seven classes or lead as many as 24 recitations, and in a six-year doctoral program 15 classes or 60 recitations. At some doctoral-granting institutions, MGTAs teach up to 68% of Calculus I courses (Selinski & Milbourne, 2015). This means that during their graduate programs, MGTAs contribute to the learning experiences of hundreds, if not thousands, of undergraduate students.

Because of their significant impact on the teaching and learning of undergraduate mathematics (Ellis, 2014; Miller et al., 2018; Selinski & Milbourne, 2015), it is important to address MGTAs' development as educators. One important piece of this preparation is addressing equity and inclusivity in their classrooms. To this end, we have designed a professional development (PD) program specifically for MGTAs focused on equitable, inclusive, and engaging teaching practices. Our vision is to have this PD program adopted as a permanent department structure, with the aspiration of creating more equitable learning environments through a larger systemic change process (Reinholz et al., 2020). Change initiatives benefit from

[1] The ELITE PD Team: Mary Beisiegel and David Fifty Oregon State University; Mary Pilgrim and Sloan Hill-Lindsay at San Diego State University; Rani Satyam and Rebecca Segal at Virginia Commonwealth University; Stacey Zimmerman at Western Carolina University; Hayley Milbourne at University of San Diego.

department-level implementation because departments represent relatively coherent units of culture (Reinholz & Apkarian, 2018). This coherence may support innovation efforts more broadly and can contribute to the sustainability and effectiveness of the implementations (Reinholz et al., 2018).

Departments are not monolithic, though, and contend with differing goals and varying cultural and institutional issues. Consequently, the first stage of this project is to investigate perspectives of faculty members serving in departmental leadership roles to learn about various supports and barriers that might impact an MGTA PD program. This requires "exploring clashes of values that may lie at the heart of institutional resistance to change" (Reinholz et al., 2020, p. 3), as transformation efforts need to unite stakeholders in an ongoing and cyclical process to adapt to changing contextual factors (Reinholz & Apkarian, 2018; Smith et al., 2021).

In this report, we address two research questions that inform the development of the MGTA PD program:

1. How do department leaders conceptualize equitable teaching?
2. What role does equity have in department leaders' positions and considerations?

Theoretical Framework

We utilize Reinholz and Apkarian's (2018) adaptation of Bolman and Deal's (2008) four-frame model of organizational change to study departmental cultures and change within them. This framework provides context for exploring how department cultures may support or hinder change initiatives by taking into consideration institutional structures and norms through a systems approach (Henderson et al., 2011; Kezar, 2011). In particular, the four constructs, or frames, explored are: *structures*, *symbols*, *power*, and *people*. Reinholz and Apkarian (2018) define these frames:

> *Structures* are the roles routines and practices of a department; their enactment and meaning are dependent on *symbols*, which are the norms, values and ways of thinking in a department; changes are ultimately enacted by *people* whose individuality impacts their intentions and perceptions; and the distribution of *power* determines who makes certain decisions and influences interactions (p. 7)

For this report, we focus on the people frame, which is also referred to as the human resources frame (Bolman & Deal, 2008). This frame highlights that each department is composed of individuals that have their own goals, agency, needs, and identities (Reinholz & Apkarian, 2018). The department needs people (and vice versa) and exists to serve human needs, especially those of students, who have their own unique goals (Bolman & Deal, 2008). Tensions within the people frame can be caused by the differing needs, goals, and values amongst various stakeholders within the department. Department chairs, in particular, are well positioned to provide leadership in navigating these differences, such as by creating an inclusive and supportive culture for faculty, staff, and students (Bystydzieński et al., 2017; Fried, 2003; Hecht et al., 1999; Wergin, 2003).

Data Sources and Methods

We report on the analysis of baseline interviews with leadership in one mathematics department at a large, public, PhD granting university prior to implementing a PD program for MGTAs focused on equitable and inclusive teaching practices. We characterized department leaders as those who we identified as powerbrokers with the ability to influence department-level decisions regarding MGTA PD. For example, the department chair, the graduate chair, experienced coordinators, and those who contribute to existing MGTA PD within the department received invitations for interviews. Our focus on these important figures is in recognition that "change effort requires sanction from the appropriate power holders to succeed" (Reinholz & Apkarian, 2018, p. 5). Three of the six department leaders accepted our invitation and were interviewed. These interviews were audio recorded and transcribed. For this report, we refer to these interviewees by their roles: (1) the department chair whose role is to oversee the entire department in its research and teaching missions, (2) the associate chair whose role is mostly to support the department chair and take on tasks delegated by the department chair, and (3) the graduate chair whose role is to oversee graduate admissions and the progress of graduate students through either the master's or doctoral program. Each of the participants are mathematicians who had either been invited to serve in their specific role or volunteered to serve in their role when the position became vacant.

The semi-structured interview protocol focused on learning about how equity and inclusivity were addressed by the department as a whole and with respect to the MGTAs' teaching practices, how equity and inclusivity were understood by department leadership, and the department leaders' perceptions of the department and its direction. Some questions that were asked included: How would you like MGTAs to engage with students?; What is most important to you for MGTA professional development?; How would you describe equitable teaching?; and, In what ways does the department or college value equity and inclusivity?

Analysis

For our first phase of analysis, we used an inductive open-coding approach (Miles & Huberman, 1994) for interview responses that related to our research questions regarding department leaders' conceptualization of equity and the role of equity in their position. Our second phase of analysis incorporated an axial coding method to surface emerging patterns and distinctions from our codes. Additional consideration in this phase was given to the department leaders' positions in the department. Lastly, we utilized the people frame as an explanatory construct to make sense of what we were seeing in the data. In particular, we referenced this frame to conceptualize how department leaders were contending with the varying needs, goals, and values of the stakeholders that their position was responsible for addressing, as well as their own.

Findings

There was a noteworthy difference in department leaders' responses to questions about equitable and inclusive practices. In particular, the department chair expressed significant institutional pressures, such as how the department would be 'measured' in whether and how it supported diverse groups of students. In comparison, the other department leaders expressed little, if any, institutional pressures in their respective roles. Thus, we saw their different leadership roles as moderating their views and descriptions of equitable and inclusive teaching practices. We address each leader's responses to questions related to equity and inclusivity and the context of their roles in those responses.

The Department Chair

When the department chair was asked "*How would you describe equitable teaching?*" he did not provide a direct answer, instead describing the pressures he felt from society as a whole as well as those from the institution. With regard to societal pressures, he stated "our society is just rigged in a way [that] makes it really hard for some people to succeed… I'm keenly aware from my position right here that the forces that affect all of those things are way deeper than anything that happens when they set foot in my classroom." With this in mind, he went on to describe "equitability" in his own teaching as "[making] sure that I'm not accidentally saying something that's going to make someone doubt whether they belong … And so, for me, it's just, you know, don't inflict more damage."

The department chair shared that much of his work and how he directs some of the focus of the department is mandated by the dean's and the institution's focus on DEI. One example of this was the requirement that each department in the college needed to establish a DEI committee. The department chair shared, "I can't claim … that I was the instigator of this, although, you know, eventually we would do it. But the dean requires this." Because of the mandate, the department chair had assembled a committee of faculty and graduate students who had each expressed interest in DEI issues in the department. The department chair had given the committee an assignment to write a mission statement for the department that focused on DEI.

In comparison to the two other department leaders, the department chair experienced significant institutional pressures and scrutiny. His voiced his frustrations several times during the interview, noting:

> Every now and then I get held accountable. You know, math departments are very vulnerable in this regard because, you know, D-F-W rates and graduation rates. I can't tell you how many times I've sat around a table with [de-identified]... And we get going on student success or failure rates in classes and next thing I know, everybody in the room is looking at me going what are you going to do about that? […] And-and one of the things that has upset me most is that every now and then a crusading high-level administrator walks into my office and throws some statistics down in front of me about how achievement gaps are particularly bad in mathematics. And you know, I don't deny or debate any of that.

Consequently, the department chair's focus on equity was mostly based on "hard-nosed measurables" because that was the lens by which his performance would be measured by the institution.

As a result, he described some of the work in his leadership position consists of advocating for the mathematics faculty and the DEI work and training that they engage in. These efforts seemed to come with disheartenment over his view that the university's president reiterates the existing inequities instead of acknowledging the work that is done:

> [I]t's extraordinarily demoralizing. Because I know that my faculty are … going to the seminars and the workshops where they are being trained … in developing the ways to be inclusive and to implement teaching practices that promote equitability and inclusivity. I know they're doing that work. But when I hear my president saying those things over and over again, as if it's … endemic to the institution, it's a blow … But I think it's important … to see and acknowledge the people who are doing the work … even if it's not showing up in the measurables yet, to acknowledge that it is happening.

This focus on measurables led to his recognition of a need to collect and track data for his department. He shared that this would enable him to guide his department with regard to PD to ensure they are collectively "making any progress." Consequently, he suggested that when evaluating equity related efforts, "one looks for impacts and sort of measurable effects."

The Graduate Chair

When the graduate chair was asked "*How would you describe equitable teaching?*" he shared that he perceived equity to mean equality, that "every student is treated in the same fashion in a classroom." He characterized inclusive teaching as "you can't leave anybody out." However, he described that "the system" should not be changed for students, stating "the kid that for whatever reason doesn't show up half of the time, or, … for whatever reason [is] the one that, you know, looks funny to you or act[s] weird. No you can't … change the system for any of those people." The graduate chair did not view DEI as a principal component of his role and he expressed little accountability towards equity in his position. This was interesting given that the department chair proudly noted that the department had worked "very hard to make sure that our graduate admissions policies and practices are inclusive and holistic. And we are keeping track of our progress in terms of diversity and in terms of, you know, what-what the demographics of our incoming classes [of graduate students] are."

When asked about whether he encourages faculty and MGTAs to engage in opportunities focused on inclusivity and equity, the graduate chair responded:

> I personally don't do it … I really don't think in these terms, so it's funny. I try to run the graduate program, okay.. [laughter]. You know, which is a very different thing than leading a college or department. So personally, I do not do that. I do not encourage faculty members or TAs to go and engage in those types of opportunities. But I see the leaders, you know, above … me. I see those do it definitely, and I think it's their role. I mean I'm not trying to duck responsibility.

In this statement, he implicitly acknowledged that his role as graduate chair had different requirements than the department chair's role, and so he did not feel similar pressures to attend to DEI-related issues. Yet, he eventually recognized that there are aspects of MGTAs' potential future careers that could involve DEI, such as academic tenure, promotion, and hiring practices; he expressed, "I think those have really tightened up already and have at least in my opinion become equitable and inclusive."

The Associate Chair

Unlike the department chair and graduate chair, when the associate chair was asked "*How would you describe equitable teaching?*", he was able to explicitly reference various student supports, such as the need to create a welcoming environment that would promote access and accommodate different learning styles. He suggested the need to contend with various barriers to student learning. One point of emphasis, his response contrasted with the graduate chair's conceptualization because he suggested providing flexibility for students, especially during the COVID-19 pandemic. He added that flexibility was also needed due to other aspects of students' lives, such as other jobs and pressures that might interfere with their learning.

The associate chair described the various ways in which DEI related to different aspects of his position prior to becoming associate chair. He referred to advocacy training that was required to participate in various hiring committees, and similar training that was needed for his previous role as department chair. He also described roles he had served in that were directly related to department's vision for DEI, such as being co-chair of a committee that established department acknowledgements regarding DEI values. Part of this responsibility required finding common ground amongst department members with regard to DEI values.

The associate chair also discussed his efforts at promoting diversity within mathematics to the department community and emphasized the importance of this with graduate students. He acknowledged the overrepresentation of while male mathematicians and the need to "make a sustained effort to say this is not all there is, and bring up a contribution of female mathematicians." These efforts included organizing department colloquia and events, such as one to show a movie on a prominent mathematician from an underrepresented background, which was followed by “long conversations.” Similarly, he discussed the memberships that the department has with various regional and national alliances and described his participation.

Department Leaders’ Views of PD for MGTAs

The leaders were asked questions about MGTAs’ teaching roles and professional development. The associate chair was the only department leader to mention equitable teaching when discussing the important teaching outcomes for MGTA PD. He described the need for MGTAs’ teaching to be responsive to the needs and backgrounds of the student body, and to be able to provide a classroom environment that is welcoming and accessible:

> For professional development I think-well two things come to mind: one is, you know, pedagogical tools to-to-to address different backgrounds in the students, to accommodate different teaching styles. And together with that is, you know, tools for providing a welcoming environment and an environment that promotes the access and equity.

This sentiment was in opposition to what the graduate chair suggested. To him, an ideal outcome would be a successful continuation of the status quo. He described this as: “[W]e want to give it along, pass it along to the next generation … you know, try to see yourself as a little link in a chain, you know, where you pass on from the next, to the next generation. And then they’ll do it again, hopefully.” Lastly, the department chair had a more individualized conceptualization than the other two leaders. He suggested that the most important outcome of MGTA PD would be for better career preparation, sharing: “I have to view graduate study as a professional pathway. You know, this really is something that you’re doing because you think it leads to a career. You’re not doing it for personal growth or, you know, to become a better citizen or stuff like that.”

Discussion

While the college and department have explicit diversity action plans and statements, and despite awareness of various department and institution DEI initiatives, the three department leaders shared very different visions of equity and equitable teaching. This embodies Gutiérrez’s (2002, 2017, 2018) description of the word equity as being complicated by a long history of conceptualization. In her work, she shares that this often results in the word *equity* retaining simplistic or superficial definitions that refer to a wide range of meanings and contexts that have led many to believe that they are speaking of the same ideas/topics despite having different definitions. The differing viewpoints, values, and goals of these department leaders highlight possible tensions that change initiatives might have to contend with, as described by the people frame.

Because of his unique position, we were particularly interested in the department chair’s views on equity. The department chair's communications about equity seemed largely influenced by other university officials, including his dean. He adopted language that he attributed to these leaders, especially his focus on “measurable outcomes.” Similarly, the department chair’s view on equity seems to be shaped by, or aligned with, his description of the university president’s focus on “endemic” deficiencies instead of the work that faculty are doing. He shares a bleak view that efforts within the classroom could only hurt students instead of help. Yet, this understanding of equity does not serve the students of the department, nor does it respond to the students’ needs and identities. The department chair’s seeming indifference towards explicitly addressing equity in the classroom does not bode well for departmental change and innovation efforts in this regard; especially as he is uniquely positioned to provide leadership in directing support towards such efforts.

One key aspect of the people frame that surfaced with the individual goals and agency amongst the three department leaders. A range of accountability towards integrating or encouraging DEI efforts emerged in our analysis. On one end of the spectrum, the graduate chair seemed to shun any responsibility for advocating for DEI, sharing that he thought it was the responsibility of those "above" him, saying that "I think it's their role." Meanwhile, the department chair felt it was his responsibility to advocate for the DEI-related work of those in the department. And lastly, the associate chair recounted various ways that they supported DEI in their work, both explicitly in their actions or implicitly by participating in related PD.

Conclusion

We conclude with thoughts about how this work may influence the development of our MGTA PD program focused on equitable teaching practices. Through our analysis we found several tensions within the people frame that were caused by the differing needs, goals, and values amongst the department leadership. Our analysis yielded several barriers and supports for our planned implementation of the MGTA PD program. First, the different understandings of equity and equitable teaching demonstrate challenges of explaining the program and its learning outcomes to various leaders and stakeholders within the department, including leaders, faculty, and MGTAs. The MGTA PD program focuses more on qualitative measures of teaching (e.g., engaging students and creating welcoming environments) and might not be accepted and supported by a department chair who feels under pressure to produce "hard-nosed measurables." Second, given that the graduate chair does not view it as his responsibility to encourage opportunities focused on equity, such a PD program may experience challenges with MGTA recruitment and sustainability. Similarly, his view that ideal PD outcomes would enable the perpetuation of current teaching practices is in opposition to our project's vision of changing teaching practices.

The interviews also surfaced various supports for the implementation of our PD program. Department leaders shared that college and university leadership are focused on avenues to leverage DEI efforts. Although the graduate chair demonstrated disinterest in advocating for equity-related opportunities, the other department leaders highlighted that there are faculty members who value these experiences and efforts. Leveraging such college- and university-level support will be instrumental in building, and maintaining, a sustainable MGTA PD program.

Acknowledgments

This work is supported by the National Science Foundation under Grant No. 2013590, 2013563, and 2013422. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Bolman, L. G., & Deal, T. E. (2008). *Reframing organizations: Artistry, choice, and leadership*. San Francisco, CA: Jossey-Bass

Bystydzieński, J., Thomas, N., Howe, S., & Desai, A. (2017). The leadership role of college deans and department chairs in academic culture change. *Studies in Higher Education*, *42*(12), 2301-2315.

Ellis, J. (2014). Preparing future college instructors: The role of graduate student teaching assistants (GTAs) in successful college calculus program. Unpublished doctoral dissertation, San Diego State University and University of California San Diego.

Fried, L. P. (2003). Organizational change to support success of women: A model and its lessons. In S. Horning (Ed.) *Equal rites, unequal outcomes: Women in American research universities* (pp. 227–44). New York: Kluwer Academic/Plenum.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, *4*(2-3), 145-187.

Gutiérrez, R. (2017). *Equity: How the E-Word helps and hurts our cause in mathematics education.* A plenary address given in the Critical Issues in Mathematics Education Workshop. Mathematical Sciences Research Institute. March 15, 2017. Berkeley, Calif. (2017a). https://www.msri.org/workshops/836/schedules/21847.

Gutiérrez, R. (2018). The need to rehumanize mathematics. In I. Goffney, R. Gutiérrez, and M. Boston (Eds.) *Annual perspectives in mathematics education: Rehumanizing mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). Reston, VA: National Council of Teachers of Mathematics.

Hecht, J., Higgerson, M. L., Gmelch, W. H., & Tucker, A. (1999). *The Department Chair as Academic Leader.* Phoenix, AZ: Orys Press.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: An analytic review of the literature. *Journal of Research in Science Teaching*, *48*(8), 952-984.

Kezar, A. (2011). What is the best way to achieve broader reach of improved practices in higher education? *Innovative Higher Education*, *36*(4), 235–247. https://doi.org/10.1007/s10755-011-9174-

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis* (2nd ed.). Thousand Oaks, CA: Sage Publications, Inc.

Miller, E. R., Uhing, K., Hass, M., Zigterman, R., Quigley, K., Wakefield, N., & Lai, Y. (2018). Graduate student instructors' growth as teachers: A review of the literature. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.) *Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education*, 185-197. San Diego, CA.

Reinholz, D. L., & Apkarian, N. (2018). Four frames for systemic change in STEM departments. *International Journal of STEM Education*, *5*(1), 1-10.

Reinholz, D. L., Rasmussen, C., & Nardi, E. (2020). Time for (research on) change in mathematics departments. *International Journal of Research in Undergraduate Mathematics Education*, *6*, 147-158.

Selinski, N. E., & Milbourne, H. (2015). The institutional context. In Bressoud, D., Mesa, V., & Rasmussen C. (Eds.), *Insights and recommendations from the maa national study of college calculus* (pp. 31-44). MAA Press. Retrieved from https://www.maa.org/sites/default/files/pdf/cspcc/InsightsandRecommendations.pdf

Smith, W. M., Voigt, M., Ström, A., Webb, D., & Martin W. G. (2021). *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning*. AMS.

Wergin, J. F. (2003) *Departments that work: Building and sustaining cultures of excellence in academic programs*. San Francisco, CA: Anker/Jossey-Bass.

EMOTIONAL KNOWLEDGE OF MATHEMATICS TEACHER

CONOCIMIENTO EMOCIONAL DE DOCENTES DE MATEMÁTICAS

María S. García González
Autonomous University of Guerrero, Guerrero, Mexico
msgarcia@uagro.mx

Emotions come into play as teachers make decisions, act, and reflect on the different purposes, methods, and meanings of math teaching. In order to learn more about the emotions of mathematics teachers, this paper shows the emotions experienced by 81 Mexican teachers who teach mathematics in middle school and high school. Emotions are contextualized by the theory of the cognitive structure of emotions, narrative and drawings were used to express their emotions. According to the results, three frequent and common emotions were identified in the teachers, happy-for, satisfaction, and disappointment. Fear was the only frequent emotion in high school teachers; it occurred at the beginning of their teaching, due to the lack of pedagogical content knowledge.

Keywords: Emotion, Teacher Educators.

Emotions and Mathematics Education

In recent years there has been a growing interest in the study of emotions in education (Graesser, 2020). In particular, the field of mathematics education classifies the emotions of teachers into two groups: negatives and positives. Negative emotions imply unpleasant experiences during the teaching moment; stress, demotivation and burnout syndrome are among these emotions. Burnout syndrome is an emotional disorder caused by job stress which leads to experience anxiety or depression (Schutz & Zembylas, 2009; Rodríguez, Guevara & Viramontes, 2017). Experiencing high levels of intensity of this group of emotions could even lead to abandoning the profession (Hannula, et. al, 2007). The most frequent negative emotion experienced by mathematics teachers is mathematics anxiety (Fennema & Sherman, 1976; Brady & Bowd, 2005; Bekdemir, 2010). On the contrary, positive emotions imply pleasant experiences when leading a class; some of these emotions are enthusiasm, joy, satisfaction, and interest (Di Martino & Sabena, 2011; Anttila, et. al, 2016).

According to the results of the investigations, there are two triggering situations for negative emotions in mathematics teachers: (a) emotional experiences as students: generally, those who experienced negative emotions related to mathematics continue to experience them after they become mathematics teachers, retaining the belief that mathematics is difficult (Coppola, et. al, 2012); and (b) the knowledge of the academic discipline: many of the teachers in charge of teaching mathematics are not specialists in the contents of the school curriculum (Philipp, 2007).

In order to expand and communicate what teachers feel while teaching mathematics, this paper presents 81 cases of mathematics teachers who communicate their emotions through narrative and drawings. This communication was carried out collectively in four different workshops, coordinated by the author; these workshops were held outside the teachers' working hours. The aim of the workshop was to create a place where the mathematics teachers could talk about their emotions during teaching with colleagues from other places and from the same or different school levels. Feeling heard and listening to other teachers, they find similar emotional histories and are allowed to reflect on how emotions affect their teaching.

OCC Theory and Emotional Knowledge

Because language is a functional means to talk about emotions, this work makes use of the Theory of the Cognitive Structure of Emotions (Ortony, Clore & Collins, 1987), also named OCC theory after the initials of the authors, to study emotions. OCC theory analyzes emotions from the narrative of a person about their emotional experience; it ignores completely the behavioral and physiological evidence that are also recognized as sources to investigate emotions. OCC theory defines emotions as “valenced reactions to events, agents or objects, with their particular nature being determined by the way in which the eliciting situations is construed” (Ortony, Clore & Collins, 1987, p.13). This definition implies that an emotion appears when a person values a specific situation; this situation is recognized as a triggering situation, and the appraisal is expressed by an emotion word.

The analysis of emotions in OCC theory is not only focused on the emotion words, although linguistic evidence is taken into account, but also considers the triggering situations. This is because daily language has several words that could be used to refer to different aspects of the same type of emotion. For example, the word distress refers to a moderate fear, while the word panic gives evidence of an intense level of fear, but they definitely refer to same type of emotion, fear. The OCC considers some emotion types (OCC typology of emotions, Table 1), and gives a generic definition for each of them, focusing on the triggering situation and the emotion word. In this way, the analysis of an emotion depends on its interpretation based on the proposed definition.

Table 1. Emotion types according to the OCC theory.

Class	Group	Types (sample name)
Reactions to events	Fortunes-of-others	Pleased about an event desirable for someone else (happy-for) Pleased about an event undesirable for someone else (gloating) Displeased about an event desirable for someone else (resentment) Displeased about an event undesirable for someone else (sorry-for)
	Prospect-based	Pleased about the prospect of a desirable event (hope) Pleased about the confirmation of the prospect of a desirable event (satisfaction) Pleased about the disconfirmation of the prospect of an undesirable event (relief) Displeased about the disconfirmation of the prospect of a desirable event (disappointment) Displeased about the prospect of an undesirable event (fear) Displeased about the confirmation of the prospect of an undesirable event (fears-confirmed)
	Well-being	Pleased about a desirable event (joy) Displeased about an undesirable event

		(distress)
Reactions to agents	Attribution	Approving of one's own praiseworthy action (pride) Approving of someone else's praiseworthy action (appreciation) Disapproving of one's own blameworthy action (self-reproach) Disapproving of someone else's blameworthy action (reproach)
Reactions to objects	Attraction	Liking an appealing object (liking) Disliking an unappealing object (disliking)
Reactions to events and agents simultaneously	Well-being /Attribution	Approving of someone else's praiseworthy action and being pleased about the related desirable event (gratitude) Disapproving of someone else's blameworthy action and being displeased about the related undesirable event (anger) Approving of one's own praiseworthy action and being pleased about the related desirable event (gratification) Disapproving of one's own blameworthy action and being displeased about the related undesirable event (remorse)

Ortony, Clore and Collins (1987).

On the other side, Emotional Knowledge is a construct that refers to the information teachers have about their own emotions while teaching mathematics (García-González, 2020). This emotional knowledge is developed, as the mathematical and didactic knowledge, and the development implies the following abilities:

1. Recognize that we feel, because emotions are part of our human nature.
2. Be able to recognize the emotion we experience in a specific situation.
3. Put a name to the emotion; this is knowing the emotion word that clearly represents what we feel.
4. Recognize the triggering situation for what we feel.
5. Distinguish the negative emotions from the positive ones.
6. Regulate the emotions we experience, being able to act in consequence.
7. Be capable of helping others, like our students, on their self-knowledge emotionally.

In this paper, the focus is on the first five abilities of emotional knowledge.

Methodology

This study involved 81 Mexican teachers who teach in middle school (students from ages 12 to 15), and high school (from ages15 to 18). These teachers attended voluntarily four different virtual workshops carried out from July 2020 to November 2021. Attendance to the workshop was as follows: workshop 1, 30 teachers, workshop 2, 15 teachers, workshop 3, 15 teachers, and workshop 4, 21 teachers. The data collection was carried out through two techniques, drawing and narrative, both of which have been tested for their functionality to express emotions through written and pictorial language (García-González & Martínez-Padrón, 2020).

Teachers are asked to draw, individually, a drawing of an emotional experience while teaching mathematics, but this drawing should not include emotion words. Then, they show it to other teachers who will identify the emotion expressed. After the observers talk about the emotion in the drawing, the teacher that draws should say if it was the actual emotion he or she wanted to communicate. Most of the times the observers match the correct emotion. Finally, the teacher that draws is asked to write the emotion word associated to the drawing, and to describe the situation that triggered the emotion. In other activity, teachers are asked to take considerable time to write their histories as mathematics teachers. Each one of them reads it in an intended session; from this narrative the listeners should comment on a positive or negative emotional experience. Both techniques were carried out in the workshops to develop emotional knowledge.

The drawings were collected in drive files for later analysis. The teachers were asked to authorized the dissemination of their products for research purposes, taking care of anonymity.

According to OCC theory, a type of emotion is identified by 2 specifications: 1) Concise phrases that express all the eliciting conditions of the emotional experiences. The evidence highlights these phrases in bold. 2) Emotion words that express emotional experience. The emotion words are highlighted in italics.

Therefore, emotion words and triggering situations were identified in the drawings and narratives. Then, with the help of the OCC typology, a reinterpretation of the evidence was carried out based on the definitions of the emotions, and the emotion type that better describes the evidence was selected.

The teachers are listed from 1 to 81, and it is highlighted whether they are male (M) or female (F). In the evidence, the code T1-M means male teacher listed with number 1, and T48-F means female teacher listed with number 48.

Results

Figure 1 shows twelve emotions identified along with their frequency (number of teachers having that emotion), the same teacher experienced several types of emotions. No distinction is made between educational levels because fear was the only emotion experienced just by the high school teachers and not by the middle teachers. Table 2 shows the triggering situations of these twelve emotions.

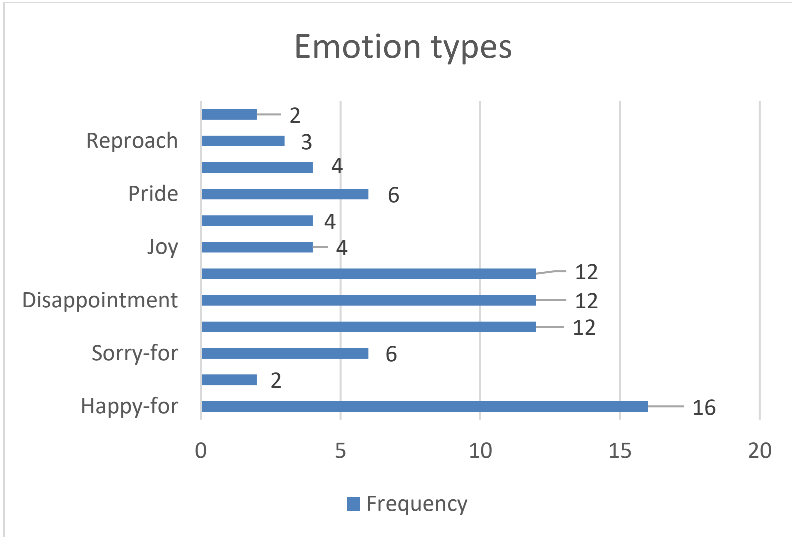


Figure 1. Teachers' emotions.

Table 2. Triggering situations of the emotions.

Emotion type	Triggering situation
Happy-for	Students' learning, Scores/text, concept understanding.
Resentment	Problem solving.
Sorry-For	Students' indifference to the understanding of concepts, internet access, students' problem-solving
Satisfaction	Students' problem-solving, participation/paying attention in class, appropriate teaching resources.
Disappointment	When students' learning is not achieved, the lack of students' paying attention/participation online, low students' scores, excess of work in virtual mode.
Fear	Lack of pedagogical content knowledge.
Joy	The management of the classroom, the attention of students.
Distress	The resolution of the test by the student.
Pride	Acknowledgement of the work of the teacher, students attending to recommendations, achievement of personal academic goals.
Self-reproach	The lack of pedagogical content knowledge, students' lack of understanding.
Reproach	Distraction in virtual class, dissatisfaction with the score, the unwillingness of students.
Liking	Teach using technology, the application of mathematics.

Below it is focused the presentation of results on the two most frequent emotion types, satisfaction and fear.

Satisfaction

All teachers have expectations of their students. Two expectations were identified in the participants, solving problems and participation or attention in class. When these expectations become real, teachers experience the emotion of satisfaction (pleased about the confirmation of the prospect of a desirable event). In addition to their expectations of students, also were identified expectations for themselves, such as those associated with online teaching. With respect to students' problem-solving the drawing of T48-F expresses her happiness (pleased) because the student solved the problem; it can be noted the smile that is drawn on her face.

T48-F: *I feel happy* when the **student managed to solve the problem** (Figure 2).

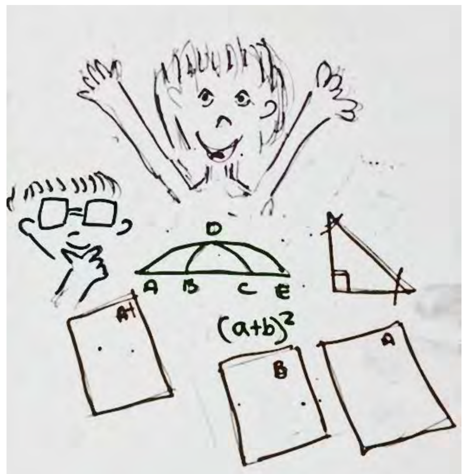


Figure 2. Satisfaction emotion-problem solving. T48-F.

Fear

Fear is defined as being "displeased about the prospect of an undesirable event". This emotion was only identified in high school teachers, and it was a consequence of the lack of Pedagogical Content Knowledge (PCK, Shulman, 1987) at the beginning of their teaching practices. The teachers were surprised when they recognized that teaching mathematics required more than just knowing mathematics (T15-M).

> T15-M: I felt a bit of *fear* when I knew that **I was in charge of a large number of students (50),** *I was uncertain* if **I would be able to control them,** … *I was attacked by the fear of knowing* if **my ideas and my way of teaching could be correctly understood and assimilated by my students,** … I thought I had a huge advantage: I knew mathematics, but my didactic was weak.

Discussion and Conclusions

Figure 1 shows the range of emotions that a teacher can experience when teaching mathematics, and is made up of positive and negative emotions. The case of the emotions of reproach and self-reproach is striking, where the agents that trigger them are the teacher himself, in the case of self-reproach, due to low PCK, and the student is reproached for his distraction in class, and his unwillingness to learn. Recognizing what we feel, putting a name to the emotion, and knowing the situation that triggers it, account for our emotional knowledge.

Previous studies have documented mathematical knowledge as a triggering situation of negative emotions in a Mexican middle teacher (Diego's history García-González & Martínez-

Sierra, 2020) due to his teaching training. In this case, were identified PCK as the triggering situation of fear in high school teachers; we can also explain it from teacher formation. In Mexico, most of the high school teachers have a strong mathematical training because they study for a mathematical career or a related college degree as physics or actuaries. However, PCK is not always part of the curriculum for these degrees.

Fear was triggered at the beginning of their teaching practices by their poor PCK. This group of teachers recognized their extensive mathematical knowledge but they felt powerless regarding to the teaching part. So, they had to look at how to change this triggering situation by means of courses and their own practice, which gave teachers more confidence, as T15-M said.

Having emotional knowledge becomes as important as having mathematical knowledge and PCK, because emotions guide the actions of the teachers. For example, teachers who are afraid because of their lack of PCK may, in the worst case, abandon their teaching labor. On the other hand, the results make us reflect on the need to develop emotional knowledge in their training, as the mathematical and didactic knowledge is developed. In this sense, Marbán, Palacios & Maroto (2020) underline the need to establish specific mathematics affective intervention programs and incorporate them into the didactic training process of pre-service teachers.

The previous excerpts confirm that teaching mathematics requires not only mathematical knowledge but specialized knowledge to teach it, in the words of Carrillo-Yañez, et. al (2018). This matches with Pekrun (2021) who points out that teachers need more than knowledge in the discipline to teach, referring specifically to the emotion and motivation of the teacher. With regard to this result, it is relevant to mention that the lack of mathematical knowledge triggers negative emotions in primary educators because many of the teachers in charge of teaching mathematics are not specialists in the contents of the school curriculum (Philipp, 2007). The data on this study point out that the triggering situation of negative emotions in high school is the PCK and not the mathematical knowledge; as mentioned earlier, these cases can be associated with the formation of both groups of teachers.

References

Anttila, H., Pyhältö, K., Soini, T. & Pietarinen, P. (2016). How does It feel to become a teacher? Emotions in teacher education". *Social Psychology of Education*, 19(3), 451-47

Bekdemir, M. (2010). The pre-service teachers' mathematics anxiety related to depth of negative experiences in mathematics classroom while they were students. *Educational Studies in Mathematics*, 75(3), 311–328. http://doi.org/10.1007/s10649-010-9260-7

Brady, P., & Bowd, A. (2005). Mathematics anxiety, prior experience and confidence to teach mathematics among pre-service education students. *Teachers and Teaching: Theory and Practice*, 11(1), 37–46. http://doi.org/10.1080/1354060042000337084

Carrillo-Yañez, J., Climent, N., Montes, M., Contreras, L., Flores-Medrano, E., Escudero-Ávila, D., Vasco, D.,Rojas, N., Flores, P., Aguilar-González, P., Ribeiro, M. y Muñoz-Catalán, C. (2018) The mathematics teacher's specialised knowledge (MTSK) model. *Research in Mathematics Education*, 20 (3), 236-253, DOI:10.1080/14794802.2018.1479981

Coppola, C., Martino, P. Di, Pacelli, T., y Sabena, C. (2012). Primary teachers' affect: a crucial variable in the teaching of mathematics. *Nordic Studies in Mathematics Education*, 17(3–4), 107–123.

Di Martino, P. & Sabena, C. (2011). Elementary pre-service teachers' emotions: shadows from the past to the future. En: K. Kislenko (ed.). *Current state of research on mathematical beliefs XVI* (pp. 89-105), Tallinn: University.

Fennema, E., & Sherman, J. A. (1976). Fennema-Sherman mathematics attitude scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *JSAS: Catalog of Selected Documents in Psychology*, 6(2), 31.

García-González, M. S. (2020). Desarrollo del conocimiento emocional en matemáticas. *Uno. Revista de didáctica de las matemáticas*, 88(1), 17-23.

García-González, M. S. y Martínez-Sierra, G. (2020). The history of a teacher's relief of his mathematics anxiety: The case of Diego. *Educational Studies in Mathematics*, 103(2020), 273-291. DOI: 10.1007/s10649-020-09941-8.

García-González, M.S. & Martínez-Padrón, O. J. (2020). Conocimiento emocional de profesores de matemáticas, *Revista Educación Matemática*, 32, 1. DOI: 10.24844/EM3201.07.

Graesser, C. (2019). Emotions are the experiential glue of learning environments in the 21st century. *Learning and instruction*, 1-5. https://doi.org/10.1016/j.learninstruc.2019.05.009

Hannula, M. S., Liljedahl, P., Kaasila, R., & Rösken, B. (2007). Researching relief of mathematics anxiety among pre-service elementary school teachers. In J.-H. Woo, H.-C. Lew, K.-S. P. Park, & D.-Y. Seo (Eds.), *Proceedings of 31st Annual Conference for the Psychology of Mathematics Education, vol. 1* (pp. 153–156). Seoul, Korea.

Marbán, J. M., Palacios, A., & Maroto, A. (2020). Desarrollo del domino afectivo matemático en la formación inicial de maestros de primaria. *Avances de investigación en educación matemática*, (18), 73–86. https://doi.org/10.35763/aiem.v0i18.286

Ortony, A., Clore, G. L., & Collins, A. (1987). *The cognitive structure of emotions*. Cambridge, UK: Cambridge University Press.

Pekrun, R. (2021) Teachers need more than knowledge: Why motivation, emotion, and self-regulation are indispensable. *Educational Psychologist*, 56(4), 312-322. DOI: 10.1080/00461520.2021.1991356

Philipp, R. A. (2007). Mathematics teachers' beliefs and affect. In F. Lester (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 257–315). Charlotte, NC: Information Age Publishing.

Rodríguez, J., Guevara, A. y Viramontes, A. (2017). Síndrome de burnout en docentes. *IE Revista de Investigación Educativa de la Rediech*, 7(14), 45-67.

Schutz, P., & Zembylas, M. (2009). Introduction to advances in teacher emotion research: The impact on teachers lives. En P. Schutz, M. Zembylas (Eds.), *Advances in teacher emotion research*. New York: Springer

Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57, 1–22.

EXPLORING THE IMPACT OF DISCUSSION-LEADING PROFESSIONAL DEVELOPMENT ON TEACHERS' PRACTICE

Nicole Garcia
University of Michigan
nmgarcia@umich.edu

Meghan Shaughnessy
Boston University
mshaugh@bu.edu

Merrie Blunk
University of Michigan
mblunk@umich.edu

Jillian P. Mortimer
Albion College
jbpet@umich.edu

D'Anna Pynes
University of Notre Dame
kpynes@nd.edu

Darrius Robinson
University of Michigan
darriusr@umich.edu

This paper examines the impact of an intensive professional development on practicing teachers' mathematics discussion-leading practice. A tool for examining specific discussion-leading moves was used to measure change in teachers' practice as observed through submitted video recordings of mathematics discussions. Participants included 33 teachers from three school districts who submitted a total of 193 videos across the study. The findings reveal that the professional development had differential impact on participating groups. We explore group attributes that could contribute to these differential outcomes.

Keywords: Professional development, classroom discourse

Improving the teaching of mathematics remains a national priority and is critical for improving the learning opportunities for mathematics students. Reforms designed to improve teaching and learning have often centered on curriculum and standards, though studies demonstrate that "no in-school intervention has a greater impact on student learning than an effective teacher" (p.1, NCATE, 2010), indicating that closer attention to teaching quality is essential. This is no small task, given the scale of the teaching force. There are simply not enough teachers with the knowledge and teaching skills needed to reach all students (NCTQ, 2008; Ball & Forzani, 2009). This challenge requires robust professional development that can contribute to improving the teaching and learning of mathematics.

We argue that teachers profit from well-designed opportunities to develop new visions for practice, learn more about students' thinking, or work on specific mathematical topics. However, such opportunities are often insufficient to support teachers with the complexity of classroom teaching. These kinds of professional opportunities focus on critical resources for instruction, but not on the details of practice itself. The professional development (PD) that is the focus of this research was an effort to address the challenge of supporting the learning of practice, specifically mathematics discussion-leading practice. We situated the PD in a "common text" for working on practice, where participants engage in a form of "legitimate peripheral participation," followed by structured opportunities to enhance their capabilities with discussion-leading practices.

Theoretical Framework

Our work is grounded in three bodies of work – conceptions of teaching and learning, teacher learning, and discussion-leading practice. We conceive of quality teaching as critical in advancing mathematics education. Mathematics teaching is something that people do; it is not merely something to know. Teachers must use knowledge flexibly and fluently as they interact in specific contexts with students, with the aim of helping those students become proficient with mathematics. Conceptualizing the work of teaching as interactions among teachers, students, and content, in environments (Cohen et al., 2003) has important implications for the design of PD. It means that simply knowing a lot of math is not enough to be able to help students learn mathematics. Teachers must understand mathematics content and practices in specialized ways that are attuned to their learners. Simultaneously, teachers must be able to engage in teaching practices that enable them to interact with subject matter and students in multiple organizational formats, and support students' interactions with the mathematics being studied. Furthermore, the complex nature of mathematics teaching and the ever-changing nature of the environments in which teaching happens, also require a commitment to, and means of, professional learning in, from, and for practice. Thus, teaching requires the integrated use of specific knowledge and skills in particular contexts of instruction. This emphasis on (a) knowledge use, (b) attention and responsiveness to learners, and (c) context forms the basis of our approach.

Decades of research have demonstrated that many approaches to PD do not adequately support increases in teachers' capabilities (e.g., Coalition for Evidence-Based Policy, 2013; Cohen & Hill, 2001; Garet et al., 2001; Jacob et al., 2017). Often, teachers participate in a patchwork of sessions where presenters share ideas and raise enthusiasm without sufficient focus on teachers' opportunities to develop mathematical knowledge (Little, 2001; Wilson & Berne, 1999), adequate engagement in practice (Darling-Hammond, 1998), or connection with practice (Ball & Cohen, 1999; McLaughlin, 1990; Olson & Craig, 2001). Typically, U.S. teachers' PD opportunities are neither sufficiently "curricular" nor effective for improving practice. In contrast, the structure of our PD reflects the five elements of effective PD identified by Desimone and Garet (2015). First, our PD had a "content focus," --the practice of leading discussions in elementary mathematics, focusing on number and operation, which spans elementary grades. Second, the PD was designed to be "active" with opportunities to engage in leading discussions. Third, the PD was "coherent" in focus, centered on building skill with leading mathematics discussions. Fourth, the 36 hours of PD is aligned with the "sustained duration" associated with positive outcomes for teachers and students (Lynch et al., 2019; Yoon et al., 2007). Finally, "collective participation" was embedded across the PD.

The PD was designed to support the learning of teaching practice by situating learning inside of elementary mathematics classroom instruction. The PD used a classroom as a "common text" for working on practice, where participants were not only watching and discussing, but were engaged in developing and learning practice. This common text creates opportunities for participants to 1) examine lesson plans and student tasks, 2) discuss the plans and tasks with the teacher of record, making adjustments together, 3) observe the enactment of the discussed lesson, including deviations from and adjustments to the planned lesson, 4) collaboratively review students' daily work in response to the teaching, and 5) debrief the teaching of the lesson, unpacking together adjustments and deviations from the plan, the mathematics that unfolded during the lesson, and moves the teacher made to engage students in mathematics discussion (see Shaughnessy et al., 2017).

In addition to engaging in this collective legitimate peripheral participation, the PD involved a focused work designed to improve skill with leading mathematics discussions. Our definition of mathematics discussions entails a period of sustained dialogue among students and the teacher where students respond to and use one another's ideas to develop collective understanding. This requires teachers to engage in areas of work, including launching and concluding the discussion, recording and representing content, and orchestrating the discussion by eliciting and probing student thinking, orienting students to one another's thinking, and making contributions. For a detailed description of this work, see Shaughnessy et al. (2021). Our study investigates the impact of the described PD on teachers' enacted discussion-leading practice.

Methods for Studying Professional Development Impact on Practice

Participants

The full study spanned three iterations of the professional learning involving variations of the setting of the PD (on site or remote) and the components of the PD (peripheral participation with and without the focused discussion-leading session). This paper focuses on three participant groups from iterations 1 and 2 and reports on the impact of the full PD (both peripheral participation and the focused discussion-leading session) on their discussion-leading practice.

Group H1 consisted of 12 teachers from one school district in the midwestern United States, 10 of whom had completed more than three years of mathematics teaching. This group attended on site. Two researchers co-facilitated the practice-based discussion session.

Group H2 consisted of 10 teachers from one school district in the midwestern United States (different from the district in Group H1), seven of whom had completed more than three years of mathematics teaching. This group attended on site and one of the facilitators from Group H1 led the practice-based discussion session.

Group A2 consisted of 11 teachers from a large urban district in the northeastern United States, nine of whom had completed more than three years of mathematics teaching. This group gathered in one location with the other Group H1 facilitator to live-stream and engaged via a chat function in the peripheral participation portion before engaging in the practice-based discussion session with the facilitator.

Data Sources

The PD goals were multi-faceted, taking on many aspects critical to the teaching of mathematics. These goals included teaching practice, mathematical knowledge for teaching, language for talking about teaching, and skill in close observation of teaching and students. However, the primary goal was to increase participants' skills with leading mathematics discussions. The additional foci of enhancing language for talking about teaching, building skill with close observation, and building mathematical content knowledge were in service of this primary goal of improved discussion-leading practice.

This focus on improving mathematics discussion-leading practice necessitated the collection of participants' mathematics discussion-leading practice before and after the intervention. Six videos were collected from each of the 33 participants in the months immediately preceding and immediately following the PD, three pre-PD (in the last two months of school prior to the PD) and three post-PD (in the first two months of school). The set of three videos consisted of two lessons identified by participants as examples of mathematics discussions and one lesson provided by the research team. The selection of mathematical tasks used in a discussion and the content focus impacts the moves that teachers can make when leading discussions as well as the overall trajectory of the discussion, so the third video record of a mathematics lesson used a provided task accompanied by a detailed lesson plan. The video recording of the provided lesson also allowed for a more direct comparison across participants. The topic, task, and lesson structure were constant across participants and across time periods (pre and post). A small number of videos were not included in the analysis. Four videos from Group H1 and one video from A2 were not included in the analysis due to the format of the lesson (lecture only, completing practice problems, or small group instruction only).

Data was also collected around the supporting foci, both to measure incoming skill in these areas and to measure potential changes resulting from the intervention. One such measure was utilized for examining changes in teachers' mathematical knowledge for teaching. The Learning Mathematics for Teaching survey (LMT, 2008), was selected as a pre- and post-intervention measure of teachers' mathematical knowledge for teaching. The PD itself had multiple content strand foci as the PD encompassed both peripheral participation in a grade 5 mathematics program focused on fractional reasoning, operations with integers, and mathematical argumentation as well as a practice-based discussion session focused on number and operations. The trajectory of the classroom content is naturally dependent on students' progression and uptake of the content alongside the teachers' decisions about the trajectory of the work. For that reason, an LMT form focused on the Number and Operations was used. The LMT was administered in the same time frame as the video collection.

Selection of Tools for Data Analysis

Many measures exist for evaluating student outcomes of mathematics lessons, but fewer measures exist for measuring mathematics teaching practice, specifically mathematics discussion-leading practice. To determine changes in both the overall quality of the mathematics and in the specific moves that teachers were using in their discussion-leading practice, tools that are oriented on these qualities were needed.

Mathematical Quality of Instruction. The Mathematical Quality of Instruction instrument was selected to measure the overall mathematical quality of the lessons (Hill et al., 2008). This tool is not aligned with any particular orientation to the teaching of mathematics (e.g., reform-oriented instruction) or to a particular lesson structure, but rather to the depth and quality of the mathematics content available to students during instruction allowing us to discern, for each lesson, the richness of the mathematics, the precision and accuracy of the mathematics instruction, students' involvement with content through Common Core-aligned mathematical practices, and the teacher's ability to interpret and respond to students' mathematical ideas.

Discussion Leading Checklist. A complimentary tool was designed to examine teachers' specific discussion-leading moves. In prior work, a discussion-focused checklist tool was developed to formatively assess beginning teachers' practice. Using the tool entailed recording the presence or absence of particular discussion-leading moves to support the noticing of the work done by teachers in a specific discussion. A previous study showed that the tool was able to reveal variations in practice across teachers and provide fine-grained detail about the skill of teachers in leading mathematics discussion while also accounting for existing classroom norms (Shaughnessy et al., 2021). Because our PD involved practicing teachers, we expanded the list of possible moves in the checklist to more accurately capture the range of moves made by experienced teachers. These additions included advanced areas of work, such as supporting students to make connections and extending and revising student ideas. The checklist tool was designed to capture the presence or absence of particular moves. The tool was not designed to judge the skill with which the move was enacted or students' responses to the move. This is in tension with our desire to capture the quality of teaching. However, the decision was made to account for teachers' attempts to enact different discussion leading moves as we are considering the learning of teachers. Our main goal is to determine whether teachers are trying out new work to improve their practice. We know that trying out new work often results in less than perfect outcomes, particularly in early implementation. We did, however, want to capture problematic implementation to determine whether problematic enactment increased, decreased, or stayed the same before and after the intervention. To that end, the checklist tool also included a section for "issues" such as consistently ineffective probes, as well as an overarching rating indicating whether the instruction in the video was aligned with our definition of a discussion. This overarching rating is meant to serve as an acknowledgement that discussion-leading is more than the sum of the parts that can be captured in a checklist tool.

The tool necessarily focuses on specific observable moves that teachers make while leading a discussion, categorized both by the phase of the work (e.g., task set-up, discussion launch, orchestration, conclusion) and by the buckets of work that teachers must do (e.g., eliciting and probing student thinking, orienting students to one another's thinking, supporting students to make connections). This organization allowed for observations about the density of work in a particular area (e.g., the number of different moves a teacher used to orient students to one another's thinking) and the spread of work across a discussion (e.g., whether a teacher was doing work in all possible areas or if the work was concentrated in one or two areas). Each observable move was marked as present, not present, or not applicable (NA). The code of NA was used judiciously in cases where the move was unnecessary. For example, NA could be used if the task selected precluded a range of responses or methods to be shared or if the teacher did not need to further probe student contributions due to the level of detail of the contribution. Many moves had possible available codes of "once," "more than once," and "student initiated" to support the ability to analyze the frequency and the extent to which established norms seemed to be influencing the work of the teacher. For our analysis, these three codes were collapsed to present.

Data Analysis Methods

Mathematical Quality of Instruction. The 193 videos in this study were coded using the discussion checklist tool and the MQI. MQI coders attended MQI training and were certified as raters. Videos were double-coded for both instruments using associated coded books. Coders met to reconcile discrepancies for each video prior to entering final codes and resolved differences via discussion and referencing the codebooks. MQI coding was applied to equal-length chapters of the full video (approximately 7.5-minute segments), with the number of chapters determined by the length of the video. Each chapter was scored on five domains. The full video was scored on nine factors and assigned one overall score for the lesson. Chapter scores were used to create aggregate video scores for the chapter-specific points. In addition, aggregated pre- and post-intervention MQI scores were created for each participant using mean scores.

Discussion Checklist. The discussion checklist coding was applied to videos. Composite scores were created for each area of work represented in the discussion checklist tool. For those areas where work was consistently possible, sum scores or scaled mean composite scores were used to represent the amount of possible work that occurred relative to the available work. For example, an Orienting composite variable was created using the sum of the "present" moves. There were five possible moves - posing questions to students about others' ideas, asking students to restate another's idea, responding to another student's idea, adding to an idea, and interpreting the strategies of others – that could be observed. For eliciting, a scaled mean score was used to represent that there were four possible moves - eliciting multiple ideas, eliciting a range of responses, engaging several students in sharing ideas, and eliciting students' mathematical processes. A scaled mean was used to enable comparison between teachers who had all four moves available and those who only had three moves available due to task selection (e.g., a range of student responses is unlikely given the task). Composite variables included task setup, discussion launch, eliciting, probing, orienting, generic orienting, connecting/ extending/ revising, concluding, and issues. A detailed description of the checklist items associated with each composite score can be found in Table 1.

Table 1: Composite Variable Descriptions

Variable	*Description*
Task Set Up	Composite score takes into account the work the teacher does to help students make sense of the task (including reading, restating, and unpacking context), whether the teacher maintains cognitive demand of the task, and whether the students are sufficiently prepared to begin the mathematical work.
Launch	Sum score takes into account whether the teacher had the attention of all students before launching the discussion, is concise and on-task during the launch, and explicitly states the goal of the discussion.
Eliciting	Scaled mean score takes into account the work of eliciting multiple ideas, eliciting a range of responses, engages several students, asks about processes. Because not all tasks allow for a range of responses, a conditional mean was calculated based on whether a range was possible, then the mean was scaled.
Probing	Score takes into account the work of posing questions to get students to explain their understanding of relevant mathematical content or processes, or follow-up questions focused on why a student did particular work.
Orienting	Sum score included posing questions to students about others' ideas, asking to restate another's idea, responding to another's idea, adding to an idea, and interpreting the strategies of others.
Generic Orienting	Sum score accounted for generic orienting moves such as encourages the class to attend/listen/respond, uses turn-and-talk to encourage discourse, elicits student to student discourse, and uses moves that require all to respond to others' work.
Revising	Score focuses on moves to supports students to revise their work.
Connecting/ Extending/ Revising	Composite mean takes into account moves including asking students to identify similarities/differences, supporting students to connect to past work, supporting students to connect to the problem context, supporting students to connect between representations, examining efficiency, generating hypothetical situations to extend thinking, and supporting students in revising their thinking.
Concluding	Sum score accounting for the teacher making a closing statement, supporting students in remembering a key idea, and taking stock of the discussion.
Issues	Sum score accounting for problematic moves such as inaccurate revoicing, mathematically problematic contributions, over-scaffolding, consistently using ineffective probes.

Aggregated pre- and post-intervention discussion checklist scores were created for each participant using mean scores. Kruskal-Wallis tests and independent samples tests were performed on pre-data to determine whether differences between participants by study group existed. Significantly different distributions were found for study groups, indicating that separate analysis of these groups is necessary. For each group of teachers, independent samples tests were examined for the discussion checklist score pre and post PD. These tests included Levene's test for equality of variances and *t*-tests for equality of means. Paired samples *t*-tests for aggregated case data were examined by study.

Findings

Examining the data disaggregated by study group, we noticed differences in outcomes of three groups who received the full PD. Next, we unpack the features of these groups, their experiences, and their outcomes to investigate factors that influence participant outcomes.

Group H1. The first group of participants received PD in the first iteration of the study. The content of the peripheral participation experience focused squarely on setting up tasks for productive work, eliciting and probing student thinking, and supporting students to engage with others' ideas, the foundation for orienting students to one another's thinking. The participants observed minimal full-class mathematics discussion-leading work. The student group was not accustomed to engaging in discussions and the teacher had extensive groundwork to do to move toward productive discussions. The focused discussion-leading session supported participants in building their knowledge and skill with launching discussions, orienting students to one another's thinking, representing and recording content, and concluding discussions.

This group entered the professional development with a group LMT score of approximately 0.51 standard deviations above the mean. Overall, few discussions were led prior to the PD, with approximately 20% of the submitted pre-videos rated as aligned to our definition of discussion. Despite a low number of discussions being led, the mean score of the submitted lessons (M = 3.222, SD = 0.43) on the MQI Whole Lesson Mathematical Quality of Instruction was slightly above the middle rating of 3.0 on a 1-5 scale.

Following the PD, this group showed a nearly significant (p = 0.053) increase on paired t-tests in the number of discussions led, with approximately 36% of submitted post-videos demonstrating a discussion. This group also showed significant increases in task set-up (p = 0.012), eliciting (p = 0.009), and concluding (p = 0.016). Nearly significant increases were shown in probing (p = 0.077), generic orienting (p = 0.084), and revising (p = 0.056).

Group H2. The second group of participants received PD in the second iteration of the study. The content of the peripheral participation focused on the full spectrum of discussion-leading practice, beginning with work on setting up tasks for discussion and eliciting student thinking, followed by orienting students to the thinking of others, with the sophistication of orienting moves increasing across the experience. The content of the focused discussion-leading session was identical to the first iteration.

This group entered the professional development with a group LMT score of approximately 0.27 standard deviations below the mean. Overall, few discussions were led prior to the PD, with approximately 15% of the submitted pre-videos rated as aligned to our definition of discussion, a slightly lower percentage than Group H1. The mean score of the submitted lessons (M = 2.7407, SD = 0.49) on the MQI Whole Lesson Mathematical Quality of Instruction was slightly below the middle rating of 3.0.

Following the PD, the group showed no significant difference in the number of discussions (p = 0.347) or any category captured by the checklist tool. Near significant increases were seen in generic orienting (p = 0.069).

Group A2. The third group of participants received PD in the second iteration of the study. The peripheral participation was completed via live-stream and the focused discussion-leading session was delivered by the facilitator who co-facilitated the session in the first iteration of the study. The content of the peripheral participation was identical to that of Group H2.

This group entered the PD with a group LMT score of approximately 0.65 standard deviations above the mean. A high number of discussions were led prior to the PD, with approximately 89% of the submitted pre-videos rated as aligned to our definition of a discussion.

Additionally, the mean score of the submitted lessons (M = 3.8611, SD = 0.54) on the MQI Whole Lesson Mathematical Quality of Instruction was well above the middle rating of 3.0.

Following the PD, the group showed no significant difference in the number of discussions or categories captured by the tool. Nearly significant increases were seen in connecting/extending (p = 0.055) and revising (p = 0.080).

Discussion and Implications

There were clear differences in the outcomes of the three study groups, with study group H1 showing the most gains from the PD. We hypothesize three potential reasons for these differences. The first potential hypothesis focuses on the content of the peripheral participation. Group H1 experienced peripheral participation where the discussion-leading work of the teacher was slower in pace, more repetitive, more deliberate, and more challenging than that of the peripheral participation of Group H2 and A2. It is possible that the pacing and repetition made this work more accessible to participants than the smooth, quickly progressing work of iteration 2, thereby allowing participants to consider the specific moves the teacher was making and how they might be incorporated into their own classrooms. They may also have identified directly with the challenges of this work and were ready for the same challenges in their own classrooms.

A second possible hypothesis for the differences is related to incoming knowledge and skills. Groups H1 and A2 both entered the PD with above average mathematical knowledge for teaching and above average mathematical quality of instruction as measured by the whole lesson MQI score. Group H1, however, was not engaged in consistent discussion-leading, unlike Group A2. One explanation for the changes seen in Group H1 is that they had the mathematical knowledge to lead discussions and strong general mathematical instructional skill, but did not yet have discussion-specific skills, but improved in this area following the PD. In contrast, Group A2 brought mathematical knowledge, general mathematical instructional skill, and high discussion-leading skill and so they had less room for improvement. Group H2, on the other hand, entered with low mathematical knowledge, below average mathematical quality of instruction as measured by the whole lesson MQI score, and the lowest level of discussion leading practice of the three groups. It is possible that low mathematical knowledge combined with below average mathematical quality of instruction meant that these teachers were not yet ready for the pace of the PD or to focus on discussion-leading practice.

One final hypothesis is related to contextual factors, which were not measured by the instruments described in this paper. The teachers in Groups H1 and A2 were in consistent school environments where their grade level assignments were stable and their administrators appeared to support their involvement in the PD. Participants in Group H2, on the other hand, discovered during the PD that they would all be changing grade-level assignments due to administrators' decisions based on student outcome data. Learning about this change likely distracted their focus and a grade change can be challenging for implementing new techniques.

Further analysis is needed to determine which of these factors most impacted the PD outcomes. However, the current results do point to the importance of teachers' incoming knowledge and skills to the outcomes of mathematics professional learning.

Acknowledgements

This work was supported by the National Science Foundation under Grant No. 1621104. Any opinions, findings, and recommendations expressed are those of the authors and do not reflect the views of the National Science Foundation.

References

Ball, D. & Cohen, D. (1999). Developing practice, developing practitioners. In L. Darling-Hammond and G. Sykes (Eds), *Teaching as the learning profession* (pp. 3–22). Jossey-Bass.

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. *Journal of Teacher Education, 60*(5), 497–511.

Coalition for Evidence-Based Policy. (2013). Randomized controlled trials commissioned by the Institute of Education Sciences since 2002: How many found positive versus weak or no effects. Retrieved from http://coalition4evidence.org/wp-content/uploads/2013/06/IESCommissioned-RCTs-positive-vs-weak-or-null-findings-7-2013.pdf.

Cohen, D. K., & Hill, H. C. (2001). *Learning Policy: When State Education Reform Works*. Yale University Press.

Cohen, D. K., Raudenbusch, S., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis*, *25*(2), 119–142.

Darling-Hammond, L. (1998). Teacher learning that supports student learning. *Educational Leadership, 55*(5), 6–11.

Desimone, L. M., & Garet, M. S. (2015). Best practices in teacher's professional development in the United States. *Psychology, Society, & Education*, 7(3), 252-253. http://repositorio.ual.es/bitstream/handle/10835/3930/Desimone%20En%20ingles.pdf?s

Garet, M. S., Porter, A. C., Desimone, L., Birman, B., & Yoon, K. S. (2001). What make professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915–945.

Hill, H.C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G.C., Sleep, L., & Ball, D.L. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, *26*, 430–511.

Jacob, R., Hill, H., & Corey, D. (2017). The impact of a professional development program on teachers' mathematical knowledge for teaching, instruction, and student achievement. *Journal of Research on Educational Effectiveness, 10*(2), 379–407.

Learning Mathematics for Teaching. *LMT measures of Mathematical Knowledge for Teaching: Elementary Number Concepts and Operations*, 2008. Ann Arbor, MI: Authors.

Little, J. W. (2001). Professional development in pursuit of school reform. *Teachers caught in the action: Professional development that matters*, *3*, 23–44.

Lynch, K., Hill, H.C., Gonzalez, K.E., & Pollard, C. (2019). Strengthening the Research Base That Informs STEM Instructional Improvement Efforts: A Meta-Analysis. *Educational Evaluation and Policy Analysis, 41*(3), 260–293

McLaughlin, M. (1990). Staff development and school change. In A. Lieberman (Ed.), *Schools as collaborative cultures* (pp. 213–232). Teachers College Press**.**

National Council for Accreditation of Teacher Education (NCATE). (2010). *Transforming Teacher Education through Clinical Practice: A National Strategy to Prepare Effective Teachers.* Author.

National Commission on Teacher Quality (2008). *No common denominator: The preparation of elementary teachers in mathematics by America's education schools*. Author.

Olson, M., & Craig, C. (2001). Opportunities and challenges in the development of teachers' knowledge: The development of narrative authority through knowledge communities. *Teaching and Teacher Education, 17*, 667–684.

Shaughnessy, M., Ball, D. L., & Garcia, N. (2017). A laboratory approach to the professional development of elementary mathematics specialists. In M. B. McGatha & N. R. Rigelman (Eds.), *Elementary mathematics specialists: Developing, refining, and examining programs that support mathematics teaching and learning* (pp. 123–131). IAP.

Shaughnessy, M., Garcia, N. M., O'Neill, M. K., Selling, S. K., Willis, A. T., Wilkes II, C. E., Salazar, S. B., & Ball, D. L. (2021). Formatively assessing prospective teachers' skills in leading mathematics discussions. *Educational Studies in Mathematics*, *108,* 451–472.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. *Review of research in education*, 173–209.

Yoon, K. S., Duncan, T., Lee, S. W.-Y., Scarloss, B., & Shapley, K. (2007). Reviewing the evidence on how teacher professional development affects student achievement (Issues & Answers Report, REL 2007-No. 033). Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southwest.

RESPONSIVE RESEARCH AND PROFESSIONAL LEARNING: COMING TO KNOW ADULTS AND CHILDREN IN A NEW PARTNERSHIP

Lynsey Gibbons
University of Delaware
lgibbons@udel.edu

Anne Wilhelm
Southern Methodist University
awilhelm@mail.smu.edu

Latrice Marianno
University of Delaware
latricem@udel.edu

Robyn Pinilla
Southern Methodist University
rpinilla@mail.smu.edu

The analysis reported here examined the initial relationship-building activities a research team engaged in with educators and children at a local elementary school, as they began to develop a partnership. In our initial interactions and activities with educators, we had three goals: 1) to develop relational trust, 2) to get to know the institutional settings in which teachers worked, and 3) to design responsive professional learning alongside teachers and leaders. In this brief report, we focus on the second goal and examine the following question: How do you come to know the adults and children in a school in order to design professional learning? We discuss our findings in relation to understanding school leaders' vision for teaching and teacher learning; teacher support structures and accountability; and teachers' experience, dispositions, and practices. Implications for improving learning conditions for each and every learner are discussed.

Keywords: professional development, elementary school education

In this brief research report, we discuss findings from an analysis that examined the initial relationship-building activities a research team engaged in with educators and children at a local elementary school, as the practitioners and researchers began to develop a partnership. The school reached out to the university for assistance in supporting teachers' instructional practices in mathematics. Simultaneously, the researchers were interested in finding a school partnership where they could study how teachers learn to facilitate classroom discussions. The larger goal is to develop a long-term collaboration with mutual aims to engage in school-wide learning and transform educational outcomes for students. Thus, we had three goals in our initial activities with teachers and school leaders: 1) to develop relational trust (Bryk & Schneider, 2002), 2) to get to know the institutional settings in which teachers worked, and 3) to design responsive professional learning alongside teachers and school leaders. In this brief report, we focus on the second goal and examine the following question: How did we come to know the adults and children in a school in order to design professional learning in partnership?

Perspectives

Research-practice partnerships (RPPs) are defined as "long-term collaborations aimed at educational improvement or equitable transformation through engagement with research." (Farrell et al., 2021, p.5). Scholars have identified five principles of RPPs: 1) RPPs are long-term collaborations, 2) RPPs work toward educational improvement or equitable transformation, 3) engagement with research is a leading activity in RPPs, 4) RPPs intentionally organized to bring together a diversity of expertise, 5) partners in RPPs employ strategies to shift power relations in research endeavors to ensure all participants have a say (Farrell et al., 2021).

Aligned with each of these principles in different ways, our goal was to get to know an elementary school and build relationships to support the development of a research-practice partnership. This is important for three reasons. First, by learning about the school and working to build relationships, we aimed to build trust and encourage the partnership to become a long-term partnership that is mutually beneficial to ourselves and our school partners (Henrick et al., 2017). Second, becoming familiar with the students and teachers supported our endeavors to design professional learning aimed at more equitable instructional practices. Third, developing relationships with teachers and school leaders helped exemplify our desire to center and honor the expertise of school staff, which helps amplify teachers' voice and shift power relations.

The analytic approach that we take views schools as lived organizations (Cobb & Smith, 2008), which means we are trying to understand the work that teachers and school leaders are accomplishing, and the interconnections between them. It is important to uncover teachers' workplace practices to understand possibilities for the ways in which teachers can interact with students and with the content they teach (Cobb et al., 2003; Cohen et al., 2003; Resnick, 2010). Examining the institutional setting can include understanding: 1) for what teachers are held accountable; 2) what set of supports teachers currently receive, 3) the curriculum materials and messages around what is valued in teaching and learning; 4) how the instructional program is set up (e.g., whether there is tracking for mathematics); 5) teachers current opportunities for collaboration and access to expertise and key resources; and 6) individual teacher's current stances, dispositions, and instructional practices (Cobb & Smith, 2008; Diamond, 2007).

Methods

Context and Data Collection

This study is part of a larger project focused on understanding how elementary school teachers facilitate classroom discussions. As part of this project, we partnered with Partner Elementary School (PES; a pseudonym) to provide job-embedded professional learning experiences using the Learning Labs structure (see Kazemi et al., 2018) centered on enacting classroom discussions. For this study, we leveraged data from pre-interviews with teachers and school leaders, observations of weekly grade-level meetings, and observations of Learning Lab planning meetings and events. Interviews with school leaders were conducted that focused on their goals for students' learning and teaching, conceptions of teacher learning, and instructional leadership. Interviews were conducted with teachers who agreed to participate in the research activities prior to their first Learning Lab. Interviews lasted approximately 45 minutes and included questions regarding teachers' perceptions of classroom discussions and about the settings within which they work. Learning Labs were conducted in half-day sessions with grade-level teams; each team participated in two Learning Labs (for a total of 12 Labs). Prior to each Learning Lab, in order to be responsive to teachers' current instructional realities, the research team met with the teachers during their grade-level team meetings to understand the mathematics content in the upcoming unit. Field notes were captured after each weekly meeting. With information gathered from the teachers, the research team then met to plan the upcoming Learning Lab, identifying goals for teacher learning based on our understanding of teachers' needs. Finally, each Learning Lab was video- and audio-recorded and teachers were asked to complete exit tickets at the end of the Learning Lab.

Data Analysis

We examined a subset of the data through iterative rounds of qualitative data analysis to explore teacher support structures and teachers' experiences, dispositions, and instructional practices within the institutional setting. First, we read all sources in the subset, recording preliminary jottings (Saldaña, 2016) as a form of pre-coding. We next re-read the sources and applied a priori provisional codes informed by Cobb & Smith's (2008) proposed support structures to improve mathematics teaching and learning. This a priori scheme provided structure to the data before we engaged in open coding to locate emergent themes within and across provisional codes (Saldaña, 2016). Open coding was conducted by teams of two, during which we convened in small groups recursively to engage in consensus conversations. We finalized the codebook and used axial coding to create substantive themes that aligned with and illustrated components of getting to know a school partner (Maxwell, 2005; Saldaña, 2016).

Results

Understanding School Leaders' Vision for Teaching and Learning

Understanding how and why principals enact leadership within particular school improvement and change efforts is important (e.g., Leithwood, 2019). Thus, we wanted to understand how the school leaders at PES considered goals for students' learning in mathematics, the envisioned mathematics instruction they wanted taking place in classrooms, and their vision for how teachers improve their mathematics teaching and how they conceived of teacher learning. We also were interested in understanding how they enacted instructional leadership practices that were intended to support the teaching and learning of mathematics, including what they were holding teachers accountable for and their role in supporting teachers to make sense of policy messages from the district and state. While we do not have room to detail responses here, we learned that the school leaders' main mechanism of direct support for teachers was "doing walk-throughs and giving feedback, being in [grade-level meetings], so that I can hear what the teachers are saying and feeling and talking about, but then also I can help throw in those questions to help them think about some ideas…." (Principal Interview, 34:30) The principal also talked about the utility of a notice and wonder protocol for pushing teachers to think differently.

Teacher Support Structures and Accountability Expectations

Through our interviews with teachers and the school leaders, we learned about a number of supports and related expectations, including professional learning opportunities, weekly collaborative time, and curricular support. When we inquired about teachers' current opportunities for professional learning, we found out that the school received university-supported literacy coaching, with visits approximately 3-4 days per month. Most teachers reported not having support in mathematics during the current nor prior school years. There were no school, university, or district-based mathematics coaches, nor were there district-wide professional development opportunities in mathematics.

With respect to collaborative time, we learned that teachers had one 45-minute block of time each week to work in grade-level teams. This was a shift from prior years when grade-level teams met twice per week. During this time, there was an expectation that teachers would follow a particular agenda and discuss particular ideas, guided by the school leaders. There appeared to be time for teachers to discuss current problems of practice.

The teachers were expected to use the district-adopted Bookworms curriculum for English language arts and the online version of Eureka Math curriculum (called "Zearn") in mathematics. Teachers described the challenge around returning to the classroom in Fall 2021 amidst the pandemic and the requirement that each day students would spend time on the computer engaging in fluency practice, watching an actor explain the mathematical ideas of the lesson and solving a related task online. One teacher described:

> Our Math block is 70 minutes, which sounds unbelievable. Woo. But really, the amount of time that I have to do the instruction is really 30 minutes… I've got to get through the word

> problem and my little lesson and then Zearn [online computer program] starts at 1:00. And then they do their Zearn. And during the time that they're doing their Zearn, then I'm pulling my RTI small group instruction. So I look at that 30 minutes of time. (Ms. C, 18:43)

Other teachers similarly expressed concern with the amount of time students' spent on computers with the online curriculum, and whether and what they had opportunities to learn about mathematics. However, teachers shared with us that their teaching evaluations were tied to the amount of time each week students completed online mathematics lessons (a district mandate) and most felt that they had no choice but to do what was being asked of them by district leaders.

Teachers' Histories and Instructional Approaches

Through our one-on-one interviews with teachers and visits to classrooms, we learned about their histories, their current instructional practices as well as their dispositions related to teaching. While many of the teachers we interviewed were quite experienced in the classroom, they were greatly influenced by the district expectations for using the online Zearn curriculum. Despite these constraints, teachers were committed to "doing what was best for the students", a refrain we heard often across teachers. We were especially curious how whole-class discussions were happening in classrooms, which was part of our project goals. For example, we saw an opportunity for more whole class discussion in the context of Ms. C's explanation of a "little lesson" that she leads ahead of the students' work time on the computers. Other teachers similarly felt that they could not lead whole class discussions because of the demands of the math block and because their curriculum was not organized around students engaging in whole class discussions.

We also learned about interesting variation in the extent to which teachers were adhering to district expectations around curriculum, and wondered about how to capitalize on those local productive adaptations (Russell et al., 2020). For example, one teacher chose to use a more inquiry-oriented curriculum to support her mathematics instruction instead of the district-adopted curriculum. She was asking her other same grade-level teachers to consider joining her.

We analyzed teachers' responses to instructional vision questions to get a sense for whether teachers shared a vision for mathematics teaching. We also attended to common language that they used so that we could make explicit connections to their ways of talking and their goals when appropriate. This allowed us to consider initial teacher learning goals that are appropriate for different groups of teachers in the school.

Discussion

We set out to provide other mathematics education researchers with ideas for the types of activities and interactions that we used and found helpful in getting to know our school partners. Through our interactions, we identified aspects of the current institutional setting where there were synergies through existing expertise or initiatives we could take into account (e.g., weekly grade-level team meetings, principal instructional leadership through pressing teachers during planning or feedback after watching instruction). We also identified points of tension where we would need to actively think with our partners about how to navigate the competing demands (e.g., teacher evaluation system emphasizing computer curriculum coverage which was perhaps at odds with leaders and teachers' desires to support student discussion). While the analysis examined one case of relationship building, our findings have broader impacts on society as we consider the relationships necessary in partnerships that work to improve learning conditions for each and every learner. In a subsequent analyses, we plan to examine how we used these relationship building activities and what we learned about teachers' institutional settings to design professional learning that will challenge a settled mathematics learning status quo.

References

Bryk, A., & Schneider, B. (2002). *Trust in schools: A core resource for improvement*. Thousand Oaks, CA: Sage Publications, Inc.

Cobb, P., McClain, K., de Silva Lamberg, T., & Dean, C. (2003). Situating teachers' instructional practices in the institutional setting of the school and district. *Educational Researcher*, *32*(6), 13-24.

Cobb, P., & Smith, T. (2008). The challenge of scale: Designing schools and districts as learning organizations for instructional improvement in mathematics. *International Handbook of Mathematics Teacher Education*, *3*, 231-254.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational evaluation and policy analysis*, *25*(2), 119-142.

Diamond, J. B. (2007). Where the rubber meets the road: Rethinking the connection between high-stakes testing policy and classroom instruction. *Sociology of Education*, *80*(4), 285-313.

Farrell, C. C., Penuel, W. R., Coburn, C. E., Daniel, J., & Steup, L. (2021). Practice Partnerships in Education: The State of the Field. *William T. Grant Foundation*.

Henrick, E. C., Cobb, P., Penuel, W. R., Jackson, K., & Clark, T. (2017). Assessing Research-Practice Partnerships: Five Dimensions of Effectiveness. *William T. Grant Foundation*.

Kazemi, E., Gibbons, L. K., Lewis, R., Fox, A., Hintz, A. B., Kelley-Petersen, M., & Balf, R. (2018). Math Labs: Teachers, teacher educators, and school leaders learning together with and from their own students. *Journal of Mathematics Education Leadership*, *19*(1), 23-36.

Leithwood, K. (2019). Characteristics of effective leadership networks: A replication and extension. *School Leadership & Management*, *39*(2), 175-197.

Maxwell, J.A. (2005). *Qualitative research design: An interactive approach* (2nd edition). Thousand Oaks, CA: Sage Publications, Inc.

Resnick, L. B. (2010). Nested learning systems for the thinking curriculum. *Educational Researcher*, *39*(3), 183-197.

Russell, J. L., Correnti, R., Stein, M. K., Bill, V., Hannan, M., Schwartz, N., ... & Matthis, C. (2020). Learning from adaptation to support instructional improvement at scale: Understanding coach adaptation in the TN Mathematics Coaching Project. *American Educational Research Journal*, *57*(1), 148-187.

Saldaña, J. (2015). *The coding manual for qualitative researchers*. Thousand Oaks, CA: Sage Publications, Inc.

WHAT DO THE EMERGING THEMES IN HIGH SCHOOL TEACHERS' JOURNALS TELL US ABOUT THEIR THINKING?

Beste Güçler
University of Massachusetts Dartmouth
bgucler@umassd.edu

Jane Chunjing Ji
University of Louisiana Monroe
ji@ulm.edu

We examine in-service high school teachers' journals to explore the emerging themes in a mathematics content course for their professional development. We use a sociocultural perspective and characterize journals as signifying teachers' communication and written discourses about their thinking and experiences in the course. We use applied thematic analysis to analyze the emerging themes. Our results demonstrate the complexity of teacher thinking and suggest that teachers do not necessarily separate their thinking about themselves or mathematics from their thinking about their students; similarly, they can take different roles as teachers and learners in a given context. Our results indicate exercising caution about potentially operating with an oversimplified picture of teacher thinking via compartmentalized pieces, especially if such frameworks are used to measure teacher thinking, knowledge, and development.

Keywords: Communication, discourse, professional development, calculus

Introduction

Journal writing has been used in mathematics education for various purposes. Our literature review about journaling revealed that, in the context of mathematics education, journals were mostly used with students. Such work examined students' mathematical thinking and cognitive skills; problem solving; their beliefs and attitudes about mathematics, and rarely, their mathematical communication (e.g., Baxter, 2008; Farmer et al., 2003; Liljedahl, 2007; Rolka et al., 2006). Journals have also been used in teacher education, particularly in relation to reflective thinking and practice (e.g., Schön, 1987). Considered mainly as reflection tools,—although what is meant by reflection is rarely defined in these works— researchers used journals to examine teachers' professional knowledge and experience; professional identity; dispositions and beliefs in teaching and learning; pedagogical and professional development; their reflective and critical thinking on various issues; and experiences in their teacher education programs and courses (e.g., Joseph & Heading, 2010; Garmon, 2001; Mewborn, 1999; Snyder, 2012). Compared to the literature on journaling in the context of general teacher education, the literature on journaling specific to mathematics teacher education is sparse and primarily focuses on pre-service teacher education at elementary levels. Our work focuses on examining in-service high school mathematics teachers' journals in the context of a mathematical content course they took as part of their professional development since what teachers take from their experiences in professional development remains a challenging issue to address in mathematics education research (Farmer et al., 2003).

Regarding the theme of PME-NA, our work challenges (a) the dominant cognitive approach that is often used while examining reflection as well as student and teacher journaling, and (b) the traditional psychological approach that views participants' accounts of events and their experiences as unreliable, implying a mistrust towards "subjects" of a study. We will use a sociocultural approach to conceptualize reflection and journals and consider our participants' accounts of their experiences as authentic sources through which we can gain more information about their thinking and development in professional development settings. We address the following questions: What themes emerge in high school teachers' journals in relation to their thinking in a calculus content course they took for professional development? What do the emerging themes in these teachers' journals indicate regarding their thinking, knowledge, and development? Our findings challenge the dominant trends in mathematics teacher education research that view teacher knowledge as consisting of distinct types or pieces of knowledge that can be identified and measured through multiple-choice tests, mainly conceptualized from the perspective of the researchers (rather than those of teachers), and mainly utilizing cognitive perspectives.

Theoretical Framework

The study uses a sociocultural perspective that conceptualizes thinking as communicating and discourse as a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions…discourses in language are distinguishable by their vocabularies, visual mediators, routines, and endorsed narratives" (Sfard, 2008, p. 297). From this lens, thinking and communicating are not viewed as separate but connected activities; examining learners' discourses (communication), is tantamount to examining their thinking. Consistently, we characterize journal entries as written endorsed narratives involving the reflections (defined as meta-level discourses, i.e., discourses about discourses) of the teachers with respect to mathematics, their pedagogical approaches and practices, and their overall experiences and thoughts in the course. We view journals as written discourse with a communicational function (with one's self or others). By examining journals, we explore the (emergent) thinking of teachers in relation to the professional development course. Rather than viewing journals as signifying potentially "unreliable" accounts of participants' thinking, we view them as consisting of narratives that provide us with authentic information, from the participants' perspectives, of what they consider to be an honest account of their thinking and experiences.

Sociocultural approaches also highlight the interpretive aspect of research and the importance of context in meaning-making (Denzin & Lincoln, 2000). Consistently, we do not consider the narratives of the teachers in their journal entries as proxies for (often decontextualized and generalized) entities such as their cognitive schema, knowledge (or lack thereof), metacognition, beliefs, or attitudes. We consider these narratives as contextual indicators of teachers' discourses and thinking, keeping in mind that, while these narratives signify the authentic voices of the teachers, research is also an interpretive process that includes the researchers' presuppositions, theoretical perspectives, and personal stories (Denzin & Lincoln, 2000).

Background

This work is part of a larger study that took place in a postsecondary mathematics content course on calculus taken by in-service high school teachers as part of their professional development requirements (in our case, there was also one pre-service teacher taking this course, which was a programmatic exception). The researcher was the instructor of the course. The larger study hypothesized that an instructional approach that specifically attends to the tacit aspects of the mathematical discourse on various calculus concepts could support teachers' learning of those concepts (Güçler, 2016). A critical component of instruction was to elicit rich classroom discourse as well as reflection on and explication of teachers' mathematical discourses to promote learning.

Since we wanted reflection (meta-discourse) to be a continuous and consistent aspect of the course, the teachers also kept weekly journal entries throughout the course where we asked them to reflect on any aspect of the course that they wanted. We collected these journals thinking that they would give us more information about how the teachers' thinking about calculus concepts evolved over time and help us triangulate the data we obtained. Further, we wanted to elicit teachers' authentic voices about their own experiences independent of the potential constraints of the questions we posed to them in the pre- and post- explorations. Although the primary focus of the original study was on teachers' mathematical discourse, the journals gave us richer information that went beyond teachers' mathematical thinking and learning. This paper only focuses on the discursive themes that emerged from the teachers' journals as they reflected on their experiences in the course and the implications of the results in terms of teacher thinking, knowledge, and development.

Methodology

The participants of the study were 1 pre-service, 7 in-service high school teachers taking a mathematics content course on calculus over the course of 13 weeks. Except for the pre-service teacher, the participants' experiences ranged 4–12 years. The journal entries were collected in the course of a semester and consisted of 11 entries for each teacher. In these journals, the teachers were asked to reflect on any aspect of the course or classes without any specific prompt from the researchers. This task was deliberately left open-ended so that the teachers could write about whichever issue interested them or puzzled them from week to week, at their discretion, to elicit their authentic voices. Most teachers kept their journals electronically, as a text document. Three kept physical journals in the form of written notes, which were scanned and transcribed (verbatim) to electronic text documents at the end of the course. These electronic journals were later transferred to the NVivo 9 software package and all the emergent coding took place in this software environment.

We used applied thematic analysis (Guest et al., 2012) to examine the emerging discursive themes in the teachers' journals. Unlike the initial focus of the study which put teachers' mathematical discourses at the forefront through an a priori coding structure (we do not report on those in this paper), the emerging themes helped us identify the discursive patterns which put teachers' mathematical discourses at the background. In other words, in the context of these emerging themes, the teachers could still be communicating about mathematics, but our focus was not on their mathematical discourse per se but on the larger theme that included the mathematical communication.

In addition to the researcher, four doctoral students were involved in the coding and analysis of the data. The project team met each week; reported on and discussed the emerging themes; and compared the coding of the previous week and elaborated on the specific instances and cases in which our coding differed until we reached agreement. This iterative, generative, and interpretive process was repeated until saturation, eventually resulting in about 90% interrater agreement.

When coding, our first focus was on segmentation, which concerns how to bound the text (Guest et al., 2012). To assure our coding did not ignore the contexts in which the discourse emerged, we characterized a segment as consisting of an excerpt that signified a complete thought so that the meaning of the segment can "clearly be discerned when it is lifted from the larger context" (Guest et al., 2012, p. 52). Therefore, in our analysis, segments often consisted of multiple sentences, which also helped us avoid overrepresentation of themes and promoted exploring relationships among the themes.

The first themes in our initial examination of the journals emerged from contexts where the teachers referred to themselves as teachers (SAT: self as teacher) or learners (SAL: self as learner). Future iterations of coding resulted in the emergence of subcategories under these two categories. We noted that, within the context of SAL, the teachers discussed their own content-related (mathematical) difficulties (CD) and communicated about their enhanced learning (EL) in the course. Under the theme of SAT, three sub-categories emerged: TC, which refers to the contexts in which teachers wrote about the factors they thought constrained their teaching; SD, which refers to the contexts in which the teachers wrote about their students' difficulties about mathematical content they either experienced before or can now anticipate; and TS, which refers to the contexts in which the teachers wrote about their teaching strategies. After further examination, two subcategories emerged under TS: ES, where the teachers wrote about their existing strategies, and FS, where they wrote about the strategies they would use in the future as a result of their experiences in the course. Other emerging themes included the contexts in which the teachers' talked about the influence of the instructor (II) and the influence of their peers (PI) in the classroom in shaping their thinking and experiences in the course.

Results

Space constraints do not allow us to elaborate on all the emerging themes and their relationships in detail, so we primarily focus on the themes SAL, SAT, and some of their subcategories and relationships since they were the most dominant themes in teachers' discourses in their journals. We want to highlight that we do not make claims about teacher identity in the context of our study because, although some aspects of the teachers' discourses in the themes SAL and SAT may give us some insights about their identities, we believe that the data we have about SAL and SAT is insufficient to provide a rich and meaningful depiction of the teachers' identities, which are multi-faceted and require a data collection process that goes beyond merely examining their journal entries.

In teachers' journal entries, we identified 166 occurrences of the theme SAL and 156 occurrences of the theme SAT. We identified 71 occurrences in which the participants referred to themselves as teachers and learners in the same context (SAL ∩ SAT). Therefore, in about 43% of all SAL contexts, the participants also referred to themselves as teachers and in about 46% of all SAT contexts, the participants also referred to themselves as learners. The following excerpts provide examples for the themes SAL, SAT, and SAL ∩ SAT. All the names used in the study are pseudonyms and Steve is the only pre-service teacher in the course.

[1] Lea: I now understand that 'one to one' means simply that each *x* value is paired with exactly one *y* value, with no two x's being paired with the same *y* (injective). Also, 'onto' is when all elements in both sets are used (surjective). I am glad that I learned about these terms and their definitions. (SAL)

[2] Carrie: As a teacher, my experience shows that modeling is very challenging for students. They struggle analyzing graphs to create a story and creating graphs for verbal models. Today's class activity would lead into a great discussion regarding modeling, slopes, displacement, velocity, acceleration, etc. This is an activity that I will use in many of my classes as well as share with many of my colleagues. (SAT)

[3] Martin: I am continuing to struggle with the concept of derivative. I am grasping the rules, but I am having trouble with the concepts and applications. I can now understand where students can have trouble when I teach this. The main issue I am having deals with the concept of continuity. There was a question on homework that dealt with whether the first derivative was defined but the second derivative was not. I still am having trouble

understanding that concept, so as a teacher I can now understand that my students may have trouble with similar issues as well. (SAL ∩ SAT)

As can be seen from these excerpts, the teachers' discourses indicate that they can take different roles as teachers and learners in a given context. In addition, their discourses also indicate that there may also be other themes or subthemes embedded in a given context in addition to SAL, SAT, and SAL ∩ SAT. For example, in excerpt [1], which is coded as SAL, Lea also mentions that she "now understand[s]" the terms and "learned about these terms and definitions", which indicate enhanced learning and/or understanding as a component of SAL. Therefore, we also coded excerpt [1] as EL, based on our definition of this theme. In excerpt [2], Carrie refers to herself as a teacher (SAT) but she also mentions the activity in her professional development course as an activity she will use in her own teaching as a future teaching strategy as a component of SAT. Therefore, we also coded excerpt [2] as FS. In excerpt [3], Martin refers to himself both as a teacher and a learner (SAL ∩ SAT). When writing about himself as a learner (SAL), Martin mentions his content-related struggles and difficulties, so this excerpt also provides a context for CD as a component of SAL. On the other hand, he associates the difficulties he has with derivative with the difficulties his students would have about the concept, so we also see the theme SD as a component of SAT in the same excerpt. The examinations of the segments we identified as signifying a complete thought reveal the complexity and dynamic characteristics of teacher thinking, knowledge, and development. Excerpts [1-3] show that, even in such a small set of exemplars, we can see the emergence of themes, subthemes, and how they can be intertwined in complex but crucial ways.

In this paper, we want to also focus on CD, SD, and contexts in which both themes occurred (CD ∩ SD). Note that excerpt [3] shows how a teacher can think about his own difficulties and his students' difficulties in the same context, and provides an example for the theme CD ∩ SD. In what follows, we provide examples for the themes CD and SD:

[4] Fred: Oh, I have to get out of the habit of thinking of a function as a single rule or rigid model describing a situation – think piecewise! I struggled with the functions defined piecewise in today's class. (CD)

[5] Sally: This week, we talked about why continuity and limits were connected and how they related to each other. Most students have difficulty understanding that if you have a continuous function then the limit exists but that doesn't necessarily mean that if the limit exists then the function is continuous. It is a one-sided relationship and tends to be misconstrued as relationship that works both ways. (SD)

We identified 91 occurrences of the theme CD, 86 occurrences of the theme SD, and 50 occurrences of the theme CD ∩ SD in the teachers' journals. Therefore, in about 55% of all CD contexts, the teachers also mentioned their students' difficulties and in about 58% of all SD contexts, they also mentioned their own content-related struggles in the course. This major overlap may be interpreted in various ways. This finding may be an indicator that teachers' own difficulties with calculus concepts can significantly inform their thinking about their students' difficulties and vice versa in a context. If that is the case, a potential association of CD with teachers' mathematical content knowledge and the association of SD with teachers' pedagogical content knowledge can be very problematic. The characterization of teachers' own mathematical thinking and knowledge as aspects of their mathematical content knowledge and their knowledge of student thinking as an aspect of their pedagogical content knowledge is quite common in mathematics education research focusing on teachers, where these "types" of knowledge are characterized as distinct, albeit related, components of teacher knowledge. Our results, however, show that these constructs may be almost inseparable within a given context in a way that may make it very challenging to identify or measure which "type of knowledge" teachers may have at a given time, especially if such measurement is being made through one-shot multiple-choice tests or decontextualized assessment problems and/or instruments .

Another potential explanation for the major overlap may be due to the teachers' profound struggles in this calculus content course and, in the absence of familiarity with the conceptual aspects of calculus and rich thinking about student difficulties about calculus concepts, the teachers mainly referred to their own struggles as resources and means for anticipating most of the student difficulties about calculus. In fact, there is some evidence in our data that the teachers who struggled most with the content were also those whose discourses included a larger number of occurrences for the theme CD ∩ SD.

Within the context of SAL, the relationship between CD and EL may also be worthy of some elaboration. In teachers' journal entries, we identified 91 occurrences of the theme CD, 53 occurrences of the theme EL, and 27 occurrences of the theme CD ∩ EL. In about 30 % of all the CD contexts, the teachers also mentioned enhanced learning or understanding and in about 51% of all the EL contexts, they also mentioned their content-related difficulties. The following excerpts provide examples for the themes CD, EL, and CD ∩ EL.

[6] Milo: I had a bit of confusion today when we started discussing derivative. If I recall, finding the derivative of basic functions is not too difficult, but understanding the meaning behind it and why-that's created a headache for me today since I don't know those meanings. (CD)

[7] Steve: I believe I am leaving this class with a better understanding of every calculus concept we covered, and for that I feel like I am prepared to be an efficient and effective teacher of calculus. Thank you for helping me attain a greater understanding about these topics, topics I already thought I was knowledgeable in. (EL)

[8] Lea: As we continue our discussion of limits in class, it is clear that I struggle and have much to learn conceptually about limits and other concepts of calculus. Although I am thoroughly enjoying the challenges brought forth each week, at times I am very humbled in the realization that there is a lot that I still struggle and need to learn about these calculus concepts. For example, this week I learned that we can view a limit being a process or a product or both depending on context! (CD ∩ EL)

The results and the excerpts indicate that challenging teachers mathematically can be a good strategy to promote teacher development (enhanced learning). Although teachers may not realize every occasion where they experience a difficulty as an opportunity to learn, our results suggest that when they reflect on their learning and explicitly mention enhanced learning, they seem to have a tendency to think about the contexts in which they struggled with mathematics. Professional development environments which do not only put teachers in a teacher role reflecting on student thinking but also in a learner role where they genuinely struggle with content and reflect on their own mathematical thinking have the potential to provide rich learning opportunities for teachers. In addition, the existence of the theme EL as an emergent theme in the journals, which is elicited through the teachers' own voices and discourses, can also suggest that journal entries may be useful resources for teacher educators in assessing teacher learning or development in professional development situations. We promote the use of journals as potential teacher assessment tools because we believe they may provide context regarding teacher thinking, which is a critical component when examining their development and also because of the connections we observed between the themes EL and FS as will be discussed next.

In teachers' journals, we identified 53 occurrences of the theme EL (as a component of SAL), 43 occurrences of the theme FS (as a component of SAT), and 21 occurrences of EL ∩ FS (as a component of SAL ∩ SAT). In about 40% of all the EL contexts, the teachers also mentioned pedagogical strategies they would use in the future and in about 49% of all the FS contexts, the teachers mentioned enhanced learning or understanding. Excerpt [7] provides an example for the theme EL. The following excerpts provide examples for the themes FS and EL ∩ FS.

[9] Ron: I liked in class how you asked the class to use one word to define function and then asked the class to use those words to form a definition. I feel that in teaching algebra II, I will use this approach since this could really get the students thinking. Students can take their previous knowledge of functions or any other concepts and reason through that when defining the concept. (FS)

[10] Sally: I can now say that when I start teaching functions next year, I will approach this topic completely different. I will now open my students up to a more discussion-based approach where we can openly look at the definition of function and talk about what makes something a function and what makes something not a function. This way they aren't trained to only know linear or quadratic functions. They won't consider a function just the "equation" or "graph" or "table" but know why the table represents a function or why the rule represents a function. My own learning and understanding of the topic in a deeper way helped me think about changing how I teach functions. (EL ∩ FS)

The results and the excerpts suggest that, although the teachers did not necessarily consider all the contexts in which they mentioned enhanced learning as opportunities to also reflect on the teaching strategies they would use in their future practice, almost half of their discussions about their future pedagogical strategies included teachers referring to their enhanced learning in the context of the course. This also suggests that changing teacher practice may be closely related to teachers changing their thinking (enhanced learning) about mathematical concepts.

Discussion

The results of the study demonstrate the complexity as well as the dynamic and situational nature of teacher knowledge, thinking, and development. A contextual analysis of teachers' (meta) discourses based on their authentic voices indicate that the teachers in our study did not separate their thinking about themselves or mathematics from their thinking about their students; similarly, they took different roles as teachers and learners in a given context. These findings challenge the existing trends in defining and measuring teacher knowledge, where such knowledge is often characterized as consisting of compartmentalized pieces (e.g., mathematical content knowledge, pedagogical knowledge, pedagogical content knowledge, mathematical knowledge for teaching). Contextually, these distinctions can be blurred, much more integrated, and complex. The emerging themes in the teachers' journals provide a cautionary tale regarding the potential dangers of providing an oversimplified picture of teacher thinking, knowledge and development (Beswick et al., 2012).

Schoenfeld (2007) highlights the importance of exploring which type(s) of knowledge that the measures assess and notes that "much needs to be done in fleshing out the relevant knowledge base to be tested and determining how well the items used actually reflect the desired competencies" (p. 204). Fauskanger (2015) found that the multiple-choice responses teachers provide in the assessments of their mathematical knowledge for teaching may not reflect their knowledge as expressed in their open-ended, constructed responses and noted that a teacher could give an incorrect response to the test item but demonstrate conceptual understanding in the latter context. There may also be clashes between the researchers' and teachers' expectations and interpretations of the test items, their goal, and what they are supposed to measure (Fauskanger, 2015). Beswick et al. (2012) note that the precise way in which different types of teacher knowledge are conceived and "how aspects of such a conception beyond 'facts that are known' is incorporated" in such models is not clear (p. 133). It is important to note that, despite their problematization of the current approaches towards assessing teacher knowledge, researchers may still operate from only a cognitive view and may still provide attempts to "refine" or "expand" the characteristics of different types of teacher knowledge for better and more reliable assessment (e.g., Fauskanger, 2015; Beswick et al, 2012; Schoenfeld, 2007).

Characterization and assessment of teachers' knowledge or development based on knowledge compartmentalization and on the cognitive "knowledge-as-acquisition" metaphor (Sfard, 2001) can be problematic since this may not reflect the multi-faceted, dialogical, social, cultural, and contextual nature of teachers' thinking, knowledge and development, especially when viewed from sociocultural lenses which often characterize learning and development as occurring through enhanced participation in mathematical communities of practice (Lave & Wenger, 1991) through the "knowledge-as-participation" metaphor (Sfard, 2001). Our findings suggest that the theoretical frameworks that go beyond cognitive constructivism can be useful in our explorations and (re)interpretations of what we mean by teacher knowledge, thinking, and development and whether, and how, to assess them through standardized instruments. We believe sociocultural frameworks have a lot to offer to the field as we continue thinking about these critical issues, particularly if we want to put the voices of the teachers at the center of our discussions about teacher education in mathematics education research.

References

Baxter, J. A. (2008). Writing in mathematics: Alternative form of discourse for academically low-achieving students. *Perspectives on Language and Literacy*, *34*(2), 37-40.

Beswick, K., Callingham, R., & Watson, J. (2012). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education*, *15*(2), 131–157.

Denzin, N. K., & Lincoln, Y. S. (2000). Introduction: The discipline and practice of qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed., pp. 1-28). Thousand Oaks, CA: Sage.

Farmer, J. D., Gerretson, H., & Lassak, M. (2003). What teachers take from professional development: Cases and implications. *Journal of Mathematics Teacher Education*, *6*(4), 331-360.

Fauskanger, J. (2015). Challenges in measuring teachers' knowledge. *Educational Studies in Mathematics*, *90*(1), 57-73.

Garmon, M. A. (2001). The benefits of dialogue journals: What prospective teachers say. *Teacher Education Quarterly*, *28*(4), 37-50.

Guest, G., MacQueen, K. M., & Namey, E. E. (2012). *Applied thematic analysis*. Sage publications.

Güçler, B. (2016). Making implicit metalevel rules of the discourse on function explicit topics of reflection in the classroom to foster student learning. *Educational Studies in Mathematics*, *91*(3), 375-393.

Joseph, D., & Heading, M. (2010). Putting theory into practice: Moving from student identity to teacher identity. *Australian Journal of Teacher Education*, *35*(3), 75-87.

Lave, J. & Wenger, E. (1991). Situated learning: Legitimate peripheral participation. New York: Cambridge University Press.

Liljedahl, P. (2007). Persona-based journaling: Striving for authenticity in representing the problem-solving process. *International Journal of Science and Mathematics Education*, *5*(4), 661-680.

Mewborn, D. S. (1999). Reflective thinking among preservice elementary mathematics teachers. *Journal for research in mathematics education*, *30*(3), 316-341.

Rolka, K., Rösken, B., Liljedahl, P. (2006). Challenging the mathematical beliefs of preservice elementary school teachers. *Proceedings of the 30th International Conference for Psychology of Mathematics Education* (Vol. 4, pp. 441-448). Prague, Czech Republic.

Schoenfeld, A. (2007). The complexities of assessing teacher knowledge. *Measurement: Interdisciplinary Research and Perspectives*, *5*(2), 198–204.

Schön, D. A. (1987). *Educating the Reflective Practitioner*. San Francisco: Jossey-Bass.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. New York: Cambridge University Press.

Sfard, A. (2001). There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning. *Educational Studies in Mathematics*, *46*(1), 13-57.

Snyder, C. (2012). Finding the "royal road" to learning to teach: Listening to novice teacher voices in order to improve the effectiveness of teacher education. *Teacher Education Quarterly*, *39*(4), 33-53.

CONNECTING MATHEMATICS TO THE REAL WORLD: ONE GROUP OF ELEMENTARY MATHEMATICS TEACHERS' EXPERIENCES

Simon Byeonguk Han
Portland State University
byeonguk@pdx.edu

Eva Thanheiser
Portland State University
evat@pdx.edu

Amanda Sugimoto
Portland State University
asugimo2@pdx.edu

Molly Robinson
Portland State University
molrob2@pdx.edu

Courtney Koestler
Ohio University
koestler@ohio.edu

Mathew D. Felton-Koestler
Ohio University
felton@ohio.edu

Howard E. Bartels
Ohio University
hb134218@ohio.edu

Jennifer A. Thompson
Ohio University
jh859394@ohio.edu

Danny Penner
Portland State University
aubrap@pdx.edu

Laura L. Wolfe
Ohio University
wolfel2@ohio.edu

In this study, we examined one group of elementary mathematics teachers' experiences in connecting mathematics to the real world. The professional development collaboration was designed to connect mathematics to the real world, particularly connecting to students' funds of knowledge. We interviewed seven elementary teachers about their experiences in connecting mathematics to the real world and to what they knew about their students. We categorized the teachers' responses resulting in the Connecting Mathematics to the Real World framework. Our findings show that while almost all teachers had implemented typical word problems as a way to connect mathematics to the real world, none had incorporated contexts relevant to their students' lives. However, most of the teachers expressed interests in developing lessons that were more relevant to their students' lives.

Keywords: Social Justice, Culturally Relevant Pedagogy, Elementary School Education, Equity, Inclusion, and Diversity.

Purpose of the Study

The purpose of this study is to examine one group of elementary mathematics teachers' experiences in connecting mathematics to the real world before engaging in a professional development. Research question: *What are teachers' experiences in connecting mathematics to the real world at one elementary school?*

Literature & Theoretical Framing

What Does Connecting Mathematics to the Real World Mean?

Extant scholarship has explicated several ways that teachers can connect mathematics to the real world. These include a) situating problems in story contexts (typical word problems), b) analyzing real-world data or real-world situations (that may or may not matter to the students per se), or c) connecting to students' funds of knowledge (both mathematically and with the context). We discuss each of these below.

Word/Story problems. One common way to connect mathematics to the "real" world is through story problems (Gainsberg, 2008). However, these story problems are often created with common but unimportant contexts, for example, a focus on “apples, puppy dogs, and ice cream” (Koestler, 2012) rather than on issues students really care about. These word problems might be about 2 trains that meet somewhere or purchasing 42 watermelons and so forth. Though the math is put in a story context, these problems are often simply exercises to do, not real problems that students are motivated to solve. When asked to create word problems with real-world connections, most teachers tend to focus on the math rather than the context (Lee, 2012).

Analysis of real data, bring in real-world issues (not necessarily relevant to the students in the class). Teachers can bring in real data or real-world issues to connect mathematics to the world (Gainsberg, 2008; Gutstein, 2006). However, this data may not always be meaningful to students or focus on problems they care about solving. These data might focus on measuring physical objects, census data, examining a newspaper article, etc. The problem contexts or data may be designed to support students’ development of a sociopolitical disposition (Frankenstein, 2012; Gates & Jorgensen, 2009). However, teachers may not use a critical lens when implementing such tasks (Bartell, 2013; Lee, 2012)

Connecting to students’ funds of knowledge (FoK). Another approach is creating problems designed to build on children’s mathematical thinking (CMT) and Cultural Funds of Knowledge (CFoK) (González et al., 2005; Turner & Drake, 2016). Problem types that fall into this category connect to the students’ mathematical knowledge bases as well as to the students’ and their communities’ interests, experiences, and backgrounds (Civil, 2007; Wager, 2012). Turner and colleagues (2012) studied how prospective teachers (PTs) develop their understanding of students’ multiple mathematical knowledge bases. As PTs develop meaningful connections between students’ multiple knowledge bases and their instruction, they can “facilitate ongoing and purposeful incorporation of multiple mathematical knowledge bases” (p. 78).

Prior frameworks (Gainsberg, 2008) for connecting math to the real world have distinguished between the first two but have not yet incorporated the third: an explicit focus on the students’ FoK. We build on the first two categories and add students’ FoK to emphasize the relevance of students’ lives to create the framework for this paper (see Table 1).

Methods

Participants and Contexts

Seven elementary teachers from one public elementary school in the Pacific Northwest of the United States participated in the study. These teachers were at the beginning of a professional development collaboration designed to connect mathematics to the real world based on connecting to students' funds of knowledge. The teachers spanned grades K-5, a student success coach, and a special education teacher.

Data Collection

The research team conducted individual 30-minute semi-structured interviews with each participant. Interview questions focused on the teachers’ experiences in connecting mathematics to the real world and what they knew about their students, the students' families, and the community. For example, we asked:

- Tell me what you know about your students’ interests and experiences outside of the classroom.
- Tell me about your students as math learners.
- What experiences, if any, do you have with connecting math to the real world?
- What are some real-world topics that you are interested in exploring with kids in math?

The interviews were recorded and transcribed.

Analysis

Analysis began with one research team member carefully reading all the transcripts and highlighting keywords and sentences from each transcript to fit into existing categories: Typical Word/Story Problems, Analysis of Real Data/Bringing Real-World Issues, Connecting to Students Funds of Knowledge. During this same initial reading, the researcher also highlighted new keywords that indicated new categories. For example, several teachers highlighted that they followed the curriculum and as such, we added "following curriculum" as a category. Another category that emerged was "cross-curricular connection to the real-world outside mathematics." Once the initial categorization was completed, additional research team members discussed the categorizations and came to an agreement on all categorized segments. Below we discuss each theme.

Results

Table 1 shows the summary of the teachers' experiences in connecting math to the real world. Below we discuss each theme.

Table 1: Teachers' Experiences in Connecting Mathematics to the Real World

Teachers' Experiences in Connecting Mathematics to the Real World	Number of Teachers
Following Curriculum	4
Typical Word/Story Problem	6
Analysis of Real Data/Bringing Real-World Issues That Might Not be Relevant to Students Personally	2
Connecting to Students' Funds of Knowledge	0
Bringing the World into the Classroom (Cross-Curricular Connection to the Real World Outside Mathematics)	3

Following Curriculum

Four teachers began their responses by mentioning that their curriculum provides connections between math and the real world and they followed it. Teacher A mentioned, "I kind of just go with like… the stories in the curriculum." Teacher F said, "I know that the curriculum attempts to do somewhat of it." These four teachers' experiences with typical word/ story problems were closely related to the curriculum. The three teachers who did not mention the curriculum each shared at least one example in another category.

Typical Word/Story Problem

Six teachers shared their use of the typical word/story problems as their experiences in connecting math to the real world. Teacher A and Teacher F used bunk beds and apple boxes for 10 frames since the curriculum provided them. Teacher A said, "Our first unit was like bunk beds and apple boxes, so we are really focusing on 10 frames and manipulating numbers within a 10 frame." Some teachers considered using students' names in the word problems as a way of connecting math to students: "I'll pick like some student names and put it in the word problems" (Teacher C) and "I've also tried to connect problems to students lives actually using students in the problems just throwing out names" (Teacher D). Teacher E shared that he used ice cream shops and shopping mall contexts for decimal operations. He said, "ice cream math and shopping mall math or whatever with the decimals to kind of like help them through things a little bit."

Analysis of Real Data/Bringing Real-World Issues That Might Not be Relevant to Students Personally

Two teachers shared analysis of real data or bringing real-world issues as their experiences in connecting math to the real world. Teacher E said he has implemented the 'Million Dollar Project' many times. He said, "I sent them to a site for houses… gave them million-dollar… they had to spend exactly a million dollars or else they don't get their million dollars." Teacher G shared her experiences in creating a small classroom economy: "They have to come in and they have to do the work, do the math, show their work, explain, help someone, and then you are paid. You are a math consultant, and so they have their own banks… at the end of the week, they can buy a goodie."

Connecting to Students' Funds of Knowledge

None of the teachers shared their experiences in connecting math to students' funds of knowledge. However, six teachers shared that they want to find and implement real-world connections that are relevant and connected to their students on a personal level (their interests), as well as family and cultural levels.

Bringing the World into the Classroom (Cross-Curricular Connection to the Real-World Outside Mathematics)

Three teachers shared their experiences in bringing the real world into the classroom outside mathematics. When Supreme Court Justice Brown Jackson was confirmed, Teacher A introduced the confirmation debate to her students. Students read about the story of the justice and saw the picture of the Supreme Court Justices. She commented that "I had not thought about how to bring math into that but I bet we could figure out a way." Teacher C shared her experience in scavenger hunts: "I've done scavenger, like a nature scavenger hunt, like finding shapes in nature." Teacher E's students were interested in Pokémon Cards, so he regularly provided time to his students for Pokémon cards trading.

Conclusion and Discussion

In response to our research question '*What are teachers' experiences in connecting mathematics to the real world at one elementary school'* we found that while almost all teachers had implemented typical word/story problems as a way to connect mathematics to the real world, yet none had used contexts relevant to their students' lives. However, most of the teachers were interested in developing lessons that would be more relevant to their students' lives. Teachers A and D explicitly stated that they were interested in making their mathematics lessons relevant to their students. Teacher B was interested in culturally relevant and social justice related mathematics lessons. Teacher C wanted to develop lessons related to time and money because her students showed interest in time and money. Teacher E was aware of his students' interests outside of mathematics, but he was more interested in how to connect mathematics to his own interests to make it real-world mathematics. Teacher F expressed her interest in connecting math lessons to students' cultural and family heritages. Teacher G shared that she wants to help her students to understand how math is in their daily lives.

Understanding how teachers think about connecting mathematics to the real world was important as we began our professional development collaboration. Just as we wanted our teachers to build on their students' funds of knowledge (González et al., 2005), we wanted to build on the teachers' funds of knowledge when working with them. Knowing where they started helped us develop an opening activity to develop a shared understanding of how we thought about connecting mathematics to the real world.

Acknowledgments

This research was supported by the National Science Foundation (Grant No. 2010463). Opinions, findings, and conclusions or recommendations expressed here are the authors' and do not necessarily reflect the views of the National Science Foundation.

References

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education, 44*(1), 129-163.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. *Improving access to mathematics: Diversity and equity in the classroom,* 105-117.

Frankenstein, M. (2012). Beyond math content and process: Proposals for underlying aspects of social justice education. In A. A. Wager & D. W. Stinson (Eds.), Teaching mathematics for social justice: Conversations with educators (pp. 49–62). Reston, VA: NCTM.

Gainsberg, J. (2008). Real-world connections in secondary mathematics teaching. Journal of Mathematics Teacher Education, 11(3), 199-219.

Gates, P., & Jorgensen, R. (2009). Foregrounding social justice in mathematics teacher education. Journal of Mathematics Teacher Education, 12(3), 161-170.

González, N., Moll, L. C., & Amanti, C. (2005). Funds of knowledge: Theorizing practices in households, communities, and classrooms. Mahwah, NJ: Lawrence Erlbaum Associates.

Gutstein, E. (2006). Reading and writing the world with mathematics: Toward a pedagogy for social justice. New York, NY: Routledge.

Koestler, C. (2012). Beyond apples, puppy dogs, and ice cream: Preparing teachers to teach mathematics for equity and social justice. In A. A. Wager & D. W. Stinson (Eds.), Teaching mathematics for social justice: Conversations with educators (pp. 81097). National Council of Teachers of Mathematics.

Lee, J-E. (2012). Prospective elementary teachers' perceptions of real-life connections reflected in posing and evaluating story problems. Journal of Mathematics Teacher Education, 15(6), 429-452.

Turner, E. E., Drake, C., Roth McDuffie, A., Aguirre, J., Bartell, T. G., & Foote, M. Q. (2012). Promoting equity in mathematics teacher preparation: A framework for advancing teacher learning of children's multiple mathematical knowledge bases. *Journal of Mathematics Teacher Education, 15*, 67-82.

Turner, E.E., & Drake, C. (2016). A review of research on prospective teachers; learning about children's mathematical thinking and cultural funds of knowledge. *Journal of Teacher Education, 67*(1). 32-46.

Wager, A. A. (2012). Incorporating out-of-school mathematics: From cultural context to embedded practice. Journal of Mathematics Teacher Education, 15(1), 9-23.

EXPLORE TEACHERS' COLLECTIVE LEARNING THROUGH LESSON STUDY FROM A PERSPECTIVE OF NETWORKING THEORIES

Rongjin Huang
Middle Tennessee State University, USA
rhuang@mtsu.edu

Xingfeng Huang
Shanghai Normal University, China
xfhuang@shnu.edu.cn

Although the effects of Lesson study (LS), a teacher collaborative professional development approach, on teacher professional learning and students learning have been widely documented, the theories for understanding of LS has just emerged as a research field. Interconnected Model of Teacher Professional Growth (IMPG) and Documentational Approach to Didactics (DAD) have been used individually to document teachers' professional learning. In this study, a networked framework is proposed by locally integrating these two theories. A lesson study facilitated by a researcher was conducted in Shanghai China. The data sets including all videotaped meetings and research lessons are collected and analyzed. The results show that the teachers' resource system evolved from adoption to adaptation with consideration of student learning. This study contributes to networking theories and its usefulness in LS context.

Keywords: Professional development, Teacher educators, teacher knowledge and beliefs.

Introduction

Lesson study (LS), a collaborative teacher professional approach, originated in Asia, has been adopted around the world (e.g., Huang et al., 2019). Although the effects of LS on teacher professional learning (knowledge, beliefs, disposition), teacher professional learning community, and students learning have been widely documented (Cheung & Wong, 2014; Willems & Bossche, 2019), the theories for understanding of process and products of teacher learning through LS have become an emerging research field (Huang et al., 2019; Lewis et al., 2009); Specifically, various theories have been utilized to frame and understand LS (Borko & Potari, 2020). Yet, most previous studies use a single theoretical lens for understanding LS (Widjaja et al., 2017; Clarke & Hollingsworth, 2002; Pepin et al., 2017; Trouche et al., 2019). Due to the complexity of teacher professional learning in collaborative LS settings, a networking theories approach may provide an alternative way to deepen understanding of the complex phenomena (Prediger et al., 2008; Shinno & Mizoguchi, 2021; Trouche et al., 2019). In this study, we aim to frame our study from a networking theories approach through which the Documentational Approach to Didactics (DAD) model (Trouche et al., 2020) is enriched by incorporating the ideas from Interconnected Model of Teacher Professional Growth (IMPG) (Clarke & Hollingsworth, 2002).

Research Background and Theoretical Framework

Chinese lesson study and teacher learning

Chinese LS has some unique features (Huang et al., 2017), including (1) the repeated teaching of the same content to different students, with immediate feedback from their colleagues and knowledgeable others; (2) a focus on refining the teaching of particular content or improving teachers' particular instructional skills; and (3) knowledgeable teaching researchers' involvement who are officially responsible for organizing school-based teaching activities and evaluating teachers' teaching and students' learning. Overall, Chinese LS provides a platform for improving teaching practice (Chen, 2020; Wei & Huang, 2022), developing teachers (Fang, 2017), implementing new curriculum (Qi et al., 2021), and linking theory and practice (Huang et al., 2016).

Theoretical framework

Both DAD and IMPG models are rooted in the sociocultural perspective. They have been used to capture different aspects of teacher learning (Clarke & Hollingsworth, 2002; Trouche et al., 2019).

DAD theory. Resources refer to what has the potential of 're-sourcing' teacher activity (Adler, 2000) at the core of instruction. To capture the features of mathematics teacher professional development through the lens of their interactions with resources, a specific conceptual framework, named Documentational Approach to Didactics (DAD) has been developed over the past decade (Gueudet & Trouche, 2009; Trouche et al., 2020). This study focuses on dialectical relationship between the teacher and the resources: the teacher permanently adapts the resources s/he is working with (this is the *instrumentalization* process), while these resources, their constraints and affordances, contribute to configuring teacher's activity (this is the *instrumentation* process). In DAD, a *document* is defined as the outcome of joint resources combined with knowledge guiding usage, and the process of developing the document is called documentational genesis (Pepin et al., 2013), which occurs through iterative processes of preparation/design and enactment (Trouche et al., 2019). Understanding teacher's documentation work needs to consider resources evolving process through successive implementations for addressing different issues which is named as *documentational trajectory* (Trouche et al., 2020).

IMPG model. Situated in a social and cultural perspective, the IMPG model (Clarke & Hollingsworth, 2002) posits that teacher change happens through enactment and reflection in four domains: Personal Domain (PD), Domain of Practice (DP), External Domain (ED), and Domain of Consequence (DC). PD includes teacher knowledge, beliefs, and attitudes and/or orientation, DP refers to teacher's experimentation (classroom practices), ED involves the external source of information or stimulus, and DC refers to the salient outcomes of experimentation perceived by the teachers. To emphasize collaborative learning during professional development programs, Prediger (2020) proposed an adapted IMPG model for portraying teacher professional development in collaborative contexts. In her model, the Personal Domain is extended to a Collective Domain (CD, Shared practice and orientation) and the Domain of Practice is extended to a Domain of Inquiry (DI, collective inquiry of new teaching practice).

A way for networking the two theories. We use integrating locally as a strategy for networking the theories of DAD and IMPG. The DAD model portrays the dynamic relationship between teachers and resources to generate documents, but the mechanism of documentational genesis remains unknown. The feature of IMPG model which emphases the dynamic among the different domains including the collective domain and external domain could be integrated into the DAD model to strengthen the model. Yet, based on the modified IMPG model by Prediger (2020), the documentational genesis can be seen as a dual process between the Collective Domain and Resources (External Domain, and Domain of Consequence) through Domain of Inquiry. Thus, we proposed a model for networking DAD and IMPG (see Figure 1): Teachers in the DAD theory are regarded as the collective domain from the IMPG perspective and resources in the DAD theory are the same as the External Domain (ED) and Domain of Consequence (DC), which includes textbooks, online resources, concrete models, students' performance in pretest, as well as advice and materials provided by researchers outside the teacher community.

Documents in the DAD theory are regarded as the outcomes (resources and ways of using them) of Domain of Inquiry, which mainly include the agreed ideas in meetings and resulting products of lesson plans and enacted lessons. The Documentational Genesis in the DAD theory therefore triggered by the dynamics (enactment and reflection) between the Domain of Inquiry and other three domains. The Domain of Inquiry is a collective and collaborative process during lesson planning meetings, research lesson observation, and post-lesson debriefing meetings in an LS Group.

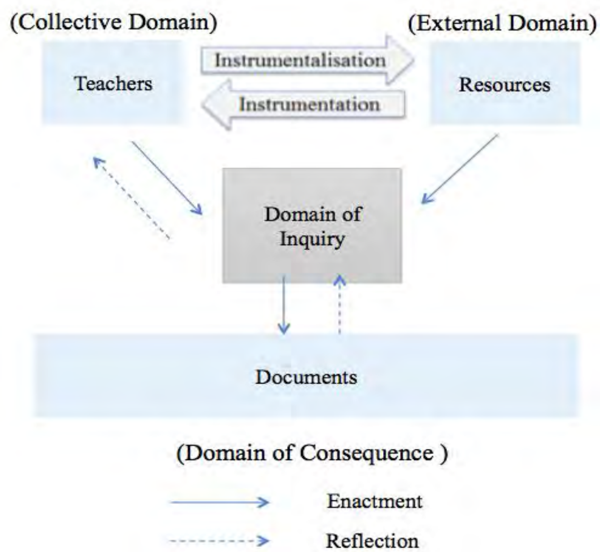


Figure 1. An Adapted DAD Model by incorporating IMPG ideas

From this adapted framework, this study aims to explore how teachers' growth by their documentation works by addressing the following research questions:

(1) How does the documentational genesis work through the enactment and reflection between the Domain of Consequence and the others?
(2) How does teachers' document system develop during the lesson study process?

Methods

The participants and the process of the LS

The researcher, Dr. Huang, from a teacher education university and seven mathematics teachers in an urban elementary school in Shanghai form an LS group. Dr. Huang works with pre-service mathematics teacher education program and research on mathematics teacher professional development, and teachers are from a Teaching Research Group (TRG, an institutionalized Chinese teacher professional learning community, see Yang, 2009) in the school. Mrs. Miao, the lead designer of teaching plans and enactor of research lessons, and the other six teachers Mr. Bao, Mr. Tan, Mr. Li, Mrs. Ji, Mrs. Wen, and Mrs. Zhao were all experienced teachers with teaching experience from 6 to 15 years. This lesson study went through four phases over four weeks. The first phase-*study* (29 April, 2021) involved the researcher and his graduate students sharing relevant research literature and the presentations of angle measures in different mathematics textbooks with the teachers. The teachers discussed the possible and specific content and task design on the size of angles. After the meeting, the researcher designed a pretest for the teachers to understand their students' learning readiness and provide the foundation for designing the lesson. The second phase-*plan* (6 May, 2021) began with a brief presentation by the researcher on students' misconceptions about angles and the size on the pretest and their strategies for comparing angle size. Then, Mrs. Miao presented her lesson plan and the TRG discussed the lesson plan. In the third phase -*teaching and reflection*, based on the feedback from the LS group, Mrs. Miao revised the lesson plan and enacted the lesson in her own class (12 May, 2021). The researcher and other teachers in the LS group observed the lesson. A post-lesson discussion followed immediately. The teachers and the researcher evaluated the enacted lesson and identified issues that occurred and provided specific recommendations for improving the lesson. In the fourth phase – *re-teaching and reflection* (19 May, 2021), Ms. Miao again enacted the lesson in another class, which the researcher and other teachers observed together. The post-lesson reflection followed. Finally, a post-test was conducted for students' learning outcomes in the second class.

Data sources. In this study, the data consisted of three components. The first is the four videos of teachers' group meetings; the second is the two videotapes of enacted research lessons; and the third is the lesson plans and slides designed by teachers before the lesson study, and the lesson plans and slides developed during the lesson study.

Data analysis. Teachers' lesson plans and slides and the meeting videos at each stage were used to analyze and describe teachers’ document system and paint documentational trajectory. Based on the research questions and the adapted DAD model, Dr. Huang generated an analytical frame for identify participating teachers’ documentational genesis. The frame focuses on the actors’ roles in meetings and classrooms (researcher, teachers, and students) surrounding the interactions (enactment or reflection) between the Domain of Inquiry and each of other three domains. Four research assistants watched the videos (meetings and lessons) and took notes independently. After watching each video, the researcher and the assistants had a group meeting to identify the major features of document work and document system at each individual phase. Any disagreements during the meeting were resolved through extensive discussion. The documentational genesis is summarized in a Table 1. The same process was applied to analyze all the videos in different stages. Finally, a diagram for presenting teachers' documentation work during the LS (Trouche et al., 2019) was created, which includes five major stages for the teachers' documentation works in line with the four phases of the LS. In the Results that follow, the document system and documentational genesis will be presented in narrative and documentational trajectory development over the stages will be summarized.

Results

The results are presented in five stages in line with the four lesson study phases. These are: 0)(pre-lesson study): Previously developed materials by TRG and e-resources identified by the teachers to support their inquiry into lesson design before the lesson study; 1): the resources provided by the researcher and misconceptions occurred on the pretest promoting the teachers’ consideration of concrete manipulatives used for developing a lesson plan; 2): The resources used to produce and revise the lesson plan through the collective inquiry, and the questions from the researcher eliciting the teachers’ rethinking about learning goals of the lesson; 3): The enactment of research lesson facilitating the teachers’ reflections on students’ learning, resulting in further refinement of the lesson plan; 4): Reenacting the research lesson facilitating the teachers’ reflection on how to use students’ misconception as teaching resources. The following narratives demonstrate how the results are supported by empirical data.

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

Stage 2: The resources used to produce and revise the lesson plan through the collective inquiry, and the questions from the researcher eliciting the teachers' rethink the lesson goals.

Documentational genesis. At this stage, the researcher's ideas (ED) impacted on the teachers' inquiry. The researcher questioned whether recognizing acute, right and obtuse angles should be included as one of the learning objectives in this lesson. He believed that the most challenging part is to establish the concept of the size of the angle, which lays the foundation for comparing the size of the angle and identifying acute, right and obtuse angles. Inspired by the researcher's question, the teachers further discussed and agreed that after developing the concept of the size of the angle, angles could be compared, and acute and obtuse angles could be developed through comparing with right angles. Mr. Tan believed that students could perceive that the size of angle is related to the openness of the angle but are difficult to understand that it is irrelevant the length of the two sides. They finally agreed on the use of strips to construct angles and to help students develop the concept of the measure of angles through manipulative activities: (1) using the strips (with same length) helps students to experience that the rotation of one side of an angle creates different angles, and to recognize the two perspective toward an angle: a shape (formed by two rays from the same vertex, static) and a rotation (process of generating the angle, dynamic); (2) using the strips with different lengths to construct angles with in the same and/or different sizes to help students realize that the length of the sides of angle is nothing to do with the measure of the angle.

In summary, the triggered by the researcher's questioning, the teachers collectively discussed their ways of helping students' development of the concept, finally resulted in a new design of lesson plan. Thus, the collective domain (teachers' knowledge and beliefs about teaching the topic) changed and the domain of consequence developed, which contributed to the development of their document systems (see Table 1).

Table 1. The teachers' documentational genesis during the Stage 2

Documentational Genesis			
Domain of Inquiry (DI)	Teacher (CD)	Reflection	The teachers' inquiry facilitated them to develop their knowledge of the measure of an angle and beliefs concerning with teaching and learning of the topic.
	Resource (ED)	Enactment	The researcher's question resulted in the teachers' inquiry into the objective of the lesson. The concrete model also impacted teachers' inquiry into designing tasks for student learning
	Document (DC)	Enactment	The teachers' inquiry resulted in the new design of the lesson plan (new objectives of learning and newly designed tasks).

Document system. In this stage, the analysis of documentational genesis (Table 1) shows the teachers' documentation work. Firstly, there is the *instrumentation* process, where the researcher's question as a Resource directly triggers the Domain of Inquiry (of the LS group), and then Domain of Inquiry facilitated the teachers' rethinking and reflection. Second, it is the *instrumentalization* process, where Mrs. Miao with her colleagues further improved their documentation system with refining the lesson plan. In the revised lesson plan, the learning goals of the lesson focused on recognizing and comparing the size of angles only while removing the content of acute, right and obtuse angles to the next lesson. Correspondingly, they modified the activities to explore the size of the angle based on a more refined learning progression, i.e. from constructing angles in different sizes through manipulatives, to comparing two angles that are close in size using concrete tools.

Summary. Based on the adapted DAD model, this study paints the learning of a group of teachers in a lesson study as the process of their development of document system and uncovers their learning mechanism through documentational genesis. The results show that the teachers' documentational trajectory went through four stages. Teachers become active resource adapters and producers. The teachers' document system has been developed through the reflections and enactment. Both the *instrumentation* and *instrumentalization* process are reflected during the dynamics between the Domain of Inquiry and the other three domains. Overall, the teachers' document system developed from adopting traditional resources (e.g., the textbook and shared materials in TRG) with a focus on delivering of subject content to designing the lesson creatively by adapting both traditional and additional online resources (e.g., online videos and manipulatives) with a focus on student-centered active learning.

Conclusion and discussion

In the section, we highlight the contribution of this study in three aspects as follows:

Document system and its development as an effect of LS

The Chinese LS process provide an effective mechanism for teachers to develop their capacity in instruction design through the enrichment and refinement of their document system as demonstrated in this study. Thus, this study contributes teachers' learning through LS by extending their learning as document system development. Moreover, this study also demonstrates the critical roles played by the researcher. For example, how to provide the resources (both ideas and resources) with teachers, and how to facilitate teacher discussion and reflection by raising meaningful questions. This finding about the critical role of knowledgeable others played in the lesson study is aligned with other studies (Gu & Gu, 2016; Li, 2019).

Enrichment of DAD by uncovering the documentational genesis process

According to DAD, teachers' learning is regarded as the development of document system through the interactions between teachers and resources. The interactions are featured by *instrumentation and instrumentalization* process. Although DAD's theory recognizes a collective perspective and explores the development of teacher document systems within a community of practice (Wenger, 1998), the process of interaction between teachers and resources are not appropriately explored (Trouche et al., 2019). In this study, we examined documentational genesis as a process of enactment and reflection between Domain of Consequence and the other from the IMGP perspective. With the cycle of LS, the documentation genesis process is described in detail, thus, both document system and its development through lesson study are revealed vividly. Therefore, his study contributes to enriching DAD by illustrating the documentational genesis.

Networking theories by locally integrating two theories

The strategies of synthesis and local integration focus on developing theory by combining a small number of theoretical approaches into a new framework (Prediger et al., 2008). In this study, we adopted a locally integrating strategy by incorporating the interactions among four domains in the IMPG model into DAD theory to explore the mechanisms of documentational genesis. As demonstrated in this study, by employing the adopted DAD model, teachers' learning through LS has been extended and described in great detail. Therefore, the locally integrating framework in this study could, on the one hand, explain the mechanisms of teacher documentational genesis and, on the other hand, describe teachers' professional development through the changes of the document system.

Limitations and further studies

Since there is no follow-up data (after the lesson study) nor interviews with the teachers, we are unable to make any claims about how teaching this specific topic can sustain the teaching of other topics. In further studies, these two data sets should be purposefully collected and analyzed to explore teacher document system development through LS.

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224. doi:10.1023/A:1009903206236.

Borko, H., & Potrai, D. (2020). *Teachers of mathematics working and learning in collaborative groups*. The proceeding of ICMI STUDY 25, Feb. 3-7, 2020. Lisbon, Portugal. http://icmistudy25.ie.ulisboa.pt (offline).

Chen, X. (2020). Teacher learning in boundary-crossing lesson study. *Journal of Educational Studies, 16*(02): 47-58. (In Chinese).

Cheung, W. M., & Wong, W. Y. (2014). Does lesson study work? *International Journal for Lesson and Learning Studies, 3*(2), 137-149.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18* (8), 947-967.

Fang, Y. (2017), School-based teaching research and lesson-case Ssudy in mediating the second-cycle curriculum reform in Shanghai. *International Journal for Lesson and Learning Studies, 6* (4), 293-305.

Gu, F., & Gu, L. (2016). Characterizing mathematics teaching research specialists' mentoring in the context of Chinese LS. *ZDM- Mathematics Education, 48* (4), 441-454.

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3), 199-218.

Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven LS. *ZDM Mathematics Education, 48*(3), 425–439.

Huang, R., Takahashi, A., & De Ponte, J. (2019). *Theory and practices of LS in mathematics: An international perspective.* Springer, New York.

Lewis, C. (2016). How does LS improve mathematics instruction? *ZDM- Mathematics Education*, 48 (4), 571-580.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through Lesson Study: a theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Li, X. (2019). An analysis of Chinese lesson study from historical and cultural perspective. In R. Huang, A., Takahashi, & J. da Ponte (Eds.), *Theory and practices of lesson study in mathematics: An international perspective* (pp.201-228). New York, NY: Springer.

Pepin, B., Gueudet, G., & Trouche, L. (2013). Re-sourcing teachers' work and interactions: a collective perspective on resources, their use and transformation, *ZDM Mathematics Education, 45*(7), 929-944

Pepin, B., Xu, B., Trouche, L., & Wang, C. (2017). Developing a deeper understanding of 'mathematics teaching expertise': Chinese mathematics teachers' resource systems as windows into their work and expertise. *Educational studies in Mathematics, 94*(3), 257–274, http://rdcu.be/koXk.

Prediger, S. (2020). Content-specific theory elements for explaining and enhancing teachers' professional growth in collaborative groups. In H. Borko & D. Potari (Eds.), *ICMI Study 25 Conference Proceedings: Teachers of Mathematics Working and Learning in Collaborative Groups* (pp. 2-14). Lisbon, Portugal: University of Lisbon.

Prediger, S., Angelika Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM Mathematics Education, 40*, 165–178. DOI 10.1007/s11858-008-0086-z

Qi, C., Cao, C., & Huang, R. (2021). Teacher learning through collaboration between teachers and researchers: A case study in China. *International Journal of Science and Mathematics education,* 1-20. https://doi.org/10.1007/s10763-021-10241-7

Shinno, Y., & Mizoguchi, T., (2021). Theoretical approaches to teachers' lesson designs involving the adaptation of mathematics textbooks: two cases from Kyouzai Kenkyuu in Japan. *ZDM-Mathematics education, 53*, 1387-1402.

Trouche, L., Gitirana,V., Miyakawa,T., Pepin, B. ,& Wang, C. Y. (2019). Studying mathematics teachers' interactions with curriculum materials through different lenses: Towards a deeper understanding of the processes at stake, *International Journal of Educational Research*, 93, 53-67.
Trouche, L., Gueudet, G., & Pepin, B. (2020). Documentational approach to didactics. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (2nd edition, pp. 307-313). Cham: Springer. doi: 10.1007/978-3-319-77487-9_100011-1
Wei, G., & Huang, R. (2022). Research-practice partnerships in lesson and learning studies: A review from Asian experiences. *Asia Pacific Journal of Education*, https://DIO.org/ 10.1080/02188791.2022.203187.
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. Cambridge, England: Cambridge University Press.
Widjaja, W., Vale, C., Groves, S. and Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education*, 20 (4), 357–383.
Willems, I., & Bossche, P. V. D. (2019). Lesson study effectiveness for teachers' professional learning: A best evidence synthesis. *International Journal for Lesson and Learning Studies, 8*(4), 257-271.
Yang, Y. (2009). How a Chinese teacher improved classroom teaching in Teaching Research Group: a case study on Pythagoras theorem teaching in Shanghai. *ZDM Mathematics Education, 41*, 279–296.

DEVELOPING EMPATHY THROUGH EPISTEMIC ACTIVITIES: TEACHERS' EXPERIENCES LIVING AND LEARNING IN BASE 7

Marc Husband	Evan Throop Robinson	Lisa Lunney Borden
St. Francis Xavier University	St. Francis Xavier University	St. Francis Xavier University
mhusband@stfx.ca	erobinso@stfx.ca	lborden@stfx.ca

This research report describes new insights into how teachers develop empathy for learners and build capacity for more responsive teaching. To develop empathy, we introduce an epistemic activity for teachers involving an unfamiliar base system that promotes the construction, communication, and critique of knowledge. Teacher responses from two professional learning groups provided data for analysis. A phenomenographic approach was used to analyze survey data and report teachers' experiences. This study suggests that teachers developed empathy—putting themselves into a students' sense-making experience and seeing the value in their work. Additionally, specific interactions that arose from the investigation influenced teachers' future teaching practices. Results show that teachers recognize and appreciate students' feelings and desire further sense-making opportunities for their future practice.

Keywords: Professional Development, Teacher Noticing, Affect, Emotion, Beliefs, and Attitudes

Objectives

What does it feel like to be a learner of mathematics? How can teachers build empathy for the learning experiences of their students? This research report answers calls in the literature for more course activities that provoke pre-service teachers' expression of empathy—seeing the merits of how students make sense of mathematics (e.g., Jaber, Southerland, & Drake, 2018). In our own work, we have also considered activities that aim to build empathy with in-service teachers. The purpose of this research report is to describe a study involving teachers' experiences solving one addition question: 13 + 6 using a base 7 place-value system. In our activity, we used the context of "living on Mars" to create dissonance and support teachers as they transitioned to thinking and working in an unfamiliar base system. This is important because normally teachers are familiar with the math they are aiming to teach in their classrooms. Mathematics education researchers (e.g., Lin & Hsu, 2018; Zazkis, 1999) recommend teachers work on questions in different base number systems because the experience puts them in a students' shoes and thus can develop empathy for mathematics learners. PMENA has invited us to consider notions of dissonance and harmony. This study aims to help teachers develop empathy through experiencing their own sense of dissonance (working in an unfamiliar base system) to appreciate the journey to harmony (meaningful learning). We believe such experiences help to build teacher capacity to support the learning conditions for all students.

Developing empathy for how students grow mathematics understanding is an important part of teacher education because of its promise for supporting teachers to be more responsive in their practice (Jaber et al., 2022). Research in mathematics education often associates responsive teaching with developing a practice of noticing. Teacher noticing has become a highly valued practice for effective teaching of mathematics because of its potential to reframe what a teacher looks for, interprets, and takes up in the classroom (Jacobs, Lamb & Philipp, 2010). For example, responding to student thinking by noticing student strengths (Jilk, 2016) or identifying moments in a lesson that are conducive to building on student thinking (Leatham, Peterson, Stockero, & Van Zoest, 2015). Yet, responding to students' ideas in the moment is challenging for teachers because of the unpredictability of student thinking (Foster, 2014; Rowland & Zazkis, 2013). The question guiding this research is: *How do educators respond to activities that aim to help them to develop epistemic empathy?*

To prepare for lessons that are responsive to mathematical thinking emerging from activities, teachers are encouraged to anticipate multiple solution strategies, listen to and observe students' ideas during activities, and respond by selecting student samples that orchestrate discussion around a lesson goal (Smith & Stein, 2018). While a lesson goal may be the focus when aiming to develop content, in this study, educators were more inclined to think about the kinds of emotions taking place in the classroom and the teacher moves to facilitate rich learning experiences in their own work. While the lesson activity elicited empathy for students' sense-making experiences in number, it also provoked ways that teachers might respond to student thinking by anticipating actions they can use with future learners.

Conceptual Perspective

To frame this study, we draw on the concept of epistemic empathy. Educational researchers (e.g. Horsthemke, 2015) have previously used this concept to talk about knowledge development. Specifically, for mathematics and science educators, Jaber et al., (2018) define *epistemic empathy* as "the act of understanding and appreciating someone's cognitive and emotional experience within an epistemic activity, meaning an activity aimed at the construction, communication, and critique of knowledge" (p.14). A conceptual perspective involving empathy fits well with the objective of elementary teachers learning in a different number system because the experiences aim to grow their perspectives on how students develop mathematical understanding in our base 10 system. For example, teachers working on questions in a base 7 place value system can help build a teacher's appreciation for students' "cognitive and emotional experience" by putting themselves in students' shoes to learn how adding works in an unfamiliar base system. In terms of the epistemic activity, the context of learning in a base 7 world (e.g., Mars) for adding 13 and 6 aimed to provoke the following actions: a) working in small groups to construct multiple solution strategies and representations, (the construction) b) engaging in peer-to-peer and peer-to-instructor dialogue (the communication) and c) developing and critiquing mathematically convincing arguments (the critique).

Methods

This study used a phenomenographic approach that collects data to study participants' experiences and perceptions of that experience. Mason (2002) says that "[t]he aim [of phenomenography] is to describe and characterize different ways of experiencing" (p.162). Phenomenography is a qualitative research approach with a long history in higher education. The approach develops qualitatively different categories of experiencing a phenomenon from the collected data through an iterative process (Kinnunen & Simon, 2012). Data is derived from a wide range of experiences/conceptions of the same phenomenon. This is accomplished by asking people to describe, in their own words, the phenomenon and how they experienced/conceptualized it personally. In the sessions, we provided the teachers with the following reflective question: *How does working in a base 7 context influence your practice?* The first author started the analysis by identifying statements related to the research question. As we examined the data, we began to see trends in what teachers shared. Our first iteration of the analysis showed largely that responses were either emotional or pedagogical in nature. These themes are described in this paper; yet we intend to deepen our analysis through further iterations to examine themes within these broader categories.

Data Sources:

We generated data from two sources: one group of 20 math educators, coaches and mentors participating in monthly virtual professional learning (VPL) meetings and another group of fifteen elementary teachers participating in a 10-day in-person professional learning (IPL) course. All teachers in both groups agreed to participate in the study. In the second year of the VPL group, teachers worked individually on 13 + 6 in base 7 before sharing solution strategies with the whole group. The teachers in the IPL group were randomly assigned to blackboard workspaces in five groups of three on the third day of the course. Teacher reflections on their experiences working with base 7 were collected as written data (IPL) and audio/video data (VPL) and analyzed for this research.

Results

Many teachers in the VPL and IPL mingled in base 10 while working with base 7 during the investigation. In other words, the task challenged them to use what they knew about adding in base 10 in novel ways. In both groups two strategies emerged as possible solutions (see figure 1): a) Mingling in base 10, the teacher moves one unit from the 6 to make 14—two 7's in 14 and 5 more is 25; and b) The teacher uses regrouping and/or counting strategies in base 7 to get 22.

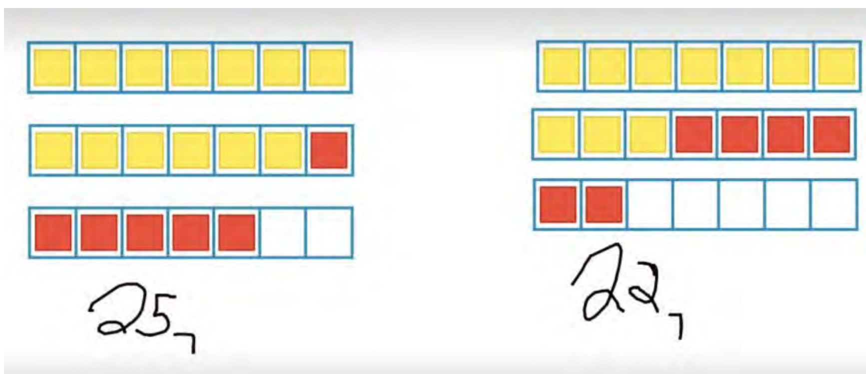


Figure 1: Two solution strategies for 13 + 6 in base 7

Based on both groups of teachers' oral and written reflections working with base 7, the following two categories emerged from analyzing the data: (1) recognizing and appreciating students' feelings in learning mathematics, and (2) desiring similar sense-making opportunities for future students. These categories are related insofar as both involve valuing students' mathematical thinking, but they have distinct features. The first category focuses on, identifies, and talks about feelings that emerge through their process of determining an answer, often aligned with dissonance. For example, the following comment made by a teacher in the IPL group talks about the disappointment she felt from realizing her answer was incorrect, "After finding out my answer was actually incorrect, I experienced a sense of disappointment because I guess I really wanted to be right and was proud of my group for going through so many steps to get to where we did." She elaborated on how her own feelings of disappointment made her think about the ways "students will feel disappointed when they have worked so hard and end up getting the incorrect answer." Here, she recognizing the dissonance her students experience as she builds empathy. In a similar way, a teacher in the VPL group said that the experience "put [her]self in the place of kids and what they might be struggling" to understand. She went on to connect her feelings about struggling in the base 7 activity with "what [students are] going through every day when we are trying to teach place value". Again, this recognizes dissonance.

The second category is distinctly different because comments under this category identify teaching actions that support sense-making and share the characteristic of teachers' desiring similar learning opportunities for future students. In other words, they are seeking harmony. For example, a teacher in the IPL group commented that peer "dialogue, disagreements, and prompts from our teacher" provided opportunities to "self-correct…misconceptions" that their group faced. This teacher goes on to say that she intends to provide "similar experiences for [her] students to engage in discourse with their peers with building new mathematical learning." Engaging with peer ideas that are different from your own was also important to a teacher in the VPL group. After comparing his work to how his colleagues "approached the [base 7] problem differently," he said that students also need "opportunities to see how someone did something different [because] it gives you a more holistic approach to solving math problems in general." These moves promote harmony seeking.

Discussion

Through creating dissonance for teachers, we put them in a situation where they needed to strive for harmony. Working in base 7 allowed teachers to develop empathy for appreciating students' cognitive (e.g., getting an incorrect answer, understanding) and emotional experiences (e.g., feeling disappointed, struggling). Teachers' comments also valued the interactions they experienced while solving the question (e.g., peer dialogue) and expressed a desire to provide similar experiences for their future students (e.g., comparing different strategies). This experience provoked teachers to envision themselves enacting practices like the ones they experienced as learners during the activity. Instead of preparing to respond to students' ideas by anticipating multiple solution strategies (Smith & Stein, 2018), teachers in this study prepared to respond to students' mathematical ideas by anticipating teaching actions like peer dialoguing and using different strategies for comparing and analyzing student work. Of particular significance in this study is how the nature of living and learning in base 7 supported Jaber et al.'s, (2018) call for more course activities that provoke expressions of epistemic empathy and extended this to in-service teachers. That is, the base 7 task initiated the construction of multiple solution strategies, communication between peers and their instructor, and a critique of two different solutions that emerged during the investigation. Overall, we see that this work put teachers in a state of dissonance that invited them to seek harmony in much the same way their students experience learning. This develops epistemic empathy. While the epistemic empathy provides teachers with a perspective on how their students are experiencing learning, engaging in this task also provided teachers with insight into pedagogical practices that can move their students from dissonance to harmony.

References

Foster, C. (2014). Minimal interventions in the teaching of mathematics. *European Journal of Science and Mathematics Education, 2*(3), 147-154.

Horsthemke, K. (2015). Epistemic empathy in childrearing and education. *Ethics and Education*, *10*(1), 61-72. https://doi.org/10.1080/17449642.2014.998025

Jaber, L. Z., Southerland, S., & Dake, F. (2018). Cultivating epistemic empathy in preservice teacher education. *Teaching and Teacher Education*, *72*, 13-23. https://doi.org/10.1016/j.tate.2018.02.009

Jaber, L. Z., Dini, V., & Hammer, D. (2022). "Well that's how the kids feel!"—Epistemic empathy as a driver of responsive teaching. *Journal of Research in Science Teaching*, *59*(2), 223-251. https://doi.org/10.1002/tea.21726

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for research in mathematics education, 41*(2), 169-202.

Jilk, L. M. (2016). Supporting teacher noticing of students' mathematical strengths. *Mathematics Teacher Educator*, *4*(2), 188-199. https://doi.org/10.5951/mathteaceduc.4.2.0188

Kinnunen, P., & Simon, B. (2012). Phenomenography and grounded theory as research methods in computing education research field. *Computer Science Education, 22*(2), https://doi.org/10.1080/08993408.2012.692928

199-218.Leatham, K. R., Peterson, B. E., Stockero, S. L, & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for research in mathematics education, 46*(1), 88-124. https://doi.org/10.5951/jresematheduc.46.1.0088

Lin, F. L., & Hsu, H. Y. (2018). Using mathematics-pedagogy tasks to facilitate the professional growth of pre-service elementary teachers. In *Research Advances in the Mathematical Education of Pre-service Elementary Teachers* (pp. 3-17). Springer, Cham. http://dx.doi.org/10.1007/978-3-319-68342-3_1

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.

Rowland, T., & Zazkis, R. (2013). Contingency in the mathematics classroom: Opportunities taken and opportunities missed. *Canadian Journal of Science, Mathematics and Technology Education*, *13*(2), 137-153. https://doi.org/10.1080/14926156.2013.784825

Smith, M. & Stein, M. K. (2018). *5 Practices for orchestrating productive mathematics discussion*. National Council of Teachers of Mathematics.

Zazkis, R. (1999). Challenging basic assumptions: Mathematical experiences for pre-service teachers. *International Journal of Mathematical Education in Science and Technology*, *30*(5), 631-650. https://doi.org/10.1080/002073999287644

ACKNOWLEDGMENT AND ACTION: TEACHERS' FIRST ACT TOWARDS AVENUES FOR SOCIAL JUSTICE WORK

Candace Joswick
The University of Texas at Arlington
candace.joswick@uta.edu

Robin Anderson
North Carolina State University
randers6@ncsu.edu

Melissa Troudt
University of Wisconsin – Eau Claire
troudtml@uwec.edu

Lisa Skultety
University of Central Arkansas
lskultety@uca.edu

As mathematics teacher educators working to better understand how to train teachers to enact justice-oriented practices, we situate our work in the guiding question: "How does your work have an impact on society more broadly, beyond individual mathematics classrooms and school districts?" Results from our development and implementation of a semester-long course for teachers shows that teachers can be supported to find reasonable avenues for action to take up in their classrooms, though they begin without a sense of empowerment or understanding of resources to do so. Our teacher participants are eager to leverage their unique knowledge and role in their contexts to advocate for the changes they envision—changes that create more socially just contexts for learning and teaching mathematics.

Keywords: Social Justice, Teacher Educators, Equity, Inclusion, and Diversity, Instructional Vision

Introduction

Reform documents (e.g., NCTM, 1989; 1991; 2000; 2006) call for making classrooms more equitable in the sense of making mathematics accessible to all students. Many have argued (e.g., Gutstein, 2003; NCSM & TODOS, 2016), it is also necessary to address broader social, political, and ideological issues present in classrooms, schools, and communities to reach these content goals. Mathematics teacher educators (MTEs) are tasked with fostering well-prepared teachers who "are equipped to question existing educational systems that produce inequitable learning outcomes for students" and are able "to advocate for themselves and challenge the status quo on behalf of their students (AMTE, 2017, p. 21-24)." Practicing classroom teachers bring an abundance of expertise and knowledge of their students, content, profession, and communities. However, due to their perceptions of their role and responsibilities, some teachers feel ill-equipped or powerless when working to make changes within and beyond their classrooms (Picower, 2011; Raygoza, 2016). MTEs can come alongside such teachers to support them to leverage their unique knowledge and positions to advocate for the changes they envision.

We describe a graduate-level course for in-service teachers intended to interrogate issues of justice in teaching. Teachers entered the course with a desire to make their classroom instruction more equitable but had not always found opportunities to do so. As described by one teacher-participant, Josh (all teacher names are pseudonyms):

> Given the structures of state-mandated high stakes testing, funding that is tied to those test scores, and the tyranny of centrally designed scope & sequence requirements, teachers barely have time to find new research-based ideas for their students let alone the freedom to implement strategies on their own.

Given this starting point, the course aimed to support the teachers' inquiry and actions into their own contexts. The purpose of this brief report is to disseminate the opportunities teachers found to envision and act after being supported with structures and tools to investigate their positionality, settings, and goals. Specifically, we ask, *when teachers are provided support to investigate their positionality, settings, and goals in relation to social justice, what avenues for action do they take up in their classrooms?*

Framing

NCSM and TODOS (2016) propose a three-stage cycle of acknowledgement, action, and accountability (AAA) to tackle issues of inequity. They call for those working in the field of mathematics education to use the AAA cycle to first acknowledge that mathematics education is unjust, then they call for action to "re-frame, re-conceptualize, intervene, and transform" mathematics education, and finally they demand accountability to ensure change endures (p. 1). By implementing an AAA cycle to promote change, teachers look to address harmful practices that range from issues in their individual classroom to school or district policies.

When addressing harmful practices teachers must understand how their positionality is impacted by contextual and historical factors (Tien, 2019). Teachers must be supported to not only understand their individual identities and their impact within the context they teach, but they must also know how the socio-political histories have shaped the experiences and issues they seek to dismantle. By developing political knowledge for teaching (Gutiérrez, 2013), teachers begin to see how historical, political, and ideological factors interact. This awareness allows for teachers to implement justice-oriented pedagogies to help students become advocates for social justice. The Learning for Justice Social Justice Standards (2018) provide a framework that teachers draw upon when planning to address injustices within their context. In this paper we draw upon the AAA cycle as a structural framing to analyze how teachers take up the social justice standards to imagine new possibilities in their contexts.

Methods

Positionality

We are all White-female mathematics teacher educators in tenure-track positions teaching undergraduate and graduate mathematics and mathematics education courses. We intentionally name our Whiteness because we are aware of the privilege given to us in a society dominated by White supremacy. Our work has evolved to better understand how to train teachers to enact justice-oriented practices, especially in areas of the United States where such practices may be unwelcomed by people in power. We have continually reflected on how our positionality has influenced the creation and analysis of the course presented in this paper.

Context and Participants

We draw from the AAA cycle for this work, in a three-level, nested way (see Figure 1). At the first level, is our (as the teacher educator-researchers and instructor) use of AAA. Our acknowledgement includes our positionality and activities to determine the needs of our teacher participants. This knowledge, coupled with our analysis of the current climate towards addressing inequities in schools from a global and geographically specific lens, informed our action: the intentional design and implementation of a master's level course for PK-12 practicing teachers. This course–described as an advanced study of and design of curriculum models for diverse learners–was taken by eight students pursuing a master's in education in curriculum and instruction, all of whom are current or former teachers. Roughly split into thirds, each portion of the class concentrated on either the concept of acknowledgement, action, or accountability, as detailed in the middle column of Figure 1. And as a culmination of learning and experience (and course accountability), students in the course ended with an acknowledgment-action-accountability project of their own.

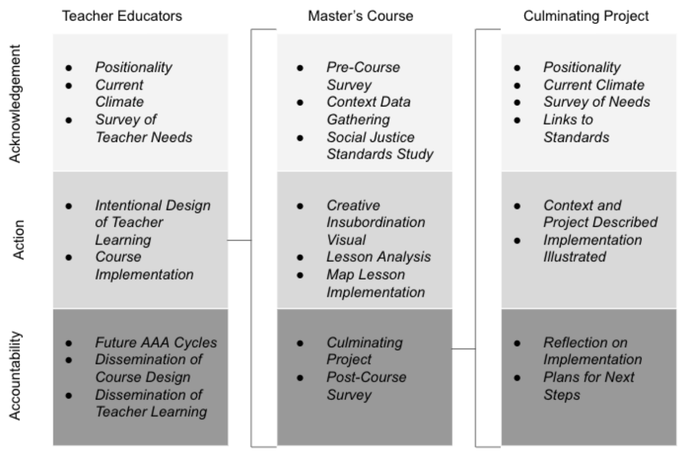


Figure 1: Nested Levels of Acknowledgement, Action, and Accountability

Data and Analysis

For this analysis, we draw data from our teachers' culminating projects (see Figure 1), specifically their acknowledgement and action pieces. The acknowledgement includes a write-up of their positionality, a problem statement, and articulated relationship between their work and the Learning for Justice Social Justice Standards (2018). Using narrative inquiry, "research [which] aims to explore and conceptualize human experience as it is represented in textual form…aiming for an in-depth exploration of the meanings people assign to their experiences" (Salkind, 2010, p. 1004), we look at the interplay between teachers' acknowledgement and action. Due to time constraints of the course, the action and accountability were mostly theoretical and not enacted. Our choice to exclude analysis of the accountability piece is due to the purely theoretical nature of many of the submitted reports. While teachers had time in a single semester to consult peers, administrators, and other community stakeholders to develop their action plans, they did not have time to implement them.

Results

To understand the ways that teachers imagined possible actions of social justice in their teaching, we first identified the Learning for Justice Social Justice Standards (2018) that were named within their culminating projects. Next, we focus on two teachers' specific projects to illustrate how the same standards were addressed in different contexts. Table 1 displays the standard addressed in each teacher's culminating project. The standards are organized by Domain, with most teachers choosing to address diversity in their project, and standard 6 being the most common standard, "Students will express comfort with people who are both similar to and different from them and engage respectfully with all people" (p. 3).

To highlight the different approaches to the standard, we share findings from two teachers, Gloriana and Amy. Gloriana's culminating project proposed the school-wide solution of using restorative circles to address her school's infraction system of discipline. Her experience as a classroom teacher has led her to believe that "student-teacher relationships are heavily involved in students' disciplinary issues and in how they are addressed. A mistake in addressing these issues may harm the student-teacher relationships..." Consequently, Gloriana sought to set up restorative circles to provide students opportunities to express themselves with their peers and teachers in ways that are respectful, productive, and contribute to positive classroom relationships.

Amy's culminating project addressed the same standard, but within her individual classroom. As a high school geometry teacher, Amy shared that her students struggled with conditional statements. Additionally, as an African American woman who grew up in a predominately White space, Amy never had the opportunity to integrate her culture and lived experience into her schooling. She chose to address these two concerns by creating a lesson on conditional statements that guided students to explore the relationship between a statement's hypothesis and conclusion through a cultural lens. Students were directed to create their own statements, such as, "If you were Cesar Chavez, then you dedicated your life to la causa." In a Socratic seminar, students share their statements and share their culture with their classmates.

Table 1: Target Learning for Justice Social Justice Anchor Standard and Domain

Teacher	**Identity**					**Diversity**					**Justice**					**Action**				
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Holly								x				x	x	x			x			
Kida	x	x	x	x	x	x	x													
Gloriana				x		x			x											
Amy						x		x												
Sarah						x				x										
Kaitlyn																	x			
Josh										x										
Caroline	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x

Discussion

We envisioned teachers' first act towards taking up justice-oriented work in their classrooms and contexts as what could be reasonably accomplished in a semester-long gradate course. Using NCSM and TODOS's (2016) AAA cycle as a frame, teachers' culminating projects were an acknowledgement and plan for action. We found teachers' foci on social justice standards varied by their own positionality, contexts, and identified needs of their students and schools. Close examination of teachers' culminating projects show that mathematics educators seek to address social justice at both the classroom level (e.g., Amy) and system levels (e.g., Gloriana), which would directly impact the culture and climate of their mathematics classrooms as well. Teachers' second acts are to come; enactment of their action and accountability pieces. Through this work, we anticipate our teachers will move from being "equipped to question existing educational systems that produce inequitable learning outcomes for students," which we found them to be at the end of the course, to being able "to advocate for themselves and challenge the status quo on behalf of their students" (AMTE, 2017, p. 21-24). While these results are preliminary, we are encouraged by the possibilities of teacher learning experiences that are framed using the AAA cycle towards combating harmful practices and promoting justice-oriented change at both the classroom and systems level.

References

Association of Mathematics Teacher Educators (2017). Standards for Preparing Teachers of Mathematics. Available online at amte.net/standards.

Gutiérrez, R. (2013). Why (urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education*, *6*(2), 7–19.

Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education, 34*(1), 37-73.

Learning for Justice (2018). Social justice standards: The learning for justice anti-bias framework. https://www.learningforjustice.org/sites/default/files/2021-11/LFJ-2111-Social-Justice-Standards-Anti-bias-framework-November-2021-11172021.pdf

National Council of Supervisors of Mathematics and TODOS: Mathematics for ALL. (2016). Mathematics education through the lens of social justice: Acknowledgment, actions, and accountability. [Position statement]. https://www.todos-math.org/assets/docs2016/2016Enews/3.pospaper16_wtodos_8pp.pdf

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics.* NCTM.

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics*. NCTM.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and Standards for School Mathematics.* NCTM.

National Council of Teachers of Mathematics (2006). *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence*. NCTM.

Picower, B. (2011). Resisting compliance: Learning to teach for social justice in a neoliberal context. *Teachers College Record, 113*(5), 1105-1134.

Raygoza, M. C. (2016). Counting the experiences and beliefs of secondary teachers striving to teach mathematics for social justice in urban schools. *Journal of Urban Education, 55*(8-9), 1-30. https://doi.org/10.1177/0042085916672289

Salkind, N. J. (2010). *Encyclopedia of research design*. Thousand Oaks, Calif: SAGE Publications.

Tien, J. (2019). Teaching identity vs. positionality: Dilemmas in social justice education, *Curriculum Inquiry*, *49*(5), 526-550, DOI: 10.1080/03626784.2019.1696150

CORRELATING TEACHERS' ENGAGEMENT IN ONLINE DISCUSSIONS WITH THEIR PERSISTENCE IN PROFESSIONAL DEVELOPMENT

Anthony Matranga
California State University San Marcos
amatranga@csusm.edu

Jason Silverman
Drexel University
Js657@drexel.edu

This paper reports on a correlation analysis of two cohorts of mathematics teachers' patterns of participation in online asynchronous discussions and their persistence in a sequence of professional development workshops. Findings indicate that increased access to colleagues' knowledge resources and more frequent dispersion of these resources across the social network related to persistence. We discuss how variations in the design of our online workshops may have impacted teachers' potential to persist in the professional development. The findings have implications for network-based instructional strategies that could increase mathematics teachers' potential to persist in online professional developments.

Keywords: Professional Development, Online and Distance Education

Objectives and Purpose

Developing discourse-rich instructional practices that are responsive to student mathematical thinking is a challenging process. Learning to implement such practices takes time and is unlikely to occur in "one-shot" professional developments or even a short workshop. Indeed, there is consensus that sustained duration is a feature of effective professional development that supports teacher learning and instructional improvement (Darling-Hammond et al., 2017; Desimone, 2009; Dille & Røkenes, 2021; Sztajn et al., 2017; Sancar, Atal, & Deryakulu, 2021). Online settings afford sustained professional development because they allow for flexibility in when, where, and how teachers participate (Fletcher et al., 2007). Further, online settings are less susceptible to local norms and instructional practices that can influence the focus of teacher collaboration (Cobb et al., 2001) and have counterproductive effects on their instruction (Munter & Wilhelm, 2020). Despite the importance of sustained professional development and the benefits of online settings for supporting teacher collaborative learning, little is known about the extent to which teachers persist in online professional development and how various factors, events, and conditions impact their persistence. In this paper, we report on our examination of relationships between patterns in teachers' interactions in our online professional development and their persistence in the workshops.

Theoretical Framework

We draw from social capital theory to frame our investigation of teachers' persistence in online professional development. A central assumption of social capital theory is that individuals have social, cultural, and intellectual resources and increased access to these resources through participation in a social network has many benefits for "upward mobility" (Burt, 1992; Coleman, 1988). Resources flow through a social network as individuals interact with one another (Light & Moody, 2020) and different positions in a network can support or constrain individuals' actions and access to resources (Burt, 2004). For example, those who emerge as *central* members of a network by interacting with a large proportion of colleagues have increased access to resources and a role in dispersing them throughout a network (Freeman, 1979).

Although there is no research we are aware of investigating relationships between teachers' centrality and persistence in professional development, studies of undergraduate education have illuminated such relationships (Dawson, 2008; Thomas, 2000; Tinto, 1975). For example, Zwolak et al. (2017) found that students with higher centrality in an undergraduate physics class social network were more likely to enroll in the next class in the program. Furthermore, in a randomized control trial, Turetsky et al. (2020) implemented an intervention that supported students in becoming more central in a "gateway" bioscience course social network. Students in the intervention group that increased their centrality were more likely than the control group students to take the next course in the bioscience sequence. Drawing from these studies and social capital theory, we conjecture that teachers' increased access to and engagement with colleagues' resources in online discussions could impact teachers' overall interest in professional development and potential to persist. The current study takes a first step at investigating this conjecture by correlating teachers' centrality in online asynchronous discussion forum conversations with their persistence in a sequence of three online workshops.

Methodology

This research was conducted as part of an online synchronous and asynchronous professional development that featured a sequence of three 6-week workshops. The workshop goals included supporting mathematics teachers in shifting towards more student- and discourse-centered instructional practice by engaging in problem-solving, examining student math work, and connecting these experiences to models of mathematics lessons, including the "Launch, Explore, Summarize" model, 5 Practices for Orchestrating Productive Mathematics Discussions, and the "5-E" model. Each workshop built on the previous one by further connecting teachers' work to their classroom practice. Workshop one included four week-long online asynchronous discussion forum conversations that connected teachers' collaborative problem-solving and analysis of student work to their classroom practice. These discussions prompted participants to make an initial post and to submit at least two response posts to their colleagues. Workshops two and three used teachers' collaboration in workshop one as a context for thinking about how to provide effective feedback to students.

We recruited two cohorts of participants to complete the professional development workshops. The first cohort (C1) included 16 participants who completed workshop one together in the spring of 2021. The second cohort (C2) included 10 participants who completed workshop one together in the summer of 2021. The primary difference between the spring and summer version of workshop one was that the spring 2021 version was six weeks and the summer version was two weeks. We implemented the accelerated summer workshop so that participants could potentially take two workshops (one and two) in the summer. We offered the second workshop in summer 2021 to both cohorts and 13 of the 26 participants completed the workshop. We offered the third workshop in the fall of 2021 to the 13 participants who completed workshop two. Nine of these 13 participants completed the third workshop. The 26 participants have between 4 and 9+ years of teaching experience across K-16 grade bands and their geographical locations vary across the United States. 75% of the participants identified as female and 25% identified as male. The racial breakdown of participants is as follows: 70% White, 18% Black, 2% Latino, 8% Asian, 2% Native Pacific Islanders.

The two data sources for this study included participants' interactions from workshop one discussion forums and persistence data across the three workshops. We defined an interaction as an occasion where a participant responded to their colleague's post on the discussion forum. We modeled this collaboration with two social networks (one for C1 and one for C2), where the nodes represented participants and directed edges represented a response post from one participant to another. Using social network analysis, we generated the following basic network characteristics for C1 and C2. C1: *edges*=153, *density* (the proportion of edges to possible edges)=52%, and *reciprocity* (the proportion of interactions that are reciprocated)=50%. C2: *edges*=77, *density*=56%, and *reciprocity*= 48%. We used a three-point scheme to denote participants' persistence in the workshops. Specifically, if a participant only completed workshop one (did not persist), we denoted their persistence with a zero. We denoted completion of workshop one and two with a one and completion of workshop one, two, and three with a two. We imported the interactional and persistence data into R (R Core Team, 2021) and used the statnet package (Butts, 2020) to conduct the analysis.

The analysis involved measuring participants' centrality in workshop one social networks and then correlating centrality measures to their persistence. Given the lack of research on teachers' persistence in online professional development, we used a range of centrality procedures such as out-degree (the number of posts sent), in-degree (the number of posts received), betweenness (a measure of dispersing information across the network), out-closeness (a measure of access to other's posts), and in-closeness (a measure of other's access to one's posts). Table 2 provides an overview of several participants' centrality measures (top and bottom rank ordered according to outdegree). Finally, we calculated the Pearson correlation coefficient and corresponding significance levels between participants' centrality measures according to their cohort and their persistence in the sequence of workshops.

Table 1: Centrality and Persistence Data

Cohort	Participant	Outdegree	Indegree	Betweenness	Out-Closeness	In-Closeness	Persistence
C1	P1	17	9	0.100788	0.761905	0.571429	1
C1	P2	15	16	0.079033	0.695652	0.666667	2
C1	P15	2	6	0.016052	0.457143	0.551724	0
C1	P16	1	2	0	0.410256	0.470588	0
C2	P17	18	7	0.131638	0.916667	0.55	0
C2	P18	15	9	0.389121	0.846154	0.785714	1
C2	P26	1	4	0	0.37931	0.578947	2
C2	P27	1	4	0.004798	0.478261	0.578947	0

Results

Table 2 presents results from the correlation analysis. We found a positive and significant correlation ($p<.05$) between cohort one's betweenness centrality and their persistence in the sequence of workshops. Betweenness centrality quantifies the number of times a participant falls on the shortest path (in terms of connections) between two other, unconnected participants in a network. In online discussions, this means that a participant with high betweenness centrality is responding to posts authored by participants who are not communicating with one another and increasing the dispersion of knowledge resources across the network. Thus, this correlation means that cohort one participants who more frequently dispersed knowledge resources across the network in workshop one were more likely to persist in the sequence of workshops.

We also found a positive and significant correlation between cohort one's out-closeness centrality and their persistence in the sequence of workshops. Out-closeness centrality is calculated by taking the sum of the reciprocal of participants' shortest outwardly directed path lengths (the number of "jumps" one makes in a network to get to their peers). In online discussions, this means that a participant with high out-closeness centrality has increased access to colleagues' knowledge resources through direct engagement with their posts or engagement in a thread initiated by a peer who they did not directly interact with. Thus, this correlation means that cohort one participants who had increased access to and engagement with colleagues' knowledge resources in workshop one were more likely to persist in the sequence of workshops.

Table 2: Correlation of SNA Measure and Persistence

SNA Measure	Cohort 1	Cohort 2	SNA Measure	Cohort 1	Cohort 2
Out-degree	0.4953055	-0.4804152	Betweenness	0.5720269*	-0.5694836
In-degree	0.4300999	0.04415746	Out-Closeness	0.5010315*	-0.5337503
			In-closeness	0.4885712	-0.2476399

*p<.05

Discussion

This report provides initial support for our conjecture regarding the importance of teachers' social capital through access to and engagement with peers' knowledge resources for their persistence in online professional development. These findings suggest that there are implications for central participation in a network that are in addition to those documented elsewhere (e.g., academic success (Saqr et al., 2020), development of MKT (Silverman, 2012)). Further, our findings extend research correlating centrality in discussion networks with persistence in college to the context of online mathematics teacher education. Establishing a research base that documents factors, conditions, and events that impact teachers' persistence in professional development can inform design and implementation strategies. For example, we plan to test a strategy for impacting teachers' persistence that includes increasing peripheral participants' betweenness centrality by prompting them to share connections they notice in colleagues' ideas on a discussion forum.

Our findings also showed that centrality was not correlated to persistence for cohort two. One explanation for this could be differences in the timeframes of workshop one. As noted above, cohort one participated in a 6-week version while cohort two participated in a 2-week version of workshop one. Our ongoing analysis indicates that the structure of cohort two's network was less reflective of a community than cohort one's network. Even though there was a range in centrality measures for cohort two, the accelerated timeframe may have reduced opportunities for teachers to engage deeply with their peers' resources; thus, impacting how they perceived the potential benefits of participation and perhaps their "sense of belonging" (Baumeister, & Leary, 1995). This opens up a question about how the length and quality of initial professional development experiences impact teachers' potential to persist in the professional development.

As the need remains for sustained and effective professional development that supports mathematics teachers in developing discourse-rich and student-centered instructional strategies, additional research is required to gain deeper insight into factors, events and conditions that impact teachers' persistence in professional development. One future area of inquiry that could extend this work might include investigating relationships between teachers' centrality in online discussions, individual attributes (i.e., gender, MKT, etc.), network structures (e.g., cliques (Joksimović et al., 2016)), and persistence. We are currently exploring statistical network models such as exponential random graph models to investigate these relationships (e.g., see Bjorklund & Daly, 2021; Lusher et al., 2013).

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. 1222355 and 2010306. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The Authors would like to thank Valerie Klein and Wesley Shumar of Drexel University for their ongoing collaboration and support of the research described in this paper.

References

Baumeister, R. F., & Leary, M. R. (1995). The need to belong: desire for interpersonal attachments as a fundamental human motivation. *Psychological Bulletin*, *117*(3), 497.

Bjorklund Jr, P., & Daly, A. J. (2021). The ties that belong: tie formation in preservice teacher identification networks. *Teaching and Teacher Education*, *97*, 103223.

Burt, R. S. (1992). *Structural holes*. Harvard University Press.

Burt, R. S. (2004). Structural holes and good ideas. *American Journal of Sociology, 110*, 349-399.

Butts, C. T. (2020). *Network: Classes for Relational Data*. The Statnet Project (http://www.statnet.org). https://CRAN.R-project.org/package=network.

Cobb, P., McClain, K., Lamberg, T., & d Dean, C. (2003). Situating teachers' instructional practices in the institutional setting of the school and district. *Educational Researcher*, *32*(6), 13–24.

Coleman, J. S. (1988). Social capital in the creation of human capital. *American Journal of Sociology*, *94*, S95-S120.

Darling-Hammond, L., Hyler, M. E., Gardner, M. (2017). *Effective teacher professional development*. Palo Alto, CA: Learning Policy Institute.

Dawson, S. (2008). A study of the relationship between student social networks and sense of community. *Journal of Educational Technology & Society*, *11*(3), 224-238.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181–199.

Dille, K. B., & Røkenes, F. M. (2021). Teachers' professional development in formal online communities: A scoping review. *Teaching and Teacher Education*, *105*, 103431.

Fletcher, J. D., Tobias, S., & Wisher, R. A. (2007). Learning anytime, anywhere: advanced distributed learning and the changing face of education. *Educational Researcher, 36*(2), 96–102.

Freeman, L. C. (1978). Centrality in social networks conceptual clarification. *Social networks*, *1*(3), 215-239.

Joksimović, S., Manataki, A., Gašević, D., Dawson, S., Kovanović, V., & De Kereki, I. F. (2016, April). Translating network position into performance: importance of centrality in different network configurations. In *Proceedings of the sixth international conference on learning analytics & knowledge*(pp. 314-323).

Light, R., & Moody, J. (2020). *The oxford handbook of social network analysis*. Oxford University Press, New York.

Lusher, D., Koskinen, J., & Robins, G. (2013). *Exponential random graph models for social network analysis: Theory, methods, and applications*. Cambridge University Press, New York.

Munter, C., & Wilhelm, A. G. (2021). Mathematics teachers' knowledge, networks, practice, and change in instructional visions. *Journal of Teacher Education*, *72*(3), 342-354.

R Core Team (2021). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. URL https://www.R-project.org/.

Thomas, S. L. (2000). Ties that bind: A social network approach to understanding student integration and persistence. *The Journal of Higher Education*, *71*(5), 591-615.

Tinto, V. (1975). Dropout from higher education: A theoretical synthesis of recent research. *Review of Educational Research*, *45*(1), 89-125.

Turetsky, K. M., Purdie-Greenaway, V., Cook, J. E., Curley, J. P., & Cohen, G. L. (2020). A psychological intervention strengthens students' peer social networks and promotes persistence in STEM. *Science Advances*, *6*(45), eaba9221.

Sancar, R., Atal, D., & Deryakulu, D. (2021). A new framework for teachers' professional development. *Teaching and Teacher Education*, *101*, 103305.

Saqr, M., Viberg, O., & Vartiainen, H. (2020). Capturing the participation and social dimensions of computer-supported collaborative learning through social network analysis: which method and measures matter?. *International Journal of Computer-Supported Collaborative Learning*, *15*(2), 227-248.

Silverman, J. (2012). Exploring the Relationship between Teachers Prominence in Online Collaboration and the Development of Mathematical Content Knowledge for Teaching. *Journal of Technology and Teacher Education, 20*(1), 47-69.

Sztajn, P., Borko, H., & Smith, T. M. (2017). Research on mathematics professional development. In C. J. (Ed.), *Compendium for research in mathematics education* (pp. 793–823). Reston, VA: National Council of Teachers of Mathematics.

Zwolak, J., Dou, R., Williams, E., & Brewe, E. (2017). Students' network integration as a predictor of persistence in introductory physics courses. *Physical Review: Physics Education Research 13*(1), 010113.

JAPANESE INSTRUCTIONAL CIRCLES

Matthew Melville
Purdue University Fort Wayne
Melbob@udel.edu

The purpose of this report is to elucidate a common professional learning opportunity in Japan, instructional circles, that has not yet been studied systematically or described in detail. Through an in-depth look at these instructional circles, I provide a detailed description of how Japanese teachers identify instructional problems and how they analyze them as a group to improve their teaching practices. Data was gathered through an ethnographic approach to better understand how these instructional circles function for the teacher participants. Categories for observation were based on literature about professional development and my own experiences with Japanese professional learning. Learning about instructional circles can also provide a pool of new ideas about professional learning. Understanding new ways of doing things, and the conditions in which they operate might help us whether and how we want to adapt them for our own practices.

Keywords: Teacher Knowledge, Instructional Activities and Practices, Mathematical Knowledge for Teaching

Introduction

In the last two decades, Japan has had a strong performance in international mathematics assessments. Beginning with early international studies of mathematics and science (FIMSS, SIMSS), to more recent iterations of the study International (TIMSS), Japan has scored in the top five spots in both the fourth grade and eighth-grade mathematics assessments (McKnight, 1987; Hiebert et al., 2005, McFarland et al., 2017). Because Japan has consistently performed high in these assessments, its instructional systems have become the focus of many studies.

Researchers have attempted to understand instruction through studying various aspects of Japanese educational systems. Some of the aspects that have been studied include examining classroom structure (Stigler & Hiebert, 1999), looking at the influences and usage of textbooks and other curriculum materials (Melville, 2018), and highlighting the quality of instructional features (e.g., Hiebert et al., 2005; Corey et al., 2010). Although instruction is only one factor that might account for high achievement, researchers have identified patterns in instruction in these countries that are often aligned with recommendations for best practices (Stigler & Hiebert, 1999; Takahashi, 2006; Watanabe et al., 2008).

Learning about the teaching methods in Japan that include many practices recommended for U.S. classrooms has led to an interest in how teachers *learn* these methods. Most studies have pointed to lesson study as the driving factor of Japanese teachers learning these high-leverage teaching practices. Based on my experiences living in Japan and continuing to interact with Japanese teachers, I have identified other professional learning opportunities that might contribute significantly to Japanese teachers' practices. One opportunity that could be especially influential I will call *instructional circles.* These are voluntary meetings of teachers in which participants learn about and discuss instructional problems they bring to the group.

The goal of this report is to elucidate a common professional learning opportunity in Japan that has not yet been studied systematically or described in detail. Because this study is embedded in Japanese culture, the aim is not to provide descriptions that can be imported to other cultures. However, descriptions of professional learning in other cultures can be used to reflect on one's practices and provide new ideas or insights that can be adapted to achieve professional learning goals in local settings.

To better understand these instructional circles, this study was guided by the following research questions:

1. How are Japanese instructional circles designed to provide learning opportunities for teachers? What structural features do they include that could contribute to teachers' learning opportunities?
2. How do teachers engage in the features of instructional circles and with each other to support their learning opportunities?

Through better understanding how instructional circles are designed to provide learning opportunities, and how teachers can support those learning opportunities through their engagement in the professional learning opportunity, I have been able to find evidence of practices that differ from normal professional learning practice.

Literature Review

The purpose of this literature review is to determine what aspects of Japanese instructional circles should be investigated. I first looked at what features of professional learning opportunities (PLOs) other researchers have found to be important to study. Then I considered additional features of PLOs I identified from participating in learning circles. Using these features, I can develop guiding features to look for when answering the first research question. I then looked at different features of teacher learning to guide what evidence I should pay attention to when addressing the second research question.

Consensus Features of Effective Professional Development

Although there is no complete agreement about what constitutes effective professional development, there are a small set of features around which consensus is building. Desimone (2009) and Gusky (2003) have synthesized research and have identified the following five features: Content, Active Learning, Coherence, Duration, and Collective Participation. These five categories have often been used to determine the types of features of effective professional development; however, further research has been done to better define some of these features. Gibbons and Cobb (2017) state that the feature of Active Learning covers a broad range of activities and further defines that category as the pedagogies of investigation and pedagogies of enactment. Whereas pedagogies of investigation focus on the analysis and critiquing of classroom practice, pedagogies of enactment focus on planning for, enacting, and rehearsing high-quality teaching practices. Furthermore, there has also been a greater emphasis on attending to student thinking during professional learning opportunities (Gibbons & Cobb, 2017; Grossman et al., 2009).

Observed Features of Instructional Circles.

My previous experiences have allowed me to become familiar with many different types of professional learning in Japan. The professional learning opportunities in Japan have some features that differ from those commonly found in the United States. I am interested in whether and how Japanese instructional circles include these features.

Knowledgeable Other. In Japanese professional learning opportunities, the role of "knowledgeable other" is commonplace (Takahashi et al., 2013; Takahashi & Yoshida, 2004; Yoshida, 1999). Some knowledgeable others that I have observed being present include the district math specialist, vice principals, principals, the superintendent, or university professors. I am interested to know if and how the role of knowledgeable other is important in Japanese instructional circles. I am also interested to know if the role of the knowledgeable other is like the other types of professional learning found in Japan.

Teacher preparation. When attending a PLO, Japanese teachers often prepare extensively (Melville, 2021). The planning that they conduct during lesson study is one of the main features that transfers to teachers' everyday practices (Melville, 2017). This idea that teachers need to prepare to attend professional development is something that is not commonly discussed in the research.

Administrative aspects. The most well-known form of professional development found in Japan is lesson study (Takahashi et al., 2013). One interesting feature of lesson study is the complex organizational structure. Lesson study can be operated at the school level, district level, regional level, and national level. Lesson study takes on a different form depending on these levels as well. Because of the intricacies of the organizational structure of lesson study, I am interested to learn about the organizational structure of Japanese instructional circles.

Features of Teacher Learning

De Jong et al. (2019) studied aspects of teacher groups that enable teacher learning that I thought were pertinent to instructional circles based on my familiarity with this specific learning opportunity. The categories include the content of teachers' conversations, roles and responsibilities teachers adopt, community features that typify the teacher groups, interdependency of the teacher interactions, and motivation for teachers to attend the learning activity.

Methods

Data was gathered through an ethnographic approach. Specifically, I followed the ethnographic guidelines for mathematics education outlined by Eisenhart (1988), namely, participant observations, interviews, artifact collection, and researcher introspection. From the literature on effective features of professional development and features of teacher groups that encourage teacher learning, I developed a guidebook of important features to look for when coding the observations, interviews, and artifacts. While coding, I separated claims about important features of instructional circles from the structural aspects of the professional learning opportunities that enabled learning opportunities.

Results

For this proposal, I provide one exemplary instance of an instructional circle to provide a rich description of both the structural aspects, as well as how the teachers engage with these features of instructional circles.

Teacher-Led Improvement Practices

Instructional circles prioritize teachers in all aspects of the professional learning opportunity. For each instructional circle session, one teacher provides a problem of practice to be investigated, while the other teachers offer previous experiences or research to suggestions solutions to that specific problem. The problems of practice selected by the teachers are also very specific, or at a very fine grain-sized problem. The discussions around these problems of practice lead to collective interpretation from all those involved, which connections to their future work (Horn et al., 2017). For example, Figure 1 is a board work plan provided by one of the teachers to set up his problem of practice. Mr. Kinjo included his question of the day, possible student solution methods, and the learning goal he wanted to achieve. Mr. Kinjo posed two problems of practice for discussion from the group, A) He wanted to know how to better launch his task through asking a better thought-provoking question (Hatsumon) and B) he was worried that his

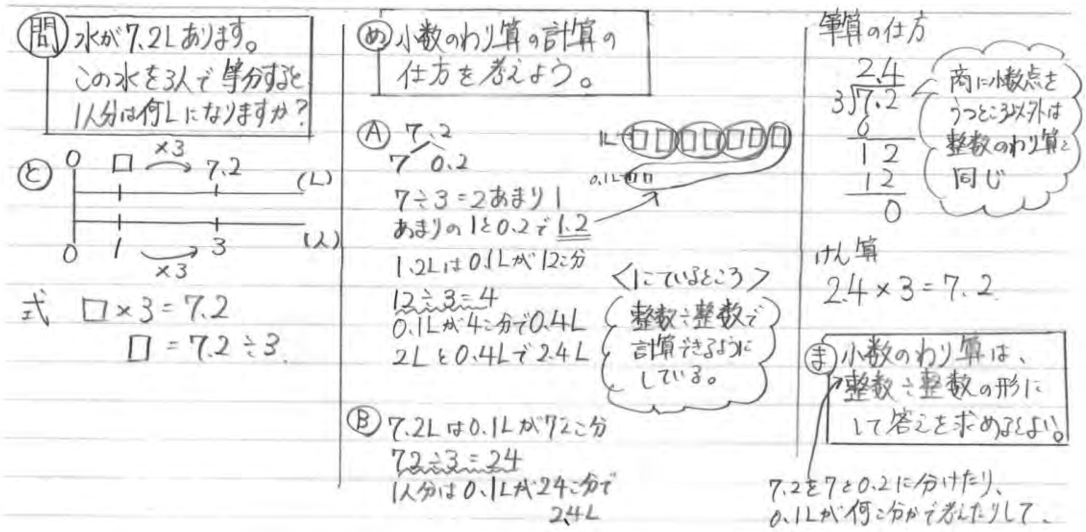


Figure 1: Boardwork Plan

students would not be able to understand the second potential student solution method. The teachers then took turns discussing potential ways to engage students in the lesson through a good thought-provoking question, as well as providing insight as to why students should be able to engage in the potential solution method. In conclusion, the teachers did not come up with mandatory questions Mr. Kinjo should use, or even if he should include the second solution method should be used. Instead, they provided their thinking and rationale and then left Mr. Kinjo to make the decisions that will best serve his class.

Educational Significance

In general terms, I see two main contributions of this study. The field of education has researched professional learning/professional development rigorously. New ideas, such as coaching, have emerged as interesting features that can also be effective methods of professional development. Learning about instructional circles will provide an opportunity to step back from the current research and look at professional learning from a different perspective. The benefit of taking this reflexive step is to learn new and interesting ways to benefit our teachers. Learning about instructional circles can also provide a pool of new ideas about professional learning. Understanding new ways of doing things, and the conditions in which they operate might help us whether and how we want to adapt them for practice in U.S. settings.

References

Corey, D. L., Peterson, B. E., Lewis, B. M., & Bukarau, J. (2010). Are there any places that students use their heads? Principles of high-quality Japanese mathematics instruction. *Journal for Research in Mathematics Education*, *41*(5), 438-478.

de Jong, L., Meirink, J., & Admiraal, W. (2019). School-based teacher collaboration: Different learning opportunities across various contexts. *Teaching and Teacher Education*, *86*, 102925.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher*, *38*(3), 181-199.

Eisenhart, M. A. (1988). The ethnographic research tradition and mathematics education research. *Journal for Research in Mathematics Education*, *19*(2), 99-114.

Grossman, T., & Hirsch, E. (2009). State policies to improve teacher professional development. Issue brief. *NGA center for best practices*.
Gibbons, L. K., & Cobb, P. (2017). Focusing on teacher learning opportunities to identify potentially productive coaching activities. *Journal of teacher education*, *68*(4), 411-425.
Guskey, T. R. (2003). What makes professional development effective? *Phi Delta Kappan*, *84*(10), 748-750.
Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., ... & Gallimore, R. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 video study. *Educational Evaluation and Policy Analysis*, *27*(2), 111-132.
Horn, I. S., Garner, B., Kane, B. D., & Brasel, J. (2017). A taxonomy of instructional learning opportunities in teachers' workgroup conversations. *Journal of teacher education*, *68*(1), 41-54.
McFarland, J., Hussar, B., De Brey, C., Snyder, T., Wang, X., Wilkinson-Flicker, S., ... & Hinz, S. (2017). The Condition of Education 2017. NCES 2017-144. *National Center for Education Statistics*.
McKnight, C. C. (1987). *The underachieving curriculum: Assessing US school mathematics from an international perspective. A national report on the second international mathematics study*. Stipes Publishing Co., Champaign, IL.
Melville, M. D. (2017). Kyozaikenkyu: An in-depth look into Japanese educators' daily planning practices. (Publication No. 6515) [Master's thesis, Brigham Young University]. Theses and Dissertations.
Melville, M. D. (2018). An international comparison of MKT and educative curriculum materials found in teachers' manuals. *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 132). Greenville, SC: PME-NA.
Melville, M. D., & Corey, D. L. (2021). Kyouzaikenkyuu: an exploration of Japanese mathematics teachers' daily planning practices. Journal of Mathematics Teacher Education, 1-13. https://doi-org.udel.idm.oclc.org/10.1007/s10857-021-09493-5
Stigler, J.W., and Hiebert, J. (1999). *The teaching gap*. New York: The Free Press.
Takahashi, A. (2006). Implementing lesson study in North American schools and school districts. In *Makalah yang dipresentasikan pada seminar APEC International symposium*.
Takahashi, A., Lewis, C., & Perry, R. (2013). A US lesson study network to spread teaching through problem solving. *International Journal for Lesson and Learning Studies*.
Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. *Inquiry into Mathematics Teacher Education*, *5*, 131-142.

PROFESSIONAL IDENTITY DEVELOPMENT OF MATHEMATICS TEACHING ASSISTANTS

T. Royce Olarte
University of California Santa Barbara
tolarte@ucsb.edu

Micah Swartz
Texas State University
mas762@txstate.edu

Sarah A. Roberts
University of California Santa Barbara
sroberts@education.ucsb.edu

This study examined how the social interactions that mathematics teaching assistants (TAs) have within their institution influenced their professional identity development as early-career undergraduate instructors. We drew on a sociocultural perspective of professional identity development in higher education to examine TAs' interactions with students, faculty, and other TAs. We qualitatively analyzed five mathematics TAs' responses to semi-structured interviews and found that some dimensions of their identities were more frequently situated within specific relationships, while others were evident in multiple relationships. Overall, the social interactions were sites for professional identity development. Identity is a complex construct, and a better understanding of how professional identity is developed can inform higher education institutions on ways to support positive identity development of future mathematics instructors.

Keywords: Professional Identity, Teaching Assistants, Higher Education

Introduction

Research on classrooms and pedagogies in recent decades has elucidated the central role of professional identity in teachers' learning and development (Beijaard et al., 2004). We draw on this research, particularly the notion that identities are complex, dynamic, social, contextual, multi-faceted constructions (Solari & Ortega, 2020) and that they underscore teachers' pedagogies, affect their motivations to teach, inform their instructional approaches, and guide the way they navigate their profession and execute their roles (Berger & Lê Van, 2019; Sachs, 2005). A large body of research has explored the identity development of mathematics teachers in teacher education programs and K-12 contexts (e.g., Sachs, 2005), however, there is still limited work examining the professional identity development of individuals involved in higher education mathematics instruction - particularly mathematics teaching assistants (TAs).

Mathematics TAs play a crucial role in undergraduate instruction and are, in many ways, early-career mathematics instructors. After their graduate studies, many TAs continue to teach in higher education, but historically, STEM graduate programs have inadequately prepared their students to teach and have done little to help them develop their identities as educators (Hancock & Walsh, 2014). Thus, examining graduate students' development of their professional identities is especially important and can inform the ways mathematics departments, and broadly, institutions, prepare their graduate students to be future educators. We drew on a sociocultural perspective of professional identity (Beauchamp & Thomas, 2009; Solari & Ortega, 2020) development in higher education (Clarke et. al, 2013) to frame this study. A sociocultural lens accounts for the institutional contexts that TAs navigate, the social domains that exist, interactions with members of their community that occur, the roles individuals perform, and the intrapersonal domain, or identities they assigned to themselves. Our research questions were: (1) How did mathematics graduate students reflect on their developing professional identities? (2) How did TAs' professional identities develop within social interactions?

Framing and Literature Review

Researchers have previously defined and classified the construct of identity in numerous ways. Identities can be institutional, based on an affiliation with a group, or even discursive (Gee, 2001). Additionally, identities can be independent or interdependent (Fryberg & Markus, 2003), and they can be based on others' perceptions (Markus & Wurf, 1987; Sfard & Prusak, 2005). Drawing on these notions, we view identity as: "a dynamic view of self, negotiated in a specific social context and informed by past history, events, personal narratives, experiences, routines, and ways of participating" (Bishop, 2012, p. 38). Framing these ideas in a professional environment, we define professional identity as an ever-changing view of one's obligations, role, and attitudes situated in institutional, social, and cultural contexts. A professional identity includes emotional components such as attitudes and beliefs (Wenger, 1999), which are often left out when considering one's self-perceptions. Factors that influence professional identity include the direct work environment, the wider context of higher education, interactions with students, and professional development activities (van Lankveld et al., 2017), which serve as a basis for examining how situated contexts can affect graduate students' professional identities.

More specifically, we drew on a sociocultural perspective of professional identity development in higher education to frame this study (Beijaard et al., 2004; Gee, 2001; Clarke et al., 2013). The formation of professional identity in higher education is a social and contextual process, and the other actors with whom an individual interacts play salient roles in helping an individual build identities and meanings for themselves as a professional (Solari and Ortega, 2020). Mathematics TAs navigate a complex interpersonal network that involves students, faculty, and other TAs, all situated within a broader institutional context and the mathematics discipline. The expectations of this community, the institution, and the discipline can shape the roles that TAs perform, thus affecting their professional identity development as higher education instructors. Situated within a specific institutional context and the mathematics discipline, we examined emerging professional identities that resulted from four key relationships (See Figure 1). We acknowledge that factors outside of the institutional context, mathematics discipline, and the four key relationships we have identified here (e.g., family, relationships, stress of the profession, etc.) can also affect professional identity.

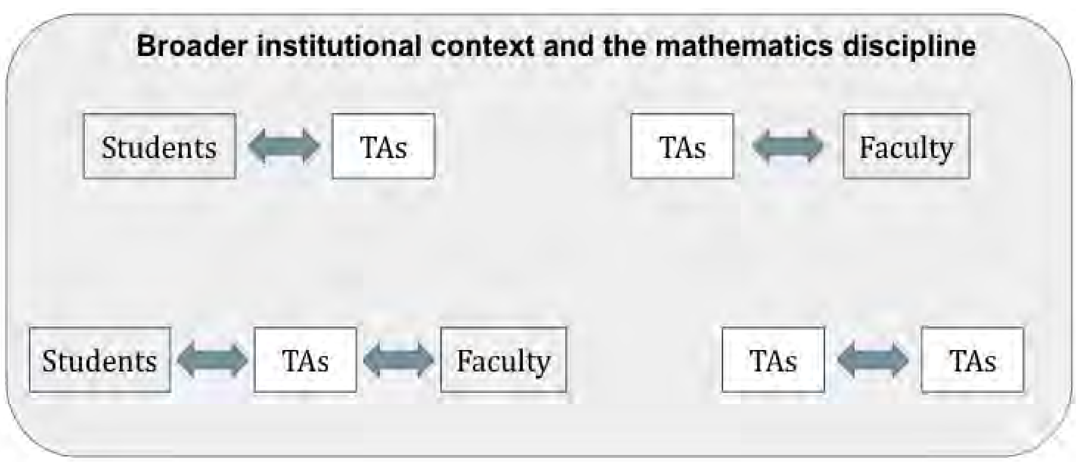


Figure 1: Teaching Assistants' Relationships within an Institution

Method

This study was conducted at a Minority-Serving Institution in California. It was a part of a larger project that examined the experiences of transfer students enrolled in a set of courses designed to develop proof-construction competencies and support transfer students' transition to a four-year university. Purposeful sampling (Creswell, 2013) was used to recruit five PhD students in the mathematics department who served as TAs for an introductory proof course, which was situated in number theory and set theory. All participants facilitated online sections of the course, were in the 2nd or 3rd year of their doctoral program and had prior experiences as TAs. Three self-identified as male and two self-identified as female. We use the pseudonyms Federico, Nestor, Wyatt, Kaitlyn, and Lisa to refer to the five TAs interviewed for this research project. We conducted semi-structured interviews (Rubin & Rubin, 2011) over Zoom that focused on the TAs' experiences enacting sections remotely, the ways the course supported underrepresented and non-traditional (i.e., transfer) students, and the TAs' perceptions of their identities and roles as TAs and early-career educators. The research team first open-coded the TAs' responses and identified themes in the TAs' reflections. The team recoded the corpus of data using the following second round codes to describe the complexity and multidimensionality of their professional identities: *content-deliverer, sensemaker, community-builder, assistant, supporter, mentor, resource sharer, learner, beliefs and values,* and *demeanor.* The research team discussed and wrote memos about emergent themes in how the TAs described their professional identities, and also coded for the individuals that TAs described as salient to each facet of their identities (i.e., students, faculty, and other TAs).

Findings

We found that graduate students reflected on their developing professional identities in a number of different ways, with their professional identities developing within certain social contexts (e.g., Student-TA relationship, TA-Faculty relationships, Student-TA-Faculty relationships, TA-TA relationships) in which the TAs were situated (e.g., *content deliverer* was most associated with the Student-TA relationship). It is important to note that professional identity is a complex construct, and all interactions and relationships are influential in developing identity. However, we only present the most salient professional identities that emerged from the TAs' reflections on the four relationships.

Professional Identities That Developed from the Student-TA Relationships

In this section, we highlight the important aspects of the TAs' professional identities most situated in the relationship between students and TAs: *content-deliverer*, *sensemaker, community-builder*, *supporter*, and *beliefs and values*. The TAs noted that they were engaged in these different relationships with students as part of their roles. We describe each aspect of their identity and highlight examples of instances when the TAs revealed their perceptions of their identities as it relates to students.

Content-Deliverer. As *content-deliverers,* the TAs viewed themselves as individuals who delivered mathematics content to students, whether presenting the same content professors lectured on or reviewing material from previous classes. For instance, Wyatt explained, "I found that I'm more effective as a TA when I can just cover a lot of material, because my strength is being able to explain it in a digestible way." Here, Wyatt identified himself as someone who delivers content and he saw his effectiveness in this role. Another TA, Nestor, described his identity as a *content-deliverer,* saying that "I should just be like a - not literally but, like a bit of a copy paste of what the professor has said in lecture. Sometimes just repetition." Nestor believed his role is to reinforce the same content to students in section that the professor delivered during lecture. We found that *content-deliverer* was an important aspect of TAs' professional identities illustrated by all participants.

Sensemaker. The professional identity of *sensemaker* emerged as TAs described engaging in sensemaking efforts of the circumstances and individuals around them. In the social context of Student-TA relationships, the TAs engaged in sensemaking to better understand their students' experiences including students' content knowledge, emotional well-being, challenges related to their university experiences, and difficulties associated with the remote-instruction brought upon by the COVID-19 pandemic. For instance, Kaitlyn made sense of the challenges the students in her class faced noting: "There's also an intensity that comes with Math A that maybe make students feel a little bit more, I don't know fearful about doing perfect in the class or whatever." In another instance, Lisa remarked about the unique challenges students of a particular gender face when she said, "The people who speak out are male, but there [are] definitely a few female students who were talkative. But yeah so that's all I really noticed." In this moment, Lisa made sense of the gender and social dynamics in her classroom and how those affected their participation. We found many instances of sensemaking centered around students' experiences and this illustrates that a salient part of TAs' professional identities involved how they noticed, made meaning, and took action to attend to their students.

Community-Builder. *Community-builder* referred to the instances when TAs described taking action to build community with students or among students. Most instances of a TA building community took on one of two forms. The TA either worked to create community among students or the TA would talk with students about non-mathematics related topics during office hours. For example, Lisa commented, "I tried to do a lot more of like breakout sections and like try to get students to talk to each other, because I was like you don't ever see each other, trying to talk to each other." Lisa engaged in community-building efforts to get her students to interact with one another on Zoom and to become more familiar with one another. Later, Lisa commented that she often tried to start class with a question to get to know what and how her students were doing outside of the class. She expressed that she was able to best engage with students on a personal level during office hours: "I'll mostly be, like 'Anyone do anything fun on the weekend?' and then usually be met with silence. But in office hours, I had students like linger around and talk to me, so that was fun." Many of the efforts related to building community were in response to the challenges presented by the pandemic, as TAs would often express difficulties with building community in the online learning environments in which they taught.

Supporter. TAs reflected on how they supported students through encouragement, offering emotional support, and advocating for their students; these encompassed the professional identity of *supporter.* TAs reported capitalizing on the interactions with students in office hours and sections to support their students. They recognized that part of their professional identities and roles included attending to students' affect and emotions, and they were cognizant of providing students with support throughout the quarter. For example, during the remote instruction brought upon by COVID-19, TAs recognized that it was an extremely difficult time for students and described themselves as being someone to whom students could talk, beyond just discussing the mathematics content of the course. For example, Nestor described an interaction during one of his synchronous office hours when he conversed with a student about aspects of each other's lives that did not directly relate to the mathematics content. He said, "I don't know exactly how we ended up there, but I guess she just needed someone to talk to that night, and I was the one for whatever reason, because in Zoom you can't really reach out to anybody else. Right?" He recognized a student's need and was able to provide her with some emotional support. Other ways that TAs provided support for the students was to advocate for them. Federico described listening to students' requests and complaints during office hours and relaying that information to the faculty. Through advocacy, offering encouragement, and overall, showing care, we found that TAs were active *supporters* of their students.

Beliefs and Values. As they reflected on their relationships with students, we also found a set of *beliefs and values* that described what TAs believed about their students, how they understood their students learned, and how TAs thought they should help their students engage with mathematics. Federico, for example, shared that his students did not learn in the same ways, and TAs should be able to support and teach all students. This belief affected his approach to teaching. In another example, Wyatt described how in his past experiences as an undergraduate student, he was taught mathematics through a narrow lens, and he believed that students needed to be allowed to be creative in mathematics for them to learn well. A common belief that many of the participants held was that a TA should help students engage with mathematics by creating learning spaces where students would feel comfortable to participate, make mistakes, ask questions, etc. This idea of fostering a comfortable space for students was the most prevalent belief that the TAs held. For example, Kaitlyn said, "I think there's a little bit of gaining that trust in your students. That they could come, they feel like they can come to you with all their questions…You want them to feel comfortable." This belief about how TAs should teach mathematics can influence their pedagogies and the ways they interact with students.

Professional Identities That Developed from the TA-Faculty Relationships

Next, we focus on the interactions between TAs and faculty. In this social context, faculty were the instructors on record for the sections the TA managed. The three key facets that TAs reflected on related to their professional identities that were most evident in this relationship were *assistant*, *learner*, and *sensemaker*. Evidence of these professional identities included the TAs assisting with grading, answering students' questions, sharing their own learning of content (and enjoyment in learning this content), and making sense of their relationships with faculty. The professional identities of *assistant* and *learner* only occurred in the TA-Faculty relationship, while *sensemaker* also occurred in the Student-TA relationship.

Assistant. Quite frequently, the TAs would position themselves as individuals whose purpose and role was to assist the faculty. This notion underscores the professional identity of *assistant*. The TA used words like "helper" or "grader" to signify their role as an assistant to the faculty member with whom they worked. For instance, Lisa shared, "[I] just do some of the grunt work of the grading and background work that needs to get done. You know, it shouldn't lie fully on the professor's hands. Just a helper, a solid helper." In this instance, Lisa identified and gave merit to helping and assisting the faculty she was working with through grading exams and doing background work. Additionally, Kaitlyn, reflected on her role and expectations as an assistant and said, "As far as the professor goes, you know, they expect you to go through homework problems and answer questions that students have and grade a little bit." In this moment, Kaitlyn acknowledged that many faculty members had specific expectations of their teaching assistants. Additionally, Federico described his relationship as an assistant to his faculty member, noting, "He told me things to do. That's what I did." The consistent use of language, such as "He told me things to do," and "grunt work," and the description of TAs' self-described roles frequently occurred across the data set. The professional identity of an *assistant* was only observed in the context of TA-Faculty relationships.

Learners. We observed another dimension of professional identity that only appeared in the social context of relationships involving TAs and faculty: *learners*. TAs described their experiences as learners, both past and present. This illustrated that they were not only teachers and individuals meant to deliver content or support students. TAs saw themselves as learners of content, learners of pedagogical methods, and learners of resources. Nestor described his experience of learning content in a new way, explaining, “I feel like every time I revisit a concept… I always learned something new, just for myself or like a different perspective, different angle, because the professor looks at it from a different, in a different way.” Nestor identified himself as a learner through his exposure to former content through a new lens, expressed his enjoyment of having the opportunity to be a learner, and engaged with the content differently from how he previously learned it. This professional identity signifies an important role that TAs can take up and it is their capacity to see themselves as both instructors and learners.

Sensemaker. TAs reflected on their professional identity as *sensemakers* within Student-TA relationships in instances when the TAs tried to better understand students’ experiences. Within the social context of TA-Faculty relationships, *sensemaker* describes how TAs noticed and made meaning of their relationships, interactions, and experiences with faculty. For example, Wyatt made sense of a faculty’s pedagogy and instructional practices, saying, “Well, you know, in an ideal world, of supposing that they had more time to concentrate, I think one thing they could do is just help TAs better know how to grade assignments.” This moment highlighted Wyatt making sense of how a faculty member interacted with him and could support him as a TA. He was forming thoughts on the pedagogical practices of faculty from past and current experiences while making sense of what could be done to better assist TAs. In another instance, Lisa commented on her experience with faculty and what she felt would best benefit her as a TA. She explained, “I think just more communication of like what is actually expected of us. Sure, you're supposed to teach four sections, do certain amount office hours, work the math lab, but, how do you do all those jobs?” Lisa was actively making sense of her past and current experiences with faculty, while also considering how to improve that relationship. She identified areas where faculty could improve their practice through understanding TAs’ challenges and reflecting on their experiences working with these students. Broadly, the professional identity of *sensemaker* entails the active process of understanding, making meaning, and taking action within a TA’s social context.

Professional Identities That Developed from the Student-TA-Faculty Relationships

We distinguish the Student-TA-Faculty relationship from just the Student-TA relationship and the TA-Faculty relationship because we observed that TAs often described themselves as individuals who bridged students and faculty. Therefore, these aspects of their professional identities and roles were the result of understanding how they mediated the various expectations, perceptions, and needs of both students and faculty. As Kaitlyn noted, “it’s kind of exactly what it seems: like you’re this middle ground between the professors and the students.” TAs identified two aspects of their professional identities in relation to *both* faculty and students: *content-deliverer* and *demeanor*. TAs delivered content in order to meet their students’ needs, in light of what the faculty member had taught, and they presented themselves to students often in response to how they understood their students perceived the faculty.

Content-Deliverer. Although TAs reflected on their professional identities as *content-deliverers* within the Student-TA social context, TAs revealed that faculty delivered content in one way, but as TAs, they should deliver the content differently to better attend to the needs and expectations of students. Often, TAs reflected that offering the content in ways that were different from the faculty would be beneficial to the students. Nestor shared, “If you revisit the…same concept, maybe through a different angle, then you can understand it better.”

Demeanor. The second aspect of TAs' professional identities within the Student-TA-Faculty social context was *demeanor*. TAs acknowledged students' perceptions of faculty (usually in a negative way), and TAs portrayed themselves in ways that differed from faculty. When asked about her primary role, Lisa said, "Just to be someone who is less scary than the professor. My job is just to be approachable." Here, Lisa noted that approachability was an important consideration for TAs because of the common perception that professors were "scary." Similarly, Kaitlyn shared, "I think part of our role as a TA is just to, sort of, be that middle ground between the scary professor and any intimidation." The ways in which Lisa and Kaitlyn portrayed themselves, particularly as caring and approachable to students, shed light on their understanding of *demeanor*. How TAs presented themselves to students was an important part of their professional identities, particularly in relation to mediating students and faculty.

Professional Identities That Developed from the TA-TA Relationships

Lastly, the ways that TAs navigated their responsibilities and roles alongside other TAs highlighted their capacities to be *resource sharers* and *supporters* of each other. Reflecting on her experiences with remote instruction, Kaitlyn revealed how the TAs in the mathematics department supported each other by giving advice, and they used various online platforms to share resources that may be useful to address the challenges of conducting sections online. She explained, "I think we gave advice to each other, and we have a Discord [online platform] for our math grads, so people talked about teaching advice and…what programs they like to use, apps they like to use to teach." TAs supported each other by offering advice and directing each other to teaching resources. These relationships illustrate the dimensions of their professional identities best described as *resource sharers* and *supporters.*

Discussion

Through a sociocultural perspective on professional identity development, we found that the relationships and interactions that TAs have with members of their mathematics community and institution were sites for identity development. Beauchamp and Thomas (2009) wrote that identity "is shaped and reshaped in interactions with others in a professional context," and we found that TAs reflected on a range of professional identities. Certain aspects of their identity were more frequently situated within specific relationships such as *supporter* most often occurring within the Student-TA relationships, and *resource sharer* most often occurring within the TA-TA relationships. However, other dimensions of their professional identities such as *content-deliverer* and *sensemaker* were evident in multiple relationships such as the Student-TA, TA-Faculty, and Student-TA-Faculty relationships. We found that the Student-TA relationship was not only the most frequently mentioned interaction but also through which multiple dimensions of their professional identities were revealed.

While the key tenets of each professional identity remain the same across the relationships (e.g., *sensemaker* refers to noticing, making meaning, and taking action), the ways TAs perceived themselves as such depended on their company (e.g., the student or the faculty). For example, in the context of a Student-TA relationship, being a *resource sharer* may look like a TA providing resources to students to support and engage learning. However, in the context of a TA-TA relationship, being a *resource sharer* may entail a TA sharing instructional resources with other TAs. This is consistent with existing research that acknowledges that the construction of professional identity development is complex, and that different relationships and interactions influence different aspects of their identity (Berger & Lê Van, 2019). Moreover, as the TAs reflected on their experiences and interactions, we noted how their professional identities were linked to their instructional practices and pedagogies. Each TA exhibited a unique set of identities, and this was revealed through the ways they individually reflected on their roles and interactions.

We extend the literature on professional identity development in higher education by examining mathematics TAs as early-career undergraduate instructors. A sociocultural perspective on professional identity development affords an exploration of the key interactions that TAs engage and the dimensions of their identity developing and intersecting across these relationships and interactions. Limitations of this study include the limited number of participants (five) and that the participants of this study served as TAs for the same introductory proof course. In addition, all five participants were TAs for online sections of the proof course and this environment could elicit different aspects of TAs' professional identities than in-person instruction. Future research can examine more mathematics TAs across a broader range of courses, such as calculus, linear algebra, etc. Furthermore, for this study, we only focused on four key social context relationships, but TAs professionally interact with many other individuals including peers in other departments, staff, various organizations, and professionals outside of the institution. Future research can more closely examine the larger, complex network that TAs navigate and how identity is developed within this network, can focus on just one type of interaction (e.g., TA-TA), or can focus on one aspect of professional identity (e.g., *community-builder*).

Conclusion

The professional identities of mathematics instructors in higher education is still a largely under-researched area of mathematics education. With many mathematics graduate students continuing to become teaching faculty themselves, it is imperative that we examine how their professional identities are developed, particularly related to their experiences as TAs. The professional identities that mathematics TAs develop through their social interactions and relationships with members of their professional community not only influence their current positions as TAs but provide a foundation for their future careers, pedagogies, and practices. Identity development is a constant process throughout the careers of mathematics educators, and it is important to acknowledge that even prior to formal appointments as a teaching faculty, their experiences – such as serving as TAs during their graduate programs – have already shaped their professional identities. A sociocultural perspective of identity development affords a lens through which researchers can identify and examine the extent to which experiences related to being a teaching assistant impact the development of professional identity as a mathematics instructor. Understanding the nuances and complexities of how professional identities of future educators are developed can inform higher education institutions on how to better develop positive professional identities of mathematics graduate students.

Acknowledgments

This project was funded by the UCSB Academic Senate Council on Research and the Instructional Resources Faculty Research Grant.

References

Beauchamp, C., & Thomas, L. (2009). Understanding teacher identity: An overview of issues in the literature and implications for teacher education. *Cambridge journal of education*, *39*(2), 175-189. https://doi.org/10.1080/03057640902902252

Beijaard, D., Meijer, P. C., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and teacher education*, *20*(2), 107-128. https://doi.org/10.1016/j.tate.2003.07.001

Berger, J. L., & Lê Van, K. (2019). Teacher professional identity as multidimensional: mapping its components and examining their associations with general pedagogical beliefs. *Educational Studies*, *45*(2), 163-181. https://doi.org/10.1080/03055698.2018.1446324

Bishop, J. P. (2012). “She's always been the smart one. I've always been the dumb one”: Identities in the mathematics classroom. *Journal for Research in Mathematics Education*, *43*(1), 34-74. https://doi.org/10.5951/jresematheduc.43.1.0034

Clarke, M., Hyde, A., & Drennan, J. (2013). Professional identity in higher education. In B. M. Kehm, & U. Teichler (Eds.), *The academic profession in Europe: New tasks and new challenges* (pp. 7-21). Springer.

Creswell, J. W. (2013). *Qualitative inquiry and research design: choosing among five approaches*. Sage Publications.

Fryberg, S. A., & Markus, H. R. (2003). On being American Indian: Current and possible selves. *Self and Identity*, *2*(4), 325-344. https://doi.org/10.1080/714050251

Gee, J. P. (2001). Chapter 3: Identity as an analytic lens for research in education. *Review of research in education*, *25*(1), 99-125. https://doi.org/10.3102/0091732X025001099

Hancock, S., & Walsh, E. (2016). Beyond knowledge and skills: rethinking the development of professional identity during the STEM doctorate. *Studies in Higher Education*, *41*(1), 37-50. https://doi.org/10.1080/03075079.2014.915301

Markus, H., & Wurf, E. (1987). The dynamic self-concept: A social psychological perspective. *Annual review of psychology*, *38*(1), 299-337. doi:10.1146/annurev.ps.38.020187.001503

Rubin, H. J., & Rubin, I. S. (2011). *Qualitative interviewing: The art of hearing data*. Sage.

Sachs, J. (2005). Teacher education and the development of professional identity: Learning to be a teacher. In P. Denicolo & M. Kompf (Eds.), *Connecting policy and practice: Challenges for teaching and learning in schools and universities* (pp. 5–21). Routledge.

Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational researcher*, *34*(4), 14-22. https://doi.org/10.3102/0013189X034004014

Solari, M., & Martín Ortega, E. (2020). Teachers’ professional identity construction: A sociocultural approach to its definition and research. *Journal of Constructivist Psychology*, 1-30. https://doi.org/10.1080/10720537.2020.1852987

Van Lankveld, T., Schoonenboom, J., Volman, M., Croiset, G., & Beishuizen, J. (2017). Developing a teacher identity in the university context: A systematic review of the literature. *Higher Education Research & Development*, *36*(2), 325-342. https://doi.org/10.1080/07294360.2016.1208154

Wenger, E. (1999). *Communities of practice: Learning, meaning, and identity*. Cambridge University Press.

INCREMENTAL CHANGE AS AN ALTERNATIVE TO AMBITIOUS PROFESSIONAL DEVELOPMENT

Samuel Otten
University of Missouri
ottens@missouri.edu

Zandra de Araujo
University of Florida
zdearaujo@coe.ufl.edu

Amber G. Candela
University of Missouri, St Louis
candelaa@umsl.edu

Courtney Vahle
University of Missouri
courtneyvahle@mail.missouri.edu

Maria E. N. Stewart
University of Missouri
mntkd@umsystem.edu

F. Paul Wonsavage
University of Florida
wonsavagef@ufl.edu

Faustina Baah
University of Missouri
fa6nz@missouri.edu

Mathematics professional development (PD) has had many small victories but has not brought about a widespread change in what constitutes typical mathematics instruction. This theoretical essay argues that many PD projects have been based on an assumption that the aims of the PD should be ambitious, but ambitious PD requires that a large set of criteria be satisfied (active learning, coherence, duration, teacher buy-in, etc.). Even then, ambitious PD may only reach a minority of teachers who are ready to make the transformation. An alternative approach is incremental PD, which starts with a teacher's contextual constraints and ubiquitous practices, offering modest but meaningful "nudges" for their instruction. These nudges are intended to be easily taken up by teachers, providing a sense of success that leads to them sustaining the practices and being portable enough to be easily shared with other teachers, allowing for scale.

Keywords: Professional Development, Instructional Activities and Practices

There have been vast investments of both time and money for teacher professional learning, yet despite these investments, mathematics instruction in the United States has remained largely unchanged (Karp & Schubring, 2014; Stigler & Hiebert, 1999). Professional development (PD) focused on teachers' instructional practice often faces three overarching challenges: (1) making a direct impact on the teachers in the PD, (2) sustaining outcomes beyond the life of the PD project, and (3) scaling up the PD (Heck et al., 2019; Karsenty, 2021). In attempts to overcome these challenges, scholars have made progress studying the characteristics of effective PD (e.g., Desimone, 2011; Wilson & Berne, 1999) and mechanisms by which the PD might spread (e.g., Morris & Hiebert, 2011). Still, despite these efforts, the lack of widespread reform in mathematics instruction persists (Wilkie, 2019).

We contend there is an important assumption underlying these challenges—namely, that PD should be targeted at instructional change that is ambitious or transformational (e.g., Kazemi et al., 2009; Kraft & Hill, 2020). Ambitious instruction, characterized by being disciplinarily rich and student-centered (Cohen, 2011), is something many mathematics education scholars strive toward and make the centerpiece of their PD for teachers. But what if we instead start with a more modest goal? Like Star (2016) and Litke (2020), we argue incremental changes could also be the goal of PD. In this paper, we discuss how this shift from ambitious to incremental goals for PD changes the instructional foci (Litke, 2020; Star, 2016) and leads to redefining success, rethinking the starting points for PD, and new mechanisms for sustainability and scaling.

Starting Points for Incremental PD

Teachers' Constraints

Many professional developers adhere to researchers' recommendations for effective PD in format, duration, and contextual features (e.g., Desimone, 2011, Feiman-Nemser, 2001). It is generally agreed upon, though with some dissent (Sims & Fletcher-Wood, 2021), that attending to criteria such as active learning, coherence, substantial duration, and teacher buy-in are necessary for implementing PD aimed at promoting ambitious instruction; however, it is unclear that PD aimed at modest incremental improvements needs to meet these same criteria. Central to the idea of incremental PD is that professional developers offer small suggestions for improvement—which we refer to as nudges—based on a teacher's current practice. Thus, we argue the starting place for incremental PD is teachers' constraints and contextual realities, which typically include limited planning time, conventional curriculum and resources, conventional expectations from students and parents, limited professional collaboration structures, and more.

Many of these constraints are often at odds with the criteria for effective PD. For example, Desimone (2011) and others have suggested PD requires a substantial investment of time, but we also know that many teachers do not have enough time for planning (Steen-Olsen & Eikseth, 2010). An incremental approach to PD would be designed with such constraints in mind. For example, the PD may simply offer a small nudge that does not take much time and that supplements existing curriculum materials, nudging toward a conceptual connection or a brief opportunity for student choice on their problems. These nudges, though small, could be more impactful than a teacher whose constraints prevent them from engaging with ambitious PD at all. If the nudge fits within the teacher's time constraints and aligns with the curriculum and their community expectations, and seems to be a slight shift in a positive direction, we hypothesize that this will yield high levels of buy-in, which is perhaps a necessary criterion for ambitious PD and incremental PD alike, but the latter can generate the buy-in from more teachers.

Ubiquitous Practices

Often PD is designed to promote ambitious practices such as robust classroom discourse (Herbel-Eisenmann et al., 2017), enacting cognitively-demanding tasks (Smith et al., 2008), argumentation, and reasoning (NCTM, 2009), and many others. These PD efforts aimed at ambitious practice are worthwhile. Yet, ambitious practices are difficult to achieve, typically resulting in pockets of success because they often require major shifts in teachers' existing practice. This discrepancy between current practice and ambitious practice can potentially position teachers in a deficit manner. We bear in mind Spangler's (2019) call to "consciously respect teachers and…avoid deficit approaches in our thinking about teachers" (p. 3).

Thus, in contrast, incremental PD would focus on the commonplace, daily pedagogical practices teachers are already comfortable with. For mathematics teachers, such practices might include demonstrations with worked examples (Atkinson et al., 2000), students' independent work time on problem sets (Otten et al., 2021), or the teacher going over homework at the board (Otten et al., 2015). An example might be comparing two procedural worked examples instead of simply doing one after the other (Litke, 2020). Another example is adding a single reversibility problem (Dougherty et al., 2015) to an existing exercise set. The intention is that teachers can immediately understand how they would incorporate these nudges into the practices they already employ, but also understand the nudges bring some modest yet meaningful benefit.

Sustaining and Scaling Incremental PD

Sustaining Modest Changes

Sustaining teacher change, especially ambitious change, has long been a challenge in teacher education (Liu & Phelps, 2020). We contend that an incremental approach can lead to sustainable change in at least two ways. First, incremental PD will focus on changes close enough to teachers' existing practices that the change is less disruptive and can be incorporated regularly. As Thaler and Sunstein (2008) have found regarding human change in general, if we want a change to take hold, we need to make it easy for people to do. Moreover, by attaching the incremental changes to common occurrences in the classroom, those changes can carry forward for the teacher by becoming their new instructional "habits," so to speak. As Star (2016) noted, it can be powerful to make a 10% improvement in a typical practice a teacher uses in 90% of their lessons rather than seeking a 90% improvement in a practice they only use 10% of the time.

Second, sustainability can come from the incremental PD delivery mechanism itself. When grant-funded (ambitious) PD projects are completed, the resources to continue teachers' learning are often depleted. The PD may involve summer workshops and follow-up sessions throughout the school year. Such models may impact teachers' instruction, but sustaining the change is challenging as projects end and teachers, administrators, and curriculum materials come and go. Instead, by aiming for incremental improvements to PD, delivered in small packages, teachers can learn them in a short time or more flexible formats (e.g., videos) and can try them relatively quickly. This would also allow the creation of artifacts that can be reviewed on demand. Thus, sustainability is gained from the low resource cost associated with delivering incremental PD, which also allows it to be shared among practitioners, leading us to the issue of scale.

Scaling Modest Changes

Another long-standing challenge for the field is the ability to scale PD initiatives (Heck et al., 2019; Karsenty, 2021). Here, we consider scale in terms of reaching a large number of practitioners. Because of the profound change entailed for most teachers concerning ambitious mathematics instruction and the confluence of factors needed for this change to occur, it seems optimistic to expect that ambitious PD might eventually scale to 10% or more of mathematics teachers. But again, as Star (2016) pointed out, it can also be beneficial to many students if 90% of (conventional) teachers make some small improvements to their instruction. As mentioned above, PD aimed at incremental changes would not require the same extensive resources as more ambitious efforts. The freedom to deliver specific, actionable practices via on-demand formats, such as videos, downloadable documents, or social media suggestions will allow for the scaling of those efforts by the PD creators and practitioners themselves. Moreover, by offering an incremental improvement to a ubiquitous teaching practice, we are increasing the likelihood that the improvement can be shared from person to person because many other teachers will also recognize the applicability of the suggestion to their own teaching.

A second notion of scale is the consideration of whether or not the incremental practices will scale toward future, substantial changes. It seems plausible that when teachers enact small improvements, they may gain confidence and feelings of success, which may lead them toward the addition of more small improvements. These small improvements may eventually lead them quite far from the conventional instruction they began with. This is not to say that the accumulation of incremental changes is necessarily equivalent to an ambitious change, but sustained improvements that have the potential to spread to other teachers should be welcomed, nonetheless. And the relationship between multiple incremental changes and larger ambitious changes can be explored empirically, which is where we now turn.

Questions and Final Considerations

What are some of the empirical questions that we can ask about incremental PD for mathematics instruction? There are a few that are of immediate interest to us:

1. Because incremental PD should be rooted in a teachers' unique contexts and built upon their existing ubiquitous practices, to what extent will certain nudges work across multiple contexts and for various teachers? There are some commonalities in many teachers' situations, so we hypothesize some generality for certain nudges. Still, we have to examine the balance between the general applicability of nudges and the individualized preferences of teachers (what aspects of their teaching do they want to be nudged on?).
2. What are some of the characteristics of teachers who are likely to respond positively to incremental PD, and what are the characteristics of those who are able to proceed directly to ambitious PD? As discussed above, we hypothesize that a majority of mathematics teachers can be well served by incremental PD, particularly those who may be resistant to ambitious PD. Still, if teachers respond positively to ambitious efforts, then we would support ambitious PD in those particular cases. Further research is needed, as well as further input from teachers themselves, to help us understand the differences.
3. In what ways can we measure the incremental changes in teachers' instructional practice when some of the changes are designed to be subtle or modest? Some of our current methods for assessing the impact of PD hinge on larger changes, but with incremental PD, we will need to be able to track the small changes and whether they are sustained.
4. Can deep, systemic issues of inequity and injustice in mathematics education be meaningfully addressed in an incremental manner? We agree that, in many ways, transformational change is needed and time is of the essence, but is there a role for incremental change in immediately bringing about some equitable practices or in setting the stage for later, more profound changes?
5. Given the multitude of small changes that could be proposed to a teacher, how will we decide which to employ and when? Relatedly, what are the sequences of incremental changes in instruction moving toward? Perhaps incremental instruction could be guided by the larger goals of ambitious instruction. It is an open question, however, whether and how an accumulation of such small changes yield (or not) a more substantial change long-term. Munter (2014) found teachers' visions of high-quality mathematics instruction can gradually increase in sophistication, but can one type of vision or pedagogy gradually shift into a different type altogether? One hypothesis is that incrementally improving conventional instruction will eventually increase the likelihood that teachers will respond positively to ambitious PD. A second hypothesis is that improving conventional instruction will lead to teachers feeling more confident and affirmed in what they do, so they will not transform away from conventional instructional models. A third hypothesis is that the incremental changes are simply independent from ambitious goals for instruction.

In embracing an incremental approach to PD, meant as a complement to the ongoing ambitious PD efforts, we contend that many of the persistent challenges in relation to the limited impact, sustainability, and scale of efforts to improve mathematics teaching may be addressed, or at least explored and understood from a perspective distinct from the ambitious approach.

Acknowledgments

This work was supported by the National Science Foundation (awards #2101508 and #2206774) though any opinions, findings, and conclusions expressed here are those of the authors and do not necessarily reflect the views of the NSF.

References

Atkinson, R. K., Derry, S. J., Renkl, A., & Wortham, D. (2000). Learning from examples: Instructional principles from the worked examples research. *Review of Educational Research*, *70*(2), 181–214.

Cohen, D. K. (2011). *Teaching and its predicaments*. Harvard University Press.

Desimone, L. M. (2011). A primer on effective professional development. *Phi Delta Kappan*, *92*(6), 68–71.

Dougherty, B., Bryant, D. P., Bryant, B. R., Darrough, R. L., & Pfannenstiel, K. H. (2015). Developing concepts and generalizations to build algebraic thinking: The reversibility, flexibility, and generalization approach. *Intervention in School and Clinic*, *50*(5), 273–281.

Heck, D. J., Plumley, C. L., Stylianou, D. A., Smith, A. A., & Moffett, G. (2019). Scaling up innovative learning in mathematics: Exploring the effect of different professional development approaches on teacher knowledge, beliefs, and instructional practice. *Educational Studies in Mathematics*, *102*, 319–342.

Herbel-Eisenmann, B., Cirillo, M., Steele, M., Otten, S., & Johnson, K. R. (2017). *Mathematics discourse in secondary classrooms (MDISC)*. Math Solutions.

Karp, A., & Schubring, G. (Eds.). (2014). *Handbook on the history of mathematics education* (pp. 303–322). Springer.

Karsenty, R. (2021). Implementing professional development programs for mathematics teachers at scale: What counts as success? *ZDM Mathematics Education*, *53*, 1021–1033.

Kazemi, E., Franke, M., & Lampert, M. (2009, July). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 12–30).

Kraft, M. A., & Hill, H. C. (2020). Developing ambitious mathematics instruction through web-based coaching: a randomized field trial. *American Educational Research Journal*, *57*(6), 2378–2414.

Litke, E. G. (2020). Instructional practice in algebra: Building from existing practices to inform an incremental improvement approach. *Teaching and Teacher Education*, *91*. https://doi.org/10.1016/j.tate.2020.103030

Liu, S., & Phelps, G. (2020). Does teacher learning last? Understanding how much teachers retain their knowledge after professional development. *Journal of Teacher Education*, *71*(5), 537–550.

Morris, A. K., & Hiebert, J. (2011). Creating shared instructional products: An alternative approach to improving teaching. *Educational Researcher*, *40*, 5–14. https://doi.org/10.3102/0013189X10393501

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for research in mathematics education*, *45*(5), 584–635.

National Council of Teachers of Mathematics. (2009). *Focus in high school mathematics: Reasoning and sense making*.

Otten, S., Cirillo, M., & Herbel-Eisenmann, B. A. (2015). Making the most of going over homework. *Mathematics Teaching in the Middle School*, *21*, 98–105.

Otten, S., Ellis, R. L., Wang, Z., & de Araujo, Z. (2021). Comparing motivations for flipped instruction to data on flipped implementations in algebra. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the 43rd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1554–1558). Philadelphia, PA: PME-NA.

Sims, S., & Fletcher-Wood, H. (2021). Identifying the characteristics of effective teacher professional development: a critical review. *School effectiveness and school improvement*, *32*(1), 47–63.

Smith, M., Bill, V., & Hughes, E. (2008). Thinking through a lesson: Successfully implementing high-level tasks. *Mathematics Teaching in the Middle School*, *14*, 132–136.

Spangler, D. A. (2019, February). *Fundamental commitments of my work as a mathematics teacher educator*. Judith E. Jacobs lecture at the annual conference of the Association of Mathematics Teacher Educators. Orlando, FL.

Star, J. R. (2016). Improve math teaching with incremental improvements. *Phi Delta Kappan*, *97*(7), 58–62.

Steen-Olsen, T., & Eikseth, A. G. (2010). The power of time: Teachers' working day—Negotiating autonomy and control. *European Educational Research Journal*, *9*(2), 284–295.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. Simon and Schuster.

Thaler, R. H., & Sunstein, C. R. (2008). *Nudge: Improving decisions about health, wealth, and happiness*. Penguin.

Valoyes-Chávez, L. (2019). On the making of a new mathematics teacher: Professional development, subjectivation, and resistance to change. *Educational Studies in Mathematics*, *100*, 177–191.
Wilkie, K. J. (2019). The challenge of changing teaching: Investigating the interplay of external and internal influences during professional learning with secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *22*(1), 95–124.
Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. *Review of Research in Education*, *24*, 173–209. http://www.jstor.org/stable/1167270

ELEMENTARY TEACHERS' CONCEPTIONS OF ARGUMENTATION AND THEIR ARGUMENT-BASED LESSON DESIGNS FOR TEACHING MATHEMATICS AND PROGRAMMING

Hyejin Park
Drake University
hyejin.park@drake.edu

Tugba Boz
University of Jyväskylä
tugboz@jyu.fi

ChanMin Kim
Pennsylvania State University
cmk604@psu.edu

We explored how elementary teachers designed argumentation-based lessons in the context of robotics-integrated mathematics classrooms. We analyzed data collected from two teachers who taught elementary mathematics and robot programming during their engagement with a professional learning course. In the course, teachers learned to use argumentation in teaching mathematics, science, and programming. We analyzed their lesson plans to understand how they structured their lessons to use argumentation in robotics-integrated mathematics classrooms. We also examined the teacher interview data collected after the course ended to capture their understanding of argumentation. Our data analysis results showed that the ways that teachers designed argumentation-based mathematics lessons with robotics and programming were largely aligned with their interpretations of argumentation. The study findings are discussed and future studies are suggested.

Keywords: Professional Development, Integrated STEM / STEAM, Computing and Coding, Classroom Discourse

Current K-12 standards and mathematics education researchers emphasize the importance of learning and teaching argumentation in mathematics classrooms (e.g., National Council of Teachers of Mathematics, 2014; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Staples & Newton, 2016). Collective argumentation occurs in the classroom when students and teachers work together to make a claim and support it with evidence (Conner et al., 2014). The benefits of using argumentation in the classroom include the development of student conceptual understanding of mathematical ideas, mathematical autonomy, communicative skills, and authentic mathematical practice (e.g., Andriessen, 2006; Staples & Newton, 2016; Yackel & Cobb, 1996). We believed that teaching programming through argumentation also could have benefits such as conceptual understanding of programming ideas, especially when teachers use pedagogical practice already in use in their mathematics classrooms to teach programming, thus increasing the possibility that teachers incorporate programming in mathematics classrooms.

In this paper, we report our study of elementary teachers who learned to teach mathematics and programming through argumentation. The following research question was addressed: *How do elementary teachers design argumentation-based lessons for robotics-integrated mathematics instruction at the end of a professional learning course in which they were introduced to use argumentation in teaching programming?*

Related Literature and Theoretical Framing

There is a natural connection between programming and mathematics (Savard & Highfield, 2015). The merits of using programming and robotics in mathematics education have been well-documented. For example, research has shown that mathematics activities integrated with programming and robotics can support student understanding of mathematics (e.g., Wei et al., 2011), develop student spatial abilities (e.g., Keren & Fridin, 2014), and provide students opportunities to express their mathematical thinking and reasoning (e.g., Ke, 2014). However, there is little research on the experiences of teachers who teach both programming and mathematics in their classrooms. The present study explored how teachers designed argumentation-based mathematics lessons with programming and robotics. To understand teacher learning about the use of argumentation in teaching programming and mathematics, we analyzed their lesson designs and interviews considering the context that they were situated as elementary teachers using a situative perspective (Peressini et al., 2004).

Methods

This study examined how elementary teachers designed argumentation-based mathematics lesson plans integrated with robotics toward the end of a semester-long professional learning course. The course occurred in a school district in the southeastern United States. Nineteen elementary teachers recruited from five different elementary schools were enrolled in the course. The course consisted of four face-to-face meetings (approximately 10 hours in total) and eight online exercises. Three instructors co-taught programming and robotics as well as the use of argumentation in teaching mathematics, science, and programming.

During the first two face-to-face class meetings, participating teachers discussed *argumentation* in mathematics and science classrooms and analyzed examples of classroom episodes identifying components of an argument (e.g., claim, data, and warrant) and focusing on teacher actions in argumentation. Throughout the course, teachers learned a variety of robots using block-based programming language. Before the last face-to-face class meeting, teachers were assigned to design and teach a mathematics lesson focusing on argumentation in their classrooms. They video-recorded a part of their lesson and discussed how they supported student argumentation during the fourth face-to-face class meeting with other teachers. After this meeting, teachers were asked to design a unit plan including three to five lessons integrating robotics and argumentation into mathematics classrooms. Our focused data to answer the research question was these integrated lessons.

We focused on two teachers, Cari and Malia (pseudonyms), as they implemented robotics and argumentation in their mathematics classrooms. Cari was an experienced teacher, and Malia was a novice teacher (see Table 1 for teachers' background information). Their robotics-and-argumentation-integrated mathematics lesson plans and post-interviews were analyzed. Using open coding (Strauss & Corbin, 1990), we first examined their robotics-and-argumentation-integrated mathematics lesson plans line by line, focusing on the parts in which they described student argumentation activities. We then made a memo for each teacher describing the ways that they designed lessons to create argumentation in their classrooms. Finally, we examined how each teacher understood and interpreted argumentation using the interview data. We also made a memo for each participant describing their views on argumentation.

Table 1: Teachers' Background Information

Teacher	Grade	Years of Teaching Experience	Previous Experience with Programming and Robotics	Previous Experience with Argumentation
Cari	3rd	16-20 years	Little	Little
Malia	5th	3-5 years	Intermediate	Little

Results

This section reports how Cari and Malia designed their argumentation-based robotics-integrated mathematics lessons and their conceptions of argumentation in elementary classrooms.

Cari's Lesson Designs for Creating Argumentation in the Robotics-Integrated Mathematics Classrooms

Cari designed 2-day-long robotics-integrated mathematics lessons for 3rd graders focusing on the concepts of perimeter and area. The learning goal of the first-day lesson was that students would be able to program robots to trace the perimeter of a given square. Cari created the second-day in-class activity to engage students in programming their robots, drawing different shapes with the same perimeter but different areas. Cari expected her students to work on these activities in pairs and make claims for each movement of the robots. She anticipated that argumentation would occur between and among students when they work together to figure out the area or test out their code with their robots and also when they disagree with each other. She emphasized that when team members agree on a claim suggested by a team member, the team can move on to the next step for solving a given task.

Cari's Conceptions of Argumentation

For Cari, argumentation was perceived as the process of arguing, proving, justifying, convincing others, and building a consensus within a classroom community. She believed that the benefit of engaging students in argumentation was to help students gain the confidence of speaking out their voices in public places (e.g., classrooms) and sharing their ideas with others. She believed that the more confident students become, the more student participation occurs in argumentation. She also noted that student claims might not always be correct but giving them opportunities to experience defending their claims in class helps students develop their skills of arguing and convincing others, which she saw as necessary life skills.

Malia's Lesson Designs for Creating Argumentation in the Robotics-Integrated Mathematics Classrooms

Considering her 5th graders, Malia designed 2-3 day class activities for her mathematics and robotics integrated lessons. The lesson goal of the first-day activity was to have students graph points on a coordinate plane and program their robots to move from one point to another. Malia expected students to find the area and perimeter of a geometrical shape they created by connecting the points on the plane. She designed the second- and third-day activities to have students explain their knowledge of graphing points on the plane and using the robots. Malia expected students to work with their groups collaboratively during the activities and specified how she wanted students to complete the activities. However, she did not discuss when, how, and what arguments would occur during student participation in the activities within her written lesson plans.

Malia's Conceptions of Argumentation

Malia saw argumentation as a process of explaining and agreeing or disagreeing with others' ideas. She thought that the critical component in argumentation was the justification part. Malia saw the use of argumentation in teaching programming would be relevant when prompting students to explain why their code works or does not. She noted that making arguments might be challenging for students when they did not know how to explain or justify their answers (claims). She also thought that students might not see justifications necessary, particularly when they think something is obviously true. She believed that engaging students in argumentation could improve their writing skills as they learn to justify with evidence.

Discussion and Conclusion

Both Cari and Malia saw argumentation as a transferable skill that students can develop in both mathematics and programming classes. They thought that engaging students in argumentation would help students learn how to argue and reach a consensus in the classroom. We also found the way that the teachers designed argumentation-based robotics-integrated mathematics lessons was largely aligned with their conceptions of argumentation. This finding is aligned with many prior studies that have shown the close relationship between teacher pedagogical conceptions and classroom practice (e.g., Kim et al., 2013). Another finding was that Cari specified when, what, and how student argumentation would occur during the robotics-integrated mathematics activities that she designed. In contrast, Malia, a novice teacher did not. We conjecture that for a novice teacher, it might be challenging to anticipate student argumentation in the classroom.

In their lesson plans, both teachers did not clearly describe their expectations of how students would use mathematics or programming knowledge when justifying their claims about each robot movement during the activities. They also did not plan on how they would facilitate student argumentation. Future studies are needed to see how teachers actually enact argumentation in their classrooms based on their lesson plans and explore the support teachers need in planning integrated lessons for productive classroom argumentation.

Acknowledgments

The work was supported by the National Science Foundation under Grant No. 1741910. Opinions, findings, and conclusions in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Andriessen, J. (2006). Arguing to learn. In K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 443–459). Cambridge University Press.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, *86*(3), 401–429.

Ke, F. (2014). An implementation of design-based learning through creating educational computer games: A case study on mathematics learning during design and computing. *Computers & Education*, *73*, 26–39.

Keren, G., & Fridin, M. (2014). Kindergarten Social Assistive Robot (KindSAR) for children's geometric thinking and metacognitive development in preschool education: A pilot study. *Computers in Human Behavior, 35*, 400–412.

Kim, C., Kim, M. K., Lee, C., Spector, J. M., & DeMeester, K. (2013). Teacher beliefs and technology integration. Teaching and Teacher Education, 29, 76–85. https://doi.org/10.1016/j.tate.2012.08.005

National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards: Mathematics standards*. Washington, DC: Author.

Peressini, D., Borko, H., Romagnano, L., Knuth, E., & Willis, C. (2004). A conceptual framework for learning to teach secondary mathematics: A situative perspective. *Educational Studies in Mathematics, 56*(1), 67–96.

Savard, A., & Highfield, K. (2015). Teachers' talk about robotics: Where is the mathematics? In M. Marshman, V. Geiger, & A. Bennison (Eds.). Proceedings of the 38th annual conference of the mathematics education research group of Australasia (pp. 540–546). MERGA.

Staples, M., & Newton, J. (2016). Teachers' contextualization of argumentation in the mathematics class- room. *Theory into Practice, 55*, 294–301.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. London, UK: SAGE.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal*

for Research in Mathematics Education, 27, 458–477.
Wei, C. W., Hung, I. C., Lee, L., & Chen, N. S. (2011). A joyful classroom learning system with robot learning companion for children to learn mathematics multiplication. *Turkish Online Journal of Educational Technology, 10*(2), 11–23.

UNPACKING THE FIVE PRACTICES USING AN ONLINE PROFESSIONAL DEVELOPMENT

Amanda Reinsburrow
Drexel University
Alr352@drexel.edu

Jason Silverman
Drexel University
Js657@drexel.edu

Valerie Klein
Drexel University
Vek25@drexel.edu

Wesley Shumar
Drexel University
shumarw@drexel.edu

Anthony Matranga
California State University
San Marcos
amatranga@csusm.edu

Understanding and utilizing the Five Practices for Orchestrating Productive Mathematics Discussions (Smith et al., 2009; Stein et al., 2008) has grown in importance as teachers continue to be challenged to engage their students in student-centered, discourse-based instruction. We argue that by engaging in rehearsals of the five practices and decomposing practice, the teachers are not only able to advance their understandings of each of the five practices, but also gain knowledge of its value in their classroom practice and its role in the supporting student development. Preliminary results from analysis of our online professional development suggest that participants learned more about utilizing the five practices within their classrooms and are able to share the role they believe it has in their classroom practice.

Keywords: Instructional Activities and Practices, Professional Development, Teacher Knowledge

Objectives and Purposes

Our work focuses on the design and implementation of a three-part online professional development devised to support teachers as they (1) work collaboratively to develop mathematical and pedagogical knowledge and (2) seek to integrate this knowledge and experience into their classroom. Specifically, the professional development activities feature a continual interplay between activities focused on the development of mathematics knowledge for teaching (Renninger et al., 2011; Shumar, 2017) and classroom practice, increasing the likelihood that teachers' online experiences will connect to and support changes in their instructional practices. In this paper, we report on one component of this work which seeks to help teachers unpack the Five Practices for Supporting Productive Mathematical Discourse (Smith et al., 2009; Stein et al., 2008) using rehearsals (Anthony et al., 2015, Kazemi et al., 2016), which provide "a space where teachers can deliberately experiment with practices in less complex settings" (Webb & Wilson, 2022, p. 129). The current paper documents teachers' participation and experiences in rehearsals and explores their use of the five practices within their classroom practice.

Theoretical Framework

According to Smith et al. (2009), the five practices for mathematical thinking and discussion are:

> (1) anticipating student responses to challenging mathematical tasks, (2) monitoring students' work on and engagement with the tasks, (3) selecting particular students to present their mathematical work, (4) sequencing the student responses that will be displayed in a specific

order, and (5) connecting different students' responses and connecting the responses to key mathematical ideas. (p. 550)

These practices help teachers to think about and utilize their students thinking to advance mathematical understandings of the entire class. The practices encourage teachers to spend more time in the planning phase to be better able to facilitate productive discussions among the students (Smith & Sherin, 2019). Stein et al. (2008) encourage teachers to utilize the five practices as a "method for slowly improving the quality of discussions over time" (p. 26).

In this project, teachers' engagement with the five practices within the professional development was situated in a rehearsal (Anthony et al., 2015, Kazemi et al., 2016) of the five practices during the course of the workshop. The design of the professional development workshop was built on allowing the teachers time to practice and experiment with the five practices in a non-threating way (Webb & Wilson, 2022) and allowed the teachers to analyze and reflect on each of the Five Practices without the constraints or pressure of time or "live" students. Each module in the workshop was designed to allow teachers to engage with a different practice:

Module 1 – Talking Notice/Wonder and Doing the Math represents the work of anticipating student thinking;
Module 2 – Looking at Student Work represents monitoring students' work;
Module 3 – Reacting to Student Work represents selecting and sequencing; and
Module 4 – Connecting to the Classroom represents connecting responses to others and mathematical ideas.

This design essentially decomposes each of the Five Practices into 1–2-week online modules. Therefore, we argue that by engaging in these structured rehearsals and with the support of both colleagues and facilitators, teachers can learn about the practice, use it to analyze a set of student work, and develop conjectures for potential instructional moves. Thus, the teachers are better positioned to do this work in their own classrooms.

Methods

We investigated teachers thinking about and unpacking of the five practices in two iterations (C1 & C2) of an online professional development workshop. The workshop included four modules designed to scaffold the rehearsal process as well as utilize the five practices. Teachers engaged in solving and discussing solutions to a problem (anticipating), noticing and wondering about a set of student work (monitoring), organizing their notices and wonders in meaningful ways (selecting and sequencing), and making decisions about what to do next for individual students and whole class discussions (connecting). The two iterations of the workshop differed in length and methods of engagement. C1 (n=13) participated in a two-week workshop consisting of six discussion boards, three synchronous sessions, and four journal entries. C2 (n=9) participated in a six-week workshop consisting of five discussion boards, six synchronous sessions, and four journal entries. In this research, we explore the research question: *In what ways do teachers see the role of the five practices for mathematical thinking as a means to providing feedback to students?* In particular, we seek to understand teachers thinking about and use of the five practices in their classrooms.

Data for this paper consisted of participating teachers' posts to one discussion board, one journal entry and transcripts from two synchronous sessions (see Table 1 for prompts provided to the teachers for each data set). Data analysis included a three-phase process, beginning with pre-coding, where the entire project team reviewed the entire data corpus multiple times to become familiar (Ravitch & Carl, 2015). The second phase featured open coding of the data by at least two members of the research team. In this phase, every effort was made to capture the theoretically significant ideas (such as terms from the Five Practices and student-centered, research-based classroom practices). In the final phase, the research team resolved any inconsistencies in the coding and developed and tested conjectures regarding themes in the coded data. They met to generate a theoretical memo that was be based on both the field notes and the patterns and themes from the coded data (Miles et al., 2014).

Table 1: Analyzed Prompts

	C1 (summer 2021)	C2 (winter 2022)
Discussion Board #5	Share something from the article (Smith et al., 2009) that connects to what we have been working on. How might you implement a version of it in your classroom?	Imagine that you are using EnCoMPASS to help you implement the 5 practices in your classroom. Brainstorm ways in which you think this might be accomplished. (You might share examples using the work you have done with the Driving to Work student responses.)
Journal #4	What do you see as the role of the 5 practices for mathematical thinking in providing feedback to students?	What do you see as the role of the 5 practices for mathematical thinking in providing feedback to students?

Results

While we continue to code additional data and revise and refine our conjectures regarding emergent themes, preliminary findings reveal a connection to the five practices, a desire to utilize them more within their classrooms, and challenges they have faced. Four themes are discussed.

The first theme that emerged was teachers beginning to see the how and why they might use the five practices in their classrooms. Examples of excerpts in this category included, (1) GA noted, “I have also learned that structuring class discussions around solutions (incorrect and correct) should work towards the larger learning goal. The five practices serve the role of structuring discussions around challenging tasks and providing a framework for building complex mathematic ideas.” (2) MJ noted, “I think their [the five practices] role is to keep us teachers focused on the ‘mathematical ideas at the heart of the lesson.’ It's a sturdy, efficient, organized framework on which to build our feedback and plan our lessons and direct the course of mathematical discourse during class.” These excerpts indicate that while math teachers are aware of the five practices, they need more support to build their foundation for success and professional development is the space for this work to be conducted.

The second theme was that while five practices are hard to do, it was even harder to learn, especially in real time. As an example, KL stated, “I like that we are getting this practice in this scenario as selecting is one step of the 5 practices that I find difficult to do in real-time in the classroom. Practicing in isolation is helpful.” Teachers appeared to recognize the benefit of engaging in practice or rehearsal of the five practices so that they can gain understanding of their use in low-stakes environments. By participanting in professional development that encourages and supports this practice, teachers can feel more confident utilizing the five practices in their classrooms.

The third theme indicated an awareness that in addition to learning more about the five practices, teachers needed support with how to get access to student thinking (and not just student answers). As an example, MD noted:

> If I were to implement a version of this [marble problem] in my classroom… I'd do this initial notice/wonder activity at the end of a lesson on Friday. I'd collect the students' notices/wonders, and then use those over the weekend to build my anticipated answers/approaches and prepare for the actual lesson on the problem for the following week. Out of all the five practices, anticipating the responses is probably the trickiest, and I think it really gets better with experience or as they mentioned, using work produced by prior classes, publishes responses, etc. No one has the actual time in school to have several teachers work the problem and get feedback (what a pipedream), so I think this prelude with notice/wonder would be very useful for this purpose.

This category highlights the importance of students engaging in the tasks assigned before the five practices can be thought about and that utilizing noticing and wondering as an engagement practice is useful for planning and implemenation.

The fourth theme was the need for expanding the five practices to support equity in the classroom. As an example, SS stated:

> I work with students with disabilities so explaining their thinking can be challenging. Sometimes I see students struggle to find the right words to explain their work/answers. I am going to try to highlight what they did correctly to alleviate some of the anxiety they may be feeling. In our group discussion the other night it came up that written feedback should be both specific and doable. I am thinking these same guidelines apply to classroom conversations with students. I am going to try to ask more specific questions about their work.

While the five practices provide a structure of guiding productive and supportive classroom discussions, there are still students who may be left behind depending on the implementation utilized by the teacher. It is important for teachers to realize that while monitoring, selecting, and sequencing, they also need to be aware of students who may not be in a place to be chosen and how to they keep those students engaged in the discussion.

Discussion

Based on the preliminary findings, the teachers not only enjoyed learning more about the five practices and their implementation, but also felt as if it provided thoughtful ways to continue the engagement within their classrooms. They felt that by participating in the rehearsals or "practice in isolation" they were able to build their skills before working with their students. They demonstrated the need to engage all students by allowing them entry into the problem utilizing notice and wonder processes. Finally, they noted that to be implemented equitably teachers need to be aware of the students and how they are engaging in the discussions.

These findings demonstrate the potential for online professional development to provide experiences that allow teachers to think deeply about common frameworks for student-centered instruction, such as the Five Practices, and for them to reflect on the details of implementation. Current work includes further analysis of these online workshops as well as analysis of teachers' reflections on their post-workshop goals for instruction. We expect to see additional support of these themes within the teachers' reflections as well as the possibility for new themes to emerge.

Acknowledgements

This work was supported by NSF Discovery Research K-12 Grant # 2010306. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation. The Authors would like to thank Sienna Medina and Mikhail Miller of Drexel University for their ongoing collaboration and support of the research described in this paper.

References

Anthony, G., Hunter, J., & Hunter, R. (2015). Supporting prospective teachers to notice students' mathematical thinking through rehearsal activities. Mathematics Teacher Education and Development, 17(2), 7-24.

Kazemi, E., Ghousseini, H., Cunard, A., & Turrou, A. C. (2016). Getting inside rehearsals insights from teacher educators to support work on complex practice. Journal of Teacher Education, 67(1), 18-31.

Miles, M. B., Huberman, A. M., & Saldana, J. (2014). *Qualitative Data Analysis: A Methods Sourcebook* (3rd edition). SAGE Publications, Inc.

Ravitch, S. M. & Carl, N. M. (2015). *Qualitative Research: Bridging the Conceptual, Theoretical, and Methodological* (1st edition). SAGE Publications, Inc.

Shumar, W. (2017). Inside Mathforum.org: analysis of an Internet-based education community (Inside Math Forum). New York, NY: Cambridge University Press.

Renninger, K. A., Cai, M., Lewis, M. C., Adams, M. M., & Ernst K. L. (2011). Motivation and learning in an online, unmoderated, mathematics workshop for teachers. Educational Technology Research and Development, 59(2), 229-247.

Smith, M. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. Mathematics Teaching in the Middle School, 14(9), 548-556.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical thinking and learning*, *10*(4), 313-340.

Webb, J. & Wilson, P. H. (2022). Designing rehearsals for secondary mathematics teachers to refine practice. Math Teacher Educator, 10(2), 129-142.

"IT'S A DIFFERENT MINDSET HERE": FACILITATION CHALLENGES IN A PRACTICE-BASED PROFESSIONAL DEVELOPMENT

Gil Schwarts
University of Michigan
gils@umich.edu

Irma E. Stevens
University of Michigan
istevens@umich.edu

Patricio Herbst
University of Michigan
pgherbst@umich.edu

Amanda Brown
University of Michigan
amilewsk@umich.edu

In this paper we examine how facilitators' prior experiences as mathematics teachers frame their work when facilitating a practice-based professional development (PD) for the first time. We focus on the experiences of a novice facilitator of StoryCircles, a professional learning process in which teachers collectively script and visualize a problem-based lesson, arguing about their rationales for different decisions connected to discussions of students' work. We situate the challenges the facilitator encountered by identifying the expectations that were not met while she was facilitating and offer possible connections to the PD design. We close by suggesting a perspective to account for facilitation challenges.

Keywords: Problem-Based Learning; Professional Development; Teacher Educators.

Background and Theoretical Framing

In recent years, research on the facilitation of professional development (PD) programs has received growing attention due to facilitators' key role in the implementation of educational initiatives (Rösken-Winter et al., 2021). Within this emerging field, scholars have discussed the preparation of facilitators (Lesseig et al., 2017), their knowledge and practices (Borko et al., 2014; Karsenty et al., 2021), and facilitators' professionalization processes (Schwarts et al., 2021). When referring to novice facilitators who are also practicing mathematics teachers, the main difficulties described in the literature are related to their limited capacity to lead in-depth discussions with teachers (Borko et al., 2014) and to the complex navigation between their multiple identities as facilitators, teachers, and colleagues (Knapp, 2017). PD facilitators find it difficult to gauge the extent of their involvement in managing participant discussions (Lewis, 2016) and they struggle to enact their ambitious goals when designing and implementing activities (Jackson et al., 2015). Overall, these results support the argument that "being a good teacher does not necessarily imply the ability to help others develop their teaching" (Even, 2005, p. 334). In spite of this work, there is still much to learn regarding the underlying factors that constrain facilitators working in the context of practice-based PD. In particular, it is important to understand in what ways complexities of the activity of facilitation rather than deficits of the facilitators themselves, might account for difficulties observed.

This paper seeks to frame the challenges of facilitation as rooted in facilitators' implicit expectations about learning in practice-based PD settings. For that purpose, we refer to facilitators' challenges in the context of the PD triangle offered by Carroll and Mumme (2007, see Figure 1). This triangle embeds the well-known instructional triangle (Cohen et al., 2003), including its claims about the interconnections between the different components of instruction, into the facilitators' level. The PD triangle includes the three vertices: the facilitator, the teachers (hereafter, referred to as participants or practitioners), and the practice of teaching and learning mathematics in the place of the content (see Figure 1). As in the case of the instructional triangle, we can hypothesize this PD triangle to be situated in environments, particularly the institutional environments that enable practitioners to participate. This framing allows us to situate facilitators' expectations with respect to the other components of the PD environment. The expectations we are interested in refer to how and what participants are supposed to learn (participants-practice arrow), and the facilitator's role with respect to this learning (how the facilitator relates to practice, or the facilitator-practice arrow, and how the facilitator relates to the participants, or the facilitator-participants arrow). We hypothesize that facilitators' years-long experiences as practicing mathematics teachers who have also participated in PD inform these expectations and influence facilitators' practices in ways that are not always discernible for them and for developers of PD environments.

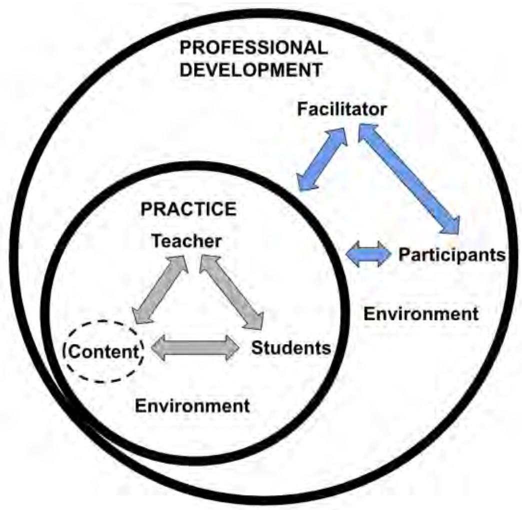


Figure 1. The PD triangle (adapted from Carroll & Mumme, 2007, p. 11)

This paper focuses on facilitation in a practice-based PD, where practitioners "learn in and from practice" (Cohen & Ball, 1999, p.18) by collaboratively inquiring on artifacts of practice, analyzing them and arguing about them. In such settings (i.e., ones that center on practitioners' reflections and where there is no specific content to be taught), the facilitator's role as a moderator of practitioners' discussions is even more complex (Schwarts et al., 2021). In this context we ask, how do facilitators' expectations based on their prior experiences shape their work facilitating a practice-based PD?

Context: The Story*Circles* process

The context of this study is Story*Circles* (Herbst & Milewski, 2018), a process of teacher collaboration that aims to engage practitioners in collective scripting, visualizing of, and arguing about a problem-based lesson. Story*Circles* has evolved over its various iterations, but it has consistently maintained the goal of having practitioners represent their practice through storyboarding a collective lesson (see Brown et al., 2021; Milewski et al., 2018, 2020 for examples). Inspired by Japanese lesson study (see Herbst & Milewski, 2018), the main design concepts of Story*Circles* include opportunities for rapid prototyping of lessons and a user-centered design (Herbst & Milewski, 2020). Thus, the discussions that are at the core of Story*Circles* position practitioners as experts and support them in talking with one another about their rationale for making certain, sometimes competing, decisions in the classroom.

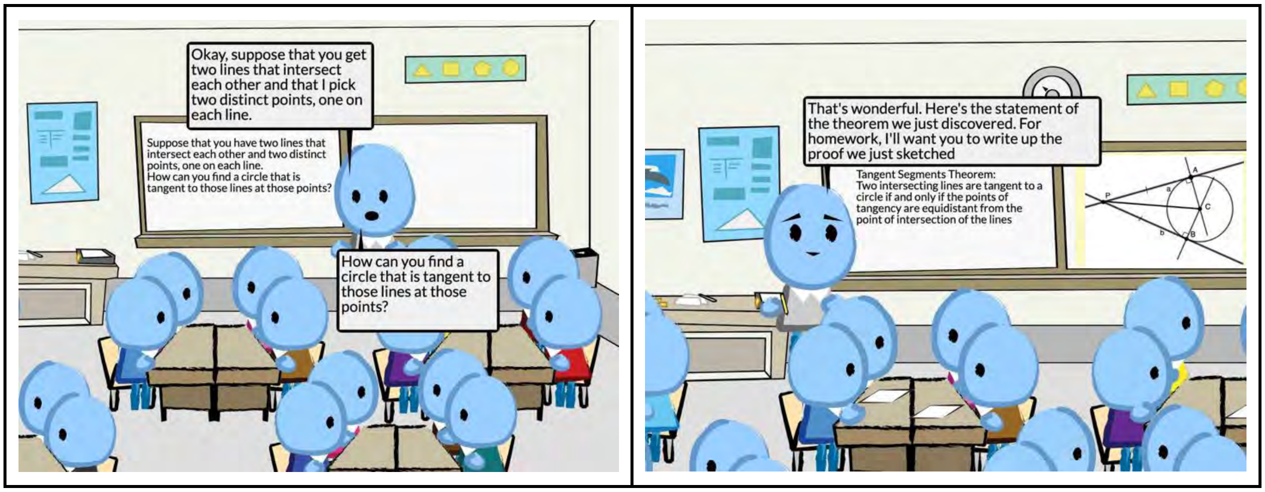


Figure 2. Scenes at the beginning and the end of the Tangent Circle lesson (© 2021, The Regents of the University of Michigan, used with permission)

In the iteration of Story*Circles* considered in this report, that took place between February and April 2021, secondary geometry teachers engaged over six weeks in discussions focused on a problem-based lesson aimed at introducing the tangent segments theorem. This lesson starts with the posing of a problem (see Figure 2, left) and ends with the statement of the instructional goal of the lesson (Figure 2, right). The lesson is illustrated in a storyboard using cartoon characters to represent a teacher and their students. Given the problem and the instructional goal of the lesson and one potential instantiation of the lesson provided to them in advance, participants started this Story*Circle* at the visualization phase, annotating what the provided version of the lesson looked and felt like to them and inserting comments where they thought the teacher might have done something differently. During six weeks they prototyped alternatives to those moments including alternative moves to manage whole class discussions and review students' work. Participants were expected to collectively argue about alternative ways in which the teacher could have handled events in the lesson, scripting the alternative scenes, which would also be storyboarded and visualized. As they scripted potential instantiations of how the lesson might unfold, it was expected they would engage in argument about decisions the teacher would make in the lesson. Connecting it to the role that experimentation plays in Clarke and Hollingsworth's (2002) model of teachers' professional growth, Milewski et al. (2020) have described Story*Circles* as "virtual professional experimentation" (p. 624). During Story*Circles*, practitioners worked on the different segments of the lesson in a combination of synchronous and asynchronous activities. These different forms of interaction permit practitioners to dedicate time to tinker with particular moments in a lesson and consider alternative ways of handling various contingencies that might arise in an instance of the lesson.

The avowed goal of each cycle of Story*Circles* is to produce a representation of a collaboratively developed lesson. Although the particulars of the entire design of Story*Circles* are beyond the scope of this paper, we describe three different design features unique to the present iteration of the Story*Circles* design. First, participants were not expected to attend every synchronous meeting – each participant was scheduled to attend only three or four out of six such meetings, so every synchronous meeting included a subset of participants (and not always the same people in each subset). Second, the meetings were not intended to build on one another; instead, each of them focused on a specific segment of the lesson. Third, by design, the facilitator is not responsible to use the software to prototype the lesson scenes the participants propose; rather two assistants (undergraduates skilled in the storyboarding software) are present in every synchronous meeting. These storyboarders represented suggestions participants made in real time during discussions. The resulting storyboards served as collective artifacts the participants could use to visualize what the lesson looked and felt like so as to argue about decisions the virtual teacher had made later on and possibly occasioning new scripting or revisions.

Method

This study is part of a five-year research project focused on teachers' learning about discussion-based teaching of Algebra and Geometry lessons. The facilitators for the 2021 iteration of Story*Circles* were practicing secondary school teachers in their content area. For Geometry, the facilitator was Quincy, an experienced teacher who had been a participant in a previous iteration of Story*Circles*, and this was her first time facilitating the work. Fourteen participants took part in this cycle.

Data collection and analysis

To identify how the facilitator's prior experiences in teaching and professional development framed her management of the Story*Circles* process, we searched in our records for evidence that her expectations were not being met. This approach builds on the idea that individuals' tacit expectations can be revealed through their reactions to deviations from customary practices (Herbst et al., 2011). The data corpus for the 6-week cycle includes, among other things, recordings and transcriptions of six synchronous meetings and follow-up debriefs (6*90 minutes). Using thematic analysis (Braun & Clarke, 2006), records were inspected for evidence that the facilitator's expectations were not fulfilled. This evidence included moments during facilitation or in the follow-up debriefs in which the facilitator expressed (using statements or gestures) surprise, confusion, conflict, discomfort, or puzzlement. We also searched for the sources of these expressions as could be inferred from the facilitator's expression (e.g., when she said "It's a different mindset here, having help", we inferred that her expectation to work individually, rather than have storyboarders representing the lesson that participants script, is related to her classroom experience in which she does not have assistants). After the identification of these moments, they were organized according to themes and were mapped to highlight the interrelations between the facilitator's expectations, the experiences that were likely to shape those expectations, and the design principles of Story*Circles*.

Findings

The following are two prominent themes from the analysis that are representative of the facilitator's expectations.

Theme 1: The traditional turn taking that animates the PD triangle

The first theme illustrates the facilitator's difficulty in adjusting herself to work in collaboration with the storyboarders and to manage time. To illustrate this difficulty, we provide some more details on the work of storyboarders during PD discussions. Story*Circles* synchronous meetings usually open with a question that the facilitator poses about a scene in the lesson at hand (for example, "Are there things that the teacher might say in their introduction of that student work?", Turn 242, second meeting). While participants raise alternatives, the storyboarders create images that represent the participants' suggestions and incorporate them into the representation of the lesson (see example in Figure 3). These images are an essential component of the argumentative process in Story*Circles* since they provide a shared representation on which participants can argue. However, after Quincy's facilitation of the first meeting, she reflected that the use of these new images was in fact overwhelming for her:

> So I guess the part that I felt most awkward about was like transitioning between that, and then like jumping down to the depictions [...]. Partly I was worried like I… did I give them [the storyboarders] enough time to depict before jumping down? so I was stalling on that a little bit (Turns 546-547, first meeting debrief).

This quote reveals the facilitator's difficulties in managing a discussion while simultaneously considering the materials the storyboarders were creating and the time they needed to create them. In addition, she mentions that moving between slides and referring to ad-hoc ideas was "awkward". She expressed similar feelings two weeks later, after the third meeting:

> Not sure how to like really utilize what you guys [talking to the storyboarders] are doing on the depicting, like I can see you're like going crazy, I don't know if I need to like just dive into the depiction earlier (Turn 601, third meeting debrief).

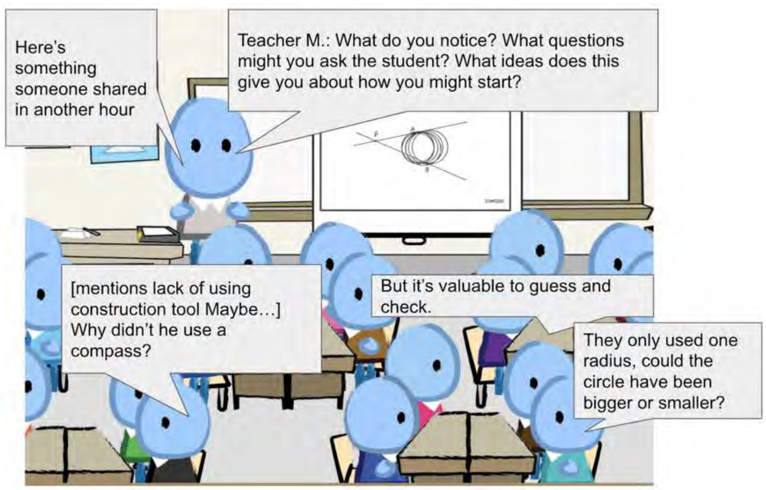


Figure 3. Image created by the storyboarders during the third synchronous meeting (© 2021, The Regents of the University of Michigan, used with permission)

Beyond knowing when to use the images, another source of tension for Quincy was how to use them: In this meeting, the storyboarders represented almost every comment made by participants. Doing so resulted in many new images, illustrating different pathways in the lesson, that Quincy felt obliged to "utilize". All of the above evidence points to Quincy's expectations of having control, both on the timing and on the mediation and interpretation of participants' contributions. This claim is supported by another comment made in the debrief of the third meeting, where Quincy compared facilitation to teaching:

> When I'm teaching off of the Google slides right now, like every day, I'm in control of them or I'm having the students interact with them, so it's very [...] It's a different mindset here, having help (Turn 681, third meeting debrief).

These moments suggest that Quincy's expectation of having full control is informed by her experiences as a mathematics teacher, used to working individually ("It's a different mindset here, having help", Turn 681). That is, "having help" is something she is unaccustomed to, and perhaps, at that moment, she felt that the work of the storyboarders did not help her at all. This challenge points to a disruption in the traditional turn taking between teachers and students (e.g., McHoul, 1978), or between facilitators and participants. In the latter case, the facilitator is commonly the only one who is responsible for the interpretations of the participants' contributions and their mediations with the PD content. Even if participants raise unexpected ideas, the facilitator knows (even if only tacitly) that it is her job to address them. However, in Story*Circles*, the participants' ideas are also interpreted by the storyboarders. The expectation that her facilitation role is similar to that of a teacher seemed to activate in Quincy the expectation that her job included managing more interactions—namely ensuring that: (1) participants' ideas are explicit enough for the storyboarders to depict, (2) the storyboarders have accurately captured participants' ideas. In addition, as described above, she appeared to take responsibility for pacing the discussion according to the time depicting took, even though storyboarders were not part of the conversation. It follows that even for practitioners who are familiar with managing discussions and allocating turns of talk, facilitating a Story*Circles* discussion is a challenging task as it may involve a decreased sense of control. We posit that such disruption, although challenging for facilitators, is an important feature of the learning environment offered in Story*Circles*: It requires participants to explicate their arguments, specify them, and explain how they relate to previous comments. That is, because the participants are able to see their own and others' contributions represented visually in real time, the participants are in an environment in which they have greater opportunities, and perhaps feel more accountable, to engage with and reflect on one another's comments. This, along with the facilitator's prompting, recruit participants' attention to others' arguments, and sharpens the ways they communicate and argue about their practice. For the facilitator, however, it adds layers of complexity.

Theme 2: The agreed-upon goal of the meetings

The second theme alludes to the ways the facilitator envisioned the goals of Story*Circles,* which were sometimes in conflict with the program design and the participants' goals. This tension was manifested in Quincy's goal to improve the storyboarded lesson in a certain way, including her expectation to build on previous meetings while doing so. As mentioned above, a main design feature of Story*Circles* is that only subsets of the enrolled participants engage in each synchronous meeting. Accordingly, all but one of the participants who attended the second synchronous meeting had not been present in the previous meeting. Quincy planned to work in this meeting on part of the lesson where the students were stuck and the teacher redirected them by discussing with them pre-selected students' work, as had been decided by the group who participated in the first meeting. However, when she asked the participants which pieces among the pre-selected student work they would like to present, they did not answer her question, but instead wanted to work on improving the beginning of the lesson. The facilitator, while surprised by this initiative, followed the participants' lead. Nonetheless, in the end of the meeting, she told the participants:

> I sort of… I mentally… I guess [I] was expecting to pick up the conversation where we left off with a totally different group of people, and maybe that wasn't the most realistic expectation. You guys brought different ideas to tonight, and so we had a little different conversation (Turns 540-541, second meeting).

The facilitator's expectation was to continue visiting the lesson chronologically picking up where it had been left by participants the prior week, however, the participants in this second week might not be ready to deliver to that expectation inasmuch as they had not necessarily reviewed what the prior week's group had done. Quincy followed the participants' ideas, although her expectation had been that the participants would go along with her goal, and contribute to the group's common attempt to improve the lesson. When reflecting on this meeting, she noted that, "I just didn't expect it with adults, for some reason" (second meeting debrief, Turn 703), "it" refers to the derailing from the original plan. This expectation is also evident in the following reflection, taken from the debrief of the fourth meeting:

Project leader:	So you started saying that you were worried at the beginning, what do you mean? (Turn 553, fourth meeting debrief).
Quincy:	Well, when they were like "we don't do proofs".[...] Oh well, that's the premise of this lesson. If we're not gonna play that game I don't know where we're going. But they came around, I think it was nice that Quintin [one of the participants] could help bring everybody in (Turns 554-557, fourth meeting debrief).

This statement indicates that Quincy was disrupted by the participants' divergence from the path she envisioned (i.e., discussing proving), which suggests that she had in mind the expectation that their scripting of the lesson during the meeting should contribute to the direction that had been set in the prior meeting ("If we're not gonna play that game I don't know where we're going"). This highlights a tension present in Story*Circles* and related to the role of the lesson as a motif for the work (representing the lesson is the avowed goal of the activity) but not the outcome of activity (learning with and from colleagues is the outcome of the activity). Along those lines, the lesson serves as a resource that enables participants to learn how to work together and communicate on their practice, under the premise that the lesson itself can evolve in multiple ways, each having its own merits, and participants can ponder on their decisions without being encumbered by the expectation to enact best practices. The facilitator's uneasiness with the takeover by the participants showed that she expected to be able to maintain alignment between the avowed goal and the expected outcome, as a teacher usually does. Interpreting this tension with the PD triangle, the facilitator seemed to take for granted the equivalence between avowed goal (which refers to content) and expected outcome (which refers to having conversations and arguments about practice among colleagues) and found that equivalence disrupted by the participants' desire to construct yet an alternative lesson. We hypothesize that her expectations may stem from her experience as a teacher who is used to having students agree with her on the purposes of lessons.

Discussion

Above we described two implicit expectations a facilitator had when leading Story*Circles* for the first time. These results corroborate previous findings about facilitators' difficulties to lead PD activities (Borko et al., 2014; Jackson et al., 2015; Jacobs et al., 2017). The method used in this study shows that such challenges can stem from the expectations facilitators bring with them from their prior experiences as mathematics teachers in the classroom and as participants in other PD programs. Although some of the disruptions described above (such as the presence of storyboarders) represent idiosyncratic characteristics of Story*Circles*, situating them in the PD triangle (Carroll & Mumme, 2007) allows us to generalize into broader themes. That is, the first theme illustrated a disruption in the communication among participants and facilitator about practice, while the second showed disruptions in the facilitator-practice edge of the triangle. We highlight that both disruptions were features of the original design, aiming at defamiliarizing practice in a way that would encourage practitioners to collaborate (Herbst & Milewski, 2020). StoryCircles is not purposefully designed to disrupt facilitators, yet the observation that disruptions happened offers insights into how the innovative nature of practice-based PD programs is complicated and cannot be simplified to a train-the-trainer model. The analysis above leads us to suggest that the implicit expectations the facilitator held, which surfaced only when she facilitated Story*Circles* for the first time, point to implicit norms that shape facilitating and participating in PD settings. These norms are related to the instructional norms that are obtained in mathematics classrooms. The analysis suggests that teachers-become-facilitators carry with them the norms of the instructional triangle as they strive to make sense of the activity of running professional development. The facilitator's implicit expectations seem to align to the expectations instruction imposes on classroom teachers and that have been described using the theory of the didactical contract (Brousseau, 1997; Herbst, 2003).

This framing can help situate and explain the challenges of novice facilitators, by revealing the complexity involved in the work of facilitation. In the same way that improvement or change in teaching is bounded by regularities that are difficult to depart from (Herbst, 2003), we showed how facilitators' practices are constrained by implicit expectations that exist even if they are at odds with the explicit design of the intervention. These results can contribute to the current discussions on issues of fidelity, integrity, scaling up, and implementation (e.g., Jacobs et al., 2017; Karsenty, 2021): Rather than assuming that the PD design principles and goals are transparent for facilitators, that all they need is training and good will to implement PD programs with fidelity, this study shows that considerations of implementation require attending to the background expectations teachers-become-facilitators bring with them to the job. The same capacities and experiences that give them street credibility to lead practice-based professional development can hamper their capacity to manage practitioners' learning of practice.

Acknowledgment

This work has been done with the support of grant 220020524 from the James S. McDonnell Foundation. All opinions are those of the authors and do not necessarily represent the views of the Foundation.

References

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of professional development. *Journal of Mathematical Behavior, 33,* 149–167.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101.

Brown A., Stevens, I., Herbst, P., & Huhn, C. (2021). Confronting teachers with contingencies to support their learning about situation-specific pedagogical decisions in an online context. In Hollebrands, K., Anderson, R., & Oliver, K. (Eds). *Online Learning in Mathematics Education*. (pp. 291-316). Springer Cham.

Brousseau, G. (1997). Foundations and methods of didactique. In G. Brousseau, *Theory of didactical situations in mathematics: Didactique des Mathématiques 1970–1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. and Trans.). Kluwer Academic. (Original work published 1986).

Carroll, C., & Mumme, J. (2007). *Learning to lead mathematics professional development*. Corwin Press.

Cohen, D. K., & Ball, D. L. (1999). *Instruction, capacity, and improvement (CPRE Research Report Series RR-43)*. Consortium for Policy Research in Education, University of Pennsylvania Graduate School of Education.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis*, *52*(2), 119–142.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, *18*(8), 947-967.

Even, R. (2005). Integrating knowledge and practice at MANOR in the development of providers of professional development for teachers. *Journal of Mathematics Teacher Education*, *8*(4), 343–357.

Herbst, P. (2003). Using novel tasks to teach mathematics: Three tensions affecting the work of the teacher. *American Educational Research Journal*, *40*, 197–238.

Herbst, P. & Milewski, A. M. (2018). What StoryCircles can do for mathematics teaching and teacher education. In R. Zazkis, & P. Herbst (Eds.), *Scripting approaches in mathematics education: Mathematical dialogues in research and practice* (pp. 321–364). Springer.
Herbst, P. & Milewski, A. M. (2020). Using StoryCircles to inquire into the social and representational infrastructure of lesson-centered teacher collaboration. In H. Borko, & D. Potari (Eds.), *Teachers of Mathematics Working and Learning in Collaborative Groups: Proceedings of the 25th ICMI Study* (pp. 629–639). University of Lisbon. http://icmistudy25.ie.ulisboa.pt.
Herbst, P., Nachlieli, T., & Chazan, D. (2011). Studying the practical rationality of mathematics teaching: What goes into "installing" a theorem in geometry?. *Cognition and Instruction*, *29*(2), 218–255.
Jackson, K., Cobb, P., Wilson, J., Webster, M., Dunlap, C., & Appelgate, M. (2015). Investigating the development of mathematics leaders' capacity to support teachers' learning on a large scale. *ZDM – Mathematics Education, 47*(1), 93–104.
Jacobs, J., Seago, N. & Koellner, K. (2017). Preparing facilitators to use and adapt mathematics professional development materials productively. *International Journal of STEM Education*, *4*(30). https://doi.org/10.1186/s40594-017-0089-9.
Karsenty, R. (2021). Implementing professional development programs for mathematics teachers at scale: What counts as success? *ZDM – Mathematics Education*, *53*, 1021–1033. https://doi.org/10.1007/s11858-021-01250-5.
Karsenty, R., Pöhler, B., Schwarts, G., Prediger, S., & Arcavi, A. (2021). Processes of decision-making by mathematics PD facilitators: The role of resources, orientations, goals and identities. *Journal of Mathematics Teacher Education*. Advanced online publication.
Knapp, M. C. (2017). An autoethnography of a (reluctant) teacher leader. *The Journal of Mathematical Behavior, 46,* 251–266.
Lesseig, K., Elliott, R., Kazemi, E., Kelley-Petersen, M., Campbell, M., Mumme, J., & Carroll, C. (2017). Leader noticing of facilitation in videocases of mathematics professional development. *Journal of Mathematics Teacher Education*, *20*(6), 591–619.
Lewis, J. M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM – The International Journal on Mathematics Education, 48*(4), 527–540.
McHoul, A. W. (1978). The organization of turns at formal talk in the classroom. *Language in Society*, *7*(2), 183–213.
Milewski, A. M., Herbst, P., Bardelli, E., & Hetrick, C. (2018). The role of virtual spaces for professional growth: Teachers' engagement in virtual professional experimentation. *Journal of Technology and Teacher Education*, *26*(1), 103–126. Retrieved from https://www.learntechlib.org/p/181094/
Milewski, A. M., Herbst, P. G., & Stevens, I. (2020). Managing to collaborate with secondary mathematics teachers at a distance: Using storyboards as a virtual place for practice and consideration of realistic classroom contingencies. In Ferdig, R.E., Baumgartner, E., Hartshorne, R., Kaplan-Rakowski, R. & Mouza, C. (Eds.) *Teaching, technology, and teacher education during the COVID-19 pandemic: Stories from the field*. (pp. 623–630). Association for the Advancement of Computing in Education (AACE). https://www.learntechlib.org/p/216903/
Rösken-Winter, B., Stahnke, R., Prediger, S. & Gasteiger, H. (2021). Towards a research base for implementation strategies addressing mathematics teachers and facilitators. *ZDM – Mathematics Education*, *53*, 1007–1019.
Schwarts, G., Pöhler, B., Elbaum-Cohen, A., Karsenty, R., Arcavi, A., & Prediger, S. (2021). Novice facilitators' changes in practices: From launching to managing discussions about mathematics teaching. *The Journal of Mathematical Behavior*, *64*, 100901.

TENSIONS ABOUT EQUITY: INSTRUCTORS' NEPANTLA ATTENDING TO IDENTITY AND POWER IN MATHEMATICS

Brinley Poulsen Stringer
San Diego State University
bstringer@sdsu.edu

Gabriela Maria Hernandez
San Diego State University
gmhernandez@sdsu.edu

Charles E. Wilkes II
San Diego State University
cwilkesii@sdsu.edu

Mary E. Pilgrim
San Diego State University
mpilgrim@sdsu.edu

In this paper, we discuss the case studies of two professional development (PD) participants: Nhung Tran and Stella Miller. This data comes from an NSF-funded grant studying inquiry and equity within mathematics instruction at a two-year college. We analyze participant conceptualizations of equity using Gutiérrez' (2009) equity framework. By considering the dominant axis (Access and Achievement) and the critical axis (Identity and Power), we identify places of dissonance, or Nepantla, that emerge for each participant. We discuss how these tensions give us valuable information as we consider future iterations of PD on the MPIE project.

Keywords: Equity, Professional Development

Community colleges often do not have the infrastructure and culture for supporting professional development that focuses on instruction (Edwards, Sandoval, & McNamara, 2015). Further, professional development that is offered are often one time workshops that do not provide meaningful and sustained opportunities for faculty to improve their practice (Bailey & Smith Jaggers, 2015; Huber, 2008). This places much of the responsibility of instructional improvements on the shoulders of the instructors themselves. This issue coupled with state mandates to redesign remedial mathematics courses (e.g., California, Tennessee, Florida, etc.) and initiatives to increase participation of *mintorized learners* in STEM raises two important questions: 1) How do faculty shift their practices as it relates to equity in response to attending PD and 2) How should professional development be designed to support faculty in two-year college practice? In this paper, we take up the former question and characterize how participants describe their beliefs about equity and how they shift over time.

Theoretical Framework

We drew upon Guttieréz' (2009) framework of equity. This framework is composed of two axes – the dominant axis concerning Access and Achievement, and the critical axis concerning Identity and Power. These elements help us understand how to "play the game" of education in order to "change the game". In other words, in order to move towards more equitable and liberatory systems, one must understand and leverage the current systems and structures in place. The framework also discusses the important concept of *Nepantla*, which resides at the intersection of the dominant and critical axes, highlighting the dissonance that inherently occurs between dominant and critical axes (Gutierrez, 2009; Gutierrez, 2017).

The dominant axis is so labeled as it is perceived as representing the most fundamental aspects of equity; in other words, it represents the elements often required for students to "play the game" in traditional education settings. The dominant axis is composed of Access, which is a "precursor" for the other pole of the dominant axis–Achievement. Access represents the opportunities that students have to engage in their education. Equitable access implies that all students have the resources they need to to learn. Achievement cannot occur without this Access, but Achievement also represents a distinct and important element of this framework. The reality is that providing students with the resources to learn (i.e., access) does not guarantee successful outcomes (i.e., achievement). Students deserve support along their journeys, in addition to access, to help them accomplish tangible results–to reach achievement. Research also shows that even when controlling for measures like access and achievement, equity gaps still exist (Oakes, 1986; Spencer et al., 2016). This introduces the necessity of the critical axis.

The critical axis is made up of Identity and Power, where Identity is a "precursor" for Power. The goal of this axis is to utilize the unique frames of references, backgrounds, and resources of students in a way that empowers them to become critical citizens who can ultimately change the game. Oftentimes, many students feel that they have to give up parts of their identity in order to assimilate in the classroom. In order to ensure that students, particularly students who are historically discriminated against, are not leaving parts of themselves out of the classroom, issues surrounding identity are essential to consider. Identity can be attended to by leveraging the history of the students as well as their cultural and linguistic resources. When students are given the opportunity to fully leverage their identities, they are afforded a greater wealth of Power from which to draw upon. The dimension of Power considers the social transformation at different grain sizes ranging from moment-to-moment interactions within the classroom to helping to create societal agents of change. Additionally, the Power dimension can be measured in the reconceptualization of education as a humanistic pursuit.

Within the Gutiérrez (2009) equity framework, tensions exist at the intersection of the dominant and critical axes. To address this, Anzaldúa (2015) discusses the Aztec concept of Nepantla, which refers to "a space of tensions, of multiple realities" (Gutierrez, 2017, p. 13). In this space, several conflicting views are held simultaneously. For example, instructors may see the need to change their practices while also preferring their current routines, and being resistant to implementing new ideas in the classroom. This allows the opportunity to leverage ideas from each perspective and to collate them, creating a "third space." Within this coalescent space, an individual is opting to exist within the dissonance, rather than simply choosing one view over the other. This dissonance may be uncomfortable, but if one is able to stay with the tensions long enough, new perspectives and knowledge have the potential to emerge. While Access, Achievement, Identity, and Power are each important, attending to one alone does not constitute equity; yet, leveraging all four of these is no easy feat. Therefore, Nepantla gives perspective to the messy process that occurs along the journey towards equity. More broadly, considering Nepantla in the work that we do allows us to more fully understand the ideas and perspectives across and within individuals that at first glance might seem contradictory. By considering Nepantla within professional development, we are more able to leverage the beliefs and ideas that participants currently hold, and use them to guide further growth along each individuals' journey.

Methods

This study is part of the Mathematics Persistence through Inquiry and Equity (MPIE) project at a local two-year college– Southern Hispanic Serving Institution (SHSI). The MPIE project is studying SHSI's response to a state-mandated change regarding mathematics courses. Specifically, the state mandate requires that students enroll in transfer level mathematics by the end of their first year. Using cycles of design research, the MPIE project aims to build the capacity of math instructors in the two-year college to foster student success, and investigate the effects of the capacity-building effort. The primary focus of this project's capacity building effort is on the professional development (PD) for two-year instructors. The MPIE project is a regional partnership between a two-year and four-year institution, both Hispanic Serving Institutions (HSIs) located in the same region and serving the same student population. A key outcome of our work will be improved understanding of how department-based mathematics course reform can be accomplished at a two-year HSI.

There were a total of 7 participants in the PD. Participants were math instructors at the college, most of whom taught a variety of courses including gateway courses. Participants met six times, two hours each in the fall (for a total of 12 hours). The PD consisted of activities that shared resources that could be used in their courses (e.g. Desmos, mathematical tasks, norms, etc.) and practices that could be implemented during classroom whole group discussions. The five practices from Smith & Stein (2018) and Gutiérrez's four dimensions of equity (2009) were discussed during PD and provided a guide for understanding the intersection of equity and inquiry in participants' practices. Instructors also had assignments outside of PD centered on inquiry and equity, such as taking an equity lens to their syllabi, as well as modifying and implementing tasks that were inquiry-oriented.

For this paper we conducted case studies focusing on two participants, Stella Miller, a middle-aged white woman, and Nhung Tran, a middle-aged Vietnamese man. We selected them as cases because they appeared to have similar views about equity and worked frequently together during the PD sessions. Where race and gender played a role in these conversations, we find it relevant to include the identities of these two participants. Given their similar perspectives on equity, we found their cases to provide insight about their trajectories during the course of PD. Further, each provided a unique perspective from their instructional roles as part-time (Stella) and full-time (Nhung) instructors.

The data for this study are comprised of professional development session artifacts, participant reflections, interviews, Application, Collaboration, and Exploration (ACE) assignments, Zoom recordings of PD sessions and PD facilitator and observer debriefs. Drawing upon our theoretical framework we analyzed the data of our two participants, Nhung Tran and Stella. We split into pairs to analyze the PD artifacts. Both pairs then split up, such that each individual was focused on one participant's data (Stella or Nhung). Pairs then met to compare and verify analysis. Our analysis process consisted of looking through the PD artifacts in chronological order to understand how the participants were thinking about equity over the duration of the PD. We then identified open-coded themes related to the participants' trajectories and presented data to support the themes, which were shared with all of the authors (Miles & Huberman, 1994). Below we unpack the themes for each participant.

Findings

Case 1.

Stella has been an educator for nearly 30 years. She has been an instructor at SHSI for over 20 years. She is, by choice, an adjunct instructor. She teaches introductory math courses including liberal arts mathematics, college algebra, and trigonometry. During PD, Stella actively participated and was willing to share her ideas with the group; she also consistently engaged with the in-between session ACE assignments, giving the different prompts attention, though not in the way we had anticipated. We illuminate her story over the course of professional development next.

Stella Miller at the start: Equity is Equality (Pre-PD, Session 1). In an interview prior to professional development, Stella described herself as an "organized, consistent, rule follower".

When asked about equity, she said that students would describe her as equitable because regardless of student situations, she treats everyone the same. Because of this, Stella described her class as being a place where regarding personal beliefs and identities:

> there's going to be a time where we've got to just turn that off and go with what you're doing and what am I doing in this classroom? I am teaching all the students here…And I'm going to do that to the best of my ability. And that's how I feel like I'm being equitable.

Stella in these instances describes her beliefs of equity as equality, which, from her perspective, justifies looking over various aspects of student experiences or identities. So long as she is doing her best to engage all of her students in mathematics, and to treat them all equally, that is what she constitutes as a learning environment that attends to equity.

While there are certain aspects of students' identity that Stella perhaps does not prioritize in the classroom, in early professional development sessions, she also shared thoughts informed by her personal experiences with students by discussing the importance of "getting to know your audience". She demonstrated an example of how this has impacted her practice by sharing that she chooses not to talk about traveling with her students, because through experience of getting to know her students over the years, she recognizes that this topic may not be relatable for them, implying a recognition of socioeconomic differences. In a debrief survey after session one, Stella expressed an eager interest to learn from her peers by listening to their ideas and experiences, also stating the need "to recognize our own personal strengths and weaknesses and see how we can best work with/around them. That take[s] a lot of listening to others for sure."

Stella Miller at middle: Resistance emerges (PD sessions 4 & 5). When Stella was introduced to the student profiles in the fourth PD session, she demonstrated resistance to accounting for student profiles for selecting and sequencing students' mathematical work. She describes it as "an extremely hard task," because reading the profiles did not in any way inform who she would choose. She asserts that "the reality is I am not colorblind, but I try to think that the profile of the students, their color, their gender, their this or that, does not matter to me!" To Stella, the dimension that she finds relevant with respect to equity is how students participate in the class– "the quietness versus the outspokenness." In the next PD session, facilitators followed up on the same task and asked participants specific questions regarding identity in relation to student profiles. We continued to observe resistance during this activity. When asked what identities the PD participants noticed about student profiles, Stella responded that the "identities were that half the students talked and half did not." When we pair this response with her earlier response about not being colorblind, it becomes more clear that Stella values verbal participation of students in her class when considering equity rather than gender or race. For Stella, student participation relies heavily on the agency of each individual student– from her perspective, students engage to learn in the way that best serves them. If a student is participating in a class discussion, they should not be discouraged from doing so. Similarly, if a student is silent and chooses not to participate in class, she assumes that is best for their learning style and she does not want to force them to speak up.

When Stella was asked if she was familiar with narratives that are relevant to student profiles, she stated that if there is a certain profile that is congruent with historical narratives, it should not be ignored. She states that "...after being in the classroom for 25 years, that is just reality… I think it would be silly to not use prior experience, it's like ignoring history." However, she also mentions that each individual is a new person coming into each class and that they may not conform to the dominant narrative. This tension between expecting students to behave a certain way versus breaking the mold highlights Stella's resistance but also willingness to change her mind based on experiences.

Stella Miller at end: Equity is justification for existent practices (Session 6 and after PD interview). Towards the end of the semester, Stella shared some of her thoughts about PD in an exit interview. When asked about equity, she shared, "now this one, I have to say, this one has been more of a learning for me". Stella went on to share how she had not previously considered equity in relation to participation within her classroom. While Stella had discussed previously how she "works the crowd" by working with students who are louder and more quiet in the class, she now saw and described how these practices were related to equity because it was attending to various and differing student needs. While this marks a shift in how Stella describes equity compared to her first interview, a shift which Stella herself recognized was present, there was not strong evidence of any major shifts in her actual teaching practices. Rather, she saw how her existing practices could be viewed as equitable practices.

Stella also shared in the interview a recent personal experience in the classroom that had her thinking about shifts that could occur within her teaching practices. She discussed how she had recently come to realize that a student in her class did not speak English fluently, and had been utilizing a peer in the class to translate what was going on during class time throughout the semester. When she realized towards the end of the semester that this was occurring, she expressed feeling "embarrassed" and "ashamed" for not knowing the student better. She then discussed how had she known sooner, she could have made some adjustments to her teaching to better accommodate this student. While Stella did not articulate these ideas in connection to equity, we again see how her personal experiences do impact her teaching practices, and that they also are slowly moving her towards a direction of more equitable practices in the classroom.

Case 2.

Nhung Tran has been teaching at the two-year college level for over 10 years. He is currently a full time instructor at SHSI where he teaches a variety of introductory mathematics courses including college algebra and statistics. In addition, he often teaches courses for preservice mathematics teachers. Throughout the PD sessions Nhung constantly asked questions and was vulnerable with his own practice. His inquisitiveness and curiosity help us in understanding his views about equity in practice. Below we unpack Nhung's trajectory with respect to his equity development into three phases, the beginning of the PD, the middle of the PD, and the end of the PD.

Nhung Tran at the start: Inquiry is easier to talk about than equity. (Sessions 1 & 2). The PD began with having participants share their ideas about inquiry and equity. For Nhung Tran, inquiry was easier to speak to than equity. He had concrete ideas about teaching practices that fostered inquiry, but only vague notions about equity. This was reflected in PD artifacts, where Nhung brought up questioning strategies and the use of multiple representations to promote inquiry. However, for equity, he did not generate any ideas when reflecting on equity on his own, and only offered sufficient work time for all students as a way to attend to equity as "not everyone works at the same pace" when Nhung was thinking about equity with his PD partner. The reason for this was Nhung's belief that mathematics is universal, meaning that it is objective and accessible by anyone. This lack of consideration for equity was reinforced in a post classroom observation of Nhung. During the interview Nhung described how he had not thought about equity specifically, instead he just focused on aspects of his practice that he thought teachers should provide, such as equal opportunities and ensuring the content was accessible. After the third author raised different questions about decisions Nhung made during instruction that were equity related, Nhung began to reflect on why he had not considered them. This marked a shift in how Nhung described equity in the following sessions.

Nhung Tran at the middle: Tensions begin to surface (Sessions 3 & 4). Tensions began to surface in Nhung's mind about how to both attend to equity oriented practices simultaneously with students' identities and work. This point is evidenced by two activities in session 4. The first of which asked participants to consider classroom norms that were inquiry- and equity-oriented. During this activity Nhung agreed that norms such as students asking questions, getting help from each other, and not being afraid to make mistakes are important. In the next activity PD participants were provided with a collection of student work with student identity profiles. Stella and Nhung were paired for this activity, and when Stella shared her thoughts about student identity, and this activity being challenging, Nhung agreed. Nhung expressed struggling to see how an instructor could select and sequence mathematical work for sharing with the class using students' mathematical work while simultaneously attending to the students' identities. Despite some of the student profiles including characteristics such as being afraid to share ideas during whole-group or making mistakes, which he named in the previous activity as important, he still struggled to see how those aspects and students profiles are connected to equity (e.g., students' identities, their work, and past experience should play a role in who is selected and when during whole-group discussion). Additionally, his personal verbalized biases about students' identities (e.g., Asian females being quiet), reflect this disconnect.

Nhung Tran at the end: (Session 5 & 6). In the last two sessions of PD we continued to focus on the intersection of student profiles and teacher practice. We revisited the activity that had participants select and sequence students' work. Instead of selecting and sequencing the work of the students, Nhung and Stella, who were partnered together again, decided to have all of the students share their work at the board and have their peers ask questions and make comments. In practice this move is not necessarily bad, however, in the context of the activity it is an example of the participants not using their power as instructors to think about how and why they may call on students. Additionally, when probed, Nhung and Stella mentioned if they had to call on a student they would call on John, a White male in the class who is always the first student to have his hand raised, has an A in the class, likes to work by himself and only seems engaged when he is talking. In Nhung's last interview he continued to talk about equity in the ways he described in the beginning (e.g., good teaching is equitable), but he also described needing to understand some of the biases he had and wanting to be more intentional about his practice.

Discussion

We first discuss our findings as they relate to the elements of the dominant axis from Gutiérrez' (2009) equity framework. For Stella and Nhung, Access plays a key role in both of their conceptions of equity. Both believe that if they provide quality mathematics instruction for students and establish classroom environments that encourage participation, then students have equitable access to learning. However, this places the agency upon the students to engage in class. For Achievement, it is not clear from the professional development sessions or interviews how Stella and Nhung make sense of this element, other than the fact that it is apparent that they care for their students to succeed. Participants' lack of attention to Achievement could be a result of the PD facilitators' emphasis on inquiry and equity using the five practices from Smith & Stein (2018), which focuses more on teacher practices that lead to more productive classroom discussions. While we see these conceptions of Access as a starting place for understanding equity, the lack of attention towards Achievement raises questions for the accountability that Nhung and Stella take with regards to their students' success. Findings related to the critical axis further expand upon this idea.

When considering the critical axis, we see mostly superficial or partial attention from Nhung and Stella towards Identity and Power. With respect to Identity, Nhung does briefly describe the importance of addressing his biases; however, he struggled in making a connection between the identities of students and how those identities can impact their classroom experiences. Stella sees Identity differently, wherein she seems to conflate or reduce the identities of her students to their type of classroom participation. Instead of considering the identities and experiences of her students, she focuses more on her own lived experiences with students and how that relates to her teaching. The second part of the critical axis is the Power dimension which is leveraged differently by Nhung and Stella. While Nhung acknowledges and describes the power he has in practice to be more equitable, Stella focuses more on the power that students have–they must choose to engage in a way that is best for their own learning. Stella relinquishes the power and responsibility she has as an instructor and focuses instead on the power of student agency as a means for success in the classroom, which aligns with Stella's consistent focus on the Access dimension. With regard to Identity, we again see both Nhung and Stella struggling to take responsibility for attending to the identity of their students in the classroom. For Power, we see Nhung acknowledging the role he plays in the classroom, but is unsure of how to leverage his power to support his students. Stella acknowledges her role as well, but highlights she only provides students with resources, leaving them to exercise their agency and control over their own success.

With respect to the dimensions of the equity framework, Nhung and Stella hold several conflicting views. Nhung and Stella both focus heavily on Access, and not as much on Achievement, Identity, and Power. In Gutiérrez's framework, Nepantla refers to the tension between these different dimensions. With Nhung, in particular, we see Nepantla with his tendency to be easily convinced by Stella during breakout sessions, which leads to shifts in his thinking. Due to his uncertainty, Nhung had a difficult time exploring and establishing beliefs regarding equity. There is a dissonance for Nhung between his open-mindedness to hearing others' ideas, and his own personal beliefs regarding what equity should entail within the class. For Stella, Nepantla is most evident in her resistance to acknowledge Identity. Her strong emphasis on student agency seems to serve as a barrier to considering other ways of being responsible for her students and their success, creating a dissonance between her sincere care for student success, and her responsibility to supporting them in their mathematical endeavors. By identifying these instances of Nepantla, we see these tensions for Nhung and Stella as opportunities for future growth within our continued PD efforts, which we discuss next.

When considering these findings in relation to what they mean for PD efforts, a few questions and reflections arise. When considering our first research question– how participants shift their practices from PD– these case studies of Stella and Nhung highlight an important follow-up question: how does the resistance of one participant affect the progress of other participants? While we see growth in both Stella and Nhung over the course of professional development, we also identified resistance, particularly from Stella, when attending to ideas relating to the critical axis (Gutiérrez, 2009). Because Stella and Nhung were paired together in several of the PD sessions for breakout room discussions, we wonder how this might have impacted Nhung's experience, and how the conversations around equity and identity might have looked different if he had been paired with a less resistant partner. There were multiple moments during PD where Nhung raised an idea and Stella pushed back, which in response, Nhung would walk back on his thoughts to agree with Stella. How might Nhung's thoughts regarding equity have evolved otherwise if paired with a partner that took up his ideas in ways that pushed him to think about the ideas he raised and how they might impact his practice? Where Stella often expressed the value in hearing her peers' experiences, we also wonder how pairing her with various other partners might have differently impacted her perspectives regarding equity. We plan to explore this research thread further, and use these findings in the design of PD in upcoming semesters.

This reflection also relates to our second research question–how should PD be designed to support two-year college faculty. As we continue to use design cycles to improve our PD efforts, we can strategically pair participants with partners in future sessions to see if we notice any changes in how participants shift their practices compared to this first semester of PD. These cycles of design also allow us to reflect on the growth that we have observed of participants thus far, and to adjust our efforts and plans to better foster further growth in coming semesters. As these two case studies have outlined, we see dissonance between where participants currently think and engage with the concept of equity and where we hope they will eventually conceive equity. We see this dissonance as an important opportunity to reflect upon, and thus an opportunity to improve our efforts with PD moving forward.

Conclusion

The case studies of Nhung Tran and Stella Miller provide insight about two-year college instructors' beliefs about equity and ways such beliefs might shift. Using Gutiérrez' (2009) dimensions of equity as an analytical lens we gained further understanding about how these instructors think about equity and the influence they can have upon each other's thinking. In particular, this work highlights tensions, or Nepantla, that emerges from the discussion of these topics. We see such dissonance as opportunities for further growth as we work towards building harmony between equity and mathematics education. This work will inform future PD design that attends to activities and group pairings which can support instructors to see equity in ways that attend to both the critical and dominant axes.

Acknowledgments

This project is funded by the National Science Foundation under Grant Nos. DUE #1953713 and #1953753.

References

Anzaldúa, G. (2015). Light in the Dark/Luz en lo oscuro. In *Light in the Dark/Luz en lo Oscuro*. Duke University.

Bailey, T., Smith Jaggars, S., & Jenkins, D. (2015). Redesigning America's community colleges. Cambridge, MA: Harvard University Press.

Edwards, A. R., Sandoval, C., & McNamara, H. (2015). Designing for improvement in professional development for community college developmental mathematics faculty. Journal of Teacher Education, 66(5), 466-481.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game.". *Teaching for Excellence and Equity in Mathematics, 1*(1), 4-8.

Gutiérrez, R. (2017). Living mathematx: Towards a vision for the future. *Philosophy of Mathematics Education*, *32*(1), 1-34.

Haynes & F. Cobb (Eds.), *Interrogating Whiteness and relinquishing power: White faculty's commitment to racial consciousness in STEM classrooms*, (pp. 253-282). New York: Peter Lang.

Huber, M. (2008). The promise of faculty inquiry for teaching and learning basic skills. Stanford, CA: Strengthening Pre-Collegiate Education in Community Colleges.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed). Sage Publications.

Oakes, J. (1986). Tracking, Inequality, and the Rhetoric of Reform: Why Schools Don't Change. *Journal of Education*, *168*(1), 60–80. https://doi.org/10.1177/002205748616800104
Smith, M., & Stein, M. K.. (2018). *5 Practices for Orchestrating Productive Mathematics Discussions*. National Council of Teachers of Mathematics.
Spencer, S. J., Logel, C., & Davies, P. G. (2016). Stereotype Threat. *Annual Review of Psychology*, *67*(1), 415–437. https://doi.org/10.1146/annurev-psych-073115-103235

ELEMENTARY MATHEMATICS SPECIALISTS: PREPARATION AND ENGAGEMENT IN TEACHER LEADERSHIP

Susan Swars Auslander
Georgia State University
sauslander@gsu.edu

Kayla D. Myers
Georgia State University
kmyers@gsu.edu

Carla L. Tanguay
Georgia State University
ctanguay@gsu.edu

Gary E. Bingham
Georgia State University
gbingham@gsu.edu

Shani Jackson
Gwinnett County Public Schools
shanijackson1213@yahoo.com

This 5-year mathematics professional development project involves 27 elementary teachers prepared and supported as Elementary Mathematics Specialists (EMSs) in high-need urban schools. They complete a university's K-5 Mathematics and Teacher Supporting & Coaching Endorsement programs and participate in Professional Learning Communities and individual mentoring. Described here are data collected at the end of Year 1, illuminating the ways in which they are engaging in teacher leadership, especially coaching. The EMSs are a distinctive population as informal teacher leaders, with a primary responsibility of teaching students. Central to the project is the university-school-community partnership, with findings illuminating reciprocity with mutual benefits, such as high quality clinical experiences for teacher candidates, coaching for novice teachers, and engagement with families and caregivers.

Keywords: Professional Development, Instructional Leadership, Elementary School Education

Purpose of the Study

The context of this study is a 5-year professional development project for 27 elementary teachers in high-need urban schools who are prepared and supported as Elementary Mathematics Specialists (EMSs). They are completing a university's K-5 Mathematics and Teacher Supporting & Coaching Endorsement programs and participating in Professional Learning Communities and individual mentoring. This study specifically examined the ways in which they engaged in teacher leadership during Year 1 of the project. This question guided the inquiry: How are the EMSs providing teacher leadership, including self-reported coaching practices?

Theoretical Perspectives and Related Research

The specific roles and responsibilities of EMSs vary (Baker et al., 2021), dependent upon the contextual needs and plans of schools, school systems, and states (McGatha et al., 2015). EMSs can serve as classroom teachers, instructional interventionists, and informal or formal teacher leaders. Within these wide-ranging responsibilities, EMSs' work as a teacher leader often involves coaching other teachers. This coaching can occur in a variety of one-on-one and group contexts. Productive activities for mathematics coaches with groups include: engaging in the discipline (e.g., mathematics through worthwhile instructional tasks), examining student work, analyzing classroom video, and participating in lesson study, while those with individuals include co-teaching and modeling instruction (Gibbons & Cobb, 2017). Further, McGatha (2015, 2017) examined related studies and determined that coaches' ways of interacting with individual teachers could be considered on a continuum from more-directive (e.g., modeling lessons, providing resources) to less-directive (a process of collecting data from observed lessons, providing feedback, and engaging teachers in thought reflection), with the latter more powerful for prompting changes in teachers' instructional practices. There are a number of coaching models evident in the extant literature, including content-focused coaching, instructional coaching, and cognitive coaching (Yopp et al., 2019). Cognitive coaching, which was emphasized in this project, is a particularly powerful approach that relies heavily on coaches' use of reflective questions to encourage teachers to refine their professional knowledge base through self-assessment and self-direction (Costa et al., 2016). All in all, context and need drive EMSs' highly varied ways of working, providing a warrant for studying their differing roles and responsibilities.

EMS preparation programs should focus on the in-depth and multidimensional development of content knowledge for teaching, pedagogical knowledge for teaching, and leadership knowledge and skills (AMTE, 2013). Programs should have a two-fold emphasis: fostering expertise as a teacher of mathematics and as a teacher leader who serves as a *more knowledgeable other*, supporting colleagues' instruction and other efforts within mathematics education such as curriculum development and community connections. When it comes to leadership knowledge and skills, specialized courses should prepare EMSs to "take on collegial, non-evaluative leadership roles within their schools and districts. They must have a broad view of the many aspects and resources needed to support and facilitate effective instruction and professional growth" (AMTE, 2013, p. 8). Several years of program development, implementation, and evaluation have revealed the salience of program experiences being embedded in practice, with strong connections and enactment within EMSs' classrooms, schools, and/or districts (Reys et al., 2017). When considering states offering pathways for advanced specialist certification, there are notable differences in EMS preparation programs related to duration, number of course hours, course emphases, field practicum experiences, and delivery (EMS and Teacher Leaders Project, 2022; Spangler & Ovrick, 2017). This variability is linked to differences in program goals and provides a warrant for study of EMS preparation programs. More inquiry needs to focus on the development of teacher leadership and how EMSs engage in this work, including coaching and associated practices (Yopp et al., 2019).

Methodology

The design of this study includes a descriptive, holistic singular-case approach (Yin, 2014). The inquiry occurred during the COVID-19 health pandemic, which had changed the functioning of schools and classrooms, thus influencing the EMSs' efforts as teachers and teacher leaders. The researchers were mindful of this contextual element throughout the study.

This study's context is a mathematics professional development project focused on the development of 27 elementary teachers as EMSs in high-need urban schools. Multiple partners are involved, including a university, school district, and non-profit organization. Project goals include the development of EMSs who deliver ambitious mathematics instruction and serve as mathematics teacher leaders in a variety of ways, such as coaching university teacher candidates (henceforth called teacher candidates), providing professional development to their peer teachers, mentoring novice teachers at their school sites, supporting the non-profit's after-school tutoring program, and engaging in community connections that promote key relationships and shared responsibility for students' learning. The project also aims to promote equity and access in mathematics, support teacher retention in high-need schools, and situate teacher candidates in a hiring pipeline. Since the EMSs' primary responsibility is teaching students, their role as a mathematics teacher leader is an informal one. The project is 5 years in duration and at the time of this study, the participants had completed 1 year and were emergent teacher leaders.

The teachers were selected to participate in the project based on criteria that identified them as successful, experienced teachers of mathematics with interest in and aptitude for teacher leadership. All are employed in a large, urban school district in the southeastern USA. They teach in 22 elementary schools, which collectively serve 91% students of color, with the largest populations being 44% Hispanic and 36% Black; 69% of students are eligible for the federally-funded free and reduced lunch program. The participants self-described as 24 females and 3 males, with 70% self-identifying as persons of color (41% Black, 7% Hispanic, 7% Asian, 7% Hispanic/White, 4% Hispanic/Black, 4% Black/White). They were a highly educated group, with 100% having a master's degree and 33% holding an educational specialist degree; further, they were experienced teachers, on average having 10.5 years of teaching experience (range of 5-22 years). Teaching positions varied widely and included: three kindergarten, one first grade, two second grade, five third grade, one fourth grade, seven fifth grade, four STEM/Math Specials, one English to Speakers of Other Languages, one Special Education, one Early Intervention Program, and one Accelerated Content. Of these participants, two taught in Dual Language Immersion settings, including Spanish (2nd grade) and French (5th grade). Within these differing grade levels and foci, all taught mathematics, including some for part of the day and some for all of the day. Notably, this group of participants represents the diversity of teachers from which students learn mathematics in elementary schools.

In this project, the participants are prepared and supported through completion of a university's K-5 Mathematics Endorsement (K-5 ME) and Teacher Support & Coaching Endorsement (TSCE) programs during the first 2 years, along with participation in Professional Learning Communities (PLCs) and individual mentoring for the entire 5 years. See Table 1 for these elements, along with the timeline. The endorsement programs include four elementary mathematics content courses integrating pedagogy, one course focusing on teacher leadership and coaching, and two internship courses, with one focusing on mathematics and the other coaching. Overall goals of both programs (AMTE, 2013, 2017) are development of: effective and equitable mathematics instructional practices (NCTM, 2014, 2020); deep and broad knowledge of elementary mathematics, including *specialized content knowledge* (Ball et al., 2008); productive mathematical beliefs and professional agency; and teacher leader capabilities, including coaching skills.

Table 1: Timeline and Project Elements Aimed at Preparing and Supporting EMSs

Year 1		Year 2			Years 3-5
Fall 2020	**Spring 2021**	**Summer 2021**	**Fall 2021**	**Spring 2022**	
1 TSCE course (Teacher Leadership & Coaching)	1 K-5 ME course (Number & Operations)	1 K-5 ME course (Data Analysis & Probability, 2-week summer institute)	1 K-5 ME course (Algebra & Rational Number) and 1 TSCE course (the Internship)	2 K-5 ME courses (Geometry & Measurement and the Internship)	
PLC and Mentoring	PLC and Mentoring		PLC and Mentoring	PLC and Mentoring	PLC and Mentoring

In addition to preparation for teacher leadership in the endorsement programs, support for the EMSs as they serve as teacher leaders is provided through a PLC and individual mentoring, both facilitated by the project's program director. PLCs and individual mentoring generally focus on: building a community of learners within each PLC, augmented support for developing effective and equitable mathematics instruction, and targeted support for what is called in this project *teacher leader activities*. The three PLCs are clustered around grade levels/teaching focus, with each having nine EMSs, and meet monthly eight times across the school year.

To lead instructional change and support wide-ranging improvements, the EMSs engage in a number of teacher leader activities across the 5 years in their school, district, community, and other contexts, applying their teacher leader strategies and skills learned in the K-5 ME and TSCE programs and the PLC. Two primary teacher leader activities include coaching a teacher candidate each year and supporting the nonprofit's after-school tutoring program for at least 1 of the 5 years. Other teacher leader activities are selected based upon the needs of the school and in consultation with school leadership. The PLC serves as a context for collaborative selection, planning, and reporting on teacher leader activities, in addition to individual conferences with the program director. Toward the beginning of the school year, each EMS proposes 3-6 specific teacher leader activities in writing to the program director, describing in detail the activities anticipated content, duration, frequency, and outcomes. The program director consults with the project's leadership and collaboratively refines with each EMS a plan for specific teacher leader activities to accomplish across that school year. Check-ins related to progress across the school year are included in both PLC meetings and individual conferences. Each EMS provides documentation at the end of each year of this work in a Teacher Leader Record.

Data were collected at the end of Year 1 in two ways. All participants completed a Teacher Leader Record (TLR), documenting various aspects (i.e., content, duration, frequency, and outcome) of their teacher leader activities across the year. For this study, only the aspect of content is included as data. The content section includes a detailed description of the teacher leader activity and the rationale for implementation, including what exactly the activity was and why they chose to do that activity. Data were also collected from all participants via a survey of mathematics coaching practices (Coaching Practices Survey [CPS], Yopp et al., 2019). The CPS is designed to capture the extent to which a coach uses certain practices related to instructional coaching in mathematics that were drawn from coaching models in the extant literature (Yopp et al., 2019). The instrument contains 20 items and uses a 7-point Likert type scale, with a higher rating indicating more self-report of particular coaching practices (ranging from very descriptive of my coaching to not at all descriptive of my coaching). This collection of 20 items exhibits good internal consistency (Cronbach's alpha estimated at 0.81).

The research team includes four university professors and the project's program director, collectively holding expertise in a variety of methodologies, along with two doctoral students. The analysis of the TLR focused on the content of the teacher leader activities, which largely involved examination for frequency of activities and clustering into categories when possible. Data from the CPS were dichotomized for analysis in order to identify practices that are descriptive or not of participants' coaching. For an item, if the response was descriptive at all (rating of 7, 6, or 5) it was assigned a 1, and the other responses of not descriptive or equally not descriptive/descriptive (rating of 4, 3, 2, or 1) were assigned a 0.

Results

The analysis of the TLR shows all participants provided teacher leadership in a number of ways, with each participant reporting 3-6 distinct teacher leader activities, dependent upon the scope and scale of each activity. Each participant coached a teacher candidate, serving as a classroom mentor teacher and/or university coach, with a total of 27 teacher candidates impacted.

The analysis of the TLR also shows that during Year 1 over one-third (n=10) of the EMSs supported the non-profit's after-school tutoring program. This support was driven by the needs of the after-school tutoring program, based upon consultation with the program's leaders. The EMSs' initial efforts largely focused on collecting and organizing tools and resources to support remote learning, then broadened to include curriculum analyses with revisions. All of this work had an intentional focus on supporting mathematical content that students were concurrently learning in their classrooms. The curriculum analyses involved careful review of the existing guidelines and materials used in after-school tutoring sessions, starting in August and spanning the entire school year, and providing feedback on how to increase cognitive demand during instruction, implement tasks that are worthwhile and engaging for students, and utilize more manipulatives and tools to improve conceptual understanding. Further, the EMSs provided additional resources and supplements to that curriculum, with the continued aim of increasing rigor, conceptual understanding, and enjoyment of mathematics.

Additional teacher leadership is evident from the analysis of the TLR. Eleven of the participants reported leading professional development of some kind for fellow teachers at their schools that focused on mathematics education (e.g., PLC, grade level planning sessions, district-wide and school-wide workshops). Ten formally mentored new teachers at their schools, in addition to coaching a teacher candidate. Other teacher leader activities focused on outreach to parents and families. Twelve EMSs facilitated a Math or STEM Community Event for families and students in their respective schools. Twelve led workshops or created resources for parents focused on mathematics as a direct response to remote learning struggles or language barriers (e.g., instructional videos, bilingual resources). While the fore-mentioned categories were the most frequently reported, the EMSs engaged in a number of other mathematics-focused activities. Examples include co-presenting at national conferences, serving on leadership teams within the school district, creating original content for use with teachers and students, facilitating after-school boot camps or tutoring for students, and writing grants to procure resources.

Based on these findings from the TLR the participants were engaged in coaching in a number of ways, and a descriptive analysis of data from the CPS provides insights into their mathematics coaching practices. Since these participants were informal teacher leaders and serving as EMSs in a variety of ways, with their coaching focusing on both teacher candidates and colleagues, these descriptive data show variability and provide contrast with those who serve in a mathematics coach role of primarily working with fellow teachers. The analysis shows that three participants rated 18 of the 20 items as descriptive of their coaching practices and at the other extreme, four participants rated five or fewer of the 20 items as descriptive. Twenty of the 27 participants rated 10 or more items as descriptive of their coaching practices, with seven participants rating fewer than 10 items. The distribution of the dichotomized data is near normal with a negative skew.

An item analysis comparing the identification ratings indicated that 56% of the items are descriptive of mathematics coaching practices. Interestingly, the four lowest-rated items, which were rated by 26%, 26%, 30%, and 33% of participants as descriptive of practices, all focus on collaboration and communication with the principal or other school administrators about mathematics coaching (e.g., discussing the school's vision for mathematics instruction, progress being made toward that vision, and teachers' needs; collaborating to ensure a clear message about effective mathematics instruction). This seems to indicate that the principal or administration at these school sites is not actively involved in the coaching practices of the participants. Since the participants are informal teacher leaders, with a focus in part on coaching teaching candidates, it is not surprising that there is variability in how much they work with school leaders in their coaching responsibilities. In contrast, Table 2 shows the seven highest-rated items, which 70% or more of the participants indicted as descriptive of their coaching practices.

Table 2: Most Prevalent Coaching Practices on the CPS (≥70%)

3. I coach teachers on needs that I observe in the teacher, even when the teacher is unaware of these needs
5. I always make sure that coaching conversations with mathematics teachers are grounded in the mathematics content
9. I try to provide the teachers I coach with an understanding of how the mathematics they teach supports learning beyond the grade level they teach
11. I encourage the teachers I coach to reflect on similarities and differences among mathematics topics in the curriculum
12. I help teachers plan their lessons
16. I reflect on state assessment data to identify curriculum areas that need to be strengthened
19. I encourage teachers to set personal improvement goals for mathematics

Discussion

Given the notable variability of EMS preparation programs across the USA (EMS and Teacher Leader Project, 2022), there has been a call for "developing a knowledge base for the preparation of EMSs", including how "elements of an EMS program are necessary for productive outcomes" (Reyes et al., 2017, p. 231). Teacher leadership, including coaching and associated practices, have not been widely studied (Yopp et al., 2019). Further, EMSs ways of working are highly varied, driven by need and context, which provides a warrant for study of their roles and responsibilities. Accordingly, this inquiry focused on how EMSs in a preparation program are engaging in teacher leadership, especially coaching. The participants' primary responsibility is teaching students, thus they are a distinctive population as *informal* teacher leaders. Further, they were largely teachers of color, working in urban schools that served students historically marginalized in mathematics. Notably, these EMSs were selected for this project based upon a rigorous process and are subsequently participating in a rigorous preparation program, which contrasts with the too often practice of those who are simply the most effective mathematics teacher being selected to serve as a teacher leader or coach.

Our project provides an example of a preparation program guided by standards and research, with this study's findings illuminating the important outcome of teacher leadership. The findings of the CPS show mathematics coaching practices that they were and were not using, providing considerations for how they can better communicate and collaborate in their coaching with school administration. The findings of the TLR provide insights into their teacher leader efforts. Each EMS coached a teacher candidate, serving as a classroom mentor teacher and/or university coach. A total of 27 teacher candidates were impacted, strengthening the university-school partnership, contributing to high quality clinical experiences for teacher candidates, and building teacher capacity for coaching and mentoring others at the school sites. The project provides a pipeline of teacher candidates to be hired at the high-need, urban schools, addressing a teacher shortage in school district. For these teacher candidates, program data show 70% are from underrepresented groups in the teaching profession, contributing to the much needed diversity of the teacher workforce as recent data show 78% of public school teachers in the USA are White (National Center for Education Statistics, 2021).

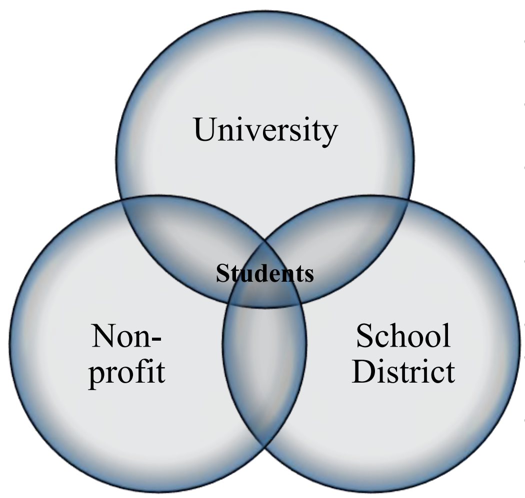
In addition, their teacher leader endeavors supported community connections and intentional interactions with parents and caregivers, which are important for fostering these key relationships and shared responsibility for students' learning in mathematics. Notably, their work focused on curriculum development in the after-school tutoring program aims to improve mathematical learning experiences for students. Since some of these students are in the EMSs' classrooms, the EMSs should receive direct benefits from this work via their students. Additionally, their teacher leader efforts focused on coaching and facilitating professional development, positioning them as a *more knowledgeable other* for a community of practice within a school, aiming to influence teachers and the school's mathematics program as a whole (Campbell & Malkus, 2014). All in all, their teacher leader efforts across the 5 years of the project and beyond aim to have a wide-ranging effect on mathematics teaching and learning at their school sites. Further, their support for a novice teacher at their schools should foster teacher retention during the novice teachers' induction period in the profession. A body of research shows that individual mentoring of those in the first 3 years of teaching is critical for retention in the profession (Desimone et al., 2014; Ingersoll et al., 2012; Ingersoll & Smith, 2004; Stanulis & Floden, 2009).

It is hoped that the *proximal* goals of improving EMSs' mathematics instruction with them in turn supporting others (e.g., fellow teachers) in doing the same, should influence the *distal* goal of enhanced student learning and understandings in mathematics. This project intentionally supports students who have been historically marginalized and under-served in mathematics education. These students are largely from traditionally underrepresented groups, and selection criteria for the project ensured the EMSs are a diverse group, with 70% identifying as persons of color. This is significant as increasing research shows students of color benefit from having teachers of color (Carver-Thomas, 2018). Further, during the K-5 ME program the EMSs are immersed in mathematics teaching and learning that supports equity, diversity, and inclusion and provides explicit and intentional opportunities for all children to learn rigorous mathematics (Aguirre et al., 2013; Bartell et al., 2017; NCTM, 2020, 2021; National Council of Supervisors of Mathematics [NCSM] and TODOS, 2016). They learn about enacting lessons that leverage children's mathematical, cultural, and linguistic strengths, while nurturing positive student identity in mathematics. For example, not only should instructional tasks have high levels of cognitive demand, they should hold cultural relevance whenever possible and draw on students' home, cultural, and language experiences (NCTM, 2014, 2020). Emphasized are culturally responsive and sustaining pedagogies, instruction for multilingual learners, and mathematics serving as a lens for understanding and critiquing the world (Harper, 2019; NCSM and TODOS, 2016; Rubel, 2017). Equity and access within mathematics education are through-threads of this project, supporting students who have been historically under-served.

This project also aims to support teacher retention of the EMSs, an aspect that is addressed in the high-quality preparation and support as well as the community of teacher leaders being cultivated. Notably, the extant literature shows that teachers who engage in teacher leadership have feelings of an upward professional trajectory, thus increasing their own satisfaction and retention in the teaching profession (Tricario et al., 2015). With this project occurring in the context of the COVID-19 health pandemic, the sudden, unanticipated shift to emergency remote teaching followed by concurrent instruction of face-to-face and virtual learners have generated tremendous challenges and angst for K-12 teachers. Those were and continue to be trying times for teachers, testing their resilience, fortitude, and persistence in the profession. Throughout, the EMSs have found community and comradery with one another. The project is providing a space for supportive and open, safe conversations as they grapple with the tremendous demands placed upon them as educators, which has been illuminated through both anecdotal data and initial interview findings. Their passion for and commitment to mathematics education are apparent, which brings us hope that our goal of retaining EMSs in the school district will be successful.

Given the numerous and various ways the EMSs are serving as teacher leaders, the results show they are making a difference for a number of stakeholders, including fellow teachers, novice teachers, teacher candidates, students, school administrators, parents and families, and community partners. When considering these various stakeholders, critical to this project is the strong partnership between the university, school district, and non-profit organization. Notably, robust school-university-community partnerships support *simultaneous renewal* (Goodlad, 1994) of all partners. This renewal is a process of partners concurrently changing, growing, and improving, with a focus on innovative, high leverage, research-based pedagogical practices (American Association of Colleges for Teacher Education [AACTE], 2018). Central to these partnerships is *reciprocity*, where there are mutual benefits for all involved (National Association for Professional Development Schools [NAPDS], 2021). Figure 1 displays some of the fore-described mutual benefits, with supporting students and their mathematical capacity for success at the center.

- Strengthened university-community connections
- Enhanced after-school tutoring curriculum
- Improved mathematics instruction
- EMSs invested in community



- Strengthened university-school connections
- Improved mathematics instruction
- Positive influences on colleagues and school's mathematics program
- High-quality clinical experiences for teacher candidates
- Reinforced hiring pipeline
- Improved teacher induction and retention
- Engagement with families/caregivers

- Strengthened community-school connections
- Enhanced support for students' mathematics learning
- Shared responsibility for students' learning

Figure 1: Project Partners and Mutual Benefits

Acknowledgement

Support for this work was provided by the National Science Foundation's Robert Noyce Teacher Scholarship Program under Award No. 1950064. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K-8 mathematics: Rethinking equity-based practices*. National Council of Teachers of Mathematics.

American Association of Colleges for Teacher Education. (2018). *A pivot toward clinical practice, its lexicon, and* *the renewal of educator preparation.* Retrieved from https://aacte.org/resources/research-reports-and-briefs/clinical-practice-commission-report/

Association of Mathematics Teacher Educators. (2013). *Standards for Elementary Mathematics Specialists: A reference for teacher credentialing and degree programs.* Author.

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.* Available online at amte.net/standards.

Baker, C. K., Saclarides, E. S., Harbour, K. E., Hjalmarson, M. A., Livers, S. D., & Edwards, K. C. (2021). Trends in mathematics specialists literature: Analyzing research spanning four decades. *School Science and Mathematics.* Advance online publication.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59,* 389-407.

Bartell, T. G., Wager, A. A., Edwards, A. R., Battey, D., Foote, M. Q., & Spencer, J. A. (2017). Toward a framework for research linking equitable teaching with the Standards for Mathematical Practice. *Journal for Research in Mathematics Education, 48*, 7-21.

Campbell, P., & Malkus, H. (2014). The mathematical knowledge and beliefs of elementary mathematics specialist-coaches. *ZDM Mathematics Education, 46,* 213-225.

Carver-Thomas, D. (2018). *Diversifying the teaching profession: How to recruit and retain teachers of color.* Learning Policy Institute.

Costa, A. L., & Garmston, R. J. with Hayes, C. & Ellison, J. (2016). *Cognitive coaching: Developing self-directed leaders and learners* (3rd ed.). Rowman & Littlefield Publishers.

Desimone, L. M., Hochberg, E. D., Porter, A. C., Polikoff, M. S., Schwartz, R., & Johnson, L., J. (2014). Formal and informal mentoring: Complementary, compensatory, or consistent? *Journal of Teacher Education, 65*(2), 88-110.

Elementary Mathematics Specialist & Teacher Leaders Project. (2022). Retrieved from http://www.mathspecialists.org/state-certifications.html

Gibbons, L. K., & Cobb, P. (2017). Focusing on teacher learning opportunities to identify potentially productive coaching activities. *Journal of Teacher Education, 68*(4), 411-425.

Goodlad, J. I. (1994). *Educational renewal: Better teachers, better schools.* Jossey-Bass.

Harper, F. K. (2019). A qualitative metasynthesis of teaching mathematics for social justice in action: Pitfalls and promises of practice. *Journal for Research in Mathematics Education*, *50*(3), 268-310.

Ingersoll, R., Merrill, L., & May, H. (2012). Retaining teachers: How preparation matters. *Educational Leadership*, *69*(8), 30-34.

Ingersoll, R., & Smith, T. M. (2004). Do teacher induction and mentoring matter? *NASSP Bulletin, 88(*638), 28–40.

McGatha, M. B. (2017). Elementary mathematics specialists: Ensuring the intersection of research and practice. In Galindo, E., & Newton, J., (Eds.). *Proceedings of the 39th annual meeting of the North American Chapter Group for the Psychology of Mathematics Education.* Hoosier Associations of Mathematics Teacher Educators.

McGatha, M. B., Davis, R., & Stokes, A. (2015). The impacts of mathematics coaching on teachers and students. In M. Fish (Ed.), *NCTM research briefs.* NCTM.

National Association for Professional Development Schools. (2021). *What it means to be a Professional Development School: The nine essentials* (2nd ed.) [Policy statement]. Author.

National Center for Education Statistics. (2021). *Characteristics of public school teachers.* Retrieved from https://nces.ed.gov/programs/coe/pdf/2021/clr_508c.pdf

National Council of Supervisors of Mathematics and TODOS. (2016). *Mathematics education through the lens of social justice: Acknowledgment, actions, and accountability.* Retrieved from https://www.todos-math.org/assets/docs2016/2016Enews/3.pospaper16_wtodos_8pp.pdf

National Council of Supervisors of Mathematics and TODOS. (2021). *Positioning multilingual learners for success in mathematics.* Retrieved from https://www.todos-math.org/assets/images/NCSM-TODOS%20Multilingual%20Learners%20Position%20Paper%202021.pdf

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematics success for all.* Author.

National Council of Teachers of Mathematics. (2020). *Catalyzing change in early childhood and elementary mathematics: Initializing critical conversations.* Author.

National Council of Teachers of Mathematics. (2021). *Continuing the journey: Mathematics learning 2021 and beyond*. Retrieved from https://www.nctm.org/uploadedFiles/Research_and_Advocacy/collections/Continuing_the_Journey/NCTM_NCSM_Continuing_the_Journey_Report-Fnl2.pdf

Reys, B., Webel, C., & de Araujo, Z. (2017). Elementary mathematics specialist preparation: What's next? What's needed? In M. McGatha & N. Riegelman (Eds.), *Elementary Mathematics Specialists* (pp. 221-234). Association of Mathematics Teacher Educators.

Rubel, L. H. (2017). Equity-directed instructional practices: Beyond the dominant perspective. *Journal of Urban Mathematics Education*, *10*(2), 66-105.

Stanulis, R. N., & Floden, R. E. (2009). Intensive mentoring as a way to help beginning teachers develop balanced instruction. *Journal of Teacher Education, 60*(2), 112-122.

Spangler, D. A., & Orvick, R. L. B. (2017). Models for mathematics specialist program development and delivery. In M. McGatha & N. Riegelman (Eds.), *Elementary mathematics specialists* (pp. 41-52). Association of Mathematics Teacher Educators.

Tricarico, K. M., Jacobs, J., & Yendol-Hoppey, D. (2015). Reflection on their first five years of teaching: Understanding staying and impact power. *Teachers and Teaching, 21*(3), 237-259.

Yin, R. K. (2014). *Case study research: Design and methods, 5th edition.* Sage.

Yopp, D.A., Burroughs, E.A., Sutton, J.T., & Greenwood, M.C. (2019). Variations in coaching knowledge and practice that explain elementary and middle school mathematics teacher change. *Journal of Mathematics Teacher Education, 22*, 5–3

LESSONS LEARNED DURING TWO TEACHER EDUCATORS' SHIFT TO SYNCHRONOUS ONLINE MATHEMATICS EDUCATION COURSES

Lynda R. Wiest	Teruni Lamberg
University of Nevada, Reno	University of Nevada, Reno
wiest@unr.edu	terunil@unr.edu

This paper describes a self-study conducted by two K-8 teacher educators in relation to their shift to teaching synchronous online mathematics education courses. They each wrote three independent self-reflections for a total of four courses (one undergraduate, three graduate) that they later analyzed and discussed for common themes in relation to the following topics: challenges (online teaching in general and for mathematics education courses), instructional adjustments, favorable aspects of teaching mathematics education courses online, skills gained that would benefit future teaching in any mode, equity considerations in teaching courses online, and recommendations for future online mathematics education courses. They report their findings in relation to these broad categories and use structured reflection on their experiences to recommend practices for online mathematics education teaching and learning.

Keywords: Online and Distance Education; Teacher Educators; Instructional Activities and Practices

The purpose of this study was to investigate our own practices as mathematics teacher educators during a required shift to online teaching, without sufficient experience and preparation, during the COVID-19 pandemic. To do this, we engaged in self-study.

Self-study in teacher education involves investigating one's own practice from one's own perspective, with the intent to draw conclusions about both general and specific practices (Berry & Kitchen, 2020; Pinnegar et al., 2020). This self-examination, which takes place with an understanding that teacher educators' practice occurs within interactions among human, material, and non-tangible elements and within an environment that is uncertain in many ways and is always subject to change, can contribute to the research base on the pedagogy of teacher education (Hordvik et al., 2020). In particular, Berry and Kitchen (2020) note, "Self-study has important contributions to make in these times [the COVID-19 pandemic] for documenting the experiences and insights that come from radical educational change" (p. 123).

Online synchronous teaching via videoconference technology has been shown to be challenging for teachers due, in part, to lack of adequate preparation for this type of instruction in relation to such areas as possessing appropriate technology skills and facilitating interaction among course participants (Cassibba et al., 2021; Rehn et al., 2018). Further, online instruction has been shown to increase teacher workload and challenge teachers emotionally, and that it must be designed carefully in order to provide equitable education rather than exacerbating educational inequities (Cassibba et al., 2021; Kaden, 2020). In a study of mathematics teacher educators, Huang and Manouchehri (2019) found that some of the challenges include posing appropriate tasks, selecting appropriate online tools and technologies, and engaging all students. Despite these challenges, those who teach mathematics have hastily learned to adapt new technology uses, such as writing tablets and mathematics software, and to take advantage of online applets and videos in efforts to simulate in-person classes (Cassibba et al., 2021; Huang & Manouchehri, 2019).

Methods

We, two long-time mathematics teacher educators, investigated our own shift to synchronous online teaching in 2020 and 2021. The two investigators have been teaching preservice and in-service mathematics education courses for 26 and 18 years (first and second author, respectively). For this study, we reflected on four mathematics education courses, three graduate and one undergraduate. One graduate course was taught by the first author, and the other three courses were taught by the second author. The graduate course topics were advanced K-8 mathematics methods, fractions/proportional reasoning, and problem solving. The undergraduate course was K-8 mathematics methods for preservice teachers.

During our online teaching, we first individually responded in writing to a set of four questions that we considered relevant to our shift in teaching mode. In a second wave, we added three questions to which we also wrote responses individually. Finally, after examining each other's responses to look for common themes, we wrote more detailed narrative responses to the questions we determined to be most salient in our initial reflective writings. In total, the questions focused on challenges (online teaching in general and for mathematics education courses), instructional adjustments, favorable aspects of teaching mathematics education courses online, skills gained that would benefit future teaching in any mode, equity considerations in teaching courses online, and recommendations for future online mathematics education courses. The final set of questions after the first two rounds of writing were:

1. Which aspects of switching to synchronous online teaching were most challenging?
2. What were some drawbacks or challenges of teaching a *math education* course online synchronously?
3. What instructional adaptations did you make in order to conduct effective synchronous online teaching?
4. What were some favorable aspects of teaching a math education course online synchronously?
5. What are some professional skills you gained by teaching online synchronously, such as skills that might transfer to your in-person teaching or other work responsibilities?
6. What equity issues, if any, did you notice while teaching online synchronously?
7. What suggestions can you make for teaching a synchronous online math education course effectively?

In the third and final round of our data collection, we wrote more "fleshed-out" responses to numbers 3, 4, and 7 in order to focus on data-based recommendations for online mathematics education courses.

To analyze our data, we read each other's three rounds of writing and compared them to our own, looking for major themes. We revisited the data (written reflections) several times until we believed the identified main themes accurately reflected the data (Strauss & Corbin, 1998). We then discussed our understanding of those themes and specific examples – or others generated through discussion that we had neglected to include in our writings – to illustrate them.

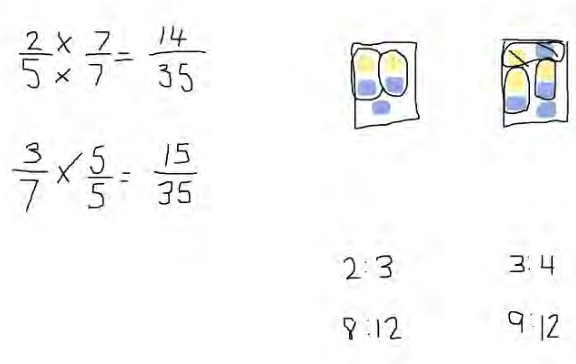
Results

Some key findings from our self-study include:

- *Challenges*: appropriate and effective use of technology; adjustments to class instruction, class assignments, and class policies and expectations; maintaining student engagement

- *Pros*: ability to see all students close up (except when spread onto more than one screen); ability to place students into groups quickly and randomly; incorporation of technology into class sessions due to its immediate access (e.g., to research information and to explore websites); greater ease of facilitating equitable whole-group participation (students muted and acknowledged in order to speak); easier access to guest speakers
- *Cons*: variable learning settings for participants; students appearing on more than one screen; joint mathematics tasks that involve drawing, use of physical materials, and sharing work that can't be shown side-by-side; limitations and potential failure of technology
- *Instructional adjustments*: greater use of technology and potentially breaking up material more by independent and group work; use of Zoom's whiteboard feature, Jamboard, Google Docs, and the video camera in doing mathematics tasks; greater pre-planning so that students can be provided needed materials in advance or asked to come to class with particular materials that would normally be housed in the classroom
- *Equity considerations*: variable settings that are more conducive or less conducive to class engagement; differing access to technology (e.g., inability to keep video on without high-speed internet) and different levels of technology skills; potential to engage all students to a more equitable degree (e.g., see earlier comment in "pros"); potential social stratification based on "window" into each other's personal lives

Our trepidation regarding how to have students engage in collaborative mathematics learning involving discussion and drawing was eased by students immediate jumping to use of tools that served those ends. In terms of drawing tools, students used the videoconference whiteboard in small groups and saved their work to show when we returned to the whole-group setting. They also used Jamboard, Word drawing tools, and other technological drawing tools, copied images from online into Word documents, and made drawings they took photos of to share in the videoconference system or simply held their work up to the camera. For example, one small group working to solve a word problem about which of two lemonade recipes would have a stronger lemonade flavor (a proportional reasoning problem) in a breakout room saved the following whiteboard work to share when we returned to the whole group to share and discuss problem solutions (they were expected to seek multiple solution methods):



Discussion and Conclusions

The findings of our self-study support some existing research results, for example, the fact that technology use can benefit teaching and learning (Alenezi, 2017) but that technology skills can be a drawback to online teaching Fransson et al., 2018) and instructors must be prepared to teach students how to use needed technologies (Bryans-Bongey & Graziano, 2016). However, the fact that our shift in practice was sudden and the nuance with which we self-reflected on our practice yielded important additional insights into online teaching for mathematics teacher educators.

One thing we observed in our writing and discussions was that the rapid and substantial learning we engaged in during our shift to online mathematics education classes is likely to influence our in-person teaching. For example, we will be more likely to consider use of online tools and resources in the classroom and supplemental online course platforms and to consider use of laptop and tablet computers and even smart phones in the physical classroom.

Despite the challenges we experienced in hastily switching to teaching mathematics education courses online, we were surprised to find that we believed this type of teaching could be highly effective. However, we noted that we needed sufficient time and preparation to learn to do this well, similar to that which has been reported by others (Cassibba et al., 2021; Rehn et al., 2018). We also learned that we must draw on our students for assistance in how to navigate good practices in online teaching and to observe and listen to their preferred methods of learning and demonstrating their knowledge online. Our own learning during this experience was favorably influenced by our students' awareness of technologies and tools to use that we had not been aware of previously. The benefits of quick and varying groupings for collaborative work and easy access to online learning resources (e.g., applets and videos) thrust us into new ways to support students' learning, which mirrors what others have experienced in teaching mathematics online (Cassibba et al., 2021; Huang & Manouchehri, 2019). However, we acknowledge that we need to continue to examine the implications of this relatively new instructional mode on different students in relation to equity considerations. Our thrust into structured, collaborative self-examination was a good start toward exploring new ways of teaching in a technological world that will benefit us in both online and face-to-face teaching now and in the future.

References

Alenezi, A. (2017). Obstacles for teachers to integrate technology with instruction. *Education and Information Technologies*, *22*(4), 1797-1816. DOI:10.1007/s10639-016-9518-5

Berry, A., & Kitchen, J. (2020). The role of self-study in times of radical change. *Studying Teacher Education*, *16*(2), 123-126, DOI: 10.1080/17425964.2020.1777763

Bryans-Bongey, S., & Graziano, K. J. (Eds.). (2016). *Online teaching in K-12: Models, methods, and best practices for teachers and administrators*. Medford, NJ: Information Today.

Cassibba, R., Ferrarello, D., Mammana, M. F., Musso, P., Pennisi, M., & Taranto, E. (2021). Teaching mathematics at distance: A challenge for universities. *Education Sciences*, *11*, 1-20.

Fransson, G., Lindberg, O. J., & Olofsson, A. D. (2018). From a student perspective, what constitutes a good (or less good) use of ICT in teaching? *Education and Information Technologies*, *23*, 2155-2177.

Hordvik, M., MacPhail, A., & Ronglan, L. T. (2020). Developing a pedagogy of teacher education using self-study: Arhizomatic examination of negotiating learning and practice. *Teaching and Teacher Education*, *88*, 1-11.

Huang, D., & Manouchehri, A. (2019). Online mathematics teacher education in the US: A status report. *Contemporary Issues in Technology and Teacher Education*, *19*(2), 171-194.

Kaden, U. (2020). COVID-19 school closure-related changes to the professional life of a K–12 teacher. Education Sciences, *10*, 1-13.

Pinnegar, S. E., Lay, C., Cutri, R. M., & Newberry, M. (2020). Exploring the contribution of self-study of teacher education practice to the conversation on research on teacher education. In C. Edge, A. Cameron-Standerford, & B. Bergh (Eds.), *Textiles and tapestries: Self-study for envisioning new ways of knowing*. Equity Press. https://edtechbooks.org/textiles_tapestries_self_study/exploring_contribution

Rehn, N., Maor, D., & McConney, A. (2018). The specific skills required of teachers who deliver K–12 distance education courses by synchronous videoconference: Implications for training and professional development. *Technology, Pedagogy and Education*, 27(4), 417-429. DOI: 10.1080/1475939X.2018.1483265

Strauss, A., & Corbin, J. (1998). Basics of qualitative research: Techniques and procedures for developing grounded theory. Thousand Oaks, CA: Sage.

NAVIGATING THE WHITE SPACE OF MATHEMATICS: LESSONS LEARNED FROM MATHEMATICS TEACHER EDUCATORS OF COLOR

Cathery Yeh
University Of Texas, Austin
catyeh@aol.com

Kathryn Bianco
Portland State University
kbianco@pdx.edu

Kayla Asato
Chapman University
kasato@chapman.edu

Eva Thanheiser
Portland State University
evat@pdx.edu

Breána Victoria
Chapman University
bvictoria@chapman.edu

Nicole Roady
Chapman University
roady@chapman.edu

Rebekah Elliott
Oregon State University
rebekah.elliott@oregonstate.edu

Ruth Heaton
Teachers Development Group
ruth.heaton@teachersdg.org

As critical race theorists would remind us, those most impacted have the greatest insight to create change. This paper applies a critical race theory framework to explore the leadership experiences of two African American and one Latinx American mathematics teacher educator and how they address issues of race, racism, and (in)justice in teacher professional development. Data analysis from semi-structured interviews, publicly available webinars and podcasts, and other published materials from the educators (eg. articles and books) reveal how they engaged teachers to attend to issues of race and racism by challenging persistent masternarratives about mathematics and mathematics ability; centering on counternarratives on the cultural identities and mathematical understanding of students of Color, and engaging in community-based pedagogies to promote coalitional resistance.

Keywords: teacher educators, social justice, professional development, equity, inclusion, and diversity

Study Purpose

Critical race theory (CRT) has seen steady growth in use as an interpretive lens to analyze and challenge racism in K-20 context and policy (Ladson-Billings & Tate, 1995; Solorzano & Yosso, 2002). Yet, a gap exists in applying CRT in mathematics education and the actions mathematics teacher educators must take to address racial and social (in)justice in teacher professional development. Martin (2019) discussed this point in his critique of the silence of anti-Blackness and white supremacy logic in the politics of mathematics. Gholson & Wilkes (2017) builds on similar claims to highlight the prevalence in mathematics education of racial scripts which divide, sort, and stratify, along a caste hierarchy placing whites at the top and Black at the bottom (Wilkerson, 2020).

As educators, scholars, and community organizers with deep commitments to equity and racial justice, our work is situated by the impact of a legacy of educational injustices (Ladson-Billings, 2006) in which the mathematics education system must be reimagined. We draw on CRT as it specifically acknowledges racism as endemic and actively confronts white supremacy within mathematics education. We extend and apply this framework to chronicle the paths of mathematics educators of Color whose scholarship addresses the racial challenges thrusted upon teachers and learners of Color and the gatekeeping nature of the mathematics discipline.

Specifically, this study explores the leadership experiences of two African American and one Laxinx mathematics teacher educator and how they address issues of race and racism in teacher education. Data include semi-structured interviews, publicly available webinars and podcasts, and other published materials (eg. articles, books) to answer the research question: What knowledge can we gain from the stories/narratives of mathematics educators of Color as they explore their experiences in, with, and for creating professional development that attend to race, racism, and racial justice in mathematics education? Study findings reveal the ways in which educational histories, cultural and ancestral roots, the students served, and current equity leadership informed their perceptions and engagement with racial justice. Our paper raises questions to the field about how we understand social justice leadership in mathematics education and support current and aspiring leaders of Color who seek to promote racial equity in their work.

Theoretical Framework and Literature Review

Our project is motivated by pressing issues in mathematics education: 1) longstanding racial injustices that have come to the fore in U.S. society and mirrored in mathematics education (Gholson & Wilkes, 2017; Martin, 2019), 2) mathematics teachers and teacher educators are majority white, while serving a disproportionate number of students of Color (deBrey, et al, 2019); and 3) mathematics serving as a gatekeeper to nearly all aspects of success. As such, we are interested in interrogating and disrupting the role mathematics education plays in perpetuating white supremacy. With few exceptions (e.g. Battey & Leyva, 2016; Martin, 2015, 2019), the literature has not significantly examined how whiteness and white supremacy culture function in mathematics education (see Martin, 2015, 2019). We draw on Critical Race Theory (CRT) (Ladson-Billings & Tate, 1995) to interrogate aspects of whiteness in mathematics classrooms, schools, and the education system. CRT has its origins from Critical Legal Studies, developed in the mid-1970s from the work of legal scholars of color, such as Derrick Bell, Mari Matsuda, and Kimberle Crenshaw, interested in investigating and transforming the injustices that were brought about due to issues of race, racism, and power in our society (Delgado & Stefancic, 2001). Although CRT began in legal studies, it expanded to other disciplines, including mathematics education. Within the field of education, CRT scholars (Davis & Jett, 2019; Solorzano & Bernal 2001) outlined several key tenets which inform our research methodology, including (a) centrality of racism; (b) intersectionality and anti-essentialism; (c) commitment to social justice; (d) voices and counterstories; and (e) interdisciplinary perspectives.

Centrality of Racism

The first tenet of CRT is the notion that racism is not a random, isolated act of racist individuals; rather, racism is a normal feature and embedded within systems and institutions. According to Delgado and Stefancic (2001), racism "is the usual way society does business" (p.7). Mathematics education by design has a legacy of systemic epistemicide of communities of Color—the erasure of knowledge systems, including languages, experiences, and interpretations of the world, and ways of coming to know and understand—through exclusion from its functions, curriculum, and pedagogy (Yeh et al., 2021; Louie, 2017; Martin et al., 2010).

Intersectionality and Anti-Essentialism

CRT recognizes multiple identities – race, class, gender, sexuality, ability - and their positions within intersecting relations of power. According to Delgado & Stefancic, "intersectionality means the examination of race, sex, class, national origin, and sexual orientation and how their combinations play out in various settings" (2001, p. 51). Identities are not fixed; anti-essentialism acknowledges the fluidity and constant shift of cultural identities within an oppressive system (Gillies, 2021). While our society and mathematics education are often organized along binaries (e.g. right or wrong, math smart or not, innate or social), CRT scholarship sees the complexities required to examine and abolish systems of oppression. For example, racism and ableism are interwoven in mathematics education, in which mathematical "smartness" is an ideological system that perpetuates whiteness and serves as a tool for stratification (Yeh et al., 2020; Leonardo & Broderick 2011). Ability is commodified and seen as property; success is measured in terms of achievement of numerical target, and such data are used to construct and constitute some students as "mathematically smart" while simultaneously constructing and constituting others as "not-so-smart," deficient, or learning dis/abled (Yeh, et al., 2020). CRT tenets of intersectionality and anti-essentialism challenge these binary systems and instead seek ways to counter these paradigms that uphold inequities in education. In challenging these definitions, we are not only incorporating intersectionality and anti-essentialism but challenging dominant ideologies themselves.

Commitment to Social Justice

CRT is about empowerment. Anzaldúa (1987) refers to the term borderland consciousness, the ways of knowing and being Peoples of Color produced in multiple and often contradictory physical, social, and political spaces. Their worldviews as students and educators of Color include homegrown ways of knowing as well as western epistemes. These approaches empower educators and students of Color in that they recognize the advantage of having insights into the experiences of the Oppressed and the Oppressors and can reveal the cracks that support social change (Anzaldúa, 1987; Collins, 2000; hooks, 1989). Our work here is to attend to and leverage the multiple ways of knowing mathematics educators of Color to offer recommendations for ways to take action for social justice.

Voices and Counterstories

Dr. Mari Matsuda (1987), one of the legal practitioners from the original movement, argues that the work of justice must center on the voices that have been left out. Matsuda refers to them as "voices from the bottom," Anzaldua (1987) calls them "mestiza," and hooks (1990) advocates for using these excluded voices to produce transformative knowledge. Counternarratives build from the belief that communities of Color hold deep resources and ways of knowing and being that are particularly important in disrupting colonial ideologies pervasive in mathematics. Yet, much of these rich and collective ways of knowing and being have not yet been made visible. It is through centering the voices of communities of Color to offer historical accounts, interpretations, and cultural practices as resources to question and challenge dominant narratives.

Interdisciplinary Perspectives

CRT scholarship challenges ahistoricism and the unidisciplinary focus of most research analyses. It is predicated on analyzing race and racism in both historical and current contexts from an interdisciplinary or transdisciplinary approach (law, sociology, education, psychology, ethnic/gender/disability studies, etc.) which is needed to better understand the systemic nature of racism and white supremacy. For example, few scholars explore mathematics as an agent of carceral state; yet, violence is enacted through institutionalized policies, such as school disciplinary practices that offer brutality, suspension, expulsion, and detention often occurs in STEM classrooms (Bullock & Meiners, 2019; Gholson & Wilkes, 2017). An interdisciplinary perspective recognizes, interrogates, and then disrupts systems of violence and trauma experienced in and beyond mathematics classrooms.

Method

Critical Race Theory served as our methodological framework. We understand that race is a sociohistorical and political construct, created to empower whites and oppress people of Color; as such, we use counterstorytelling (Delgado and Stefancic, 2001) to capture experiences in discussion and praxis that attend to race, racism, and racial justice. This study includes one African American male math educator, one African American female math educator, and one Latinx female mathematics teacher educator in which their self-selected pseudonyms are Francis Cox, Etta Falconer, and Katherine Johnson, respectively. The criteria for participant selection included the following: 1) self-identification as an educational leader of Color; 2) extensive experience (20+ years) as a classroom teacher and mathematics instructional coach; and 3) research and teaching that focuses on examining the ways education intersects with issues of race, equity, and justice in communities that have historically been, and continue to be, underserved by schools (e.g. Black, Latinx, Indigenous, or immigrant populations).

Attending to story, theory, and praxis, the data sources include three semi-structured, in-depth interviews (roughly 1 hour each), three publicly available webinars and podcasts (1 hour each), and other published materials written by the participants (e.g., articles, books). Analysis began with the webinar and podcasts. For the webinar and podcasts, we created an accompanying transcript. Each transcript was divided into talk segments, where each segment represented a central idea the interviewee focused on during a posed question or activity. At least two of the seven members of the research team coded and categorized each webinar and podcast through a critical race analytical lens and tenets of CRT (e.g., centrality of racism; intersectionality and anti-essentialism; commitment to social justice; voice and counterstories; and interdisciplinary perspectives) as well as from the K-12 and higher education scholarship on racialized experiences of teachers and students of Color cited in the literature review. Example codes included definitions of mathematics, racial hierarchy, transformational resistance, intersectionality, and color blindness. Coding allowed us to categorize the data into overall themes as codes that were repeatedly used across the three cases, which were then placed into overarching themes. After the research team member coded the podcast/transcript, coding was discussed by the research team of seven for agreement. Disagreements were resolved through discussion. After coding each educator's podcasts and workshops, we interviewed three of them . Interviews were about 60 minutes each. Interview questions began with:

1. What should mathematics instruction look like when it centers on the mathematical brilliance of Black, Indigenous, Latinx, and multilingual students and their communities?
2. How do we support the creation of school systems or structures for teachers to have the school-wide support to implement and learn equitable practices in math ed?
3. Most of our workshop consultants are white mathematics teacher educators, what resources would help them to be able to authentically engage in racial justice work?

Interviews were transcribed and analyzed using the same codes as described above. Triangulation of data allowed us to gain a better understanding of the experiences of the three educators. We used publicly available podcasts, webinars, interviews, and published articles as methodological procedures to triangulate the data.

Results

Our findings reveal three major themes. All three educators of Color discussed the need to situate education within broader societal contexts and the need to challenge white masternarratives that position mathematics and mathematics ability as singular, individualistic, and ahistoricized. They described their own and students' cultural identities and understanding and its connection to mathematics teaching and learning and the need for coalitional resistance to disrupt whiteness in mathematics teacher education.

Challenging Persistent Masternarratives.

All three described specific ways in which mathematics education itself functions as a white space in historicized context, naming the contemporary forms of institutional, structural, and everyday racism in the here and now rather than just the past. Specifically, they called to rethink what constitutes mathematical knowledge and what constitutes knowledge production in mathematics education. In his interview, Francis highlighted the need for mathematics teacher educators to explicitly shift away from the white frame:

> I would say that it should look the same or similar as when it centers on the mathematical brilliance of white students right, so it should have examples and images of people that reflect their culture and the contributions that people from their history and culture have made to the field of mathematics, it should include examples that also reflect students that are African American, Latinx, Indigenous, multilingual students.

In the quote above, Francis highlights that racial equity work in mathematics teacher education requires us to examine critically whose histories, cultures, and experiences are centered in mathematics curriculum and the ways in which whiteness is made hegemonic as it is positioned as the normative and communities of Color as the exception. He further explains that,

> Looking at mathematics from a kind of diverse Black perspective, there's a book called "The Crest of the Peacock: the Non-European Roots of Mathematics." Now [it] does get into some pretty heavy mathematics, at some point, you know so it's not for the faint of heart right, but you know I think anyone can [at] least engage in it to get an idea to dispel the myth that you know mathematics is a European invention and endeavor right? That it is a human endeavor right? Rochelle Gutiérrez says mathematics needs people, just like people need math mathematics right and, and I think we've moved too far away from that notion that mathematics is a human endeavor that all humans engage in right, whether they realize it or not.

All three math educators explicitly called to broaden curricula, classroom environments, and school cultures to explicitly attend to the voices, perspective, histories and experiences of communities of Color. In the webinar led by Francis Cox and Etta Falconer, the workshop opening mathematical task asks participants to match the professions of Black dentists, Black professional athletes, Black doctors, and Black lawyers to "the corresponding number of Black professionals in the United States". The opening task explicitly attended to participant bias and public narratives around Black professions. During the interview, Etta Falconer stated, "representation really does matter. So it's important that you make sure that all students can see themselves reflected in the curriculum." Every mathematical task and student vignette used in the professional development workshop focused on the agency and identity construction of Black youths and the ways in which social structures mediated their mathematical identity formation.

Cultural Identity and Understanding.

The mathematics educators used storytelling to center on the experiences of communities of Color and as examples to generate action. Both Katherine and Francis began interviews sharing their mathematics experiences as students and teachers in which their experiences shaped their work and commitments. As a student, Katherine "felt dehumanized in class. I felt invisible. I use the word invisible when I talk about my schooling" and Francis worked in "neighborhoods that had experienced poverty and other kinds of trauma, and, you know, large numbers of students who have been underserved." While their experiences could be viewed as "challenges", their stories testify to the normalization of such experiences for children and teachers of Color AND the strength and resilience communities of Color developed. They saw themselves in the students and teachers of Color they served.

All three educators talked about the importance of developing mathematics learning experiences that shift from damaged based narratives to student strengths by building from their cultural identities and understandings. Katherine in a podcast interview talks about the importance of recognizing the brilliance of children and embedding their culture in the mathematics classroom:

> So we expect that all students are brilliant. That's a mindset issue. We talk about growth mindset. This is more than that. This is knowing that all students and believing that they're brilliant, and they bring knowledge, mathematical knowledge into the classroom, right. So that's one of the pieces. The other piece is also bringing that cultural competence, right. I'm caring enough about the math that you're doing, and I want to learn about it. And, and I want to share the math that I do, too, you know. We talk about real world math all the time. It's really not real world, many of the problems that we see in books are contrived. Real world math is the math that kids are actually doing with their families. And so, how do we tie that into the standards and bring those into the classroom, like my colleague [anonymized] said, as tasks, as activities, because that's going to build students' cultural competence. And I think sometimes it brings, well, I know it brings the community together, cause the more I know about Maura, Joe, and Stella, the better I get to perform as a student, because now I know how they best understand math, what they care about, the math they do in their everyday life.

Katherine in a webinar describes how she was able to learn more about her students by having them draw their own multiplication array, describing context from their families and family experiences. In particular, she gives the two following examples:

> One is about a student talking about his uncle, how they played baseball, and his array was a 3 by 10. Then another student who actually was a multilingual learner and talking about her Vovo, and how they may visualize the tacos together in Brazil.

Similarly, in their workshop and interviews, Etta and Francis centered on stories of Black students to expose, analyze, and critique the racialized reality of Black youths in mathematics education. Etta described an experience where she felt challenged by a female student, and some of those challenges stemmed from her strengths being directed at Etta. Instead of conceptualizing it negatively, Etta reassessed these as strengths. During her interview, Etta shared her thinking out loud, "How can I leverage that? How can I leverage that strength, that leadership and tenacity that she has in mathematics?" The counterstories used followed a similar format of centering on the standpoint of Black students, critiquing unfair practices and ideologies, and pointing to possibilities and actions.

Coalitional Resistance

All three educators discussed the importance of organizing forms of learning for mathematics teachers and students that provided new identity pathways where people work together to critique, re-imagine, strategize, design, and re-make how they can engage with the mathematics and each other. The community was identified as a site for learning, organizing, and activism. A central theme is the importance of community responsive pedagogies in which mathematics educators must begin by learning from the community to develop curriculum and pedagogy responsive to community needs, strengths, and contexts. Francis explicitly attends to this during his interview when he shares a story how teachers were wary of having community members come to the educators' homes even though the teachers visited the community members' homes.

> It's a two way street. Right? It's like we want, you know, the community to come into the school, but we don't want to go into the community, you know? I remember in the teacher education program at UCLA, they really were encouraging the teachers to do home visits…And they have this meeting afterwards, and you know this gathering and after gathering the parents asked the teachers if they could come visit their home. And the teachers were often, "Wait, whoa wait a minute," right. And so you know we want, we, it seems like we only want this relationship to be one way right. We have, it has to be doable by, it has to go in both directions right, we have to be willing to go out into the community and into and find out what are the, the the assets.

During the interview in which Katherine was asked to share ways to create school systems or structures to implement and learn equitable practices in mathematics education, she shared:

> So when we talk about dismantling racism, the way we do that is to humanize all of our neighborhoods and our kids by bringing them there. Like our teachers need to come out of the schoolhouse and really go into places. There's so much to be learned, actually, seeing and learning and listening and then not talking. Don't talk, just ask questions and listen.

During a teacher professional development workshop, Francis stated, "And our definition of expert is anyone who can contribute to the knowledge of the group. Therefore any one of your students can be an expert, given the opportunity."

In the quotes above, Katherine and Francis describe how connecting to and learning from students' and in communities are critical to countering masternarratives of who and what mathematics is and for and who can be seen as the experts and teachers of mathematics. This also includes challenging masternarratives in teacher professional development in which the mathematics educator or consultant are seen as the expert to professional learning spaces in which the mathematics educator begins by learning in, for, and with the school teachers to move towards collective action. For example, in a podcast interview Francis talks about his PD practices.

> Anytime that I write a grant with a district or get a request from a school to do professional development, one of the first things I want to do is meet with the teachers themselves, or represent, representatives of the teachers to find out, what are their concerns? What is it that they want? What is it that they need, right, so I think that's the most, because teachers want to feel important.

Discussion

As critical race theorists (Davis & Jett, 2019; Dixson & Rousseau, 2005; Matsuda, 1987) would remind us, those most impacted have the greatest insight to create change. In this paper, we extend and apply this framework to chronicle the paths of mathematics educators of Color whose scholarship addresses the racial challenges thrusted upon teachers and learners of Color and the gatekeeping nature of the mathematics discipline. What knowledge can we gain from the stories of the three mathematics educators of Color as they explore their experiences in, with, and for creating professional development that attend to race, racism, and racial justice in mathematics education? All three mathematics educators of Color discussed the need to situate mathematics education within broader socio-political contexts and the need to challenge white masternarratives that position mathematics and mathematics ability as singular, individualistic, and ahistoricized.

As we think about creating culturally sustaining spaces for students and teachers of Color, we are also reminded to consider how mathematics teacher education is supporting mathematics educators of Color. School administrators and leaders in the United States are disproportionately white. As a field, we need to consider the practical, on-the-ground support provided to in-practice and in-preparation leaders of Color. Dr. Patel's (2017) notion of token or incremental inclusion is helpful. How much of current equity efforts are token change? For example, having a lone teacher leader of Color assigned to an all-white equity committee or to expect collaboration simply by putting a group of educators and families in a room, without acknowledging that the same hierarchical power dynamics responsible for long-standing inequities will quickly reassert themselves. How often are we socializing nondominant students and educators into norms, expectations, and agendas that have been set without their perspectives or input?

As mathematics educators and education researchers, we are trained in academic spaces in which Western intellectual culture is espoused as "objective science". When whiteness and white supremacy serve as the marker of what is considered normative and excellence, there is a deep need for unlearning and relearning of the possibilities to engage with students, teachers, and communities in more ethical ways. These activist-educators have taught us powerful lessons about historical injustice as well as community resistance, cultural practice, and ethical responsibilities that should fundamentally shape how we approach mathematics teacher education and research. This necessitates an examination of the mathematics teacher education system and the ways it reproduces racial injustice. Particularly in devising practices, structures, and policy interventions to address racial (in)justice, CRT calls for considering unintended consequences of proposed remedies, addressing intersecting policies and structures, and acting intentionally to ensure that harm is not further replicated within our mathematics education system.

References

Anzaldua, G. E. (1987). *Borderlands/ la frantera: The new mestiza*. San Francisco, CA: Aunt Lute Books.

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education*, *9*(2), 49–80.

Bullock, E. C., & Meiners, E. R. (2019). Abolition by the numbers mathematics as a tool to dismantle the carceral state (and build alternatives). *Theory Into Practice*, *58*(4), 338–346.

Collins, P. H. (2000). *Black feminist thought*. New York, NY: Routledge.Davis, A. Y. (1990). *Women, culture & politics*. Vintage Books.

Davis, J., & Jett, C. C. (Eds.). (2019). *Critical race theory in mathematics education*. New York, NY: Routledge.

deBrey, C., Musu, L., McFarland, J., Wilkinson-Flicker, S., Diliberti, M., Zhang, A., Branstetter, C., and Wang, X. (2019). Status and trends in the education of racial and ethnic groups 2018 (NCES 2019-038). U.S. Department of Education. Washington, DC: National Center for Education Statistics. Retrieved from https://nces.ed.gov/pubsearch/.

Delgado, R., & Stefancic, J. (Eds.). (2001). *Critical race theory: An introduction*. New York University Press.

Dixson, A. D., & Rousseau, C. K. (2005). And we are still not saved: Critical race theory in education ten years later. *Race Ethnicity and Education*, *8*(1), 7–27.

Gholson, M. L., & Wilkes, C. E. (2017). (Mis) taken identities: Reclaiming identities of the "collective Black" in mathematics education research through an exercise in Black specificity. *Review of Research in Education*, *41*(1), 228–252.

Gillies, C. (2021). Curriculum Integration and the Forgotten Indigenous Students: Reflecting on Métis Teachers' Experiences. *In Education*, *26*(2), 3–23.
hooks, b. (1989). *Talking back: Thinking feminist, thinking black*. South End Press.
hooks, b. (1990). *Yearning: race, gender, and cultural politics*. United States: South End Press.
Ladson-Billings, G. (2006). From the achievement gap to the education debt: Understanding achievement in US schools. *Educational Researcher*, *35*(7), 3–12.
Ladson-Billings, G., & Tate, W. F. (1995). Toward a critical race theory of education. *Teachers College Record*, *97*(1), 47–68.
Leonardo, Z., & Broderick, A. A. (2011). Smartness as property: A critical exploration of intersections between whiteness and disability studies. *Teachers College Record*, *113*(10), 2206–2232.
Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, *48*(5), 488–519.
Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 316–333.
Martin, D. B. (2015). The collective Black and principles to actions. *Journal of Urban Mathematics Education, 8*(1), 17–23.
Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, *22*(4), 459–478.
Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education*, *3*(2), 12–24.
Matsuda, M. (1987). Looking to the Bottom: Critical Legal Studies and Reparations. *Harvard Civil Rights-Civil Liberties Law Review*.
Marshall, C., Rossman, G. (2016). *Designing qualitative research*. SAGE Publications, Inc.
Mills, J., & Birks, M. (2014). *Qualitative methodology a practical guide*. SAGE Publications, Inc.
Patel, L. (2017). Rejecting a politics of inclusion. [Blog post]. Retrieved from https://decolonizing.net/2017/07/27/rejecting-a-politics-of-inclusion/
Solorzano, D. G., & Bernal, D. D. (2001). Examining transformational resistance through a critical race and latcrit theory framework: Chicana and Chicano students in an urban context. *Urban Education*, *36*(3), 308–342.
Solórzano, D. G., & Yosso, T. J. (2002). Critical race methodology: Counter-storytelling as an analytical framework for education research. *Qualitative Inquiry*, *8*(1), 23–44.
Stinson, D. (2017). Beyond white privilege: Toward white supremacy and settler colonialism in mathematics education, *Journal of Urban Mathematics Education*, 10(2), 1-7.
Tate IV, W. F. (1997). Critical race theory and education: History, theory, and implications. *Review of Research in Education*, *22*(1), 195–247.
Wilkerson, I. (2020). *Caste: The origins of our discontents*. New York: Random House Publishing Group.
Yeh, C., Ellis, M., Mahmood, D. (2020). From the margin to the center: A framework for rehumanizing mathematics education for students with dis/abilities. *Journal of Mathematical Behavior. 58,* 100758.
Yeh, C., Tan, P., & Reinholz, D. (2021). Rightful presence in times of crisis and uprising: A call for disobedience. *Equity & Excellence in Education. 54*:2, 196-209, DOI: 10.1080/10665684.2021.1951631

.

COACHES' NOTICING BEFORE, AFTER, AND DURING VIDEO CLUBS

Julie M. Amador
University of Idaho
jamador@uidaho.edu

Jeffrey Choppin
University of Rochester
jchoppin@Warner.Rochester.edu

Cynthia H. Callard
University of Rochester
ccallard@Warner.Rochester.edu

Cynthia Carson
University of Rochester
ccarson@Warner.Rochester.edu

Jennifer Kruger
University of Rochester
jkruger@Warner.Rochester.edu

Ryan Gillespie
University of Idaho
rgillespie@uidaho.edu

Stephanie Martin
University of Rochester
smartin@Warner.Rochester.edu

Adam Hanan
University of Idaho
Hana3274@vandals.uidaho.edu

Keywords: Inservice Teacher, Professional Development, Research Methods, Coaching

We have designed and implemented an innovative online video-based professional development model for mathematics coaches in rural contexts that includes an online course, one-on-one video coaching with a mentor coach, and video clubs. As a part of this project, we were interested in the coaches' professional noticing (e.g., Jacobs et al., 2010; van Es & Sherin, 2008; van Es, 2011) as evidenced primarily through video clubs (i.e., Walkoe, 2015). The intent of the poster will be twofold: (a) share the design of our online video club for coaches and (b) share our process for researching coach noticing during video clubs as well as at the beginning and end of the project.

Video clubs provide collaborative learning opportunities for teachers (Barnhart, 2016; Dobie & Anderson, 2015) and have shown to be beneficial for mathematics teachers in rural contexts (Choppin et al., 2019). Our project focused on mathematics coaches in rural contexts, so we adapted the typical video club focus on classroom instruction, to coach-teacher interactions in coaching conversations (Carlson et al., 2017). The coaches in our project watched excerpts from one-on-one coaching cycle conversations and engaged in small group discussions, facilitated by an experienced mathematics coach. We were interested in the role of video clubs to support the development of the noticing of mathematics coaches, so the following research questions guided our work: (1) How did coaches exhibit noticing in written records of video clubs?, and (2) How did coach noticing change from the beginning of their participation in the project to the end?

To answer the first research question, we collected written data from coaches as they watched video excerpts in eight video clubs across two years. We provided a specific *see, think, and wonder* protocol that prompted participating coaches, in writing, to identify (*see*) specific moments in the coach-teacher interactions, create interpretations (*think*) about the thinking of the teacher and coach within this moment, and pose questions (*wonder*). The methodology to examine coach noticing through written records is in its infancy, so the poster would report on our most current data analysis processes and findings to the point that we have results. To answer the second research question, we interviewed each coach prior to participating in project learning experiences. As a part of the interview, we provided coaches with three vignettes from transcribed one-on-one coaching conversations, each representing a range in coaching styles. Coaches were asked a series of questions about what they noticed about teacher thinking and the coach's interaction with the teacher. The analysis for this process is underway. We will report findings from a subset of our data for the poster. Our focus on video clubs for coaches adds to the existing literature base on both noticing and video clubs in mathematics education.

Acknowledgments

This work was supported by the National Science Foundation (#2006263). Any opinions, findings, and recommendations expressed are those of the authors, and do not necessarily reflect the views of the National Science Foundation.

References

Barnhart, T. (2016). Participation in a video club: Influences on teachers and teaching. UC Irvine Electronic Theses and Dissertations.

Carlson, M. A., Heaton, R., & Williams, M. (2017, September). Translating professional development for teachers into professional development for instructional leaders. *Mathematics Teacher Educator, 6*(1), 27 - 39. https://doi.org/10.5951/mathteaceduc.6.1.0027

Choppin, J., Amador, J., Callard, C., Carson, C., & Gillespie, R. (2019). Synchronous online model for mathematics teachers' professional development. *Handbook of Research on Online Pedagogical Models for Mathematics Teacher Education*. (pp. 176-202). IGI Global.

Dobie, T. E., & Anderson, E. R. (2015). Interaction in teacher communities: Three forms teachers use to express contrasting ideas in video clubs. *Teaching and Teacher Education*, 47, 230-240

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010, March). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169-202. https://doi.org/10.5951/JRESEMATHEDUC.41.2.0169

van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. Teaching and Teacher Education, 24(2), 244–27. http://doi.org/10.1016/j.tate.2006.11.005

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A.

Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 134-151). New York: Routledge.Walkoe (2019)

Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education*, *18*(6), 523-550.

BRIDGING PROFESSIONAL DEVELOPMENT AND MATH TEACHERS' CONTEXTS

Erin E. Baldinger
University of Minnesota
eebaldinger@umn.edu

Jen Munson
Northwestern University
jmunson@northwestern.edu

Keywords: Professional development, Learning theory

While teacher education research has coalesced around features of effective professional development (PD), there exists a persistent gap between PD and practice, which we argue is driven by the decontextualization of teacher learning opportunities. PD designs, even those that adhere to principles for efficacy, often provide learning opportunities out of context, away from the teachers' own students, school, curriculum, leaders, and, ultimately, the constraints and assets stemming from each. Teachers learn about practice and learn to enact practice in a more clinical setting and then must translate their emerging learning into real and complex settings that present their own challenges. Separated from their local contexts, teachers may not have the opportunity to learn to negotiate those challenges during their professional learning experiences.

Guided by Rogoff (1995), we hypothesize that the disconnect between professional learning opportunities and teachers' contexts may explain some of the failure of PD to lead to meaningful changes in practice. Teacher educators have been moving toward embedding teacher learning opportunities in classrooms (e.g., Gibbons et al., 2021; Munson et al., 2021). However, we do not yet know how deepening the connection between PD and instructional contexts might influence teacher learning, or how PD might be designed to support teachers in negotiating the practices they want to implement and the contextual constraints they face in doing so. In this study, we analyzed one practice-based PD program for 22 early career secondary mathematics teachers serving lower income communities across the US to address the following question: In what ways do teachers make their local contexts visible during practice-based PD?

Teachers participated in a two-week summer residential institute. They returned to their schools and attempted to implement the practices they learned, while participating in monthly video coaching, both one-on-one and in small groups. This design included opportunities for concentrated teacher learning separated from context (i.e., summer institute) and support in context (i.e., coaching). We analyzed recordings from the summer institute and the coaching sessions to identify instances in which the participants described features of their local teaching context when engaged in professional learning. We inductively coded these instances by topic and then categorized them based on the activity during which they occurred (e.g., rehearsal planning, one-on-one coaching). Each instance was also coded by the speaker and the listeners.

In our preliminary analysis, we found several patterns regarding when and what context became public. First, all participants were asked to describe their contexts during the summer institute, and while what they shared varied (e.g., student population, courses, administrative support), they frequently shared contextual challenges. Second, some teachers continued to bring their contexts into the summer institute's professional learning discussions, but several did not, indicating a gap in how, and possibly to what degree, teachers were grappling with their contexts as they engaged in learning away from their settings. Finally, all teachers brought their contexts into the school-year coaching, discussing barriers they faced, assets they could and did leverage, and strategies for negotiating both toward the goal of pedagogical change. These findings highlight the importance and challenge of promoting connections between teachers' pedagogical learning opportunities and the contextual realities of implementation.

References

Gibbons, L. K., Lewis, R. M., Nieman, H., & Resnick, A. F. (2021). Conceptualizing the work of facilitating practice-embedded teacher learning. *Teaching and Teacher Education*, *101*, 103304. https://doi.org/10.1016/j.tate.2021.103304

Munson, J., Altshuler, M., Baldinger, E. E., Lee, H. S., & Kadiyali, V. (2021). Integrating three planes of teacher learning: The case of side-by-side coaching. In D. E. Olanoff, K. Johnson, & S. M. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1849–1853).

Rogoff, B. (1995). Observing sociocultural activity on three planes: Participatory appropriation, guided participation, and apprenticeship. In J. V. Wertsch, P. del Río, & A. Alvarez (Eds.), *Sociocultural studies of mind* (pp. 139–164). Cambridge University Press. https://doi.org/10.1017/CBO9781139174299

CONCEPTUALIZING HIGH-UPTAKE PRACTICES

Zandra de Araujo
University of Florida
zdearaujo@coe.ufl.edu

Mitchelle M. Wambua
University of Missouri
mwqdd@mail.missouri.edu

F. Paul Wonsavage
University of Florida
wonsavagef@ufl.edu

Samuel Otten
University of Missouri
ottensa@missouri.edu

Amber Candela
University of Missouri – St. Louis
candelaa@umsl.edu

Keywords: Instructional Activities and Practices, Professional Development

Researchers have begun to explore professional development (PD) aimed at incremental improvements to instruction (Litke, 2020; Star, 2016). The intention of such an approach is to enact modest improvements to teachers' existing instruction in ways that recognize and build from teachers' current practice. If we are to seriously entertain the notion of incremental PD, we must identify and study practices that would undergird such an approach. Our purpose is to propose a definition for such practices which we term high-uptake practices.

We define high-uptake practices as pedagogical actions that are readily and frequently utilized by teachers. Consistent with three potentially overlapping dimensions of practicality theory, high-uptake practices are clearly articulated teacher actions (instrumentality), aligned with a teacher's current instructional approach and context (congruence), and the expected benefit from implementing the practice exceeds the effort and resources to enact it (cost) (Doyle & Ponder, 1977; Janssen et al., 2013). High-uptake practices are not contingent upon the quality of the action, but rather by the rate at which teachers incorporate them into their instructional repertoire. Consider if we were to suggest teachers adjust students' independent work time to include not only solving individual problems but also selecting two problems that contrast with one another. We hypothesize that the practice of choosing contrasting problems would be a high-uptake practice because it is:

- a specific action that can be implemented without delay (i.e., teachers do not have to wait for a certain lesson or content topic to try it out; instrumentality & cost);
- presented to teachers who already employ independent work time and is close enough to current practice that teachers have confidence in enacting it (congruence); and
- a change that aligns with teachers' underlying goals (e.g., higher student engagement by avoiding monotony, helping students to identify bigger ideas rather than only procedural execution; cost and congruence).

Presently we are working to determine whether practices we hypothesize as high-uptake actually are high-uptake. We will specify what separates a high-uptake practice from, say, a medium- or low-uptake practice. This process will allow us to understand what and why practices are high-uptake so we can then study how such practices impact students' learning.

Acknowledgement

This work was supported by the National Science Foundation (DRL #2206774) though any opinions, findings, and conclusions expressed here are those of the authors and do not necessarily reflect the views of the NSF.

References

Doyle, W., & Ponder, G. A. (1977). The practicality ethic in teacher decision-making. Interchange, 8(3), 1-12.

Janssen, F., Westbroek, H., Doyle, W., & Van Driel, J. (2013). How to make innovations practical. Teachers college record, 115(7), 1-43.

Litke, E. (2020). Instructional practice in algebra: Building from existing practices to inform an incremental improvement approach. Teaching and Teacher Education, 91, 103030.

Star, J. R. (2016). Improve math teaching with incremental improvements. Phi Delta Kappan, 97(7), 58-62.

TEACHER LEARNING THROUGH CROSS-CULTURAL LESSON STUDY

Rongjin Huang
Middle TN University
rhuang@mtsu.edu

Xingfeng Huang
Shanghai Normal University
xfhuang0729@gmail.com

Joshua Willson
Middle TN University
Jas2ey@mtmail.mtsu.edu

Joanna Weaver
Bowling Green State Univ
weavej@bgsu.edu

Gabriel Matney
Bowling Green State Univ
gmatney@bgsu.edu

Christine Painter
Bowling Green State Univ
cgpaint@bgsu.edu

Keywords: Teacher professional development, teacher educators, teacher knowledge.

Lesson study (LS), a teacher-centered and student-focused professional development approach, has been adopted around the world (Huang et al., 2019). Research has documented the positive effects of LS on teacher professional learning (Cheung & Wong, 2014; Willems & Bossche, 2019). Recently, cross-cultural lesson study has been utilized to expand teacher collaborative learning (Huang et al., 2021; Isoda et al., 2021). However, the machinimas of and effect of teacher learning through lesson study cross-culturally is still largely unknown (Huang et al., 2021). This study aimed to examine participating teachers' learning through a cross-cultural lesson study from the Cultural-History Activity Theory (CHAT) perspective (Engeström, 2001).

The CHAT portrays a collective and dialectical perspective of learning that included interactions between different single systems within a single activity, different tools that are mediated by the cultural and historical roots of the system such as community, the rules and division of labor that are interplayed to achieve the goals. *Contradiction* is denoted as the 'historically accumulating structural tensions within and between activity systems' (Engeström, 2001, p. 137) and CHAT views contradictions as driving forces of expansive learning when new goals/objects, concepts and motives emerge (Engeström & Sannino, 2010).

In this study, two lesson study teams collaborated. A lesson study team in Shanghai, China included six mathematics teachers from a public middle school. A lesson study team in Ohio, USA included six interdisciplinary teachers from an arts school, two of whom focus on teaching mathematics. All teachers had completed their bachelor's degree and had more than five years of teaching experience. Each team was facilitated by a researcher from a local university and cross-cultural meetings were mediated by a third researcher. The cross-cultural lesson study took place including study, plan, teach, and reflect over six months. The activities included two meetings devoted to identifying learning goals and content commonly taught across cultures; one meeting for sharing lesson plans; developing a public lesson of the identified topic by each team; reviewing the shared videos of research lessons by each individual team; and one meeting for discussing the videos of research lessons cross-culturally. The data collected were lesson plans, meeting notes, videos of research lessons and cross-culture sharing meetings, and final teacher interviews. From the CHAT perspective, the data analysis focused on identifying the contradictions and the ways of dealing with them through the different phases of LS.

The identified contradictions included: (1) concrete activity (USA) vs. variation practice (China); (2) making sense of the concept through exploring real life situation (USA) vs. applying knowledge and skills in real life situations (China); and (3) fully active engagement, communication and sense making (USA) vs. heavily mathematics knowledge and skill application and problem solving. Through addressing the contradictions over the LS process, teachers from the two countries have learned and intended to adopt some new ideas and effective strategies from each other to improve their own teaching.

References

Cheung, W. M., & Wong, W. Y. (2014). Does lesson study work? *International Journal for Lesson & Learning Studies, 3*(2), 137-149.

Engeström, Y. (2001). Expansive learning at work: Toward an activity theoretical reconceptualization. *Journal of Education and Work, 14*(1), 133-156.

Engeström, Y., & Sannino, A. (2010). Studies of expansive learning: Foundation, findings and further challenges. *Educational Research Reviews, 5*, 1-24.

Huang, R.,Takahashi, A., & De Ponte, J. (2019). *Theory and practices of lesson study in mathematics: An international perspective.* Springer.

Huang, X., Lai, M.Y., & Huang, R. (2021). Teachers' learning through an online lesson study: an analysis from the expansive learning perspective. *International Journal for Lesson and Learning Studies, 10*(2), 202-216

Isoda, M., Soledad, E., Zakaryan, D., Baldin, Y., Olfos, R., & Araya, R. (2021). Digital competence of a teacher involved in the implementation of a cross-border lesson for classrooms in Brazil and Chile. *International Journal of Lesson and Learning Study, 10*(4), 362-375.

Willems, I., & Bossche, P. V. D. (2019). Lesson study effectiveness for teachers' professional learning: A best evidence synthesis. *International Journal for Lesson and Learning Studies, 8(*4), 257-271.

.

TEACHER OPPORTUNITIES TO INTERPRET AND RESPOND TO CHILDREN'S MATHEMATICAL IDEAS

Sam Prough
University of Delaware
sprough@udel.edu

Amber Webster
University of Delaware
ambercb@udel.edu

Lynsey Gibbons
University of Delaware
lgibbons@udel.edu

Keywords: assessment, elementary school education, professional development

We see the need for professional development research to explore how teachers initially make sense of students' mathematical thinking. Using the core principles of Cognitively Guided Instruction to support teachers in understanding student thinking (Carpenter et al., 1996; Carpenter et al., 2015), we interviewed elementary students and shared the reports of their strategies and thinking with their teachers. In an analysis of the teachers' discussions about the reports, we respond to the following questions in our study: How do teachers make sense of students' mathematical thinking in reports that attend to student strategy use?

Students across an entire elementary school who were present in October (K-5, N=334) engaged in cognitive interviews (see Kazemi et al., 2016; 3-5 tasks were given depending on age) with researchers (including the authors). Results of each interview were compiled by classroom and grade level. The reports highlighted trends in student strategy use across problems as well as recommendations to support future student learning. Facilitated by the first author, sixteen teachers met as grade level teams to explore the report, discuss noticings about student work, and consider future steps. Data included audio recordings, transcripts, and researcher memos of the meetings. Analysis examined teachers' perceived constraints and imagined possibilities for taking up practices that center student sensemaking, given their context for teaching (e.g., Goh et al., 2017; Knapp & Peterson, 1995). The first two authors independently coded data then met to come to consensus on categories of teachers attending to student thinking.

With regards to student thinking, the teachers initially attended to student correctness within the different problems (e.g., "no one in my class can count to 80"). With continued focus on student sensemaking and strategies used in the reports and prompted by the facilitator, teachers shifted to consider student understanding. For example, when pointing out how her 1st graders attempted to add 6 + 7, Leah commented that one student "had some sort of understanding of 'we're putting things together.'" We found that teachers' reflections included constraints for incorporating the practices from the reports that showed a tension between district expectations and what teachers recognized as best practices for supporting student sensemaking. For example, many teachers believed that the curriculum's pace was too fast and encouraged students to rush through activities, with little emphasis placed on understanding. Overall, reports like these can help teachers collectively inquire into students' thinking and consider opportunities for students to make sense of and discuss cognitively demanding mathematics tasks in their instruction.

Acknowledgments

This work was done with the support of grant 220020522 from the James S. McDonnell Foundation. All opinions expressed are those of the authors and do not necessarily represent the views of the Foundation.

References

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, *97*(1), 3–20.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2015). *Children's mathematics:Cognitively guided instruction* (Second Edition). Heinemann.

Goh, T. L., Hannon, J. C., Webster, C. A., & Podlog, L. (2017). Classroom teachers' experiences implementing a movement integration program: Barriers, facilitators, and continuance. *Teaching and Teacher Education*, *66*, 88–95.

Kazemi, E., Gibbons, L. K., Lomax, K., & Franke, M. L. (2016). Listening to and learning from student thinking. *Teaching Children Mathematics*, *23*(3), 182–190.

Knapp, N. F., & Peterson, P. L. (1995). Teachers' interpretations of "CGI" after four years: Meanings and practices. *Journal for Research in Mathematics Education*, *26*(1), 40–65.

MATHEMATICS TEACHERS' IDENTITY FORMATION DURING THE FIRST YEARS OF TEACHING

Molly Sutter	Will Hall	Ashley Whitehead
Washington State University	Washington State University	Appalachian State University
molly.sutter@wsu.edu	w.hall@wsu.edu	whiteheadan@appstate.edu

Keywords: Affect, Emotion, Beliefs and Attitudes; Teacher Beliefs; Professional Development

This poster presents a proposal for an addition to an ongoing study in which students (n=34) in a mathematics teacher education program have completed autoethnographies over the past 3-4 years. As these students are now entering the teaching profession, we feel the autoethnographic activities from the preservice program can be extended into the experiences of the teaching profession to help in-service teachers develop their identities as mathematics teachers.

Statement of the Problem

The first years of teaching are some of the most critical years of teaching, and if teachers do not feel confident in who they are as a teacher, they are at risk for experiencing burn out and end up leaving the profession (Beauchamp & Thomas, 2009; Hodges & Hodge, 2017; Losano et al., 2018; Lutovac & Kaasila, 2014; Molfino & Ochoviet, 2019; Pillen et al., 2013; Ruohotie-Lyhty, 2013). Identity is a key component in teachers feeling confident in their role of teaching mathematics. Researchers have studied identity formation in educators but have not found an agreed upon definition of identity. However, many agree that identity is socially formed, constantly changing, and consists of multiple subidentities (Akkerman & Meijer, 2011; Beauchamp & Thomas, 2009; Chung-Parsons & Bailey, 2019; Darragh, 2016; Noonan, 2019; Ruohotie-Lyhty, 2013).

Purpose

Autoethnography is self-reflective cultural narrative requiring the author to reflect upon, and analyze experiences to understand culture and the impact it has on identity; both a process and a product (Anderson, 2006; Austin & Hicky, 2007; Duarte, 2007; Ellis et al., 2011; Hains-Wesson & Young, 2017; Hamilton, 2021; Pinner, 2018; Yung, 2020). This gives teachers space to understand how their culture and lived experiences impact how they fit into the teaching profession.

Qualitative methods, including interviews with teachers, can be informed by developing autoethnographic activities within Bronfenbrenner's Ecological System (Cross & Hong, 2012) and Gee's views of identity (Carrier et al., 2017; Tsybulsky & Muchnik-Rozanov, 2019). Bronfenbrenner's Ecological System breaks down environments in which people are situated while Gee illustrates different parts of identity (Carrier et al., 2017; Cross & Hong, 2012; Tsybulsky & Muchnik-Rozanov, 2019). Our purpose is to illustrate our theoretical framework that situates autoethnography within these two existing frameworks (Bronfenbrenner's Ecological System and Gee's views of identity) in identity formation. Ultimately, we aim to design reflective questions to engage mathematics teachers in examining their lived experiences in each level of environment and identity through our created framework.

References

Akkerman, S. F., & Meijer, P. C. (2011). A dialogical approach to conceptualizing teacher identity. *Teaching and Teacher Education*, *27*(2), 308–319. https://doi.org/10.1016/j.tate.2010.08.013

Anderson, L. (2006). Analytic Autoethnography. *Journal of Contemporary Ethnography*, *35*(4), 373–395. https://doi.org/10.1177/0891241605280449

Austin, J. & Hicky, A. (2007). Autoethnography and Teacher Development. *The International Journal of Interdisciplinary Social Sciences: Annual Review*, *2*(2), 369–378. https://doi.org/10.18848/1833-1882/CGP/v02i02/52189

Beauchamp, C., & Thomas, L. (2009). Understanding teacher identity: An overview of issues in the literature and implications for teacher education. *Cambridge Journal of Education*, *39*(2), 175–189. https://doi.org/10.1080/03057640902902252

Carrier, S. J., Whitehead, A. N., Walkowiak, T. A., Luginbuhl, S. C., & Thomson, M. M. (2017). The development of elementary teacher identities as teachers of science. *International Journal of Science Education*, *39*(13), 1733–1754. https://doi.org/10.1080/09500693.2017.1351648

Chung-Parsons, R., & Bailey, J. M. (2019). The hierarchical (not fluid) nature of preservice secondary science teachers' perceptions of their science teacher identity. *Teaching and Teacher Education*, *78*, 39–48. https://doi.org/10.1016/j.tate.2018.11.007

Cross, D. I., & Hong, J. Y. (2012). An ecological examination of teachers' emotions in the school context. *Teaching and Teacher Education*, *28*(7), 957–967. https://doi.org/10.1016/j.tate.2012.05.001

Darragh, L. (2016). Identity research in mathematics education. *Educational Studies in Mathematics*, *93*(1), 19–33. https://doi.org/10.1007/s10649-016-9696-5

Duarte, F. P. (2007). Using Autoethnography in the Scholarship of Teaching and Learning: Reflective Practice from 'the Other Side of the Mirror'. *International Journal for the Scholarship of Teaching and Learning*, *1*(2). https://doi.org/10.20429/ijsotl.2007.010221

Ellis, C., Adams, T. E., & Bochner, A. P. (2011). Autoethnography: An Overview. *Forum: Qualitative Social Research*, *12*(1), 1–13.

Hains-Wesson, R., & Young, K. (2017). A collaborative autoethnography study to inform the teaching of reflective practice in STEM. *Higher Education Research & Development*, *36*(2), 297–310. https://doi.org/10.1080/07294360.2016.1196653

Hamilton, M. (2021). An Autoethnographic Self-Study Navigating the Transition to Becoming a STEM Teacher Educator. *The Qualitative Report*. https://doi.org/10.46743/2160-3715/2021.4462

Hodges, T. E., & Hodge, L. L. (2017). Unpacking personal identities for teaching mathematics within the context of prospective teacher education. *Journal of Mathematics Teacher Education*, *20*(2), 101–118. https://doi.org/10.1007/s10857-015-9339-2

Losano, L., Fiorentini, D., & Villarreal, M. (2018). The development of a mathematics teacher's professional identity during her first year teaching. *Journal of Mathematics Teacher Education*, *21*(3), 287–315. https://doi.org/10.1007/s10857-017-9364-4

Lutovac, S., & Kaasila, R. (2014). Pre-service teachers' future-oriented mathematical identity work. *Educational Studies in Mathematics*, *85*(1), 129–142. https://doi.org/10.1007/s10649-013-9500-8

Molfino, V., & Ochoviet, C. (2019). A mathematics teacher's identity study through their teaching practices in a postgraduate training course. *The Mathematics Enthusiast*, *16*(1), 389–407.

Noonan, J. (2019). An Affinity for Learning: Teacher Identity and Powerful Professional Development. *Journal of Teacher Education*, *70*(5), 526–537. https://doi.org/10.1177/0022487118788838

Pillen, M., Beijaard, D., & den Brok, P. (2013). Professional identity tensions of beginning teachers. *Teachers and Teaching*, *19*(6), 660–678. https://doi.org/10.1080/13540602.2013.827455

Pinner, R. S. (2018). Re-learning from experience: Using autoethnography for teacher development. *Educational Action Research*, *26*(1), 91–105. https://doi.org/10.1080/09650792.2017.1310665

Ruohotie-Lyhty, M. (2013). Struggling for a professional identity: Two newly qualified language teachers' identity narratives during the first years at work. *Teaching and Teacher Education*, *30*, 120–129.

Tsybulsky, D., & Muchnik-Rozanov, Y. (2019). The development of student-teachers' professional identity while team-teaching science classes using a project-based learning approach: A multi-level analysis. *Teaching and Teacher Education*, *79*, 48–59. https://doi.org/10.1016/j.tate.2018.12.006

Yung, K. W.-H. (2020). Becoming a teacher educator through being a student teacher: An autoethnography in the practicum. *Journal of Education for Teaching*, *46*(2), 248–250. https://doi.org/10.1080/02607476.2020.1724655

Chapter 13:

Statistics, Probability, and Data Science

TEACHERS' GOALS FOR SEEING WITH DATA

Ryan Seth Jones
Middle Tennessee State University
Ryan.jones@mtsu.edu

Sara Salisbury
Middle Tennessee State University
ss2cc@mtmail.mtsu.edu

Fonya Scott
Middle Tennessee State University
fcs2h@mtmail.mtsu.edu

Lisa Shepherd
Middle Tennessee State University
Lisa.shepherd@mtsu.edu

We present an interview study of 6th grade math and science teachers' expressed goals for engaging their students with data. We explored this across disciplinary boundaries to contribute to a body of knowledge that can support the development of a more coherent experience for students across math and science classes. Our teachers were all highly motivated to engage their students with data, and all wanted their students to see things with their data models. However, we observed consequential differences in the kinds of things they wanted students to see. Here we describe these differences and discuss potential implications for practice.

Keywords: Data Analysis and Statistics, Modeling

What do students see when they build, revise, and use data models to carry out statistical investigations? After all, professional researchers use data models to set the "conditions for seeing" the phenomenon they are studying (Lehrer & Schauble, 2010). And seeing with data models is always an interdisciplinary accomplishment because it is a "meta-discipline" that stretches across many different communities (Wild, Utts, & Horton, 2018). The interdisciplinary coordination necessary for seeing the world with data models means this is a challenging learning goal for students. Although standards documents across different STEM education communities now include significant focus on data and statistics (AAAS, 2011; Franklin et al., 2007; National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010; NRC, 2012), little is known about how to coordinate these experiences for students across diverse disciplinary contexts. If students are to be supported to see with data models in ways that support them to generate knowledge about the world around them, then much work is needed to better understand how to coordinate learning across disciplinary boundaries.

Here we contribute to this need by exploring middle school mathematics and science teachers' expressed goals when they engage their students with data. We conducted this study within a larger design-based research project where we are developing and studying a coordinated approach to teaching data across math and science classes. In our early work we quickly developed a felt sense of some differences among our teachers in their motivations for using data in classes. To better understand their perceptions of their own practice we interviewed all math and science teachers in the 6th grade at our partner school. In these interviews we identified significant differences in the types of things teachers hope their students "see" when working with data, which we think are consequential for teachers' practice during these lessons. We share these findings here with two goals in mind. First, we hope that a better understanding of teachers' instructional goals related to data and statistics across different disciplinary communities will be a small contribution toward building a more coherent learning experience for students. And second, we aim to provoke a conversation about extending the contexts for research on teaching and learning data beyond mathematics and statistics classes to better understand how these ideas take shape across more diverse disciplinary communities.

Interdisciplinary Coordination in Seeing with Data

How do plants grow? Are temperatures increasing across the planet? Does this new treatment help patients recover from this illness? Empirically grounded answers to questions like these require a specialized form of seeing where practitioners work to create a data model that amplifies features of a phenomenon relevant to the question while reducing the features deemed irrelevant (Latour, 1999). This means data are not simply numbers, they provide a lens to see the world in in new ways and the correspondence between the context of the data and the numbers themselves provides justification for the meaning generated from the data (Cobb & Moore, 1997).

Although research on teaching and learning of data and statistics has almost exclusively taken place in the context of mathematics or statistics classes, data models involve coordination among a complex set of interdisciplinary ideas and decisions such as posing a question, deciding upon appropriate attributes to measure, establishing operational measurement techniques, designing sampling plans, structuring variable data to create distribution, measuring features of the distribution with statistics, and building probability models of sampling variability to inform claims about difference (e.g. Arnold, Confrey, Jones, Lee, & Pfannkuch, 2018; Petrosino, Lehrer, & Schauble, 2003; Lehrer, Kim, Ayers, &Wilson, 2014; Lehrer & Romberg, 1996; Lehrer, Kim, & Schauble, 2007; Konold & Pollatsek, 2002). Take for example the phrase commonly used to describe a critical aspect of building a data model, “collecting data.” This phrase suggests that the data are simply “out there”, and that a researcher’s job is to gather them in the same way one might “collect” leaves, or coins, or baseball cards. However, data are not *collected,* but are *created.* Mathematizing the world requires any number of decisions about measurement and sampling that can create significant differences in the data produced. And the knowledge necessary to make these decisions in justifiable ways almost always stretches across disciplinary realms.

Research Methods

In our larger project we aimed to engage students with data modeling across math and science classes in ways that are consistent with the interdisciplinary work STEM professionals engage in when they see the world through data. We conducted this work in partnership with middle school teachers, so it was important for us to begin our work by working to understand their goals for their students when they engage them with data in their classes.

We conducted individual interviews with each of the 6th grade math (N=5) and science (N=3) teachers at our partner school. We also interviewed one special education teacher that worked in math and science classes as inclusion support. The teachers have a wide range of backgrounds, with most teachers having degrees in education, and some with degrees in non-education fields such as Leadership and Ethics, Biology, Recreational Management, and Accounting who obtained alternative licensure. The teachers have between 1 and 22 years of teaching experience in subjects including mathematics, science, social studies, and English, and have taught students between elementary and middle school age ranges. The teachers work at a large middle school in the South-Central Region of the United States of America with a diverse student body. For example, 28% of students in the school are English learners and 50% are considered low-income. Our sample represents a group of teachers that are committed to improving their practice, weekly collaborate on their lessons in PLC meetings, and are motivated to partner with projects like ours to develop new approaches to teaching in their school. In addition, the math teachers had been working for over four years on improving their teaching about data and statistics using an innovative approach called Data Modeling (Lehrer, Kim, Ayers, & Wilson, 2014).

We conducted individual, semi-structured interviews over video calls in the summer of 2020. Each interview took approximately 40 minutes to complete, and asked teachers to share information about their backgrounds, classroom practices, perspectives on using data in their classrooms, and how they plan their lessons. To understand how they thought about engaging their students with data we asked them about how important they feel it is for their students to learn to use data, and what their learning goals are for students using data displays, statistics, probability, and modeling.

Members of the research team transcribed all interviews verbatim and inductively analyzed transcripts of the interviews. We used an inductive approach to create in-vivo codes, characterizing responses in the teacher's own words. Then we discussed our initial findings and identified major categories to characterize in the data. Once we coded independently, we met to agree on consensus codes and then worked together to identify themes. Throughout this process, our research team regularly came together to share and discuss coding results, and in instances where discrepancies arose between codes, the team worked until we had a consensus.

Results

Teachers often wanted students to "see" many different things with their data. Although we did not use this word in our questions, all our teachers used it extensively in describing their goals. Teachers thought that it was important for students to learn to see things with data, but there were significant differences in the types of things they want students to see. We describe these different goals here in two categories: Unproblematic Seeing and Problematic Seeing.

Unproblematic Seeing

Sometimes teachers wanted students to see things in a straightforward, unproblematic way. These comments often ignored or minimized the complexity of seeing with data to emphasize the desired goal of seeing some important learning goal. So, the goal aims for students to see things with data in unproblematic ways without the view being obstructed by complexity or uncertainty. Within this category there were two different things teachers wanted students to see in unproblematic ways.

Teachers sometimes wanted students to use their data to see conventional principles or explanations of a phenomenon under investigation. For example, one of our science teachers named Ms. Manchester described a goal of her lab in the following way.

> "Probably, a concrete example we do with that is the elephant toothpaste lab and the, recording the temperatures... through recording the, the temperatures and graphing, that they can see, see a chemical reaction taking place... they can actually track that huge increase in temperature, and then how it slowly drops off. And that kind of is the, the number form of, you know, what's going on with, with the chemical reaction."

Seeing change in temperature and interpreting them as chemical reactions requires a complex arrangement of ideas and practices. Here though, Ms. Manchester aims for her students to see the change in temperature in a straightforward manner without the uncertainty and complexity that would come with building and using a data model to do this work.

Other times teachers described a desire for students to see statistical or mathematical practice in a straightforward way that minimizes complexity. For example, another science teacher in our project named Ms. Kent articulated significant complexity in the ways her students choose to generate data, but once she began talking about data analysis, she said students would "graph or show it [the data] in some way so that they can use it as evidence to justify whatever their claim was at the end of that lab or experiment." She did not elaborate on complexity in the work students would do, although she had just emphasized the complexity in creating data.

Problematic Seeing

When teachers wanted their students to engage in problematic seeing they aimed for them to see the complexity and uncertainty in making claims with data models. Instead of wanting students to draw a straight line from their data to a phenomenon or particular analytic technique they wanted them to problematize this line by encountering challenges, uncertainty, and multiplicity in data models and claims. For example, Mr. Trumbull described the following goals for his students when working with measures of center.

> "But help them see that the average is not the only way to calculate a best guess… But like I'd really like more than to see like the power of median. … And I think that, that is made really visible in in data where you're right, you calculate the mean, but what on earth is that really telling you? It's basically telling you nothing. Or, yeah, you calculate the mean, but look at what these outliers are doing to it, they're making it inaccurate."

He wanted his students to not only use mean and median, but to encounter the complexity of multiple ways to index center, and relations among distributional characteristics and these methods. At other times the teachers described a desire for students to see and question investigation protocols to better understand the challenges associated with creating data. Mr. Houston said he wanted students to ask questions like the following with data.

> "…what happened? How does my data compare to somebody else's data? Why did that happen? I guess just basic, basically, understanding what, you know, what we did and why it happened. Could it be something in the procedure? Could it be an error? Could it be an outlier? You know, how do we make sense of that? Should we do this again, you know, to see if we get the same results?"

In these examples, and across the comments in this category, teachers described a desire for students to see challenges and uncertainty in modeling the world with data.

Discussion

We believe that these differences in goals for teachers are consequential for the ways they engage their students with data. For example, if a teacher's goal is for students to primarily see a phenomenon in an unproblematic way, then they are likely to find complexity and uncertainty as a distraction. However, if a teacher aims for students to problematize the work of seeing with data models, then these features of an activity will be seen as productive. However, more work is needed to better understand the relationship between these different goals and teaching practice.

Finally, we do not see these categories as static characteristics of a teacher. All our teachers made comments across these two categories. However, there were differences in the proportion of comments made in one category or the other. So, although we don't believe these are static characteristics, we do think that they are consequential for teachers and influence their practice. If we are going to make progress in supporting students to learn about modeling data across disciplinary boundaries, then it will be important for us to continue to develop an understanding of how teachers conceive of their teaching practice as it relates to data and statistics across disciplinary boundaries of schooling.

References

American Association for the Advancement of Science. (2011). *Vision and change in undergraduate biology education: A call to action*. Washington, DC: Author. Retrieved from http://visionandchange.org/ final report

Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. The American Mathematical Monthly, 104(9), 801-823.

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2007). Guidelines for assessment and instruction in statistics education (GAISE) report. Alexandria: American Statistical Association.

Latour, B. (1999). *Pandora's hope: essays on the reality of science studies*. Harvard university press.

Lehrer, R., Kim, M. J., Ayers, E., & Wilson, M. (2014). Toward establishing a learning progression to support the development of statistical reasoning. *Learning over time: Learning trajectories in mathematics education*, 31-60.

Lehrer, R., & Schauble, L. (2010). What kind of explanation is a model?. In Instructional explanations in the disciplines (pp. 9-22). Springer, Boston, MA.

National Governors Association Center for Best Practice & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington D.C.: Author.

National Research Council. (2012). *A framework for K-12 science education: Practices, crosscutting concepts, and core ideas.* Washington, DC: The National Academies Press.

Wild, C. J., Utts, J. M., & Horton, N. J. (2018). What is statistics?. In *International Handbook of Research in Statistics Education* (pp. 5-36). Springer, Cham.

PSYCHOANALYSIS AND PROBABILISTIC THINKING

Alexander S. Moore
Virginia Tech
asm1@vt.edu

Psychoanalysis is largely overlooked in mathematics education, yet is relevant to many aspects of the field, such as the institution of the school, curriculum and instruction, and content. Specifically, one content area where psychoanalysis is both exceedingly relevant and absent is that of probability education. In this critical literature review, I resurface important literature on the topic of psychoanalysis and probabilistic thinking, providing a Lacanian synthesis of its relevance in the future of probability education research. Important concepts such as subjectivity, epistemology, and linguistics are explored, all of which hold massive implications for probability education researchers in their interpretations of students' probabilistic thinking. The purpose of the paper is to demonstrate the relevance of psychoanalytic theory to probability education research, and to introduce the idea of such an alignment to researchers.

Keywords: Research Methods, Probability

The purpose of this literature review is double. First, literature on the topic of psychoanalysis and probabilistic thinking is resurfaced and synthesized for the mathematics education research audience. Second, a synthesis of the literature and related psychoanalytic concepts are elaborated and discussed from a Lacanian point of view apropos probability education research.

Psychoanalysis is a theory—juxtaposed against positive psychology—that acknowledges the existence of the unconscious and thus understands itself through its own limits; it can be thought of as the "first psychiatry" that existed before modern psychoactive medicines. The work of early pioneers such as Sigmund Freud is well-known in popular discourse although it is widely misunderstood by those who do not study it professionally. Perhaps the second most well-known psychoanalyst coming out of the Freudian tradition was the French psychoanalyst Jacques Lacan whose work is notoriously difficult to understand. Much of his published work comes from the transcriptions of his yearly *Seminars*. Despite Lacan being trained in medical school as a Freudian, he built on or reinterpreted many of Freud's foundational ideas after he began his own practice. As a result, Lacanianism has largely become its own school of thought in psychoanalysis, although it is based in the Freudian tradition rather than in the other major school of psychoanalytic thought—the Jungian tradition (see Bailly, 2009). The word "Lacanian" has come to represent "courageous and radical commitment to understanding the depth and vastness of the human condition, with full acceptance of the impossibility and ineffability of that task" (Gerson, n.d.).

Psychoanalysis is different from other types of psychological theories because it is largely based on speculative analysis of the analysand and their speech, and the interaction between their conscious and unconscious desires. An analysand can be an individual person, a group of people, or an institution (e.g., the state). In the case of mathematics education, all three are relevant: for example, a student or teacher, a classroom, and the curriculum or school itself as situated within a particular political and cultural context. While the case of the unconscious in a single subject (e.g., a student) might be obvious, the unconscious is also present in groups of people and institutions: in the form of spirit (e.g., Hegel's notion of world spirit), through the economy of ideology (see, for example, Žižek, 1989/2008), and by that which conditions discourses (Lacan, 1993). Additionally, psychoanalysis is not concerned with mathematical concepts' existence *as* *such* or what might be described as "out there" or as ontologically positive in-and-for themselves. Indeed, Lacan even used mathematics to develop a system of Lacanian Algebra that he used to describe concepts so that people would not fall prey to intuitive understandings of psychoanalytic concepts—a trap of the Imaginary (see next section; also see No Subject, 2019)—because there is no metalanguage that would guarantee the universality of mathematics through the conferral of meaning.[1] In other words, according to Lacan, the ingenuity of mathematical symbols *qua* signifiers is that you have "to explain what you are going to do with them" (Lacan, 1991, p. 2).

Background on Relevant Lacanian Concepts

Signifiers and Signifieds

The incorporation of Saussurean linguistics into psychoanalysis was one of Lacan's major contributions to psychoanalytic theory. Signifiers are, for Lacan, mental images of the sound of a sign. In other words, signifiers are the meaning(s) psychically assigned to a symbol, word, idea, or other sign by a subject, as interjected into an already-constituted network of meaning made up of other signifiers. Jöttkandt (2016) describes this relationship as:

> What one hears in speech is the signifier rather than the signified. The signified is not what we hear (in the auditory sense) but something that must be read. In order to signify, the signifier must undergo an act of signification. This process can be described as the signifier, *S*, becoming shot through or injected with signifieds, *s*, that have undergone a certain operation: a transfer occurs whereby a signified crosses over the bar [in Saussurean notion, *S/s*,] that separates signifier and signified to be a signifier, *S*. (p. 145)

A crucial property of signifiers is that they are meaningless and belong to a closed system of constantly re-conferred meaning, following the transference of meaning back to a signifier that signifies nothing:

> The signified is something quite different – it's the meaning [… that] always refers to meaning, that is, to another meaning. The system of language, at whatever point you take hold of it, never results in an index finger directly indicating a point of reality; it's the whole of reality that is covered by the entire network of language. (Lacan, 1993, p. 32)

However, this meaninglessness is what gives signifiers their power: "The more [a] signifier signifies nothing, the more indestructible it is" (Lacan, 1993, p 185). Moreover, the signifier is what constitutes[2] the subject: he[3] is merely a system of signifiers that, for himself, structure the ways in which he interprets the meaningfulness of signs in everyday life. Because signifiers *qua* signifiers are meaningless and closed, acting as a subject for another signifier to confer meaning onto, there is always one signifier missing at the beginning of the chain, thus rendering the whole chain meaningless. A string of signifiers only makes sense in retrospect by reaching the end by punctuating it, as evidenced by the necessity of having to hear an entire sentence before one can

1 Elaborating more on this point is outside the scope of this paper. Readers interested in more relevant discussion of philosophy of mathematics may find Jöttkandt (2016) to be of interest.
2 The field of the signifier is related to the field of l'Autre, which is the precarious Symbolic entity that—among other things—confers onto each person his subject position; it is only from this position that one begins to speak.
3 I acknowledge the failings of the English language apropos gendered pronouns. I use he/him/his in this paper in the admittedly archaic general sense to mean an individual Subject (one person), regardless of gender. I do not use they/them/their for that purpose because of the linguistic detraction that it makes from conceptualizing the Subject in their subjectivity, viz. situating the identifying voice in the radically-subjective first-person view (of the Subject). The distinction between he/him/his and they/them/there is purely linguistic.

retroactively assign/understand the meaning of the entire sentence, beginning back with the first word and the string of all words comprising the sentence.

The Structure of the Ego

In Freudian and Lacanian theory, the ego consists of three parts: the ideal ego, the ego-ideal, and the superego. The ideal-ego "stands for the idealized self-image of the subject (the way I would like to be; how I would like others to see me)" (Žižek, 2007, p. 80). The ego-ideal is "the agency whose gaze I try to impress with my ego image, the big Other who watches over me and propels me to give my best, the ideal I try to follow and actualize" (Žižek, 2007, p. 80). The Superego is

> the same agency (*l'Autre*) as the ego-ideal, but in its revengeful, sadistic, punishing aspect. It is what pushes us towards the expectation of certainty, even if certainty is absent—which it always is in actuality. Nothing forces anyone to enjoy except the superego. (Žižek, 2007, p. 80)

Further, the superego makes enjoyment an imperative. *Jouissance* qua enjoyment—in reality—brings as much pain as it does pleasure, so the superego is also responsible for things like the drive to do things that bring us pain. In late capitalism, Lacanian psychoanalysis elucidates that enjoyment is not only an imperative but has also become a "kind of weird and twisted ethical duty" (Žižek, 2007, p. 80).

The Psychic Registers: Imaginary, Symbolic, and Real

Lacanian theory is largely based on the structure of three psychic registers or realms of Lacan's theory of psychic reality: the Imaginary, the Symbolic, and the Real.

The *Imaginary* is the register of images ("picture thinking" or *Vorstellung*) and their relation to the ego. The *Symbolic* is the register of logic, meaning, and symbols as they determine the subject; for example, the Symbolic contains language and law, since they are always-already constituting the people who raise children into the same ordering structures. The *Real* is the register that resists representation, what cannot be symbolized, logically represented, or represented through pictures or *Vorstellung*, where words fail, "gut" feelings, anxiety, that residue that cannot be reduced, the foreclosed element that can be "approached but never grasped: the umbilical cord of the symbolic" (Lacan, 1979/1998, p. 280). In short, the Real is that which does not rely on my conception of it. Lacan describes it algebraically using Lacanian algebra because it is simultaneously already here and continuously emerging, and thus "cannot be conceived" (u/wokeupabug, 2015).

A newborn baby enters the Imaginary through not knowing that she is a separate person from the mother or primary caregiver; "as it gains a visual image of the world and itself"—for example, the first time she recognizes herself in a mirror—she "starts to understand that [she] is a distinct object" (u/wokeupabug, 2015).

The Symbolic can be described by the interactions experienced with other subjects: "Those we come in contact with use language to communicate and tell us how to see the world (how to Symbolize it). Eventually all experiences are filtered through language" (u/wokeupabug, 2015). This starts when the baby enters into language.

The Real cannot be defined but can be described through examples and metaphors. For example, the Real is evidenced in "[A]n experience or thought occurs that creates a response so sudden or inexplicable and that language we have does not have time or sufficiency to explain it and we experience something primal—it cannot be Symbolized" (u/wokeupabug, date).

Reddit user u/wokeupabug described the three registers as follows, using lay terminology and metaphor to craft a useful triad of examples:

> The *Imaginary* is like this: Suppose you think you're working late at the office and you think you're alone, and you start absently picking your nose while you work, then in the middle of it you realize your co-worker hasn't left yet and has totally caught you in the act. You suddenly feel very different about yourself, right? That's the Imaginary.
>
> The *Symbolic* is like this: Suppose you're at a party and you're meeting people for the first time, and after some basic introductions when you settle into the conversation, they ask what you do. Unfortunately, you've been depressed for the past year, you were doing school, but you failed a couple courses and didn't go this year, and you haven't been able to find a job in the meantime. You're kind of embarrassed and don't really know what to tell them. A couple years later you're at another party and the same thing happens, only now your situation has changed, and you report that you just finished your degree and are working a help desk job in an IT department with some good opportunities for advancement. You feel differently about yourself after giving this report about what you do, right? That's the Symbolic.
>
> The *Real* is like this: You're getting ready for work in the morning, running late, but you've got to iron your shirt. Ok, you iron it, get ready to head out, as you're putting your jacket on you remember you may not have turned off the iron. So you go check; actually, ok it's off. You head out the door but think you were actually a bit hasty, you barely glanced at the iron when you checked, and you're not really confident you saw it right. It'll only take a minute, you jump back in to check the iron; yup, it's off. So you head out, the whole time down the street you're thinking gee you probably should have unplugged it to make sure, you start imagining it catching something on fire and think about what an asshole you'd feel like if your house burnt down over something simple like this, and it drives you a bit nuts all morning. That's the Real. (u/wokeupabug, 2015)

Lacanian Synthesis of the Literature Results

Signifiers and Vindication, and the "Types" of Probability

Psychoanalysis first reveals that the misrecognition of signifiers—e.g., "probability" and "chance"—in probabilistic thinking is an important topic of study. This is not a novel focus of probability education research; quite the contrary, as probabilistic thinking is well-known to contain complicated disjunctions of meaning in the minds of students (e.g., Abrahamson, 2014; cf. Chassan, 1956; Jones et al., 2007). Chassan (1956) published one of the first papers on the matter. As with any writing, his claims must be considered within the historical context in which they were written. At the time, the emergence of quantum mechanics had created something of a philosophical crisis within the physics and mathematics communities, as the foundational tenets of determinism on which both fields had been built were up for question.[4] The development of probability theory had always been somewhat separate from the rest of the mathematics community. The philosophy that had guided the development of probability up until the middle of the 20th century had been mainly deterministic in nature before the advent of quantum mechanics created questions about determinism apropos mathematical philosophy. Chassan claimed that this existential crisis that the physics and mathematics communities were facing—the main contest of which was the philosophy of probability theory—could be solved through the use of psychoanalysis. "Probability theory is applicable when the initial state is so complicated that it is impossible in practice to ascertain it accurately enough to determine the final state uniquely…" (p. 56). This is precisely the same case as in the unconscious and thinking about

4 For example, the debate of Heisenberg's Uncertainty Principle.

probabilistic events such as causality and chance: that "nothing malfunctions more than human reality […] One is always being fooled" (Lacan, 1993, p. 82).

Chassan drew primarily on Freud's work describing the connection between determinism, chance, and superstitious beliefs,[5] in which he separates the commonly committed conflation of real chance with psychic accidents. For Chassan, the difference between "real" chance in the external world and the psychic reality of the person doing the thinking about it was not delineated in any mathematical literature. This naïve conflation was the cause of the philosophical conflicts developing over the (in)determinism of probability. Chassan, using Freud, sought instead to suss the external and psychic realities apart, one being a product of psychology and the other being a product of the conditions of the physical external reality. However, Chassan broke with Freud over his claims of causality, instead articulating a position that would later become more aligned with Lacan. Lacan's work was just becoming available in published form at this time; for example, Lacan's major break with the established Freudian psychoanalytic school had only occurred three years prior to this paper's release, in 1953, and at the time was only available as a French transcript of a talk he had given in Paris that year. That same year, in 1953, Chassan had written an analogous paper[6] to the 1956 paper, but focusing on the interrelation of *statistics* with psychoanalysis. Chassan stated that, "The acceptance of the usefulness of thinking in terms of probabilities in psychoanalysis confronts the research investigator who would seek to establish probability statements with considerations of observation and objectivity" (p. 58). The most relevant point he makes in both papers is that of intuition and *a priori* judgments—and how subjective experiences are the determining limits of them for an individual.

To shore up the relevance of signifiers to probabilistic thinking, I next discuss the concept of vindication—elucidated in the located literature— which implicates the importance of using psychoanalysis in research on students' probabilistic thinking. Moncayo and Romanowicz (2015) discuss the interesting psychological relationship between "certainty" of quantitative measurement in the soft sciences (such as education) against the—as they argue—more appropriate concept of probability. The chapter is a critique of the famous adage by Edward Deming: "In God we trust; all others bring data." The authors argue that within the quantitative paradigm, probability is the only appropriate numerical concept for measuring "the things that cannot be measured" in the soft sciences—as opposed to the "hard" causality. "Lacan distinguished between lawful regularity and causality and placed causality on the side of chance. On the side of chance, causality is the same as the absence of causality or causality in the form of a gap" (Moncayo & Romanowicz, 2015, p. 77). Here we begin to see the relationship between the unconscious and probability: the word "probability" was chosen when probability theory was developed out of the mathematical study of games as opposed to the rowdier term "chance," which connotes gambling and so forth. The signifier probability and the signifier chance ostensibly connote different things happening but in reality are the same. By my reading, the article shows that probability and the mirage of certainty are psychic products of the Real; thus, the normal distribution is a Symbolic distribution, representing regularities within the randomness.

A "personal" probabilistic understanding was further developed by van Fraassen (1983), who wrote about the difference between *personal probability* and *frequentist probability* in an edited volume that connected physics with psychoanalysis. At that time, with the advent of quantum

5 Viz. Freud (1938).
6 Viz. Chassan (1953).

mechanics becoming more salient, psychoanalysis was beginning to be seen as a useful tool in understanding the existential nature of the horizon towards which mathematics and physics were heading. In the chapter, van Fraassen (1983) attempted to establish connections between the frequentist approach to probability (counting frequencies of occurrences) and the personal approach to probability (subjectivist or Bayesian, dealing with degrees of belief in an event happening), stating that "the use of probabilistic language to express personal opinion about a single event can be understood in a way that avoids the major problems with which frequentists have struggled" (p. 295). He begins the chapter with a vignette:

> Yesterday I said, 'I promise to give you a horse.' But I did not give you anything, and today you accuse me of the heinous immorality of breaking a promise. No, I reply, I am not guilty of that at all, but only of the much lesser offense of lying. All that happened was that yesterday I stated falsely that I was promising to give you a horse. (p. 296)

Evaluating this vignette, van Fraassen argues that there are two ways to analyze attitudes: reasonableness and vindication. The former concerns the present, and the latter concerns the future. Further, reasonableness requires vindication: "Let the frequentist equate probabilistic expression of opinion with something else; and let him investigate the conditions under which such vindication is not *a priori* excluded" (van Fraassen, 1983, p. 297). Use of reasonable and vindication to analyze statements of personal or subjective probability address the failings of language apropos probabilistic concepts through what van Fraassen calls *calibration*. Calibration is a way of measuring or "frequentizing" one's subjective judgments apropos probabilistic statements "as indicators of actual frequencies" (van Fraassen, 1983, p. 300). It consists of evaluating one's personal probabilistic claims afterwards to see how well they fit the observed frequencies of events. Adjusting one's personal probability through calibration leads one towards what van Fraassen calls *coherence*. We can then assume that coherence is a frequentist's way of intentionally adjusting an individual's signifying chains apropos the language associated with the frequentist's claims and the subjectivist's claim. Through calibration, van Fraassen attempts to unify the divide between personal and frequentist probability, although the implications for psychoanalysis are not well sussed out in the chapter.

Joyce (2005) extended van Fraassen's arguments about the analytic potential of subjectivity, employing the notions of specificity, ambiguity, and sufficiency. In his paper, he focuses on evidence for probabilistic events, and claims that types, weight, and balance of evidence form a basis for subjective interpretation of probabilistic events. Consider a probability word problem that describes some conditions of an event, and perhaps an existing knowledge about how some of the events have already turned out. The problem is providing evidence in the form of those given statements. The individual subjectively interprets the statements of evidence, evaluating to what extent each piece of evidence exhibits specificity towards the probabilistic situation being considered, how much ambiguity there is around the conditions of the situation, and the extent to which the evidence given is sufficient for making a personal probabilistic statement (as informed by the non-personal evidence given) about the situation. It seems that Joyce's extensions of van Fraassen's arguments provide a finer grain of the subjectivist's reintegration to frequentism.

We can build on our understanding of Chassan through van Fraassen's intervention and Joyce's elaboration, in that psychoanalysis not only splits apart probability into what is "out there" and what is "inside us," but that psychoanalysis also allows us to reunify them through an interpretation of events *a posteriori*. In my interpretation the psychic registers of psychoanalysis reveal the following homology: Frequentist = Imaginary, Theoretical = Symbolic, Personal/Subjective = Real. Thus, psychoanalysis gives us a framework for understanding these three forms of probability in relation to each other.

The Superego and Desire

Britton (2021) argues that the "should" of superegoic expectation is used heavily in probabilistic thinking. Britton develops the notion of Belief/Doubt as an ego function that is mediated by the superego's expectations of "should." Crucially, these are psychic constructs which are separate from the material, external reality. This leads to implications for the way we understand what people think "should" happen, including the relations of causality with belief and counter-belief. The author gives an example of a case study patient: "She could not derive any security from belief unless she regarded it as knowledge; probability did not exist for her, only certain doubt or certainty" (p. 72). Thus, probability leads to anxiety because its signification is at conflict with the psychic reality of certainty and doubt. In external reality, there is only probability, but we psychically insist on certainty in our interpretation of the world (e.g., the superego's assertions). Thus, the superego filters the way in which the signifiers of probabilistic concepts (e.g., likelihood, chance) are being chained together in the unconscious.

Connecting desire and the superego, Rohy (2019) elucidates the connection between probability, psychoanalysis, and sexuality:

> Comparing the detective's investigative task to the process of psychoanalytic interpretation, Slavoj Žižek argues that 'every final product of the dream work, every manifest dream content, contains at least one ingredient that functions as a stopgap, as a filler holding the place of what is necessarily lacking in it.' This is the crucial clue, the telling symptom (p. 50).

Like a detective in one of Poe's mystery stories, the psychoanalyst's task is to transform what "appears to be accidental or random into a narrative of motivated causality. The difference is to a large extent one of perception: an anomaly that weak thinkers dismiss as 'coincidence' may in fact be nothing of the sort" (Rohy, 2019, p. 46). An example of this is the gambler's fallacy. The gambler's fallacy confronts the Real and the Symbolic through the failure of notions such as the anticipation of "being owed" or "being due." For Lacan, the Real cannot be represented or assimilated, whereas the symbolic is precisely the register of representation and assimilation. The Real, which creates the tension of the gambler's fallacy, opposes meaning, including causality. Our difficulty in thinking probabilistically might be related to the fact that we are socialized as gendered and sexualitied[7] people from childhood, assuming heterosexual necessity instead of queer accidentality, vis-à-vis suppression of the Real (see Jöttkandt, 2016; Tomšič, 2016). If true, this engendered orientation towards heteronormativity would function as a technology for predisposing us to think about "what should be the case" (heterosexuality and cisgenderism as "normal" or "standard"; what "should" be the outcome of a probabilistic event) when really there is no such thing: there is only maximum probability—viz. that most people, for biological reasons, are indeed heterosexual and cisgender—underlaid by motivated causality. My argument for this homology comes from the irreducibility of sexuality, in psychoanalysis, to meaning.[8]

---

7 Elsewhere (Moore, 2021), I have used this term in parallel with the term gendered to denote categorical determinations of the process of sexual identification, such as the same process that occurs in human socialization apropos gender.

8 Due to space limitations, I will not elaborate on this further here, although interested readers may find Zupančič (2017) of interest. I have been exploring this connection in my work elsewhere (e.g., Moore, 2022).

Conclusion: Towards a Future of Psychoanalytic Probability Education Research

Psychoanalysis possesses the potential to enhance the study of the teaching and learning of probability. There are several key concepts from psychoanalysis that provide a useful and innovative reframing of issues in probability education research. The first is the logic of the signifier and the three registers of psychic reality; they are both related to vindication and the different "types" of probabilistic thinking. The second is the superego and its complexification of the signifiers in probabilistic concepts.

To briefly recapitulate and close with an imperative takeaway, probability learning leads to anxiety because its signification is at conflict with the psychic reality of certainty and doubt. The superego drives towards certainty when it is not there. Thus, psychoanalysis affords a unique yet crucial perspective on understanding probabilistic thinking. Future studies into students' probabilistic thinking should take up these psychoanalytic concepts in theoretical orientation and methodological considerations.

References

Abrahamson, D. (2014). Rethinking probability education: Perceptual judgment as epistemic resource. In E. J. Chernoff & B. Sriraman (Eds.), *Probabilistic thinking: Presenting plural perspectives* (pp. 239–260). Springer.

Bailly, L. (2009). *Lacan: A beginner's guide*. Oneworld Publications.

Britton, R. (2021). The concept of the ego. In *Sex, death, and the superego: Updating psychoanalytic experience and developments in neuroscience* (2nd Ed., pp. 65–74). Routledge.

Chassan, J. B. (1953). The role of statistics in psychoanalysis. *Psychiatry*, *16*, 153–165.

Chassan, J. B. (1956). On probability theory and psychoanalytic research. *Psychiatry*, *19*(1), 55–61.

van Fraassen, B. C. (1983). Calibration: A frequency justification for personal probability. In R. S. Cohen & L. Laudan (Eds.), *Physics, philosophy, and psychoanalysis* (pp. 295–319). D. Reidel Publishing Company.

Freud, S. (1938). "Determinism—chance—and superstitious beliefs." In *The basic writings of Sigmund Freud* (pp. 150-178). Modern Library Publishers.

Gerson, M. J. (n.d.). *Jacques Lacan: A Real, Imaginary, and Symbolic psychoanalyst* [Blog Post]. https://www.psychstudies.net/jacques-lacan-a-real-imaginary-and-symbolic-psychoanalyst/

Jones, G. A., Langrall, C. W., & Mooney, E. S. (2007). Research in probability: Responding to classroom realities. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 909–955). NCTM.

Jöttkandt, S. (2016). Mathematics in the bedroom: Sex, the signifier, and the smallest whole number. In A. Cerda-Rueda (Ed.), *Sex and nothing: Bridges from psychoanalysis to philosophy* (pp. 139–156). Karnac.

Joyce, J. M. (2005). How probabilities reflect evidence. *Philosophical Perspectives*, *19*, 153–178.

Lacan, J. (1991). *Seminar I: Freud's papers on technique* (J. Forrester, Trans.). W. W. Norton & Company.

Lacan, J. (1993). *Seminar III: The psychoses* (R. Grigg, Trans.). W. W. Norton & Company.

Lacan, J. (1998). *Seminar XI: The four fundamental concepts of psychoanalysis* (A. Sheridan, Trans.). W. W. Norton & Company.

Lacan, J. (2007). *Écrits: The first complete edition in English* (B. Fink, Trans.). W. W. Norton & Company.

Moncayo, R., & Romanowicz, M. (2015). On probability, causality, and chance. In *The real jouissance of uncountable numbers: The philosophy of science within Lacanian psychoanalysis* (pp. 75–86). Taylor & Francis.

Moore, A. S. (2021). Queer identity and theory intersections in mathematics education: A theoretical literature review. *Mathematics Education Research Journal*, *33*(4), 651–687. https://doi.org/10.1007/s13394-020-00354-7

Moore, A. S. (2022). The Big Other and sentimentalism: A sexual critique of mathematics, its education, and the disavowed fetish of technology. *"Boundaries of Philosophy": The Annual Conference of The Society for Philosophy in the Contemporary World*. Estes Park, CO. https://doi.org/10.5281/zenodo.6406631

No Subject. (2019). Mathematics. *No Subject: The Encyclopedia of Psychoanalysis*. https://nosubject.com/Mathematics

Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). Basic Books.

Rohy, V. (2019). The calculus of probabilities: Poe, contingency, and detection. In *Chances are: Contingency, queer theory, and American literature* (pp. 43–59). Routledge.

Tomšič, S. (2016). Psychoanalysis and antiphilosophy: The case of Jacques Lacan. In A. Cerda-Rueda (Ed.), *Sex and nothing: Bridges from psychoanalysis to philosophy* (pp. 81–103). Karnac.
[u/wokeupabug]. (2015). [Online forum post.] Reddit. https://www.reddit.com/r/askphilosophy/comments/31p3xx/comment/cq3qu60/?utm_source=share&utm_medium=web2x&context=3
Žižek, S. (2007). *How to read Lacan*. W. W. Norton & Company.
Žižek, S. (2008). *The sublime object of ideology*. Verso. (Original work published 1989)
Zupančič, A. (2017). *What* Is *Sex?* MIT Press.

BRIDGING FREQUENTIST AND CLASSICAL PROBABILITY THROUGH DESIGN

Amanda Provost
Montclair State University
provosta1@montclair.edu

Su San Lim
Montclair State University
lims1@montclair.edu

Toni York
Montclair State University
yorka1@montclair.edu

Nicole Panorkou
Montclair State University
panorkoun@montclair.edu

The frequentist and classical models of probability provide students with different lenses through which they can view probability. Prior research showed that students may bridge these two lenses through instructional designs that begin with a clear connection between the two, such as coin tossing. Considering that this connection is not always clear in our life experiences, we aimed to examine how an instructional design that begins with a scientific scenario that does not naturally connect to theoretical probability, such as the weather, may support students' bridging of these two models. In this paper, we present data from a design experiment in a sixth-grade classroom to discuss how students' shifts of reasoning as they engaged with such a design supported their construction of bridges between the two probability models.

Keywords: Probability, Design Experiments, Middle School Education, Integrated STEM.

There are distinct views of probability in the literature which differ not only in the way they define probability but also in the nature of their emerging solutions to the solving of problems (Batanero et al., 2016). The classical view of probability is one of the earliest approaches and is connected to chance in games. From this view, probability is considered as a fraction of the number of favorable cases divided by the number of all possible cases. This definition was created on the assumption that "all possible elementary events were equiprobable" (Batanero et al. 2016, p. 5), which is applicable to most situations in games. However, this view has been criticized because the idea of equiprobable outcomes is not always valid in natural phenomena.

The other most common approach, the frequentist view, sees probability as a convergence of relative frequency when a random experiment is repeated infinitely many times. According to Cosmides and Tooby (1996), humans have more experience with encountered frequencies in their observation of the world, therefore students would be more receptive to frequentist probability where data is collected through experience. In contrast to the connection to real-life experiences, the frequentist ideas of probability are not appropriate when discussing a single event or when the experiment cannot be repeated multiple times under the same conditions. This frequentist view is only an estimation of probability that results from a series of repetitions.

Considering the above, Lee et al. (2010) claim that since most everyday probabilistic situations (such as weather forecasting) "do not allow for the classical approach to probability" (p. 91), it is necessary for students to examine situations where it is possible to both calculate classical theoretical probability and make the connection to the frequentist experimentally collected empirical probability. Similar to Lee et al. (2010), many researchers have claimed that developing a connection between the classical and frequentist views is helpful for students to fully grasp the concept of probability (e.g., Henry & Parzysz, 2011; Ireland & Watson, 2009). During the learning process, students need to distinguish between the two models and understand when each one can be used for solving problems (Batanero et al., 2016).

Prodromou (2012) argued that while the two models differ, they are complementary and not mutually exclusive. She engaged pre-service teachers (PTs) in dice rolling tasks related to both theoretical and experimental probability and examined how they developed a bidirectional relationship between the two, explaining that "PTs perceived the theoretical probability as the intended outcome and the experimental probability as the actual outcome. In the opposite direction, PTs considered the theoretical probability as the target towards which the experimental probability is directed" (p. 866). This notion was also highlighted by Stohl and Tarr (2002), who reported that the tasks that engaged students in bidirectionality of probability helped students to develop robust inferences about probability. To engineer this bidirectionality, they started with coin tossing and dice rolling designs, and then used three different computer simulation tasks, two of them using experimental probability as a tool to evaluate theoretical probability and the other using theoretical probability as a tool to anticipate the result of the experiment.

Many supported that the connection between classical and frequentist probability is grounded in the Law of Large Numbers (LLN) (e.g., Drier, 2000; Prodromou, 2012; Stohl & Tarr, 2002), which states that larger numbers of trials performed for a given event lead the relative frequency for that event to approach the theoretical probability. A study by Aspinwall and Tarr (2001), examined how sixth graders' understanding of experimental probability related to sample size and the LLN using designs with flipping chips, spinners, and dice. They demonstrated that probability simulations can challenge students' preexisting conceptions. However, they also noted that there were challenges in developing the understanding of the role that sample size played in determining experimental probability through the LLN. Aspinwall and Tarr (2001) claim this theorem has been shown to be challenging and nonintuitive for students. Students may not recognize when to use the LLN (Fischbein & Schnarch, 1997), or they may believe that the LLN applies to small numbers as well (Tversky & Kahneman, 1971).

Researchers found that offering students multiple representations of data supports their understanding of the connection between theoretical and experimental probability. Stohl and Tarr (2002) reported that the use of graphs and tables enables students to see bidirectionality of probability by involving them in representing and analyzing data with different forms. Similarly, Ireland and Watson (2009) claimed that multiple representations in computer simulations can help students perceive how theoretical probability is connected to experimental probability. In their experiment with digital mixers and spinners, students made a connection between theoretical and experimental probability by comparing data in various representations.

Using computer simulations as a mode of representation provides students with the opportunity to collect large amounts of data in shorter periods of time, which addresses the limitations of time constraints and resources that Biehler (1991) argued students encounter as they explore the concept of probability. Computer simulations also support students' connection-making between classical and frequentist approaches of probability (Ireland & Watson, 2009; Prodromou, 2012). Abrahamson and Wilensky (2005) used modified coin flipping computer simulations to help students develop their understanding of experimental probability by bridging the gap of theoretical probability with simulating and collecting large amounts of experimental data. Paparistodemou (2005) conducted a study where students were able to manipulate aspects of the computer environment to affect the generation of random events. The students manipulated the simulation in multiple ways to make use of the LLN to achieve the target probability.

While many studies have examined how students construct bridges between the two models of probability, their designs involved beginning with a clear connection between the two, such as coin tossing or dice rolling scenarios (e.g., Abrahamson & Wilensky, 2005; Prodromou, 2012). However, our real-life experiences of probability, such as predicting the weather, do not always have clear theoretical probabilities that can be calculated. Consequently, we started with the conjecture that it is possible to support students' bridging of these two models by first engaging them with probability scenarios situated in the context of science. We hoped that this would make use of students' realistic experiences to construct more meaningful connections between the two models. Specifically, we explored: *(a)* What kind of design that starts with the frequentist perspective without a clear connection to the theoretical probability and moves to the classical perspective would support students' bridging between the two models? *(b)* How may students' reasoning progress as they shift between the frequentist and classical perspectives?

Methods

In this paper we report on the results of a whole-class design experiment (Cobb et al., 2003) in a sixth-grade classroom in the Northeastern U.S. The class met in ten 15- to 50-minute sessions via Google Meet due to COVID-19 restrictions. To test our conjecture we designed two simulations, one based on the frequentist and one on the classical model, along with investigation and interview questions. We chose a scientific context because our previous work (e.g., Panorkou & Germia, 2021; Panorkou & York, 2020) showed that sixth-grade students can engage meaningfully with mathematical concepts in a setting integrated with science concepts. We focused on the topic of weather because research showed that students consider weather events as a relevant context for probability discussions (Chick & Baker, 2005).

The Weather Forecast simulation (Figure 1) is based on the frequentist model and generates the results of an imaginary weather forecast. The chosen Data Set determines the forecasted percentages of days that are expected to be rainy and sunny. The Run Size determines how many times the simulation runs the experiment. The larger the chosen Run Size the more accurate the resulting forecast will be for the chosen Data Set. After exploring this simulation, students were asked to gather data about the results of different Run Sizes in a table and graph these values on a log-scale plot of Percentage of Rainy Days versus Run Size to observe how the results tend towards certain probabilities for each Data Set as the Run Size increases.

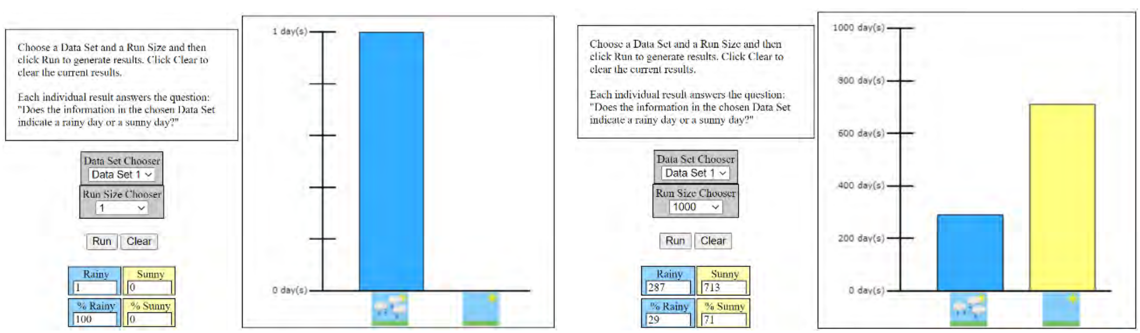


Figure 1: The Weather Forecast Simulation Showing Different Run Sizes for Data Set 1

The Chance of Rain simulation (Figure 2) is based on the classical model and represents weather data for every day during the month of June for 20 years in a certain location. The data is not based on a real location but was instead chosen to provide certain ratios of the different outcomes. Each of the 600 individual data points indicates if that day was sunny, cloudy, rainy, or stormy. The student can view a random day, a specific day, an entire specific month, or a bar chart summarizing all 600 days. As in the classical model, the probability of each weather outcome for any given day can be described by a fraction with the total number of days in the denominator. After exploring the simulation, the students were asked to gather data by clicking on the View Random Day button 100 times, recording what the weather was for each of these random days, and then comparing this to the probability fractions for each type of weather.

Figure 2: The Chance of Rain Simulation Showing the Different Data View Screens

In this paper, we focus on the analysis of one pair of students, Violet and Anne, to describe a chronological account of the progression of their reasoning about probability and the design decisions that the researcher made to support the development of their reasoning further. Of the three pairs of students recorded, this pair was chosen for an initial analysis due to completeness of data and their work with the research team member who was also part of the design team.

Findings

At the beginning of the design experiment, Violet and Anne were asked to state if they knew anything about weather reports or the concept of chance in predicting rainy or sunny weather. Violet made a connection to a recent snow day and mentioned weather forecasters talking about the chance of snow, saying, "There's a 50% of snow, 50% chance that snow might come. Well, when they say that they're not completely sure if it's going to come though." Her reasoning shows a classical understanding of probability as a percent.

Next, students were asked to explore the Weather Forecast simulation. First, they examined Run Size 1 for Data Set 1 (30% rainy, 70% sunny) and identified that it was sometimes rainy and sometimes sunny. As Violet said, "it only got rainy two times so it's not always rainy" showing the classical perspective of considering two rainy times out of the total number of runs. They then tried Run Size 5 and identified that it was now more likely for the predictions to be sunny than rainy. As Violet stated, this is because "it mostly shows that it's going to be more percent chances of being sunny than rain," illustrating that she was starting to notice a pattern in the data. Subsequently, students were asked to explore the larger Run Sizes (10, 50, 100, 500, 1000, 5000, 10000, 50000, 100000) and observed that there are usually more sunny than rainy days. However, they were not yet ready to bridge the connection to a specific theoretical probability.

In the next task, students were asked to use the simulation to record data in a table with the percent rainy and percent sunny for a single run for each size (Figure 3). When asked if they saw a pattern, Violet replied "I got 30 and 70, 3 times. … But I'm a little surprised I got the same one three times because I thought I was gonna get different more but then I got the same." This collection of data was challenging Violet's preconceptions about the expected probability.

They then graphed the data from their tables and compared their graphs to each other (Figure 3). Violet identified that their graphs were similar, but not the same: "When I got to 100 my number, her number, like the Run Size started to get a little different from 100 to 1000. But then when we got into the even bigger numbers, we had the same again." Violet was beginning to identify patterns in the Run Sizes, but the limitations of collecting random data was impeding her ability to make a claim that larger Run Sizes are more precise than smaller Run Sizes.

Run Size	% Rainy	% Sunny
1	0%	100%
10	30%	70%
100	28%	72%
1,000	31%	69%
10,000	30%	70%
100,000	30%	70%

Run Size	% Rainy	% Sunny
1	0%	100%
10	30%	70%
100	29%	71%
1,000	28%	72%
10,000	30%	70%
100,000	30%	70%

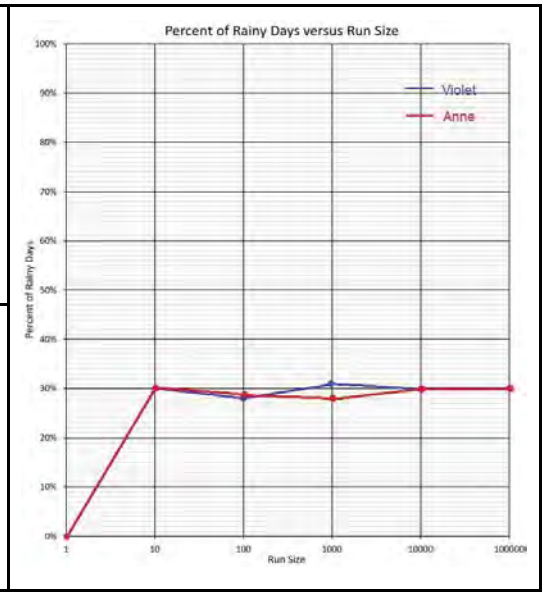


Figure 3: Recreation of Tables and Graphs of Data Set 1

Run Size	% Rainy	% Sunny
1	100%	0%
10	90%	10%
100	93%	7%
1,000	90%	10%
10,000	90%	10%
100,000	90%	10%

Run Size	% Rainy	% Sunny
1	100%	0%
10	100%	0%
100	88%	12%
1,000	90%	10%
10,000	90%	10%
100,000	90%	10%

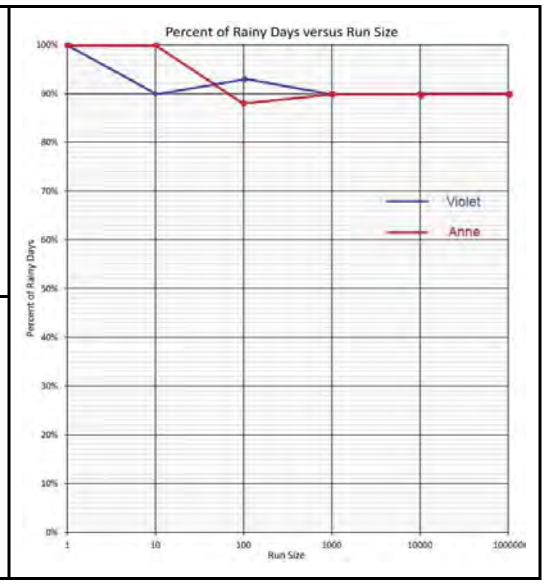


Figure 4: Recreation of Tables and Graphs of Data Set 2

The students went through the same process for Data Set 2 (90% rainy, 10% sunny). As Figure 4 shows, Anne got 100% rainy for Run Size 1 and 10, and 90% rainy for Run Sizes 1,000 to 100,000. When asked to make a numerical prediction about the chance of rain Violet replies, "there is 100% chance in the beginning of the day in the graph. But then it gets lower. So, I'll put it between, like 99 and 100%, it's gonna rain." Her reasoning shows that she erroneously considered the x-axis to represent time of day rather than Run Size and this led her to struggle in identifying what the different probabilities represented. The researcher then intervened to clarify this and asked her to explain what happens when the Run Size increases:

> Violet: I think you should run the simulator, seven times. Or a lot of times, because the number you get the most is probably the percentage of rain or the chances of rain, you're gonna get that day. Because if you run it one time, you're probably not going to get the right answer. Cause it could say that it's going to be 100% sunny when you run it one time, but it's probably going to be rainy that day. But when you run it like seven or more times, you're gonna get probably the same number like two or three times. But if you get it more than that, it means that it's probably going to be that weather that day.

Violet identified that there was a "right answer" or target theoretical probability that the experimental data should approach. She was also able to observe the limitations of small Run Sizes. However, for her a Run Size of "seven" or "a lot of times" was large enough to make a claim about the effect of the increased Run Size.

At this point the researcher modified the original design and discussed coin flipping and dice rolling aiming to examine whether the focus on the classical approach would help them in developing their frequentist understanding. Violet was able to bridge the connection between flipping a coin and its theoretical probability by saying there was "a 50% chance because there's like half a chance you're going to land on tails." They struggled with understanding the theoretical probability of the dice, but Violet used her understanding of experimental probability to bridge the connection to the theoretical probability by explaining the chances of rolling a one:

> Violet: Yeah, six possibilities, you can get it, but most of the time, you're not gonna get it. It's like, 10 out of 100, that you're probably going to get one. When you roll the dice, you're not going to get the number, most of the time, you're not going to get the number you wanted.

After this introduction to classical probability, they engaged with the Chance of Rain simulation and used the bar graph (Figure 2) to calculate the theoretical probability for each weather type in June (Figure 5, left). The students then collected 100 data points using the View Random Day button (Figure 5, right).

Weather	Days	Fraction of Total Days
Sunny	300	300/600 = 1/2
Cloudy	150	150/600 = 1/4
Rainy	75	75/600 = 1/8
Stormy	75	75/600 = 1/8

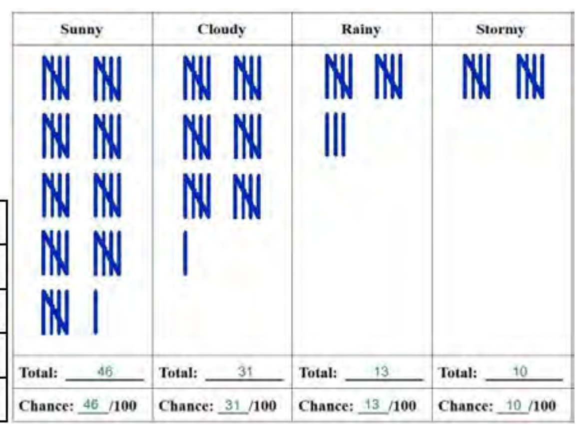


Figure 5: Theoretical and Experimental Probability for Chance of Rain Investigation

Working with the probabilities in fraction form required that they use strategies to compare fractions, which made connecting the probabilities more difficult. They discussed rainy and stormy which each had a theoretical of “one-eighth” and then created 12.5/100 as an equivalent fraction to compare it to their experimental 10/100 and 13/100. As Violet stated, “I tried simplifying some of them to see if I got closer to the same number.” They were able to connect the experimental to the theoretical probabilities. As Anne stated, “they’re all in the same thing cause like 10, if it was 11 it would have been more closer, like 11, 12, 13. But I think they’re like, equal.” They were then asked why they collected 100 data points, and not more or less:

Anne: So that you could get a good amount of like, sunny, stormy, rainy, and cloudy. And, it won’t have to only be sunny, cause maybe if you did 10, it would maybe be like 30 sunny maybe like two stormy, three rainy.

Violet: I do prefer 100 more than 10. But I tried it out. I clicked the view Random Day button 10 times. And I actually got rainy, stormy and cloudy. But I only got like two chances of sun.

They both saw the limitations of a small sample size and Violet even used experimental data to help support her point. At this point the researcher brought up the example of coin flipping again in another modification of the experimental design, to help illustrate what happens as the sample size increases. They discussed how it was more believable that they would get heads repeatedly with small sample sizes, but Violet explained:

Violet: When the number gets bigger, the more unbelievable it gets that you’re gonna get tails, or heads like that much. I mean, it could be possible to get heads or tails, like three or two times the same. … Because if I take a few minutes to flip my small watch 100 times, I’m most likely going to get tails and heads at the same time. Because like I said, there’s two sides of a coin.

Violet showed some understanding of the LLN and the advantages of larger sample sizes by using her understanding of experimentally flipping a coin and connecting it to the target theoretical probability of flipping a coin, bridging the frequentist and classical perspectives. Anne then showed similar reasoning when talking about the possibility of a computer simulation flipping heads 100 times in a row, saying, “there might be a 50% chance that the computer might just get like heads 100 in a row and then there’s like the other 50% chance,” showing that with large samples, she would expect the results to approach the target theoretical probability.

At this point they were asked to return to the Weather Forecast to continue their discussion on the LLN in a modification of the design. The researcher bridged the transition by continuing the discussion on the 100 data points collected and what they thought would happen if they collected more data. Both students agreed that collecting more than 100 would have been better with Violet saying, “if we did 200, the higher we go, I think it might be the easiest to compare them,” and Anne saying, “maybe if we did more than 100 maybe it would be like, close to each other. A little bit.” They had both explained the limitations of smaller sample sizes and agreed that larger sample sizes could make it easier to compare to the theoretical target.

However, when asked to draw conclusions about what happens as the Run Size gets larger from the graphs of the Data Sets, they still had difficulty connecting those thoughts to their analysis of the graphs. As Violet stated, “sometimes it goes higher, and then lower, and then it’s gonna go higher,” and Anne said, “it was one at the bottom, then at the top, then at the center, then at the top again.” Here they were focused on the variation of the data points and had not developed the covariational reasoning to connect their data to the Run Size. The researcher decided to prompt them to discuss the ‘gaps’ between their data points (Figure 4), which was a productive design modification:

Violet: [Looking at the right of the graph of Data Set 2, Fig. 4] Well, for the first three, there's not as big a gap because they're all 90. [Looks at the left of the graph] That's a big gap. Because for the last one here, it's 89. But this one's 100. That's sort of a big gap.
Anne: [Referring to Data Set 1, Fig. 3] Because you were comparing like there was a big gap or if they were closer to each other. Most of them were closer to each other, especially the 30 and the 29 and stuff.

This discussion was productive in helping them distinguish the pattern with smaller Run Sizes compared to larger Run Sizes. Violet was able to summarize this:

Violet: Because as larger as it gets, it shows how likely the number you picked is going to be the answer. Like in my graph as the numbers got bigger, it showed 30, two times. So that means it's gonna be 30% chance of rain. So as larger as the number is, is how it's going to be or how it may be. … Because the run number did get larger, and it did show why it's gonna be that percentage.

Violet's explanation shows her understanding of the LLN and how it can be used to approach the target theoretical probability or "answer." Anne mostly agreed with Violet's thinking but questioned what amount of data was sufficiently large. Despite this continuing limitation, Violet and Anne developed their understanding of experimental probability to predict a theoretical probability using the LLN, through their bidirectional engagement with the two simulations and the use of multiple representations of probability.

Discussion

Our results show that our design that started with a frequentist perspective without a clear connection to theoretical probability and moved to the classical perspective supported students' construction of a bidirectional link (Prodromou, 2012) between the two probabilities. While engaging in the simulations, the students' learning process was not linear but rather it was their transitions between the different representations that helped them develop bridges of the two models. Through these transitions, students developed an understanding of the LLN by emphasizing the strength of large numbers and the limitations of smaller Run Sizes.

The scientific context of the weather showed to be productive in illustrating the utility of probability for understanding phenomena in the real world. Bringing in examples of flipping coins and tossing dice helped bridge the gap for students in connecting the two probability models. Interaction with data by engaging with multiple modes of representations, including simulations, tables, and graphs, also fostered the students' understanding of this connection.

This experiment was limited as we only analyzed the reasoning of one pair of students. Thus, for future research, we would like to investigate other pairs to see how their connection of theoretical and experimental probability would develop. We would also plan to continue revising our design by exploring further how we can support students' struggles related to reading graphs, comparing fractions, and understanding what Run Size is sufficiently large.

Acknowledgments

This research was supported by a grant from the National Science Foundation (#1742125). The data presented, statements made, and views expressed are solely the responsibility of the authors. We would like to thank Jay Singh, Archana Prasad, Anthony Cuviello, and Richard Tashjian for their help in the design of the simulations and activities.

References

Abrahamson, D., & Wilensky, U. (2005). ProbLab goes to school: Design, teaching, and learning of probability with multi-agent interactive computer models. In D. Pratt, R. Biehler, M. B. Ottaviani, & M. Meletiou (Eds.), *The Proceedings of the Fourth Conference of the European Society for Research in Mathematics Education* (pp. 570-579).

Aspinwall, L., & Tarr, J. E. (2001). Middle school students' understanding of the role sample size plays in experimental probability. *The Journal of Mathematical Behavior*, *20*(2), 229-245.

Bakker, A., Hahn, C., Kazak, S., & Pratt, D. (2018). Research on probability and statistics education in ERME: Trends and directions. *Developing Research in Mathematics Education: Twenty Años of Communication, Cooperation and Collaboration in Europe* (pp. 1-15).

Batanero, C., Chernoff, E. J., Engel, J., Lee, H. S., & Sánchez, E. (2016). Introduction. In C. Batanero, E.J. Chernoff, J. Engel, H.S. Lee, & E. Sánchez (Eds.), *Research on Teaching and Learning Probability* (pp. 1-33).

Biehler, R. (1991). Computers in probability education. *Chance encounters: Probability in education*, 169-211.

Chick, H. L., & Baker, M. (2005). Teaching elementary probability: Not leaving it to chance. In P.C. Clarkson, A. Downton, D. Gronn, M. Horne, A. McDonough, R. Pierce & A. Roche (Eds.), *Building connections: Theory, research and practice: Proceedings of the 28th annual conference of the mathematics education research group of Australasia* (pp. 233-240). MERGA.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Cosmides, L., & Tooby, J. (1996). Are humans good intuitive statisticians after all? Rethinking some conclusions from the literature on judgment under uncertainty. *Cognition, 58*(1), 1-73.

Drier, H. S. (2000). Children's meaning-making activity with dynamic multiple representations in a probability microworld. In M.L. Fernandez (Ed.), *Proceedings of the 22nd annual meeting of the North American Chapter of the International Group for Psychology of Mathematics Education*.

Fischbein, E., & Schnarch, D. (1997). Brief report: The evolution with age of probabilistic, intuitively based misconceptions. *Journal for research in mathematics education*, *28*(1), 96-105.

Hawkins, A. S., & Kapadia, R. (1984). Children's conceptions of probability—a psychological and pedagogical review. *Educational Studies in Mathematics*, *15*(4), 349-377.

Henry, M., & Parzysz, B. (2011). Carrying out, modelling and simulating random experiments in the classroom. In M. Pytlak, T. Rowland, E. Swboda (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 864-874).

Ireland, S., & Watson, J. (2009). Building a connection between experimental and theoretical aspects of probability. *International Electronic Journal of Mathematics Education, 4*(3), 339-370.

Lee, H. S., Angotti, R. L., & Tarr, J. E. (2010). Making comparisons between observed data and expected outcomes: Students' informal hypothesis testing with probability simulation tools. *Statistics Education Research Journal*, *9*(1), 68-96.

Panorkou, N., & Germia, E. F. (2021). Integrating math and science content through covariational reasoning: the case of gravity. *Mathematical Thinking and Learning*, *23*(4), 318-343.

Panorkou, N., & York, T. (2020). Designing for an integrated STEM+ C experience. In Sacristán, A. I., Cortés-Zavala, J. C., & Ruiz-Arias, P. M. (Eds.), *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 2233-2237).

Paparistodemou, E. (2005). Young children's expressions for the law of large numbers. In D. Pratt, R. Biehler, M. B. Ottaviani, & M. Meletiou (Eds.), *The Proceedings of the Fourth Conference of the European Society for Research in Mathematics Education* (pp. 570-579).

Prodromou, T. (2012). Connecting experimental probability and theoretical probability. *ZDM, 44*(7), 855-868.

Stohl, H., & Tarr, J. E. (2002). Developing notions of inference using probability simulation tools. *The Journal of Mathematical Behavior*, *21*(3), 319-337.

Tversky, A., & Kahneman, D. (1971). Belief in the law of small numbers. *Psychological bulletin*, *76*(2), 105.

IN-SERVICE HIGH SCHOOL STATISTICAL TEACHERS' REASONING ON SIGNIFICANCE TESTS WITH SIMULATION

Francisco Sepúlveda Vega
Centro de Investigación y Estudios Avanzados, IPN
francisco.sepulveda@cinvestav.mx

Ernesto Sánchez Sánchez
Centro de Investigación y Estudios Avanzados, IPN
esanchez@cinvestav.mx

Guadalupe Carrasco Licea
Colegio de Ciencias y Humanidades, Plantel Sur, UNAM
gcarrascolic@gmail.com

In this paper we address the following questions: How in-service teachers use simulation to solve p-value problems? And how do teachers reason when solving significance test problems with simulation? The answers to two p-value problems by 19 in-service teachers who worked in teams on an on-line update course are analyzed. The course was conceived as a design experiment. Teachers were able to learn to use the Fathom software, but some resistance to the informal approach is observed in teachers' need to make more calculations than necessary to solve the problems. Misconceptions about significance level, and the documented verificationist conception of statistical tests were also observed.

Keywords: Professional Development, Data Analysis and Statistics, Problem Solving, Technology.

Introduction

Significance tests have been subject of controversy. Some dissonance critique comes from their widespread use in experimental areas and users' overconfidence in its results, however most of the criticisms do not regard the tests themselves, but to the misuse of tests by experimental researchers (Batanero, 2000, p.75). Resonant harmony may come from statistical education that might contribute to the better understanding and application of statistical inference. Simulated sampling distributions (SSD) offer an alternative to teach significance tests that avoid its theoretical complexity emphasizing the statistical logic of its procedure. It seems that, after some instruction, high school students can achieve well within this so-called informal approach. However in-service high school teachers are not familiarized with it and little research has been done about their related reasoning. It is well known that hypothesis tests are complex and hard to understand because they relate multiple concepts (e.g., sampling, population, *p*-value, sampling distribution) and are not intuitive in nature (Lane-Getaz, 2017; Vallecillos, 1995). This complexity is usually avoided by university teachers by means of exposing the theme to its students as a mechanical series of steps that must be followed which helps to draw 'scientific' conclusions. However, a big part of the method and important details of the whys of this process remains unknown to a big amount of the population that yields to a misuse of tests, even from experimental researchers (Batanero, 2000). For over 20 years (e.g., Hesterberg, 1998), statistical education research has promoted the use of simulated sampling distributions (SSD) to teach an informal approach of hypothesis tests because "technology allows students to be directly involved in the 'construction' of the sampling distribution focusing on the process involved, instead of being presented only with the final result" (Ben-Zvi, Bakker, & Makar, 2015, p.298). However, and maybe due to the little time allocated in the curriculum to hypothesis tests at the university level, the classical approach is still the most common way to teach it. This has led to researchers in statistics education to explore the possibility of teaching the fundamental ideas of this topic from high school level (e.g., García-Ríos, 2017; Matuszewski, 2018) to foster the reasoning of students of those grades about samples and sampling (Ben-Zvi et al., 2015), empirical sampling distributions (Noll & Shaughnessy, 2012), and statistical inference (Saldanha & Thompson, 2014). But, as previously said, high school teachers are still not familiarized with this informal approach to significance tests. With the general objective of exploring this possibility, our research questions are the next two: How in-service teachers use simulation to solve *p*-value problems? How do teachers reason when solving significance tests problems with simulation?

Conceptual Framework

Simulated Sampling Distribution (SSD)

"The term *empirical sampling distribution* (ESD) refers to a finite distribution of sample statistics from repeated samples of a population" (Noll & Shaughnessy, 2012, p.514). An important feature of an ESD for teaching is that it can be obtained through a series of repeated sampling actions that the student can perform from a few instructions done in a rudimentary way with manipulatives and pencil and paper. An SSD refers to a simulated distribution, in our case, in Fathom (Finzer, 2014). An SSD is a concrete, simplified and approximated version of a theoretical sampling distribution. It depends only on three given conditions: 1) The sampling is random and with replacement, 2) A determined sample size and, 3) The proportion in the population of the attribute under study.

Significance Tests with Simulation

An SSD allows the researcher to calculate the probability that a randomly chosen sample will have a proportion p or another more extreme proportion. Such a probability is called the *p*-value. If the *p*-value of a given sample is less than a significance level α (small) the sample is unusual, and technically, the sample is said to be significant. That is, if it is assumed that a population has a proportion for an attribute (null hypothesis) and for some reason the researcher suspects that this has changed, a significance test is a process to argue in this sense. The logic of this test is to assess how unusual a given sample is, assuming the null hypothesis to be true. To accomplish this, it is necessary to generate an SSD with the proportion indicated by the null hypothesis. This SSD will permit us to estimate the *p*-value and compare it with a significance level α. An unusual sample (p-value $< \alpha$) would support the decision to reject the null hypothesis, otherwise there would not be enough evidence to reject it. In the present report we use the expressions significance test problems and *p*-value problems as synonyms.

Note: Both random sampling and sample size are crucial to ensure the quality of the inference. However, in this study they are considered implicitly, without promoting an explicit discussion with teachers about it.

Informal Inferential Reasoning

Informal inferential reasoning (IIR) is increasingly gaining significance in statistics education research (e.g., Bakker & Derry, 2014). In this study we follow Schlinder & Seidouvy (2018) adopting the definition by Zieffler et al. (2008, p.44) who described IIR as *"the way in which students use their informal statistical knowledge to make arguments to support inferences about unknown populations based on observed samples"*. It is important to acknowledge that this is a construct to describe the way in which *"students use their knowledge to argue and make statistical inferences about an unknown population based on observed samples, and without using the formal methods or techniques of inferential statistics, such as the use of the theoretical sampling distribution, standard deviation, standard score, etc"* (García-Ríos, 2017, p. 31). One of the purposes of this research is to explore how teachers react to this paradigm switch, that is, how easy it is to them to adopt the alternative informal approach and the use of SSD to make inferences, discarding formulas and traditional approaches.

Statistical Knowledge for Teaching (SKT)

With the overall purpose of organizing the course we divided the activities into two different categories. The first kind of activities assess the teachers' subject matter knowledge, related with their own conceptions and reasoning about significance test and the second, the pedagogical content knowledge, related with the students' conceptions, the teaching strategies and the curriculum knowledge. Both categories are part of the statistical knowledge for teaching (SKT) (Groth, 2007, 2013). It is convenient to mention that this framework was used as a theoretical guide to classify the activities, and that the tasks presented on this report are only related to the subject matter knowledge, which we can subdivide into two kinds of knowledge: common content knowledge, which is represented by teachers conceptions on significance tests, and specialized content knowledge, related with the software use and the generation of an SSD within it.

Background

It is known that students can establish relationships between the sampling distribution and the parameter with which it was generated and evaluate the probability of a statistic (Jacob & Doerr, 2013) and that high school students may achieve a good understanding of the relationship between the theoretical proportion in the population and the proportion in the sample (Batanero et al., 2020). Some of the findings when working with high school students with *p*-value problems similar to the proposed on this report are that they intuitively take the decision of rejecting or not the null hypothesis in three different ways: a) They look for the majority in the SSD forgetting the sample proportion; b) They compare the mode of the SSD with the null hypothesis, or; c) They do not use the SSD to take the decision, and only evaluate the sample proportion (Sánchez, García-Ríos, Silvestre, & Carrasco, 2020). The first two could be explained with the hypothesis that students interpret the SSD as a simulated population, this means that they are learning something which they will have to repeat (repeated sampling) with real populations if they want to make inferences. However, those conceptions seem to be overcomed when confronted with similar problems and teachers' feedback.

Little research has been done with in-service teachers at high school level. Matuszewski (2018) worked with three in-service teachers and found that it is important for teachers to have *"some form of multiplicative sampling reasoning"* (p. 223) in the sense of Saldahna and Thompson (2002) and to possess statistical vocabulary before being confronted with those kinds of activities. Matuszewski also found that after a few activities with simulation, teachers can improve their understanding on the logic of hypothesis tests. In our case, to design the course, it was also assumed that teachers may have the same conceptions of students when confronted with significance test problems (Thompson, Liu, & Saldanha, 2007).

Method

A twenty-hour on-line course with 19 high school in-service teachers was carried on in December 2021 by the authors. The purpose of the course was to instruct teachers on the SKT of the informal approach to significance tests. This report discusses only the first part of the course related to the subject matter knowledge and focuses on the observed teachers' reasoning on significance tests with simulation.

The course was imparted trough the Zoom platform. Sessions were divided into two types, plenary sessions and team working sessions. Plenary sessions were useful to expose the subject and instructions, and to discuss the solutions for the problems.

The overall course has been planned as a design experiment: with pragmatic as well as theoretical purposes, with carefully designed tasks based on discussion and existing literature, and with produced data that could be analyzed at several levels within a learning ecology that "include the tasks and problems that [teachers] are asked to solve, the kinds of discourse […], the norms of participation […], the tools and related material means provided, and the practical means by which classroom [instructors] can orchestrate relations among these elements." (Cobb et al., 2003, p.9).

Participants

The course was presented to professors under an inter-semester update scheme at a public high school in México. In addition to their particular interest in the content of the course (since they have many courses to choose from), teachers had the incentive to obtain points with which they would be benefited at work. Teacher's experience varied from two to twenty-five years teaching statistics at high school level.

From the nineteen in-service teachers, sixteen claimed to have studied hypothesis tests. However, only eleven professors indicated that they had taught hypothesis tests, perhaps because it is the last topic on their agenda. Only six teachers reported being familiar with Fathom. Thirteen teachers are engineers, five are physicists or mathematicians, and one is an actuary. In addition, among the participants there was one professor with a doctorate degree, fifteen with a master's degree and three had only a major's degree.

Tasks and activities

As an introduction to the course teachers were familiarized with Fathom with an exposition about how to simulate some basic probability models. Specifically, the simulation of tossing a coin and creating the relative frequency graph and graphing the sampling distribution of tossing a die. Then, they were taught how to create an SSD with the example of tossing ten times a coin and received a step-by-step pdf manual of how to create an SSD.

This report presents the analysis of the answers to two significance test problems (see Appendix A). With the overall purpose of presenting an informal simplified version of significance test to teachers, proportion problems with categorical variables were chosen. Those kinds of problems permit the teachers to confront students with a simpler version of significance tests avoiding explanations about, for example, statistics parameters or more complex statistical tests as means of comparison.

Those problems were presented to teachers after the introduction session in the 2nd and 3rd sessions. Teachers had one hour to answer each problem. Six teams of teachers were formed. Not all the teachers were present during the sessions, so teams had two or three participants. Teams were created trying to equilibrate the experience of teachers, so almost all teams had at least one ten or more years' experienced statistics teacher. Team working served teachers to answer the problems and create a report that they were asked to upload to a shared folder. Between the first and the second problem we had a plenary discussion in which two teams of teachers exposed their answers and received feedback. Only the plenary sessions were videotaped, and only one answer was received by each team, which, with the videotaped sessions, composes the material of analysis for this report.

Analysis

Answers were analyzed at two different levels. The first one was with a focus on the use that teachers made of the software. The second with a focus on their reasoning about significance tests. For the first analysis five specific items were chosen to measure how teachers used the software. Those were the simulation of the urn, the simulation of one sample, the repeated sampling simulation, the generation of the sampling distribution, and the use that teachers made of the SSD to make inferences (see Table 2). Each team received a 1 if evidence of the item was present in their response or a 0 if not. The totals per problem were calculated to compare how they evolved between the first and the second problem. Two different phenomena were observed in the teachers' answers. Some of the teachers already had experience with Fathom, so they used more tools than necessary to make the decision. On the other hand, some teams were not familiarized with it, so they misused the software. Those characteristics were coded as "over simulated" and "under simulated" respectively (see Table 3). Finally, a brief description of how teachers used the technology was done. Combined, those three items permitted us to better understand how the teachers used technology to simulate distributions and make inferences.

For the analysis of the teachers' reasoning about significance tests, a two-way reading was carried on over the answers teachers gave to both problems. The first consisted of the description, comparison, and coding of answers to observed patterns. Then, the codes were transformed into categories that formed the basis of the review on a second way analysis. Each answer was then analyzed systematically through this lens, receiving a 1 point if the code could apply to the answer or a 0 if not. This permitted us to dismiss the codes that only appeared once.

Once this classification was made, more specific descriptions of the categories were done, and answers were chosen that were representative of each code. A theoretical interpretation of those codes is discussed in the conclusions.

Results

Table 2 shows the totals that teams obtained per item in simulation. Precise descriptions of how teachers missed the points are given down.

Table 2: Teachers' use of simulation

	Urn simplified representation	**One sample simulation**	**Repeated sampling simulation**	**Generation of SSD**	**Correct use of SSD as a measure of variability**	**Total**
Problem 1	3/6	6/6	5/6	5/6	4/6	23/30
Problem 2	6/6	6/6	6/6	6/6	4/6	28/30

On Problem 1, since the null hypothesis was a half-divided population, it was sufficient to simulate an urn with two balls. One of the teams simulated this hypothesis with a 1000 random pick of balls (p=0.5), which gives an approximation of the null hypothesis, but is more complex than the necessary. This team received a 0 in the "urn simplified representation" and was classified as "over simulation" in Table 3.

Table 3: Under and over simulation

	Under-simulation	Precise simulation	Over simulation
Problem 1	**1/6**	**2/6**	**3/6**
Problem 2	**0/6**	**3/6**	**3/6**

Table 4 shows the codes that we obtained with a short description on the second column and a brief example of a representative answer for each code in the third column.

Table 4: Codification

Code	Description	Example
Correct H0	Teachers correctly define the null hypothesis (In Problem 1 p=0.5 and in Problem 2 p=0.1)	On Problem 1, teachers simulate an urn with two balls, one with an "N" and another with a "P". This is enough to simulate a half divided population.
Correct answer	Teachers correctly decide to reject or not the null hypothesis (In Problem 1 they must reject it, and in Problem 2 not)	Team 4 responded to the first question of Problem 1: *"We consider that the information obtained from the sample is sufficient to reject the hypothesis that the population is divided in half"*
Hypothesis tests	Teachers transform the significance test problem on a hypothesis test problem proposing an alternative hypothesis	Team 3 answer to the first and second questions of Problem 1: "*Null hypothesis: p=0.5* *Alternative hypothesis: p>0.5* *The null hypothesis is rejected (p=0.5), the alternative hypothesis is accepted (p> 0.5), so what is stated in the problem is correct"*
Verificationism	Teachers understand significance tests as a process to verify or tests a hypothesis	Team 5 answer Problem 1 arguing that: "*The information obtained from the histogram proves that the probability of consumption is greater for the consumption of coca cola."*
Misconception about significance level	Teachers understand significance level as an interval, if the proportion of the sample falls within this interval they said it is 'significant', so they do not reject the null hypothesis	Team 2 answer Problem 2 arguing that: *" Of 5000 samples of size 120, drawn from a population where 10% of the items are defective, in 14.64% of them the items are defective, therefore, the number of defective parts that came out in the sample is within the level of significance"*
Mistrust on sample or sampling size	Teachers argue that 'if the sample size was bigger, results were better too'. Some of them reason that if the simulated samples were more, the results were better too.	Team 4 answered the 4th question of Problem 1 arguing: *"We cannot be certain that this is how the general population behaves. We could increase the sample size (over 180) and the number of cases in the simulation (over 500) to increase certainty"*

Another example of "over simulation" was one team that, on Problem 1, having correctly simulated the SSD also used Fathom to calculate the means and standard deviation of samples. This team took the decision of rejecting the null hypothesis based on those calculations, and not on the proportion of samples that fell in the queue of the SSD. This team also received a 0 in the "Correct use of SSD" in Problem 1. One team, in Problem 1 only simulated 100 samples in the repeated sampling simulation item, which is not enough to approximate the normal distribution, so this was classified as under-simulation. Their decision of rejecting the hypothesis was made only based on the proportion of the real sample (a misconception observed in high school students too (e.g., Sánchez et al., 2020)). In general, the plenary discussion of the first problem led the participants to have better results in the second problem as shown in Tables 2, 3 and 5. An extended discussion about those results is presented in the conclusions.

Table 5 shows the frequency of appearance of each code in each of the problems. The two first codes could be considered as qualitatively different to the others because they could be measured as a yes or no answer. However, both types of codes are presented in the same table to contrast them.

Table 5: Frequencies of Codes

Code	Problem 1	Problem 2
Correct H0	5/6	6/6
Correct answer	4/6	6/6
Hypothesis tests	2/6	2/6
Verificationism	3/6	0/6
Misconception about significance level	1/6	3/6
Mistrust on sample or sampling size	2/6	1/6

Discussion and Conclusions

Our research questions were the next two: How in-service teachers use simulation to solve *p*-value problems? How do teachers reason when solving significance test problems with simulation? Most research on statistical knowledge for teaching has been done with pre-service teachers (e.g., Batanero et al., 2018; Thompson et al., 2007) but little with in-service teachers. Part of our work was made based on the hypothesis that in-service teachers may have the same misconceptions as students when confronted with similar problems. We found that this could apply in some cases, as in the difficulties presented to understand significance value (misunderstood as an interval of acceptance of the null hypothesis), in the intuition that bigger sample sizes are more representative of the population, and in the verificationist conception of statistical tests (see Table 5). However, important differences were observed when working with experienced in-service teachers. It seems that, contrary to the students, and to pre-service teachers, in-service teachers reasoning about *p*-value problems may be strictly procedural and conceptually misunderstood, that is, their intuition is to use calculations and formulas (as standard deviations and standard scores) and specific concepts as significance level, even if sometimes misused (see Table 5). This phenomenon could generate resistance when confronted with a different approximation to the same theme. The approach to significance tests with simulation promotes the generation, visualization, and interpretation of an SSD. Fathom offers the possibility to reduce calculations to the minimum (replaced with knowledge of commands), favoring the understanding of the logic of significance tests. But in-service teachers' intuitions made them calculate the exact values, maybe, as looking for certainty to corroborate the observed in the SSD. How would this mistrustfulness about the process manifest in front of students? Positive observations are that in-service teachers already interpret SSD as a mathematical entity to measure variability that may replace the normal distribution, and that, after a little feedback, they start to substitute their formal knowledge by the alternative informal approach to significance test.

Informal and formal approaches to significance tests are not presented as contrary versions, but they are the opposite and complementary sides of a continuum in which formality and informality become relative. One of the objectives of this course was to show to teachers the way in which students can go from one side to the other, following a more intuitive way aided by technology. It was observed that the manipulation of the software was not a barrier for in-service teachers. Limitations are in the interpretation teachers made of SSD, and more research must be done in this sense to reduce the gap between students' intuitions and teachers' procedural formality. That is, it must be understood by researchers and teachers that the use of this alternative approach to significance test does not guarantee by itself the correct interpretation of the logic behind statistical inferences.

Some limitations of our design were that the chosen problems only confronted teachers with a part of the statistical cycle (the inference), and not with the overall scientific process of posing questions, planning an experiment, collecting, and organizing data, and obtaining conclusions. Since our research is centered on statistical reasoning and not on statistical thinking this is not a problem to us; yet it could be from the part of teachers since it is recommended to teach all the cycle and not only present to the students the inferential logic. One suggestion is that problems could be accompanied with theoretical text and discussions to familiarize teachers with results of research, for example, about the misinterpretation of significance value. During the last session teachers expressed their interest to use this informal approach with their students.

Appendix A

Problem1

It is traditionally believed that half of the population that drinks cola prefers Pepsi, and the other half prefers Coke. However, for advertising purposes, Coca Cola advertising presumes that the majority (more than 50%) of the population that consumes cola prefers Coke instead of Pepsi. To support this claim, he surveyed 180 randomly selected people who regularly consume soft drinks. Of the 180 participants, 104 people preferred Coca Cola. Is the sample information obtained sufficient to reject the hypothesis that the population is divided in half at a 5% level of significance?

Correct answer: Null Hypothesis (H_0): p=0.50, N=180, *p*-value=0.021, H_0 is rejected.

Problem2

If the daily production of a factory machine has more than 10% defective items, it needs to be repaired. To check the quality of the machine, the supervisor takes a random sample of 120 pieces of the day, and this contains 16 defective pieces. With these data and a significance level of 5%, should the supervisor request to repair the machine?

Correct answer: Null Hypothesis (H_0): p=0.10, N=120, *p*-value=0.143, H_0 is not rejected.

Both problems were accompanied by the following questions

1. Answer the problem
2. Explain your conclusion
3. Detail step by step how you arrived at your answer. You can use images of the program (print screen) to have a complete report.
4. How sure are you of your conclusions?

Acknowledgements

The research reported in this paper had the support of a CONACYT scholarship.

References

Bakker, A., & Derry, J. (2011). Lessons from inferentialism for statistics education. *Mathematical Thinking and Learning*, 13(1-2), 5-26. doi:10.1080/10986065.2011.538293

Batanero, C. (2000). Controversies around the role of statistical tests in experimental research. *Mathematical Thinking and Learning*, 2(1-2), 75-98.

Batanero, C., Begué, N., Borovcnik, M., & Gea, M. (2020). Ways in which high-school students understand the sampling distribution for proportions. *Statistics Education Research Journal*, 19(3), 35-52.

Batanero, C., López-Martín, M., Gea, M., & Arteaga, P. (2018). Conocimiento del contraste de hipótesis por futuros profesores de Educación Secundaria y Bachillerato. *Publicaciones*, 48(2), 73-95.

Ben-Zvi, D., Bakker, A., & Makar, K. (2015). Learning to reason from samples. *Educational Studies in Mathematics*, 88, 291-303. doi:10.1007/s10649-015-9593-3

García-Ríos, V. N. (2017). *Diseño de una Trayectoria Hipotética de Aprendizaje para la Introducción y Desarrollo del Razonamiento sobre el Contraste de Hipótesis en el Nivel Medio Superior*. Centro de Investigación y Estudios Avanzados, Matemática Educativa. Ciudad de México: Tesis sin publicar.

Groth, R. (2007). Toward a conceptualization of statistical knowledge for teaching. *Journal for Research in Mathematics Education*, 38(5), 427-437.

Groth, R. (2013). Characterizing key developmental understandings and pedagogically powerful ideas within a statistical knowledge for teaching framework. *Mathematical Thinking and Learning*, 15, 121-145.

Hesterberg, T. (1998). Simulation and Bootstrapping for Teaching Statistics. *American Statistical Association Proceedings of the Section on Statistical Education* (pp. 44-52). American Statistical Association.

Jacob, B., & Doerr, H. (2013). Student's informal inferential reasoning when working with the sampling distribution. Paper presented in Working Group 5 (A. Bakker, Chair). *The Eighth Congress of European Research in Mathematics Education (CERME 8).* Turkey.

Lane-Getaz, S. (2017). Is the P-value really dead? Assessing inference learning outcomes for social science students in an introductory statistic course. *Statistics Education Research Journal*, 16(1), 357-399.

Matuszewski, A. (2018). *High School Statistics Teacher's Understanding of Hypothesis Testing Through Simulation*. Tennessee, US: Middle Tennessee State University.

Noll, J., & Shaughnessy, M. (2012). Aspects of student's reasoning about variation in empirical sampling distributions. *Journal for Research in Mathematics Education*, 43(5), 509-556. doi:10.5951/jresematheduc.43.5.0509

Saldanha, L., & Thompson, P. (2014). Conceptual issues in understanding the inner logic of statistical inference: Insights from two teaching experiments. *Journal of Mathematical Behavior*, 35, 1-30.

Sánchez, E., García-Ríos, V., Silvestre, E., & Carrasco, G. (2020). High school student's misconceptions about significance testing with a repeated sampling approach. En A. Sacristán, J. Cortés-Zavala, & P. Ruiz-Arias (Ed.), *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1298-1306). México: Cinvestav.

Thompson, P., Liu, Y., & Saldanha, L. (2007). Intricacies of statistical inference and teachers' understandings of them. En M. Lovett, & P. Shah (Eds.), *Thinking with data* (pp. 207-231). Lawrence Erlbaum Associates Publishers.

Vallecillos, A. (1995). Comprensión de la lógica del contraste de hipótesis en estudiantes universitarios. *Recherches en Didactique des Mathématiques*, 15(3), 53-81.

Zieffler, A., Garfield, J., DelMas, R., & Reading, C. (2008). A framework to support research on informal inferential reasoning. *Statistics Education Research Journal*, 7(2), 40-58

LEARNERS' USES OF DESIGNED COMPUTER SIMULATIONS IN PROBABILISTIC AND STOCHASTIC SETTINGS

Karoline Smucker
Eastern Oregon University
ksmucker@eou.edu

Simulations play a key role in statistics and probability education, and are important tools for students and teachers. This report presents preliminary findings from design research investigating the impact of computer simulations on learners' approaches to tasks involving probabilistic modeling, sampling, and stochastic reasoning. Five preservice secondary mathematics teachers engaged with a series of seven tasks, with a goal of understanding their use of the computer simulations in the course of problem solving. The designed simulation environments were created using the Common Online Data Analysis Platform (CODAP). Analysis of one participant's approaches suggests that the simulation was used in several distinct ways; in particular it was used to confirm, to test, and to investigate/refine.

Keywords: Technology, Data Analysis and Statistics, Probability, Design Experiments

Instruction in statistics and probability has long relied on the use of both physical and virtual simulations. Research suggests that these tools can be used for a variety of purposes, including to test/develop hypotheses about probabilistic models (English & Watson, 2016; Konold & Kazak, 2008), develop understanding related to sampling and variation (Saldanha & Thompson, 2014; van Dijke-Droogers et al., 2020), and understand or apply randomization-based procedures within statistical inference (Garfield et al., 2012; Watson & Chance, 2012). In some instances, learners may utilize or design a simulation environment to model a setting where the underlying probability distribution is known (Ferreira et al., 2014; Kazak et al., 2015), while in other instances learners may be asked to make inferences in a scenario where the underlying model is "hidden" in some way (Konold et al., 2011; Stohl & Tarr, 2002).

While all of this research showcases the many uses for computer simulations in statistics and probability instruction, less is known about why learners may use a simulation in the course of problem solving. This is especially important to consider in light of recent advances in software like the Common Online Data Analysis Platform (CODAP, https://codap.concord.org) which gives students the ability to easily engage with designed simulations or create their own models. Both researchers and teachers must consider why and when students might utilize these tools. This design research used clinical interviews to investigate the following question: for what purpose do learners use a computer simulation environment to tackle tasks involving expectations, likelihoods, and drawing inferences based on both known and unknown distributions?

Theoretical Perspective

This research considers the ways that learners (specifically secondary preservice teachers) engage with computer simulation technology in probabilistic and stochastic contexts. From a sociocultural perspective, tools play a key role in mediating learning (Vygotsky, 1987); it is of interest to understand how specific simulation tools are utilized by learners in these contexts (Kazak et al., 2015), especially as these tools become more widely available. Researchers must also consider the affordances of the specific tool being used (for example, what representations are available and what variables can be manipulated), and how these may impact learners' approaches and use of the technology (Bumbacher et al., 2018). It has been suggested that computer simulations can play a key role as learners engage in inquiry based on constructivist principles including discovery learning and conceptual change (Lane & Peres, 2006). In discovery learning, learners are asked to hypothesize about what they believe will occur, test it empirically, and then attempt to explain their results, especially any conflict between their observations and predictions (Ireland & Watson, 2009; Sharma, 2016). We attempted to gain understanding of how learners use specific simulation environments in the course of problem solving, including whether/when they propose solutions or make predictions and how the simulation impacts their conclusions and/or confidence.

Methods

Design principles were utilized in this investigation. This type of research attempts "to develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb et al., 2003, p. 10). This methodology is highly interventionist, and utilizes cycles of implementation and reflection to iteratively test hypotheses about thinking and learning, often through the use of designed tasks and activities. Theory which is built within design research is expanded and/or refined as it is tested in new contexts.

A series of five semi-structured clinical interviews (Goldin, 2000) was implemented virtually via Zoom with five preservice secondary mathematics teachers in Summer 2021, investigating their approaches to a designed sequence of seven tasks using the CODAP sampler tool. This tool allows the user to adjust the type of sampler, along with the number of items/samples drawn, and create/organize various representations of data gathered by the sampler. The designed tasks fell into two primary categories. Some tasks involved known theoretical distributions (like the binomial distribution), while for others the distribution was unknown (and involved a mystery sampler). Five of the designed tasks involved known distributions, while two involved unknown distributions. The tasks were selected, modified, or created based on contexts found in prior research related to sampling and probabilistic/stochastic modeling (i.e. Ferreira et al., 2014; Kazak et al., 2016; Konold et al., 2011).

Data included video recordings of the Zoom sessions, along with still images of any written work and CODAP screenshots. Following principles of design research, analysis was both prospective and retrospective (Cobb et al., 2003). After each round of interviews, the research team met and considered changes to the order of tasks or specific prompts based on what occurred. After the interviews were complete, more in-depth analysis of the data began, highlighting when and how simulation was utilized by the participants. For this paper, we will consider the work of one participant, Bridget (a pseudonym).

Preliminary Findings

Three distinct uses of simulation emerged as Bridget engaged with the designed tasks: simulation as a tool for confirmation, simulation as a tool for testing, and simulation as a tool for investigating and refining. These uses were related to the task context (known vs. unknown distributions) and Bridget's confidence in any solution she had prior to accessing the simulation.

Simulation to confirm

In several instances, the simulation was utilized primarily as a confirmation tool after Bridget had reached a solution she was confident in. When considering Katie's Walk, a task which prompted Bridget to investigate the fairness of Katie's plan to visit one of five friends based on the results of four coin flips (traveling north or east after each flip), Bridget created a systemic list prior to accessing the designed simulation environment which included all 16 possible outcomes and which friend would be visited (Figure 1, left side). She used this information to justify her conclusion that Katie's plan is not fair by calculating the likelihood that each friend would be visited and connecting it to the number of heads flipped.

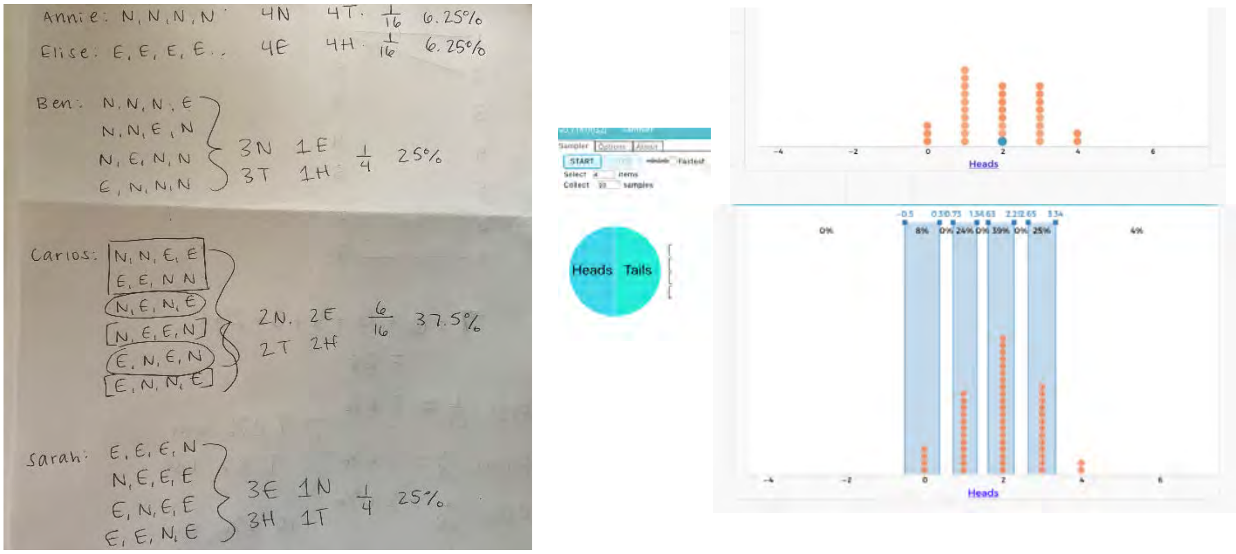


Figure 1: Bridget's initial work (left) and graphs in the simulation (right) for Katie's Walk

When Bridget finally accessed the simulation, she already had a solution that she was very confident in. In this instance, the sampler was already created with a pre-made spinner representing heads/tails, along with a graph which would showcase the number of "Heads" for each set of four flips. Bridget collected trials until her data "matched" the solution she had in mind. When her initial data (31 trials, top graph) did not have Carlos (2 Heads) visited the most, Bridget said, "I don't think this simulation is wrong, but I don't think what I wrote is wrong either . . . I would have liked for them to match". She then collected 20 more trials (bottom graph), and felt much better about this result, saying, "Aha! This is kind of the picture I had in my head". At this point, she felt no need to collect additional data because "there's nothing really that I'm questioning". She only felt the need to collect data until it confirmed her expectation.

Simulation to test

Another use of simulation occurred in situations when Bridget was testing a proposed solution. This involved contexts where she was less confident. One example of this is the Chip Bag task, which asked her to consider what would happen if two poker chips were drawn from a bag containing 3 green chips and 1 blue chip (theoretically the likelihood is 50/50 to draw two greens versus one of each color). Bridget's initial hypothesis was that she would be most likely to draw two green chips because of how many greens were in the bag, but she struggled to explain why this would be the case. She then stated that she would like to "set the situation up on the website" (CODAP), and created a mixer which modeled the context. In this instance, the simulation was not being used to confirm a solution she had high confidence in, but to test an idea that she was unable to explain outside the simulation.

Bridget first collected 20 samples, and found that getting two greens was more frequent (12 vs. 8). This appeared to align with her hypothesis, but she noted that 20 was not very many samples. She then collected another 80 samples and was surprised to see two greens become slightly less frequent than one of each color. She then collected an additional 200 trials which continued this pattern. This caused Bridget to reconsider her original hypothesis. She said, "I don't really feel like that's my hypothesis anymore; I kind of want to let that go". Utilizing the simulation gave Bridget evidence to reject her idea that drawing two greens was the most likely outcome, but the simulation did not provide her with insight on what a new hypothesis might be. She said she wanted to "go back to the math" and consider how she might model the context.

Simulation to investigate/refine

The third distinct way that Bridget utilized simulations in her task approaches was as a tool for investigating and refining her ideas. This primarily occurred as she engaged with tasks involving an unknown distribution. In these instances, Bridget was content to present an approximation as her "final" solution. She gained confidence in her ideas/approximations with increasing data.

For example, as Bridget engaged with the Mystery Marble Bag task, which involved sampling from a hidden bag of marbles of various colors in CODAP, the mystery nature of the task led Bridget to immediately begin her investigation in the simulation. She collected 10 marbles, then 15, then 30, and finally several sets of 50 as she considered what she might be able to say about the bag's contents. Based on her first sample of 50 including 31 greens, and being told that the bag had 180 total marbles, she first predicted 111 green marbles using a proportion, stating that this would be "a rough estimate", and to get a more precise solution she could "do more samples". This eventually led her to take nine additional samples, calculate the average number of greens (32.4), and use a proportion to get a more "exact" estimate of 117 greens. Bridget believed that she could continue this process and make her estimate more precise, but seemed confident that this approximation was close to the true value. For Bridget, use of the simulation as a tool for investigation and refinement was characterized by increased confidence as she collected more data, a willingness to provide approximations as solutions, and a belief that while she could continue to collect data, what she had was enough to make tentative inferences.

Discussion/Limitations/Next Steps

This paper provides initial analysis of one participant's use of the designed computer simulations. Bridget utilized the CODAP simulations in three ways. In instances where she had a clear solution, she collected data to confirm her theoretical ideas. She also utilized simulation as a tool for testing her ideas when she had less confidence; this provided evidence as to whether her hypothesis had merit or if she should let it go. Finally, in cases where Bridget considered an unknown distribution, she investigated the context and gained confidence through collecting data to refine her ideas. She was content in these instances to provide approximate values as her solutions. Understanding the ways that learners may utilize simulation technology in probabilistic and stochastic settings is important for researchers and teachers as they attempt to understand student thinking in these areas and develop appropriate simulation-based activities. For example, collecting data in pursuit of a specific results (simulation to confirm) is not acceptable practice when performing statistical analysis and inference; this is something teachers and researchers may need to consider further.

However, this analysis represents only one learner's approaches, limiting the scope of these tentative/preliminary conclusions. The research team is currently considering the other four participants and how they compare to Bridget. Initial analysis has shown some similarities to her work (especially the use of the simulation as a confirmation tool), but more in-depth analysis is needed. In addition, this study considered one specific group of learners (secondary preservice mathematics teachers). In order to create a more comprehensive model of how learners use computer simulations in probabilistic/stochastic settings, future iterations of this design research will investigate different groups (for example K-12 students, other groups of undergraduates) and contexts (for example students working in pairs/groups or in a classroom setting).

References

Bumbacher, E., Salehi, S., Wieman, C., & Blikstein, P. (2018). Tools for Science Inquiry Learning: Tool Affordances, Experimentation Strategies, and Conceptual Understanding. *Journal of Science Education and Technology*, *27*(3), 215–235. https://doi.org/10.1007/s10956-017-9719-8

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher*, *32*(1,), 9–13.

English, L. D., & Watson, J. M. (2016). Development of Probabilistic Understanding in Fourth Grade. *Journal for Research in Mathematics Education*, *47*(1), 28. https://doi.org/10.5951/jresematheduc.47.1.0028

Ferreira, R. D. S., Kataoka, V. Y., & Karper, M. (2014). Teaching probability with the support of the R statistical software. *Statistics Education Research Journal*, *13*(2), 135–147.

Garfield, J., delMas, R., & Zieffler, A. (2012). Developing statistical modelers and thinkers in an introductory, tertiary-level statistics course. *ZDM*, *44*(7), 883–898. https://doi.org/10.1007/s11858-012-0447-5

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In *Handbook of research design in mathematics and science education* (pp. 517–545).

Ireland, S., & Watson, J. (2009). Building a connection between experimental and theoretical aspects of probability. *International Electronic Journal of Mathematics Education*, *4*(3).

Kazak, S., Fujita, T., & Wegerif, R. (2016). Students' informal inference about the binomial distribution of "bunny hops": A dialogic perspective. *Statistics Education Research Journal*, *15*(2), 46–61.

Kazak, S., Wegerif, R., & Fujita, T. (2015). The importance of dialogic processes to conceptual development in mathematics. *Educational Studies in Mathematics*, *90*(2), 105–120. https://doi.org/10.1007/s10649-015-9618-y

Konold, C., & Kazak, S. (2008). Reconnecting Data and Chance. *Technology Innovations in Statistics Education*, *2*(1).

Konold, C., Madden, S., Pollatsek, A., Pfannkuch, M., Wild, C., Ziedins, I., Finzer, W., Horton, N. J., & Kazak, S. (2011). Conceptual Challenges in Coordinating Theoretical and Data-centered Estimates of Probability. *Mathematical Thinking and Learning*, *13*(1–2), 68–86. https://doi.org/10.1080/10986065.2011.538299

Lane, D. M., & Peres, S. C. (2006). Interactive simulations in the teaching of statistics: Promise and pitfalls. *Proceedings of the Seventh International Conference on Teaching Statistics: Working Cooperatively in Statistics Education*, 6.

Saldanha, L. A., & Thompson, P. W. (2014). Conceptual issues in understanding the inner logic of statistical inference: Insights from two teaching experiments. *The Journal of Mathematical Behavior*, *35*, 1–30. https://doi.org/10.1016/j.jmathb.2014.03.001

Sharma, S. (2016). Probability from a socio-cultural perspective. *Statistics Education Research Journal*, *15*(2), 126–144.

Stohl, H., & Tarr, J. E. (2002). Developing notions of inference using probability simulation tools. *Journal of Mathematical Behavior*, *21*, 319–337.

van Dijke-Droogers, M., Drijvers, P., & Bakker, A. (2020). Repeated sampling with a black box to make informal statistical inference accessible. *Mathematical Thinking and Learning*, *22*(2), 116–138. https://doi.org/10.1080/10986065.2019.1617025

Vygotsky, L. S. (1987). Thinking and Speech. In R. W. Rieber & Carton, A. S. (Eds.), *The Collected Works of L. S. Vygotsky* (Vol. 1). Plenum Press.

Watson, J., & Chance, B. (2012). Building intuitions about statistical inference based on resampling. *Australian Senior Mathematics Journal*, *26*(1), 6–18.

INVESTIGATING CRITICAL STATISTICAL LITERACY HABITS OF MIND

Nina G. Bailey
University of North Carolina at Charlotte
nbaile15@uncc.edu

As data drive our world and decisions, there has been a consistent call for statistical literacy (SL) and data science in K-12 mathematics by researchers and professional organizations over the last few decades (e.g., Craig, 2018; Gal, 2002; Gewertz, 2020; National Governors Association Center for Best Practice & Council of Chief State School Officers, 2010; Wallman, 1993). More recently, this call has shifted toward including SL from a consumer orientation and has been amplified by the need to consume statistical information and avert misinformation amidst the COVID-19 pandemic (e.g., Engledowl & Weiland, 2021; Watson & Callingham, 2020). SL is conceptualized from the consumer orientation as the skills needed to effectively make sense of statistical messages in the real world. Despite this growing emphasis on SL and data science, there is a disconnect between the skills needed to effectively consume real world statistical messages and what is taught in schools (e.g., Nicholson et al., 2019). All of which illuminates the need for articulating Critical Statistical Literacy Habits of Mind (CSLHM) for making sense of such messages, particularly using a critical lens. The purpose of this study is to explore the similarities and differences of how people enact CSLHM when presented with data representations from the media.

This study draws on Gal's (2002) SL model and Weiland's (2017) framing of critical statistical literacy (CSL). Gal's work provided a solid starting point for describing the habits of mind needed to enact CSL. Weiland's CSL framing more explicitly stressed the importance of placing emphasis on sociopolitical inequity and the actions needed to disrupt and dismantle such inequity. Specifically, I use the CSLHM (Authors, 2019, under review) which operationalizes the habits of mind needed to enact CSL. The CSLHM framework has seven components: questioning sample size and methods, recognizing appropriate statistics, desiring additional information, acknowledging alternate explanations, recognition of one's own sociopolitical consciousness, employing active citizenry, and acknowledging ethical considerations.

In this instrumental multiple case study (Yin, 2018) there were four cases; each defined as a group of individuals with similar backgrounds with respect to statistical knowledge, statistical self-efficacy (Finney & Schraw, 2003), and critical consciousness (Diemer et al., 2020). There were five participants in each case. Each participant took part in a semi-structured task-based interview (Goldin, 2000) in which they were asked to make sense of data representations shared on Twitter. All tweets included a static or interactive graph and were related to social justice issues. Data were first coded using the CSLHM framework (DeCuir-Gunby et al., 2011). Next, I looked for emerging themes across the codes to write up case summaries. Preliminary findings suggest that statistical knowledge, statistical self-efficacy, and critical consciousness appear to influence the enactment of CLSHM. The poster will share further detail on the operationalization of Gal's (2002) and Weiland's (2017) framings to develop the CSLHM, the case criteria, findings specific to each case, a cross case comparison, and implications for how to best support learners in developing CSLHM in any environment.

References

Bailey, N., & McCulloch, A. W. (2019). Undergraduates' expression of critical statistical literacy. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 375–379).

University of Missouri.
Bailey, N. G., & McCulloch, A. W. (under review). Describing Critical Statistical Literacy Habits of Mind.
Craig, J. (2018). The promises of numeracy. *Educational Studies in Mathematics*, *99*(1), 57–71. https://doi.org/10.1007/s10649-018-9824-5
DeCuir-Gunby, J. T., Marshall, P. L., & McCulloch, A. W. (2011). Developing and using a codebook for the analysis of interview data: An example from a professional development research project. *Field Methods*, *23*(2), 136–155. https://doi.org/10.1177/1525822X10388468
Diemer, M. A., Frisby, M. B., Pinedo, A., Bardelli, E., Elliot, E., Harris, E., McAlister, S., & Voight, A. M. (2020). Development of the Short Critical Consciousness Scale (ShoCCS). *Applied Developmental Science*, 1–17. https://doi.org/10.1080/10888691.2020.1834394
Engledowl, C., & Weiland, T. (2021). Data (Mis) representation and COVID-19: Leveraging misleading data visualizations for developing statistical literacy across grades 6–16. *Journal of Statistics and Data Science Education, 29*(2), 160–164. https://doi.org/10.1080/26939169.2021.1915215
Finney, S. J., & Schraw, G. (2003). Self-efficacy beliefs in college statistics courses. *Contemporary Educational Psychology*, *28*(2), 161–186. https://doi.org/10.1016/S0361-476X(02)00015-2
Gal, I. (2002). Adults' statistical literacy: Meanings, components, responsibilities. *International Statistical Review, 70*(1), 1–51. https://doi-org.librarylink.uncc.edu/10.1111/j.1751-5823.2002.tb00336.x
Gewertz, C. (2020, February). Teaching students to wrangle 'big data'. *Education Week*. https://www.edweek.org/teaching-learning/teaching-students-to-wrangle-big-data/2020/02
Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates.
National Governors Association Center for Best Practice & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*.
Nicholson, J., Gal, I., & Ridgway, J. (2019). Understanding civic statistics: A conceptual framework and its educational applications. *A product of the ProCivicStat Project. http://IASE-web.org/ISLP/PCS*.
Wallman, K. K. (1993). Enhancing statistical literacy: Enriching our society. *Journal of the American Statistical Association, 88*(421), 1–8. https://doi.org/10.1080/01621459.1993.10594283
Watson, J., & Callingham, R. (2020). COVID-19 and the need for statistical literacy. *Australian Mathematics Education Journal, 2*(2), 16–21. https://search.informit.org/doi/10.3316/informit.287208637600347
Weiland, T. (2017). Problematizing statistical literacy: An intersection of critical and statistical literacies. *Educational Studies in Mathematics, 96*(1), 33–47. https://doi.org/10.1007/s10649-017-9764-5
Yin, R. (2018). *Case study research and applications: Design and methods* (6th ed.). Sage Publications, Inc.

DATA SCIENCE STUDENTS' DEVELOPMENT OF COMPUTATIONAL ACTION: A DESIGN-BASED RESEARCH STUDY

Karle Flanagan
University of Illinois at Urbana-Champaign
kflan@illinois.edu

Keywords: Data Analysis and Statistics, Social Justice, Design Experiments

Poster Proposal

In today's age of information and data, data science literacy is necessary. Students in all disciplines need basic data science skills to be successful at their jobs. The need for data scientists has expanded well beyond the tech industry to many other disciplines, for example business and the humanities (Irizarry, 2020). There is widespread acknowledgment that the job market for people who have data science skills is strong, and there is evidence that demand for this type of labor far exceeds supply (Baumer, 2015). However, data science skills and conceptual understanding of data science topics are not only necessary to prepare students for being employees, but also for being citizens (Finzer, 2013). In other words, it is important for college students to become critical consumers and producers of data who know how to answer real world questions, think about the implications of data science, and make decisions under uncertainty both in the workforce and in their everyday life. The skills that students learn in their data science courses empower them to understand the world, question the status quo, and think about important issues of social justice and equity. However, social justice issues are typically not emphasized in data science courses. In their recent book, *Data Feminism*, D'ignazio and Klein (2020) discuss how data science needs feminism and that we need to rethink the way that data science is taught.

Using this book as motivation, I designed a curricular study using design-based research methods (Brown, 1992; Collins, 1992) to improve the labs in a large college-level introductory data science course. Using the theory of distributed cognition, I designed labs that contained scaffolds and artifacts that help students to develop communication skills and increase awareness of using data science for social justice. The goal was for students to gain computational action by engaging with coding exercises, individual reflections, and group discussion questions in the labs. Computational action is a relatively new construct for computing education that emphasizes the idea of ensuring the skills that students learn inside the classroom can help them outside the classroom (Tissenbaum et al., 2019). The labs were designed to be collaborative, and the students worked together in small groups to use data science to explore social justice issues. I specifically focused on two labs that occurred during week five and week nine of the 16-week semester. The first lab covered visual displays of data and explored salary discrepancies among different genders. The second lab covered simulation and had the students simulate jury selection and analyze a trial where the random jury selection was in question. Throughout the iterations of this study, I collected pre and post lab survey data, audio recordings to understand how the students gained computational action during the labs, and I conducted interviews with the students after completing these labs. I had two research questions guiding this study. The first looked at the design process and the second looked at how the students gained computational action. Most of the data collected was qualitative and was analyzed using different types of discourse analysis and thematic analysis (Braun & Clarke, 2006). The outcome of this study is a set of design principles for improving labs in data science courses.

References

Baumer, B. (2015). A data science course for undergraduates: Thinking with data. *The American Statistician*, *69*(4), 334-342.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology*, *3*(2), 77-101.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences*, *2*(2), 141-178.

Collins, A. (1992). Toward a design science of education. In *New directions in educational technology* (pp. 15-22). Springer, Berlin, Heidelberg.

D'ignazio, C., & Klein, L. F. (2020). *Data Feminism*. MIT press.

Finzer, W. (2013). The data science education dilemma. *Technology Innovations in Statistics Education*, *7*(2).

Irizarry, R. A. (2020). The role of academia in data science education. *Harvard Data Science Review*, *2*(1).

Tissenbaum, M., Sheldon, J., & Abelson, H. (2019). From computational thinking to computational action. *Communications of the ACM*, *62*(3), 34-36.

CREATING DATA STORIES: STUDENTS' REASONING SKILLS WHEN WORKING WITH AN ONLINE PLATFORM

Franziska Peterson
University of Maine
Franziska.peterson@maine.edu

Keywords: Data Analysis and Statistics, Integrated STEM / STEAM, Technology

The purpose of this preliminary study was to investigate how students in one 9th grade earth science classroom reason about data when constructing evidence-based scientific explanations during Data Story assignments using an online data platform. A critical component of science learning is developing the ability to obtain meaning from a set of data, critically evaluate it, make a claim, and effectively communicate scientific ideas (National Research Council, 2012). Multiple studies have shown that students find it difficult to connect data to the real world (Pfannkuch, Regan, Wild, & Horton, 2010; Whitacre & Saul, 2016).

A Data Story is an interdisciplinary, scaffolded written argumentation assignment that requires students to analyze and graph authentic, real-world data to draw their own conclusions. Students worked with datasets assigned via TuvaLabs.com, a commercial data visualization platform, to analyze authentic, real-world data, select graphs, and to draw their own conclusions (Tuva, 2022). It allows students to easily explore and manipulate data by dragging and dropping variables on axes and selecting any kind of graph or statistical calculation. Students had to select an earth science dataset and construct a Data Story for their chosen data set. All Data Story assignments were analyzed with a rubric and then four purposefully selected students were chosen for follow-up interviews who represented a range of scores from high to low. During the interview, students were asked to explain how they created their Data Stories, and to create one while the interviewer was watching and share their thinking in the moment. These interviews were transcribed and analyzed using coding techniques guided by grounded theory (Corbin & Strauss, 2008) to identify themes.

Findings indicate that how students approach the variables in the data sets from the online platform, and what preconceived notions about graphs they have influence the quality of their Data Stories and reasoning skills. The student who created the most successful story, looked through all possible variables in the data set and thought about which variables could tell an interesting story. The student also mentioned things like, "If I was the scientist who collected this data, what would I be interested in?" They would drag and drop the carefully chosen variables on the axes and interpret the graph. Students who were less successful with their Data Stories tended to drag and drop variables on the axes seemingly at random until they liked how the graph looked. They would then try and interpret their graph. This often led to variable choices that did not tell a sensible story about the science data. Thus, thinking about the variables within the science context seems to be an important step that needs to be explicit when working with online platforms. Students also shared that they 'had to find a correlation'. They wanted their graphs to fit a picture they had in mind. Students did not seem to be comfortable with data that showed relationships that differed from correlated variables. These findings are currently being used to inform teacher professional learning experiences. Some science teachers might not be aware of what types of challenges students can face when reasoning quantitatively about data from a mathematical perspective. Thus, helping science teachers develop more interdisciplinary teaching skills can enhance their students' quantitative reasoning and data literacy skills.

References

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage Publications.

National Research Council. (2012). *A Framework for K-12 Science Education: Practices, Crosscutting Concepts, and Core Ideas* (Committee on a Conceptual Framework for New K-12 Science Education. Board on Science Education. Division of Behavioral and Social Sciences and Education, Ed.). Washington, D.C.: The National Academies Press.

Pfannkuch, M., Regan, M., Wild, C., & Horton, N. J. (2010). Telling data stories: Essential dialogues for comparative reasoning. *Journal of Statistics Education, 18*(1), 1–38. https://doi.org/10.1080/10691898.2010.11889479

Tuva (2022). Retrieved from https://tuvalabs.com

Whitacre, M. P., & Saul, E. W. (2016). High school girls' interpretations of science graphs: Exploring complex visual and natural language hybrid text. *International Journal of Science and Mathematics Education, 14*(8), 1387–1406. https://doi.org/10.1007/s10763-015-9677-7

EXAMINING THE ROLE OF SUBSCRIBED IDENTITIES IN SOCIAL CONTEXT DATA ANALYSIS TASKS

Jennifer M. Seat
Middle Tennessee State University
jms2dr@mtmail.mtsu.edu

Keywords: Data Analysis and Statistics, Social Justice, Technology, Affect, Emotion, Beliefs, and Attitudes

Recent world events have focused on current systemic inequities present in society. These problems are not new to scholars in education. One way to help students both explore and potentially overcome such systematic biases is Teaching Mathematics for Social Justice (TMSJ; Wager & Stinson, 2012), which has become a well-documented practice in mathematics education. Hackman (2005) claimed that understanding multiple perspectives and narratives allowed ideas like TMSJ to be successful. One way to include multiple perspectives is to use narratives from non-dominant sources. A second way is to explore the identities and experiences present in a classroom community. Hahn Tapper (2013) described identities as composed of two types of labels: ascribed and subscribed. Ascribed labels come to an individual from others and are typically related to appearance and status. Subscribed labels are labels individuals choose for themselves. Hahn Tapper (2013) stated that subscribed labels were more influential than ascribed labels in determining the effects on the behavior of individuals. One particular way to think about students' narratives is to focus on the roles that lived experiences and subscribed labels play in interactions with complex multivariate datasets.

Findings from the statistics education literature can help understand how lived experiences can influence student interaction in a social justice context. Wroughton and colleagues (2013) found that when the study's findings aligned with participant beliefs, the students were more likely to accept the sampling method, regardless of bias. Similarly, Queiroz and colleagues (2017) explored the role of affect in students' interpretation of social statistics. In this study, one striking example documented how a student nearing completion of formal training responded negatively to a statistic about bicycle safety due to personal experience with the loss of a loved one in a bicycle accident. These are single-point evaluations of fine-grained statistical content, but they indicate that statistics content and experience cannot be separated.

This interview-based study explored how participant subscribed identity influenced their interaction with a sizeable multivariate dataset. Participants were prompted with the general statement, "Use the provided data to find a noticeable difference or demonstrate there is no noticeable difference between groups represented in the data." Participants used variables associated with their subscribed identities to serve as an entry point to the task. This choice meant that variables selected by participants were related to their identities. This relationship between identity and selected variables gave a unique insight into how participants reason with data that contradicts their personal experiences. Participants' use of data to make sense of these contradictions takes different paths depending on the level of the contradiction. For minor discrepancies that can be considered only a small fraction of the data set, participants tended to dismiss the cases represented as different from the perceived trend. For more significant discrepancies that can be thought of as entire classes, participants were more actively engaged in developing explanations for why the class may not follow the anticipated pattern. These patterns can help understand how students may engage in TMSJ tasks that are based on data.

References

Hackman, H. W. (2005). Five Essential Components for Social Justice Education. *Equity & Excellence in Education*, *38*(2), 103–109. https://doi.org/10.1080/10665680590935034

Hahn Tapper, A. J. (2013). A pedagogy of social justice education: Social identity theory, intersectionality, and empowerment. *Conflict Resolution Quarterly*, *30*(4), 411–445.

Queiroz, T., Monteiro, C., Carvalho, L., & Francois, K. (2017). Interpetation of Statistical Data: The Importance of Affective Expressions. *Statistics Education Research Journal*, *16*(1), 163–180. eue.

Wager, A. A., & Stinson, D. W. (Eds.). (2012). *Teaching mathematics for social justice: Conversations with educators*. National Council of Teachers of Mathematics.

Wroughton, J. R., McGowan, H. M., Weiss, L. V., & Cope, T. M. (2013). Exploring the Role of Context in Students' Understanding of Sampling. *Statistics Education Research Journal*, *12*(2).

Chapter 14:

Student Learning and Related Factors

RELEVANCE AS PERCEIVED BY HIGH SCHOOL STUDENTS IN DECONTEXTUALIZED MATHEMATICS LESSONS

Aamina Adil
Boston University
aamina@bu.edu

Kihoon Lee
Boston University
klukelee@bu.edu

Leslie Dietiker
Boston University
dietiker@bu.edu

When mathematics educators work towards making mathematics more relevant, they often think about including more real-world applications into mathematics lessons. But what happens when a lesson is devoid of real-world contexts? In what ways can students find it relevant? This study explores how high school students perceived relevance when they were asked to describe their experiences during decontextualized mathematics lessons. Students highlighted how they found certain characteristics of the lessons to be useful in their learning and how they perceived relevance through different feelings experienced in the lessons. This, in turn, broadens our understanding of what relevance means to students.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Curriculum, High School Education

For many years, educators have attempted to incorporate relevant-increasing strategies and interventions in the classrooms, as they have been shown to increase student motivation and performance (e.g., Lazowski & Hulleman, 2016). However, mathematics educators often assume that to be relevant, mathematical content must be connected to real-life contexts with a wide emphasis on incorporating mathematical content that students can see and apply in the world around them (e.g., NRC, 2003; Wilkerson, 2021). While this association is important in its own regard in helping students connect with mathematics and see its importance in their everyday lives, it blinds us from looking at other ways of conceptualizing relevance. One way of expanding our understanding of relevance is to examine it from the perspective of students in order to develop better intervention strategies that will help them find meaning in what they learn (Albrecht & Karabenick, 2018).

This study forms a part of a larger research project which is studying the characteristics of decontextualized high school mathematics lessons that were specially designed to increase aesthetic opportunities for the students (referred to as *Mathematically Captivating Learning Experiences*, or MCLEs). Since the lessons were decontextualized, and because relevance is tightly connected with contextualization, we predicted that students would not find MCLEs relevant. However, surprisingly, using the relevance measure from surveys given to students after lessons, no association was found between the relevance measure and the type of lesson, suggesting that students viewed the decontextualized MCLEs to be as relevant as the non-MCLEs, some of which were contextualized (Dietiker et al., in progress). These results motivated us to look deeper into the views of the students regarding the decontextualized lessons to broaden conceptions of relevance from their perspective.

In this paper, we describe the different categories that emerged from student interviews conducted after the MCLEs, giving examples of their statements that explain what or how they found some aspect of the lesson as relevant, in order to begin to answer the question: *In what ways do students perceive decontextualized high school mathematics lessons to be relevant?* We end with a discussion on how the findings of our study can help expand perceptions of relevance, and what this can mean for the mathematics education community.

Theoretical Framework

Various definitions of relevance have surfaced within the past years (Albrecht & Karabenick, 2018). While some researchers relate relevance with utility value (e.g., Hulleman et al., 2017), others frame relevance in terms of meaningfulness (e.g., NRC, 2003) or the fulfillment of intrinsic and extrinsic goals (e.g., Keller, 1983). Schamber and Eisenberg (1988) argue that most of the notions used to describe relevance, such as usefulness and satisfaction, depend on the perception of an individual. By presenting relevance as a complex cognitive phenomenon which involves the relationship between an individual and their surroundings, Schamber and Eisenberg suggest exploring relevance by questioning the individuals to describe their own perceptions of how they form connections with the topic of information (Schamber & Eisenberg, 1988). In this paper, we define *relevance* as an individual's perception of how an experience was useful in meeting their needs and desires, such as how an activity evoked a sense of satisfaction, which further facilitated a student's learning. When the experience fails to fulfill the individual's needs or desires, then the perception of the relationship is defined as irrelevant.

Recent educational studies that have incorporated student perspectives related to relevance reflect similar notions of fulfillment of an individual's needs and desires. For example, the concept of satisfaction formed the basis for a study that asked college students to produce a list of strategies used by their instructors that increased course relevance for the students (Muddiman & Frymier, 2009). Drawing from Keller (1983), Muddiman and Frymier (2009) define relevance to be a "student's perception of whether course content *satisfies* personal needs, personal goals, and/or career goals" (p. 131, italics added). They identified four ways educational activities could be viewed as relevant by students: how the content was linked to examples drawn from students' lives beyond the classroom, how the teaching style of the instructor helped students relate with the material, methods and activities that encouraged participation and group work, and course supports provided by the teacher that helped them improve their performance. Similarly, Dobie (2019) questioned middle school students about their perceptions of how mathematics lessons were "useful" (p. 28). Student responses from this study fell under two broad categories: *applicability of content* and *features of the learning experience*. Both these studies suggest that in addition to relating relevance with how the content was applicable in their outside lives, students also found relevance in certain teaching strategies, methods, and activities.

We aim to add to what is known about what can make mathematics learning relevant by looking at the perceptions of high school students on mathematics lessons which were specially designed to be decontextualized.

Methods

The current study analyzes perceptions of relevance from statements students made when asked to describe important aspects of their learning experience during an MCLE. Eighteen MCLEs were implemented in six different classrooms from three high schools in the Northeastern region of the United States. One school was a small charter school in the city with a predominantly Latinx student population, the second school was a large urban public high school with diverse racial and ethnic groups, and the third school was a large suburban public high school with a predominantly white student population. Immediately after each MCLE, two to five students were selected as a representative sample in terms of gender and ethnicity and interviewed about their aesthetic experiences during the lesson. The audio recordings of the interviews were then transcribed for data analysis. The current study analyzed a total of 44 interview transcripts from ninth to twelfth grades in the 2018-19 academic year.

During the interviews, students were asked to describe their lesson experiences with the aid of, but not limited to, a given list of positive, neutral, and negative aesthetic descriptors. While none of the questions asked by the interviewers concerned relevance, student perspectives on certain needs and desires often emerged from student remarks.

The data analysis consisted of three major coding passes through student interview transcripts. Each pass consisted of multiple cycles where two coders first independently scrutinized student utterances and then came together to align any differences. During the first pass, the researchers read through the interviews to identify student utterances that included elements of perceived relevance or irrelevance, asking, "In what ways is this student expressing how the experience met their needs and/or desires (or not)?"

The major criteria for evaluating utterances for this pass included highlighting the connections to relevance by placing primary focus on the quality of usefulness. One student's remark clearly reflected one of such instances:

> We told each other what we knew about trigonometry before we went into the lesson. ... It kind of gives you an idea of how people around you know and, like, how they understand the topic. It makes you feel more safe talking to people about it.

For this student, the interaction with their peers through the conversations on the topic prior to the main lesson was useful as a way to understand how others perceived the topic. Such an interaction fulfilled their need by rendering a sense of safety within the in-class discussions.

Also included were statements where students lamented the lesson's failure to meet their needs or desires, which in turn implied how a different aspect of the lesson could have helped fulfill those missing needs or desires. For example, a student explained how they felt less curious in the lesson as they already had an idea of what they were learning, indicating the missed opportunity where the lesson could have been relevant for this student if their desire to learn content outside of their prior knowledge was fulfilled.

On the other hand, while student utterances with positive aesthetic reactions were initially selected as potential candidates for reflecting relevance, many of them were excluded because of a lack of conclusive connection to relevance. For example, a student stated: "I actually really enjoyed this lesson because it was, like, something I never thought would actually occur during a math problem, and the calculator didn't actually have another solution." While this student expressed excitement and enjoyment over the unexpected outcome of the calculator, these reactions did not indicate how this desire was useful to the student. Therefore, this statement was excluded from further analysis.

The first pass resulted in the identification of 55 utterances where students highlighted different aspects of the decontextualized lesson as being relevant to them. After the student utterances relating to relevance were identified, the researchers made a second pass at creating categories of relevance. Since our lessons were decontextualized, Dobie's (2019) major category of *applicability of content* was excluded from our framework and we started with her four subcategories: *method of interaction*, *structure of the activity*, *representation being utilized*, and *value of learning new or important things*. However, unlike Dobie's (2019) study, our student population was not asked explicitly about the ways in which they found the mathematics lessons relevant. In addition, all the MCLEs involved particular characteristics of sequencing activities in order to spark student curiosity with some portion of collaborative small-group problem solving. With such differing characteristics of our study, all of Dobie's (2019) categories needed adjustment to accurately represent the themes that emerged from the student statements. In particular, we modified the subcategory *structure of the activity* because, while some students found particular activities to be useful, a greater number of utterances highlighted that the *sequence* of activities throughout the lesson was perceived as relevant. Since the sequence of the activities was a feature of the entire lesson and not of a particular activity, we broadened the category as *structure of lesson*.

On the occasion of encountering utterances that expressed needs and desires that were not captured by the modified categories, we developed new categories to reflect the new themes, by looking for patterns across the interviews to find expressions of similar needs and desires. For example, when we noticed that multiple students described how they valued making meaningful progress in the lesson on their own, we created the category: *sense of accomplishment*.

With a new set of categories, a final coding pass of the data was carried out to solidify categories that captured the emerging themes of the selected utterances. We then selected representative student utterances for each finalized category.

Findings

The student perceptions of the relevance of decontextualized mathematics lessons can be classified into two broad categories: *characteristics of the lesson students perceive as relevant* and *ways in which relevance is experienced by students during the lesson*. Each of the broad categories consist of multiple subcategories. While there were many instances where students talked about what or how something was relevant, there was at least one instance in each subcategory where students expressed a lack of fulfillment of their needs or desires. Many ways students discussed relevance also fell under several subcategories, both within and across the two main categories. We now describe each category in greater depth.

Characteristics of the Lesson Students Perceive as Relevant

This category describes aspects of the lesson that students identified as enabling them to develop a deeper understanding of and connection with the subject of the lesson. Four subcategories emerged under this category: *interaction with others*, *representation*, *structure of lesson*, and *learning new and unexpected ideas*.

Interaction with others. Students stated that small group interactions with their classmates or guidance from their teacher was often useful in furthering their understanding or helping them get closer to an answer. This interaction was sometimes in the form of a remark or question from either their teacher or one of their peers, or even both, that redirected them to better ideas for solving a problem. For example, a student mentioned, "[The teacher] gave me an idea when she told us how to draw the figure, she came to my group and said, 'try to describe it like you're trying to describe it over the phone.'" Many students also valued collaborative activities that helped them discuss ideas and share thoughts with each other, as evident in this statement: "it was enjoyable too, cause I got to talk about it with the people in my group, … it helped me understand it better." This utterance highlights interactions with other students as an effective medium to fulfill the student's need for greater understanding.

Some students, though, further explained how certain features curtailed the effectiveness of these peer interactions. As one student stated: "When a student describes it, it actually confuses me more. And, like, the way that they explain it is confusing because they put a lot of words into it." For this student, the wordiness and ambiguity in the explanation provided by a fellow student did not allow the interaction to be effective. Thus, having clarity in the exchange of ideas and thoughts emerged as a feature that supported the interactions to be relevant for the students.

Representation. The different ways in which mathematical ideas were presented were often perceived by students as useful in helping them build conceptual understanding and figuring out how to solve problems. The following utterance highlights a student's appreciation for physical representations in developing understanding of the meaning of a formula:

> I like using the little triangles and the diagrams because it showed you how it was going to work instead of just saying "oh, this is the formula that she showed on the board with the 2x thing." Instead of showing us that and how it works, we get to see and physically feel how that works.

This student statement reflects the fulfillment of their need of a visual and tangible representation to understand the concept rather than simply being given a formula with which to work.

In addition, students also appreciated forms of representation that facilitated pattern recognition. For example, a pattern apparent in area models helped one student make sense of polynomial division. Other students highlighted how finding patterns usually meant that "something's right." A student avoided having "holes" (i.e., gaps) in tile patterns because "it wouldn't have looked nice I feel like. … it would have been like 'oh that looks wrong or something.'" One student struggled with "letters and words" that came up in a lesson on logarithm identities, but eventually figured her way out "because of the pattern that makes sense." For these students, finding patterns in these different representations turned out to be a useful way to evaluate their strategies while solving problems.

Structure of lesson. Students expressed how they valued several features involving the way mathematical content (e.g., statements, activities) unfolded across the lesson. They explained how revisiting a topic before diving into the actual lesson, how seeing their teacher perform a calculation, or how being able to derive formulas on their own instead of having them being written on the board increased their understanding of a topic. Several students appreciated having a "hands-on" activity in the lesson which involved cutting, drawing, or playing with shapes, as it increased engagement in the activity, which in turn facilitated their learning process. For instance, a student stated, "I like what we're doing, like, all these different activities and stuff, because if you make it fun then it helps us learn even more."

While these students described particular structural aspects of the lesson, such as a fun activity or a good launch, as being useful in furthering their learning, other students highlighted how the overall flow of the lesson contributed to a meaningful lesson experience. A student particularly pointed out how the slower pace of the lesson helped him process the new concepts: "It was, just, like, a smoother class. It went, like, slower, and, like, we actually took time to explain. And I'm like, 'okay, I understand this,' and I knew the new formula and all the stuff."

Students also described how their teacher can misdirect them into mistakenly assuming something which later results in a surprise or can question them in ways that build their curiosity, as captured by the following student remark: "I felt this lesson was pretty interesting because, like, we have expectations, and she, like, leads into our expectations and then we find out, like, it's wrong. And I'm like, 'now we need to learn why it's wrong.'" This element of unexpectedness that was achieved through the ways these lessons were structured turned out to be important in captivating students as it satisfied their desire to be curious about what they were learning.

Learning new and unexpected ideas. Many students expressed a desire to learn about new and surprising content and valued lessons that presented them with such opportunities. For example, a student mentioned how not knowing about the content beforehand pushed them to want to know more about a new idea: "Today I was a little bit curious because we didn't really know a whole lot about the material." Other students described how the mathematics lessons had shown concepts that were contrary to their previous understandings, therefore finding the value of unexpected ideas. Below is one example of such an instance:

> I would say [the lesson was] surprising because we had two solutions, and the calculator said something else. I would say, for once the calculator disagrees with us, or something like that, where the calculator's right, and we're wrong, or something like that. And I guess I would say [I was] frustrated at some points because it was like, "wait, but, like, how does that work?", which made me want to listen to [the teacher] more.

Notice that, in these student statements, their curiosity (i.e., their desire to learn more) was spurred through learning *new* (i.e., not knowing about the material) or *unexpected* (i.e., discrepancy between the solutions derived by the students and their calculator) ideas. In particular, as shown in the utterance above, such new and unexpected ideas can accompany the element of surprise.

On the other hand, the inverse of such instances also emerged in one interview, where one student explained how their existing content knowledge interfered with their learning experience:

> It was interesting to kinda, like, learn them, but at the same time, I kinda had a guideline already of what I was learning ... because, to find triangles, it's always base times height divided by 2. [Having prior knowledge made me] less curious, because, like, I already knew what it was.

Had the element of new and unexpected ideas been present in this lesson, this student's desire to learn something outside of "the guideline" may have been fulfilled.

Ways in which Relevance is Experienced by Students during the Lesson

This category explains the ways in which students perceived a lesson experience as useful or satisfying. All such remarks emerged as feelings, which were subcategorized into: *sense of accomplishment* and *sense of belonging*.

Sense of accomplishment. Many students perceived resolving challenging tasks on their own or getting them right as a feeling of accomplishment. In particular, while getting the correct answer was mentioned and valued by many students, the sense of achievement was not simply derived from getting the correct answer; it also involved making meaningful progress in spite of prior negative feelings (e.g., fear, frustration). The student utterance below highlights one such instance:

> I began the class period very frustrated and uninteresting because I didn't get the answer right away, like I usually do. But then, towards the end, I was mostly amused. … I actually got an answer at the end. … Any time that I have a problem, and I don't get it right away, and it takes me some time, like it did today, I feel like once I get it right, even if it's something small, then I'll get excited."

Although this student was initially frustrated, their progress towards the solution, despite being small and time-consuming, evoked a sense of accomplishment.

Likewise, the emergence of a self-belief that they were able to "figure it out" (i.e., the belief that the fear of getting something wrong could be surmounted) was the main element that contributed to students' sense of accomplishment. One student, after explaining how they were "just kind of nervous" and worried "what if that was wrong?", continued:

> And then I was like, “look! this is, like,” I'm thinking, “this is similar to what we were looking for, even if it was wrong,” I was thinking. And I figured out something, and I was like, “oh, that's interesting” … I was proud.

This student came to a realization that even if their answer was wrong, it could work as a meaningful aspect of problem-solving. Such remarks often involved explanations of how they were “proud” that they were able to do something on their own and contribute to something worthwhile.

Sense of belonging. Having a sense of belonging within the classroom or a student group was also identified as a way for students to perceive relevance. Their desire to not be alone was fulfilled by finding common interests or sharing a wholesome and engaging class experience with their peers. One student explained how often their peers’ disengagement to the class prevented their learning: “some people are just, like, sleeping, they're just tired, ... some people have side conversations [when] I'm trying to learn.” However, they proceeded to indicate how, in this particular lesson, their classmates were more engaged, which created a “drive to actually understand [the lesson].” In other words, for this student, what they desired and needed was to be in sync with their classmates and to have a sense of community.

Other times, such a sense of belonging accompanied other aforementioned categories such as *interaction with others* or *structure of lesson*. For example:

> I felt more comfortable with [this way of teaching], and helped me learn, and like working in groups helped me, like, interact with other students, and, like, hearing their ideas and, like, make everything, like, make sense, putting it all together. … So, working, like, with other students and the teacher going back and forth, like, working in pairs and the teacher explains and then, like, discussing with the whole class. It, like, makes me feel comfortable in the class. Like, sometimes I'm in class, [it] can get kind of stressful, and it makes you feel like you're with other people and you're not alone, and it just feels like relief. … We had, like, different ideas and points of views, and then going on the paper and actually doing it out and realizing, like, what each person was thinking and seeing what they thought and what I thought, like, I feel like that really showed how people think and how you can, like, come together as a team, as a group.

The above statement captures the way in which the structure of the lesson (i.e., collaborative work followed by teacher explanations and then a whole class discussion) facilitated interaction with other students (i.e., a discussion of different thoughts and ideas), leading to a sense of belonging (i.e., feeling comfortable and relieved):

Overall, this category highlights the importance of having a community where students can come together, even with different perspectives, leading to a sense of relief and comfortableness.

Discussion

We are gratified to see that high school students can find mathematics lessons meaningful and are able to connect with them, even when the content is decontextualized. While we do not discount the importance of using real-world applications in our mathematics lessons to allow students to see mathematics in the world around them, our research sheds light on other ways mathematics can be perceived as relevant by students. This new way of looking at relevance opens up ways for mathematics educators to rethink pedagogical approaches that aim to increase relevance in classrooms. For example, teachers can structure lessons in ways that present elements of surprise, as our research suggests that students tend to value content that goes against their expectations, and that surprise often led to an increased curiosity in what they were learning. Moreover, since our study concentrated on high school lessons only, the varied perceptions of relevance pave the way for making even abstract advanced mathematics content meaningful to students, without necessarily making a connection to the real world — especially when such a connection is difficult to make or is divergent from students' actual life experiences. Furthermore, many of the students connected relevance with positive aesthetic experiences and identified irrelevance with negative or neutral feelings, suggesting a potential connection between aesthetic experiences and perceptions of relevance.

While our study identifies varied perceptions of relevance, the nature of our study presents some limitations. Although our schools had diverse student populations, our data only comes from schools located in the Northeastern region of the United States. Furthermore, since we were able to interview only a few students after each lesson, our study potentially limits the number of different views obtained. More research with varied school settings could help expand these findings.

We hope that this research study will help the mathematics education community recognize the ways relevance can be conceptualized from the perspectives of students, in turn making mathematics more meaningful for all students.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1652513. We would also like to thank our teacher participants and other research team members for their help in collecting and organizing the data and giving valuable feedback for this study.

References

Albrecht, J. R., & Karabenick, S. A. (2018). Relevance for Learning and Motivation in Education. *The Journal of Experimental Education*, *86*(1), 1–10. https://doi.org/10.1080/00220973.2017.1380593

Dietiker, L., Riling, M., Singh, R., Nieves, H. I., & Barno, E. (in progress). The Aesthetic Effects of a New Lesson Design Approach: Mathematical Stories.

Dobie, T. E. (2019). Expanding conceptions of utility: Middle school students' perspectives on the usefulness of mathematics. *Mathematical Thinking and Learning*, *21*(1), 28–53. https://doi.org/10.1080/10986065.2019.1564969

Hulleman, C. S., Kosovich, J. J., Barron, K. E., & Daniel, D. B. (2017). Making connections: Replicating and extending the utility value intervention in the classroom. *Journal of Educational Psychology*, *109*(3), 387–404. https://doi.org/10.1037/edu0000146

Keller, J. M. (1983). Motivational Design of Instruction. In C. M. Reigeluth (Ed.), *Instructional-design theories and models* (pp. 383–434). Lawrence Erlbaum Associates.

Lazowski, R. A., & Hulleman, C. S. (2016). Motivation Interventions in Education: A Meta-Analytic Review. *Review of Educational Research*, *86*(2), 602–640. https://doi.org/10.3102/0034654315617832

Muddiman, A., & Frymier, A. B. (2009). What is Relevant? Student Perceptions of Relevance Strategies in College Classrooms. *Communication Studies*, *60*(2), 130–146. https://doi.org/10.1080/10510970902834866

NRC. (2003). *Engaging Schools: Fostering High School Students' Motivation to Learn* (p. 10421). National Academies Press. https://doi.org/10.17226/10421

Schamber, L., & Eisenberg, M. (1988, October 23). *Relevance: The Search for a Definition*. 51th Annual Meeting of the American Society for Information Science, Atlanta, GA.

Wilkerson, T. (2021, October). *Mathematics with Rigor, Relevance, and Responsiveness—National Council of Teachers of Mathematics* [Blog]. President's Message. https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Trena-Wilkerson/Mathematics--with-Rigor,-Relevance,-and-Responsiveness/

TOWARDS AN ELABORATION OF CONCRETENESS FADING: REFLECTIONS ON A CONSTRUCTIVIST TEACHING EXPERIMENT

Joseph Antonides
Virginia Tech
jantonides@vt.edu

Michael T. Battista
The Ohio State University
battista.23@osu.edu

Over half a century has passed since Bruner suggested his three-stage enactive-iconic-symbolic model of instruction. In more recent research, predominantly in educational psychology, Bruner's model has been reformulated into the theory of instruction known as concreteness fading (CF). In a recent constructivist teaching experiment investigating two undergraduate students' combinatorial reasoning, we utilized an instructional approach that maintains the enactive-iconic-symbolic stages of CF, but through a gradual and much elaborated process. We found that our theory of levels of abstraction explicated the "fading" effect that is central to CF. In this theoretical report, we discuss how CF can be elaborated by our instructional approach and theoretical perspective.

Keywords: Learning Theory; Instructional Activities and Practices; Advanced Mathematical Thinking; Mathematical Representations

Introduction

Substantial research points to the potential affordances of using manipulatives in mathematics teaching (Bouck & Park, 2018; Carbonneau et al., 2013; Domino, 2010; Moyer-Packenham & Westenskow, 2013; Peltier et al., 2020). However, *how* should manipulatives be used to benefit student learning? From a constructivist perspective, manipulatives—by which we mean physical or virtual objects on which sensory-motor actions may be performed—do not carry inherent mathematical meaning (Ball, 1992; Wheatley, 1992). Students must *construct* the meanings that they come to associate with representational forms, even when, from a more knowledgeable person's perspective, those forms "look like" the concepts they are intended to represent. Thus, the shift from using manipulatives to using formal *symbols* for a given concept needs to be given careful consideration (Clements & McMillen, 1996; Fennema, 1972; Resnick & Omanson, 1987).

Research suggests one effective means for introducing abstract symbols and ways of operating on them in ways that are meaningful to students is the three-stage enactive-iconic-symbolic "concreteness fading" (CF) instructional model, originally proposed by Bruner (1966), and similarly the concrete-representational-abstract (CRA) model. Within the CF model, a concept is first represented using "concrete" materials on which students may perform sensory-motor actions, followed by "iconic" representations which may include graphic or pictorial forms, and lastly "symbolic" representations such as words or letters for the concept.

The purpose of this theoretical paper is to suggest a potential elaboration of the CF model using a theory of levels of abstraction (Battista, 2007), and an elaboration that emerged from a teaching experiment investigating two preservice teachers' combinatorial reasoning (Antonides & Battista, under review). Our instructional approach utilized concrete/enactive tasks in that students were asked to enumerate permutations represented as "towers" by constructing towers using physical, multi-colored connecting cubes (cf. Maher et al., 2011). Our students used these manipulatives to enumerate towers 3-cubes, 4-cubes, and 5-cubes-high, all the while constructing numerical symbols and computational expressions that were explicitly linked to their tower constructions. Gradually, the students' reasoning shifted from operating on towers to operating primarily on symbolic representations, which they could later use to reason in novel situations. This theoretical report seeks to establish two claims: (a) that our instructional approach represents a case of a much-elaborated instantiation of CF, and (b) that our theoretical framework focusing on students' levels of abstraction serves to explicate the "fading" effect that is central to CF.

Concreteness Fading and Related Perspectives

Bruner argued for a theory of instruction that includes three broad representational forms: enactive, iconic, and symbolic. According to Bruner (1964), "Their appearance in the life of the child is in that order, each depending upon the previous one for its development, yet all of them remaining more or less intact throughout life" (p. 2). *Enactive representations* are characterized by sensory-motor actions within experiential situations; Bruner suggests examples of riding a bicycle, tying knots, or driving a car. *Iconic representations* "[summarize] events by the selective organization of percepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images" (p. 2). To reason about an experience, such as riding a bicycle or tying a knot, a student can call forth internalized mental representations as material on which to operate. *Symbolic representations* include, in particular, words that are used to point to particular conceptual referents. Symbols, unlike icons, typically do not bear a perceptual resemblance to the objects that they represent.

Goldstone and Son (2005) introduced the term "concreteness fading" to refer to the process of successively decreasing the level of concreteness of a simulation for a scientific concept, with the eventual goal of "attaining a relatively idealized and decontextualized representation that is still clearly connected to the physical situation that it models" (p. 70). While Goldstone and Son related CF to Bruner's theory of instruction, their formulation of CF did not specify a three-stage representational sequence.

McNeil and Fyfe (2012) formulated CF as a three-step gradual fading process, similar to Bruner's recommended model. They conducted the first study to experimentally test the benefits of such a three-stage progression, specifically within the context of undergraduate students' learning and transfer of the properties of an abelian group of order three (associativity and commutativity, and the existence of inverse elements and an identity element). The students were randomly placed into one of three instructional conditions: generic, which used abstract symbols (two-dimensional shapes); concrete, which used iconic representations of measuring cups; and fading, which used a concrete-to-generic approach with an explicit linking of the two representations through an intermediary, Roman numeral-based representation. Their results showed that students in the fading condition performed significantly better than students in the generic or concrete conditions.

Notably, McNeil and Fyfe's "concrete" condition did not involve perceptual materials on which sensory-motor actions were performed—a distinction from Bruner's characterization of the enactive. However, Bruner's focus in his original formulation of the three-stage model seemed to be on *children's* intellectual development, whereas McNeil and Fyfe's study focused on undergraduates. This raises important questions about the nature of "concrete" and "enactive" representations. What do these terms mean? Fyfe and Nathan (2019) "use the term concrete representation to refer to any external representation" (p. 410), which they suggest may vary along at least two dimensions: physicality and perceptual richness. The physicality of a concrete representation refers to whether it is two-dimensional (such as a drawing, consistent with Bruner's term iconic) or three-dimensional (such as physical cubes). Perceptual richness typically refers to the visual features maintained by a representation, such as colors, patterns, or texture. Fyfe and Nathan defined CF "as the three-step progression by which a concrete representation of a concept is explicitly faded into a generic, idealised representation of that same concept," which they suggested is accomplished "by removing perceptual and conceptual information either within or across lessons until one arrives at the representation that involves the least effort to infer and generalise the invariant relation" (p. 411). Fyfe et al. (2014) also suggested "fading" may occur "by encouraging structural recognition and alignment" (p. 19).

CF is related to additional theories of instruction and learning. For instance, CF is a form of the more general notion of progressive formalization (PF), a model of instruction in which students first gain experience about a concept through concrete materials, then transition to operating on symbols that are conceptually grounded in these initial concrete experiences (Nathan, 2012). According to Nathan, the PF model combines advantages of both concrete and abstract representations:

> Concrete entities are meaningful to learners early on and so provide accessible entry points, abstractions transcend the applicability of the representations and rules from any one context, and *grounded abstractions* support learners understanding of what the formalisms "say" and how they apply widely to new application areas. (p. 139)

CF is also related to the types of mathematical activity suggested by Gravemeijer (1999; see also Gravemeijer, 2002). Students operating at the level of referential activity use models that are conceptually grounded in experientially real task settings. At the level of general activity, students transcend from operating with "models of" to operating with "models for," meaning "students' reasoning loses its dependency on situation-specific imagery," which Gravemeijer suggests "can be seen as a process of reification" (p. 164).

Context of Our Proposed Elaboration

We conducted one-on-one constructivist teaching experiments (Steffe & Thompson, 2000) with two preservice middle school teachers with the goal of developing second-order models of the students' concepts and actions/operations for enumerating permutations. To help make our students' reasoning salient, and to provide sensory-motor experiences that we hypothesized would serve to conceptually ground our students' developing mathematical meanings, the tasks included in our study generally represented permutations of n objects as n-cube "towers," each tower comprised of n different colors of cubes connected together with a vertical spatial orientation (see also Maher et al., 2011).

As noted in the Introduction, we claim that our instructional approach represents an instantiation of CF. Indeed, both students (DC and NK, neither of whom had studied combinatorics previously) initially counted tower possibilities by constructing towers 1-by-1 using the available perceptual materials—consistent with Bruner's descriptions of *enactive* representations, as well as Fyfe and Nathan's (2019) definition of the broader term, *concrete*. As the tasks became increasingly complex, DC and NK transitioned from relying on 1-by-1 construction techniques alone to constructing partial sets of towers before generalizing multiplicatively. They could mentally imagine towers that they had not yet constructed, consistent with Bruner's meaning of the term *iconic* representation. Each student progressed to constructing multiplicative operations without needing to first operate on towers, but justified using (and thus conceptually grounded in) this sensory-motor cube-towers context. This is consistent with Bruner's characterization of *symbolic* representations, as well as Nathan's (2012) description of abstract representations in the PF model of instruction. Ultimately, each student was able to construct, through a process of guided reinvention (Gravemeijer, 1999), a generalized formula for counting permutations. The instructional sequence through which our students constructed their formulas, and a brief summary of each student's reasoning, are provided in Table 1. Note that students were provided with physical cubes to construct 3-, 4-, and 5-cube towers, and more than enough cubes were available to construct all 3- and 4-square towers (but not 5-square towers, of which there were 120 possibilities).

Table 1: Instructional Sequence and Summary of Students' Reasoning

Task	Summary of DC's Reasoning	Summary of NK's Reasoning
1. Counting 3-cube towers each containing 3 colors of cubes	Constructed towers systematically one-by-one using physical cubes	Constructed towers systematically one-by-one using physical cubes
2. Counting 4-cube towers each containing 4 colors of cubes	Constructed 12 towers one-by-one, organized by base color, then multiplied $6\times4=24$	Reasoned there are six 3-cube towers for any 3 colors, and 4 possible top-cube colors, so there are $6\times4=24$ towers
3. Counting 5-cube towers each containing 5 colors of cubes	Used enactive processes to enumerate 4-cube permutations with a fixed cube in the fifth position, then multiplied 24×5	Reasoned there are 24 4-cube towers for any 4 colors, and 5 possible top-cube colors, so $24\times5=120$ towers
4. Counting 6-cube towers each containing 6 colors of cubes	Multiplied 120×6	Multiplied 120×6
5. Counting 9-Cube towers each containing 9 colors of cubes	Multiplied $720\times7\times8\times9$	Multiplied $720\times7\times8\times9$
6. Counting 20-cube towers each containing 20 colors of cubes	Multiplied $362{,}880\times10\times\ldots\times19\times20$	Described the multiplication $20\times19\times\ldots\times2\times1$
7. Counting 100-cube towers each containing 100 colors of cubes	Described the multiplication $1\times2\times\ldots\times99\times100$	Described the multiplication $100\times99\times\ldots\times2\times1$
8. Counting *n*-cube towers each containing *n* colors of cubes	Described the formula $1\times2\times\ldots\times n$	Described the formula $n\times(n-1)\times(n-2)\times\ldots\times2\times1$

Theoretical Perspective

We adopted a psychological constructivist view of mathematical knowing and learning (cf. Piaget, 1970; von Glasersfeld, 1995). Within this view, a *concept* is a mental representation of a phenomenon that is stable enough to be re-presented (e.g., visualized or described) in the absence of relevant sensory-motor input (von Glasersfeld, 1991). An *action* refers to either a physical transformation on perceptual material or a mental action on imagined/re-presented material. An action constitutes an *operation* when (a) it is internalized (so that it can be performed mentally), (b) it is reversible, and (c) it can be composed with other mental actions (Piaget, 1963). A *scheme* is a way of operating under certain situations. It consists of an assimilatory mechanism for recognizing situations along with an integrated set of abstractions used to mentally represent the situation at hand (i.e., a *mental model*); a sequence of actions/operations associated with the assimilated situation; and a set of expectations or anticipations about possible results from those actions (Battista, 1999; see also von Glasersfeld, 1995).

Levels of Abstraction

To frame and analyze our students' conceptual progressions, we used Battista's (2007) theory of levels of abstraction, a reformulation of Piaget's theory of abstraction via Steffe and von Glasersfeld (Steffe, 1998; Steffe et al., 1988; von Glasersfeld, 1995). Battista's original levels of abstraction were generally stated and could be used to analyze and interpret student understanding of concepts and operations across a wide range of mathematical domains. At the *perceptual/recognition level*, an item from one's experiential flow is isolated and entered into working memory. Sensory properties necessary to recognize future instantiations of the item are empirically abstracted. At the *internalized level*, abstracted object-concepts may be re-presented (i.e., visualized) in the absence of relevant perceptual input, or abstracted actions and action sequences may be re-enacted in the absence of relevant kinesthetic signals prompting the abstracted action sequence. At the *interiorized level*, the student's understanding of the abstracted item becomes generalized in that they can apply the item to reason in novel situations. It is at this level that structures, patterns, and abstract forms are abstracted from particular sensory-motor contexts (Steffe et al., 1988). In particular, it is at the interiorized level that actions can be come operations, as the action can be performed in thought, is reversible, and can be composed with other mental actions. At the *second interiorized level*, the student constructs symbols as conceptual "pointers" to interiorized material, and these symbols are used as substitutes for the originally abstracted material in reasoning. At the *third interiorized level*, more complex operations can be performed on these symbols, such as curtailing sequences of symbolic operations into more condensed forms.

We used the original levels to explicate students' progressively abstract conceptualizations of, and operations on, spatial material within permutation enumeration contexts. However, through our data analysis, we found a need to elaborate the theory by developing an additional, empirically supported set of levels of abstraction. This new set of levels served to explain students' abstractions of *computational operation schemes* (COS), and we found the original set of levels occurred in parallel with this new set of levels. When a sequence of symbolic computations has been perceptually abstracted, the student can re-enact the action sequence step-by-step, but only with external cues such as instructions or a formula. The student can activate this COS when they recognize a situation as being similar to the original one in which they abstracted the sequence of computations. Once internalized, the sequence of computations can be performed, step-by-step, without such external cues. At the interiorized level, a student's understanding of a COS shifts from its step-by-step performance to analyzing the meanings and results of each computational operation, enabling them to apply meaningful deviations from the original and to adapt their COS to reason about novel situations. At the second interiorized level, the student can treat the sequence of operations of their COS as a *conceptual symbol*, meaning they can operate on the COS itself. For instance, they could reverse the sequence of operations, or they could decompose its constituent components and rewrite each computation in terms of this decomposition. At the third interiorized level, the student can algebraically generalize their COS using variable expressions. At the fourth interiorized level, the student can construct and conceptually operate on written/verbal symbols as pointers to their algebraically-generalized computations (e.g., $n!$ as a symbol for $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$).

To summarize, the levels of abstraction sequence occurs first on actions/operations on objects, then reappears as an individual transitions to numerical, algebraic, and other forms of symbolic mathematical reasoning. The inter-relationship between the original set of levels for actions/operations on objects and the levels for computational operations is illustrated in Figure 1.

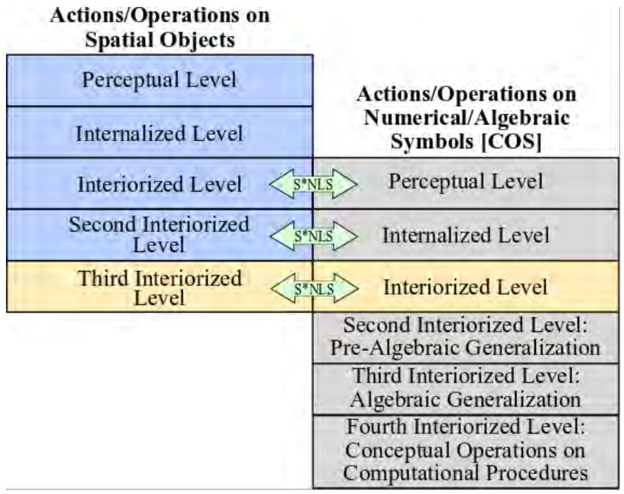


Figure 1: The Two Sets of Levels of Abstraction, Linked via S*NLS

Links between the two sets are made via S*-numerical linked structuring (S*NLS). S*-structuring is the mental process of constructing an organization or form for sequences of physical/mental actions on perceptual/imagined material; numerical (or N-) structuring is the process of constructing an organization or form for a set of numerical/algebraic symbols; and S*NLS is a form of reasoning that coordinates S*-structuring and N-structuring consistent with a learner's understanding of the relevant spatial/numerical properties (Antonides & Battista, 2022).

Discussion

As noted in the Introduction, we believe that our instructional approach and theoretical perspective provide an elaborated description of both the cognitive mechanisms that underlie CF and its instructional implementation—two sites in which Fyfe et al. (2014) called for additional research. First, incorporating multiple levels of abstraction greatly elaborates the notion of concreteness "fading," with each abstraction level fading some of the "concreteness" of the previous level, as illustrated in Figure 2.

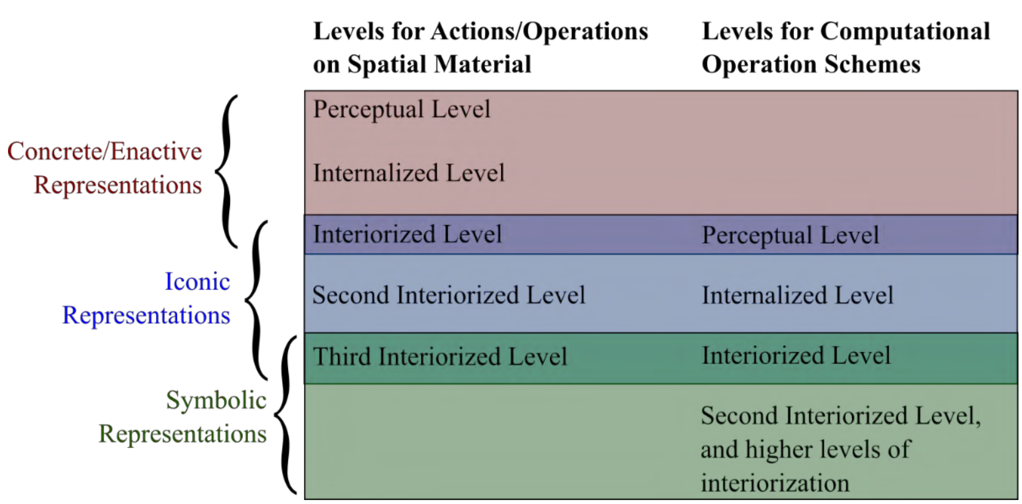


Figure 2: Elaborating Concreteness Fading Using Levels of Abstraction

At the perceptual and internalized levels of abstraction, student reasoning about a given mathematical concept is constrained to the original perceptual context in which the concept was initially encountered, with relevant figurative (or "concrete") materials perceptually available. We hypothesize that even in the shift from perceptual abstraction to internalization, many of the sensory properties initially registered into memory become "faded." Upon reaching the interiorized level, mathematical ideas can be extended beyond this initial sensory-motor context, and more abstract representations, such as drawings and motor-kinesthetic items (e.g., counting using fingers), become available to the student. Interiorization affords the construction of a more generalized structure from a student's actions on sensory-motor material, enabling a shift from concrete/enactive to iconic/enactive representations. For instance, DC and NK interiorized the structure of a 3-cube tower; for DC, this was reflected in his spatial structuring [base cube] + [reversible 2-cube tower]. He used gestures and verbal descriptions to describe aspects of his reasoning, but still strongly linked to the cube-towers context. At the second interiorized level, DC and NK constructed and operated on symbols, such as NK's enumeration of 4-cube towers by multiplying 4×6 *without* needing to construct towers one-by-one.

At the third interiorized level, students' focus of attention begins to shift from actions/operations on the context-specific spatial material to their computational processes themselves, thus emerging upon the symbolic stage within CF theory. DC and NK, at this level, performed more complex symbolic computations, such as enumerating 9-cube towers by multiplying $720\times7\times8\times9$. Our students' reasoning then focused entirely on their computational operation schemes, without reference to cube-towers (though they could, if asked, explain their reasoning in terms of towers).

Second, our theory uses S*NLS (Antonides & Battista, 2022) to elaborate Fyfe et al.'s (2014) discussion of structuring and linking to elaborate the connection between concrete and formal symbolic representations. This connection is illustrated by the levels of abstraction occurring in parallel in Figure 2, with concrete representations on the left and formal symbolic representations on the right, linked via S*NLS as illustrated in Figure 1. Our S*-structuring perspective emphasizes the importance of providing students with opportunities to act on sensory materials when forming their initial combinatorial conceptualizations and reasoning; for us, S*-structuring represents a cognition-based elaboration of Lockwood's (2014) set-oriented perspective, as students draw on their concrete/enactive S*-structuring experiences to conceptually ground symbolic computational formulas and expressions.

Lastly, our investigation and theoretical framework which draws on mental models elaborates Fyfe et al.'s claim that "the concrete stage enables learners to acquire a store of images that can be used when abstract symbols are forgotten or disconnected from the underlying concept" (p. 13). In fact, while mental images may be a constituent part (such as stored mental representations of particular towers or sets of towers), it is *mental models* that enable students to conceptually operate with the combinatorial composites and symbolic manipulations (Battista, 2007).

Furthermore, returning to Fyfe et al.'s (2014) call for describing ways to optimize the fading technique, using the theory of levels of abstraction to guide CF seems to have great potential for optimizing its instructional use. As opposed to the representations used by Braithwaite and Goldstone (2013) in their concreteness-fading study of combinatorics with undergraduates (specifically, letter sequences followed by arithmetic explanations for factorials), we provide a very different and much elaborated interpretation of concrete versus abstract representation, as well as CF, for combinatorial reasoning. Indeed, our "fading" is accomplished by starting with small-number permutations of physical cubes and incrementally increasing this number so that actual manipulation of cubes becomes impractical and necessarily needs to fade into the background as symbolic representations are abstracted and come to the fore. At this point, students start operating on symbolic representations, first with simple arithmetic operations, then with pre-algebraic to algebraic operations, with the fading being accomplished by increasing levels of abstraction, generalization, and consequent symbolic representation. All-the-while during our fading, the students were encouraged and supported, via S*NLS, to directly connect reasoning on each new problem to reasoning on the previous problem using the powerful mathematical reasoning of recursion.

Finally, we view the process of generalization to be a critical component of CF. Drawing on Ellis et al.'s (2021) research synthesis of generalizing actions, CF often involves "deriving broader results from particular cases to form general relationships, rules, concepts, or connections" as well as "extending one's reasoning beyond the range in which it originated" (p. 2), which the authors connected to the process of abstraction. Thus, in our teaching experiments, an essential component of our CF is progressive formalization (Nathan, 2012) in which students *incrementally generalized* their reasoning, again via S*NLS, as they moved from considering small-number to large-number towers. Consistent with our instructional approach, we hypothesize that the enactive stage of instruction may be more effective when it becomes impractical or impossible for the student to completely model the task situation using concrete materials, which would create an intellectual need for a transition to a different symbol system. This hypothesis is one potential avenue for future research. Fyfe et al. (2014) similarly argued that CF enables concepts to be "generalized in a manner that promotes transfer" (p. 12). Consistent with this claim, in later teaching-experiment sessions, DC and NK transferred their cube-based reasoning about permutations to permutation tasks not involving cubes, with both students referring back to their reasoning about permutations of cubes. This suggests the perceptual context of constructing and enumerating permutations of cubes provides powerful mental models for applying and transferring one's reasoning about permutations.

Conclusion

In this theoretical report, we have suggested a potential elaboration of CF theory using our recent teaching-experiment research and theoretical perspective. Our elaborated theory of levels of abstraction explicates the fading mechanism central to CF, with multiple levels of abstraction occurring within enactive-iconic-symbolic representational stages and with specific levels of abstraction at the transition between stages. Our investigation provides a case of implementing a much-elaborated instantiation of CF in the context of one-on-one instruction. We acknowledge that the one-on-one nature of our investigation is a limitation of our study, and it could explain our positive findings. However, significant research has found CF to be a powerful instructional method for supporting transfer of mathematical and scientific concepts (Bouck & Park, 2018; Fyfe et al., 2014). In future research, we intend to investigate how our instructional approach (with appropriate adaptations) may support student learning in classroom-based contexts.

Acknowledgements

We extend our appreciation to Emily Fyfe and to the anonymous reviewers for their helpful comments and suggestions.

References

Antonides, J., & Battista, M. T. (2022). Spatial-temporal-enactive structuring in combinatorial enumeration. *ZDM Mathematics Education*. https://doi.org/10.1007/s11858-022-01403-0

Ball, D. L. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator*, *16*, 14–18.

Battista, M. T. (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *Journal for Research in Mathematics Education*, *30*(4), 417–448. https://doi.org/10.2307/749708

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Information Age Publishing.

Bouck, E. C., & Park, J. (2018). A systematic review of the literature on mathematics manipulatives to support students with disabilities. *Education and Treatment of Children*, *41*(1), 65–106.

Braithwaite, D. W., & Goldstone, R. L. (2013). Integrating formal and grounded representations in combinatorics learning. *Journal of Educational Psychology*, *105*(3), 666–682. https://doi.org/10.1037/a0032095

Bruner, J. S. (1964). The course of cognitive growth. *American Psychologist*, *19*(1), 1–15. https://doi.org/10.1037/h0044160

Bruner, J. S. (1966). *Toward a theory of instruction*. Belknap.

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*(2), 380–400. https://doi.org/10.1037/a0031084

Clements, D. H., & McMillen, S. (1996). Rethinking "concrete" manipulatives. *Teaching Children Mathematics*, *2*(5), 270–279.

Domino, J. (2010). *The effects of physical manipulatives on achievement in mathematics in grades K-6: A meta-analysis* [Unpublished doctoral dissertation]. University at Buffalo, State University of New York.

Ellis, A. B., Lockwood, E., Tillema, E., & Moore, K. (2021). Generalization across multiple mathematical domains: Relating, forming, and extending. *Cognition and Instruction*, 1–34. https://doi.org/10.1080/07370008.2021.2000989

English, L. D. (1993). Children's strategies for solving two- and three-dimensional combinatorial problems. *Journal for Research in Mathematics Education*, *24*(3), 255–273. https://doi.org/10.2307/749347

Fennema, E. H. (1972). The relative effectiveness of a symbolic and a concrete model in learning a selected mathematical principle. *Journal for Research in Mathematics Education*, *3*(4), 233. https://doi.org/10.2307/748490

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational Psychology Review*, *26*(1), 9–25. https://doi.org/10.1007/s10648-014-9249-3

Fyfe, E. R., & Nathan, M. J. (2019). Making "concreteness fading" more concrete as a theory of instruction for promoting transfer. *Educational Review*, *71*(4), 403–422. https://doi.org/10.1080/00131911.2018.1424116

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *Journal of the Learning Sciences*, *14*(1), 69–110. https://doi.org/10.1207/s15327809jls1401_4

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155–177. https://doi.org/10.1207/ s15327833mtl0102_4

Gravemeijer, K. (2002). Preamble: From models to modeling. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 7–22). Springer.

Lockwood, E. (2014). A set-oriented perspective on solving counting problems. *For the Learning of Mathematics*, *34*(2), 31–37.

Maher, C. A., Powell, A. B., & Uptegrove, E. B. (Eds.). (2011). *Combinatorics and reasoning: Representing, justifying, and building isomorphisms*. Springer.

McNeil, N. M., & Fyfe, E. R. (2012). "Concreteness fading" promotes transfer of mathematical knowledge. *Learning and Instruction*, *22*(6), 440–448. https://doi.org/10.1016/j.learninstruc.2012.05.001

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments*, *4*(3), 35–50.

Nathan, M. J. (2012). Rethinking formalisms in formal education. *Educational Psychologist*, *47*(2), 125–148. https://doi.org/10.1080/00461520.2012.667063

Peltier, C., Morin, K. L., Bouck, E. C., Lingo, M. E., Pulos, J. M., Scheffler, F. A., Suk, A., Mathews, L. A., Sinclair, T. E., & Deardorff, M. E. (2020). A meta-analysis of single-case research using mathematics manipulatives with students at risk or identified with a disability. *The Journal of Special Education*, *54*(1), 3–15. https://doi.org/10.1177/0022466919844516

Piaget, J. (1963). The attainment of invariants and reversible operations in the development of thinking. *Social Research*, *30*(3), 283–299.

Piaget, J. (1970). *Genetic epistemology* (E. Duckworth, Trans.). W. W. Norton & Company.

Piaget, J., & Inhelder, B. (1975). *The origin of the idea of chance in children* (L. Leake, P. Burrell, & H. D. Fishbein, Trans.). W. W. Norton & Co.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 3, pp. 41–95). Lawrence Erlbaum Associates.

Steffe, L. P. (1998, April 1). *Principles of design and use of TIMA software* [Symposium, Constructive software: Developing computer environments based on theoretical models]. Research Presession of the 76th Annual Meeting of the National Council of Teachers of Mathematics, Washington, D.C.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies*. Springer-Verlag.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Lawrence Erlbaum Associates.

Tillema, E. S. (2013). A power meaning of multiplication: Three eighth graders' solutions of Cartesian product problems. *Journal of Mathematical Behavior*, *32*(3), 331–352. https://doi.org/10.1016/j.jmathb.2013.03.006

von Glasersfeld, E. (1991). Abstraction, re-presentation, and reflection: An interpretation of experience and Piaget's approach. In L. P. Steffe (Ed.), *Epistemological foundations of mathematical experience* (pp. 45–67). Springer-Verlag.

von Glasersfeld, E. (1995). *Radical constructivism: A way of knowing and learning*. Falmer Press.

Wheatley, G. H. (1992). The role of reflection in mathematics learning. *Educational Studies in Mathematics*, *23*(5), 529–541. https://doi.org/10.1007/BF00571471

COLLEGE STUDENTS' INPUT ON THE DESIGN OF WORKED EXAMPLES FOR ONLINE ENVIRONMENTS

Avery H. Closser
Worcester Polytechnic Institute
ahclosser@wpi.edu

Jenny Yun-Chen Chan
Worcester Polytechnic Institute
jchan2@wpi.edu

Hannah Smith
Worcester Polytechnic Institute
hsmith2@wpi.edu

Erin Ottmar
Worcester Polytechnic Institute
erottmar@wpi.edu

Worked examples have been shown to improve student learning in algebra. However, less is known about how to design worked examples to support student learning in online settings. We explore how college students react to worked examples that vary in their degree of extensiveness and dynamicness. In an online, within-subjects study, 109 college students viewed six worked example presentations: 1) static concise, 2) static extended, 3) sequential concise, 4) sequential extended, 5) dynamic history, and 6) dynamic no history. Students rated the helpfulness of each worked example and explained their rating. We found that students rated the static concise presentation as the most helpful and the dynamic no history presentation as the least helpful example. Responses were coded by researchers for common themes and revealed insights that may inform how researchers and teachers design worked examples for online environments.

Keywords: Learning Theory, Instructional Activities and Practices, Metacognition, Technology.

Purpose of the Study

Worked examples are an effective means of instructional support for math education (e.g., Booth et al., 2015; Carroll, 1994; Foster et al., 2018). However, worked examples have primarily been presented the same way, with research focusing on how static images of worked examples that display major derivations align with students' underlying cognitive mechanisms and in turn support learning (e.g., Sweller, 2006, 2020). Recently, Scheiter (2020) called for broadening the research on example-based learning beyond pure cognitive mechanisms to incorporate non-cognitive theories. While some affective factors have been investigated in relation to example-based learning (Hartmann et al., 2020; Tempelaar et al., 2020), we propose that analyzing students' perceptions of online worked examples may also a) advance our understanding of how worked examples impact student thinking and b) inform future iterations of worked examples that optimize support for student learning in online settings.

Previously, we tested the effectiveness of six worked example formats that varied in presentation but matched in content and found that grade-school Algebra I students improved in their ability to simplify equations after completing instructional practice with any of the six worked example formats (Smith et al., 2022). Since all of the worked example formats led to comparable learning gains among Algebra I students, we now aim to understand how different features of worked examples affect students beyond learning gains alone. In this study, we assess college students' perceptions of the six worked example formats on simplifying equations to gather student feedback that might inform future designs of worked examples for online math learning environments. Specifically, we ask: 1) *Which worked examples are rated as the most vs.*

least helpful? 2) *What themes emerge in students' explanations for their rating?* 3) *How do students' explanations for the most helpful and least helpful worked example presentations provide further insights into the features to which students attend?*

Theoretical Framework

Frameworks of self-regulated learning (SRL; e.g., Bjork et al., 2013; Dunlosky & Ariel, 2011; Nelson & Narens, 1990) broadly model learning as a cycle between students monitoring and controlling learning tactics based on self-assessments of their progress in developing content knowledge or skills. Recent evidence showed that students were able to self-regulate learning by deciding if and when to use worked examples or practice problem solving (Foster et al., 2018). However, students tended to underutilize worked examples, and were more likely to study worked examples *after* problem solving. These student behaviors contrast with evidence that studying worked examples is more effective *prior to* problem solving (Leppink et al., 2014; Van Gog et al, 2011). Although students are able to self-regulate learning, Foster and colleagues (2018) call for more research on how students' self-regulation decisions are related to learning outcomes. More broadly, we posit that students' perception of worked examples may influence the ways in which they regulate learning and impact learning outcomes.

Here, we compare students' reactions to six different formats of worked examples varying in their extensiveness and degree of dynamicness based on cognitive load theory (Sweller, 2006) and perceptual learning theory (Gibson, 1969; Goldstone et al., 2017), respectively. We reason that there are two primary competing hypotheses as to which worked example format students will perceive as most helpful. First, based on cognitive load theory, students may prefer the *concise static* worked example as it provides only the major derivation steps to a problem without splitting their attention across multiple sources of information in worked examples. Second, based on perceptual learning theory, students may prefer the *dynamic* worked examples that show fluid transformations between each derivation step. Aligned with perceptual learning theory, the additional visual cues may help direct students' attention towards relevant actions for problem solving and raise awareness of perceptual cues to aid with problem solving. However, it is unclear how much detail and animation may be helpful, prompting us to also explore the effect of studying *dynamic* compared to *static* and *sequential* (an intermediate version that presents each equation line as a new animation) worked examples that vary in their amount of detail. As a first step towards understanding students' perception of the worked examples, we asked college students to rate the helpfulness of the worked examples as well as to explain their ratings.

Methods

A total of 109 students from a private university in the Northeastern U.S. participated for partial course credit. The sample afforded 80% power to detect the effect of $f > 0.35$ at $p < .05$.

Students completed a 30-minute online, within-subjects study designed as an assignment in ASSISTments, an online homework and research platform (Heffernan & Heffernan, 2014). Within the assignment, students completed six pairs of worked examples and practice problems, with each pair followed by two survey items, in a randomized order. For each worked example, students were instructed to study the worked example then enter the solution as an answer. On the following page, students completed a practice problem that matched the equation structure of the worked example without any instructional support or feedback. Immediately following each pair, students were asked how much they agree with the following statement: "*The worked examples were helpful for learning how to solve equations*". Students rated their perceived helpfulness on a 6-point Likert scale (1=Strongly Disagree; 6=Strongly Agree). They were then prompted to explain their rating in an open-response textbox that appeared below on the same screen which said, “Please use the open response to explain your answer.”

We designed six worked example presentations that varied in their visual features but not in content. Specifically, we manipulated the degree of *dynamicness* (static, sequential, or dynamic) and *extensiveness* (concise or extended) of each worked example. We adapted six worked examples that were designed for seventh-graders from Rittle-Johnson and Star (2007), so they were similar in content and difficulty. Previously, we also used these worked example presentations to investigate their differential effects on learning among Algebra I students; we found that students did improve from pretest to posttest on simplifying equations after completing instructional practice with any of the six worked example presentations (Smith et al., 2022). In this study, students viewed all six of the following formats in a randomized order.

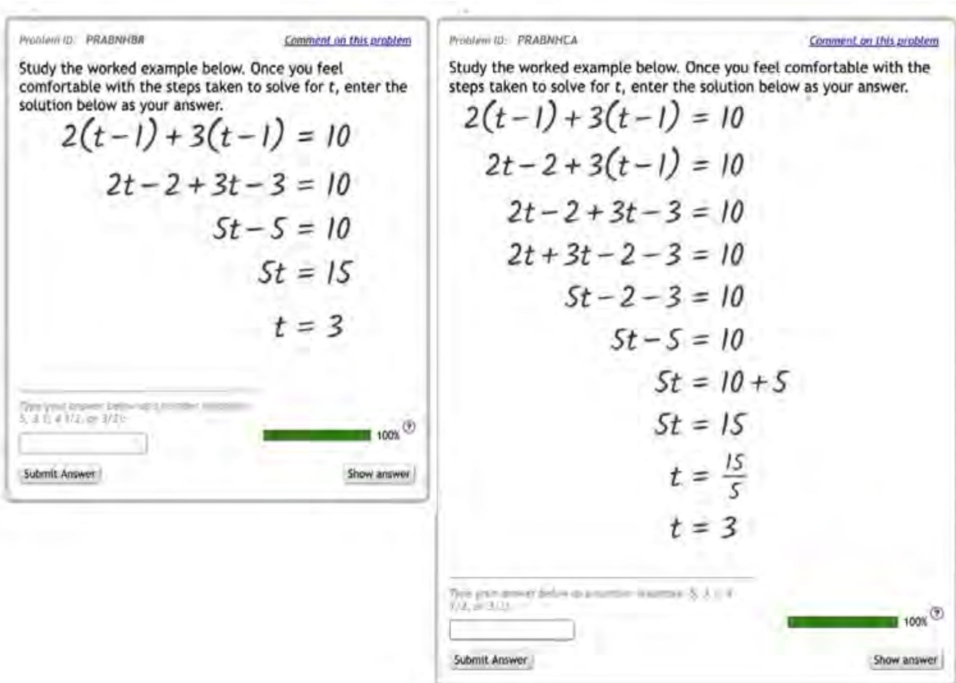


Figure 1: Static Concise Presentation (Left) and Static Extended Presentation (Right)

The first four conditions were based on a 2×2 design, differing in (a) length (i.e., concise or extended), and (b) the dynamicness of the worked example, (i.e., static or sequential). The *concise* worked examples displayed only the major steps in the derivation while the *extended* worked examples showed every step in the derivation. Static presentations mirrored the most commonly used design of worked examples in math (e.g., Rittle-Johnson & Star, 2007), presenting each worked example as images (Figure 1). The sequential worked examples were displayed as a looping GIF video that presented the static worked examples line-by-line in approximately 3-second intervals, creating a step-by-step history of the derivation over time. By varying the length and dynamicness, we created four different worked examples: *static concise*, *static extended*, *sequential concise*, and *sequential extended*.

Additionally, we created two presentations which displayed dynamic transformations of the expression and varied in whether the history of the derivation was displayed or not. The *dynamic history* worked examples showed looping videos of the transformation process through a screen recording. For example, to transform $2(t-1)+3(t-1)=10$ to $2t-2+3(t-1)=10$, students watched as the 2 was dragged over the parentheses to enact the transformation and record the result of the action on a new line (Figure 2). Each step of the problem was shown on the following line and a history of the steps taken was displayed sequentially. The transformations in the *dynamic no history* presentation were identical to those in the *dynamic history* presentation, but all occurred on one line of the equation without creating a derivation history.

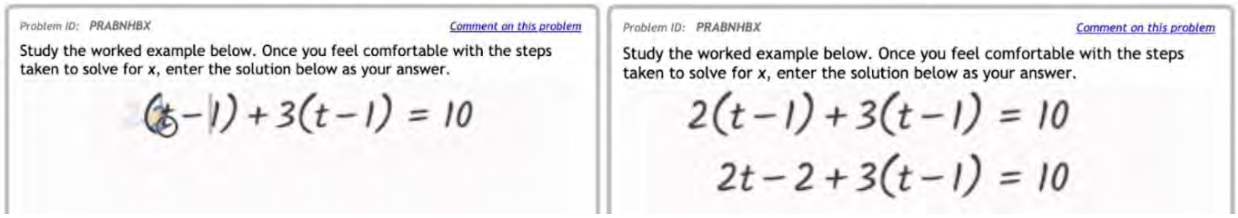


Figure 2: The *Dynamic History* Presentation Shows the Process of Each Transformation

Approach to Analysis and Results

Research Question 1: Perceived Helpfulness Ratings of Worked Example Presentations

To first compare how students rated the different worked example formats, we treated students' ratings (on a 6-point scale) as a continuous variable (Robitzsch, 2020). A one-way ANOVA revealed a significant effect of worked example presentation on students' rating, $F(5, 648) = 5.76$, $p < .001$. Post hoc comparisons using Bonferroni correction indicated that the mean rating for the *static concise* presentation ($M = 5.17$, $SD = 1.21$) was significantly higher than the *dynamic no history* ($M = 4.22$, $SD = 1.56$), *sequential extended* ($M = 4.59$, $SD = 1.44$), and *dynamic history* ($M = 4.60$, $SD = 1.53$) presentations, ps $< .05$. Further, the mean rating for the *static extended* presentation ($M = 4.92$, $SD = 1.31$) was significantly higher than the *dynamic no history* condition ($M = 4.22$, $SD = 1.56$), $p < .01$ (Figure 3). There were no significant differences in students' rating between the *sequential concise* presentation ($M = 4.61$, $SD = 1.38$) and any of the other presentations.

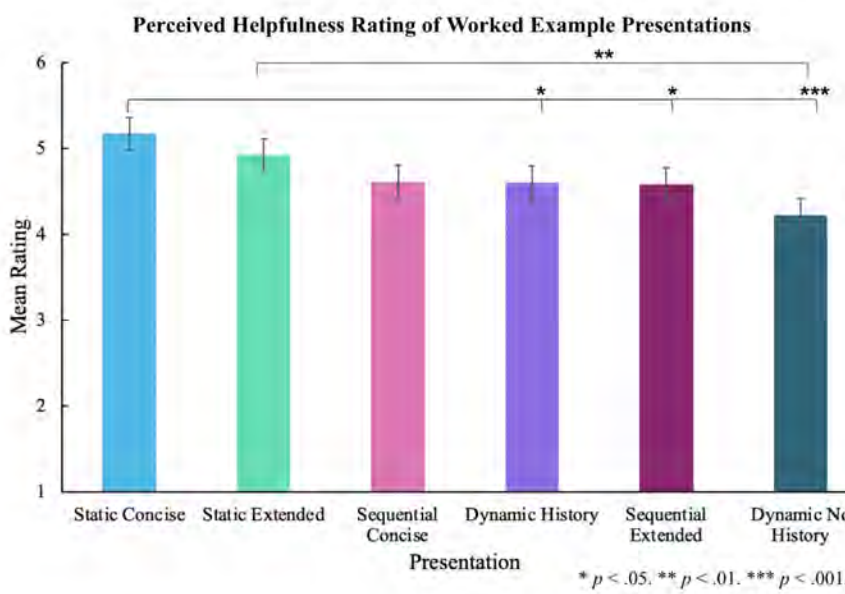


Figure 3: Average Perceived Helpfulness Rating of Each Worked Example Presentation

Research Question 2: Themes Among Students' Explanations

Next, we identified common themes in students' explanations of why they found each worked example helpful or unhelpful. Two researchers independently coded the 654 explanations with 79% initial agreement and then resolved discrepancies together to create the final codes. The following themes emerged: amount of content, video speed, dynamic coherence, and visual features (Table 1). Importantly, students' explanations were coded as whether they included a reference to one or more of the themes rather than whether students found each feature to be a positive or negative feature of the worked example presentation.

Table 1: Themes Identified Across Students' Explanations

Theme	Definition	Student Example
Amount of Content	Comments on the quantity of information presented in the worked example	*"I like seeing each step in the problem (regardless of how "trivial" the step is). It helps me see the train of thought that I'm supposed to be having while solving the problem."*
Dynamic Coherence	Comments on the presence or absence of fluid transformations and transitions that connect steps and actions.	*"The transitions may have helped but made it more visually confusing"*
Speed	Comments on the speed of the animations	*"It was very slow and hard to focus on"*
Visual Features	Comments on visual features such as font, spacing, styling, and format	*"…The font is very clear…"*

The themes identified were not exhaustive or mutually exclusive: students' explanations could be coded as pertaining to none, one, or more than one theme. As a result, 303 of the 654 explanations were not labeled as matching any of the above themes. For example, one student explained the problem-solving process rather than describing why they found the example to be helpful or unhelpful: "*By using the distributive property, the 3 was multiplied to the h and the −2, and the 5 was multiplied to the h and the −2. Like terms were added together, and h was isolated by simple algebra*." Conversely, another student's reaction to the *sequential concise* worked example referenced amount of content, video speed, and dynamic coherence:

> *While this animation laid out all the necessary steps to solve the equation, the steps were revealed somewhat spastically and it was almost startling to the eye to watch the equation being solved. Additionally, this animation left the viewer with little time to slowly work through the problem at their own pace and go back to a previous step if they were confused. This is because the format of this animation was a bit stressful.*

Table 2 presents the frequency of explanations that were labeled as containing one or more of the described themes by condition and overall. Students commented most often on the *amount of content* presented followed by the *dynamic coherence* of the worked examples and *speed* of the worked example. Miscellaneous visual features such as font and format were less often noted.

Table 2: Frequency of Themes Across Conditions

Worked Example Condition	Theme			
	Amount of Content	Dynamic Coherence	Speed	Visual Features
Static Concise	15	20	18	6
Static Extended	30	11	5	6
Sequential Concise	33	17	30	9
Dynamic History	16	36	12	9
Sequential Extended	16	24	19	6
Dynamic No History	31	31	21	7
Total	141	139	105	43

Research Question 3: Students' Explanations for Most and Least Helpful Presentations

Finally, to delve deeper into how students attend to features of worked examples and to contribute findings to cognitive load and perceptual learning theories, we explored how students' reactions to the *static concise* and *dynamic no history* presentations may explain their highest and lowest helpfulness ratings, respectively (Table 3). A closer inspection revealed that students found the *static concise* worked examples familiar and helpful because students could self-pace how they studied them. Twelve students praised the opportunity to self-pace themselves; further, four students noted that they would have preferred to also have instructional explanations accompany the *static concise* worked example. On the other hand, nine students positively noted that the *dynamic no history* presentations were helpful by providing explicit transitions between derivation steps while 38 students noted that the format caused confusion, and the derivation disappearing as each new line appeared required more effort and time to process.

Table 3: Sample Explanations for Static Concise and Dynamic No History Presentations

Static Concise	Dynamic No History
"the worked example was very helpful because steps were concise and understandable and the numbers didn't move around or disappear."	"Having all of the steps on the same line during the animation can be cumbersome if a student trying to learn the process has a specific question about a particular set and has to wait for the full animation to finish for it to restart."
"The example showed the steps, so I could understand the process. However, if I did not know how to do it already I may have been confused because there were no	"This worked example really helped as it would physically move each number around so you would physically see what was happening and then show what it would produced so you would

written instructions of what the person did.”	know where each number was going. This helped with visualizing what was taking place.”
“With few steps, this is not too hard to follow. But with larger numbers or more written steps, this would easily become overwhelming!”	“I think it would have been more helpful to see the steps written out statically rather than in a gif. It could be difficult to keep up, especially if you didn't understand a step right away.”
“I could look at the whole problem at my own pace.”	“Was way harder to follow what was happening without everything written out; once the directions disappeared I couldn't follow the solution anymore”

Discussion

This study is a foundational effort to explore how students respond to worked examples in ways that may not be reflected through learning gains alone. To integrate students’ perspectives on worked example designs for algebra, we asked college students to rate the helpfulness of six worked example presentations and to explain their ratings. We found that students rated the *static concise* presentation as the most helpful and the *dynamic no history* presentation as the least helpful. Students’ explanations for their ratings revealed that they commonly attended to the following elements of the worked examples: *amount of content*, *dynamic coherence*, *speed*, and other *visual features*. On the one hand, students revealed that the *static concise* presentation was familiar to previous classroom experiences and allowed for self-pacing, potentially supporting students’ chances to self-regulate their learning. On the other hand, the *dynamic no history* presentation required more effort to process, rendering it less helpful. These findings advance cognitive theories of mathematics learning and provide implications for researchers and teachers.

Our finding that students rated the *static concise* presentation as the most helpful aligns with cognitive load theory and explanations for the worked example effect. Specifically, students preferred the presentation with the least amount of information and content which likely prevented students from feeling as though they were studying redundant information or splitting their attention between different areas of the worked example (Sweller, 2020). This presentation is also a commonly used format for algebra worked examples (e.g., Rittle-Johnson & Star, 2007) and multiple students commented that they found it familiar with their past experiences.

Further, the finding that students found the *dynamic no history* presentation the least helpful aligned with prior work comparing different formats of worked examples. In particular, Lusk and Atkinson (2007) exposed college students to worked examples on proportional word problems that were: *fully-embodied* with an animated parrot that used gesture and gaze, *minimally embodied* with a static parrot who could only talk, or *voice-only* worked examples that provided only an audio description for the worked example. They found that while learning gains were the highest in the *fully-embodied* condition, students who studied the *minimally embodied* worked examples had the lowest levels of cognitive load. Lusk and Atkinson’s (2007) results suggest that extra information presented through animation and dynamic videos may support learning but may also present more challenges to students as they study worked examples with animations. Although ample research has demonstrated the benefits of perceptual scaffolding in math to direct students’ attention towards important cues in notation (e.g., Goldstone et al., 2017), perhaps the fluid transformations displayed in the *dynamic no history* worked example presentation provide extraneous distractions that increase demands on, rather than offload, students' working memory.

Students' explanations for the static concise and dynamic no history presentations indicated important visual features to consider when modifying these worked example presentations in the future. Namely, students indicated that they would find the worked examples more helpful if accompanied by written explanations for each step in the derivation. Further, if researchers or instructors choose to use a dynamic version of worked examples in classroom instruction or online practice, slowing the video speed or providing students with the autonomy to pause videos may reduce the cognitive effort and time that students indicated needing to process the *dynamic no history* presentations.

Importantly, we acknowledge that students' ratings and explanations for the helpfulness of the worked examples may be influenced by their own content knowledge. Specifically, the study materials were developed based on middle-school math content and might have been too easy for our sample of college students. However, by conducting this study with college students, we received thoughtful reactions to the worked example formats that still provide insights that may inform future iterations of worked examples used for online instructional practice.

Looking ahead, conducting comprehensive studies with algebra students may help delineate cognitive and non-cognitive mechanisms of learning and inform the design of worked examples for online learning environments. For instance, the six worked example presentations here were modeled after principles of cognitive load theory and perceptual learning theory; future studies should measure how students' cognitive load is impacted and how students may be impacted by studying different worked example presentations at different stages of learning. Further, it may be worthwhile to investigate the relation between students' learning gains, cognitive load, and perceptions of helpfulness. For instance, Whitehill et al. (2019) found that learners' subjective ratings of various tutorial videos (i.e., how helpful the videos would be for others) were highly correlated with their later learning gains. Participants who found the videos helpful learned more than those who did not find the videos helpful, confirming their initial bias towards the videos. These findings suggest that learners might be able to gauge the effectiveness of instructional support or that learning may be modulated by learners' perceived helpfulness. Regardless of the directionality of the influences, students' perceived helpfulness of the instructional materials may be closely related to learning outcomes, warranting further investigations of how this effect may impact the effectiveness of worked examples.

Analyzing students' perceptions of six worked example formats revealed that students prefer worked examples with minimal instructional content in a static image rather than worked examples with animations and fluid transformations between steps. These findings provide insights for researchers and teachers as they design algebra worked examples for online settings.

Acknowledgments

We thank Nellie Rushton and Lilly-Beth Linnell for coding students' responses. Additionally, the research and development reported here was supported by the Institute of Education Sciences, U.S. Department of Education, through the Small Business Innovation Research (SBIR) program contracts 91990019C0034 and 91990018C0032 to Graspable Inc. The opinions expressed are those of the authors and do not represent views of the Institute or the U.S. Department of Education.

References

Bjork, R. A., Dunlosky, J., & Kornell, N. (2013). Self-regulated learning: Beliefs, techniques, and illusions. *Annual Review of Psychology*, *64*, 417-444.

Booth, J. L., Oyer, M. H., Paré-Blagoev, E. J., Elliot, A. J., Barbieri, C., Augustine, A., & Koedinger, K. R. (2015). Learning algebra by example in real-world classrooms. *Journal of Research on Educational Effectiveness*, *8*(4), 530-551.

Carroll, W. M. (1994). Using worked examples as an instructional support in the algebra classroom. *Journal of Educational Psychology*, *86*(3), 360.

Dunlosky, J., & Ariel, R. (2011). Self-regulated learning and the allocation of study time. B. Ross (Ed.). *Psychology of Learning and Motivation, 54*, 103–140.

Foster, N. L., Rawson, K. A., & Dunlosky, J. (2018). Self-regulated learning of principle-based concepts: Do students prefer worked examples, faded examples, or problem solving? *Learning and Instruction*, 55, 124– 138.

Gibson, E. J. (1969). *Principles of perceptual learning and development*. Englewood Cliffs, NJ: Prentice Hall.

Goldstone, R. L., Marghetis, T., Weitnauer, E., Ottmar, E. R., & Landy, D. (2017). Adapting perception, action, and technology for mathematical reasoning. *Current Directions in Psychological Science*, *26*(5), 434-441.

Hartmann, C., Van Gog, T., & Rummel, N. (2020). Do examples of failure effectively prepare students for learning from subsequent instruction? *Applied Cognitive Psychology, 34*, 879–889. https://doi.org/10.1002/acp.3651

Heffernan, N. T., & Heffernan, C. L. (2014). The ASSISTments ecosystem: Building platform that brings scientists and teachers together for minimally invasive research on human learning and teaching. *International Journal of Artificial Intelligence in Education, 24*, 470-497.

Leppink, J., Paas, F., Van Gog, T., Van Der Vleuten, C. P. M., & Van Merriënboer, J. J. G. (2014). Effects of pairs of problems and examples on task performance and different types of cognitive load. *Learning and Instruction, 30*, 32–42.

Lusk, M. M., & Atkinson, R. K. (2007). Animated pedagogical agents: Does their degree of embodiment impact learning from static or animated worked examples?. *Applied Cognitive Psychology: The Official Journal of the Society for Applied Research in Memory and Cognition, 21*(6), 747-764.

Nelson, T. O., & Narens, L. (1994). Why investigate metacognition? In J. Metcalfe, & A. P. Shimamura (Eds.). *Metacognition: Knowing about knowing* (pp. 1–25). Cambridge, MA: MIT Press.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561.

Robitzsch, A. (2020, October). Why ordinal variables can (almost) always be treated as continuous variables: Clarifying assumptions of robust continuous and ordinal factor analysis estimation methods. In *Frontiers in Education* (Vol. 5, p. 177). Frontiers.

Scheiter, K. (2020). Embracing complexity in research on learning from examples and from problem solving. *Applied Cognitive Psychology*, *34*(4), 906-911.

Smith, H., Closser, A. H., Ottmar, E. R., & Chan, J. Y. C. (2022). The impact of algebra worked example presentations on student learning. *Applied Cognitive Psychology, 36*(2), 363-377.

Sweller, J. (2006). The worked example effect and human cognition. *Learning and Instruction, 16*(2), 165–169. https://doi.org/10.1016/j.learninstruc.2006.02.005

Sweller, J. (2020). Cognitive load theory and educational technology. *Educational Technology Research and Development*, *68*(1), 1-16.

Tempelaar, D. T., Rienties, B., & Nguyen, Q. (2020). Individual differences in the preference for worked examples: Lessons from an application of dispositional learning analytics. *Applied Cognitive Psychology, 34*, 890–905. https://doi.org/10.1002/acp.3652

van Gog, T., Kester, L., & Paas, F. (2011). Effects of worked examples, example-problem, and problem-example pairs on novices' learning. *Contemporary Educational Psychology, 36*, 212–218.

Whitehill, J., Aguerrebere, C., & Hylak, B. (2019). Do Learners Know What's Good for Them? Crowdsourcing Subjective Ratings of OERs to Predict Learning Gains. *International Educational Data Mining Society*.

A 2D SHAPE COMPOSITION LEARNING TRAJECTORY OF A STUDENT WITH DIFFICULTY IN MATHEMATICS

Angela Crawford
Boise State University
Angelacrawford1@boisestate.edu

Aysia Kernin
Boise State University
aysiakernin@u.boisestate.edu

This study explores a spatial reasoning learning trajectory of a student with difficulty in mathematics. Using a teaching experiment methodology across 15 instructional sessions, we observed how the student responded to instruction based on an established 2D shape composition learning trajectory (Sarama & Clements, 2009). A narrative microgenetic analysis identified conditions that were likely to have promoted learning. The analysis shows the student's actual trajectory was similar to that of the hypothesized learning trajectory. Challenges to progress emerged around teacher-guided components of instructional support. We describe how a student-centered approach and mathematizing through specific praise was generative of learning, while explicit guidance was not. We discuss how the guiding conjecture of the teaching experiment evolved and how it is situated within the broader literature base.

Keywords: Learning Trajectories and Progressions, Students with Disabilities, Geometry and Spatial Reasoning.

Students with learning disabilities and math anxiety often experience dissonance in mathematics learning spaces. The "clamor" created for the individual as they interact with mathematics and with others in the mathematics environment can, at times, be intense. This work centers the learning experience of Eva, a nine-year-old student with learning disabilities and anxiety. We wish to take an anti-deficit stance and refer to Eva as a student who has experienced considerable difficulty in mathematics learning situations.

An approach to instruction that has potential for harmonizing the learning environment for students who have experienced difficulty in mathematics is the use of learning trajectories (LTs). We conducted a teaching experiment with Eva that investigated spatial reasoning LTs. Here, we present a narrative microgenetic analysis of one of those trajectories, Eva's actual 2D shape composition LT and describe features of the experience that were supportive of and hindrances to her progress.

Theoretical Framework

In the United States, mathematics interventions are commonly supplemental programs intended to provide more intensive and individualized support (Powell et al., 2013). There is a consensus among researchers that students in interventions benefit most when this support is designed to be systematic (Fuchs et al., 2021). Systematic instruction includes features such as connecting to previously learned mathematics, using accessible number and visual representations, and gradually increasing complexity.

However, there are disagreements around instructional delivery format and what should structure the increasing complexity across lessons. Many researchers contend that instructional delivery must incorporate explicit, direct methods (e.g., Grigorenko et al., 2020). In contrast, other researchers are using carefully structured sequences of tasks that build on students' thinking and direct their attention to mathematical relations without initial explicit guidance (e.g., Hunt et al., 2020; Xin et al., 2016).

LTs offer another potential way to provide structure to support the learning of students who have experienced difficulty in mathematics learning situations. Research has shown positive effects of mathematics instruction based on LTs for young students at risk for later difficulty in mathematics (Clements & Sarama, 2007; Clements et al., 2019; Clements, Sarama, Baroody et al., 2020). Further, research indicates that instruction based on geometry trajectories can support learning in other mathematical domains and support executive functions which have been shown to be important to mathematics achievement (Clements et al., 1997; Clements, Sarama, Layzer et al., 2020; Schmitt et al., 2018).

LTs structure learning through developmental progressions, sequenced learning goals based on the progression, and tasks designed to support progress toward the goals (Sarama & Clements, 2009; Simon, 1994). The systematicity provided by the developmental progression is something that distinguishes instruction based on LTs from instruction that is more typical of intervention contexts for students who have experienced difficulty in mathematics. Whereas a typical intervention approach is to identify components of the target knowledge or skill and provide instruction for each component, instruction based on an LT focuses on intermediary phases of reasoning or acting that culminate in the desired knowledge. Simply put, rather than breaking a mathematical task down into smaller parts in order to learn how to do it, the ability to perform the task is built up "naturally" based on a students' current knowledge. Tasks can be designed to draw on a variety of instructional delivery formats–explicit instructional principles, incorporate multi-modal delivery and response, or first elicit student thinking before guiding that thinking toward more sophistication and abstraction.

2D Shape Composition as a Developmental Progression

The 2D shape composition LT (Clements & Sarama, 2021; Sarama & Clements, 2009) involves children gradually developing understanding of and attending to geometric attributes and mentally anticipating results of combining shapes. At an early stage of the trajectory, Piece Assembler, when children begin to put smaller shapes together to compose larger shapes, they do so through trial-and-error. At the next level, Picture Maker, children can combine several shapes to make part of a picture but still use trial-and-error. They select a shape based on a general shape or approximate side length, and can fill in puzzles that indicate the placement of the shape (interior lines are present). When children reach the Shape Composer level, they can anticipate shapes that are needed to fill in a larger region and choose shapes considering angles and side lengths. They can also intentionally use rotation and flipping to correctly orient shapes. This is followed by Substitution Composer in which children use trial-and-error to make multiple versions of a composite shape by combining shapes in different ways. The subsequent levels, Shape Composite Repeater and Shape Composer--Units of Units, involve understanding composite shapes themselves as part of larger shapes and working with patterns that combine composite shapes.

Method

We describe a teaching experiment with Eva, followed by a narrative microgenetic analysis. Teaching experiments apply methodological consistency within a naturalistic setting (Confrey & Lachance, 2000; Steffe et al., 2000). A guiding theory or conjecture is tested through intentional teaching, data collection, and on-going analysis. The conjecture that guided this teaching experiment is that an established LT supplemented with individualized supports can facilitate learning in a student who has experienced difficulty in mathematics learning situations.

This narrative microgenetic analysis had two important elements: (a) focusing on identifying conditions that might promote learning (Lavelli et al., 2005; Siegler, 2006), and (b) identifying a plot that links data as unfolding temporal development (Polkinghorne, 1995). The first element, conditions that promote learning, is based on a density of observations during a period of time during which we hope to see change. These observations can describe types of statements or behaviors, how the statements or behaviors change, the rate at which change occurs, possible sources of change, and how widely generalized the knowledge (Siegler, 2006). The second element, the resulting "story," gives shape to a multitude of factors that are at play.

Participants

Eva is a multi-racial girl living in the western United States. At the time of this teaching experiment, Eva was nine years old and in grade 3. Two years prior to this study, Eva underwent a neuropsychological evaluation which identified the following challenges: attention deficit with unspecified impulse-control and conduct disorder, speech-sound disorder, specific language disorder with impairments in written language and mathematics, and generalized anxiety. Eva attends a public charter school in a small city and has received special education services in a pull-out program ("resource room"). Upon meeting with Eva's mother, we discussed Eva's difficulties in mathematics and talked about ways to provide her with additional academic support without increasing her anxiety. We gained informed parent consent and student assent following guidelines established by the university's Institutional Review Board.

We recognize our roles as participants in this research. As a teacher, one of us conducted the teaching experiments in relationship with Eva and her mother, who was present for all the sessions. This relationship involved negotiating goals, activities, what was to be attended to and what was not, as well as negotiating interpersonal interactions and trying to establish trust. As researchers, we both continue this relationship with Eva through our interpretations of the videos, transcripts, and memos. We take the position of bracketing ourselves into this narrative, rather than attempting to position ourselves outside (Connelly & Clandinin, 2006).

Teaching Experiment Procedure

The first author, Angie, conducted all of the teaching experiment sessions. The sessions took place in the winter and spring of 2020 amidst the COVID-19 pandemic. The meeting dates were flexible to allow for family needs and local quarantining requirements. All sessions took place at Eva's home, and Eva's mother was present to support Eva's well-being and engagement. There were 15 teaching experiment sessions, each ranging from 30-45 minutes. Due to the pandemic and to limit factors that might contribute to Eva's anxiety, it was not possible to have an additional researcher present during the sessions. During each session, Angie asked Eva's permission to video record the activity and only recorded when Eva stated she was comfortable with it. We recorded nine of the 15 sessions.

Tasks described in this paper come from the 2D shape composition LT (Clements & Sarama, 2021; Sarama & Clements, 2009). These tasks involved placing pattern blocks (colored, wooden blocks in shapes of triangles, squares, trapezoids, hexagons, and two different rhombuses) into "puzzles" that provided outlines of a composite shape (puzzles may contain all, some, or no interior lines). We were open to adjusting the sequence of tasks and delivery format as needed to support Eva's engagement and documented the nature of and rationale for any changes to the LT.

We used a planning protocol for documenting instructional decisions and reflections on Eva's activity. The protocol included prompts for observations, outcomes, adjustments made, and rationale for adjustments. These plans were shared with two colleagues. One colleague, with expertise in early mathematics instruction for students with learning disabilities, advised on the tasks and supports. A second colleague, with experience with children with anxiety, advised on the plans in light of Eva's affective responses.

Data Sources

Data sources were the planning protocols previously described, field notes from each session, photographs of student work, videos and transcriptions, and a post-hoc observation protocol. The second author, Aysia, completed the post-hoc observation protocol as a way to triangulate observations and interpretations of Eva's responses. This protocol recorded several features of the interactions: the task description, student behaviors (strategies, demonstrations, comments), teacher behaviors (explanations, providing time to work without interruption, additional supports), and any other observations. Aysia completed the protocol for each video. It was common to pause and re-watch segments of a video file multiple times to capture all necessary data.

Data Analysis

We used a form of narrative analysis adapted for microgenetic studies that can account for the multidimensional nature of learning in individuals (Lavelli et al., 2005). This approach comprises five stages. The first involves viewing videos repeatedly to identify a list of potential "frames," the specific viewpoints or phenomena which could be a focal point for analysis. In the second stage, we constructed descriptive, chronological narratives, one for each session of the teaching experiment to document the sequence of events. To ensure accurate and comprehensive narratives, we each wrote an initial narrative for half of the sessions and then reviewed and revised the narratives written by the other. Thoughts related to frames, explanations, interpretations, questions or reflections were recorded in a parallel set of memos. In the third stage, we used the chronological narratives to discuss possible configurations of frames that seemed profitable ways of portraying learning. This study presents the frames of Eva's activity within the 2D shape composition LT and Eva's response to components of instructional support. The fourth stage involved re-reading the chronological narratives to look for evidence of stability or change over time and to develop stories that synthesize a preliminary "plot." Evidence to confirm or refute this plot was then gathered through an in-depth analysis of the data sources with particular attention to overt behaviors and strategies. In the fifth and final stage of analysis, the stories and evidence were used to create a narrative that synthesizes our views on Eva's experience.

Findings

We conducted a microgenetic study of Eva's learning experience with LTs. We present the frames of 2D shape composition activity and Eva's response to components of instructional support in narrative form, highlighting Eva's responses we believe are representative of Eva's experience.

The Story of Eva's 2D Shape Composition Trajectory

Sessions 1—4: Piece Assembler. In the first session, Eva did not want to solve puzzles, and she abandoned the activity. She returned when asked if she wanted to create her own picture, and she created a figure from a favorite cartoon with each block playing a unique role in the picture (i.e., one shape for each part of the picture, no shared sides). In the next sessions, Eva's activity suggested she recognized triangles, squares, and hexagons, but she did not name them. She used trial-and-error to fill shape outlines. When asked why she picked a particular shape, she would describe the shape as "skinny" or point at an angle or single side without further responses. We used "mini-puzzle" task cards with outlines of 2 shapes to control complexity and encourage Eva's intentional selection of shapes and her use of turning to orient them appropriately. Eva created her own picture of a dog in which three shapes combined to make a part of the picture (dog's head and ears). See Figure 1 for photos of Eva's work at this level.

Figure 1. Images of Eva's work at Piece Assembler level (sessions 1—4).

Sessions 5–7: Picture Maker. Eva placed several like shapes together to create sections of a puzzle when the interior lines were provided. She found this "easy" and asked for harder puzzles. We provided puzzles that had fewer interior lines but with some exterior line "cues" such as clear indicators of a triangle or rhombus. She continued to select recognizable shapes first. For example, she filled a rhombus with two triangles, thus composing without an interior line, but also shared she could have used "the red piece" instead. See Figure 2 for images of Eva's work during these sessions.

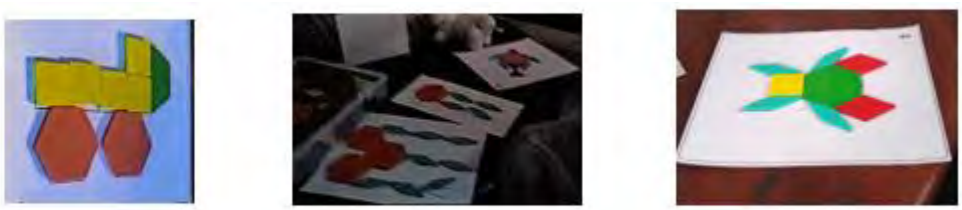


Figure 2. Images of Eva's work at Picture Maker level (sessions 5—7).

Sessions 8-14: Shape Composer. Eva described puzzles with interior lines provided as easy, and now she began to fill in those with fewer interior lines and more shared sides. During the first sessions at this level, Eva was showing more evidence of intentional actions by looking at the picture and then selecting a particular block. When a selected shape did not work at first, rather than try to turn and orient it differently, Eva would return to the container and look for something else. She showed more perseverance, trying several blocks before abandoning the task. Once she placed a block that appeared to fit but in combination with additional blocks no longer did so. Rather than replace any, she added a block that overlapped others to fill the empty space.

In later sessions at this level, Eva spent more time thinking about what would fit before selecting a block. Eva began to intentionally turn the blocks to orient them. Also, she began to see herself as an expert and wanted harder puzzles. We introduced pentomino puzzles, and Eva returned to trial-and-error and resisted guidance or suggestions. In session 14, though she still used trial-and-error, she did use rotations and flips to try different orientations of the pentomino to see if it fit. Images of Eva's work at Shape Composer level are shown in Figure 3.

Figure 3. Images of Eva's work at Shape Composer level (sessions 8—14).

Session 15: Substitution Composer. Despite her previous confidence, on this day Eva did not feel ready for "expert" puzzles. She did complete a pentomino puzzle with two pieces and no interior lines which shared a two-unit long side by applying turns and flips. She filled a pattern block puzzle two different ways, but she became frustrated on the third attempt and said she "tried her hardest."

Figure 4. Image of Eva's work at Substitution Composer level (session 15).

The Story of Eva's Response to Instructional Supports

Having learned that Eva was very anxious about math and that math instruction usually focused on numbers, we decided to focus on spatial reasoning tasks. Perhaps unsurprisingly, a challenge arose during the first session. Eva tried a few pattern blocks to fill in a puzzle and did not find they fit and said she wanted to do something else. Angie suggested looking at another section of the puzzle, but Eva got mad, stood up from the table, grabbed a scooter, and rode around her home's patio. Eva returned when Angie said they would do something different and brought out a whiteboard and markers. At the end of the session, Angie asked if Eva would make any picture she wanted with the pattern blocks. Eva made an image of a cartoon character she liked.

Based on interactions like that just described, Eva's engagement was a primary factor in task selection and instructional decisions. We selected activities that would be accessible to her and provide her with quick successes. For example, in the second session, we gave Eva a puzzle with no shared sides and easily recognizable shapes (i.e., square, triangle, and hexagon). For the next few sessions, we gave Eva mini-puzzle cards, each with two blocks, gradually increasing the proportion of the shared sides and gradually introducing less recognizable shapes (i.e., trapezoid and two rhombuses). Eventually these cards only provided the outline with no interior line. After Eva completed a few of these mini-puzzles, we offered her a larger puzzle that incorporated the shapes she had just worked with and asked if she wanted to try it. Using this approach of offering accessible puzzles and choices, Eva began filling in more complex shapes, perhaps due to increasing confidence or awareness of attributes or both.

We initially planned to discuss mathematics involved in tasks and to mathematize her activity to support her naming of shapes, noticing and describing attributes, and orienting shapes to compose a larger shape. However, when Angie discussed how many blocks were used in a section or tried to show the difference between angles of different shapes, it elicited avoidant behaviors–stopping participation, leaving the table, and expressions of frustration, anger, or sadness. This was even more evident when Angie asked Eva for a verbal response in return. Eva expressed that she didn't know how to answer Angie's questions or answered by repeating what

Angie had just said. The following transcript illustrates one of the times when Eva was upset by Angie's approach to providing guidance:

Angie: But you know what I see? I see lots of gaps and holes. I wonder if there's another way that might work better. Is there another way that will work better?... [Eva starts making humming noises] I just want you to show [Angie pauses]…. Does that work? …[Eva makes a noise like "grrr"]
Eva: This is right.
Angie: This is right? [Eva makes a noise like "grrr"] How do you know that?
Eva: [sounds as though she is growling out the words] It looks just like it.
Angie: It looks just like it?
Eva: [loudly] Yes it does.
Angie: Okay. Okay, all right.
(Simultaneously) Angie: Eva, I'm gonna use that description that you just used… Eva: I'm in the yellow.
(Simultaneously) Angie: I'm going to ask you to…. Eva: I'm in the yellow!
Eva: I…am...in…the…yellow!
Angie: Yes, I heard you say that.
Eva: You don't continue if you're in the yellow.

Angie stopped and looked to Eva's mother who explained that when at school 'in the yellow' means that Eva is upset, and it is a time to bring in tools to help Eva regulate her emotions.

Eva consistently responded negatively to guidance in the form of explicit explanations, discussion of math involved in the tasks, or corrective feedback. However, if mathematizing Eva's activity came in the form of specific praise, such as, "Wow! What great problem solving! I saw you *turn* the block to find out if it would fit another way," it had a much more positive effect. Angie would mimic Eva's activity while giving the praise, emphasizing important terms and motions. Eva would then continue to draw on that behavior or strategy in future puzzles. Eva also appeared to learn the names of some of the pattern block shapes after hearing them in the context of praise across several sessions.

We settled on an approach that offered Eva choices of puzzles that very gradually increased in complexity and which were likely to elicit some of the behaviors or strategies we hoped to see. We then used praise to draw her attention to those elements. Thus, our conjecture needed to be revised: engagement and progress would be supported by a gradual, systematic increase in complexity allowing for student choice and mathematizing through specific praise.

Over the course of the sessions, Eva began to express confidence. In session 7, Eva said she wanted "a very, very hard one." When Angie asked what she would do if it was too hard, Eva said she would "think hard about the shape." In session 8, she said, "Give me all challenge." In session 11, she picked a puzzle she said "would be hard for other people." By session 14, Eva said she was an expert and didn't know why she had to do the puzzles again. For this session, we had also planned to switch to puzzles that involved pentominoes rather than pattern blocks to encourage her attention to "turning" and "flipping." Eva's interest in the new puzzles was piqued, but it was clear when these were introduced, that her persistence with these still depended on early successes.

Discussion

This microgenetic study provides information on several different aspects of Eva's learning: presence of change, rate, stability, sources of change, hindrances, and generalization (Siegler, 2006). Eva's actual trajectory in 2D shape composition aligned with the established LT (Clements & Sarama, 2021; Sarama & Clements, 2009). Eva demonstrated increasing ability to use multiple pattern blocks to compose larger composite shapes with fewer lines as interior cues. She spent more time working in Shape Composer level than in the previous levels as she became more intentional in selecting and rotating blocks to complete the puzzles. Rather than suddenly demonstrating multiple actions that comprise a higher level in the progression in a single session, Eva demonstrated gradually more sophisticated actions-on-objects across sessions. These changes also appeared to be stable. In fact, Eva found tasks of a similar level of difficulty "too easy." Eva also appeared to be close to generalizing some use of attributes and transformations (i.e., turning and flipping) to solve problems with pentominoes.

A teaching experiment intentionally applies and tests a conjecture in real teaching and learning interactions (Confrey & Lachance, 2000; Steffe et al., 2000). Teaching, data collection, and analysis are responsive to the students, and conjectures are evaluated and revised accordingly. We found that Eva quickly abandoned tasks for which she did not possess a ready solution strategy. We also found that directing Eva's attention prior to activity, providing explicit explanations or demonstrations, or drawing attention to challenges were ineffective in engaging her with the task. Our conjecture evolved to state that a gradual, systematic increase in complexity which allowed for student choice and mathematizing the student's activity through specific praise would support engagement and progress. As we adjusted instruction to align with this conjecture, we found Eva expressed growing confidence and willing engagement in 2D shape composition tasks. We ensured she had the opportunity to choose puzzles and begin to solve them without any guidance at first, allowing her the opportunity to approach the task in whatever way made sense to her. We ensured systematicity by carefully selecting or creating puzzles that gradually increased in difficulty. We supported her continued use of strategies or vocabulary by providing specific praise when strategic thinking was observed and naming the shapes she had used.

A noteworthy aspect of these findings is that the revised conjecture problematizes recommendations that instruction intervention contexts should incorporate explicit explanations and specific corrective feedback (Grigorenko et al., 2020; Fuchs et al., 2021). A direct, explicit instructional approach did not encourage Eva's engagement with mathematical tasks. Rather, this study adds to other research which has shown students who have experienced difficulty in mathematics learning situations can benefit from instruction which elicits and builds on student thinking (e.g., Hunt et al., 2020; Xin et al., 2016). In Eva's case, that guidance was most successful when it also celebrated her successes. We are continuing to analyze Eva's responses to better understand how guidance impacted her engagement in other LTs. We believe these findings add to the literature which suggests LTs can provide a structure for establishing more "harmonious" mathematics learning situations for students like Eva.

References

Clements, D. H., Battista, M. T., Sarama, J., & Swaminathan, S. (1997). Development of students' spatial thinking in a unit on geometric motions and area. *The Elementary School Journal*, *98*(2), 171-186.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. Journal for Research in Mathematics Education, 38(2), 136-163. DOI: 10.2307/30034954

Clements, D. H., & Sarama, J. (2021). *Learning and teaching early math: The learning trajectories approach* (3rd ed.). Routledge.

Clements, D. H., Sarama, J., Baroody, A. J., Joswick, C., & Wolfe, C. B. (2019). Evaluating the efficacy of a learning trajectory for early shape composition. *American Educational Research Journal, 56*(6), 2509-2530.

DOI: 10.3102/0002831219842788
Clements, D. H., Sarama, J., Baroody, A. J., Kutaka, T. S., Chernyavskiy, P., Joswick, C., Cong, M., & Joseph, E. (2020). Comparing the efficacy of early arithmetic instruction based on a learning trajectory and teaching-to-a-target. *Journal of Educational Psychology.* DOI: 10.1037/edu0000633
Clements, D. H., Sarama, J., Layzer, C., Unlu, F., & Fesler, L. (2020). Effects on mathematics and executive function of a mathematics and play intervention versus mathematics alone. *Journal for Research in Mathematics Education, 51*(3), 301-333. DOI: 10.5951/jresemtheduc-2019-0069
Confrey, J. & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In A. E. Kelly & R. A. Lesh (Eds.), Handbook of research design in mathematics and science education (pp. 231-265). Lawrence Erlbaum Associates.
Connelly, F. M., & Clandinin, D. J. (2006). Narrative inquiry. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.) Handbook of complementary methods in education research. (pp. 477-487). Lawrence Erlbaum Associates.
Fuchs, L.S., Newman-Gonchar, R., Schumacher, R., Dougherty, B., Bucka, N., Karp, K.S., Woodward, J., Clarke, B., Jordan, N. C., Gersten, R., Jayanthi, M., Keating, B., and Morgan, S. (2021). Assisting Students Struggling with Mathematics: Intervention in the Elementary Grades (WWC 2021006). Washington, DC: National Center for Education Evaluation and Regional Assistance (NCEE), Institute of Education Sciences, U.S. Department of Education. Retrieved from https://ies.ed.gov/ncee/wwc/PracticeGuide/26
Grigorenko, E. L., Compton, D.. L., Fuchs, L. S., Wagner, R. K., Willcutt, E. G., & Fletcher, J. M. (2020). Understanding, educating, and supporting children with specific learning disabilities: 50 years of science and practice. *American Psychologist, 75*(1), 37-51.
Hunt, J. H., Martin, K., Khounmeuang, A., Silva, J., Patterson, B., & Welch-Ptak, J. (2020). Design, development, and initial testing of asset-based intervention grounded in trajectories of student fraction learning. Learning Disability Quarterly, 1-14. DOI: 10.1177/0731948720963589
Lavelli, M., Pantoja, A. P. F., Hsu, H.-C., Messinger, D., & Fogel, A. (2005). Using microgenetic designs to study change processes. In D. G. Teti (Ed.), Handbook of research methods in developmental psychology. (pp. 40–65). Blackwell Publishers.
Polkinghorne, D. E. (1995). Narrative configuration in qualitative analysis. *Qualitative Studies in Education, 8*(1), 5-23.
Powell, S. R., Fuchs, L. S., & Fuchs, D. (2013). Reaching the mountaintop: Addressing the Common Core Standards in Mathematics for Students with Mathematics Difficulty. *Learning Disabilities Research & Practice 28*(1), 38-48.
Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. Routledge.
Schmitt, S. A., Korucu, I., Napoli, A. R., Bryant, L. M., & Purpura, D. J. (2018). Using block play to enhance preschool children's mathematics and executive functioning: A randomized controlled trial. *Early Childhood Research Quarterly,* 44, 181-191. DOI: 10.1016/j.ecresq.2018.04.006
Siegler, R. S. (2006). Microgenetic analyses of learning. In. D. Kuhn, R. S. Siegler, W. Damon, & R. M. Lerner (Eds.) *Handbook of child psychology: Cognition, perception, and language* (pp. 464–510). John Wiley & Sons Inc.
Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.
Steffe, L. P., Thompson, P. W., & von Glaserfeld, E. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly, & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education*, Laurence Erlbaum Associates, 267-306.
Xin, Y. P., Liu, J., Jones, S. R., Tzur, R., & Si, L. (2016). A preliminary discourse analysis of constructivist-oriented mathematics instruction for a student with learning disabilities. *The Journal of Educational Research, 109*(4), 436-447. DOI: 10.1080/00220671.2014.979910

FIFTH GRADE STUDENTS' MATHEAMTICS IDENTITY, STUDENT ACHIEVEMENT, AND CLASSROOM EXPERIENCES

Jennifer Cribbs
Oklahoma State University
Jennifer.cribbs@okstate.edu

Juliana Utley
Oklahoma State University
Juliana.utley@okstate.edu

Drawing from a sample of 63 fifth grade students in one school setting, we conducted a pilot study exploring students' self-reported mathematics identity and how it correlated with math achievement and experiences in their mathematics classroom. Data were analyzed using linear modeling to determine which variables were predictive of mathematics identity. Results indicate that achievement and students' classroom experiences associated with assessments, a focus on memorization, and teacher strategies were predictive of their mathematics identity. This work informs the field by expanding on our understanding of how important aspects of students' classroom experiences contribute to their mathematics identity development.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Mathematics Identity

Decades of research has shown that non-cognitive factors matter when it comes to student persistence in mathematics (e.g., Boaler & Greeno, 2000; Maltese & Tai, 2011; Sasson, 2021). More specifically, mathematics identity is connected to students' career choice in mathematics and mathematics-related fields (Cribbs et al., 2020; Godwin et al., 2016). Recent research suggests that students' ninth grade mathematics performance, high school STEM GPA, and their mathematics identity are a strong predictor of twelfth grade students' mathematics achievement (Brohrnstedt et al, 2021). Further Zhang and colleagues (2021) found that when compared with other affective measures (self-efficacy, math interest), "mathematics identity had a significant relationship with mathematics achievement" (p. 60). These studies highlight the essential role that mathematics identity has on students' engagement and continued participation in mathematics. However, little research explores this construct quantitatively to better understand trends in identity development throughout students' K-12 schooling experience or how teachers are supporting students' mathematics identity development for students in general (expanding beyond small groups of students). Graven and Heyd-Metzuyanim (2019) indicate that all but two studies in their review of literature on mathematics learner identity had eight or less participants. Additionally, middle grades mathematics (5^{th}-8^{th} grade) is a pivotal time in a student's schooling as it is often a time when attitude and interest toward mathematics begins to drop (Harter, 1981; Marsh, 1989; Zhang et al., 2021). However, there have been calls for teachers to help students' develop a productive mathematics identity, seen through various NCTM publications (Aguirre et al., 2013; Fernandes et al., 2017) as well as in the new CAEP standards for preparing secondary mathematics teachers (NCTM, 2020).

Theoretical Framework

In our study, mathematics identity draws from Gee's (2001) work framing identity in terms of *being a certain kind of person* and Cobb and Hodge's (2011) extension of that work into specific aspects of identity – with this study focusing on core identity. Core identity is described as being a more enduring sense of identity, which becomes more stable as experiences accumulate (Cribbs et al., 2022). The construct of mathematics identity is comprised of three sub-constructs, recognition, interest, and competence/performance, and has been shown to be a valid and reliable measure with adults (Cribbs et al., 2015). Interest is defined as an individual's desire or curiosity to think and learn about mathematics. Recognition is defined as how the individual views themself and how the individual perceives others view them in relation to mathematics. Competence/performance is defined as an individual's beliefs about their ability to understand and perform mathematics.

With these studies and calls from NCTM in mind, the purpose of this study was to develop a mathematics identity instrument that could be used with middle grade students and explore how this measure could be used to examine how mathematics identity correlates with other variables of interest, such as achievement and teacher practice. For the purpose of this paper, the instrument development component is not included. Aligned to this purpose, the following research question guided the current study: Are students' mathematics achievement and classroom experiences in their mathematics classroom predictive of their mathematics identity?

Methods

Participants

This study include 63 students in grade 5 in one school in a mid-western state. These students were in different blocks with the same mathematics teachers. Sixty-three percent of the students identified as male, 35% identified as female, and 2% did not respond. Forty-nine percent of the students identified as White, 14% as multicultural, 13% as American Indian/Native American, 13% as Other, 5% as Black/African American. Additionally, 5% indicated they did not know their race. The average age of students was 11 with a range of 10 to 12 years old.

Data Collection and Analysis

Paper and pencil surveys were administered at the end of the spring semester by one member of the research team. Survey items included demographics (e.g., age, race), a measure for mathematics identity, student math achievement, and items related to students' experiences in their mathematics classroom.

The measure for mathematics identity included 11 items on a Likert-scale (1= not at all, 5=very much so). Four items aligned with the sub-construct recognition (e.g., I see myself as a math person). Three items aligned with the sub-construct interest (e.g., Math is interesting), and four items aligned with the sub-construct competence/performance (e.g., I understand the math I have studied). Mathematics identity items drew from prior research indicating that these items were valid and reliable for adult students (Cribbs et al., 2015). The mathematics achievement variable was measured with the question "on average what grade did you make in math this year" with responses ranging from 1=A to 5=F. Students' responses to classroom experiences included items related to technology (e.g., I used a calculator on tests), conceptual versus procedural learning (e.g., …during math class this year…required memorization of procedures in math class), mathematics teacher characteristics (e.g., my teacher was enthusiastic about mathematics), collaboration and discussion (e.g., we worked in small groups), connections and relevance of content (e.g., my teacher makes connections to real-life math applications), and classroom environment (e.g., students were disrespectful to you). These items were on a Likert scale (1=Never, 5=Always) and drew from prior research (Cribbs et al., 2020).

Linear modeling was conducted to address the research question, exploring if students' mathematics achievement and classroom experiences in their mathematics classroom predicted students mathematics identity.

Results

Table 1 details the results of the linear model. The dependent variable was the mathematics identity construct and the independent variables included students' mathematics achievement and perceptions of classroom practices. Only significant variables, with a significance level of $p < 0.05$, were included in the final model. Gender was also tested in the model as a potential control variable but was also removed as it was not significant.

Table 1 Linear Model Predicting Students' Mathematics Identity

(n=63)	Estimate	SE	t value	p-value	Effect size
Intercept	-4.21	5.56	-0.76	0.45	
Mathematics Achievement[1]	6.16	0.95	6.46	***	0.59
Classroom Experiences					
Required memorization of procedures in math class	2.56	0.63	4.03	***	0.40
Tests or quizzes required new insight and creativity	-1.97	0.70	-2.83	**	-0.30
My teacher highlighted more than one way of solving a problem	3.55	0.79	4.48	***	0.44
My teacher made mathematical errors	-1.68	0.75	-2.23	*	-0.21

*p<0.05 **p<0.01 ***p<0.001

Note: [1] this variable was reverse coded so that a higher value indicated a higher grade in the mathematics course

The results indicate that students' mathematics achievement (as shown by their math grade for the current academic year) was highly predictive of students' mathematics identity. Additionally, four classroom experiences reported by students predicted mathematics identity. Two of these practices were negative predictors "tests and quizzes required new insight and creativity" and "my teacher made mathematical errors." Positive predictors were "required memorization of procedures in math class" and "my teacher highlighted more than one way of solving a problem." These results will be discussed in more detail in the next section.

Discussion

While achievement predicting student outcomes is not a new finding, this study adds to literature by providing evidence of the correlation between student achievement and mathematics identity. Of the variables in the model, mathematics achievement had the largest effect size at .59, indicating a moderate to large effect. This finding is consistent with other research noting that students' prior achievement was predictive of 12th grade students' mathematics identity (Bohrnstedt et al., 2021). While that study explored prior achievement from 9^{th} grade, this study indicates that this relationship exists even at the elementary level.

The predictive classroom practices are more challenging to interpret. Three of the significant items, "required memorization", Tests and quizzes", and "mathematical errors" might align to the idea that students experiencing clear instruction could help them feel more competent with the math concepts. For example, prior research notes that students who experience instruction where teachers "explained ideas clearly" was predictive of students mathematics identity (Cribbs et al., 2020). These results could be representative of students need for a foundational understanding mathematical concepts in order to build their sense of competence and reduce their sense of anxiety. The last significant item related to the students' teacher highlighting more than one way of solving a problem. It is worth noting that this item had the largest effect size after student achievement, indicating the influence it had on students' mathematics identity. Having a chance to see different ways of thinking could be correlated to students' mathematics identity in several ways. While seeing these various methods may aid in students understanding (competence) with the content, this practice could also help students feel as if their ideas are being represented and shared with the class – potentially receiving validation from the teacher and/or their peers (recognition).

It is important to note that these results are very preliminary as the mathematics identity instrument has undergone additional revisions based on the findings from this pilot study. In addition, mathematics achievement and classroom experiences were self-reported by the students. However, students' perceptions of what occurs in the classroom are essential to exploring and understanding students' mathematics identity.

Connecting to the Conference Theme

This line of research, along with other work connected to mathematics identity, helps to provide a picture for how classrooms can serve as spaces for students to feel valued and supported. It is my hope that this unique approach to exploring identity through quantitative methods will serve to move the field forward and provide a lens for improving "learning conditions for each and every mathematics learner." A measure for mathematics identity could provide an efficient way for researchers and/or educators to assess influences on students' identity development. This could include interventions such as informal STEM programs for students or teachers embedding recommended equity-based practices based on professional development or literature.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The Impact of Identity in K-8 Mathematics Learning and Teaching: Rethinking Equity-Based Practices*. NCTM.

Boaler, J., & Greeno, J. G. (2000). Identity, agency, and knowing in mathematics worlds. *Multiple perspectives on mathematics teaching and learning,* 171-200. Ablex Publishing.

Bohrnstedt, G. W., Cohen, E. D., Yee, D., & Broer, M. (2021). Mathematics identity and discrepancies between self-and reflected appraisals: Their relationships with grade 12 mathematics achievement using new evidence from a US national study. *Social Psychology of Education*, *24*(3), 763-788.

Cobb, P., & Hodge, L. L. (2011). Culture, identity, and equity in the mathematics classroom. In A. J. Bishop (Managing Ed.), E. Yackel, K. Gravemeijer, & A. Sfard (Vol. Eds.), Mathematics Education Library: Vol. 48. *A journey in mathematics education research: Insights from the work of Paul Cobb* (pp. 179-195) New York, NY: Springer. doi:10,1007/978-90-9729-3.

Cribbs, J., Hazari, Z., Sadler, P.M., & Sonnert, G. (2015). Establishing an explanatory framework for mathematics identity. Child Development. 86(4), 1048-1062. doi:10:1111/cdev.12363.

Cribbs, J., Hazari, Z., Sonnert, G., & Sadler, P. (2020). College students' mathematics-related career intentions and high school mathematics pedagogy through the lens of identity. *Mathematics Education Research Journal*. https://doi.org/10.1007/s13394-020-00319-w

Cribbs, J., Tassell, J., Hazari, Z., Sadler, P., & Sonnert, G. (2022). Unpacking mathematics identity: Exploring undergraduate students' enduring experiences. *International Journal for Research in Mathematics Education* (RIPEM).12(2), 68-91. https://doi.org/10.37001/ripem.v12i2.2923

Fernandes, A., Crespo, S., & Civil, M. (2017). *Access and equity: Promoting high-quality mathematics in grades 6-8*. NCTM.

Gee, J. P. (2001). Identity as an analytic lens for research in education. *Review of Research in Education*, 25, 99-125.

Godwin, A., Potvin, G., Hazari, Z., & Lock, R. (2016). Identity, critical agency, and engineering: An affective model for predicting engineering as a career choice. *Journal of Engineering Education*, *105*(2), 312-340.

Graven, M., & Heyd-Metzuyanim, E. (2019). Mathematics identity research: The state of the art and future directions. *ZDM*, *51*(3), 361-377.

Harter, S. (1981). A new self-report scale of intrinsic versus extrinsic orientation in the classroom: Motivational and informational components. *Developmental Psychology*, 17(3), 300-312.

Maltese, A. V., & Tai, R. H. (2011). Pipeline persistence: Examining the association of educational experiences with earned degrees in STEM among US students. Science education, 95(5), 877-907.

Marsh, H. W. (1989). Age and sex effects in multiple dimensions of self-concept: Preadolescence to early adulthood. *Journal of Educational Psychology*, 81(3), 417.

National Council of Teachers. (2020). Standards for the preparation of middle level mathematics teachers. https://www.nctm.org/Standards-and-Positions/CAEP-Standards/

Sasson, I. (2021). Becoming a scientist—Career choice characteristics. *International Journal of Science and Mathematics Education*, *19*(3), 483-497.

Zhang, J., Bohrnstedt, G., Park, B. J., Ikoma, S., Ogut, B., & Broer, M. (2021). *Mathematics motivation and its relationship with mathematics performance*. American Institutes for Research.

MODELS BUILT BY UNIVERSITY STUDENTS WHEN SOLVING AN MEA ASSOCIATED WITH THE STEP FUNCTION

MODELOS CONSTRUIDOS POR ESTUDIANTES UNIVERSITARIOS AL RESOLVER UNA MEA ASOCIADA A LA FUNCIÓN ESCALONADA

Carlos Enrique Flores-Gasca
Universidad de Guadalajara
cefg.mat@gmail.com

Verónica Vargas-Alejo
Universidad de Guadalajara
veronica.vargas@academicos.udg.mx

This article presents the results of a research related to identifying and analyzing how university students built, modified, and refined their models to make sense of a situation where the step function underlies. A Model Eliciting Activity [MEA] was designed and implemented in an online environment. The participants in this study were 12 college-level first-term students. The theoretical framework that was used for the design of the activity, the implementation and the analysis of the results was the Models and Modeling Perspective. As a result, it was observed that the students were able to modify and extend their histogram models to step functions.

Keywords: Problem solving, modeling, precalculus, technology.

Introduction

Among the different types of functions, we can find the step function. This type of function is important in the economic and social areas, among others, since it allows describing, manipulating, and predicting various phenomena close to everyday life (Kaput & Roschelle, 2013). Furthermore, in calculus courses, the concept of a step function is a preamble to topics such as limits and continuity.

Researchers such as Doerr, Ärlebäck & Stanic (2014), Kaput & Roschelle (2013), Mochón and Maroto-Cabrera (2002) have identified that students –at different educational levels– present difficulties in modeling situations close to real life associated with piecewise functions. As well as interpret graphs of this type (Mochón & Maroto-Cabrera, 2002).

The objective of this research was to identify and analyze, based on models and modeling perspective [MMP] by Lesh & Doerr (2003), the models that university students built, developed, and externalized during the resolution of an MEA associated with the step function. It was interesting to know how the MEA could contribute to the development of students' knowledge about this function. The research was carried out through the virtual environment Zoom, as a consequence of the COVID-19 pandemic. The research question was, what models did the students build and externalize when solving the MEA "The Tokyo 2021 Olympic Games" in which the concept of step function underlies?

Conceptual Framework

Learning mathematics, based on the MMP of Lesh & Doerr (2003), involves a process of building, modifying, and refining models by students by engaging them in solving MEAs (Brady & Lesh, 2021; Sevinc, 2021). The models that are built at the beginning are often unstable, but these are continually refined and evolved to become shareable and reusable. Models are "conceptual systems (consisting of elements, relationships, operations, and rules that govern interaction models) that are expressed by external notational systems, and are used to construct, describe, or explain the behaviors of other systems" (Lesh & Doerr, 2003, p.10).

From the MMP, the design and implementation of MEAs are proposed as tools to engage students in the need to build models that enable them to describe, manipulate or predict phenomena close to everyday life. MEAs are designed based on the six principles for designing activities (Lesh, Cramer, Doerr, Post & Zawojewsky 2003): a) The Reality Principle, b) The Model Construction Principle, c) The Self-Evaluation Principle, d) The Model externalization Principle, e) The Simple Prototype, and e) The Model Generalization Principle.
Lesh (2010) proposes a *Quality Assessment Guide* to analyze some criteria as shareability and reusability of tools and artifacts that students produce during MEAs. Shareability and reusability "are among the most important criteria which are emphasized implicitly or explicitly in MEA problem statements" (p. 17). The guide permits analyze the types of models that students build, develop, and externalize when solving an MEA. These can be of five levels of action: 1) the model requires redirection, 2) it requires major extensions and refinements, 3) it requires only minor editings, 4) it is useful for this specific data given, and 5) it is shareable and reusable.

Based on the results of an investigation, based on the MMP and the covariational reasoning framework (Carlson et al., 2002), Montero-Moguel & Vargas-Alejo (2022) proposed a "Guide for the evaluation of models related to the concept of function [GEMF], which allows classifying the type of models built by students when solving MEAs that encourage the construction, modification, extension, and refinement of the function concept" (p. 226). Two of the models that the researchers proposed as part of the GEMF are as follows (Table 1): T1 Model, and T2 Model.

Table 1: Classification of models (extract taken from Montero-Moguel & Vargas-Alejo, 2022, p. 227)

Model T1. The model requires direction
The model is not associated with the function (exponential, in this case) that allows to better describe, interpret, predict, and control the situation. Students associate a linear behavior to the situation. Students need additional comments from their classmates or questions that encourage reflection by the teacher, that allow them to redirect their way of thinking. In relation to covariational reasoning, students show level 1 of Carlson et al. (2002, p. 358): "the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1)".
Model T2. The model requires major extensions or refinements
The model is associated with the (exponential) function that best describes the situation; however, students are unable to dissociate linear behavior. The student needs to work further to obtain greater extensions or refinements. Regarding covariational reasoning associated with the function that best describes the situation, the student shows coordination and direction of the variables. Students' reasoning relates to level 2 of Carlson et al. (2002, p. 358): " the images of covariation can support the mental action of coordinating the direction of change of one variable with changes in the other variable ".

In this classification, the results of Carlson et al. (2002) about that the learning of functions requires students develop their covariational reasoning are considered. Covariational reasoning is defined as "the cognitive activities involved in the coordination two varying quantitative while attending to the ways in which they change in relation to each other" (p. 357). Carlson, Larsen, and Lesh (2003) identified that involving students in solving MEAs allows revealing the development of their covariational reasoning. The MEAs "can be designed to promote a significant development of covariational reasoning in a short period of time, they have the potential to provide valuable information for researchers and teachers" (Carlson et al., 2003, p. 470). That is, they allow observing the processes that students use to differentiate, integrate, and refine relevant constructs (Lesh & Doerr, 2003).

Methodology

The research was qualitative because one of the objectives was to know the type of models that the students built and how these were modified and refined when solving the MEA. In particular, the teaching experiments research (Lesh & Kelly, 2000) was used. This type of research focuses on "focus on development that occurs within conceptually rich environments that are explicitly designed to optimize the chances that relevant developments will occur in forms that can be observed". (Lesh & Kelly, 2000, p. 192. One of the characteristics of the teaching of experiments is that the teacher is an integral part of the system he studies. By interacting with it, the teacher can observe how the student modifies, expands, and refines her knowledge when solving MEA (Lesh & Kelly, 2000).

The participants were 12 students, from 18 to 20 years of age. They were starting the Mathematics I course in the first semester of the Bachelor of Economics, so they had not seen, as part of the curriculum, the concept of step function. The modality was online.

The implemented activity (Figure 1) was designed based on the six principles for developing MEAs (Lesh & Doerr, 2003). The context was close to the students' career. The MEA was made up of three parts: a journalistic note of the context of the problem, questions about the context, and one problem. The mathematical idea behind the MEA is the step function. The students had to write a letter addressed to Alexa Moreno, as well as to gymnasts with similar difficulties to get a hotel accommodation which expense was the minimum possible from certain assigned budget as a loan. Data from three different hotels were provided (Figure 2).

> The directors of CONADE, to support the athletes, have agreements with three hotels: Shinjuku Tsukuyami, Tokyo Grand Palace and Dragon Palace. The hotels have accessible costs for the stay of the athletes (page 4, page 5 and page 6). In addition, the directors of CONADE offered a loan of $25,000 pesos. This is so that the athletes can stay during the two weeks of the Olympic Games.
>
> However, the interest rate on the loan is high. Therefore, it is necessary to plan the lodging of the athletes very well (that they spend the least amount of money possible) in any of the three hotels.
>
> Write a letter to the athletes explaining how they should plan their stay at any of the three hotels. You can write the letter in a way that helps athletes plan their accommodations at other hotels in the future.

Figure 1: Extract from the MEA "The Tokyo 2021 Olympic Games"

Figure 2: Extract from the MEA. Data provided of the situation

The implementation of the MEA was carried out in four phases, in an online environment.

1. **Resolution of the warm-up activity.** During the class, the teacher sent the file with the journalistic note to the students through Zoom. Individually they read and answered the questions. The answers were discussed in a group session.
2. **Resolution of the Situation.** The students were grouped into teams of four students to solve the MEA (teams A, B, and C). The teacher interacted through questions, constantly, with each team.
3. **Group discussion and closing of the session.** The teams presented their letters, discussed, and evaluated them with the support of the teacher and the group.
4. **Resolution of the MEA as an extra-class task.** The teacher asked the teams to build a new letter based on the final contributions –models–.

The duration of the MEA implementation was approximately 110 minutes. The data sources were a) the video recording of the session, b) the letters, c) procedures by the students and d) the teacher's notes. For data analysis, two categories of the GEMF (Montero-Moguel and Vargas Alejo, 2022) were adapted to the characteristics of the step function. Two researchers analyzed the results.

MT1, the model requires redirection. The model is not associated with the step function that allows describe and manipulate the situation. Students identify the variables and how they vary but have difficulty describing the relationship between them. Students can draw polygon graphs or histograms, even they can do only operations. They need additional feedback from their peers or the teacher to allow them to redirect their thinking.

MT2 The model requires major extensions and refinements. The model is associated with the function that best describes the situation; however, students have difficulty dissociating the histograms from the step function. Even when they observe the relationship between the variables, they do not identify or find difficult to describe and interpret the relationship between the variables, for example in the points associated with the extremes of each interval of the step function in its different representations.

Results and Discussion

This section describes and discusses the results obtained by implementing the MEA. In particular, the types of models that were built, modified, and refined by the students of team B are described and discussed. The evolution of the models of this team B is representative of what happened in the group.

First Model

Team B initially analyzed how much it would cost the gymnast to stay in each hotel for two weeks. The team found that she would have to pay $25,500 to stay at the first hotel, $24,500 at the second hotel, and $25,000 at the third hotel. As a result of the results obtained, the team carried out the following conversation.

b_1: I think gymnasts can't stay in just one hotel… how much budget do we have?
b_2: It is $21,500… if we consider that Alexa stays the first three nights in the first hotel, the next three nights in another hotel and so on?

After these questions, the team redirected its way of proceeding. The team considered that the budget could be optimized if the gymnasts booked in two or more hotels during their stay.

Observations to the first model. Team B related data based on their initial interpretation to the question posed in the problem. The model was not associated with the step function, it was a MT1. According to MMP "early models are expected to be relatively barren and distorted compared with later models" (Lesh, 2010, p. 17), "the things students see in a given activity often varies a great deal from one moment to another—as attention shifts from one perspective to another, from the big picture to details, or from one type of detail to another" (Lesh, Cramer, Doerr, Post & Zawojewsky 2003, p. 54). Also, the models that students built are shaped through interactions with their peers and the teacher.

Second Model

Team B rethought how the gymnasts would be housed and considered proposing a stay that included more than one hotel to fit the budget.
b_4: Look, if Alexa and the others stay three nights at the Shinjuku hotel, they are only going to pay $6,000, then if she stays another three nights now at the Dragon Palace then she only will pay $4,500… that is already $10,500.
b_3: Then you must consider the price for every three nights in the hotels and that it is… just see what happens from Sunday to Monday.
b_1: From Sunday to Monday, you only pay $500 at the Tokyo Hotel… let is take that for the first week and that it is, we have week one and we do the same with week two.

Once the team had finished comparing the accommodations costs for the gymnasts during their stay, team B plotted the number of nights on the x-axis and the price on the y-axis (Figure 3). The team first plotted the points associated with the price per night of the plan of lodging and then joined the points using line segments. The team related the variables (number of nights and the price) by means of a polygonal-type graph, without giving meaning to the line segments drawn. That is, the team had difficulty coordinating the change in one variable with changes in the other variable (Carlson et al., 2002).

Observations on the second model. According to Vargas-Alejo et al. (2016), it is common that students plot points and join them to represent situations of this type. Team B's model was not associated with the step function. The students needed to interact with their teacher, who through questions made them reflect on the relevance and meaning of the segments that joined the dots. The model was MT1 but compared to the first model, it has more representations where the identification of variables and how they are related is observed.

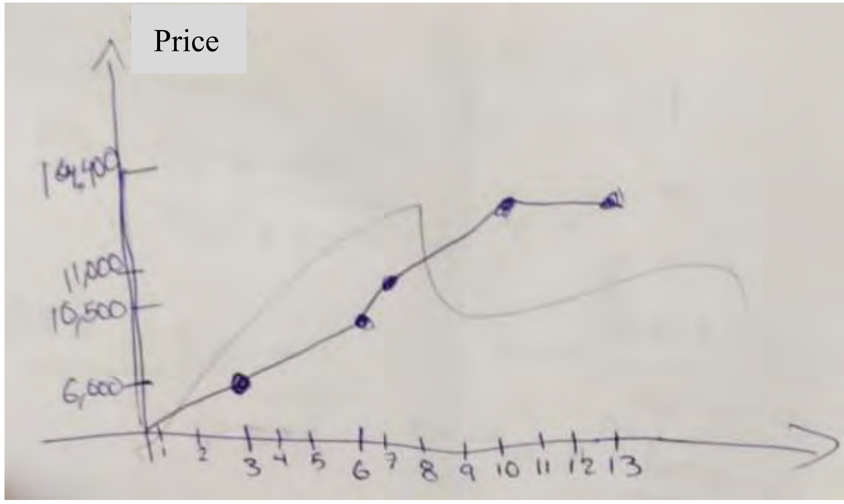


Figure 3: Polynomial type graph built by Team B

Third Model

Team B changed its way of thinking, since the teacher intervened in the discussion by asking the following questions: what is price? What do the plotted points mean in the context of the problem? What do the segments connecting the dots mean in the context of the problem? The team answered the first question as follows.

b_3: Well, the price that Alexa is going to pay due to the hotel accommodation, right, teacher?
b_2: The price is not... it would be the cost, right? It's just that the microeconomics' teacher told us that the cost is what the good cost or the product [*the student was trying to explain the concepts*] and the price is the cost plus the profit.
b_3: You are right teacher; it is the cost.

The team answered the second question as follows.

b_2: Look, where it costs three on the axis below [He pointed to the x-axis] there are three nights teacher, and there the gymnasts must pay \$6,000, where there are six nights there they must pay \$10,500 and so on.

To answer the third question, the team had difficulties.

b_4 : we don´t know teacher, they always come together, right?

Based on this interaction, Team B changed their thinking and wrote their gymnasts' accommodations proposal in terms of cost per number of nights. The Team constructed a histogram (Figure 4).

Observations to the third model. According to De Villiers (1988) and Vargas-Alejo et al. (2016) the students often create histograms to model problems like the one discussed in this document. It was necessary for team B to interact with their classmates and the teacher, during the group discussion (phase 3), so that they could analyze, redirect their way of thinking, and modify their model, which was still MT1. The team still had difficulty representing the coordination of the change of one variable with changes in the other variable (MA1, Carlson et al., 2002).

According to the data provided, you have a loan of $21,500 pesos. In addition, we consider that your stay could be like this:

In the first week, you could stay (from Monday to Thursday) at the "Shinjuku Tsukuyami" hotel with a cost of $6,000, the following 3 nights, you will stay at the "Dragon Palace Hotel" with a cost of $4,500. The last night, you will stay at the "Tokyo grand Palace hotel" with a cost of $500. In total, the first week you will pay $11,000.

In the second week you will stay at the "Tokyo Grand Palace" hotel at a cost of $8,400.

In total, you will pay $19,400 for two weeks.

Figure 4: Extract from Team B Letter

Fourth Model

During phase 3, the teacher asked the students to present their models and discuss them for review, comparison, and evaluation. The students identified that the step function model was the best to describe the situation. Team B presented the following model as an extra-class task (Figure 4 and Figure 5).

Dear athletes,

We provide you with the following proposal for your accommodation in Tokyo, Japan. With this proposal, you will be able to participate in the Olympic Games this year. We know that CONADE's resources are scarce. We also know that CONADE lent them the amount of $21,500 pesos.

The amount of money that CONADE lent you is limited, and unfortunately it must cover all your expenses. To help you, we carried out an exhaustive review of the packages offered by hotels (which have an agreement with CONADE); We detail below the plan that we consider beneficial for you and your present.

Week 1

Three nights (Monday-Tuesday, Tuesday-Wednesday, Wednesday-Thursday) you will stay at the "Shinjuku Tsukuyami" Hotel, you will pay $2,000 pesos per night. The following two nights (Thursday-Friday, Friday-Saturday) you will stay at the "Grand Palace" Hotel. In that hotel you will pay $1,500 pesos. The last night of this week (Saturday-Sunday) you will stay at the same hotel (Grand Palace). You will pay $500 pesos. In total, you will pay a cost of $9,500 pesos in the first week.

Week 2

Three nights (Sunday-Monday, Monday-Tuesday, Tuesday-Wednesday) you will stay at the "Grand Palace" hotel at a cost of $2,000 pesos per night. The following two nights (Wednesday-Thursday, Thursday-Friday), you will stay at the "Shinjuku Tsukuyami" hotel at a cost of $1,200 pesos per night. The last night (Friday-Saturday) you will stay at the "Tokio Grand Palace" hotel. In that hotel they will pay $600 pesos. In your second week, you will pay a total cost of $9,000 pesos.

Based on our proposal, you will pay a total of $18,500 pesos for two weeks. With this proposal, you will save $3,000 pesos. With this money, you can pay off the CONADE loan faster. In addition, the interest on the loan would be lower. We take the opportunity to add our graphic proposal for a better understanding, so that you can visualize and examine it. Greetings athletes. Good luck.

Figure 4: MT2 of Team B

It can be seen in Figure 4 that team B modified the plan of lodging, in such a way that the gymnasts could pay a lower amount. The team proposed a stay of $18,500, that is, $1,000 less than the budget described in the third model. Figure 5 shows how the team described the planning through graphical and tabular representations. It considered that the gymnasts should pay $6,000 for the first three nights, then they should pay $3,000 for the next two nights, and finally they should pay $500; $9,500 for the first week. While, for the second week, the team considered that for the first three nights they should pay $6,000, for the next two nights they should pay $2,400 and, finally, for the fifth, and sixth nights they should pay $600.

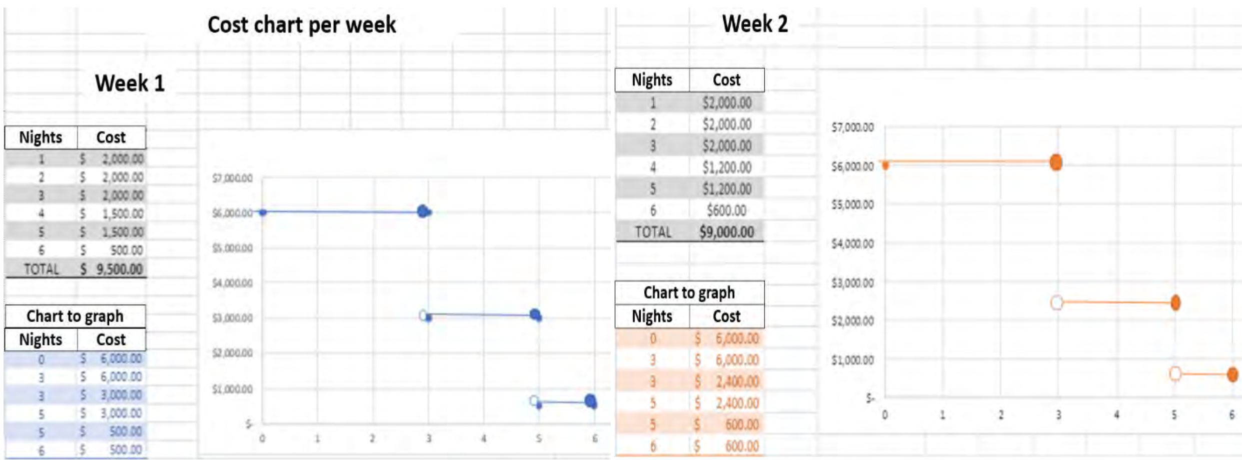


Figure 5: MT2 of Team B

Observations to the fourth model. Team B identified and related the variables that were involved in the situation through verbal, written, tabular, and graphic representations. The total cost was considered constant for each interval of number of nights. The team optimized the total cost of the gymnasts' stay compared to its previous proposal. It had difficulty denoting the endpoints of each interval of the step functions on their graph; however, the team managed to coordinate the direction of change of one variable with respect to the other variable (Carlson et al., 2002). The model that the team B built was associated with the step function, it can be classified as MT2. According to Kaput and Roschelle (2013), it is natural that students have difficulties in constructing graphs of the step function. Especially when they have not been taught in their math courses to model problems involving the use of constant and discrete functions.

Final thoughts and conclusions

In response to the question, what models did the students build and externalize when solving the MEA "The Tokyo 2021 Olympic Games" in which the concept of step function underlies? Team B students built and externalized two types of models: MT1, which was modified and extended from group interaction to become MT2. Although this last model still requires refinement, the model was no longer a histogram, it had characteristics of the step function. The reported findings show us how the students developed knowledge about the step functions and gave it meaning in terms of the proposed situation.

Acknowledgment

The research was supported by CONACYT through postgraduate student scholarships.

References

Brady, C., & Lesh, R. (2021). Exploring Mathematical Modeling with Young Learners. In J. M. Suh, M. H. Wickstrom & L. D. English (Eds.), *Development in Mathematical Modeling* (pp. 95-110). Springer.

Carlson, J. S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: a framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Carlson, J. S., Larson, S., & Lesh, R. (2003). Integrating a Models and Modeling Perspective with Existing Research and Practice. In R. Lesh, & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp-465-478). Mahwah, NJ: Erlbaum.

Doerr, H. M., Ärlebäck, J. B., & Stanic, A. C. (2014). Design and effectiveness of modeling-based mathematics in a summer bridge program. *Journal of Engineering Education*, *103*(1), 92–114.

Doerr, H. M. (2016). Designing sequences of model development tasks. In C. R. Hirsch & A. R. McDuffie (Eds.), *Annual Perspectives in Mathematics Education 2016: Mathematical modeling and modeling mathematics* (pp. 197-205). National Council of Teachers of Mathematics.

Kaput, J. J., & Roschelle, J. (2013). The mathematics of change and variation from a millennial perspective: new content, new context. In S. J. Hegedus, & J. Roschelle (Eds.), *The SimCalc Vision and Contributions* (pp. 13–26). Springer.

Lesh, R. (2010). Tools, researchable issues and conjectures for investigating what it means to understand statistics (or other topics) meaningfully. *Journal of Mathematical Modeling and Application*, *1*(2), 16-48.

Lesh, R., & Doerr, H. M. (2003). Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving. In R. Lesh & H. M. Doerr (Eds.), *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching* (pp. 3-34). Routledge.

Lesh, R., Cramer, K., Doerr, H. M., Post, T. & Zawojeswski, J. S. (2003). Model Development Sequences. In R. Lesh & H. M. Doerr (Eds.), *Beyond Constructivism. Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 35-58). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000), Principles for Developing Thought-Revealing Activities for Students and Teachers. In A. Kelly & R. Lesh (Eds.), *Research Design in Mathematics and Science Education* (pp. 591-646). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lesh, R., & Kelly, A. (2000). Multitiered teaching experiments. In A. E. Kelly y R. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum associates.

Mochón, S., & Maroto-Cabrera, P. (2002). A study of the conversion from table to graph: selecting the type of graph according to the context of the problem. In D. Mewborn (Ed.), *Proceedings of the twenty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vols. 1-4, pp. 444-454). Columbus.

Montero-Moguel, L. E., & Vargas-Alejo, V. (2022). Ciclos de modelación y razonamiento covariacional al realizar una actividad provocadora de modelos. *Educación Matemática*, *34* (1), 214-248.

Sevinc, S. (2021). Toward a Reconceptualization of Model Development from Models-and-Modeling Perspective in Mathematics Education. *Educational Studies Mathematics*, 102, 193-219.

Vargas-Alejo, V., Reyes-Rodríguez, A., V. & Cristóbal-Escalante, C. (2016). Ciclos de entendimiento de los conceptos de función y variación. *Educación Matemática, 28* (2), 59-83.

UNDERGRADUATE STUDENTS' CONCEPTIONS ABOUT COMPLEX NUMBERS: A TRAJECTORY OF THEIR MENTAL STRUCTURES

Diana García Caro
Universidad de Guadalajara
diana.garcia2183@alumnos.udg.mx

Carlos Valenzuela García
Universidad de Guadalajara
carlos.valenzuela@academicos.udg.mx

María T. Sanz
Universitat de València
m.teresa.sanz@uv.es

María S. García González
Universidad Autónoma de Guerrero
msgarcia@uagro.mx

This paper describes the conceptions about complex numbers that a group of university students has, these were built from the application of an activity sequence centered on these numbers. This sequence is based on the APOS theory, some aspects of semiotic representation theory, and the use of digital technology. Particularly, both the general results of a pretest and a posttest are shown and compared. Additionally, the example of a student is analyzed to show evidence of how the mental structures and mechanisms that define the students' conceptions are built through the implementation of the sequence. The results show how the activity sequence allowed students to coordinate algebraic and geometric processes on complex numbers to improve their conceptions.

Keywords: Complex Numbers, APOS Theory, Conceptions, Undergraduate.

Introduction

Researchers who have taken the teaching-learning of complex numbers as an object of study have reported various difficulties in students and trends in the teaching of this concept (Panaoura et al., 2006; Pardo and Gómez, 2007; Randolph and Parraguez, 2019). In teaching, they have identified an emphasis on the algebraic register, which results in a loss of essential characteristics of complex numbers (Distéfano et al., 2012). This teaching approach limits students in the construction of mental structures and mechanisms related to the geometric representation of complex numbers in their different forms. Therefore, the need to include the geometric register in teaching is established to favor the conversion and coordination of processes (Aznar et al., 2010) for the characterization of the elements of the concept under study. Derived from this problem, the research objective was to improve the conceptions about the complex numbers of students who start their university studies from the design and implementation of a teaching sequence based on the APOS theory, the use of digital technology, and with a focus on the geometric representation of those numbers. In this way, we provide an answer to the following question: how do students' conceptions of complex numbers change when they participate in a teaching sequence with the aforementioned characteristics?

Theoretical Framework

In order to become the objective of this research, we consider the elements of the APOS theory, initially developed by Dubinsky and a group of researchers (Arnon et al., 2014), as well as some elements of the theory of semiotic representations (Duval, 2006). From the APOS theory, a student's conceptions are developed in a context when he is faced with solving problems that involve the use of mental structures and mechanisms (Dubinsky, 2014). The structures are actions, processes, objects, and schemes that are transformed from the use of mechanisms such as: internalization, coordination, reversion, encapsulation, decapsulation, among others. These transformations and their relationships are shown in Figure 1, which is adapted from the work of Arnon et al. (2014).

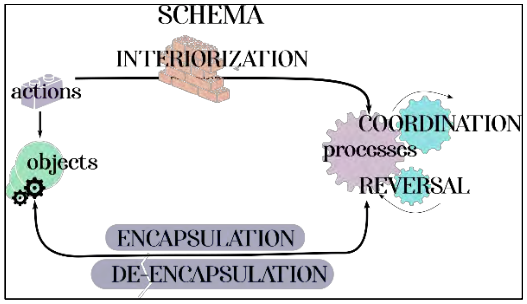


Figure 1: Mental structures and mechanisms cycle

The cycle begins with actions on previously constructed mental objects. In this conception, the student needs external help, which is why they require explicit steps for execution. When the actions are reflected, they are internalized and result in a process. Another way to build processes is through the coordination and reversal of two or more previous processes. In this conception, the student can explain the generality of the concept and describe its characteristics. To favor the development of this mental structure, we consider, like Duval (2006), that mathematical activity needs the articulation between the different semiotic representation systems, and this is achieved when the student can make treatments and conversions, these transformations allow the construction of processes and consequently when these processes are encapsulated, mental objects are achieved.

The object is established when the process can be seen in its entirety, that is, the student can perform actions on this object through encapsulation and it can be returned to the processes that made up the object through decapsulation. Thus, if the set of these structures is consistent, it allows the construction of the scheme. This cycle of structures and mechanisms constitutes the theoretical aspect of the APOE theory, but the same theory contemplates methodological elements, which is called the research cycle.

Methodology

For the development of the research, the research cycle proposed from the APOE theory (Asiala et al., 1996) was followed. This cycle is made up of three elements (Figure 2). The first element is the theoretical analysis, which is developed from the epistemology of the mathematical concept and results in the construction of a genetic decomposition. Arnon et al. (2014) defines genetic decomposition as a hypothetical model that describes the structures and mechanisms that an ideal student requires to understand a concept. The second element is the design and implementation of the instruction, this design addresses the characteristics of genetic decomposition, and leads and transforms the previous conceptions of the students, which is explained from the cycle of mental structures and mechanisms. The third element is the collection and analysis of data, this enables the comparison of the preliminary genetic decomposition and the developments obtained in the analysis. It is worth mentioning that this cycle is repeated with the intention of perfecting the genetic decomposition.

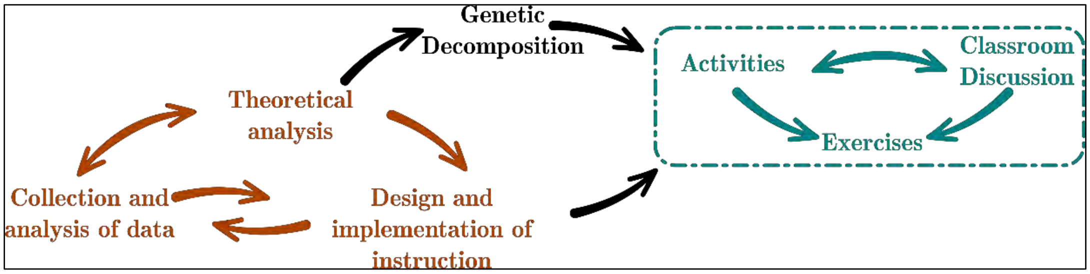


Figure 2: Research cycle and ACE Teaching Cycle

The implementation of the activities was developed from the ACE teaching cycle (Arnon et al., 2014). The acronym for this cycle (Figure 2) presents its elements: (A) activities, (C) class discussion and (E) exercises. In the first moment of the cycle, the activities are guided and elaborated tasks to favor the construction of mental structures arranged in the genetic decomposition, from this moment it is suggested to incorporate teamwork. The second moment is the discussion in class, here it is possible for the students to work in small groups, and that the discussions are guided by the teacher and complemented through worksheets and technological tools. The third moment considers the exercises, whose purpose is to help to consolidate and evaluate the conceptions that have been developed in the activities and the discussion in class. Therefore, the activities of this research were designed taking into account the genetic decomposition and the ACE cycle, for reasons of space and the focus of this writing, no details are given about the design, but they can be consulted at the link: https://www.geogebra.org/m/zx5ww8bf.

Population and method

The investigation began with the application of a pretest, later the sequence of activities was implemented considering the ACE teaching cycle, and finally a posttest was applied. The questionnaires have the same characteristics and are made up of two parts, their application was made individually. The sequence has ten activities and has two teaching cycles. The first cycle comprised of activity 1 to activity 6 with discussions every two or three activities, as required by the group, and ended with activity 7. The second cycle is made up of activities 8 and 9, a general discussion, and ended with activity 10. Activities 7 and 10 were considered exercises. The activities were carried out in teams. The general study was developed with a group of 15 first-semester students of the degree in electrical mechanics from a public university in Mexico. The application of the questionnaires and the sequence of activities was carried out in the hybrid modality.

To carry out the analysis, initially a general comparison was made between the pre- and post-test results of the entire group. Subsequently, and to account for how the conceptions of the students changed throughout the implementation of the sequence, one student (E1) was chosen. According to the teacher's criteria, E1 was characterized by having a performance below the group average, but was responsible, participative, showed interest in learning mathematics and, in addition, showed interest during the implementation of the sequence. Given that E1 teamed up with another student who was characterized by having an above-average performance (E2), this last student will also be mentioned in the results.

Result

In the first part of the pretest, students were inquired about previous objects related to complex numbers. That is, to solve a quadratic equation whose solutions are complex, and perform two operations in which the imaginary unit had to be recognized. In the general results, the students did not perform actions on the imaginary unit object from the square root of a negative number, so there is no imbalance in their mental structures. This is because 9 of 15 students recognized the square root of a negative number as a solution to the equation (Figure 3a). Another 4 of 15 students recognized the imaginary unit, but had errors in the algebraic treatment (Figure 3b).

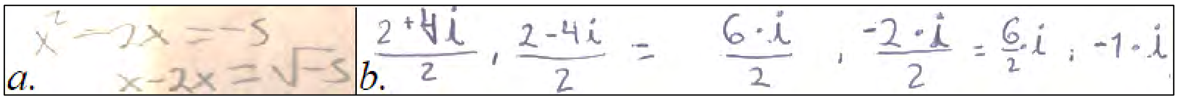


Figure 3: students' responses of the first part of the pretest

The second part of the pretest includes explicit elements of complex numbers. 3 reagents were proposed, in the first and second it was asked to determine the module and the argument of a complex number: first from its representation in the complex plane, and then from its binomial form. In question 3, it was asked to determine the real part and the imaginary part of a complex number given its module and argument. The results show that 13 of 15 students did not answer that part of the questionnaire (Figure 4a) and only 2 of the 15 presented actions on the object prior to distance to establish the value of the module (Figure 4b). In general, most of the students did not present previous conceptions related to complex numbers.

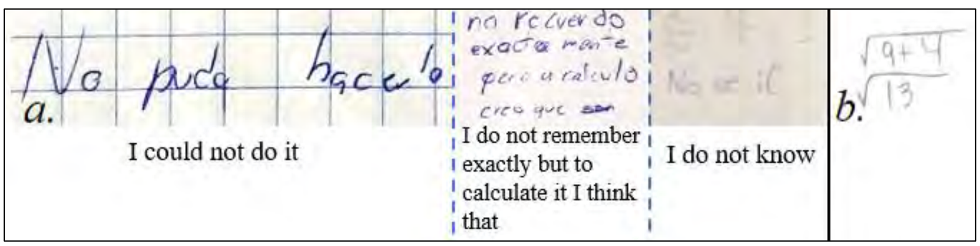


Figure 4: students' responses of the second part of the pretest

In general, the results in the pretest allowed us to conclude that the students had difficulties and did not associate their previous objects with the complex numbers, and therefore did not perform actions on them. On the other hand, in the post-test there is evidence of the construction of new mental structures such as actions, processes and objects. In the first part, 11 out of 15 students presented approximations to the imaginary unit more frequently than in the pretest (Figure 5). There are processes on complex numbers (Figure 5a); however, in the first question, 8 of 15 students showed difficulties in operating with fractions associated with the imaginary part of the complex number (Figure 5b). In the operative part, second question, 6 of 15 students presented difficulties in the properties to operate with the imaginary unit (Figure 5c).

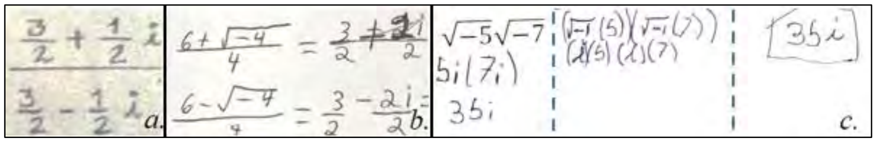


Figure 5: students' responses of the first part of the postest

In contrast, in the second part of the post-test, 14 of 15 students presented an important evolution in the construction of new mental structures through the characteristics of complex numbers in their algebraic and geometric registers. Geometric and algebraic actions are evidenced on objects such as: coordinates, angle, distance between two points, trigonometric ratios, among others (Figure 6a). The students identified the explicit elements of each notation and used the different complex number notations in their treatments, in particular ordered pair, polar and trigonometric (Figure 6b).

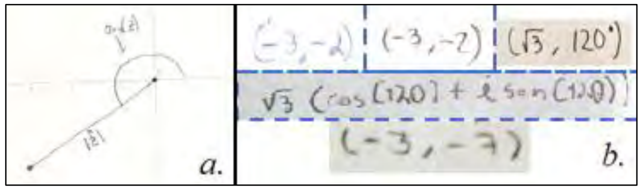


Figure 6: students' responses of the first part of the postest

In the second part of the post-test there is evidence of algebraic and geometric processes and their combinations that allowed the students (13 out of 15) to develop generalities for the representations of complex numbers. It is worth mentioning that the group discussion and teamwork guaranteed these generalities, since in the post-test questionnaire the students worked with specific cases. The geometric processes allowed the students to represent the elements of the complex number in the different quadrants of the complex plane, as well as to convert to the algebraic register with the use of the trigonometric ratios and the distance between two points. The students performed treatments on the algebraic register to find the values of the module and argument. They used generality to find the argument from the arctangent with the real part and the imaginary part, the students added 180° when considering that the complex number is located in the second or third quadrant (Figure 7a). Similarly, these processes are also explicit in the item in which the module and the main argument are explicitly given to obtain the real and imaginary part and represent the number in the complex plane (Figure 7b).

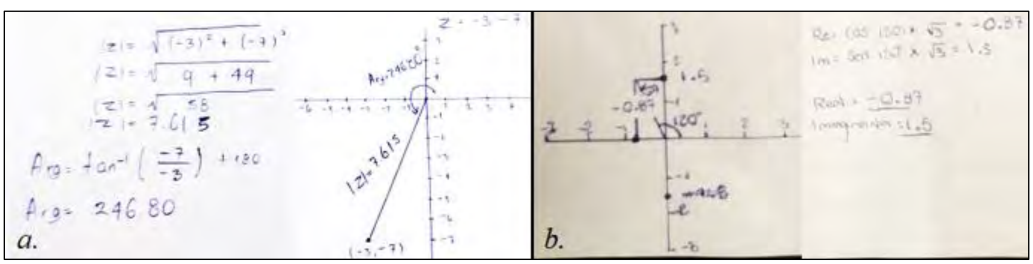


Figure 7: Algebraic and geometric processes in student's responses

The results allow us to establish that, in general, the students built new mental structures from the development of the sequence of activities. To detail these structures, the case of a student (E1) is exemplified, for this the trajectory of it is established and a description of its conceptions is made. In the results of the first part of the pretest, the student did not present actions on previous objects of complex numbers and presented difficulties on objects such as solutions of quadratic equations and properties of real numbers. E1 did not answer the second part of the pretest.

As mentioned, in the development of the sequence of activities E1 worked as a team with E2. In activities 1 and 2, the students built actions on previous objects such as: angle, distance between two points, coordinates and trigonometric ratios. They identified in the complex plane each one of the elements of the complex numbers, these actions were established when they represented different complex numbers in the quadrants of the plane through the applet. An example of these algebraic actions is shown in Figure 8, in which the students worked in the second quadrant. Also shown are the worksheets in which they determined the modulus and argument values for a specific complex number.

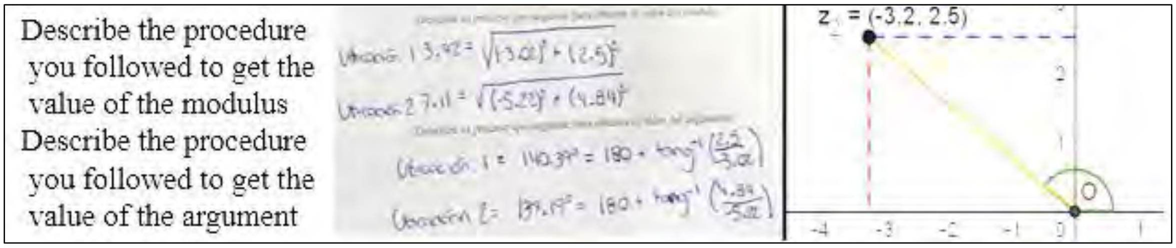


Figure 8: E1 and E2's response in activity 2 of the sequence

The geometric actions of the students considered specific complex numbers within the complex plane, identified the different elements of the complex numbers, and found their values by means of the trigonometric ratios and the distance between two points (Figure 8). In particular, E1 participated in the team discussion mentioning the differences of the complex numbers when they are in each quadrant. From the reflection of the actions in each of the quadrants, the students established the generality to find the module, an argument (Figure 9), the real part and the imaginary part, which gave way to algebraic and geometric processes.

¿Cómo hallaste los argumentos? Explica tu desarrollo

[sic]

como el valor de x es 0 en ambos tenemos que tomar el valor de y. y así podemos opservar sus angulos. si es positivo son 90 grados, si es negativo son 270 grados.

How did you find the arguments? Explain your procedure

since the value of x is 0 in both we must take the value of y, and thus we can observe their angles. If it is positive, it is 90 degrees and if it is negative, it is 270 degrees.

Figure 9. E1 and E2's response in activity 3 of the sequence

Algebraic and geometric processes helped students build and relate the elements of complex numbers in each of these representations. The students started from the geometric register in its polar form and made the conversion to the algebraic register, this allowed them to characterize and find the real part and the imaginary part. These constructions were based on their previous knowledge about trigonometric ratios, in this way E1 and E2 found the generality through the reflection of examples that they chose with the applet. E1 systematized the particular cases raised by E2, this allowed E1 to raise the generality, which he expressed in the discussion with E2 (Figure 10). This generality was achieved from the coordination of geometric and algebraic processes. Regarding the processes related to the module and an argument, the students worked in activities 5 and 6 with sets of complex numbers. The students were able to describe different sets of complex numbers with their elements in the cases of a constant modulus and argument.

PARTE REAL= COS ARGUMENTO X MODULO	PARTE IMAGINARIA = SENO ARGUMENTO X MODULO
PARTE REAL = COS ALFA X R	PARTE IMAGINARIA = SENO ALFA X R
Real part= cos argument x modulus	Imaginary part= sine argument x modulus

Figure 10. E1 and E2's response in activity 4 of the sequence

In activity 7, E1 worked individually, evidencing processes that involve reversal, since in previous activities they usually started with the geometric register of complex numbers, but here they started with the algebraic register. E1 presented in the development of the questions algebraic and geometric processes, when using and describing the generalization of the characteristics of the complex number. Additionally, E1 presented a characterization of the representations, in particular the ordered pair and trigonometric form, and performed conversion and treatment to determine the values of the module and the argument of a number (Figure 11).

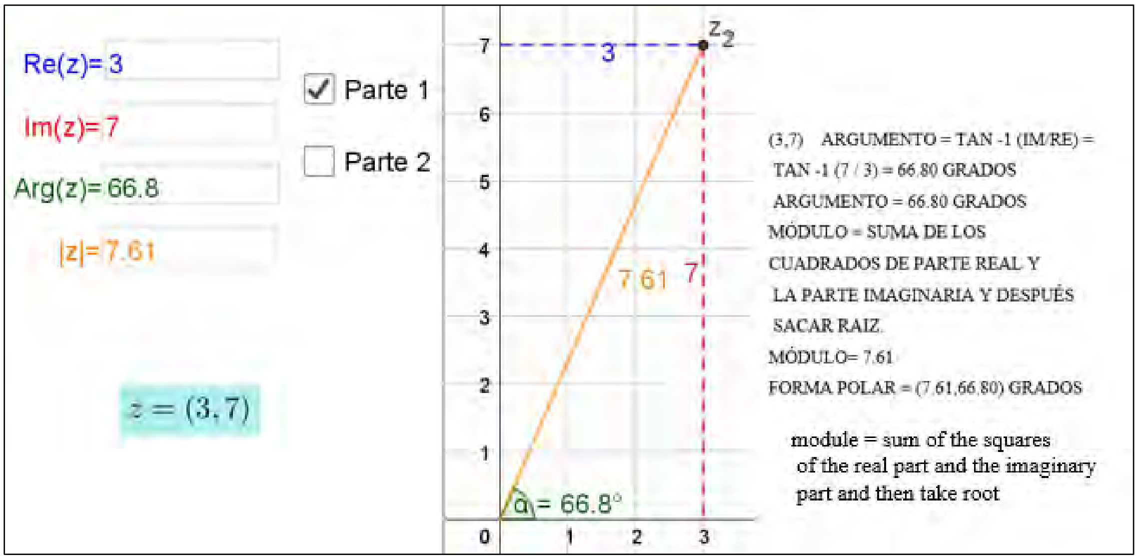


Figure 11: E1 and E2's response in activity 7 of the sequence

The second cycle of teaching is made up of activities 8 and 9. These activities are focused on the arguments and the imaginary unit. In activity 8, students E1 and E2 presented difficulties in establishing the generality of the main argument when they worked as a team. However, in the general discussion some students presented different results, and in this way students E1 and E2 reflected and were able to generalize the arguments from the main argument, and from additional questions asked by the teacher. In this sense, Oktaç et al. (2019) affirm that there is no division between the teacher and the researcher since they have the common goal of enabling the construction of mental structures. Additionally, the authors establish that the construction of mental structures does not always occur in a linear fashion.

In activity 9 the students established the relationship between the modules and the arguments from the geometric register, and the students understood the binomial and trigonometric notations of complex numbers. This cycle concluded with activity 10, which was worked on individually and is called an exercise. Here, E1 coordinated previous processes to build new algebraic and geometric processes that allowed him to characterize elements of the new representations (Figure 12a), as in the post-test questionnaire where the construction of these mental structures in the student was evidenced, he identified and related those elements of the complex number, in addition, used and understood the different representations and notations of complex numbers (Figure 12b).

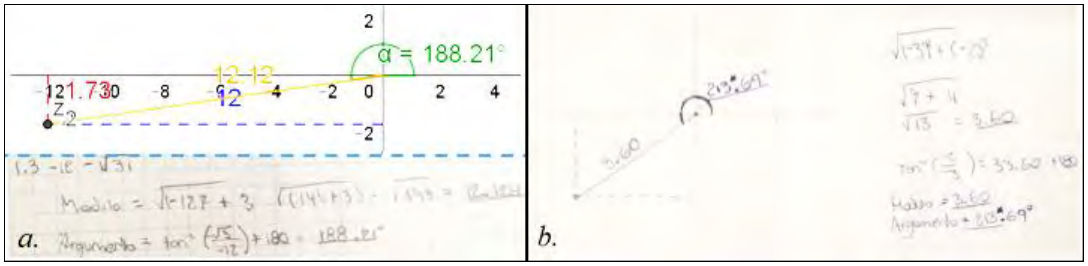


Figure 12. E1 and E2's response in activity 10 of the sequence

Conclusions

The results show that most of the students built mental structures related to complex numbers from the development of the proposed sequence of activities, thus improving their conceptions. The sequence guided students to start from previous objects that led them to activate their mechanisms and articulate the different representations of complex numbers. This allowed generating and coordinating algebraic and geometric processes that in some students can be characterized as objects due to their argumentation in the discussions and their development in the worksheets.

An important element in the design of the sequence of activities is the approach that was given to the registry of geometric representation of complex numbers to later make a conversion to the algebraic registry, and in this regard, evidence was given of how the conversion and treatment in this record allowed students to reach higher level mental structures. In addition, the interaction that the students had with the applets that make up the sequence gave rise to the first actions that led the students to internalize them to build processes on complex numbers. Similarly, the general discussions contemplated in the teaching cycle allowed the students to present their results and argue based on the work with their classmates. According to the results, it is essential that the teaching of complex numbers involves different representations where students have the possibility of expressing generalities about the different elements of these numbers.

References

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Roa-Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS Theory: A framework for research and curriculum development in mathematics education*. Springer.

Asiala, M., Brown, A., DeVries, D., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput., A. Schoenfeld., & E. Dubinsky, (Eds.), *Research in Collegiate Mathematics Education* (2nd ed., pp.1-32). The American Mathematical Society.

Aznar, M., Distéfano, M., Prieto, G., & Moler, E. (2010). Análisis de errores en la conversión de representaciones de números complejos del registro gráfico al algebraico. *Revista Premisa*, *12*(47), 13-22.

Distéfano, M., Aznar, M., & Pochulu, M. (2012). Errores asociados a la representación geométrica-vectorial de números complejos: un análisis ontosemiótico. *Unión. Revista Iberoamericana de Educación Matemática*, 30, 61-80.

Dubinsky, E. (2014). Actions, Processes, Objects, Schemas (APOS) in Mathematics Education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education* (pp. 8-11). Springer Netherlands.

Duval, R. (2006). Un tema crucial en la educación matemática: La habilidad para cambiar el registro de representación. *La Gaceta, 9*(1), 143-168.

Oktaç, A., Trigueros, M., & Romo, A. (2019). APOS theory: connecting research and teaching. *For the learning of mathematics*, *39*(1), 33-37.

Panaoura, A., Elia, I., Gagatsis, A., & Giatilis, G. (2006). Geometric and algebraic approaches in the concept of complex numbers. *International journal of mathematical education in science and technology*, *37*(6), 681-706.

Pardo, T., & Gómez, B. (2007). La enseñanza y el aprendizaje de los números complejos: un estudio en el nivel universitario. *PNA*, *2*(1), 3-15.

Randolph, V., & Parraguez, M. (2019). Comprensión del Sistema de los Números Complejos: Un Estudio de Caso a Nivel Escolar y Universitario. *Formación universitaria*, *12*(6), 57-82.

SELF-EFFICACY DURING THE COLLEGIATE TRANSITION AND THE IMPORTANCE OF TASK ALIGNMENT: A CASE STUDY

Jordan R. Gill
Lansing Community College
gillj12@star.lcc.edu

This case study examined how a period of transition could affect a student's mathematics self-efficacy (MSE) and their ability to productively struggle on challenging mathematics problems. This study focused on a single student named Candice who was making the transition from high school to college. Candice initially reported high levels of MSE to start the semester but experienced difficulty adjusting to her first college mathematics class, which was taught in an active learning style. Interviews with Candice revealed that her initially reported high MSE was associated with her ability to successfully complete her high school mathematics courses. The structure of her active learning style classroom presented new challenges that did not align with this MSE. This potentially resulted in a lowered state of MSE and a lack of engagement with productive struggle both inside and outside the classroom.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Undergraduate Education; Computational Thinking; Problem-Based Learning

Mathematics self-efficacy (MSE), or a person's belief in their ability to perform or learn mathematics, has repeatedly been linked to positive outcomes such as achievement in mathematics (Hackett & Betz, 1989; Multon et al., 1991; Thomas et al., 1987; Usher et al., 2018) and persistence on mathematical tasks (Lent et al., 1984; Multon et al., 1991, Ruch et al., 2014). Despite the link to positive outcomes, little work has been done to examine how MSE changes during the transition from high school to college. This is especially important when considering that the transition to college can bring new levels of expectations and different types of tasks than what students may be used to. This case study focuses on a single first-year college student who reported high MSE to start the semester but struggled to adapt to the collegiate environment.

Theoretical Framework

MSE refers to peoples' beliefs in their ability to perform or learn mathematics successfully (Bandura, 1986; Zimmerman & Cleary, 2006). Research suggests that judgements of self-efficacy are task-specific (Bandura, 1986; Pajares & Miller, 1995), that is, self-efficacy for a class or the field of mathematics (generalized MSE) might not match MSE for solving specific problems (task-specific MSE). However, those with high MSE for problem solving may also be more likely to hold high MSE for being successful in mathematics courses, which potentially explains why measures of MSE that are not task specific (generalized MSE) often predict problem solving outcomes. As put by Pajares and Miller (1995), "prediction is enhanced as self-efficacy and performance correspondence more closely matches."

Productive struggle is defined by Hiebert and Grouws (2007) as the effort a student must put forth in order to understand a difficult mathematical concept. Hiebert and Grouws emphasize that the word struggle does not relate to any feelings of frustration or despair when facing overly difficult problems but is instead a reflection of an individual's willingness to "muddle through" (p. 388). Self-efficacy is thought to play a major role in student's willingness to engage with productive struggle. Research agrees that those with higher levels of self-efficacy will be more likely to persevere in the face of obstacles (Lent et al., 1984; Multon et al., 1991, Ruch et al., 2014). Research also agrees that students with high MSE are often more effective when facing difficult tasks that require them to analyze situations and choose appropriate mathematical techniques (Holenstein et al., 2021).

Literature Review

Research suggests that MSE is important as students with high levels of MSE are commonly associated with higher achievement (Multon et al., 1991; Thomas et al., 1987, Usher et al., 2018) along with the choice to pursue mathematics-related majors (Hackett & Betz, 1989). Additionally, research suggests that those with higher MSE are more likely to persist in both long-term and short-term situations. Research suggests that those with high MSE are more likely to persist within mathematics related majors (Lent et al., 1984) and are more likely to persist on mathematical tasks (Multon et al., 1991). Much of this previous research has been quantitative in nature and there is a need for qualitative support.

In addition, for students undergoing the transition from high school to college mathematics courses, self-efficacy may play a vital role in this transition (Chemers et al., 2001); in particular, high MSE may be vital to help students overcome the struggle of transitioning to college mathematics. However, while MSE may aid transition, it should also be noted that the transition itself can have potential effects on MSE. Bandura (1986) suggested that periods of transition may be the time when self-efficacy levels are most likely to be disrupted. While research generally agrees that MSE is stable long-term (Bandura 1986; Middleton & Spanias, 1999), the issue of stability is rarely addressed in research (Lane & Lane, 2011).

Research also puts particular importance on the role of productive struggle in generating deep mathematical understanding. Research suggests that the process of productive struggle is vital to understanding new conceptual ideas (Heibert & Grouws, 2007; Handa, 2003). Students who engage in productive struggle may develop higher levels of creativity and a greater ability to reflect on their own ideas (Nadjafikhah et al., 2012). Like with self-efficacy, periods of transition could affect a student's willingness to engage in productive struggle. Previous research has mostly been devoted to conceptualizing the process of productive struggle or identifying whether it occurred (Zeybek, 2016). Little work has been done to examine how student engagement with productive struggle may change in new settings. In the present study, I will examine both MSE and productive struggle during a period of transition, specifically for a first-semester college student who experiences new task expectations.

Methods

This study focuses on a first-year calculus student named Candice who took part in a larger study examining transitionary effects on MSE and mindset for collegiate mathematics students. Students from three courses (intermediate algebra, calculus, and discrete) were invited to take a survey that focused on MSE and mathematical mindset. The survey was adapted from the self-report questionnaire found in Pintrich and DeGroot (1990), the May (2009), and Blackwell (2003). Students responded on a 5-point Likert scale to MSE statements that included a mix related to mathematical tasks (such as "I am sure I can do an excellent job on the problems and tasks assigned for my math classes") and success in math courses (such as "I expect to do very well in my mathematics classes"). Two interviews were conducted during the semester with students who scored significantly higher or lower (one standard deviation above or below the mean) than their classmates on the MSE portion of the survey. The present study focuses on Candice, whose survey results showed she held higher MSE than most of her classmates. Candice is a unique and illuminating case because her high MSE was rooted in her ability to be successful in high school mathematics courses and this did not align well with the structure of her active learning style course. Over the course of the two interviews, this misalignment potentially caused a lowered sense of MSE and contributed to a lack of engagement with productive struggle.

Interview Protocols

The first interview served two purposes. The first purpose was to confirm what students reported in their surveys and establish baselines for each student's MSE. The second purpose was to ask participants to solve two challenging mathematics problems and to examine whether they engaged in productive struggle. The two problems are listed below in Table 1.

Table 1: Interview Session problems.

Problem 1
In a certain state, there are four towns, situated at the vertices of a square of side 10 miles. The department of transportation wants to construct a road network connecting these towns, so that it is possible to travel from each of the four towns to the other three towns. Constructing a mile of road costs two million dollars. Due to a budget crisis caused by tax cuts, the department of finance can only grant 55 million dollars for the project. Is this amount sufficient?
Problem 2
Find two numbers and such that $x - y$, x/y, and xy are all equal. (from Posamentier & Solkind, 1996).

The two problems asked were ideal for determining whether students engaged with productive struggle. First, when students proposed solutions, students could easily verify whether their answers were correct. This eliminated the scenario where a student stopped working because they believed they had solved the problem. Secondly, these problems were non-typical, and participants chose the types of mathematical approaches they took in solving the problems. As previously discussed, MSE is expected to play a significant role in student success when this level of analysis is required (Holenstein et al., 2021).

Participants were given each problem separately with Problem 1 being presented first. To allow for productive struggle, students were told that they could work on the problem for as long as they liked. Students could ask clarifying questions to help them understand the situation being presented. In Candice's case, Candice asked clarifying questions like "We don't know the size of each town right?" and "Does each town have to connect to a single point or can there be one long road?" The interviewer answered these clarifying questions to ensure students like Candice could grasp the problem being asked and choose whether to engage in productive struggle. After the first interview students were later invited back to a second interview to learn if their current courses or the transition to a college environment had affected their view of mathematics, their MSE, or their study habits.

Coding and Reliability

Candice's interview (along with those from the larger study) was audio recorded and then transcribed. Four interviews were randomly selected from the larger pool and coded using codes developed inductively from the data (Seidman, 2013). To establish coding reliability, a method outlined by Campbell and colleagues (2013) was used. Coded sections were labeled with the appropriate code as well as brackets. The codes were then removed from the interview, leaving the bracketed sections visible on the transcribed interview. A second qualified individual then coded the bracketed sections using the developed codebook. Coders initially agreed on 71% of the bracketed sections. Differences between codes were then negotiated and several codes were either combined or deleted. A fifth interview was then coded using the same method and the agreement percentage rose to 93.5%. The remainder of the interviews were coded by the original coder using the established codebook.

For the problem-solving section of interview one, each student's written work was saved in addition to the audio recordings. Each solution attempt was coded as showing no evidence, some evidence, or high evidence of productive struggle. To establish trustworthiness for the problem portions, both coders each initially suggested the level of evidence they felt was present and then differences were debated until both coders agreed.

Results

In this results section, Candice's initial MSE will be discussed along with some of the background information she provided about herself. From there, Candice's solution attempts to Problem 1 and Problem 2 will be discussed and contrasted with the results seen from other calculus students in the study. Finally, Candice's results from interview 2 will be discussed, specifically how her course had impacted her view on her MSE and how this had impacted her solution attempts during the first interview.

Introduction to Candice

Candice was one of seven calculus students recruited and was interviewed during her first semester of college in the fall of 2018 at a large midwestern university. Her first collegiate mathematics class was a calculus course that used discovery-based methods and active learning as its main method of delivery.

Candice's initial survey results suggested that she held relatively high MSE compared to many of her classmates. During her first interview, Candice confirmed her high MSE when she said, "I would say I'm pretty confident. I really enjoy math, so I like the challenge of it." Candice prided herself on being a responsible student and favorably compared her study skills to other students. It became apparent during this first interview that Candice's high MSE was rooted in her ability to successfully navigate mathematics courses, but not necessarily in her ability to complete mathematics problems. Candice said, "I expect to do well overall. I think there are some students who don't know how to study, or they don't want to study. So, I think compared to certain students, I'm going to do better." Candice could not ever recall a time where she had not studied for an upcoming mathematics test or exam and saw her study skills as something that separated her from other students.

Despite her high MSE, Candice acknowledged that she had been having difficulties to start the semester in her calculus course. Candice said, "I'm struggling a lot right now, but it is only the first couple of weeks… I know I'll eventually get it. I feel like I just have to get back into school and get back into math, I guess." Candice suggested that she was actively trying to rectify her early struggles but noted that she had less opportunity to interact with her college professors. Candice said, "In college you don't get as much time with your teachers, so I've emailed my teacher already a couple times."

Candice's Lack of Productive Struggle

Candice showed no evidence of productive struggle on either of the problems presented during the first interview. This was already a surprising result considering Candice's high MSE, but her results stood out even further when comparing her solution attempts to those provided by calculus participants in the broader study. Specifically, Candice showed no productive struggle in situations where calculus students were generally more likely to (see problem 2).

Problem 1. On the first problem, Candice provided a solution attempt that showed no evidence of productive struggle. Candice gave an initial attempt at the problem that connected all four towns in a square configuration. Candice's initial configuration can be seen below in Figure 1.

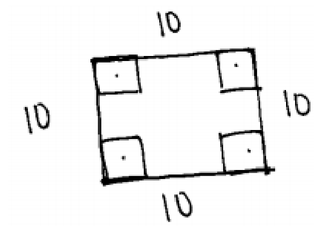


Figure 1. Candice's initial attempt at problem #1.

Candice calculated the cost of her initial configuration and concluded that it would be too expensive. Candice spent a few minutes thinking before drawing a triangle on her paper. Candice then started entering numbers into her calculator. When the interviewer asked what she was entering in Candice said, "I'm just trying to figure out, I guess, trying to make a line through the middle somehow. I'm trying to use the area of a triangle to figure out how long the line would be." Candice had drawn a right triangle and incorrectly used the formula for the area of a triangle in an attempt to find the length of the hypotenuse (see figure 2). Candice then laughed and said, "This is a tough problem." Candice spent approximately two more minutes in silence thinking about the problem before deciding to stop her attempt.

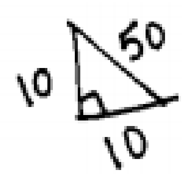


Figure 2. Candice's second drawing.

Candice's attempt was labeled as showing no evidence of productive struggle as she did not progress past the most common configurations (square and X configurations) that most students initially tried. Candice's acknowledgement of the difficulty of the problem seemed to be a point where she had the option to "muddle through" but instead chose to end her attempt rather quickly.

Problem 2. On problem 2, Candice again showed no evidence of productive struggle. Her entire attempt consisted of using her calculator to test values for x and y and at no point did Candice attempt to set up an equation or employ any algebraic techniques. Candice did not write down anything on her paper and stopped her attempt after about five minutes of working. Candice's results here are especially interesting when considering the other calculus students interviewed in the broader study. Of the seven calculus students interviewed, Candice was the only participant who showed no evidence of productive struggle on problem 2. Further, all six solution attempts provided by the other calculus students incorporated algebraic techniques (such as setting up equations). It may be the case that the algebraic nature of the problem and the presence of variables caused the problem to feel familiar to calculus students and had a positive effect on their willingness to productively struggle. The fact that Candice's attempt did not incorporate these same techniques is curious and is perhaps explained by the source of Candice's high MSE as well as by the difficulty she experienced in her calculus course.

Possible Explanations for Candice's Lack of Productive Struggle

Candice's results stand in contrast to what previous research has found. She entered the semester with higher MSE than many of her peers but showed less evidence of productive struggle on the problems presented in the first interview. This contradiction is potentially explained by the task-specific nature of Candice's MSE, specifically that her high MSE was rooted in her ability to complete mathematics courses (and potentially only traditional lecture style courses). The problems presented in the first interview may not have aligned with Candice's high MSE since the problems required Candice to choose her own mathematical techniques without the backdrop of a classroom lesson or unit. In other words, since the problems given in the first interview were devoid of a classroom context, Candice may not have felt that she had the resources she needed to complete the problems. Candice was asked if a lack of context affected her attempts to which she responded, "I think it does because say if that one (problem 1) is geometry, then I would know some basic equations to use versus if that one (problem 2) is algebra, then I would know the basic equations for algebra." These responses may indicate that despite Candice's lack of productive struggle, she may have been more confident in her ability to complete them in a setting with a more narrow focus (such as a classroom).

There may have been additional misalignment between Candice's MSE and the tasks presented in her calculus course. This potentially resulted in lowered MSE and engagement with productive struggle on mathematics problems, including the problems presented in the first interview. During her second interview (six weeks after the first interview), Candice said, "The majority of this semester, I haven't known what's going on." Candice pointed to the active learning style of the course as a major reason for the difficulty she experienced. Candice said, "I don't think the material is more difficult, I think the way it's being taught is more difficult… I don't think the rest of the math classes are taught in this way." Candice elaborated and said "The teacher will talk about it conceptually and try to explain it conceptually. I can understand that part of it, but when it comes to the math, I don't actually know how to do the math." Candice backed up her claim by explaining:

> So, say we're looking at a graph and we're taking the second derivative of a function. I'll know that the second derivative is like the concavity of the original function, but I won't necessarily know how to find the concavity of the original function. I'll know what it means, but not necessarily how to find it.

This seemed to suggest that Candice was able to understand many of the concepts and definitions but did not have a full understanding of the procedures needed to solve problems.

Interestingly, Candice made an unprompted link between her struggles in calculus to her lack of productive struggle during the first interview. Candice said:

> I've felt unsuccessful a lot. Sometimes, I guess it affected how I worked on the problems. You know how we did that problem in the first session? I just wasn't getting it and I was just like, I give up now. I feel like when I come to problems where I just don't know where to start or what to do, I just kind of accepted my failure.

While the other calculus students interviewed may have been encouraged by the algebraic nature of problem 2, Candice may have had the opposite reaction. Specifically, since the procedural aspects of calculus seemed to be the primary points of difficulty for Candice, the presence of an algebraic problem may have actively dissuaded her from engaging in productive struggle.

While Candice recognized that she did not fully understand the calculus content, she still seemed to pride herself on her ability be successful in a classroom setting. Candice said, “I don’t think because I’m doing bad in this math class, that I’m not a good math student anymore.” Candice’s choice of phrasing here is interesting in that she chose to say, “good math student” instead of something like “good at math.” This may provide further evidence that Candice’s MSE remained rooted in her ability successfully navigate mathematics courses. Further, her positive outlook for the long-term may be based on the idea that future courses may revert to a typical lecture style and her belief that she would be able to successfully navigate that type of course.

Discussion

While high MSE is generally associated with higher levels of persistence (Lent et al., 1984; Multon et al., 1991, Ruch et al., 2014), this study provides evidence that instances of transition (that potentially bring new task expectations) may disrupt this relationship. While Candice entered the semester reporting high MSE, it became clear that this was linked to her ability to succeed in mathematics courses and not necessarily to her ability to solve mathematics problems. Candice did not show evidence of productive struggle on either problem, but her effort level on problem 2 was noticeably less than that of her peers. This seems to be related to her lowered MSE for the procedural aspects of problems. This provides further evidence that self-efficacy is task-specific (Pajares, & Miller, 1995) and may also provide some insight into why previously successful students may struggle in environments that have different expectations (such as active learning). Students like Candice, whose MSE is tied to previous success in high school courses, may struggle adapting to the levels of analysis required in an active learning style class.

This study may also provide insight into the stability of MSE. In this instance, a student making the transition to college experienced negative results which negatively impacted her MSE. This study provides support for Bandura (1986), who theorized that periods of transition could cause the stability of MSE to be disrupted. This study provides an example of a decline in MSE that occurred during the transition from high school to college and it is possible this period of instability had negative long-term ramifications as well. Candice reported that she was “accepting her failure” when encountering difficult problems, which inhibited her from developing a full understanding of certain procedures. This suggests that students’ first collegiate courses could potentially have long-lasting impacts on both MSE and achievement.

Conclusion

The results of this study elevate the theoretical importance of task-specific self-efficacy. While generalized MSE seems to hold positive relationships with academic success and higher levels of persistence, these results may be dependent on students remaining in environments that fostered the high levels of MSE to begin with. When students transition to new environments (such as high school to college or a lecture style to an active learning classroom), success may be directly tied to the specific tasks that will be required and their self-efficacy for those specific tasks.

This in-turn has important implications for educators. The first implication is that educators need to appropriately model the ways in which students should engage with course content, especially if the course is “non-traditional.” Students may enter a classroom with limited experience analyzing advanced mathematical tasks and educators need be accommodating to novices if this is the expectation. As put by Holenstein and colleagues (2021), educators should make sure that "tasks are taught in a way that allows for individual levels" (p.14). This also underscores the need for available resources (tutors, office hours, etc.) for students when they encounter difficulty. As Candice mentioned, face-to-face time with professors is often limited, so students need avenues by which they feel they can accomplish the work when outside the classroom.

The second implication for educators is to be aware of the effect that transition can have on student MSE. Students who report traits that generally produce favorable outcomes (such as high MSE or growth mindset) can still experience declined states of MSE that affect their engagement with course material. As seen here, Candice had identified that she was struggling as early as three weeks into the course but had not yet decided that the course was out of her control. Six weeks later, however, she had come to decide that many components of the course were lost on her. This underscores the need for intervention at multiple points throughout the semester as it appears students may write off their chances well before the end of a course.

References

Bandura, A. (1986). Social foundation of thought and action: A social-cognitive theory. Englewood Cliffs, NJ: Prentice Hall.

Blackwell, L. S. (2003). Psychological mediators of student achievement during the transition to junior high school: The role of implicit theories.

Campbell, J. L., Quincy, C., Osserman, J., & Pedersen, O. K. (2013). Coding in-depth semistructured interviews: Problems of unitization and intercoder reliability and agreement. *Sociological Methods & Research*, 42(3), 294-320. doi:https://doi.org/10.1177%2F0049124113500475

Chemers, M. M., Hu, L. T., & Garcia, B. F. (2001). Academic self-efficacy and first year college student performance and adjustment. *Journal of Educational Psychology*, 93(1), 55. doi:https://doi.org/10.1037/0022-0663.93.1.55

Hackett, G., & Betz, N. E. (1989). An exploration of the mathematics self-efficacy/mathematics performance correspondence. *Journal for Research in Mathematics Education*, 20(3), 261-273. doi:https://doi.org/10.5951/jresematheduc.20.3.0261

Handa, Y. (2003). A Phenomenological Exploration of Mathematical Engagement: Approaching an Old Metaphor Anew. *For the Learning of Mathematics*, 23(1), 22–29. Retrieved from: http://www.jstor.org/stable/40248406

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. Second handbook of research on mathematics teaching and learning, 1(1), 371-404. Charlotte, NC: Information Age Publishing. Retrieved from: http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.405.3591&rep=rep1&type=pdf

Holenstein, M., Bruckmaier, G., & Grob, A. (2021). How do self-efficacy and self-concept impact mathematical achievement? The case of mathematical modelling. *British Journal of Educational Psychology*, e12443. doi:https://doi.org/10.1111/bjep.12443

Lane, J., & Lane, A. (2001). Self-efficacy and academic performance. *Social Behavior and Personality: An International Journal*, 29(7), 687-693. https://doi.org/10.2224/sbp.2001.29.7.687

Lent, R. W., Brown, S. D., & Larkin, K. C. (1984). Relation of self-efficacy expectations to academic achievement and persistence. *Journal of Counseling Psychology*, 31(3), 356-362. doi:https://doi.org/10.1037/0022-0167.31.3.356

May, D. K. (2009). Mathematics self-efficacy and anxiety questionnaire (Doctoral dissertation, University of Georgia). Retrieved from: https://getd.libs.uga.edu/pdfs/may_diana_k_200908_phd.pdf

Middleton, J. A., & Spanias, P. A. (1999). Motivation for achievement in mathematics: Findings, generalizations, and criticisms of the research. *Journal for Research in Mathematics Education*, 30(1), 65-88. doi:https://doi.org/10.2307/749630

Multon, K. D., Brown, S. D., & Lent, R. W. (1991). Relation of self-efficacy beliefs to academic outcomes: A meta-analytic investigation. *Journal of Counseling Psychology*, 38(1), 30-38. Retrieved From: https://www.academia.edu/29900606/Relation_of_self_efficacy_beliefs_to_academic_outcomes_A_meta_analytic_investigation?from=cover_page

Nadjafikhah, M., Yaftian, N., & Bakhshalizadeh, S. (2012). Mathematical creativity: some definitions and characteristics. *Procedia-Social and Behavioral Sciences*, 31, 285-291. doi:https://doi.org/10.1016/j.sbspro.2011.12.056.

Pajares, F., & Miller, M. D. (1995). Mathematics self-efficacy and mathematics performances: The need for specificity of assessment. *Journal of Counseling Psychology*, 42(2), 190. doi:https://doi.org/10.1037/0022-0167.42.2.190

Pintrich, P. R., & De Groot, E. V. (1990). Motivational and self-regulated learning components of classroom academic performance. *Journal of Educational Psychology*, 82(1), 33. https://doi.org/10.1037/0022-0663.82.1.33

Posamentier, A. S., & Salkind, C. T. (1996). Challenging problems in algebra. Courier Corporation.

Ruch, W., Weber, M., Park, N., & Peterson, C. (2014). Character strengths in children and adolescents: Reliability and initial validity of the German values in action inventory of strengths for youth (German VIA-Youth). *European Journal of Psychological Assessment*, 30(1), 57-64. doi:https://doi.org/10.1027/1015-5759/a000169

Seidman, I. (2013). Interviewing as qualitative research: A guide for researchers in education & the social sciences. New York: Teachers College.

Thomas, J. W., Iventosch, L., & Rohwer Jr, W. D. (1987). Relationships among student characteristics, study activities, and achievement as a function of course characteristics. *Contemporary Educational Psychology*, 12(4), 344-364. doi:https://doi.org/10.1016/S0361-476X(87)80005-X

Usher, E. L., Li, C. R., Butz, A. R., & Rojas, J. P. (2018). Perseverant grit and self-efficacy: Are both essential for children's academic success? *Journal of Educational Psychology*, 111(5), 877. doi:https://doi.org/10.1037/edu0000324

Zeybek, Z. (2016). Productive struggle in a geometry class. *International Journal of Research in Education and Science*, 2(2), 396-415. Retrieved from: https://files.eric.ed.gov/fulltext/EJ1110272.pdf

Zimmerman, B. J., & Cleary, T. J. (2006). Adolescents' development of personal agency: The role of self-efficacy beliefs and self-regulatory skill. *Self-efficacy Beliefs of Adolescents*, 5, 45-69.

COLLEGE ALGEBRA STUDENTS' ATTITUDES TOWARD MATH AND GRAPHS: AN EXPLORATORY FACTOR ANALYSIS

Livvia Bechtold	Courtney Donovan	Heather Lynn Johnson
CU Denver	CU Denver	CU Denver
Livvia.bechtold@ucdenver.edu	Courtney.donovan@ucdenver.edu	Heather.johnson@ucdenver.edu

We report on results from a mixed methods study investigating a measure of students' attitudes toward math and graphs. At the beginning of eight consecutive fall and spring semesters, we distributed a fully online attitude survey, adapted from Pepin (2011), to undergraduate College Algebra students. Our report includes two samples, Validation (n=1256) and Calibration (n=712). Our research team qualitatively coded students' responses into five categories: positive, mixed, ambiguous, negative, detached (Gardner et al., 2019). Next, we quantitized those qualitative codes into a four category scale, which condensed the mixed and ambiguous categories. Conducting an Exploratory Factor Analysis, we found that students' attitudes grouped by topics (math and graphs). We conclude with implications for research and practice.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Data Analysis and Statistics, Research Methods, Undergraduate Education

We report on a mixed methods study in which we examine undergraduate students' attitudes toward math and graphs. Our population is students who enroll in College Algebra, one of the earliest credit-bearing mathematics courses at U.S. institutions (Blair et al., 2018), with a history of challenges for student success (Gordon, 2008; Tunstall, 2018). Analyzing students' responses to a fully online, open-ended survey, we investigate a measure of students' attitudes. We aim to contribute to the field's knowledge about the attitudes of College Algebra students.

A key aspect of our study is the quantitizing of qualitative data (Sandelowski et al., 2009). Quantitizing is a process of assigning numerical values to non-numerical data. In so doing, researchers can illuminate patterns and peculiarities arising within large data sets via quantitative analysis strategies. Our quantitizing has allowed us to transform qualitative attitude codes into a four-category scale, which then afforded an exploratory factor analysis.

Theoretical Framework

We adopt Di Martino and Zan's (2010) multidimensional perspective on students' attitudes toward math. That is, students' attitudes toward math comprise three interrelated dimensions: their emotional disposition toward math (e.g., how they like or dislike math), their perceived competence toward math (e.g., how they feel about their capabilities when it comes to math), and their vision of what math is (e.g., what they view math to be). This stance blurs boundaries between McLeod's (1992) categories of beliefs, attitudes, and emotions as distinct components of mathematical affect.

Drawing on this perspective, Pepin and colleagues (Ding et al., 2015; Pepin, 2011) developed a survey including three open-ended response questions, addressing each of these dimensions. In their analysis, they coded students' responses according to three categories: positive, negative, and neutral. Yet, students' attitudes toward math could have complexities beyond just a positive or negative attitude (e.g., Di Martino, 2019). For example, a student could feel that mathematics is "boring and cool at the same time." Hence, future studies could make room for complexities in students' attitudes, via instruments and coding mechanisms.

Methods

The Attitude Survey

We adapted the attitude survey that Pepin and colleagues (Ding et al., 2015; Pepin, 2011) to include five open-ended response questions and to be in a fully online format (Johnson et al., 2019). Table 1 shows the survey questions. The first three questions were the same as those used by Pepin and colleagues. We added the last two questions to address students' attitudes toward graphs in two areas: emotional disposition (like/dislike) and perceived competence (can/cannot). We decided to include students' attitudes toward graphs, because students worked with graphing activities as part of the broader research projects of which this study was part. To allow for a range of responses, we did not require students to select like/dislike or can/cannot. Students were to type in responses to each of the questions; they could complete the survey on a mobile phone, tablet, or computer.

Table 1: The Attitude Survey

Attitude Survey Questions
I like/dislike math because____
I can/cannot do math because_____
Mathematics is______
I like/dislike graphs because _____
I can/cannot make sense of graphs because_____

Data Collection

We collected data in conjunction with two U.S. National Science Foundation funded research projects, examining undergraduate college algebra students and instructors. One aspect of both projects was to investigate students' attitudes toward mathematics. The first project took place at a single institution and concluded in Summer 2020. The second project began in Fall 2020, and extended the efforts to four institutions.

Across eight consecutive spring and fall semesters (Spring 2018 through Fall 2021), we administered the attitude survey to students enrolled in college algebra. In our broader study, we administered the attitude survey in both the beginning and end of the semester. For this analysis, we drew on only those responses from the beginning of the semester. We made this choice to allow for a greater sample size.

We separated our data into two samples. The validation sample (n=1,256) was collected from students who responded between Spring 2018 and Spring 2021. The calibration sample (n=712) was collected from students who responded in Fall 2021. The Fall 2021 sample was greater than other semesters because student responses included a large, lecture style course at one of the institutions. Hence, we decided to use responses from that semester to calibrate our findings with the validation sample.

Data Analysis

We mixed qualitative and quantitative methods for data analysis. First, we engaged in qualitative coding, following the coding scheme put forward by Gardner et al. (2019). Second, we quantitized the qualitative data (Sandelowski et al., 2009), to turn qualitative codes into a scale via Rasch Analysis (Bond et al., 2015). Third, we conducted an Exploratory Factor Analysis (EFA) to examine construct validity.

Qualitative Coding. We drew on students' responses to the five questions in Table 1 as sources of data for their attitude towards mathematics. We coded their responses to extend beyond binary choices of positive or negative (Gardner et al., 2019). Table 2 lists the five categories, including a brief description and sample response. Along with positive and negative, we added the codes of Mixed and Ambiguous, to indicate when student responses evidenced more than one code (Mixed) or when student responses could be coded as positive or negative (Ambiguous). In addition, we included the code of Detached, to indicate when a student separated the content of math or graphs from their connection to it. As cautioned by Di Martino (2019), students' statements about the utility of mathematics (e.g., "math is useful" which we coded as detached) pointed to something other than a positive attitude toward mathematics.

Table 2: Attitude Survey Qualitative Codes

Code	Description	Sample Response
Positive	Like/Can	I like graphs because they represent what a function will look like
Mixed	More than one of these codes: positive, ambiguous, and/or negative	Mathematics is boring and cool at the same time.
Ambiguous	Code could be positive or negative	It is hard for me.
Negative	Dislike/Cannot	I cannot do mathematics because I forget the steps
Detached	Separates the content from their connection to it.	Mathematics is filled with a lot of rules!

For the validation sample, each question was coded by two trained coders. Each coder received training with a coding rubric, then independently coded responses. After coding, they met to identify disagreements, and then calibrated the disagreements via discussion, consulting with an expert coder if needed. As the data set grew, our team trained a machine to assist with the qualitative coding process. Beginning in Spring 2021, we used the machine learning program to assist with the qualitative coding process, verifying with human coders via the earlier process if there was less than 70% confidence by the machine coding.

Quantitizing the qualitative codes: Rasch analysis. Via Rasch analysis (Bond et al., 2015), we transformed the qualitative codes into a mathematically supported scale, ordering the codes according to the level of positivity expressed by each category. This resulted in a collapsing of the Mixed and Ambiguous codes. Our scale was as follows: 0-Detached, 1-Negative, 2-Mixed/Ambiguous, 3-Positive. Category probability curves indicated an even distribution of the four categories with clearly advancing steps. Rasch-Andrich thresholds increased with category values with no evidence of step misfit. Per Linacre (2015), Mean Square (MNSQ) infit values should be less than 2.0; our values ranged from 0.91 to 1.24.

Exploratory factor analysis (EFA). We conducted EFA to examine construct validity. Specifically, to explore how items grouped together mathematically compared to the intended theoretical groupings. We used Bartlett's test of sphericity and the Kaiser–Meyer–Olkin (KMO) measure of sampling adequacy to assess the suitability of the items for factor analysis. We assessed dimensionality using principal components analysis.

Results

Our analysis revealed that the samples were adequate and the items were suitable for factor analysis. The Validation sample (KMO = 0.549) and the Calibration sample (KMO= 0.524) were over the 0.5 minimum threshold. Bartlett's test was significant at $p < .001$ with a $\chi^2 = 548.89$ validation sample, and a $\chi^2 = 282.17$ for calibration sample.

For the validation sample, two factors were retained, the first with eigenvalues of 1.698 and 1.192, combined explaining 57.8% of the total variance (Table 3). Two factors were likewise retained for the Calibration sample (eigenvalues of 1.60 and 1.24), explaining 56.71% of the total variance. Using a Varimax rotation, the first factor we called Attitude Towards Math; it included the first three questions from Pepin (2011, Table 1). The second factor we called Attitude Towards Graphs; it included the two new questions about graphs.

Table 3: Attitude Survey Item Statistics

Item	Calibration Mean	Calibration Factor Loading	Validation Mean	Validation Factor Loading
I like/dislike math because____	2.09	0.68	2.07	0.66
I can/cannot do math because_____	2.39	0.61	2.35	0.61
Mathematics is______	1.81	0.13	1.85	0.21
I like/dislike graphs because _____	2.10	0.71	2.12	0.70
I can/cannot make sense of graphs because_____	2.30	0.71	2.28	0.69

Discussion

The results of our EFA analysis reveal that items loaded by topic: math and graphs. Hence, students' emotional disposition or perceived competence toward math and graphs may not align with each other. Our analysis points to interrelationships between dimensions of attitudes put forward by Di Martino and Zan (2010).

We found a difference between the attitudes coded in the Validation (n=1256) and Calibration (n=712) samples. Interesting, the Validation sample had more negative codes while the Calibration sample had more positive codes. In the Validation sample, 28% of responses were coded positive and 41% negative. In the Calibration sample 46% of responses were coded as positive and 23% coded negative. We conjectured that the Calibration sample had more positive attitudes in part because students were returning to in-person learning that semester.

We address two limitations. Because our data sources were limited to students' responses to the attitude surveys at the beginning of the semester, our results report only students' attitudes at the start of the course. Furthermore, our resulting attitude scale condenses Mixed and Ambiguous to a single code, potentially dampening some complexities in students' attitudes.

Our study contributes to research investigating attitudes toward math and graphs in an under-studied population, undergraduate students in early credit bearing mathematics courses, such as College algebra. Students' mean response codes (Table 3) indicate that they entered College Algebra with more positive than negative emotional dispositions and perceived competence toward both math and graphs. Future studies can investigate links between students' attitudes toward math and graphs and their engagement in the course.

Acknowledgement. This research was supported by U.S. National Science Foundation awards 1709903, 2013186. Opinions, findings, and conclusions are those of the authors.

References

Blair, R., Kirkman, E. E., & Maxwell, J. W. (2018). Statistical abstract of undergraduate programs in the mathematical sciences in the United States. Fall 2015 CBMS survey. Washington, DC: American Mathematical Society.

Bond, T. G., & Fox, C. M. (2015). Applying the Rasch model: Fundamental measurement in the human sciences (3rd ed.). Mahwah, NJ: Erlbaum.

Di Martino, P., & Zan, R. (2010). 'Me and maths': Towards a definition of attitude grounded on students' narratives. Journal of Mathematics Teacher Education, 13(1), 27-48. https://doi.org/10.1007/s10857-009-9134-z

Di Martino, P. (2019). Chapter 9: The complex relationship between mathematical modeling and attitude towards mathematics. Affect in mathematical modeling (pp. 219-234). Springer International Publishing. https://doi.org/10.1007/978-3-030-04432-9_14

Ding, L., Pepin, B., & Jones, K. (2015). Students' attitudes towards mathematics across lower secondary schools in Shanghai. In B. Pepin & Roesken-Winter (Eds.), *From beliefs to dynamic affect systems in mathematics education* (pp. 157-178). Switzerland: Springer International Publishing.

Gardner, A., Smith, A, & Johnson, H. L. (2019) Humanizing the coding of college algebra students' attitudes towards math. In Weinberg, A., Moore-Russo, D., Soto, H., & Wawro, M. (Eds.). Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education (pp. 1113-1114). Oklahoma City, OK: RUME.

Gordon, S. P. (2008). What's wrong with college algebra? PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 18(6), 516-541. https://doi.org/10.1080/10511970701598752

Johnson, H. L., Gardner, A., Smith, A., Olson, G., & Wang, X. (2019). Interacting with dynamic computer activities impacts college algebra students' math attitudes and performance. In Otten, S., Candela, A., de Araujo, Z., Haines, C., & Munter, C. (Eds.). Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1424-1433). St Louis, MO: University of Missouri.

Linacre, J. M. (2015). A user's guide to Winsteps/Ministep: Rasch-model computer programs. Beaverton, OR: Winsteps.com.

McLeod, D. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 575–596). New York: McMillan.

Pepin, B. (2011). Pupils' attitudes towards mathematics: A comparative study of norwegian and english secondary students. Zdm, 43(4), 535-546. https://doi.org/10.1007/s11858-011-0314-9

Sandelowski, M., Voils, C. I., & Knafl, G. (2009). On quantitizing. Journal of Mixed Methods Research, 3(3), 208-222. https://doi.org/10.1177/1558689809334210

Tunstall, S. L. (2018). College algebra: Past, present, and future. PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 28(7), 627-640. https://doi.org/10.1080/10511970.2017.1388315

OFFLOADING COGNITIVE DEMANDS OF FRACTIONAL TASKS ON WORKING MEMORY THROUGH DRAWINGS

Rachael Stryker
Virginia Tech
rach35@vt.edu

Vladislav Kokushkin
Virginia Tech
vladkok@vt.edu

Anderson Norton
Virginia Tech
norton3@vt.edu

Sarah Kerrigan
Virginia Tech
stk123@vt.edu

This study examined the role of student generated drawings to offload cognitive demands of a mathematical problem. We used Unit Transformation Graphs to compare students' thought processes when they had to solve the problem mentally, and when they were allowed to use pen and paper. The results indicated that the possibility to rely on drawings helped the participants to free up working memory resources and complete a cognitively demanding fractional task.

Keywords: Cognition; Learning Theory; Number Concepts and Operations; Problem Solving.

Background

A growing body of research within psychology of mathematics education has been aimed at understanding students' cognitive affordances and constraints in doing mathematics (e.g., Bull & Lee, 2014; De Smedt et al., 2009). In particular, prior literature revealed an important role of working memory (WM) – a psychological construct for human's ability to simultaneously store and process information (Baddeley & Hitch, 1974; Daneman & Carpenter, 1980; Ma et al., 2014). WM is limited and varies widely among individuals. If the complexity of a cognitive task requires to operate with multiple items at the same time, students' WM may be overloaded and they become unable to process the requested amount of information. In this case, it is natural for an individual to seek ways to offload the cognitive demands on their WM. Once information is offloaded into the environment, a student then has more mental space available to progress on task (Kirsh, 2009).

A typical example of cognitive offloading is hand gesturing. For instance, Alibali and DiRusso (1999) investigated the role of gestures on the counting abilities of children. They suggested that finger counting allows children to physically instantiate some aspects of the task and, therefore, offload cognitive demands on their WM. Goldin-Meadow et al. (2001) measured the number of items students can hold in their mind when they are explaining a mathematical task. The results indicated that the participants are able to recall more items when they are allowed to gesture. Similar results supporting the beneficial effect of gesturing on WM were achieved in other studies (Cook et al., 2012; Ping & Goldin-Meadow, 2010; Wagner et al., 2004).

Drawings are considered to be a mediation tool for thinking and for meaning making. Vygotsky and Cole (1978) pointed out that human beings' mental activities are supported and developed by means of signs that are the products of their internalization processes and are called psychological tools. Vygotsky (1981) then suggests a list of examples of psychological tools, including drawings. Therefore, as drawings are composed of systems of signs and, thus, physically represent the original properties of an object of thought, they allow an individual to transmit some pieces of information into the environment and, therefore, reduce cognitive demands, playing the role of an important offloading mechanism.

Empirical research was done to examine the impact of student generated drawings with regards to WM and cognitive offloading on college students' understanding of science text. Lin et al. (2017) explored the levels of WM and the cognitive load that students bear when learning college level science material. In their study, three groups of students were instructed to study in particular ways: one group of students was told to create representations of the material being studied, another had a repeated reading method, and the other group simply had to imagine the relationships and reason through ideas in their heads. Evidence supported the conclusion that the study method of using learner-generated drawings facilitated schema construction to integrate prior knowledge with the new information that allows for easier transfer to storage in long term memory instead of simply with their WM. Learners who exhibited a lower prior knowledge of the course content demonstrated a deeper and higher level of understanding through the use of learner-generated drawings. It also found that students bore less of a cognitive load when they were able to draw out and connect their ideas on paper. By being able to draw out the relationships and make proper connections, students were able to offload the information learned during the study rather than attempting to make all of the connections mentally.

Another study explored the components and effects of learner-generated drawing on subsequent testing. During the learning process, internalizing and externalizing the information can be difficult for students to retain, but once this information is offloaded into the environment it is not as difficult to keep track of (Schmidgall et al., 2019). The process of drawing out representations allows learners to engage in generative learning processes. This study found that the students who utilized the drawing methods for studying outperformed those who simply did a summary of the information to study.

Despite the prior work and findings, drawings have received considerably little attention as being a reliable method for offloading cognitive demands on students' WM when they are engaged in mathematical reasoning. The purpose of the present study is to examine the impact of using student generated drawings to offload the cognitive demands of mathematical tasks. Understanding mechanisms underlying the phenomenon of cognitive offloading will help to improve learning conditions for mathematics learners, which addresses one of the major goals of PME-NA. Specifically, we contribute to the existing body of research by answering the following research question: *How do drawings lighten the load of students' working memory when they are engaged in fractions tasks?*

Theoretical Framework

In answering our research question, we adopt a Piagetian perspective on the construction of mathematical knowledge and a neo-Piagetian approach to WM. Following Piaget, we conceptualize mathematics as a coordination of mental actions (e.g., Beth & Piaget, 1966). In the context of fractional knowledge, these actions are well researched (Boyce & Norton, 2016; Hackenberg & Tilemma, 2009; Steffe & Olive, 2010; Steffe, 1991, 1992). The operations most germane to our study include *unitizing* (U_n; taking a collection of n units as a whole), *iterating* (I_n; copying a unit n times to form a new unit), *partitioning* (P_n; breaking a whole into n identical units); *disembedding* (D_n; taking n equal units out of a whole without losing their connection to the whole), and *distributing* ($T_{m:n}$; inserting a collection of m units of a whole into each of the n units of another whole to create a unit of units of units).

Pascual-Leone (1970) proposed a neo-Piagetian characterization of WM through a mental-attentional operator, which is known as M-operator. Its capacity is called M-capacity and represents "the number of separate schemes (i.e., separate chunks of information) on which the subject can operate simultaneously using his mental structures" (p. 302). It has been shown that an average adult's *M*-capacity ranges from 5 to 7, meaning that they can activate 5-7 schemes at once, without offloading cognitive demands onto the environment. In the context of fractional tasks, we hypothesize that WM involves manipulating sequences of actions used to construct and transform units.

Although an average person cannot activate more than 7 schemes during a cognitive task, these schemes differ in complexity and may contain other, "smaller" schemes. In particular, some of the units and unit transformations can be organized within "larger" units coordinating structures (Boyce & Norton, 2016; Hackenberg, 2007; Ulrich, 2016). Consider, for example, the construction of 1/12 as a unit, which has a one-to-twelve relationship to a whole unit. 1/12 may be obtained by partitioning the whole into 12 equal units (see the left part of Figure 1). Vice versa, the whole may be built by the mental action of iterating 1/12 twelve times. In the absence of structures to assimilate multiple levels of units, each unit or unit transformation poses separate demands on a student's WM. However, the whole, the unit fraction, and the mental transformations between them may be chunked into a single cognitive unit, thus reducing cognitive demand of the task from three to one (see the right part of Figure 1).

Figure 1: Units coordinating structures

Our integrated framework explicitly accounts for students' available structures in lightening cognitive load on their WM when solving fractional tasks. However, the increase in the number of units and mental transformations required to operate on may overwhelm one's WM even for students with high WM capacity and advanced level of unit coordination. The utility of drawings in handling the situations of cognitive overload is the primary interest of this study.

Data and Methods

The data used in this paper is part of a larger project aimed to explore behavioral and neurological aspects of mathematical development. Here, we report on video recorded behavioral data and students' written work.

Participants

The participants recruited for this project were pre-service elementary teachers (PSTs) at a large public research university in the southeastern United States, enrolled in one of two sections of Mathematics for Elementary School Teachers, taught by the same instructor. We chose to invite PSTs because they were encouraged to explain and reflect on their reasoning while solving elementary school mathematics tasks. In total, 12 students agreed to participate. In this paper, we report on two of them.

Data Collection and Analysis

Each PST participated in an individual clinical interview (Clement, 2000) with a member of the research team, lasting approximately 75 minutes. The interviews consisted of an assessment of students' available structures (Norton et al., 2015), a WM assessment (backward digit span; Morra, 1994), and a set of fraction tasks (modified from Hackenberg & Tillema, 2009). In this study, we focus on one of those tasks: *Imagine cutting off 1/4 of 5/6 of a cake. So, how much is that of the whole cake?* The analysis of cognitive demands of this task can be found in (Kerrigan et al., 2020).

The participants were given the fraction task verbally and asked to initially solve it without using any figurative materials, so that we could determine the number of units and unit transformations each student could hold in their WM without relying on drawings. The participants were sometimes asked follow-up questions to clarify their reasoning. None of the PSTs were able to properly solve the task mentally. After they attempted to solve the fraction task in their heads, the participants were given the opportunity to use a pen and paper. All interviews were video and audio recorded, and all drawings were captured using LiveScribe pens.

We qualitatively analyzed the behavioral video data, transcripts of students' verbal reasoning, and data collected via LiveScribe pens. We built unit transformation graphs (UTGs; Norton et al., in press) to illustrate the sequences of actions the students' used in attempting to solve the task mentally, and being able to rely on drawings. The comparisons of these graphs helped us to draw inferences about the impact of student generated drawings on cognitive offloading for fraction tasks.

Results

We report on two participants: PST A (with WM capacity of 6, operating at unit coordination stage 3) and PST B (with WM capacity of 7, operating at units coordination stage 2). We decided to choose these two students because both of them were assessed with high WM capacities, made the most progress on the task in the initial phase, and could come up with the correct answer when they were allowed to use drawings. Here, we present our analysis of their reasoning before and after drawing.

PST A Before Drawing

Researcher: This is the next task. Imagine cutting off one fourth of five sixths of a cake. So, how much is that of the cake. First just try to do it in your head then you can draw.

PST A: Ok, so [motions with hands] three fourths of the five sixths.

Researcher: One fourth of five sixths.

PST A: One fourth. One fourth of the five sixths. Ok, so I see six bars, but only five. So, one fourth of one… One fourth of the five sixths… Ok…. So, that would be… [pauses for twenty-six seconds.] Umm… Five… Twenty fourths? But you're not using all six… It's really hard to do in your head. I'm not sure. I'm like really messing up your experiment, I'm sorry.

Researcher: No, this is… I mean this is… We have to have ones where you're…

PST A: Just complete dud. Um, so you have five sixths [Student motions with hands]. And one fourth of that… The fifths don't go in evenly with the fourths. So that's what is confusing to me. Is there some easy way to multiply these? Am I just, am I overlooking something?

Researcher: No.

PST A: Um…

Researcher: Do you want to draw it?

PST A: Yeah.

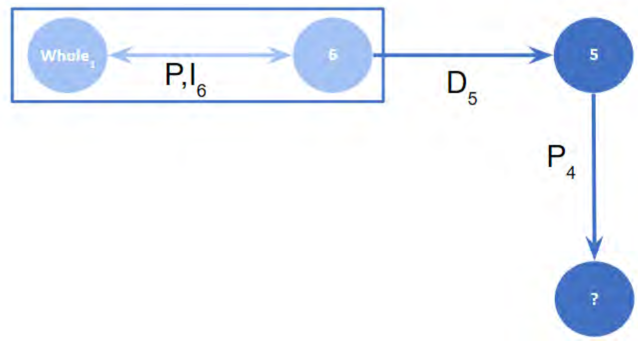


Figure 2: UTG for PST A before drawing.

As evident in UTG (Figure 2), we inferred that PST A used a two-level unit coordinating structure. This means that she could partition the whole into 6 equal parts, while being simultaneously aware of the whole, a piece of the whole, and a 1:6 relationship between them (see the rectangle in Figure 2). Therefore, mental actions of partitioning and iterating, as well as the two quantities they connect, allowed the student to chunk these three units, reducing the cognitive demand of the task by two. She could further disembed 5/6 and attempted to operate on it through partitioning into four parts. However, the latter action led to considerable cognitive overload, which might impede the student's progress on the task.

PST A After Drawing

PST A: [Picks up pen and begins to draw.] So, you have six things… Or six… But you're only using five of them. So, you need one fourth of this.
[Student thinks and draws for fourteen seconds.] Split that into fours.

Researcher: Mhm.

Student: Ok so that means it's one-fourth of that, so… One two three four five. Of the whole thing? No, just five sixths. So, five…

Researcher: Out of the- uh- original cake.

PST A: So, six sixths?

Researcher: Yes.

PST A: So, five out of, five twenty-fourths?

Researcher: Yeah.

PST A: Five twenty-fourths.

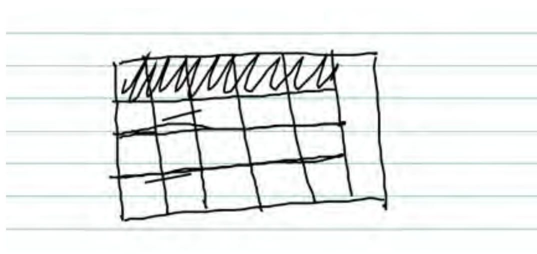


Figure 3: LiveScribe Pen drawing of PST A after drawing.

Being able to draw the problem out allowed PST A to complete the task. The student first drew a whole and split it into six pieces. She then cut each of the five pieces into four parts. After that, she shaded in a fourth from each fifth, as shown in Figure 3. Finally, the participant counted the total number of fours in each sixth to find the total number of pieces to base her shaded portion out of.

PST B Before Drawing

PST B: You're cutting off one fourth of five sixths of a cake?
Researcher: Yes.
PST B: [uses hands to show number of pieces on the desk and begins talking to herself] So, you'd have, so you'd have six pieces…and out of those five…you want to cut off one fourth of that. Um…I guess you would…I mean I guess you could split those five pieces into four and get one of those, but I'm trying to think like numbers-wise what that would...I. well… [pauses for seven seconds] I guess of those five pieces you could…Split them into…Like you could get a…Split them into twenty pieces because five times four is twenty and then, um, you would take one fourth of that… I guess it would be five pieces. Yeah, it would be five pieces of that twenty to find the one fourth of the five sixth. Is that, do I need to explain it more?
Researcher: Okay, uh let's…
PST B: Which would be, do you want me to draw it? [reaches towards paper]
Researcher: Well tell me the final answer and then we can draw it.
PST B: Um, oh gosh it would be… [pauses for four seconds] Splitting twenty, it would be five…Well it would be five twentieths, which would equal one fourth, so like five of those, but then I don't know how to figure that out into sixths. I think that's my…
Researcher: Yeah that's cool, I like the way you're reasoning. Let's draw it, and I think you will figure it out.

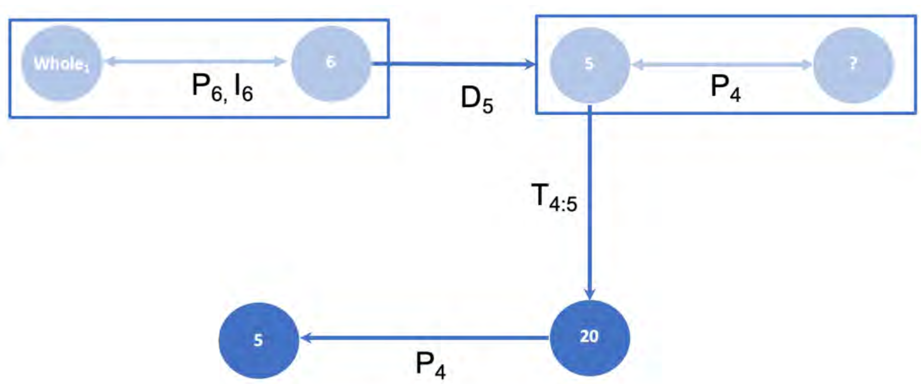


Figure 4: UTG for PST B before drawing.

Similar to PST A, PST B began with the use of a two-level structure to conceptualize 1/6, then disembedded 5 copies of it ("so you'd have six pieces… and out of those five…"), and attempted to split them into four pieces (Figure 4). However, when constructing 1/4 of 5/6, the student lost track of the sixth part making up the whole. She, thus, ended up distributing four parts into each of the 5 pieces, producing 20 parts in total. Taking a quarter out of 20, with no relation to the whole, resulted in her final answer of 5/20, or 1/4.

PST B After Drawing

PST B: Okay, so there's…This and then…Five, and we're trying to find… We have five sixths and we're trying to find one-fourth, and you… [Student works for fifteen seconds.] Okay, wait can I… can I cross this out?

Researcher: Sure. [Student crosses out picture.]

PST B: Okay this is five sixths, we want five of those and then you would, I guess divide…

[Student writes on paper for twenty-four seconds.] So then yeah you would have twenty little pieces and then…One fourth of that…Would just be, one two three four five. [Student marks pieces.] So… so I guess in terms of sixths it would be like…Well, oh wait I guess, wait…In total it would be, twenty-five twenty-fifths…Okay so in terms of six that would be… [Student works for six seconds.] Five…Wait, yeah it would be five twenty-fifths would be one fifth. I don't think that's right.

Researcher: Why don't you think it's right?

PST B: I just think, I just don't think one fourth of… I don't know it just doesn't…

Researcher: How did you get the twenty-five?

PST B: Um, I shaded in… Well I shaded in five of these because that's one fourth of the five sixths, because there are… [Student counts on drawing.] Because that's one fourth of that and then, when you turn the five sixths into terms of…Um…Twentieths, well, then I just… [Student reexamines paper.] …There are actually twenty-four…There's… So, it's out of twenty-four then, I guess, so then it would be, five twenty-fourths.

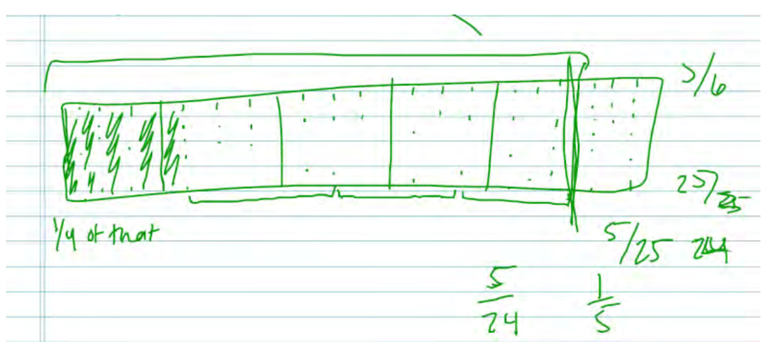


Figure 5: LiveScribe Pen drawing of PST B.

Consistent with her original strategy, the participant partitioned the whole into six pieces and marked off five of them. From there, she went through her previous approach of fourthing 5/6 by dividing each fifth piece into four smaller pieces to make the total of twenty within the fives.

Taking 1/4 of that, the student shaded in five pieces of that twenty, aligning with her previous final answer (Figure 5). However, this time, she was able to refer back to comparing that new piece to the whole cake by dividing the remaining 1/6 into four parts as well and counting the total number of pieces. The ability to draw the problem out and to refer to the depiction allowed PST B to maintain her perspective on the whole without losing the extra 1/6, and eventually come up with the correct answer of 5/24.

Discussion

The partial solutions, presented by our participants before drawing, exhausted their WM capacities. Both of the students employed two-level units coordinating structures, which allowed them to alleviate the cognitive demands on their WM. However, the experienced cognitive demand of the task was too high, leaving no cognitive resources required to complete the task. PST A had potential to make greater progress on the task because she was assessed as having WM capacity of 6 and operating at unit coordination stage 3. Nevertheless, it is known that students do not necessarily construct all possible mental structures by virtue of operating at stage 3 (Izsák, 2008). Similarly, it appears that people do not always use all their available WM resources (Eysenck & Cavo, 1992; Norton et al., in press).

After being able to represent their thought processes pictorially, both students could continue with their original method of solving the problem. Although PST A and PST B produced different drawings, their final thought processes converged (see Figure 6). They both partitioned the whole into 6 pieces, disembedded 5 of them, which constituted a new whole. After that, they distributed 4 pieces into each of the 5 pieces, producing 20 parts, which were further partitioned into 4 parts, arriving at 5 smaller pieces. In order to conceptualize how much of the whole cake these 5 pieces were, the participants had to repeat the process for the remaining 1/6 (indicated by dashed lines, ovals, and cross-hashed circles in Figure 6). Because the first part of the solution was already drawn out, the students could focus on the second part without losing any of the previously obtained information. Thus, the possibility of relying on the picture allowed the participants to save mental resources and eventually come up with the correct answer, as predicted by the prior literature (Kirsch, 2009; Schmidgall et al., 2019).

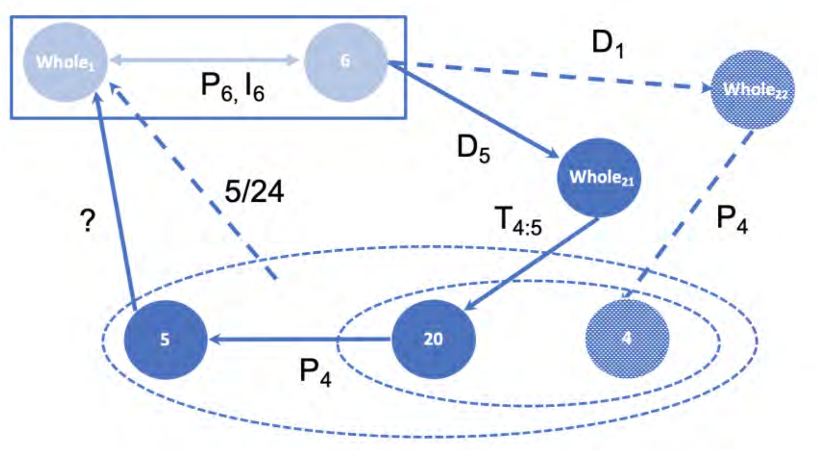


Figure 6: UTG after drawing for both PST A and PST B.

Notably, in absence of other offloading tools, the participants actively produced hand gestures when reasoning through the task. Gesturing was used as a natural method to physically represent some properties of the object of thought and make the task less demanding (e.g., Ping & Goldin-Meadow, 2010; Wagner et al., 2004). However, the use of hand gestures did not help our participants to overcome the experienced cognitive overload. Future research should be conducted to compare the effects of gesturing and drawing on managing the cognitive demands of mathematical tasks.

Acknowledgments

We thank the Adaptive Brain & Behavior Destination Area at Virginia Tech for their support of this project.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development*, 14, 37–56.

Baddeley, A. D., & Hitch, G. (1974). Working memory. In *Psychology of learning and motivation* (Vol. 8, pp. 47–89). Academic press.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (Trans. W. Mays). New York: Gordon and Breach.

Boyce, S., & Norton, A. (2016). Co-construction of fractions schemes and units coordinating structures. *The Journal of Mathematical Behavior*, *41*, 10–25.

Bull, R., & Lee, K. (2014). Executive functioning and mathematics achievement. *Child Development Perspectives*, *8*(1), 36–41.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. *Handbook of research design in mathematics and science education*, 547–589.

Cook, S. W., Yip, T. K., & Goldin-Meadow, S. (2012). Gestures, but not meaningless movements, lighten working memory load when explaining math. *Language and cognitive processes*, *27*(4), 594–610.

Daneman, M., & Carpenter, P. A. (1980). Individual differences in working memory and reading. *Journal of verbal learning and verbal behavior*, *19*(4), 450–466.

De Jong, T. (2010). Cognitive load theory, educational research, and instructional design: Some food for thought. *Instructional Science, 38*(2), 105–134.

De Smedt, B., Janssen, R., Bouwens, K., Verschaffel, L., Boets, B., & Ghesquière, P. (2009). Working memory and individual differences in mathematics achievement: A longitudinal different levels of representing unknowns. *Journal for Research in Mathematics Education*, 52–80.

Eysenck, M. W., & Calvo, M. G. (1992). Anxiety and performance: The processing efficiency theory. *Cognition & Emotion*, *6*(6), 409–434.

Goldin-Meadow, S., Nusbaum, H., Kelly, S., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological Science*, 12, 516–522.

Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *The Journal of Mathematical Behavior*, *26*(1), 27–47.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior*, *28*(1), 1–18.

Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and instruction*, *26*(1), 95–143.

Kerrigan, S., Norton, A., & Ulrich, C. (2020). Ranking the cognitive demand of fractions tasks. In *Mathematics Education Across the Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). Mazatlán, *Mexico.*

Kirsch, D. (2009). Interaction, external representations and sense making. In *Proceedings of the 31st Annual Conference of the Cognitive Science Society*, N. A. Taatgen and H. van Rijn, Eds., Cognitive Science Society, 1103–1108.

Lin, L., Lee, C. H., Kalyuga, S., Wang, Y., Guan, S., & Wu, H. (2017). The effect of learner-generated drawing and imagination in comprehending a science text. *The Journal of Experimental Education*, *85*(1), 142–154.

Ma, W. J., Husain, M., & Bays, P. M. (2014). Changing concepts of working memory. *Nature neuroscience*, *17*(3), 347–356.

Morra, S. (1994). Issues in Working Memory Measurement: Testing for M-Capacity. *International Journal of Behavioral Development*, *17*(1), 143–159.

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. (2015). A written instrument for assessing students' units coordination structures. *Mathematics Education*, *10*(2), 111–136.

Norton, A., Ulrich, C., & Kerrigan, S. (in press). Unit Transformation Graphs: Modeling the Cognitive Demands of Mathematical Tasks. *Journal for Research in Mathematics Education.*

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta Psychologica*, *32*, 301–345.

Ping R.M., Goldin-Meadow S. (2010). Gesturing saves cognitive resources when talking about nonpresent objects. *Cognitive Science,* 34, 602–619.

Schmidgall, S. P., Eitel, A., & Scheiter, K. (2019). Why do learners who draw perform well? Investigating the role of visualization, generation and externalization in learner-generated drawing. *Learning and Instruction*, 60, 138–153.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences*, *3*(1), 61–82.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences*, *4*(3), 259–309.

Steffe, L. P., & Olive, J. (2010). *Children's fractional knowledge*. New York: Springer.

Ulrich. (2016). Stages in constructing and coordinating units additively and multiplicatively (part 2). *For the Learning of Mathematics*, *36*(1), 34–39.

Vygotsky, L. S., & Cole, M. (1978). *Mind in society: Development of higher psychological processes*. Harvard university press.

Vygotsky, L.S. (1981). The instrumental method inp sychology. In J.V.Wertsch (Ed.), *The concept of activity In Soviet psychology.* Armonk, NY: M.E. Sharpe.

Wagner, S. M., Nusbaum, H., & Goldin-Meadow, S. (2004). Probing the mental representation of gesture: Is handwaving spatial? *Journal of Memory and Language*, *50*(4), 395–407.

IDENTIFYING PERSISTENT UNCONVENTIONAL UNDERSTANDINGS OF ALGEBRA: A CASE STUDY OF AN ADULT WITH DYSCALCULIA

Katherine E. Lewis
University of Washington
kelewis2@uw.edu

Gwen Sweeney
University of Washington
gwens@uw.edu

Grace M. Thompson
University of Washington
graciemt@uw.edu

Rebecca Adler
Vanderbilt University
rebecca.adler@vanderbilt.edu

Kawla Alhamad
Imam Abdulrahman bin Faisal University
kawlaalhamad@gmail.com

Research on dyscalculia has focused almost exclusively on elementary-aged students' deficits in speed and accuracy in arithmetic calculation. This case study expands our understanding of dyscalculia by documenting how one college student with dyscalculia understood algebra during a one-on-one design experiment. A detailed case study of 19 video recorded sessions revealed that she relied upon unconventional understandings of algebraic quantities and notation, which led to persistent difficulties. This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra and suggests the utility of documenting the persistent understandings that students with dyscalculia rely upon, particularly in understudied mathematical domains, like algebra.

Keywords: Students with Disabilities, Algebra and Algebraic Thinking, Design Experiments

Although many students may have difficulties with mathematics, the 6% of students with dyscalculia (Shalev, 2007) have a neurological difference in how their brains process quantity (Butterworth, 2010). Research on dyscalculia has identified that students have difficulty processing both symbolic (e.g., 5) and pictorial (e.g., *****) representations of quantity (Butterworth, 2010). This neurological difference in number processing may render standard mathematical tools, like symbols or representations, less accessible for students with dyscalculia (Lewis, 2014; 2017). Currently, research on dyscalculia has predominantly focused on elementary-aged students engaged in basic arithmetic (Lewis & Fisher, 2016). It remains largely unknown what kinds of difficulties students may experience when encountering more complex mathematics, like algebra. This is a critical omission because algebraic reasoning is qualitatively different than arithmetic (e.g., Carraher & Schliemann, 2007; Kaput, 2008; Kaput et al., 2008; Kieran, 1992; Stephens et al., 2013), quantities are represented abstractly in a variety of forms (Kaput et al., 2008; Kieran, 1992), and failure to pass algebra can limit students' academic and career opportunities (Adelman, 2006).

Large-scale studies of students with dyscalculia in algebra are not currently feasible because of difficulties in accurately identifying students with dyscalculia. Researchers emphasize the importance of differentiating between students with dyscalculia and students who have mathematical *difficulties* that are due to environmental, language, instructional, or affective factors (Lewis & Fisher, 2016; Mazzocco, 2007; Mazzocco & Myers, 2003). Researchers also argue that it is essential to differentiate students with dyscalculia from other disabilities (e.g., dyslexia) who may struggle with math, because these students have different cognitive profiles (Lyon et al., 2003) and conflating these groups of learners may mask unique characteristics of each (Mazzocco & Myers, 2003). To establish whether students' low mathematics achievement is due to cognitive or noncognitive factors, researchers often use longitudinal designs (e.g., Geary et al., 2012; Mazzocco & Myers, 2003; Mazzocco et al., 2013) or work with adult learners (e.g., Lewis, 2014; Lewis & Lynn, 2018). For example, in the context of fractions, longitudinal research has found that the difficulties experienced by students with dyscalculia are qualitatively different than low achieving students (Mazzocco & Devlin, 2008), and that these difficulties have been found to persist over years (Mazzocco et al., 2013). Detailed analyses of adults with dyscalculia have demonstrated that these difficulties may be due to persistent, unconventional understanding and use of standard mathematical tools, which suggests that all mathematical tools are not equally accessible for students with dyscalculia (Lewis, 2014; 2016; 2017; Lewis et al., 2020). Although both studies of adults with dyscalculia and those with a longitudinal design have identified characteristic patterns of reasoning students with dyscalculia demonstrate in fractions (Lewis 2016; Lewis et al., 2022; Mazzocco et al., 2013), no similar studies have been conducted in algebra.

To extend work on dyscalculia into algebra, we conducted a detailed analysis of an adult learner with dyscalculia ("Melissa") as she engaged in a weekly videorecorded one-on-one design experiment focused on algebra. We adopt an anti-deficit theoretical orientation to disability (Vygotsky 1929/1993), and we identify the understandings she relied upon rather than interpreting her data through a deficit frame. A detailed analysis of 19 weekly hour-long videorecorded sessions suggests that the student relied upon unconventional understandings of algebraic symbols. This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra and suggests the utility of documenting the unconventional understandings that students with dyscalculia persistently rely upon.

In this section we review research on algebra teaching and learning which has established both the common misconceptions experienced by all students when learning algebra, as well as instructional approaches intended to address these issues. We then present our theoretical framework – grounded in an anti-deficit Vygotskian framing of disability. We conclude by considering how this framing influenced the design decisions for our one-on-one learning environment.

Prior Research on Algebra

In this study, we aimed to extend research on dyscalculia to the mathematical topic of algebra. Algebra is a particularly appropriate content area to explore dyscalculia because algebra is representationally and conceptually far more complex and abstract than arithmetic (Kaput, 2008). Kaput (2008) defines algebraic reasoning as generalizations within a conventional symbol system and syntactically guided action on those symbols. Because students with dyscalculia have difficulty both using symbols to represent quantities and manipulating those quantities in arithmetic (Piazza et al., 2010) – it is critical that we begin to explore how these difficulties emerge in algebra when symbol use and manipulation is core to the mathematical activity. Fortunately, research with *nondisabled* students offers considerable insight into the nature of common student difficulties and a wealth of instructional approaches for addressing these difficulties (e.g., Carraher & Schlieman, 2007). For example, using real world problems, manipulatives, and two-sided scale models have been recommended to support students' understanding of unknowns, equality, and algebra (e.g., Common Core State Standards, National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; van de Walle et al., 2016). Although common difficulties and effective instructional approaches have been identified for nondisabled students, it is unclear what kinds of unique difficulties students with dyscalculia may experience, as well as which mathematical representations and tools may be inaccessible.

Theoretical Perspective – Reconceptualizing Dyscalculia as Difference

Although dyscalculia is typically conceptualized in terms of cognitive *deficits* (e.g., Geary, 2010), we argue that it is more productive to conceptualize dyscalculia in terms of cognitive *difference.* Our perspective is derived from a Vygotskian perspective of disability (Vygotsky, 1929/1993). Vygotsky argued that mediational signs and tools (e.g., language, symbols), which developed over the course of human history, were often incompatible with the biological development of children with disabilities (Vygotsky, 1929/1993). For example, the mediational tool of *spoken* language is not accessible to a Deaf child, and therefore does not serve the same role in supporting the child's development of language as it would for a hearing child. In the case of students with dyscalculia, it is possible that standard mathematical mediational tools (e.g., numerals, graphs, equations), which support the mathematical development of most students, may be incompatible with how a student with dyscalculia cognitively processes numerical information (e.g., Piazza et al., 2010). Students may have difficulties accessing and using these standard tools and may understand representations and symbols in unconventional ways. Although all students may use standard tools in unconventional ways as they are first learning a topic, we propose that students with dyscalculia may experience *persistent* incommensurability because of the inaccessibility of these mathematical mediators. Therefore, in this study, we identify unconventional use or understanding of standard mathematical tools that persist across problems and contexts – we term these *persistent understandings*.

Disability Through the Lens of a Design Experiment

In this study, we capture the student's attempts to learn during a design experiment (Cobb et al., 2003). A design experiment involves engineering learning environments and systematically studying the forms of learning (Cobb et al., 2003). In this design experiment not only do we capture the student's unconventional understandings as she is engaged in attempts to learn, but we attempt to design instructional approaches which address her difficulties. It is through the iterative cycles of design, enactment, and analysis that we can understand both the student's unconventional understandings and what instructional approaches are accessible for the student. The outcome of this design experiment is not a recommendation for a particular sequence of instructional activities or tools. Instead, the design experiment serves as the context through which we are able to better understand the kinds of inaccessibility this student with dyscalculia experienced in mathematics and what kinds of tools were more accessible. In this paper we focus on identifying the kinds of unconventional understandings of mathematical mediators that the student persistently relied upon during the design experiment (persistent understandings).

Methods

Case Study Participant History

Melissa was a 31-year-old, woman, native English speaker, who identified as half Black and half White. We recruited Melissa from her pre-college mathematics class at community college. All students in the pre-college mathematics class were given a written fractions assessment (Lewis et al., 2022), which has been shown to identify students who demonstrate unconventional understandings, characteristic of dyscalculia (Lewis et al., 2022). Melissa demonstrated unconventional understandings on this assessment and was invited to participate in an interview, formal assessment, and design experiment focused on algebraic concepts. On the Woodcock-Johnson Test of Achievement IV (WJ-IV; Schrank et al., 2014), Melissa composite math score was at the 19th percentile – which is below the 25th percentile – the most commonly used cutoff for determining dyscalculia eligibility (Lewis & Fisher, 2016). An interview revealed that

Melissa had a long history of difficulties with mathematics through school, despite having sufficient resources (e.g., private tutor). She had repeatedly failed pre-college mathematics classes at the community college. She reported doing all her homework and practicing problems “over and over and over again,” but she still struggled to understand the content. She did not pass this class. She explained, “how my mind processes it, is quite different than the average person. It seems easy for other people, but for me you have to explain it in a different way.” She explained that she did well in all her other classes, “it’s just math that gets me.” Given Melissa’s unconventional fraction understandings, low mathematics achievement score, and her history of continued mathematics failure despite sufficient resources, Melissa meets the dyscalculia criteria.

One-on-One Design Experiment

We conducted 19 videotaped design experiment sessions with Melissa. This design experiment involved iterative microcycles of design, enactment, and analysis (Gravemeijer & Cobb, 2006). Each microcycle involved designing and enacting an individual, one-on-one instructional session and then analyzing a video of the session in order to design the subsequent session. The first and third authors participated in the design microcycles and the first author (“Kelly”) was the tutor for all sessions. The goal of these sessions was to identify the ways in which Melissa used mathematical tools in unconventional and problematic ways and to provide Melissa with alternative mediational tools to support her understanding (for more details about this iterative approach to design see Lewis et al., 2020). In designing alternative mathematical mediators we (a) drew upon prior research on the teaching and learning of algebra with nondisabled students (e.g., Kieran, 2007), (b) leveraged instructional recommendations offered by an adult with dyscalculia who developed ways of compensating (Lewis & Lynn, 2018) and (c) built upon Melissa’s intuitive notations about mathematics and what she reported was more or less effective for her. In our design we aimed to provide Melissa with mediators that would help support a conventional understanding of algebra, specifically solving for an unknown, which is a core algebraic concept.

Retrospective Analysis

After the conclusion of data collection, we began our retrospective analysis. We transcribed all video recordings and scanned all written artifacts. We parsed each transcript into individual problem instances, which began with a question and ended with a student answer. The first and second author iteratively reviewed videos of each of the sessions and generated and refined analytic categories that captured the nature of the student’s understanding. We produced operational definitions for persistent understandings, specifying inclusion criteria and identifying prototypical examples of each. Five persistent understandings were related to Melissa’s understanding of algebra (see Lewis et al., 2020 for a description of persistent understandings associated with integer operations). Three coders (first, fourth, and fifth authors) systematically coded each problem instance for correctness, problem type, instructional approach, mediational tools, and any persistent understandings. Each problem instance was coded by at least 2 coders. Reliability for the coding of the 5 algebraic persistent understandings was 95.4%. Any discrepancies in coding were resolved during our weekly research team meetings by rewatching the video and discussing whether there was sufficient evidence to warrant the attribution of that operational definition (for a similar approach see Schoenfeld et al., 1993; Lewis, 2014).

Findings

The detailed analysis of video recordings revealed a collection of five persistent algebraic understandings that reoccurred, were unconventional, and led to difficulties. These persistent understandings were related to (1) the value of unknowns, (2) the equal sign, (3) coefficients, (4) the meaning of “x=”, and (5) the value of zero. We begin by providing a high-level overview of the unconventional understandings and data about their prevalence throughout the sessions, which indicate that these unconventional understandings were often associated with an incorrect answer. We then dive into detail for the first unconventional understanding to demonstrate how this unconventional understanding emerged in the data. We then present a problem instance taken from the first session which shows how multiple unconventional understandings sometimes occurred in tandem and led to significant difficulties.

We identified five persistent unconventional understandings based on the detailed analysis of Melissa’s video recorded data, these include:

1. **Expansive and static view of unknowns** - When working with unknowns/variables, Melissa asserted that any non-numeral symbol (e.g., +, =) was a variable, that x could be different values in the same problem, and that x was a static value equal to 1.
2. **Equal sign as a bridge** - When working with equations, Melissa treated the equal sign as a symbol which indicated the result of a calculation, so she often used intermediate equal signs between solution steps or had equations with more than 1 equal sign. She often made invalid transformations of the equations moving terms from one side to the other across “the bridge”, and did not object to invalid equalities (e.g., -4=5).
3. **Unconventional manipulation of coefficients** - When working with coefficients, Melissa often assumed an additive relationship between the coefficient and the unknown and would subtract the coefficient away from the unknown (e.g., 6x-6=x). At other times, she would resolve coefficients by dividing by the entire term, rather than the coefficient value (e.g., to solve 6x=12, diving by 6x rather than 6).
4. **x = the answer** - When working with unknowns, Melissa understood x to equal the answer, so would often ignore the location of x in an equation and tack “x=” in front of the answer she calculated, regardless of whether it represented the value of the unknown.
5. **Zero is not a value** - When solving algebra problems, Melissa treated the value 0 as if it were not a quantity. At times she treated like any other constant, (subtracting it from both sides of the equation) and at other times she argued that it was not a valid value.

Within these 19 individual sessions, 427 problem instances involved algebra content. Of the 427 problem instances, 186 were coded as incorrect (44%) and 90% of these incorrect answers were associated with an unconventional persistent understanding. Indeed, in problems where Melissa relied upon these unconventional persistent understandings, she often produced an incorrect answer (see Table 1).

Table 1: Prevalence of Persistent Understandings and Correctness of Problem Instances

Persistent Understanding	Number of problem instances	Associated with an incorrect answer
1. Expansive and static view of unknowns	80	80%
2. Equal sign as a bridge	110	59%
3. Unconventional manipulation of coefficients	70	74%
4. x = the answer	46	74%
5. Zero is not a value	13	61%

These five persistent understandings (along with the integer persistent understandings, Lewis et al., 2020) provided a relatively comprehensive explanatory frame for the difficulties that the student experienced. We now provide details for one persistent understanding, to illustrate how this emerged over the course of the sessions.

Persistent Understanding #1 - Expansive and Static View of Unknowns

The first persistent understanding – ***expansive and static view of unknowns*** – involved an unconventional understanding of algebraic unknowns and their values, that was both overly expansive and overly static. Melissa was overly *expansive* in her definition of unknowns, in that she used the term "unknown" or "variable" to refer to any non-numeral mathematical symbol. She explained, "a variable is a… an unknown number," and that a variable could be an "addition or subtraction problem" and then identified a whole range of different mathematical symbols (e.g., $+, \pi, [, x, m, \div, <, =$) as variables. She explained, "[The equal sign] is a variable, as well as an *x* is a variable, or a plus is a variable." In addition to treating all symbols as variables, she was also overly expansive about her understanding of unknowns in that she believed that an unknown (e.g., *x*) could be different values within the same problem (e.g., $2x+4=3x$), and argued that even after solving for x, that that unknown could still be anything.

Although she was often overly expansive in her understanding of unknowns, she also demonstrated a *static* view of unknowns, and often asserted that unknowns were equal to 1. She explained, "The rule of x is 1, that's most common unknown, in other words, for x to be 1." For example, during one session Kelly asked her to write a value that was greater than x. She incorrectly determined that 2 was larger than x and explained, "because I look at x, or a letter, as 1. And 2 is just bigger."

Believing that unknowns were static values equal to 1 was sometimes problematic when she attempted to solve for *x*. For example, when asked to solve $12=x+5$, she replaced the x with a 1 simplifying the equation to 12=6, then divided both sides by 6 to get an answer of 2. Melissa's static understanding of x, being equal to 1, was used in this example to create an invalid equation 12=6. She did not find this to be problematic and continued to procedurally manipulate the values, as if she was still solving for x, determining that the answer was 2.

In another example, when she solved the problem x+3=8, she correctly determined that x=5, but when Kelly asked her what it meant that x=5 (a standard question), she explained that "one equaled five [*writes 1=5; see Figure 1*] because I see x as 1." In this instance, even though she had just determined that x was equal to 5, her understanding that x=1 emerged. This resulted in her creating an invalid equality 1=5, which she did not find problematic.

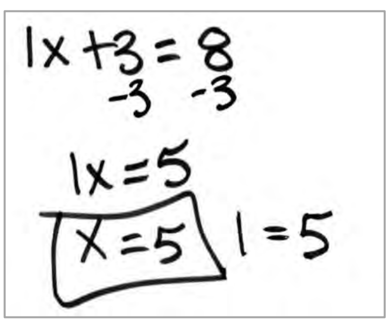


Figure 1. Melissa's written work to solve the problem x+3=8

Both Melissa's understanding of x as a static value, equal to 1, and her overly expansive understanding of unknowns, which involved believing that x could be any value, even after determining the value of x, led to her unconventional use of unknowns and errors across the sessions. This persistent understanding was evident in 80 problem instances across the sessions, and 80% of the time was associated with an incorrect answer. A similar pattern of prevalence and errors were found for the other 4 persistent understandings (see Table 1).

Persistent Understandings Occurring in Tandem

To illustrate how these persistent understandings often appeared together in the same problem, we illustrate how Melissa relied upon several persistent understandings as she solved the problem x/2+7=10. This episode was taken from the first instructional session as Kelly tried to assess her existing strategies for solving for x. The prevalence of these persistent understandings in this first section suggests that Melissa came to this design experiment with these persistent understandings. In Figure 2 we present the student's work, along with the persistent understanding identified and the rationale for that attribution. This example illustrates how Melissa's persistent understandings often occurred together and resulted in difficulties.

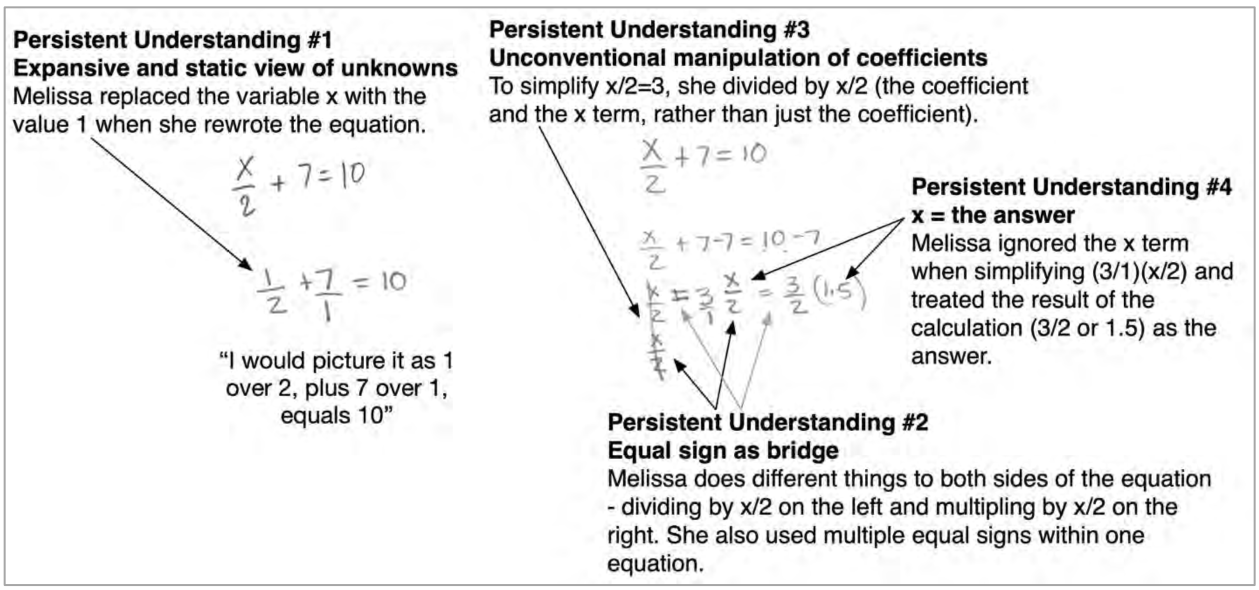


Figure 2. Melissa's written work, and illustration of multiple persistent understandings within one solution process.

Discussion

This detailed case study of Melissa, an adult student with dyscalculia, found that she had persistent unconventional understanding of standard mathematical symbols (e.g., unknowns, the equal sign, coefficients, x=, and zero). Unlike the kinds of difficulties that all students experience when first learning a topic, these unconventional understandings were persistent. We argue that the persistence of these unconventional understandings suggests that these standard mathematical tools – used to represent quantities and relationships between quantities – were at least partially inaccessible to the student. In other work (Lewis et al., in press) we explore how Melissa's persistent understandings interacted with tools specifically designed to provide her with increased access. Here we note that the persistent understandings continued to emerge as Melissa engaged with new tools, but reorientation to the tools enabled Melissa to reason in more conventional ways.

This detailed case study extends prior research on dyscalculia in several important ways. First, this research demonstrates how number processing difficulties found in younger students with dyscalculia (Landerl, 2013; Rousselle & Noël, 2007), occur in older students engaged in algebraic reasoning. Just as prior research has demonstrated that students with dyscalculia are slower and more error prone when asked to compare or manipulate *arithmetic* quantities (e.g., Desoete et al., 2012), Melissa often made errors (44% of algebra problems were incorrect) and she experienced persistent difficulties understanding, comparing, representing, and manipulating *algebraic* quantities. This study, therefore, extends findings that have been documented in students with dyscalculia, and begins to identify how these difficulties would emerge in an algebraic context. This kind of detailed case study can enable researchers to begin to explore mathematical topic domains beyond basic arithmetic, and provides much needed insight into dyscalculia across mathematical topic domains.

Second, this study offers an anti-deficit framing of dyscalculia by providing a detailed depiction of a student engaged in the process of learning and doing mathematics, rather than describing the student's performance from a deficit frame in terms of speed and accuracy. Unlike prior research on dyscalculia which infers learning difficulties based on patterns of errors on outcome measures (e.g., Bouck et al., 2016; Mazzocco et al., 2008), this study explored the student's reasoning underlying these errors. This study documented the ways in which Melissa was understanding, representing, and manipulating quantities, while engaged in problem solving. This anti-deficit framing is critical for making progress in the field towards accurate identification of dyscalculia and offers new avenues to explore for re-mediation.

Future research is needed to determine whether these understandings identified in this study are unique to Melissa, or if they are typical of students with dyscalculia. Building from case study research to large scale studies to examine the prevalence of these characteristics and to develop screening measures to accurately screen for dyscalculia has been demonstrated in the domain of fractions (Lewis et al., 2022).

Conclusion

This exploratory case study provides new insights into the character of difficulties that arose and persisted for one student with dyscalculia in the context of algebra. Findings suggests the utility of documenting the persistent understandings that students with dyscalculia rely upon to understand how dyscalculia may impact students learning of algebra. Beginning to understand dyscalculia in algebra is critical, as algebra often acts as a gate keeper, like it did for Melissa, limiting students' academic and career opportunities.

Acknowledgments

This research was supported by a post-doctoral grant from the National Academy of Education / Spencer Foundation. The opinions expressed are those of the authors and do not represent views of the National Academy of Education / Spencer Foundation.

References

Adelman, C. (2006). *The toolbox revisited: Paths to degree completion from high school through college*. Washington, DC: U.S. Department of Education.

Booth, L. R. (1984). *Algebra: Children's strategies and errors: A report of the strategies and errors in secondary mathematics project*. NFER-Nelson Publishing Co.

Bouck, E. C., Bouck, M. K., Joshi, G. S., & Johnson, L. (2016). Understanding problem-solving errors by students with learning disabilities in standards-based and traditional curricula. *Learning Disabilities: A Multidisciplinary Journal*, *21*(1), 14–26.

Butterworth, B. (2010). Foundational numerical capacities and the origins of dyscalculia. *Trends in Cognitive Sciences*, *14*(12), 534–541. https://doi.org/10.1016/j.tics.2010.09.007

Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669–705). Reston, VA: National Council of Teachers of Mathematics.
Cobb, P., Confrey, J., & diSessa, A. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13. https://doi.org/10.3102/0013189X032001009
Cobb, P., Jackson, K., & Dunlap, C. (2017). Conducting design studies to investigate and support mathematics students' and teachers' learning. In J.Cai (Ed.), *Compendium for research in mathematics education (pp. 208-236)*. Reston, VA: National Council of Teachers of Mathematics.
Desoete, A., Ceulemans, A., De Weerdt, F., & Pieters, S. (2012). Can we predict mathematical learning disabilities from symbolic and non-symbolic comparison tasks in kindergarten? Findings from a longitudinal study. *British Journal of Educational Psychology*, *82*(1), 64–81. https://doi.org/10.1348/2044-8279.002002
Duane, D. D. (1979). Toward a definition of dyslexia: A summary of views. *Bulletin of the Orton Society*, *29*, 56–64. https://doi.org/10.1007/BF02653733
Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*(2), 19–25.
Geary, D. C. (2010). Mathematical disabilities: Reflections on cognitive, neuropsychological, and genetic components. *Learning and Individual Differences, 20*(2), 130–133. https://doi.org/10.1016/j.lindif.2009.10.008
Geary, D. C., Hoard, M. K., Nugent, L., & Bailey, D. H. (2012). Mathematical cognition deficits in children with learning disabilities and persistent low achievement: A five year prospective study. *Journal of Educational Psychology, 104(1)*, 206–223. https://doi.org/10.1037/a0025398
Gravemeijer, K., & Cobb, P. (2006). Design research from a learning design perspective. In J. Van den Akker, K. Gravemeijer, S. McKenney & N. Nieveen (Eds.), *Educational design research* (pp. 17-51). London: Routledge.
Grünke, M., Hord, C., & Macdonald, S. J. (2021). A rationale for the use of case reports in special education: The significance of detailed descriptions of assessment and intervention scenarios for bridging the research-to-practice gap. *Insights into Learning Disabilities*, *18*(1), 91–101.
Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). New York, NY: Erlbaum.
Kaput, J. J., Blanton, M. L., & Moreno, L. (2008). Algebra from a symbolization point of view. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 19–55). New York, NY: Erlbaum.
Kieran, C. (1981). Concepts associated with the equality symbol. *Educational studies in Mathematics*, *12*(3), 317-326.
Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York, NY: Macmillan.
Kieran, C. (2006). Research on the learning and teaching of algebra: A broadening of sources of meaning. In A. Gutiérrez, P. Boero, A. Gutiérrez (Ed), & P. Boero (Ed) (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future.* (pp. 11–49). Rotterdam, Netherlands: Sense Publishers.
Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669–705). Reston, VA: National Council of Teachers of Mathematics.
Landerl, K. (2013). Development of numerical processing in children with typical and dyscalculic arithmetic skills—A longitudinal study. *Frontiers in Psychology*, *4*. doi: 10.3389/fpsyg.2013.00459
Lange, K. W., Reichl, S., Lange, K. M., Tucha, L., & Tucha, O. (2010). The history of attention deficit hyperactivity disorder. *ADHD Attention Deficit and Hyperactivity Disorders*, *2*(4), 241–255. https://doi.org/10.1007/s12402-010-0045-8
Lewis, K. E. (2014). Difference not deficit: Reconceptualizing mathematical learning disabilities. *Journal for Research in Mathematics Education*, *45*(3), 351–396. https://doi.org/10.5951/jresematheduc.45.3.0351
Lewis, K. E. (2016). Beyond error patterns: A sociocultural view of fraction comparison error patterns in students with mathematical learning disabilities. *Learning Disability Quarterly 39*(4), 199-212. https://doi.org/10.1177/0731948716658063
Lewis, K. E. (2017). Designing a bridging discourse: Re-mediation of a mathematical learning disability. *Journal of the Learning Sciences*, *26*(2), 320–365. https://doi.org/10.1080/10508406.2016.1256810
Lewis, K. E., & Lynn, D. M. (2018). Access through compensation: Emancipatory view of a mathematics learning disability. *Cognition & Instruction*, *36*(4), 424–459. https://doi.org/10.1080/07370008.2018.1491581
Lewis, K. E., Sweeney, G., Thompson, G. M., & Adler, R. (2020). Integer number sense and notation: A case study of a student with a mathematics learning disability. *Journal of Mathematical Behavior, 59, 1-27.* doi: 10.1016/j.jmathb.2020.100797

Lewis, K. E., Sweeney, G., Thompson, G. M., Adler, R. M., and Alhamad, K. (in press). Dyscalculia in algebra: A case study. *Insights into Learning Disabilities.*

Lewis, K. E., Thompson, G. M. & Arvey, S. A. (2022). Screening for characteristics of dyscalculia: Identifying unconventional fraction understandings. *International Electronic Journal of Elementary Education 14*(3), 243-267. doi: 10.26822/iejee.2022.242

Lyon, G. R., Shaywitz, S. E., & Shaywitz, B. A. (2003). A definition of dyslexia. *Annals of Dyslexia*, *53*, 1–14.

Mazzocco, M. M. M. (2007). Defining and differentiating mathematical learning disabilities and difficulties. In D. B. Berch, & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 29–48). Baltimore, MD: Paul H. Brookes.

Mazzocco, M. M. M., Devlin, K. T., & McKenney, S. J. (2008). Is it a fact? Timed arithmetic performance of children with mathematical learning disabilities (MLD) varies as a function of how MLD is defined. *Developmental Neuropsychology, 33*(3), 318–344. https://doi.org/10.1080/87565640801982403

Mazzocco, M. M. M., & Myers, G. F. (2003). Complexities in identifying and defining mathematics learning disability in the primary school-age years. *Annals of Dyslexia*, *53*, 218–253. https://doi.org/10.1007/s11881-003-0011-7

Mazzocco, M. M. M., Myers, G. F., Lewis, K. E., Hanich, L. B., & Murphy, M. M. (2013). Limited knowledge of fraction representations differentiates middle school students with mathematics learning disability (dyscalculia) versus low mathematics achievement. *Journal of Experimental Child Psychology, 115*(2)*,* 371–387. https://doi.org/10.1016/j.jecp.2013.01.005

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common Core State Standards for Mathematics. Washington, DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Piazza, M., Facoetti, A., Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., … Zorzi, M. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition*, *116*(1), 33–41. https://doi.org/10.1016/j.cognition.2010.03.012

Rousselle, L., & Noël, M.-P. (2007). Basic numerical skills in children with mathematics learning disabilities: A comparison of symbolic vs non-symbolic number magnitude processing. *Cognition*, *102*(3), 361–395. https://doi.org/10.1016/j.cognition.2006.01.005

Schoenfeld, A. H., Smith, J. P., III, & Arcavi, A. (1993). Learning: The microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (Ed.), *Advances in instructional psychology* (pp. 55–175). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schrank, F. A., Mather, N., & McGrew, K. S. (2014). Woodcock-Johnson IV tests of achievement. *Rolling Meadows, IL: Riverside*.

Shalev, R. S. (2007). Prevalence of developmental dyscalculia. In D. B. Berch & M. M. M. Mazzocco (Eds.), *Why is math so hard for some children? The nature and origins of mathematical learning difficulties and disabilities* (pp. 49–60). Baltimore, MD: Paul H. Brookes.

Stephens, A. C., Knuth, E. J., Blanton, M. L., Isler, I., Gardiner, A. M., & Marum, T. (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. *Journal of Mathematical Behavior*, *32*(2), 173–182. https://doi.org/10.1016/j.jmathb.2013.02.001

Van de Walle, J., Karp, K., & Bay-Williams, J. (2016). *Elementary and Middle School Mathematics: Teaching Developmentally* (9th ed.). Boston, MA: Pearson.

Verhoeff, B. (2013). Autism in flux: A history of the concept from Leo Kanner to DSM-5. *History of Psychiatry*, *24*(4), 442–458. https://doi.org/10.1177/0957154X13500584

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics*, *49*(3), 341–359.

Vygotsky, L. S. (1993). Introduction: Fundamental problems of defectology. In R. W. Rieber & A. S. Carton (Eds.), *The collected works of L. S. Vygotsky: Vol. 2. The fundamentals of defectology (Abnormal psychology and learning disabilities)* (J. E. Knox & C. Stevens, Trans., pp. 29-51). New York, NY: Plenum Press. (Original work published 1929)

Wolff, S. (2004). The history of autism. *European Child & Adolescent Psychiatry*, *13*(4), 201–208. https://doi.org/10.1007/s00787-004-0363-5

PROBLEM SOLVING AND PERSEVERANCE IN GEOMETRY: REVELATIONS FROM THINK ALOUDS WITH MIDDLE GRADES STUDENTS

J. P. Mannix
Ball State University
jpmannix@bsu.edu

Brianna Bentley
North Carolina State University
blbentle@ncsu.edu

Erin E. Krupa
North Carolina State University
eekrupa@ncsu.edu

Problem solving is a very important skill for students to learn (e.g., Bonilla-Rius, 2020; NGA, 2010), and part of developing problem solving skills is learning to persevere. One strategy for learning how to persevere is by providing students with materials that allow them the opportunity to engage with challenging problems (e.g., Kapur, 2010; Middleton et al., 2015). This study of the Volume unit of the AC²inG materials analyzes students' strategies for problem solving and persevering. Findings from these think-aloud interviews indicate that different students will utilize one or more methods for solving challenging problems, such as asking clarifying questions, talking themselves through the problem, and attempting various mathematical approaches.

Keywords: Problem solving, middle school education, geometry and spatial reasoning, curriculum

Theoretical Framework

The first Standard for Mathematical Practice outlined in the Common Core State Standards (NGA, 2010) states that students should be able to "make sense of problems and persevere in solving them" (SMP.1). Perseverance and problem solving are also emphasized in standards outside the United States (e.g., Canada (Ontario, 2020); Mexico (Bonilla-Rius, 2020)). The ability to problem solve and persevere are highlighted as part of the skillset required to be successful in our modern society. According to Bonilla-Rius (2020), the recently re-envisioned national standards in Mexico outline eleven skills and characteristics necessary to be a successful citizen in the 21st century, including critical thinking and problem solving.

Students also recognize the importance of perseverance when faced with a challenging problem. When asked what it takes to solve problems, one group of elementary students offered the following responses: "Be open minded."; "Be able to defend your thinking."; and "Don't give up – persevere!" (Costello, 2020). However, while many stakeholders understand the importance of perseverance and problem solving abilities, researchers understand that these crucial skills are not always innate for students and, thus, must be modeled for them (Colgan, 2020). One way to model these skills is through the materials and resources teachers provide for their students (Kapur, 2010; Middleton et al., 2015). Materials that utilize contrasting cases are one such set of resources that provide opportunities for students to practice problem solving (Loibl et al., 2020), an important part of which is persevering.

The Animated Contrasting Cases in Geometry (AC²inG) Materials

The Animated Contrasting Cases in Geometry (AC²inG) project has recently developed a set of materials that utilize contrasting cases to offer middle-grades students the chance to reason about various geometric topics in a relatively novel way. Contrasting cases are materials that present two or more methods for solving the same or similar problems, and research has shown that these types of activities can be effective for developing both procedural fluency and conceptual understanding (Rittle-Johnson & Star, 2009). Covering topics across the CCSS-M

Grade 8 standards, including angles, transformations, the Pythagorean theorem, and volume, the AC^{2}inG materials are organized into four units, each covering one of the aforementioned topics, that include five or six Worked Example Pairs (WEPs) per unit. Each WEP presents students with two fictitious characters' methods for solving a problem and allows students to explore the reasoning of these two characters side by side, providing them the opportunity to compare and contrast each character's method. Furthermore, each WEP consists of five parts: (1) the first character's method alone; (2) the second character's method alone; (3) both methods side by side; (4) a series of discussion questions and practice problems about the concept covered in the WEP; and (5) a Thought Bubble page where one of the characters shares a revelatory idea about the concept. This design presents students with myriad opportunities to critique the reasoning of others (another of the Standards for Mathematical Practice, SMP. 3 (NGA, 2010)) while practicing their problem solving skills and challenging themselves to persevere.

Methods

Due to the constraints of the current pandemic, our team conducted individual think aloud interviews (Piaget, 1976) with 42 students in lieu of piloting these materials in classrooms. In each interview, students worked through one or more of the WEPs from a particular unit while discussing their thoughts about the methods of each character as well as how the students themselves would solve the problems and answer the discussion questions. These interviews were transcribed and independently coded by two members of our research team. After one researcher coded an interview, a second researcher then coded the interview to determine agreement. In the case of disagreements, a third researcher made a final determination about the code(s) in question.

This report focuses on codes from the Volume unit pertaining to students' original geometric thinking, particularly those where students struggled and persevered. The code 'Persevered' was a subcode of the code 'Struggle,' which was used when a student showed us they were having trouble working through a problem. We analyzed quotes tagged with the 'Struggle' code for similarities and differences and realized that, when students were confused, they took one of two possible pathways. First, the student could stop working and give up, an option students exercised approximately one-third of the time. Second, the student could choose to continue working through the problem, an option we observed the other two-thirds of the time and coded as 'Persevered.' These codes were further analyzed for similarities and grouped into two broad approaches, as outlined below.

Findings

Throughout the 14 think alouds covering WEPs from the Volume unit, students worked to solve a variety of problems. Many of these problems caused them to struggle, as indicated by the 41 times we used the code 'Struggle' when analyzing transcripts from this unit. Given that perseverance was observed in eight of the 13 students who completed WEPs from the Volume unit, it is no surprise that this trait manifested itself in several different ways. Primarily, we noticed two broad approaches to persevering when problem solving: (1) the student tried to make sense of the problem; or (2) the student tried to use an alternate strategy.

Making Sense of Problems

Persevering through a challenging problem was seen when students attempted to make sense of the problem before proceeding. They did this in three distinct ways: (1) asked clarifying questions; (2) talked themselves through the problem; and (3) tried to recall something they had previously learned.

Ask clarifying questions. When students encountered difficult problems while working through the materials, one strategy they used to persevere was to ask the researcher clarifying questions about the problem. For example, when posed with the question, "Can two cylinders with different dimensions have the same volume?", one student asked about the word dimensions. "Is that like height or depth or something?" Rather than give up on this problem due to a vocabulary issue, this student chose to ask a clarifying question and was able to craft a cogent, correct response.

Other students asked clarifying questions and were able to correct their misconceptions through the act of asking their question aloud. One WEP prompt asked how much paint is necessary to paint the walls of Rachel's room. Two of her friends, Damien and Sydney, offered to help her solve this problem. Damien used surface area to calculate his answer, and Sydney used volume. One student originally said that volume would be best because "Rachel painted the entire room, and it said that she used [volume]." This student had misassigned Sydney's method to the fictitious character, Rachel, who was the friend in the scenario who needed help. This student asked the researcher to clarify what the question was asking and realized that they had made a mistake when interpreting the methods and solutions. They then changed their answer to surface area, realizing that volume would fill the room with paint.

Talk through the problem. Another strategy students used was to talk themselves through the problem at hand. One student initially struggled to understand the method employed by one of the characters when attempting to find the volume of a cylindrical container. "It seems weird to me," said the student. "I kept running through it. I was wondering why it was like that." Then revelation struck. "Oh wait! I see now. This is B [the area of the base] - you're trying to find the circle." By walking through the problem and provided method several times, this student was able to understand the reasoning of the character in the WEP on their own.

Recall previous learning. A third strategy we observed students using was to dig into their memories to recall concepts and information they had previously learned. Sometimes they were successful in remembering. One student knew that they had learned how to calculate the volume of a cylinder previously and thought their teacher would be "so angry with [them] if [they] forget these things." Consequently, the student tried to think back to their previous work and finally recalled the correct strategy. Another student initially misread a problem about calculating the volume of composite figures and struggled to remember a specific formula before remembering that the figure could be separated into more common figures, the formulas for which the student was able to recall.

Other students attempted to remember what they had learned but were not as successful. One such student stated that they "kind of forgot how to do the area of a cylinder" and cited this as the reason they were confused when trying to reason through the characters' methods. Even though this student could not recall the knowledge necessary to solve the problem, we identified this as perseverance because they attempted and did not give up on the problem until they had attempted several times.

Trying an Alternate Strategy

When students would get stuck in the middle of a problem, we noticed that some of them: (1) tried a different mathematical approach; or (2) took an educated guess.

Tried a new mathematical approach. Students sometimes decided to take a different mathematical approach than the one they originally thought of or tried. One student was confused by a problem that asked them to find the volume of a cylinder if the radius were scaled by a factor of three. The first thing they attempted was to find the height of the cylinder so they could use the volume formula. When they realized the problem did not give them the height, they attempted to plug in what they were given and solve for the height. This strategy was more complex than they were expecting, and even though they did not find the correct answer, they persevered through their selected strategy and found a solution, albeit one that was incorrect.

Took an educated guess. After exhausting other efforts, some students settled on offering a best guess. After talking through the problem of scaling the radius of a cylinder, one student offered an incorrect solution but was able to justify their guess and the steps they took to arrive at their guess. Another student, when trying to use one of the characters' methods to find the volume of a cylinder, admitted that they did not fully understand the fictitious character's method, and while they did not come to the correct solution, they did offer an explanation of their thinking and a guess as to what they thought it might be.

Several times, students initially did not have any idea what to do with a problem, but when the researcher asked what they were thinking, they were able to offer a guess. When thinking about why scaling the radius by some number n causes the volume to scale by n^2, one student tried to reason through the problem, after initially stating that they did not know how to answer it. "Um, I mean, you didn't change the height, I guess, so that's why the volume multiplied. I don't know." Even though this student was not able to completely articulate their thinking, they persevered and demonstrated a preliminary understanding of the relationship between the radius and the volume.

Discussion and Future Research

The math education community places great importance on problem solving and perseverance, as witnessed by the standards we set for our students (e.g., CCSS Standards for Mathematical Practice (NGA, 2010)). These skills are essential in the 21st century (Bonilla-Rius, 2020), and students themselves recognize the importance of persevering when faced with challenging problems (Costello, 2020). In this study, we witnessed students using a variety of problem solving strategies and persevering to overcome questions that caused them confusion. They asked clarifying questions, talked through the problem, dug into their memories, tried multiple strategies, and even took educated guesses when necessary.

The think aloud interviews allowed us to uncover ways in which students persevered when working with middle-grades volume concepts. Future research might analyze what type of materials best encourage students to persevere. It also might be pertinent to compare the students who persevered with those who did not and try to understand the differences between these students.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1907745. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Bonilla-Rius, E. (2020). Education truly matters: Key lessons from Mexico's educational reform for educating the whole child. In F. Reimers (Ed.), *Audacious Education Purposes: How Governments Transform the Goals of Education Systems* (pp. 105-151). Springer.

Colgan, L. (2020). Elementary math matters: If math were an animal. *Ontario Mathematics Gazette, 59*(2).

Costello, D. (2020). Linking literacy and math: Going beyond surface level when problem-solving. *Ontario Mathematics Gazette, 59*(1).

Kapur, M. (2010). Productive failure in mathematical problem solving. *Instructional Science*, 38(6), 523–550.
Loibl, K., Tillema, M., Rummel, N., & van Gog, T. (2020). The effect of contrasting cases during problem solving prior to and after instruction. *Instructional Science, 48*, 115-136.
Middleton, J.A., Tallman, M.A., Hatfield, N., & Davis, O. (2015). Taking the severe out of perseverance: Strategies for building mathematical determination. In N. Alpert & C. Kurose (Eds.), *Mathematical Instruction for Perseverance.* Spencer Foundation.
National Governors Association Center for Best Practices, & Council of Chief State School Officers (NGA). (2010). *Common Core State Standards for mathematics.* Washington, DC: Authors.
Ontario. (2020). *The Ontario curriculum, grades 1-8*. Toronto: Ontario, Ministry of Education.
Piaget, J. (1976). *The child's conception of the world* (J. Tomlinson and A. Tomlinson, Trans.). Littlefield, Adams & Co. (1926).
Rittle-Johnson, B., & Star, J. R. (2009). Compared to what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*, 529-544. doi: 10.1037/a0014224

MATHEMATICS IDENTITY AND GENDER DIFFERENCES IN STEM PERSISTENCE: A LATENT GROWTH CURVE MODEL

Dalton D. Marsh
California State University, San Bernadino
Dalton.Marsh@csusb.edu

Mathematics and science identity has been shown to be a significant factor related to student achievement, course-taking, and college and career choices in STEM. However, less in known about how students' mathematics and science identities change over time and how these changes relate to persistence in STEM programs. This study uses data from three waves of the High School Longitudinal Study of 2009 and latent growth curve modeling to investigate how STEM students' mathematics and science identities change from high school through college. Results show that while male STEM students tend to maintain high levels of mathematics identity throughout, female STEM students' mathematics identity tends to start lower and decline over time.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Gender.

It is well established that despite having similar high school STEM achievement and coursework as their male counterparts, female students are less likely to enter and more likely to leave postsecondary Science Technology, Engineering, and Mathematics (STEM) degree programs (Chen, 2013). To explain women's underrepresentation in STEM, the extant literature points to motivational effects of gender stereotypes and observed female gender disparities in positive self-perceptions about mathematics and science. Such motivational differences are typically examined at fixed time points, however a review by Wang and Degol (2013) argues that more longitudinal research is needed.

One longitudinal study (Wang et al., 2017) employing expectancy-value theory, found that self-perceptions about ability, importance, interest, and usefulness of science tend to decline over secondary school more sharply for girls. These trajectories were linked to lower STEM achievement, course-taking, and career aspirations. More work is needed to see how these effects play out in postsecondary settings. The present study aims to address this gap by examining growth trajectories of two STEM-related self-perceptions that have garnered critical attention recently: mathematics and science identity. Trajectories for these self-perceptions were examined from high school through college and notable gender differences were found that offer explanations for female students' lower interest and persist in STEM.

Theoretical Framework

This study employs expectancy-value theory (Eccles, 2009). In terms of identity, the theory argues that students tend to persist in the achievement-related tasks they view as consistent with their personal and collective identities (i.e., sense of self and sense of belonging to a group). In the context of STEM, this means that students tend to persist in STEM as long as they perceive the activities involved in STEM to be consistent with their view of themselves and others they identify with (e.g., gender and racial groups). Empirical research has agreed, finding that students who form and maintain strong mathematics and science identities persist in STEM degrees and careers (e.g., Aschbacher, Li, & Roth, 2010; McGee, 2015). However, currently most of these studies are small-scale and qualitative. This research aims to address this gap.

Methodology

The present study seeks to answer the following research questions: (1) How do STEM degree pursing students' mathematics and science identities change from high school through college? (2) Do these trajectories differ by gender, race, or first-generation college status?

Data and Sample

To answer these questions the present study employed data from the High School Longitudinal Study of 2009 (HSLS:09; Ingles et al., 2011). HSLS:09 is the most recent in a series of surveys administered by the National Center for Education Statistics (NCES) that follow nationally representative samples of young people as they transition from high school to postsecondary years. The first wave of HSLS:09's data collection began in the fall of 2009 with over 23,000 ninth-graders from 944 public and private schools throughout the United States. Sampling involved a complex, two-stage design in which eligible schools were first randomly selected and then students within those schools were randomly selected (Ingles et al., 2011). The students were followed up in the spring of 2012, when most were in the eleventh grade (Ingles et al., 2014); in 2013, for high school transcripts and postsecondary plans (Ingles et al., 2015); in 2016, when most college-attending students were in their fourth year of college (Duprey et al., 2018); and in 2017, for postsecondary transcripts (Duprey et al., 2020).

The analytic sample consisted of HSLS:09 students who participated in all five rounds of data collection listed above and were enrolled in a STEM degree program according to the 2013 Update (Ingles et al., 2015). Majors considered STEM included computer, physical, and natural sciences; engineering; mathematics and statistics; and military and science technologies or technicians. The resulting sample is then representative of U.S. 2009 ninth graders who pursued STEM degrees after high school.

Dependent Variables

Mathematics and science identity were measured at three timepoints: 9th grade, 11th grade, and in the fourth year of college (Fall 2009, Spring 2012, and Fall 2016). In each of the three waves, two Likert-scale items measured students' mathematics (science) identity: (a) you see yourself as a math (science) person and (b) other people see you as a math (science) person. Items (a) and (b) were combined to form a composite scale via principal components factor analysis. Reliability tests suggested good internal consistency between the items in each scale ($\alpha > .80$). The scales were standardized ($M = 0$, $SD = 1$) over all (STEM and non-STEM) students.

Time-Invariant Predictors

Prior achievement in mathematics was measured using the ninth-grade algebraic reasoning assessment score. The assessment was developed by NCES using an item-response theory (IRT) design. *Prior coursework in mathematics and science* was measured using the highest-level mathematics course and highest-level science course variables gathered from the high school transcript file. The above were each standardized ($M = 0$, $SD = 1$) in the analytic sample. Additional student background variables included student's *gender* (1 = female, 0 = male); *race/ethnicity* (Asian, Black or African American, Hispanic or Latino, Native American, two or more races, and White was used as the reference group); and *first-generation college status* (1 = parents do not have a college degree; 0 = parents have at least an associate's degree).

Estimation

To estimate the mathematics and science identity growth trajectories the researcher used latent growth curve modeling (Byrne, 2011). The analysis was conducted in Mplus 8.4 with the MLR estimator which is robust against non-normality and accounts for missing data (Muthén & Muthén, 1998-2017). Due to the complex sampling design of HSLS:09, a balanced repeated replication (BRR) weighting procedure was employed, and model fit was assessed with the standardized root mean square residual (SRMR) index (Asparouhov & Muthén, 2010, 2018). SRMR values less than .05 and .08 are considered excellent and acceptable fits, respectively (Hu & Bentler, 1999). The HSLS:09 survey weight W5W1W2W3W4PSTRANS was used in combination with its corresponding 200 BRR weights (see Duprey et al., 2020).

Results

The sample of STEM degree pursing students included $N = 1{,}931$ students of which 33.3 percent were female and 66.7 percent male. 6.9 percent of the sample identified as Asian, 8.9 percent as Black or African American, 16.0 percent as Hispanic or Latino, 1.4 percent as Indigenous (Native American, Native Hawaiian, Alaska Native), and 7.5 percent identified with two or more races. 26.5 percent were reported to be first-generation college students.

The growth curve analysis began by estimating the unconditional linear model (intercept, slope, no predictors). The model fit was estimated to be excellent (SRMR = .025). The mathematics curve was estimated to have an intercept of 0.459 ($p < .001$) and a slope value of 0.001 ($p = .788$). The science curve was estimated to have an intercept of 0.489 ($p < .001$) and a slope of 0.006 ($p = .549$). These estimates indicated that STEM degree pursuers on average have about a half of a standard deviation higher sense of mathematics and science identity initially compared to other college students and then maintain these levels of identity throughout high school and college. Quadratic growth curve models (intercept, linear, and quadratic terms) were estimated but the analyses found these models to not fit the data. Therefore, the growth curves were assumed to be linear.

Next, predictors were added, all of which were modeled as time invariant. The model fit test suggested excellent fit (SRMR = .019). The covariate model analysis did not find race/ethnicity or first-generation status to be significant ($p > .05$) predictors on the intercept or slope, suggesting that students' mathematics and science identity growth trajectories did not vary by race, ethnicity, or college generational status. However, there were significant gender effects on the intercept ($p = .007$) and slope ($p = .020$) for mathematics identity but not for science. In the interest of gender differences, Table 1 displays the model estimates for mathematics identity.

Table 1: Parameter Estimates for Latent Growth Curve Model of Mathematics Identity

Predictor	Estimate	*p*-value
Intercept		
Female	–0.193	.007
Asian	–0.182	.182
Black or African American	0.168	.202
Hispanic or Latino	0.077	.514
Indigenous	0.080	.795
Two or more races	–0.209	.134
First-generation to college	0.102	.279
Prior math achievement	0.376	< .001
Prior math coursework	0.107	< .001
Prior science coursework	–0.028	.456
Slope		
Female	–0.030	.020
Asian	–0.001	.960
Black or African American	0.002	.927

Hispanic or Latino	–0.009	.677
Indigenous	–0.021	.640
Two or more races	0.011	.609
First-generation to college	0.007	.719
Prior math achievement	–0.012	.142
Prior math coursework	0.001	.949
Prior science coursework	0.002	.814

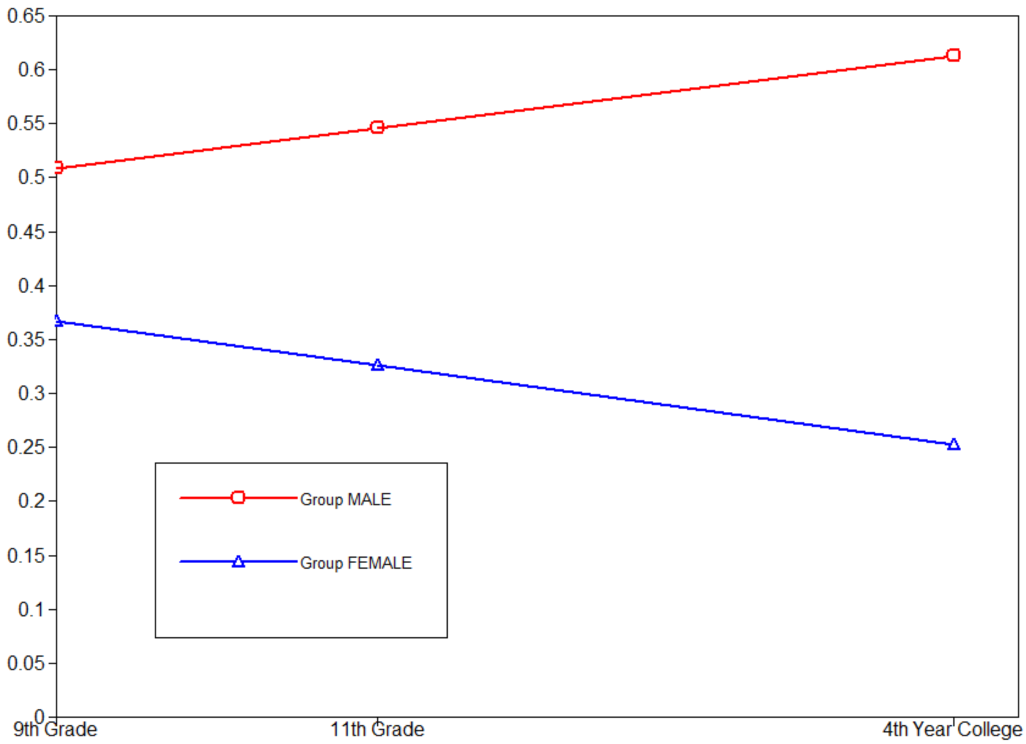


Figure 1: Mean Mathematics Identity Growth Trajectories for Male and Female Students

To further analyze gender differences in mathematics identity growth trajectories, plots were generated of the estimated mean growth curves for each gender group (Figure 1). The estimated mean mathematics identity growth trajectory for male students had an intercept of 0.508 and slope of 0.015 while the mean growth trajectory for female students had an intercept of 0.397 and slope of -0.016. Thus, while male STEM students' mathematics identity started higher and increased over time, female students' identity started lower and decreased, resulting in a further widening of the gender gap in mathematics identity in college.

Discussion

The preliminary results from this study provide large-scale evidence of gender disparities in women's identity as a math or science person that may help to explain why they are less likely to enter and more likely to leave STEM degree programs in the United States. The study is waiting on future data from HSLS:09 to directly link changes in mathematics identity with STEM degree attainment. Other concerns to follow up on include how these results can inform initiatives for broadening participation in STEM and the degree to which this quantitative measure of identity is consistent with other conceptualizations of identity in the literature.

References

Aschbacher, P. R., Li, E., & Roth, E. J. (2010). Is science me? high school students' identities, participation and aspirations in science, engineering, and medicine. *Journal of Research in Science Teaching, 47*(5), 564–582. doi: 10.1002/tea.20353

Asparouhov, T. & Muthén, B. (2010). Resampling methods in Mplus for complex survey data. Technical Report. May 4, 2020. https://www.statmodel.com/techappen.shtml

Asparouhov, T. & Muthén, B. (2018). SRMR in Mplus. Technical Report. May 2, 2018. https://www.statmodel.com/techappen.shtml

Byrne, B. M. (2011). *Structural equation modeling with Mplus: Basic concepts, applications, and programming*. New York, NY: Routledge.

Chen, X. (2013). *STEM attrition: College students' paths into and out of STEM fields* (NCES 2014–001). Washington, DC: National Center for Education Statistics.

Duprey, M. A., Pratt, D. J., Jewell, D. M., Cominole, M. B., Fritch, L. B., . . . Wilson, D. H. (2018). *High School Longitudinal Study of 2009 (HSLS:09) Base-Year to Second Follow-Up Data File Documentation* (NCES 2018-140). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC. Retrieved October 16, 2020, from https://nces.ed.gov/pubsearch.

Duprey, M. D., Pratt, D. J., Wilson, D. H., Jewell, D. M., Brown, D. S., Caves, L. R., . . . Wescott, J. D. (2020). *High School Longitudinal Study of 2009 (HSLS:09) Postsecondary Education Transcript Study and Student Financial Aid Records Collection Data File Documentation* (NCES 2020-004). U.S. Department of Education. Washington, DC: National Center for Education Statistics, Institute of Education Sciences. Retrieved October 16, 2020, from https://nces.ed.gov/pubsearch.

Hu, L. & Bentler, P. (1999). Cutoff criteria for fit indices in covariance structure analysis: conventional criteria versus new alternatives. *Structural Equation Modeling, 6*, 1–55. doi:10.1080/10705519909540118

Ingels, S. J., Pratt, D. J., Herget, D. R., Burns, L. J., Dever, J. A., Ottem, R., . . . Leinwand, S. (2011). *High School Longitudinal Study of 2009 (HSLS:09): Base-year data file documentation* (NCES 2011–328). Washington, DC: National Center for Education Statistics. Retrieved May 25, 2018, from http://nces.ed.gov/pubsearch

Ingels, S. J., Pratt, D. J., Herget, D. R., Dever, J. A., Fritch, L. B., Ottem, R., . . . Leinwand, S. (2014). *High School Longitudinal Study of 2009 (HSLS:09): Base year to first follow-up data file documentation* (NCES 2014–361). Washington, DC: National Center for Education Statistics. Retrieved July 9, 2018, from http://nces.ed.gov/pubsearch

Ingels, S.J., Pratt, D. J., Herget, D., Bryan, M., Fritch, L.B., Ottem, R., . . . Wilson, D. (2015). *High School Longitudinal Study of 2009 (HSLS:09) 2013 Update and High School Transcript Data File Documentation* (NCES 2015–036). Washington, DC: National Center for Education Statistics. Retrieved November 3, 2018, from http://nces.ed.gov/pubsearch

McGee, E. O. (2015). Robust and fragile mathematical identities: A framework for exploring racialized experiences and high achievement among black college students. *Journal for Research in Mathematics Education, 46*(5), 599-625. doi:10.5951/jresematheduc.46.5.059

Muthén, L. K. and Muthén, B. O. (1998–2017). *Mplus user's guide* (8th ed.). Los Angeles, CA: Muthén & Muthén.

Wang, M.-T., Chow, A., Degol, J., & Eccles, J. S. (2017). Does everyone's motivational beliefs about physical science decline in secondary school?: Heterogeneity of adolescents' achievement motivation trajectories in physics and chemistry. *Journal of Youth & Adolescence, 46*(8), 1821–1838. doi:10.1007/s10964-016-0620-1

Wang, M.-T. & Degol, J. (2013). Motivational pathways to STEM career choices: Using expectancy-value perspective to understand individual and gender differences in STEM fields. *Developmental Review, 33*(4), 304–340. doi:10.1016/j.dr.2013.08.001

POTENTIAL OF AN MEA TO ADVANCE BUSINESS STUDENTS' MODELING SKILLS

Luis E. Montero-Moguel
The University of Texas at San Antonio
luis.monteromoguel@utsa.edu

Verónica Vargas-Alejo
University of Guadalajara
veronica.vargas@academicos.udg.mx

Cynthia Lima
The University of Texas at San Antonio
Cynthia.Lima@utsa.edu

Guadalupe Carmona-Domínguez
The University of Texas at San Antonio
Guadalupe.Carmona@utsa.edu

The development of modeling skills in mathematics is essential for individuals to understand, describe, control, and predict phenomena around them. This article describes the results of an investigation to find out how an activity –based on a Models and Modeling Perspective– stimulates the mathematics modeling skills of undergraduate students who are in the first quarter of a business degree. As a result, it was shown that the MEA enabled students to exhibit, develop, and refine different modeling skills, such as: identification of variables, assumptions based on the real-life context, identification of patterns, and construction of mathematical representations.

Keywords: Modeling, higher education, precalculus.

Introduction

The pandemic generated by COVID 19 has highlighted the need for math education researchers and educators to reflect on the knowledge and skills that students must develop to understand, describe, control, and predict phenomena that impact humankind, and that also allows them to make decisions and take actions to relieve negative effects of these phenomena. Some educators suggest teaching strategies based on mathematics modeling that allow students to develop these skills and knowledge, as well as take active part on the solution process of these real-life problems (Lesh, 2010; Sevinc, 2021). Several authors, such as Lesh and Doerr (2003) and English et al. (2020), agree on the need to structure students' experiences in ways that allow them to develop modeling skills. According to Niss et al. (2007), learning to model implies that students develop different skills, such as "ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation" (p. 12).

Project GAIMME indicates that "to understand how students grow in their ability to do mathematical modeling is a challenging task. Useful studies of mathematical modeling have been published, but this area has not yet been researched as deeply and systematically as other areas" (Garfunkel & Montgomery, 2019, p. 24). The National Council of Teachers of Mathematics (NCTM, 2018) mentions that mathematical modeling skills are part of modeling cycle processes that involve multiple steps and iterations. However, studies by Montero et al. (2021) and Touchstone (2014) highlight that word problems in math textbooks for higher education students in business careers have a single answer and don't usually foster modeling processes. Therefore, we need to provide learning-teaching opportunities that are purposefully designed for university students in business careers to develop modeling skills to solve real-life problems.

The purpose of this study is to understand how a model-eliciting activity (MEA) can contribute to students in business careers to develop and refine modeling skills. The research question is *What types of modeling skills do undergraduate students in business careers develop when they solve an MEA called Breaking barriers with yogurt investments?*

This research study used Model and Modeling Perspectives (MMP) as proposed by Lesh & Doerr (2003) by focusing on structuring students' experiences that foster the development of modeling skills.

Theoretical Framework

Models and Modeling Perspectives propose the design of model-eliciting activities for students to develop, amplify, and refine their thinking by enhancing math and modeling skills as they solve real-life problems (Lesh & Doerr, 2003). MEAs are situations that are purposefully designed for students to generate models using specific mathematical ideas, and are situated in real-life contexts that are meaningful to students (Aliprantis & Carmona, 2003; Doerr, 2016; Sevinc, 2021). Models are conceived as

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently.
>
> A mathematical model focuses on structural characteristics (rather than, for example, physical or musical characteristics) of the relevant systems. (Lesh & Doerr, 2003, p. 10)

Models reside in the mind and representational media. These representational media can include written symbols, oral communication, diagrams, metaphors, tables, graphs, and algebraic symbols (Lesh & Doerr, 2003; Lesh & Harel, 2003).

Modeling activities based on real-life contexts usually have many possible solutions. They can also involve irrelevant ideas or lack information which creates a need for students to develop modeling processes such as generating assumptions and evaluating situations based on specific contexts (Sevinc & Lesh, 2021). Research studies based on MMP have provided evidence that MEAs promote students' development skills to solve problems related to their professional life in the field of business (Lesh & Yoon, 2007). According to Lesh (2010), MEAs foster students to show their thinking in ways that research can "to investigate the development of important aspects of students' mathematical thinking" (p. 29).

Methodology

We used qualitative methods to conduct this research study involving a total of 13 participants from a group of (male and female) students in their first quarter as freshmen enrolled in a business degree program. Students were registered in an in-person course on mathematics applied to business, which is the first mathematics course students take in the business undergraduate program. The topic on exponential functions had not been covered in this course previous to student participation in this study.

The MEA called Breaking barriers with yogurt investments (Yogurt MEA) was designed in the context of a colony of bacteria and yeast that produce Bulgarian yogurt (Figure 1). This MEA was designed using six principles for the construction of an MEA as proposed by Lesh et al. (2000). The mathematical ideas embedded in the design of the Yogurt MEA are: exponential function, equation, variation, growth, and rate of change. The solution requires that students identify the problem, variables and relationships, relevant assumptions, interpretation, and validation of the solution.

Business-Health Magazine

EL YOGURT ESTÁ ROMPIENDO LAS BARRERAS DE LAS INVERSIONES

LA CRECIENTE POPULARIDAD DEL YOGURT SE HA CONVERTIDO EN UN SALVADOR PARA LAS MAYORES LECHERÍAS. SUS VENTAS SUPERAN LAS VENTAS DE LECHE POR PRIMERA VEZ

El yogurt se está convirtiendo en la "vaca en efectivo" de la industria láctea. De acuerdo con el informe de Mordor Intelligence 2018, se esperan ingresos de $106,400 millones de dólares para el año 2024 (Whitely, 2019). Muchos empresarios están emprendiendo negocios millonarios basados en el proceso de este lácteo.

La revista Forbes de negocios ha nombrado a Hamdi Ulukaya el Steve Jobs del yogurt, ya que su empresa en tan solo cinco años se puede valuar en $1,000 millones de dólares; su empresa es rentable, dominante y crece de forma importante. En estos cinco años la empresa de Ulukaya pasó de tener cinco a 2,000 empleados, convirtiéndose en el número uno del yogurt griego en Estados Unidos.

No solo las empresas de los países del continente americano han tenido estas altas utilidades, en China la creciente popularidad del yogur se ha convertido en un salvador para las mayores lecherías del país.

Las ventas de leche china crecieron sólo un 4% en 2017 en comparación con el año 2016, mientras que las ventas de yogures aumentaron un 18%.

Una de las razones por las cuales la empresa del yogurt está creciendo es debido a los grandes beneficios que tiene este lácteo para el cuerpo humano. Contiene una gran cantidad de calcio, un mineral necesario para dientes y huesos sanos. Sólo una taza proporciona el 49% de las necesidades diarias de calcio; es alto en vitaminas del grupo B, particularmente vitamina B12 y riboflavina, que pueden proteger contra enfermedades del corazón y ciertos defectos congénitos del tubo neural. Una taza también proporciona el 38% de la necesidad diaria de fósforo, 12% de magnesio y 18% de potasio. Estos minerales son esenciales para varios procesos biológicos, como la regulación de la presión arterial, el metabolismo y la salud ósea.

Un problema es que la mayoría de las variedades compradas en las tiendas pierden el grado de calidad por tres principales razones, la calidad de la leche no es buena, contienen sabores, endulzantes, colores artificiales y le adicionan azúcar.

Por esta razón, algunos negocios están emprendiendo en el sentido de hacer yogurt con características semejantes al yogurt casero que no tiene aditivos ni conservantes, esto permite al producto mantener sus nutrientes y ser un alimento sano para el cuerpo humano.

La empresa familiar "Kéfir healthy", después de leer el artículo pretende lanzar una línea de yogurt casero, que le permita entrar en el mercado. Uno de los principales obstáculos a los cuales se enfrenta es conocer el crecimiento de los búlgaros de leche (Kéfir). La empresa ha recabado algunos datos con un experimento en laboratorio.

Resultados de laboratorio

La empresa necesita conocer el comportamiento de crecimiento de los búlgaros, de tal manera que pueda decidir en qué momento tendrá cantidades suficientes para vender.

Tu equipo es el responsable de proponer a la empresa "Kéfir healthy" una descripción del comportamiento del crecimiento de los búlgaros y, por lo tanto, un procedimiento que le ayude a predecir en qué momento empezar a vender. Escribe una carta a la empresa donde le expliques cómo puede calcular la cantidad de búlgaros (mililitros) en cualquier momento. En tu carta debes incluir los procedimientos matemáticos, de tal manera que puedan utilizarse por cualquier otra empresa.

a b

Figure 1: a) Yogurt MEA Newspaper article y b) Yogurt MEA Problem Situation

For the implementation of the Yogurt MEA, students were organized in five teams: Team 1 [T1] with two members, Team 2 [T2] with three members, Team 3 [T3] with three members, Team 4 [T4] with two members and Team 5 [T5] with three members. The Yogurt MEA was implemented in several phases:

- Phase 1. Warm-up Activity (15 min). Students read page 1 of the MEA (Figure 1a), answered the warm-up activity, and discussed the context in their teams.
- Phase 2. Team Solving the Yogurt MEA (90 min). Students worked in teams to solve the problematic situation in the MEA (Figure 1b includes the QR code with the video students watched). Students used tools such as spread sheets.
- Phase 3. Teams presented their model solutions to the whole class (30 min). Each team presented their solution model and the process they used to construct it. After each presentation, the class had the opportunity to ask questions and provide feedback about the model.

Data collection involved: video recordings of the group discussion about the model-construction processes and presentation of the final solution model; the solution letters written by each team; log notes from class observations; and the electronic files that the students submitted. Data analysis was guided by an MMP, through a qualitative lens considering the iterative modeling process, which included each team's final solution. We documented how students changed, extended, and refined their conceptual system during the solution process and during whole-class presentations and discussion. In particular, the analytical framework we used is supported in the skills as described by Niss et al. (2007), which include: the ability to identify questions, variables, relationships, relevant assumptions in a given real-life situation, interpretation, and validation of a solution.

Results and Discussion

Based on the models built by the students, we identified that the Yogurt MEA elicited modeling abilities, in particular, four: identification of variables, assumptions based on the real-life context, identification of patterns, and construction of mathematical representations (Table 1).

Table 1: Teams' Modeling Abilities

<table>
<tr><th></th><th>Identification of variables</th><th>Assumptions based on the real-life context</th><th>Identification of patterns</th><th>Representations</th></tr>
<tr><td>T1</td><td rowspan="5">They identified the relationship between the variables: amount of kefir grains, time, and temperature.</td><td rowspan="3">Assumption A. The growth of kefir grains (G) depends on the temperature (T) and time (t). The temperature and time are variables.</td><td rowspan="2">They described and predicted the growth of kefir grains. They identified a pattern of weekly cyclical growth.</td><td rowspan="2">They included verbal, tabular, and graphical representations.</td></tr>
<tr><td>T2</td></tr>
<tr><td>T3</td><td>They described the growth of kefir grains. They multiplied the amount of initial kefir grains by a constant.</td><td rowspan="2">They included verbal and tabular representations.</td></tr>
<tr><td>T4</td><td rowspan="2">Assumption B. The growth of kefir grains (G) depends on the temperature (T) and time (t). They maintained the temperature constant, and the time was variable G(t).</td><td>They described and predicted the kefir grains' growth based on the number of jars. The growth estimation was 625% in four weeks.</td></tr>
<tr><td>T5</td><td>They described and predicted the kefir grains' growth. The daily growth rate estimated was 33%.</td><td>They included verbal, tabular, and graphical representations.</td></tr>
</table>

Team's Modeling Abilities

Identification of variables. Based on the data provided and the Yogurt MEA video, all the students identified three variables related to the process of kefir grains growth. They focused their attention on the variables: growth, temperature, and time. For instance, team T3 mentioned the following in reference to their data table:

[1] T3: Tenemos, en cuanto a los días, temperatura, mililitros y… esto es la diferencia entre un día y otro en cuanto a mililitros [We have, in relation to days, temperature, milliliters and... this is the difference in milliliters between one day and another].

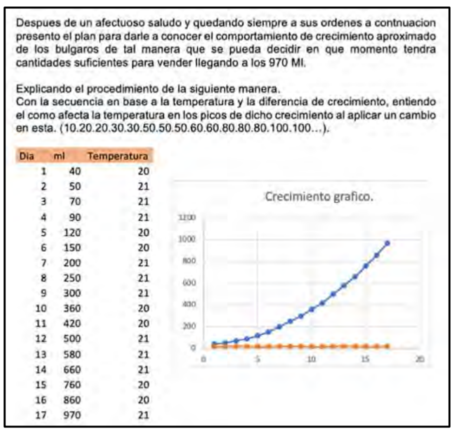
The team T1 indicated the following while they pointed to the column headings of their table (Figure 2a).

[2] T1: Nosotros intentamos resolver esto, primero, con los datos que nos habían dado [para] los primeros siete días. Viendo así, el cambio de temperatura y [cómo] los mililitros cambiaban. [We tried to solve this, first, with the data given for the first seven days. Looking at the temperature change and how the milliliters changed].

After the team's T1 presentation, T2 mentioned that they had found a different relationship among variables.

[3] T2: Yo no estoy de acuerdo… nosotros que también nos basamos en la temperatura y en el porcentaje, nos dimos cuenta que para el día seis, siete el cambio no era de 10 10. Sino se elevaba al doble, pues, de hecho, sacamos como una tablita. [I do not agree… we also considered the temperature and the percentage, and noticed that on day sixth, seventh the change was not 10. Instead, it doubled, and we actually built a table].

[4] T1: Yo también hice esa secuencia [refiriéndose a un modelo inicial que había construido y posteriormente refinó con base en el contexto de la MEA]. [I also created this sequence [referring to an initial model that they initially build and refined later based on the MEA's context].

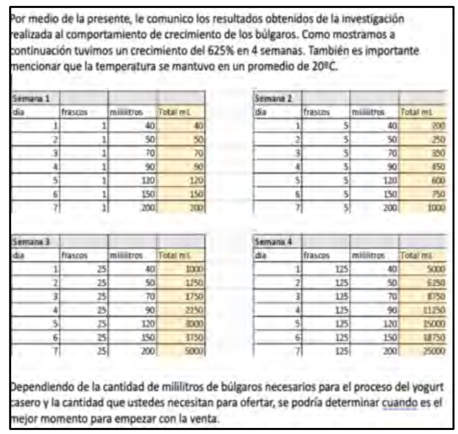


Despues de un afectuoso saludo y quedando siempre a sus ordenes a contnuacion presento el plan para darle a conocer el comportamiento de crecimiento aproximado de los bulgaros de tal manera que se pueda decidir en que momento tendra cantidades suficientes para vender llegando a los 970 Ml.

Explicando el procedimiento de la siguiente manera.
Con la secuencia en base a la temperatura y la diferencia de crecimiento, entiendo el como afecta la temperatura en los picos de dicho crecimiento al aplicar un cambio en esta. (10.20.20.30.30.50.50.50.60.60.80.80.80.100.100…).

Dia	ml	Temperatura
1	40	20
2	50	21
3	70	21
4	90	21
5	120	20
6	150	20
7	200	21
8	250	21
9	300	21
10	360	20
11	420	20
12	500	21
13	580	21
14	660	21
15	760	20
16	860	20
17	970	21

a



Por medio de la presente, le comunico los resultados obtenidos de la investigación realizada al comportamiento de crecimiento de los búlgaros. Como mostramos a continuación tuvimos un crecimiento del 625% en 4 semanas. También es importante mencionar que la temperatura se mantuvo en un promedio de 20ºC.

Semana 1			
dia	frascos	mililitros	Total mL
1	1	40	40
2	1	50	50
3	1	70	70
4	1	90	90
5	1	120	120
6	1	150	150
7	1	200	200

Semana 2			
dia	frascos	mililitros	Total mL
1	5	40	200
2	5	50	250
3	5	70	350
4	5	90	450
5	5	120	600
6	5	150	750
7	5	200	1000

Semana 3			
dia	frascos	mililitros	Total mL
1	25	40	1000
2	25	50	1250
3	25	70	1750
4	25	90	2250
5	25	120	3000
6	25	150	3750
7	25	200	5000

Semana 4			
dia	frascos	mililitros	Total mL
1	125	40	5000
2	125	50	6250
3	125	70	8750
4	125	90	11250
5	125	120	15000
6	125	150	18750
7	125	200	25000

Dependiendo de la cantidad de mililitros de búlgaros necesarios para el proceso del yogurt casero y la cantidad que ustedes necesitan para ofertar, se podría determinar cuando es el mejor momento para empezar con la venta.

b

Figure 2: a) T1's Model and b) T4's Model

Discussion. The Yogurt MEA fostered an iterative discussion of the variables related to the kefir grains growth, [3] and [4]. According to Bliss et al. (2014), identification of variables is important because it allows students to determine the factors influencing the phenomenon they are analyzing and "distinguish between independent variables, dependent variables, and model parameters" (p. 7).

Assumptions based on the real-life context. The students built their models by generating assumptions based on the real-life context; they contrasted their procedures with the actual growth of kefir grains and their personal and professional experiences. We identified two types of assumptions.

Assumption A. The initial assumption of teams T1, T2, and T3 was that kefir grains growth (G) depends on the temperature (T) and time (t). They assumed that the temperature and time were independent variables. T2's personal experience allowed them to identify the possibility of additional variables that could influence the growth. The following excerpt illustrates this idea.

[5] T2: Yo, por experiencia que he cultivado muchos búlgaros, lo que platicaba con ellas, al tener siempre la misma temperatura, los búlgaros van reaccionando... lo ideal sería mantener la misma temperatura, para mayor crecimiento. Siento que depende mucho de varios factores. [I, from my experience growing kefir grains, what I was talking with them, because they always keep the same temperature, the kefir grains react… the best scenario would be to maintain the same temperature, for a higher growth. I feel that it depends of multiple factors].

T3 showed the need to investigate the kefir growth to understand the phenomenon and the influence of time and temperature. The students made the following statement:

[6] T3: Primero que nada, no sé si alguno de ustedes investigó un poco sobre este tipo de bacterias. Por medio de la investigación te das cuenta … que van creciendo solas dependiendo de la temperatura y el ambiente, y ese rollo. Se van reproduciendo, una se convierte en dos y así. [First of all, I don't know if any of you investigated about this type of bacteria. Through investigation you realize… that they grow by themselves depending on the temperature and the environment, and all that. They reproduce, one becomes two and so on].

Assumption B. Assuming an entrepreneurial role, Teams T4 and T5 considered a constant temperature based on the MEA context and the precision of industrial refrigerators [9], and they focused on modeling the kefir grains growth depending on time, $\mathrm{G}(t)$. For instance, T5 justified considering a constant temperature by contrasting their mathematical analysis with the factors that influence the growth of kefir grains in real-life.

[7] T5: noté que cuando teníamos una temperatura menor de 20º el porcentaje de crecimiento se reducía, se volvía a elevar cuando la temperatura subía porque es una bacteria. Hace rato lo comentó, el otro Bryan; que al ser una bacteria, el ambiente, la cantidad de proteínas y lo demás que hay alrededor es influyente para que se reproduzcan. Si tienen más temperatura, igual aun no conozco mucho a que temperatura se mueran y demás, pero con los datos que nos dieron, a mayor temperatura o sea 21º se reproducen más, manteniendo el sistema de las bacterias en un estándar. Entonces, mantener una temperatura, sí es posible. Yo dije mantenemos siempre la temperatura, saco un promedio en el porcentaje del crecimiento que me daba de los días de, que teníamos 21º, pues me dio un promedio del 33%. Entonces si yo mantengo siempre mi temperatura a 21º porcentualmente, más menos, voy a tener un crecimiento del 33%. [I noticed that when we had a temperature lower than 20º [C], the growth percentage decreased, it increased again when the temperature rose because it is a bacterium. If they have a higher temperature, I still don't know a lot about the temperature at which they die and so on, but with the data they gave us, the highest the temperature, i.e., 21º they reproduce more, maintaining the system of the bacteria in standard conditions. Then, maintaining the temperature is possible. I said, we always maintain the temperature, I obtain an average of the growth percentage from the days we had 21º C, and I got an average of 33%. Then if I always maintain my temperature at 21º C percentage, more or less, I will have a growth of 33%].

During the questioning phase, team T1 hesitated about the possibility of considering a constant temperature during the kefir grains growth. Based on their experience, T5 replied the following:

[8] T1: Pero está difícil, la temperatura hace un cambio de un día para el otro. [But it is hard, the temperature changes from one day to the next].
[9] T5: sí, pero no es lo mismo el sistema de refrigeración industrial que el de un salón. [yes, but the industrial refrigeration system is not the same as a classroom].

ml	temperatura	%	promedio de crecimiento
40.00	20		
50.00	21	20	33%
70.00	21	40	
90.00	21	40	
60.00	20	30	
79.80	20	25	
106.13	21	33	
141.16	21	30	
187.74	21	33	
249.69	21	33	
332.09	21	33	
441.69	21	33	
587.44	21	33	
781.30	21	33	
1,039.12	21	33	
1,382.04	21	33	
1,838.11	21	33	
2,444.68	21	33	
3,251.43	21	33	
4,324.40	21	33	

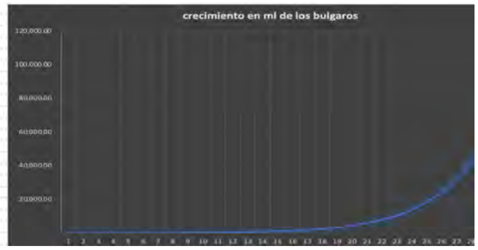


Figure 3: Mathematical Representations Included in T5's Model

Discussion. The fact that the Yogurt MEA includes three variables -time, temperature, and amount of kefir grains that influence their actual growth, allowed students to generate real-life assumptions to model the problematic situation. The whole group discussion created opportunities for students to revise and evaluate their assumptions and models. According to Garfunkel and Montgomery (2019), during the modeling process "we select 'objects' that seem important in the real-world question and identify relations between them. We decided what we will keep and what we will ignore about the objects and their interrelations" (p. 12).

Identification of patterns. The MEA allowed students to exhibit and develop their ability to identify patterns and build various strategies to solve the problem. Teams T1, T2, and T3 identified a "weekly cyclical growth" pattern.

[10] T2: Cada siete días íbamos repitiendo el ciclo, hasta terminar. [Every seventh day, we repeated the cycle until finished].

Team T3 described the kefir grains growth behavior, and teams T1 and T2 predicted the growth behavior in addition to describing it. For instance, the team T1suggested a growth pattern that established a relationship between time and temperature variation (Figure 2a). Even though the mathematical succession T1 developed requires further refinement to predict the growth in a way that is closer to the actual phenomenon, it provided a model to solve the situation.

Team T4 identified a kefir grain growth pattern that was not strongly related to the temperature. They focused on describing and predicting the growth considering the contextual factors. They suggested that the company could use jars to replicate growth whichever number of times was necessary to obtain sufficient kefir grains to produce Yogurt (Figure 2b).

[11] T4: Tomamos el crecimiento del primer día al día siete, y cerramos como semana. Volvimos a empezar en uno. Como al día siguiente teníamos 200, nosotros dividimos, 200 entre 40ml y ya fijamos los cinco frascos (segunda fila, tabla Semana 2, Figura 2b). [We considered the growth from the first to the seventh day, and we finished the week. We started again like in day one. The next day we had like 200, we divided, 200 by 40 ml and then we have five jars (second row, Week 2 table, Figure 2b)].

Team T5 identified an exponential growth pattern. They calculated the daily kefir grains growth and the corresponding percentage. Next, they obtained the average growth percentage considering only the days in which the temperature was 21ºC. The team obtained an average rate of 33% and predicted the growth (Figure 3). The students found a growth pattern that corresponds to the expression.

Discussion: The Yogurt MEA prompted students to organize and systematize information, recognize patterns, build various solution strategies and generate diverse models to estimate the growth. Sevinc and Lesh (2021) discuss that problems in realistic contexts can be solved through various solution strategies and "allow students to make mathematical and contextual inferences, an essential activity for the development of problem-solving competence” (p. 4).

Construction of mathematical representations. Given that the data included in the Yogurt MEA was included in various images embedded in a video (see QR code, Figure 1b), the students had to extract, interpret, organize and create mathematical representations to analyze the kefir grains’ growth. The students used diverse representations in their models. For instance, teams T1, T2, and T5 included verbal, tabular, and graphical (Figures 2 and 3), while teams T3 and T4 included oral and tabular representations in their model. These representations allowed them to describe, explain and estimate when the kefir grain growth was sufficient to start thinking about selling yogurt (Figure 2b). The following excerpt illustrates the way they described it verbally and in their letters:

> [12] T5: En un mes ya tengo un aproximado de 3600 litros que ya los puedo mandar a producción. Porque aquí tengo los mililitros en bacterias, más no tengo la producción en yogurt. [I have approximately 3600 liters in a month that I can send to production. Because this is the milliliters in bacteria, but I do not have the production of yogurt].

Discussion: The Yogurt MEA fostered model-building using diverse representations to explain and describe the kefir grains growth phenomenon. The creation of the models reported by the students revealed their modeling abilities and the knowledge used and developed throughout the problem-solving activity. According to Lesh and Doerr (2003), “meanings associated with a given conceptual system tend to be distributed across a variety of representational media” (p. 12).

Conclusions

The analysis of the results supports the conclusion that the Yogurt MEA has the potential to support the development of diverse modeling abilities. In this study, students developed modeling abilities through model building, team interactions, and whole-group discussion. The abilities reported included: variable identification, assumptions based on the real-life context, identification of patterns, and construction of mathematical representations.

The identification of variables was fostered through the inclusion of three variables in the Yogurt MEA. The students had the opportunity to identify, select and establish relationships among variables to solve the problem. The assumptions based on the real-life context emerged because the Yogurt MEA was situated in a context close to the students. Thus, it fostered opportunities to self-evaluate and contrast models with real-life experiences growing kefir grains. This is consistent with the fact -discussed in the MMP- that the ways of thinking or models built to make sense of situations, requiring realistic decision-making, integrate with frequency ideas from more than one discipline or theory. The teams identified patterns to describe and explain the kefir grains growth and estimate the time to decide when a sufficient amount of kefir grains would be available to start selling yogurt. The teams built mathematical representations to organize data, interpret the situation, identify patterns and explain their model.

We observed students refine their modeling abilities during the MEA solution process, which was mediated by their understanding of the phenomenon, mathematical knowledge, personal and professional experience, and interaction with the environment.

References

Aliprantis, C. D., & Carmona, G. (2003). Introduction to an economic problem: a models and modeling perspective. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and Modeling perspectives on mathematics problem solving, Learning, and Teaching* (pp. 255-264). Lawrence Erlbaum Associates.

Bliss, K. M., Fowler, K. R., & Galluzo, B. J. (2014). *Math modeling: Getting started & getting solutions.* SIAM

Doerr, H. M. (2016). Designing sequences of model development tasks. In C. R. Hirsch & A. R. McDuffie (Eds.), *Annual Perspectives in Mathematics Education 2016: Mathematical modeling and modeling mathematics* (pp. 197-205). National Council of Teachers of Mathematics.

English, L. D., Adams, R., & King, D. (2020). Design learning in STEM education. In C. C. Johnson, M. J. Mohr-Schroeder, T. J. Moore, & L. D. English (Eds.), *Handbook of research on STEM education* (pp. 76-86). Routledge.

Garfunkel, S., & Montgomery, M. (Eds.). (2019). *Guidelines for Assessment and Instruction in Mathematical Modeling Education.* SIAM. https://www.siam.org/Portals/0/Publications/Reports/GAIMME_2ED/GAIMME-2nd-ed-final-online-viewing-color.pdf

Lesh, R. (2010). Tools, researchable issues and conjectures for investigating what it means to understand statistics (or other topics) meaningfully. *Journal of Mathematical Modeling and Application*, *1*(2), 16-48.

Lesh, R., & Doerr, H. M. (2003). Foundations of a models and modelling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism. Models and Modeling perspectives on mathematics problem solving, learning and teaching* (pp. 3-34). Lawrence Erlbaum Associates.

Lesh, R., & Harel, G. (2003). Problem solving, modeling, and local conceptual development. *Mathematical thinking and learning*, *5*(2-3), 157-189.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for Developing Thought-Revealing Activities for Students and Teachers. In A. E. Kelly (Ed.), *Handbook of Research Design in Mathematics and Science Education* (pp. 35-44). Lawrence Erlbaum Associates.

Lesh, R., & Yoon, C. (2007). What is Distinctive in (Our Views about) Models & Modelling Perspectives on Mathematics Problem Solving, Learning, and Teaching? In *Modelling and applications in mathematics education* (pp. 161-170). Springer.

Niss, M., Blum, W, & Galbraith, P. (2007). Introduction. In W. Blum, P. L. Galbraith, H. W. Henn & M. Niss (Eds.), *Modelling and applications in mathematics education. The 14th ICMI study* (pp. 3–32). Springer.

Montero-Moguel, L., Vargas-Alejo, V., & Carmona-Domínguez, G. (2021). The evolution from linear to exponential models when solving a model development sequence. In D. Olanoff, K. Johnson, & S.M., Spitzer (Eds.), *Proceedings of the forty-three annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1154-1171).

Sevinc, S. (2021). Toward a reconceptualization of model development from models-and-modeling perspective in mathematics education. *Educational Studies in Mathematics*, 1-28.

Sevinc, S., & Lesh, R. (2021). Preservice mathematics teachers' conceptions of mathematically rich and contextually realistic problems. *Journal of Mathematics Teacher Education*, 1-29

Touchstone, K. (2014). Exponential problems in business courses: the translation of time units. *e-Journal of Business Education & Scholarship of Teaching, 8*(2), 76–99.

A MATHEMATICS SUPPORT PROGRAM: MODIFICATIONS AND RESPONSES TO THE PANDEMIC

Jenna Tague
Clovis Community College
jenna.tague@cloviscollege.edu

The purpose of this report is to share the facets of a support program set in a mathematics department in a mid-sized comprehensive western university. This is an ongoing project and I use the social community framework (Mondisa & McComb, 2015) to compare the students in the mentoring program to a control group of students. The current results allowed me to examine the effect of the mentoring program on their community of practice during the 2020-2021 Covid-19 pandemic. Results include overall effects of the pandemic on mathematics students, and interventions that could help encourage a sense of belonging through the pandemic environment.

Keywords: Policy, Undergraduate Education, Systemic Change.

The question of how to help support students in mathematics majors remains open (Lisberg & Woods, 2018). Research indicates that through the leaky pipeline, students intending to major in STEM are lost from high school to college (Snyder & Dillow, 2011) and 48% of students leave STEM majors during college (Chen, 2013). STEM majors are particularly vulnerable to attrition in their first year of college (Chen 2009; Higher Education Research Institute 2010). A goal of no attrition is unreasonable; undergraduates should change majors if they would like to. Unfortunately, research indicates that students may leave the major because they do not feel they belong and/or they have a poor instructional interaction (Seymour & Hewitt,1997). Additionally, both people of color and women are more likely to leave mathematics majors than white or male students (e.g. Anderson & Kim 2006; Hill, Corbett, & Rose, 2010; Griffith, 2010). These are stark disparities that indicate a need for more supportive mathematics departments.

I worked with a team of mathematicians to design a mentoring program for mathematics majors at a mid-sized western comprehensive university. The mentoring program was open to all students, but was designed specifically to address issues faced by low-income students, students of color, and female students in our mathematics department. The purpose of this presentation is to describe changes and updates to the mentoring program, and to share the third year of results of what has helped our students, and what did not. The data from the third year was collected in Fall 2020, and as such, is heavily impacted by the Covid-19 pandemic and online learning.

Literature Review

There are two important questions related to designing/studying a mathematics support program: (1) What supports are most helpful for students?, and (2) What measures, quantitative and qualitative, are important for measuring the effectiveness of such a program?

Lisberg and Woods (2018) found four areas of focus when designing support programs for STEM majors: (1) Peer & faculty mentorship, (2) Familiarity with programs & faculty, (3) Student mindset, and (4) Student learning techniques. In the first category, students are paired with a peer or faculty as a mentor; in studies of this model results indicated having mentors help students with the transition into the university community (Mondisa & McComb, 2015) and increased retention rates (Campbell & Campbell, 1997). Familiarity with programs & faculty includes helping demystify the academic structure and increase connections with faculty. Student mindset centers around providing students with evidence that intelligence is not fixed and mistakes are necessary to learning (Dweck 2008) coupled with evidence that academic setbacks can be overcome (Walton & Cohen, 2011). Lastly, student learning techniques provided students with STEM-specific study skills. In their support program, Lisberg and Woods (2018) reported that under-represented minority students in first year mathematics were more likely to pass, and if they did not pass them, were twice as likely to retake them and persist during their first year.

Karp (2011) conducted a metanalysis of non-academic supports and also found four main categories: *social relationships*, *career options*, *college structure*, and *life issues* (italics added for emphasis and to denote each category). *Social relationships* indicated programs that helped to encourage community building amongst students. Providing information about *career options* aided students in connecting their college experiences with long term goals. Similar to Lisberg and Woods (2018), Karp (2011) also found that it is necessary to illuminate the *college structure* for students. Lastly, *life issues* are the most difficult category to provide support for, however, when students were able to connect with resources, they can recover from *life issues* to remain in college. The metanalysis indicated that much of the research relied on Likert-style questions of how helpful students found particular supports, but did not delve into how or why.

Mondisa and McComb (2018) described how the majority of studies of mentorship programs tend to focus on quantitative measures exclusively (i.e. comparisons of students' GPA, attrition rates) while neglecting how these programs affect individual students on a social level. "Evidence is needed identifying what social elements contribute to the positive experiences of program members…and what characteristics may explain differences across member experiences (Mondisa & McComb, 2018, p. 94).

In summary, we designed our program around Karp's (2011) four categories: *social relationships*, *career options*, *college structure*, and *life issues*. We also integrated pieces of Lisberg and Woods (2018) categories in workshops. In creating the research design, we did include quantitative methods of comparison, such as attrition rate, but mainly focus on a qualitative understanding of how the program affects students on a social level.

The Support Program

The study was conducted at a mid-sized western comprehensive university with a student population including 67% first generation students, and 61% Pell-eligible. The overall student population demographics are: 3% African American, 14% Asian (mainly Southeast Asian: Hmong and Cambodian), 49% Hispanic, 6% non-resident students, 3% two or more races, 5% unknown, and 20% white.

Previous research (Tague, 2021) described how focus groups were held and the themes from those were used to reinforce Karp's (2011) four supports: *social relationships*, *career options*, *college structure*, and *life issues*. I worked with a team of three mathematicians, who will be referred to as mentors, to design the support program using research and the focus group data as a guide. The program has been running since Fall 2018 and includes four main facets: scholarships, advising, required office hours/tutoring, and weekly Friday meetings including workshops, problem solving challenges, homework help sessions, and socializing.

Scholarships, weekly meetings, and required office hours are included in the *social relationships* category. Scholarships are included in this category because the tuition support allowed students to work fewer jobs, which provided time for students to build connections. The weekly meetings and office hours were designed to help students build relationships with one another and also with faculty members. Advising and workshops/guest speakers were designed to meet the *career options* category. Workshops (financial aid, health/mental health center, etc.) were designed to help students navigate the *college structure*. *Life issues* are the most difficult to plan for, however, we hoped that providing students with a community and knowledge of resources would protect against this category.

Theoretical Framework and Methods

Mondisa and McComb (2018) proposed using the framework of Social Community as an answer to the gap in qualitative research results from mentoring programs in mathematics. The Social Community framework is designed to measure the effectiveness of the program based on level to which participants feel *connectedness*, *resiliency*, and *communities of practice*. Connectedness is how linked to the program each student feels (in the current study: do you have a mentor?). Resiliency is focused on students' reactions to hardships or challenges (in the current study: what was an obstacle you faced this year, and how did you overcome it?). Lastly communities of practice are "collections of like-minded individuals sharing similar experiences and social resources as they interact with and support each other (Eckert, 2006; Wenger, 2000)" (Mondisa & McComb, 2018, p. 98). An example of a communities of practice question in the current study was: Do you feel like you belong as part of the math department? I adapted the questions from Mondisa and McComb's (2018) work to form interview questions.

Table 1. Participants

	Second Year	Third Year	Total
Scholars	10	6	16
Control Group	9	5	14

The participants are shown in Table 1 according to group and also year at the university. All participants were mathematics majors, the control group was chosen to be the same level in the program, and both groups were majority people of color. All students participated in semi-structured interviews lasting 30 minutes (on average) in October-November of 2020. The audio recordings were transcribed verbatim, organized by question in a spreadsheet, and then coded according to *connectedness*, *resiliency*, and *community of practice* (Mondisa & McComb, 2018).

Results

The results will be presented categorically according to *connectedness*, *resiliency*, and *community of practice* (Mondisa & McComb, 2018).

Connectedness

The Scholars were significantly more likely (χ^2=8.6277; p=0.003) to report feeling connected than the control group. In responding to the question of if they had a mentor, 14 out of 16 (87.5%) of the scholars versus 5 out of 14 (35.7%) control group students cited a faculty member at the university as a mentor. A representative quote from one of the Scholars was: "*She's literally there to help me with whatever I'm there for. And it makes me…even more comfortable now.*" The scholars were also more likely to list one another, or a math student from the same university than someone outside of the university or from a different major. In contrast, the students in the control group did say they had mentors, but they were family members or friends outside of the university. This difference shows that the control group were less connected to members of the mathematics department community than the scholars.

Resiliency

The biggest challenges faced by all of the students were (1) time management, (2) motivation, and (3) social interactions. All three were brought on by the pandemic and the mental health fallout. The time management issue was brought on by lack of structure with not having to be a physical place, causing students from both groups to disrupt their sleep schedules, and procrastinate working on school. The motivation drop was also prevalent in interviews with scholars and control group alike. They articulated that it was difficult to stay motivated when they were going to school and sleeping in the same physical location. Lastly, all of the students were missing social interactions. For the scholars, a bit of the social interactions were eased through the Friday weekly meetings. Many of them said that although they felt forced to meet on a Friday afternoon, they appreciated connecting with one another.

Communities of Practice

To rate the community of practice level for each student, I relied mainly on the following question: Do you feel like you belong in the math department? Out of 16 scholars 14 (87.5%) said they felt like they belonged, and out of 14 of the control group, 10 of them (71.4%) said they felt like they belonged. Thus there was not a statistically significant difference (χ^2=1.2054; p=0.272). However, the reasons given for the sense of belonging were different. The scholars explained that they belonged because of their social connection and network within the department. One scholar commented, "*The whole [scholar] community really helps you feel like you belong…They're very warm, welcoming. [T]here's things that the math department are now doing like they just…did the LGBTQ plus math day*." In contrast, the control group would make statements like, "*Am I smart enough to be here? And then like, after the first test and stuff, I'm like, Oh, yeah.*" This quote indicates that the control group drew their sense of belonging from their understanding of mathematics, which will likely be challenged during this time as a mathematics major.

Conclusions

All of the students expressed resiliency in working through the difficult times of transitioning to online learning and living through a pandemic. The scholars displayed higher levels of connectedness and communities of practice. The findings in the third year are consistent with those from the first and second years of the program (Author, year; Author, year). During the first year, scholars were more comfortable seeking out a faculty member for help with academic or life issues (6 out of 7 versus 1 out of 6). Similarly, in the second year and third years, scholars were more likely to consider a faculty member a mentor and seek them out for help (12 out of 16 versus 3 out of 12; 12 out of 16 versus 5 out of 14). Additionally, the control group students persist in linking their sense of belonging in mathematics with their understanding of mathematics. In contrast, the scholar group links their sense of belonging in mathematics with their relationship with faculty members and social connections amongst the department. It seems that during the pandemic time, even having a one hour meeting each week can help students continue to feel a sense of belonging and community through the difficult issues they face.

Future research reports on the results from the program will be forthcoming, however, pieces of the current model could be implemented to see if the results also hold in different context and if more mathematics departments could be more welcoming and supportive.

Acknowledgments

This research was possible due to NSF grant # 1742236.

References

Anderson, E., and Kim, D. (2006). *Increasing the Success of Minority Students in Science and Technology*. Washington, DC: American Council on Education.

Campbell, T. A., & Campbell, D. E. (1997). Faculty/student mentor program: effects on academic performance and retention. *Research in Higher Education, 38*(6), 727–742.

Chen, X. (2009). *Students Who Study Science, Technology, Engineering, and Mathematics (STEM) in Postsecondary Education* (NCES 2009-161). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Chen, X. (2013). *STEM Attrition: College Students' Paths Into and Out of STEM Fields* (NCES 2014-001). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Crisp, G. (2010). The impact of mentoring on the success of community college students. *The Review of Higher Education, 34*(1), 39-60.

Dweck, C. S. (2008). Mindsets and math/science achievement. Carnegie Corp. of New York–Institute for Advanced Study Commission on Mathematics and Science Education, New York.

Eckert, P. (2006). Communities of practice. *Encyclopedia of language and linguistics, 2*(2006), 683–685.

Ehrich, L. C., Hansford, B., & Tennent, L. (2004). Formal mentoring programs in education and other professions: A review of the literature. *Educational Administration Quarterly, 40*(4), 518–540.

Estrada, M., Burnett, M., Campbell, A. G., Campbell, P. B., Denetclaw, W. F., Gutierrez, C. G., Hurtado, S., Gilbert, J. H. Matsui, J., McGee, R., Moses Okpodu, C., Robinson, T. J., Summers, M.F., Werner-Washburne, M., & Zavala, M. (2017). Improving underrepresented minority student persistence in STEM. *CBE-Life Sciences Education*, *15*(3).

Gershenfeld, s. (2014). a review of Undergraduate Mentoring Programs. *Review of Educational Research, 84*(3), 365–391.

Griffith, A. (2010). Persistence of women and minorities in STEM field majors: Is it the school that matters? *Economics of Education Review, 29*(6): 911–922.

Higher Education Research Institute. (2010). Degrees of Success: Bachelor's Degree Completion Rates Among Initial STEM Majors. Los Angeles: Author.

Hill, C., Corbett, C., and Rose, A.S. (2010). *Why So Few? Women in Science, Technology, Engineering, and Mathematics*. Washington, DC: American Association of University Women.

Hrabowski, F. A., & Pearson, W., Jr (1993). Recruiting and retaining talented African–American males in college science and engineering. *Journal of College Science Teaching, 22*(4), 234–238.

Karp, M. M. (2011). *Toward a New Understanding of Non-Academic Student Support: Four Mechanisms Encouraging Positive Student Outcomes in the Community College*. CCRC Working Paper No. 28. Community College Research Center, New York, New York.

Lisberg, A., & Woods, B. (2018). Mentorship, mindset and learning strategies: An integrative approach to increasing underrepresented minority student retention in a STEM undergraduate program. *Journal of STEM Education*, *19*(3), 14-20.

Maton, K. I., Hrabowski, F. A., & Schmitt, C. L. (2000). African American college students excelling in the sciences: College and postcollege outcomes in the Meyerhoff Scholars Program. *Journal of Research in Science Teaching, 37*(7), 629–654.

Mondisa, J. L., & McComb, S. A. (2015). Social Community: A Mechanism to Explain the Success of STEM Minority Mentoring Programs. Mentoring & Tutoring: Partnership in Learning, 23(2), 149–163.

Mondisa, J. L. & McComb, S.A. (2018). The role of social community and individual differences in minority mentoring programs. *Mentoring & Tutoring: Partnership in Learning*, *26*:1, 91-113.

Seymour, E., and Hewitt, N.M. (1997). *Talking About Leaving: Why Undergraduates Leave the Sciences*. Boulder, CO: Westview Press.

Snyder, T.D., and Dillow, S.A. (2011). *Digest of Education Statistics, 2010* (NCES 2011-015). National Center for Education Statistics, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Tague, J. (2021). The Social Community of a Mathematics Support Program. Proceedings of the 2021 Psychology of Mathematics Education- North America Conference, Philadelphia, Pennsylvania, USA.

Treisman, U. (1992). Studying students studying calculus: A look at the lives of minority mathematics students in college. *The College Mathematics Journal, 23*(5), 362–372.

Walton, G. M., & Cohen, G. L. (2011). A brief social-belonging intervention improves academic and health outcomes of minority students. *Science, 331*(6023), 1447-1451. doi:10.1126/science.1198364.

Wenger, E. (2000). Communities of practice and social learning systems. *Organization, 7*(2), 225–246.

IRRELEVANT DETAILS WHEN USING LEGO® BRICKS AS MANIPULATIVES

Alison TELLOS	Helena P. OSANA	Anne LAFAY
Concordia University	Concordia University	Université Savoie Mont Blanc
alison.tellos@mail.concordia.ca	helena.osana@concordia.ca	anne.lafay@univ-smb.fr

The primary aim of the present study was to examine whether the studs on LEGO® bricks act as irrelevant details when solving fraction division problems. We also investigated whether prior knowledge plays a role in the irrelevant details effect and in children's problem-solving strategies. Fifth- and sixth-grade students (N = 38) first completed a fractions test to assess their prior knowledge. A video-recorded lesson then was delivered to all students on how to solve fraction division problems with LEGO® bricks. Most of the participants were not distracted by the studs on the LEGO® bricks when solving fraction division problems on a transfer task. Prior knowledge was related to accuracy on learning and transfer performance, with low prior knowledge students generating less accurate solutions than to high prior knowledge students.

Keywords: Problem Solving, Mathematical Representations, Number Concepts and Operations

Any quick search on the Internet reveals hundreds of webpages and videos that show how to use LEGO® bricks in mathematics instruction (Tellos, 2021). Most of these illustrations target teachers and parents, who are often interested in how to make mathematics fun and interesting. Despite recent research on concrete objects (i.e., manipulatives) and their use in mathematical contexts (e.g., Carbonneau et al., 2013), it remains unclear if and how LEGO® bricks can support children's learning in mathematics.

Our research program focuses on the conditions under which children can learn mathematical concepts and procedures with LEGO® bricks. Recent research has shown that the perceptual features of manipulatives can impact learning in different ways (e.g., Lafay et al., in press; McNeil et al., 2009). LEGO® bricks have perceptual features, too; they are colorful, and the studs (i.e., the bumps that connect the bricks together) add an interesting visual feature. Children may enjoy working with LEGO® bricks because the perceptual features are appealing, but the features may be unrelated to the learning objectives ("irrelevant details"; Mayer et al., 2008), and thus detract from the concepts that the manipulatives are intended to represent (McNeil & Uttal, 2009). Prior domain knowledge has also been found to moderate the irrelevant details effect: Magner et al. (2014) found that students with high geometry knowledge were better able to overcome irrelevant details in geometric illustrations than those with low prior knowledge.

The Present Study

The primary objective of the present study was to explore whether the studs on the LEGO® bricks act as irrelevant details when fifth and sixth graders solve fraction division problems, and if prior knowledge of fractions is related to the irrelevant details effect. Because little is known about children's thinking when they use LEGO® bricks in mathematical contexts, a secondary objective was to document students' responses when using the bricks to solve fraction problems.

In individual online sessions with a researcher, participants completed a fractions test to assess their prior knowledge, followed by a video lesson on how to use LEGO® bricks to represent fractions and solve fraction division problems (e.g., $1/4 \div 1/16$). All examples in the lesson used a 1 x 1 brick (i.e., with one stud) to represent the divisor. Students were shown how to count the number of divisor bricks that could fit on the dividend brick to find the solution.

After the lesson, students solved fraction division problems with computerized images of LEGO® bricks on two tasks, namely learning and transfer (Panels A and B in Figure 1, respectively). On the learning task, only a one-stud divisor could be used to find the correct solution, as shown in the lesson. On the transfer task, the only brick available that would result in the correct solution was a 2 x 1 brick (i.e., with two studs). If participants counted the studs on the transfer task, we could conclude that the studs acted as a detractor from the measurement division concept demonstrated in the lesson. On the other hand, counting the bricks would show that the studs had no negative impact on their understanding of measurement division.

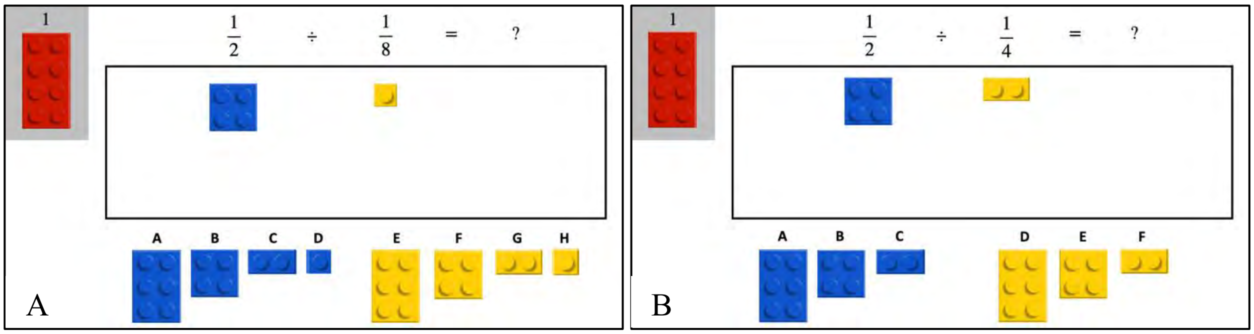


Figure 1: Sample of a Learning Item (Panel A) and a Transfer Item (Panel B)

Two research questions framed the current study: (1) Do the studs act as irrelevant details when solving fraction division problems, and is prior knowledge related to the irrelevant details effect? (2) What is the effect of prior knowledge on performance accuracy, and what is the effect of prior knowledge on the types of responses generated on the learning and transfer tasks?

Method

Participants

Participants were 38 fifth- and sixth-grade students (14 girls; age: M = 11.6 years, SD = .59) from a private French school in a large urban center in Canada.

Measures

Fractions test. The fractions test was a paper-and-pencil test with 8 items based on Saxe et al. (2001). Designed to measure knowledge of fractions concepts, the test required students to represent and estimate fractional quantities depicted in area models, and to solve problems using drawings. A median split was conducted to create low and high prior knowledge groups.

Learning and transfer. All five items on the learning task required the students to use a one-stud brick to find the solution, whereas all five items on the transfer task required a two-stud divisor (see Figure 1). To test the irrelevant details effect, we coded whether the student counted the studs or the bricks on the transfer task. Accuracy scores on each of the learning and transfer tasks were the mean number of correct responses. All online sessions were videorecorded for subsequent coding of the students' use of the LEGO® bricks on the learning and transfer tasks.

Results

Irrelevant Details

Seven students (18.4% of the sample) counted the studs rather than the bricks at least once on the transfer task, evenly distributed across the low- (3 students) and high-knowledge groups (4 students). The same pattern was found using item as the unit of analysis: Of all responses generated on the transfer task, 11.6% involved counting the studs to find the solution; in each prior knowledge group, the same proportion was observed (11.6%). Together, these results show that prior knowledge was not related to whether the students counted the studs or the bricks.

Accuracy and Response Analysis

Accuracy. Means and standard deviations for the accuracy scores are presented in Table 1.

Table 1: Means and Standard Deviations for Accuracy Scores on the Learning and Transfer Tasks by Prior Knowledge

Measure	Learning accuracy			Transfer accuracy		
	n	*M*	*SD*	*n*	*M*	*SD*
Low prior knowledge	19	.67	.25	19	.74	.31
High prior knowledge	19	.90	.14	19	.84	.29

Using students as the unit of analysis, the results showed that there was a main effect of prior knowledge, $F(1, 36) = 5.53$, $p = .02$, with the high prior knowledge group ($M = .87$, $SD = .20$) outperforming the low prior knowledge group ($M = .71$, $SD = .23$). There was no main effect of task type nor a prior knowledge by task type interaction, however. These findings suggest that children with low prior knowledge had less accurate solutions than children with high prior knowledge, regardless of task type.

Response analysis. We conducted a descriptive analysis of the types of responses generated on the learning and transfer tasks to provide additional insight on the reasons for the different accuracy levels observed between the low and high prior knowledge groups. Table 2 shows the number of times each type of response was generated on each task as a function of prior knowledge. On the learning task, participants either used the bricks correctly or chose the wrong brick to represent the dividend fraction. On the transfer task, participants either used the bricks correctly, or committed errors in three categories: (a) chose the wrong dividend brick, (b) counted the studs, or (c) chose the wrong dividend brick and counted the studs.

Table 2: Responses on the Learning and Transfer Tasks by Prior Knowledge ($N = 190$)

	Counted studs		Wrong dividend brick		Wrong dividend brick and counted studs		Correct	
Knowledge group	*n*	%	*n*	%	*n*	%	*n*	%
				Learning				
Low prior knowledge	—	—	29	30.5[a]	—	—	66	69.5[a]
High prior knowledge	—	—	8	8.4[a]	—	—	87	91.6[a]
				Transfer				
Low prior knowledge	7	7.4[a]	13	13.7[a]	4	4.2[a]	71	74.7[a]
High prior knowledge	11	11.6[a]	4	4.2[a]	0	0	80	84.2[a]

Note. Dashes refer to no data recorded. Coding of stud-errors applied only to the transfer task. [a]Percent represents the proportion of responses generated out of the total number of responses in each prior knowledge group ($n = 95$).

Using item as the unit of analysis, almost a third of the responses in the low prior knowledge group consisted of choosing the wrong dividend brick on the learning task, whereas under 10%

involved this error in the high knowledge group, $\chi^2(1, N = 190) = 14.8$, $p < .001$. On the transfer task, the data show that the proportion of items completed by participants with low and high prior knowledge also differed by response type, $\chi^2(3, N = 190) = 10.19$, $p = .02$. Participants with high prior knowledge used the bricks appropriately 84.2% of the time, whereas participants with low prior knowledge did so 74.7% of the time. In addition, participants with low prior knowledge chose the wrong dividend brick on 17.9% of all responses on the transfer task, whereas participants with high prior knowledge generated this error 4.2% of the time. Also, the data in Table 2 show that students in the low-knowledge group used the wrong dividend brick — an error that we had not anticipated — more often than counting the studs (17.9% vs. 11.6%).

Discussion

The present study was the first step in our research program to explore the effects of LEGO® bricks in mathematics learning. Contrary to our prediction, the results revealed that most of the students solved the fraction division problems on the transfer task by counting the number of bricks, and not the studs, that fit on the dividend brick. This finding could suggest that the studs on LEGO® bricks did not distract the students from learning the conceptual underpinnings of fraction division. Alternatively, other factors, including those related to the instruction that was delivered, may account for the apparent absence of the irrelevant details effect. Anecdotal evidence from the students' work on the learning and transfer tasks revealed that some repeated the instructions from the video lesson out loud to themselves (e.g., "Now, I have to count how many one-eighths are in one half"). Thus, how the conceptual underpinnings of fraction operations are emphasized in instruction and its relation to the irrelevant details effect, is a consideration for future research. Moreover, students with different degrees of prior knowledge may react differently to the perceptual features of LEGO® bricks under different instructional conditions. Finally, concrete LEGO® bricks should be used in future studies. Embodied cognition would imply that physically interacting with the bricks would generate different, and perhaps more pedagogically meaningful, results than working with images of the same objects.

The positive relations observed between prior knowledge and problem solving points to the importance of fundamental fractions concepts when learning about fraction division. In the present study, accuracy suffered in the low prior knowledge group primarily because they used the incorrect brick to represent the dividend fraction. This implies that teachers may wish to first emphasize the conceptual foundations of fractions and how they can be represented with manipulatives, to prepare students to learn new fraction operations with manipulatives.

References

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*(2), 380-400.

Lafay, A., Osana, H. P., & Levin, J. R. (in press). Does conceptual transparency in manipulatives afford place value understanding in children with mathematics learning disabilities? *Learning Disability Quarterly*.

Magner, U. I. E., Schwonke, R., Aleven, V., Popescu, O., & Renkl A. (2014). Triggering situational interest by decorative illustrations both fosters and hinders learning in computer-based learning environments. *Learning and Instruction*, *29*, 141-152.

Mayer, R. E, Griffith, E., Jurkowitz, T. N., & Rothman, D. (2008). Increased interestingness of extraneous details in a multimedia science presentation leads to decreased learning. *Journal of Experimental Psychology*, *14*(4), 329-339.

McNeil, N. M., & Uttal, D. H. (2009). Rethinking the use of concrete materials in learning: Perspectives from development and education. *Child Development Perspectives*, *3*(3), 137-139.

McNeil, N. M., Uttal, D. H., Jarvin, L., & Sternberg, R. J. (2009). Should you show me the money? Concrete objects both hurt and help performance on mathematics problems. *Learning and Instruction*, *19*(2), 171-184.

Saxe, G. B., Gearhart, M., & Nasir, N. S. (2001). Enhancing students' understanding of mathematics: A study of three contrasting approaches to professional support. *Journal of Mathematics Teacher Education*, *4*(1), 55-79.

Tellos, A. (2021). *Bricks or studs: Exploring irrelevant details when using LEGO® bricks as manipulatives* [Master's thesis, Concordia University].

THE USE OF SELF-REGULATION STRATEGIES IN FIRST-SEMESTER CALCULUS: THE CASE OF SUNNY

Kyle Turner
University of Texas at Arlington
kyle.turner2@mavs.uta.edu

James A. M. Álvarez
University of Texas at Arlington
james.alvarez@uta.edu

This case study explores self-regulation strategies used by Sunny, a high-achieving first-semester freshman student in calculus and aims to identify how she uses self-regulation strategies while in the course. Individual and group interviews were analyzed using categories of self-regulation strategies previously identified in the research literature. Findings suggest that Sunny's specific self-regulation habits for mathematics centered on mastering course concepts and shifted from focusing on memorization to focusing on understanding. This provides leverage for continued efforts to help foster productive self-regulation strategies for success in first-semester calculus.

Keywords: Calculus, Self-Regulation Strategies, Mathematics Self-Efficacy

While transitioning into a college mathematics course, undergraduate freshmen often face challenges related to their mathematics background, mathematics self-efficacy, and study habits (Hernandez-Martinez et al., 2011 and Seymour & Hunter, 2019). To address these challenges students can use self-regulation strategies to promote academic success (Zimmerman & Pons, 1986). Self-regulation strategies in mathematics can be defined as those actions that are used to overcome some obstacle when working on tasks associated with a mathematics classroom, whether in a specific problem context or in a more general response to tests and grades. High failure rates in college calculus in the United States (Rasmussen et al., 2019, Bressoud et al., 2015, Seymour & Hunter, 2019) make it critically important to explore ways to increase success rates in this gateway course to mathematics-based disciplines. This includes exploring not only the mathematical understandings that support success in calculus (see Carlson et al., 2010, Carlson et al., 2015, and Thompson & Harel, 2021), but also the self-regulation strategies employed by successful calculus students. This case study presents data analysis on a high-achieving freshman calculus student, Sunny, and the self-regulation strategies she used in a first-semester college calculus course. We explore the research question: In what ways do successful first-semester freshman use self-regulation strategies in a first-semester calculus course?

Conceptual Framework

Zimmerman and Pons (1986) produced several categories of strategies primarily used by high achieving students from a middle-class suburban high school, where high achieving was based off test scores, GPA, and teacher and counselor recommendations. The categories they found are as follows: seeking information, keeping records and monitoring, organizing and transforming information, seeking teacher assistance, seeking peer assistance, seeking adult assistance, and self-evaluating. They highlight that as students use these strategies, further instruction could improve the use of strategies and inform students of available strategies.

Wolters (1998) identified 14 separate categories of self-regulation strategies: performance goals, extrinsic rewards, task value, interest, mastery goals, efficacy, cognition, help seeking, environment, attention, willpower, emotion, other motivation, and other. These strategies were identified from open-response questionnaires administered to groups of psychology students at a large midwestern university. Wolters collected survey data from 115 college students which focused on how they would respond to different academic scenarios, gauging what motivational or self-regulatory strategies were used. Focusing on mathematics, Johns' (2020) study centered on the self-regulation strategies of 424 consenting first-semester calculus students and tested whether there were differences relating to gender and academic achievement. She found that high achievers both reported greater intrinsic motivation, task value, and self-efficacy than low achievers. High achievers also reported greater use of self-regulation strategies. From this, Johns found a correlation with the use of self-regulation strategies with student achievement.

How self-regulation strategies are used and developed by undergraduates in mathematics courses has not been widely discussed especially within this transition. Gueudet (2008) outlined several of the challenges that arise during the transition to tertiary school, especially related to individual, social, and institutional phenomena. For the individual phenomena, Gueudet describes that undergraduates beginning the transition often lack flexibility of switching between different thinking modes, such as practical and theoretical thinking. Sonnert et al. (2020) found that high school preparation was a significant factor on retention and performance for first-year students, but as the distance from high school grows the influences lessens. Though for those first-year students it was preparation in mathematics, and to a lesser extent attitudes about mathematics, which had the largest impact on performance in a calculus class. This is attributed to the highly cumulative nature of mathematics as calculus relies on earlier concepts.

Methods and Data Collection

This case study focuses on a set of five interviews and an initial questionnaire for a first-semester calculus student, Sunny. Sunny was one of six undergraduate calculus students who participated in five interviews throughout the Fall 2021 15-week semester. The study occurred at a large, urban research university in the Southwestern United States. All first-semester calculus courses meet two (for 80-minute lecture periods) or three (for 50-minute lecture periods) times a week for lecture. Two additional weekly 50-minute meetings enable students to engage in a collaborative lab activity facilitated by the instructor and graduate teaching assistant (GTA) for one class meeting and traditional recitation activities facilitated by the GTA during the other meeting. Each of the courses have departmentally coordinated exams written by the course instructors, these exams are typically offered the fourth full week of the semester, the ninth full week of the semester, and the final exam during the last week of the semester. Of the five interviews, two individual interviews occurred within a week of the first midterm and within a week of the final exam. These interviews lasted 20-30 minutes and focused on students' mathematics background, self-efficacy, and identity as well as differences between their experiences in high school and college mathematics. The other three interviews were task-based group interviews, lasting 50 minutes to an hour, occurring a week before each midterm and the final exam. Each task-based interview involved at most three participants examining the past three group activities given in the collaborative lab periods. Students provided information about how they worked as a group during those lab periods and worked a specific problem from each lab during the interview to demonstrate their processes. In the second task-based interview, students provided information about specific self-regulation strategies used in this course after being shown a list of strategies derived from Zimmerman and Pons (1986) and Wolters (1998).

Initial surveys were sent to thirteen first-semester calculus sections which included approximately 800 students. Of those invited, 193 students completed the initial survey which included background information, the Motivated Strategies for Learning Questionnaire (MSLQ) (Pintrich et al. 1991), a survey to measure mathematical identity (Kaspersen, 2016), select questions from the Factors Influencing College Success in Mathematics (Sonnert & Sadler, 2020), and a survey to measure the sources of self-efficacy (Usher & Pajares, 2009). To obtain an interview pool of first-time first-semester freshmen with differing measures of self-efficacy, mathematics identity, and self-regulation strategy usage on the initial survey, 38 of the students were invited to participate in a sequence of five interviews. Of these, seven accepted and began the interview sequence, with one student dropping out after the third interview.

A case study methodology was chosen to explore individual experiences of undergraduates' transitions to college mathematics and their uses of self-regulation strategies. A mix of qualitative data from the interviews and quantitative data from the initial surveys was used to get a vivid picture of the students' transition. For each interview an audiovisual recording was made and transcribed verbatim by the researcher. Participants were given pseudonyms. Coding identified self-regulation strategies using codes from categorizations by Zimmerman and Pons (1986) and statements related to mathematics identity, self-efficacy, and differences between high school and college mathematics experiences.

Results

Sunny was a first-year calculus student majoring in computer science. From the initial surveys collected, she scored above average for both mathematics identity and mathematics self-efficacy and reported moderate to high use of the self-regulation strategies identified on the MSLQ. Her high scores related to time and effort spent studying for a course. In particular, her highest score related to the belief that the effort given to learn course material will result in positive outcomes. Her responses from the initial survey also indicated that she used cognitive strategies to study for course material in the following ways: memorizing course material; elaborating material by connecting past ideas from the course to integrate and connect new information; and actively organizing information to understand the main concepts of the course. From both the individual and task-based group interviews we see instances of Sunny using these skills and considering how these skills aid in her understanding of the course material.

In her first individual interview, Sunny reported she had taken Advanced Placement Calculus BC in high school. Although she received an A in the high school course, she felt she did not learn as much as she needed to learn, citing the course's online format due to the COVID-19 pandemic as the reason she did not master the material. She further explained that she did not feel prepared for her current calculus course because she "didn't completely understand everything, I got kind of an idea of how things work…" but now she "can go back and really understand what's happening in the equations." From the five interviews, we noted that Sunny's view that the course was an opportunity to master the material served as a primary motivator in how she studied for exams and solved mathematics tasks. In the second individual interview Sunny reported a change in strategies, rather than relying on memorizing material she found it helpful to be shown how to derive certain formulas, citing the formulas for the derivatives of inverse trigonometric functions specifically. She conveyed that she felt "more confident now, cause [sic] I feel like I understand more… and if I see something I can actually derive it instead of just having to memorize it." She attributed the memorization method as a strategy shown in high school, while the university course required a deeper understanding of the concepts.

Across each of the three task-based interviews Sunny reported that while she worked on the collaborative lab activities, she found it helpful for each member of the group to work independently until they were stumped before turning to collaborate with another group member. She found this helpful as it allowed her to work through each problem and also get help. Since all group members were also working the same problem, she could "check and see where you did [the problem] wrong." This contrasted with her initial attitude of working mathematics tasks independently primarily, stating that that by working the problems herself "it helps [her] understand [the problem] more." When looking at specific tasks, Sunny often relied on organizing and transforming the problem into something more easily understood. For example, when shown the problem in Figure 1 below, Sunny first used a strategy that involved sketching the graphs of the functions as prompted by the problem, but then took the problem further by determining another set of functions that have the same behavior, presenting the piecewise function shown in Figure 2. Furthermore, Sunny is setting goals to master the content and leverages having successfully demonstrated the concept to extend the problem to functions with other attributes, in this case looking at finite discontinuities rather than infinite discontinuities.

Is it possible that $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x)$ does not exist, but $\lim_{x\to a}[f(x)+g(x)]$ does exist? Consider the functions $f(x)=\frac{1}{x}$ and $g(x)=-\frac{1}{x}$.

a. Sketch the graphs of $f(x)$ and $g(x)$ and find their limits (if they exist) as x approaches zero.

b. Sketch the graph of $[f(x)+g(x)]$ and find $\lim_{x\to 0}[f(x)+g(x)]$.

Figure 1: Lab Activity 2 Problem 2

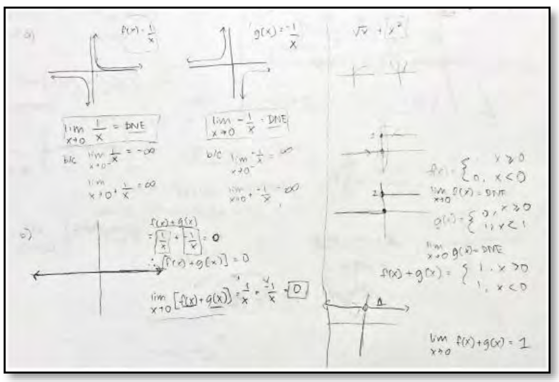


Figure 2: Sunny's written work for Lab Activity 2 Problem 2

Discussion

Sunny had a strong mathematics background, a positive mathematics identity, and valued prolonged effort to master mathematics concepts. These somewhat ideal attributes supported her demonstrated methods and strategies typically used by successful mathematics students. Sunny's intrinsic motivation for mastering the course content and enhancing her proficiency in the course content aligned with Johns' (2020) findings that high-achieving students often exhibit greater use of intrinsic motivation. However, by identifying her self-regulation habits, including reflecting on notes taken, identifying alternative methods for solving problems, creating goals for mastering the material that focused on deeper understanding rather than memorization, this informs mathematics-specific self-regulation strategies for promoting academic success for students transitioning to university mathematics courses. While some of these strategies may relate to mathematical problem solving capacity (see Alvarez, et al., 2019), these findings support the importance of incorporating self-regulation strategies in tandem with foundational mathematics concepts in the teaching and learning of calculus. Further research is needed to examine how calculus instructors can foster their students' development of these strategies.

References

Álvarez, J.A.M., Rhoads, K., Campbell, R.C. (2019). Toward Designing and Developing Likert Items to Assess Mathematical Problem Solving. In: Liljedahl, P., Santos-Trigo, M. (eds) Mathematical Problem Solving. ICME-13 Monographs. Springer, Cham. https://doi.org/10.1007/978-3-030-10472-6_11

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (Eds.). (2015). *Insights and recommendations from the MAA national study of college calculus*. MAA Press.

Carlson, M.P., Madison, B. & West, R.D. A Study of Students' Readiness to Learn Calculus. *Int. J. Res. Undergrad. Math. Ed.* **1,** 209–233 (2015). https://doi.org/10.1007/s40753-015-0013-y

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The Precalculus Concept Assessment: A Tool for Assessing Students' Reasoning Abilities and Understandings. *Cognition and Instruction*, *28*(2), 113–145. http://www.jstor.org/stable/27806346

Gueudet, G. (2008). Investigating the secondary–Tertiary transition. *Educational Studies in Mathematics*, *67*(3), 237–254. https://doi.org/10.1007/s10649-007-9100-6

Hernandez-Martinez, P., Williams, J., Black, L., Davis, P., Pampaka, M., & Wake, G. (2011). Students' views on their transition from school to college mathematics: Rethinking 'transition' as an issue of identity. *Research in Mathematics Education*, *13*(2), 119–130. https://doi.org/10.1080/14794802.2011.585824

Johns, C. (2020). Self-regulation in first-semester calculus. *International Journal of Research in Undergraduate Mathematics Education*, *6*(3), 404–420. https://doi.org/10.1007/s40753-020-00114-9

Kaspersen, E., Pepin, B., & Sikko, S. A. (2016). Measuring stem students' mathematical identities. *Educational Studies in Mathematics*, *95*(2), 163–179. https://doi.org/10.1007/s10649-016-9742-3

Pintrich, P. R., F., S. D. A., Garcia, T., & McKeachie, W. J. (1991). *A Manual for the use of the motivated strategies for learning questionnaire (Mslq)*. National Center for Research to Improve Postsecondary Teaching and Learning.

Rasmussen, C., Apkarian, N., Hagman, J.E., Johnson, E., Larsen, S., & Bressoud, D. (2019). Characteristics of Precalculus Through Calculus 2 Programs: Insights From a National Census Survey, *Journal for Research in Mathematics Education, 50(1), 98-112*. https://doi.org/10.5951/jresematheduc.50.1.0098

Seymour, E., & Hunter, A.-B. (2019). Talking about leaving revisited persistence, relocation, and loss in undergraduate Stem Education. Springer International Publishing.

Sonnert, G., Barnett, M. D., & Sadler, P. M. (2020). The effects of mathematics preparation and mathematics attitudes on college calculus performance. *Journal for Research in Mathematics Education*, *51*(1), 105–125. https://doi.org/10.5951/jresematheduc.51.1.0105

Thompson, P.W., Harel, G. Ideas foundational to calculus learning and their links to students' difficulties. *ZDM Mathematics Education* **53,** 507–519 (2021). https://doi.org/10.1007/s11858-021-01270-1

Usher, E. L., & Pajares, F. (2009). Sources of self-efficacy in Mathematics: A validation study. *Contemporary Educational Psychology*, *34*(1), 89–101. https://doi.org/10.1016/j.cedpsych.2008.09.002

Wolters, C. A. (1998). Self-regulated learning and college students' regulation of motivation. *Journal of Educational Psychology*, *90*(2), 224–235. https://doi.org/10.1037/0022-0663.90.2.224

Zimmerman, B. J., & Pons, M. M. (1986). Development of a structured interview for assessing student use of self-regulated learning strategies. *American Educational Research Journal*, *23*(4), 614–628. https://doi.org/10.3102/00028312023004614

CHANGING FIRST-YEAR MATHEMATICS STUDENTS' PERSPECTIVES: COMMUNITY, AFFECT, AND ASSESSMENT

Karina Uhing	Keith Gallagher	Nicole Infante
University of Nebraska at Omaha	University of Nebraska at Omaha	University of Nebraska at Omaha
kluhing@unomaha.edu	keithgallagher@unomaha.edu	ninfante@unomaha.edu

Student experiences in first-year mathematics (FYM) courses are crucial to their success in college. In recent years, there has been a push to improve FYM courses by centering and engaging students. The purpose of this study is to examine how students perceive their experiences with active learning and standards-based grading in FYM courses. In this paper, we present our analysis from end-of-semester Mathographies written by students who were enrolled in FYM courses in Fall 2021. We discuss three preliminary findings: community, affect, and assessment. We compare these findings to a framework relating to students' experiences with active learning and provide questions for future exploration.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Assessment, Instructional Vision, Undergraduate Education

Students' experiences in mathematics courses are a crucial part of their academic success in college (Adelman, 2006). In recent years, there has been a push to improve students' experiences in first-year mathematics (FYM) courses by increasing the use of evidence-based instructional practices (Abell et al., 2017; Smith et al., 2021; CBMS, 2016; PCAST, 2012). In this paper, we present preliminary findings on students' self-reported perceptions of their mathematics experiences after participating in FYM courses at the University of Nebraska at Omaha (UNO), a large metropolitan university. The Mathematics Department at UNO was recently tasked with redesigning the entire FYM program due to high failure rates. At UNO, "first-year mathematics" refers to any course below a standard Calculus 1 course. To support this initiative, the Mathematics Department hired a Director of FYM (Author 3) to oversee these courses. Following the Director's vision and with the support of upper administration, the Mathematics Department hired 10 additional faculty members (including Authors 1 and 2) to redesign and teach FYM courses.

The primary focus of our efforts was to transition our FYM courses from an emporium model (Twigg, 2011; Webel, Krupa, & McManus, 2017) to a more "traditional" classroom model. Our vision for these courses was to use evidence-based student-centered teaching practices to engage students in the learning process. (Henceforth, we use the term "active learning" to encompass this idea. A brief literature review of active learning is provided below.) We began redesigning two courses, Intermediate Algebra and College Algebra, in Summer 2021 and piloted our changes in Fall 2021. Data for this study come from the final assignment in these redesigned courses. Class sections were capped at 40 students, and every section was assigned a full-time faculty member, a graduate teaching assistant, and an undergraduate learning assistant to help facilitate learning. Faculty worked together along with the Director of FYM to develop curricular materials. Both courses were coordinated and used the same instructional materials and assessments. Our overall goal in implementing these changes was to help our students develop a growth mindset towards learning mathematics (Dweck, 2006).

Our course redesign involved two major components: 1) implementing active learning and 2) using standards-based grading. Active learning is a ubiquitous term in education. While several definitions of active learning exist, it is commonly used as an "umbrella term to communicate an alternative to lecture" (Lombardi et al., 2021, p. 9). For this paper, we adopt Theobald, et al.'s (2020) definition of active learning: "any approach that engages students in the learning process through in-class activities, with an emphasis on higher-order thinking and group work" (p. 6480). In addition to implementing active learning, we also used standards-based grading to assess student learning (Elsinger & Lewis, 2019; Lewis, 2020). Under this grading system, students had multiple opportunities to demonstrate skill on content standards, which we called learning outcomes, throughout the semester. After redesigning these courses, we sought to gather feedback about students' experiences in them to inform future work. The central research question guiding our study is: How do students perceive their experiences with active learning and standards-based grading in a first-year mathematics course?

Theoretical Framework

In this paper, we build on a framework that describes how students experience active learning (Uhing et al., 2021). This model was developed in the context of the SEMINAL project, a large study examining departmental change efforts to incorporate more active learning in first-year college mathematics courses. The model consists of five overarching themes relating to students' classroom experiences: student-to-student interactions, relationships with instructors, course format, assessment, and students' affective experiences. These themes were originally developed by looking at broad, open-ended survey response questions across students at multiple institutions taking a variety of precalculus and calculus courses. In our study, we used this framework in our analysis to examine written student reflections to a specific set of questions dealing with their mathematical experiences in a particular course. Our findings expand on this framework to further characterize students' experiences in first year mathematics courses.

Methods

The data in this study come from students at UNO who were enrolled in Intermediate Algebra and College Algebra during the Fall 2021 semester. Data were collected from all students who submitted a "Mathography" assignment at the end of the semester (228 submissions from College Algebra and 241 from Intermediate Algebra). For this assignment, students responded to a set of questions that had them reflect on their mathematical experiences. These prompts were adapted from Drake (2006). For example, students were asked to describe both a "high point" and a "low point" that they had experienced in Intermediate Algebra or College Algebra. They were also asked more general questions such as "What do you think it takes for a student to be good in math?". To analyze the data, we read through a subset of excerpts (54 from college algebra and 77 from intermediate algebra) and developed a list of preliminary codes using the themes from our theoretical framework as a guide. We then read back through these excerpts and systematically applied the codes. We are currently in the process of refining our codebook and analyzing the entire collection of Mathographies. We present our initial findings from the Mathographies below.

Findings

Five initial themes emerged from the data: community, affect, assessment, engagement, and personal growth. We discuss the first three themes below. In our discussion, we highlight connections between these themes and our theoretical framework.

Community

Several students commented on classroom community and culture. Students valued working in groups and getting to talk with their peers. One student said, “I liked working in groups and talking through things with my group and conversing with my peers.” Another student reflected, “I think in math it's very important to talk to others and not try to do everything on your own. I learned a lot better by asking my group questions and asking them for help when I needed it.” Overall, students appreciated the ability to work with their classmates because they liked hearing other students’ ideas and helping each other learn.

In addition to working with their peers, students commented on their relationships with instructors and the overall experience of feeling included in a community. One student explained, “The teacher helped me so much compared to other teachers, who just told me I should drop their class. It felt a lot more inclusive and much more like a community of students and teachers rather than a lecture.” Another student shared a similar comment about the classroom environment: “I always felt welcome in class. There was never a time I felt uncomfortable while I was learning. I think you should try and keep the positivity in the classroom, it really helps a learning environment in my opinion.” This sense of community helped students feel comfortable interacting with others in the classroom and is closely related to students’ affective experiences.

Affect

Affect was another significant theme that was present in several excerpts. Many students wrote about their emotions and how their experiences in these courses helped them enjoy mathematics and become more confident in their mathematical abilities. As one student wrote, “I really enjoyed my time in this class. It made me remember the fun of math.” Another student commented, “After taking [this course], I feel a lot more confident in my abilities.” For some students, the experience was transformative as explained by the student below:

> Besides learning the concepts, I got a new perspective on math. Prior to the class, I did not like math. I wanted to major in computer science but felt discouraged by the amount of math courses needed. All of that changed after [taking this course]. I found myself enjoying math. I changed my perspective of myself from not being a math person, to being one. I am no longer discouraged when I think of future math courses. I am excited!

While the majority of excerpts relating to affect involved positive emotions, there were a few students who expressed negative sentiments towards mathematics. However, these excerpts often had a “silver lining” to them. For example, one student commented, “I still do not like math, its [sic] something that I will never get fond of. Compared to my other classes I did enjoy the way it was taught.” Another student conveyed a similar message, “I’ve never been a fan of math. [...] Mostly, in part because I’m bad at it, but this class made it easier to learn.” These quotes show that even though students may have had positive experiences in their mathematics class, these experiences were not always powerful enough to make them “enjoy” mathematics.

Assessment

When discussing their experiences in these courses, students often commented on assessment methods. To help students develop growth-oriented mindsets, we implemented standards-based grading, which most students had never experienced. While some students were confused or skeptical about it at first, many of them discussed how they felt less stressed about assessment and more focused on their learning. One student commented, “I liked how for our ‘tests’ we were able to attempt to get a [learning outcome] as many times as we needed before the end of the year. It made the class a lot less stressful.” Another student shared a similar sentiment:

> The [assessments] really helped me realize that if I can't get something on the first, second, or even third try, I can always try again and not get penalized. This also really helped me feel more relaxed while I studied for the [assessments] because I wasn't constantly in fear that if I do badly on a test, my entire grade will plummet like in past math classes I've taken.

Overall, students appreciated the opportunity to progress on learning outcomes over the course of the semester and felt like they were able to focus on understanding the material rather than "earning a good letter grade".

Discussion

Many of our themes aligned with those found in the SEMINAL student experiences framework (Uhing, 2021). In this section, we discuss similarities between our findings and this framework, discuss implications of our research, and propose some ideas for future work.

In our analysis, we found that it was helpful to combine student-to-student interactions and relationships with instructors into a single theme: community. While the original student experiences framework separated these two themes, we discovered that the types of interactions that students experienced led several of them to comment about how they felt "included" and unafraid to participate in the mathematical community. It was through these interactions with peers and instructors that the classroom community and culture emerged. Thus, we felt compelled to encompass student-to-student interactions and relationships with instructors under the larger theme of community.

Our findings also paralleled the affective dimension of the student experiences framework. Students wrote about both positive and negative emotions associated with their learning experiences. From the excerpts we analyzed, students were, as a whole, more positive about their experiences in these redesigned first-year mathematics courses, although analysis on the entire data set is still ongoing. This finding reinforces the notion that affect is ever present in the classroom and can have a huge impact on student learning (Hannula, 2006).

Finally, as in the student experiences framework, assessment practices were a central focus for many students. Our findings suggest that the standards-based grading system helped (at least some) students focus less on grades and more on their learning. Compared to assessment theme discussed in the original framework, students seemed to have a more positive outlook on assessment practices, although several offered suggestions for how to improve specific class assignments.

Overall, our findings align well with the SEMINAL student experiences framework. One strength of our work is that it draws on student responses to a focused set of questions about specific FYM courses, rather than broad, open-ended survey questions. Our study also highlights areas that are important to examine, as both researchers and educators: the themes that emerged from the data come from experiences that students have deemed important enough to share in response to the Mathography prompts. Our findings suggest that teachers should incorporate learning opportunities that foster a sense of community in the classroom, attend to student affect, and consider alternative grading strategies that promote growth-oriented mindsets.

As we continue to analyze our data, we expect to find additional ways to expand the original framework. In particular, we pose the following questions for further exploration: What additional insights can be gained from analyzing students' perceptions of their classroom experiences? How do these insights further inform the student experiences framework? Through our work in answering these questions, we will better understand how to create a classroom environment that promotes success for all students taking FYM courses.

References

Abell, M. L., Braddy, L., Ensley, D., Ludwig, L., Soto, H., & Project Leadership Team. (2017). Instructional Practices Guide. https://www.maa.org/programs-and-communities/curriculum%20resources/instructional-practices-guide

Adelman, C. (2006, February). *The toolbox revisited: Paths to degree completion from high school through college.* Washington, D.C.: U.S. Department of Education.

Conference Board of the Mathematical Sciences (CBMS) (2016) *Active learning in post-secondary mathematics education.* Retrieved January 20, 2020 from https://www.cbmsweb.org/2016/07/active-learning-in-post-secondary-mathematics-education/

Drake, C. (2006). Turning points: Using teachers' mathematics life stories to understand the implementation of mathematics education reform. *Journal of Mathematics Teacher Education, 9*(6), 579-608.

Dweck, C. S. (2006). *Mindset: The new psychology of success.* Random House, Inc.

Elsinger, J., & Lewis, D. (2020). Applying a standards-based grading framework across lower level mathematics courses. *PRIMUS, 30*(8-10), 885-907. https://doi.org/10.1080/10511970.2019.1674430

Hannula, M. S. (2006). Affect in mathematical thinking and learning: Towards integration of emotion, motivation, and cognition. In J. Maasz & W. Schloglmann (Eds.), *New mathematics education research and practice* (pp. 209-232). Sense Publishers. DOI https://doi.org/10.1163/9789087903510_019

Lewis, D. (2020). Student anxiety in standards-based grading in mathematics courses. *Innovative Higher Education, 45*, 153-164. https://doi.org/10.1007/s10755-019-09489-3

Lombardi, D., Shipley, T. F., Astronomy Team, Biology Team, Chemistry Team, Engineering Team, Geography Team, Geoscience Team, & Physics Team. (2021). The curious construct of active learning. *Psychological Science in the Public Interest, 22*(1), 8-43.

President's Council of Advisors on Science and Technology (PCAST) (2012). *Engage to excel: producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Washington, D.C.: The White House

Smith, W.M., Voigt, M., Ström, A., Webb, D., & Martin, W.G. (Eds.). (2021). *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning*. American Mathematical Society.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones, L. I., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., Lowe, A., Newman, S., Okolo, V., Olroyd, S., Peecook, B. R., Pickett, S. B., Slager, D. L., Caviedes-Solis, I. W., Stanchak, K. E., Sundaravardan, V., Valdebenito, C., Williams, C. R., Zinsli, K., & Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences, 117*(12), 6476-6483. https://doi.org/www.pnas.org/cgi/doi/10.1073/pnas.1916903117

Twigg, C. A. (2011). The math emporium: A silver bullet for higher education. *Change: The Magazine of Higher Learning, 43*(3), 25-34.

Uhing, K., Hass, M., Voigt, M., Calleros, E., & Ström, A. (2021). Students' experiences with active learning mathematics. In W.M. Smith, M. Voigt, A. Ström, D. Webb, & W.G. Martin (Eds.), *Transformational change efforts: Student engagement in mathematics through an institutional network for active learning* (Chapter 12, pp. 221-241). American Mathematical Society.

Webel, C., Krupa, E. E., & McManus, J. (2017). The math emporium: Effective for whom, and for what? *International Journal of Research in Undergraduate Mathematics Education, 3*(2), 355-380.

THE SPATIAL STRUCTURE OF HUNDREDS CHARTS

Vera WAGNER
Concordia University
vcmwagner@gmail.com

Helena P. OSANA
Concordia University
helena.osana@concordia.ca

Jairo NAVARRETE
O'Higgins University
jairo.navarrete@uoh.cl

Hundreds charts are an essential tool for teachers in the mathematics classroom. The present study investigated the effects of spatial configuration in hundreds charts on children's place-value knowledge and their strategies for solving arithmetic word problems. Kindergarten and first-grade students were assigned to three instructional conditions in which they worked with a researcher to solve arithmetic problems. The charts in the conditions differed by spatial configuration. Results showed that the children improved their place-value knowledge and were more likely to use strategies based on place-value concepts when the numbers in the chart increased from top to bottom than when the numbers increased from bottom to top.

Keywords: Problem Solving, Mathematical Representations, Number Concepts and Operations

Visual representations are often used in elementary school classrooms to illustrate abstract concepts (Schraw et al., 2013). The hundreds chart is an example of a visual representation that teachers often use to teach concepts such as counting, arithmetic, and place value. These 10 x 10 grids contain the numbers from 1 to 100 (or sometimes from 0 to 99) in increasing order. A "top-down" arrangement puts the lowest numbers in the top row, with the numbers increasing with downward movement by rows in the chart. Another variation of the chart (i.e., "bottom-up") is inversely configured, with lower numbers at the bottom and larger ones at the top.

Previous research indicates that the spatial configurations of instructional materials impact how children learn and think about number. Board games that display numbers in linear arrangements provide better support for the development of children's number magnitude than games with circular arrangements (Siegler & Ramani, 2009). Moreover, Navarrete et al. (2018) found that when preschoolers viewed numbers in a linear arrangement that increased from left to right, their performance on an estimation task improved relative to those who saw the same numbers arranged from right to left. Findings from both studies can be explained using processes of analogical reasoning: Mappings made between the spatial configuration of the materials and the desired mental representation (i.e., the mental number line) supported children's numerical performance. Further, the more the structural features in the materials are highlighted, the more likely the target mental representations will be constructed (Gentner et al., 2007).

No systematic studies have been conducted to test the spatial structure of hundreds charts, however. Some anecdotal observations have suggested that moving down in the chart when the numbers increase can be confusing to children ("When we put more milk in the carton, it goes down. When we take milk out, it goes up," Bay-Williams & Fletcher, 2017 p. e3; see also Cobb, 1995). On the one hand, the upward movement in a bottom-up hundreds chart corresponds to increases in quantity, which can be mapped onto observations in the real world (e.g., water filling a glass) through the "up is more" relation. On the other hand, the fact that most young children are learning to read text in a top-to-bottom direction also provides a rationale to expect that a top-down hundreds chart may best support children's number knowledge.

Present Study

The objective of the present study was to study the effects of different spatial configurations on children's place-value knowledge and problem-solving strategies. Children were randomly assigned to one of three instructional conditions that differed by the spatial configuration of the hundreds chart used to solve arithmetic word problems. After the intervention, children completed a learning and transfer task to document their strategies for solving problems using the chart. A test of place-value knowledge was administered before and after the intervention.

We asked two research questions: (1) Does the spatial structure of the hundreds chart have an effect on children's place-value knowledge and the strategies they use when using the chart to solve arithmetic word problems? (2) Does an explicit visual analogy of the "up is more" relation provide additional benefits to their place-value knowledge and problem-solving strategies?

Method

Participants

Data were collected through online interviews with kindergarten and first-grade students in Canada and the US (N = 47; 23 boys; age: M = 81.6 months, SD = 5.9). Conditions did not differ by age, $F(2, 44) = .68$, $p = .51$; grade, $\chi^2(2) = 0.68$, $p = .71$; or gender, $\chi^2(2) = 1.08$, $p = .58$.

Instructional Intervention

In individual online sessions, a researcher showed the children how to use the Virtual Hundreds Chart (VHC; Osana et al., 2021) to solve Join Result Unknown word problems (Carpenter et al., 2014; sample problem in Figure 1). Children in the top-down condition (n = 15) worked with a hundreds chart in which the smallest numbers were in the top row and increased in value with downward movement in the chart. Children in the bottom-up (n = 16) and explicit analogy (n = 16) conditions worked with a bottom-up hundreds chart, in which the numbers increased with upward movement in the chart. The strategy demonstrated in all three conditions consisted of moving up (or down, depending on condition) the number of rows that matched the tens in the second addend and then right within the rows for the ones in the second addend.

In the explicit analogy condition, the instructional intervention included an additional visual cue to highlight the structure of number in the bottom-up hundreds chart. The visual cue was a cylinder that filled with water as the first addend in the problem increased by the second.

Measures

Place-value knowledge. Place-value knowledge was assessed at pre- and posttest using the Picture Place Value Task (PicPVT; Osana & Lafay, 2019), a judgment task designed to assess whether children understand that a digit in a numeral represents a quantity that is determined by its position. A double-digit numeral with one of the digits underlined was presented on a screen. Groups of dots were presented below the numeral and children judged whether the representation correctly matched the underlined digit. Scores were the mean number of correct responses.

Strategy use. The children's movements in the chart were videorecorded to document their problem-solving strategies using the VHC. They solved six Join Result Unknown problems (as in Figure 1) on a learning task and four Separate Result Unknown problems on a transfer task (e.g., There were 48 cars in a parking lot. 30 cars drove away. How many cars were left?), which required moving by tens in the chart in the opposite direction as was shown in the instruction. Mental computation strategies (i.e., no use of the chart) were not coded.

All strategies that were coded as mathematically appropriate involved moving vertically or horizontally in the chart in directions that were consistent with its structure. Movements were further coded as either (a) moving by tens: vertically for tens and horizontally for ones, or (b) moving by ones: horizontally by ones only. The moving by tens strategy demonstrated greater knowledge of place value and numeration concepts than the moving by ones strategy (Carpenter et al., 2014; Fuson, 1990). Errors such as starting on the wrong value in the chart or providing solutions that were off by one (either by one ten when moving by tens, or by one unit when moving by ones) were forgiven. Mathematically inappropriate strategies involved errors such as moving in the wrong direction (i.e., inconsistent with the structure of the chart); moving horizontally for the tens digit; and adding only the value of one digit. Provided no mathematical errors occurred, students were not penalized for strategies that started with the second number in the problem on either the learning or transfer task (e.g., for the problem in Figure 1, starting on 38 and moving 51 spaces to land on 89).

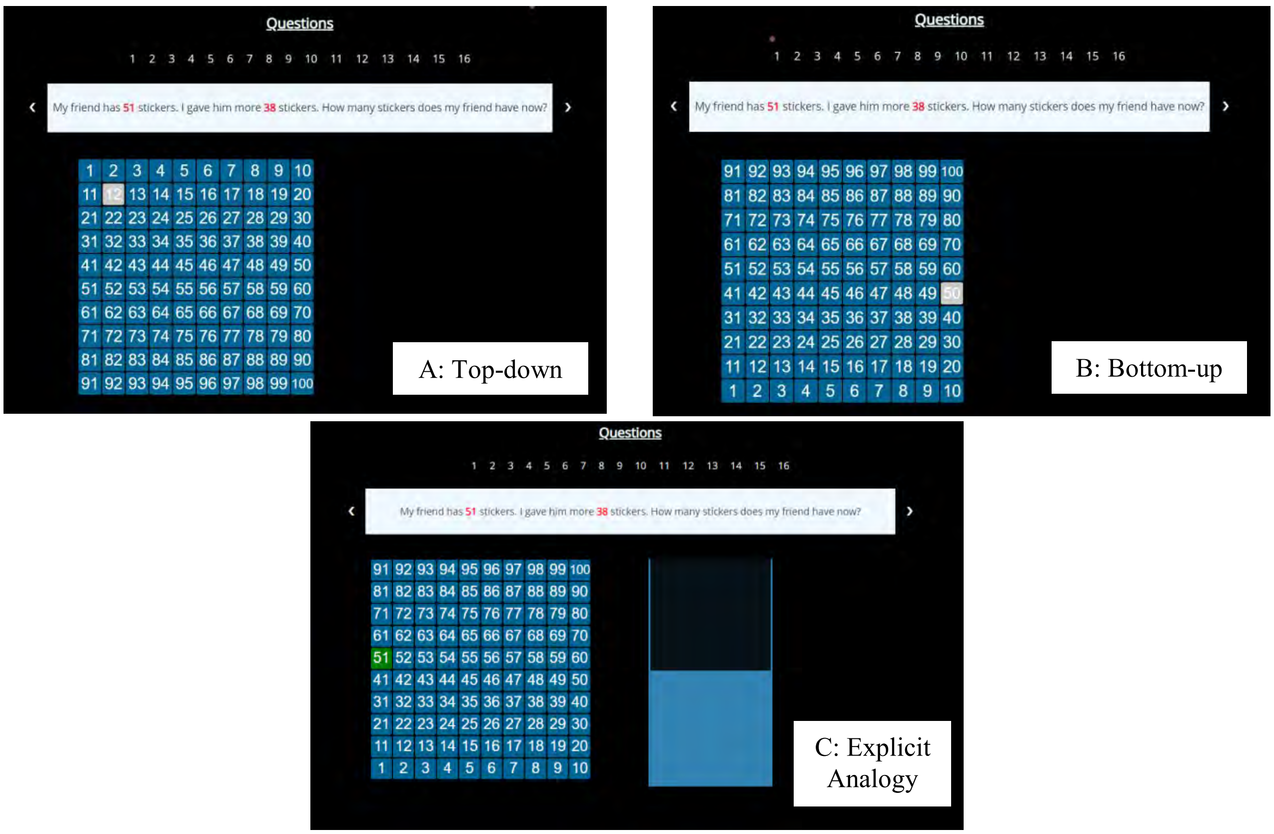


Figure 1: VHC Used in Top Down (A), Bottom Up (B), and Explicit Analogy (C) Conditions

Results

A one-way analysis of covariance (ANCOVA) was performed using PicPVT scores at posttest as the dependent measure, condition as the between groups factor, and PicPVT pretest scores as the covariate. A main effect of condition was found, $F(2, 41) = 3.68$, $p = .03$. Least Significant Difference post hoc comparisons indicated that the top-down condition ($M_{adj} = .83$; $SE = .03$) outperformed both the bottom-up ($M_{adj} = .74$; $SE = .03$) and explicit analogy ($M_{adj} = .73$; $SE = .03$) conditions (*p*s < .05), with no difference between the latter two groups.

Table 1 presents the frequency that each strategy was used as a function of condition. Using item as the unit of analysis, a significant association was found between strategy type and condition on the learning, $\chi^2(4) = 21.1$, $p < .001$, and transfer tasks, $\chi^2(4) = 18.5$, $p < .001$. A larger proportion of the strategies in the top-down condition involved moving by tens (85.6% on the learning and 82.0% on the transfer task) than in the bottom-up (55.8% and 42.0% on each task, respectively) and explicit-analogy (68.8% on each task) conditions.

Table 1: Frequency of Strategy Types on Learning and Transfer within Condition

Strategy	Condition					
	Top down		Bottom up		Explicit analogy	
	n	%	n	%	n	%
			Learning (n = 240)			
Moving tens	71	85.6	43	55.8	55	68.8
Moving ones	7	8.4	10	13.0	12	15.0
Mathematically inappropriate	5	6.0	24	31.2	13	16.2
			Transfer (n = 151)			
Moving tens	41	82.0	21	42.0	35	68.6
Moving ones	2	4.0	9	18.0	6	11.8
Mathematically inappropriate	7	14.0	20	40.0	10	19.6

Note. Data were missing on 20 items on the learning task and 12 items on the transfer task.

Discussion

As expected, the children's knowledge of place-value was dependent on the spatial structure of the chart they used to solve arithmetic word problems, as was the extent to which their strategies appropriately applied number concepts. Despite its misalignment with the "up is more" relation, the top-down configuration produced the largest benefits in performance. The instruction in the top-down condition involved movements that matched the direction in which words are read in text (i.e., left to right and down). Such movements may have freed cognitive resources to attend to the instruction, thereby facilitating the application of number concepts on the learning and transfer tasks. Alternatively, the benefits of the top-down chart may have been due to classroom exposure, a potential confound to be addressed in future research.

The effectiveness of the top-down chart may in part explain why the visual cue of a cylinder filling with water did not have the anticipated instructional effects: It explicitly highlighted the bottom-up structure, which by itself was relatively less effective. All the same, the cue may have had some instructional effect, as the children exposed to it made use of the "moving by tens" strategy more often than the children in the bottom-up condition. Although our conclusions are preliminary, the results nevertheless imply that teachers should attend to the ways students interact with the spatial configurations of mathematical representations.

References

Bay-Williams, J., & Fletcher, G. (2017). A bottom-up hundred chart. *Teaching Children Mathematics, 24*(3), e1-e6.

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2014). *Children's mathematics: Cognitively guided instruction* (2nd ed.). Heinemann.

Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362-385.

Fuson, K. C. (1990). Conceptual structures for multiunit numbers: Implications for learning and teaching multidigit addition, subtraction, and place value. *Cognition and Instruction, 7*(4), 343-403.

Gentner, D., Loewenstein, J., & Hung, B. (2007). Comparison facilitates children's learning of names for parts. *Journal of Cognition and Development, 8*, 285–307.

Navarrete, J. A., Gómez, D. M., & Dartnell, P. (2018). Promoting preschoolers' numerical knowledge through spatial analogies: Numbers' spatial alignment influences its learning. *Contemporary Educational Psychology, 54,* 112–124.

Osana, H. P., & Lafay, A. (2019). The relationship between analogical reasoning and second graders' structural knowledge of number. In S. Otten et al. (Eds.), *Proceedings of the forty-first annual meeting of the Psychology of Mathematics Education—North American Chapter* (pp. 201-205). University of Missouri.

Osana, H. P., Wagner, V., & Loiselle, A. (2021). Virtual hundreds chart. (Version 1.0) [Computer software]. *Média Techno Vision.* http://mathchart.ca/#nt

Schraw, G., McCrudden, M. T., & Robinson, D. (Eds.). (2013). *Learning through visual displays*. Information Age.

Siegler, R. S., & Ramani, G. B. (2009). Playing linear number board games—but not circular ones—improves low-income preschoolers' numerical understanding. *Journal of Educational Psychology, 101*(3), 545-560.

SECONDARY STUDENTS' AGENCY IN MATHEMATICS CLASSROOMS

Mitchelle M. Wambua
University of Missouri-Columbia
mwambua@mail.missouri.edu

Engaging students in ways that promote their agency in mathematics classrooms is seen by many scholars as pertinent in nurturing meaningful, enjoyable, and equitable learning opportunities for the students. Yet, whether teachers value and support students' mathematical agency is as important as how they do so. This study uses teacher questionnaire to investigate how secondary mathematics teachers think about students' agency and the forms of agency their students exercise in class. Results indicate that students exercise different types of agency depending on how the classroom authority is distributed.

Keywords: Classroom Discourse; Teacher Beliefs; High School Education; Equity, Inclusion, and Diversity.

Within mathematics education, there has long been a focus on the need for teachers to engage students in ways that promote the students' agency and position them in authority during mathematics classroom learning. Professional organizations (e.g., National Council of Teachers of Mathematics, 2000), policymakers (e.g., National Governors Association & Council of Chief State School Officers, 2010), as well as scholars (e.g., Bieda & Staples, 2020; Wilson & Lloyd, 2000) have underscored the need to build and sustain students' mathematical agency. Mathematical agency refers to one's belief that they can generate valid mathematical ideas and engage in a meaningful practice of doing mathematics (Bieda & Staples, 2020; Ruef, 2021). Bieda and Staples add that for students to exercise their mathematical agency, their teachers should provide them with the opportunities to do so. Mathematics teachers can promote students' agency within their classrooms through practices like having them share ideas and methods that steer class discussions, engaging them in constructing arguments, justifying their thinking, and critiquing/assessing their peers' work (Amit & Fried, 2015; Bishop et.al., 2021). Scholars see these practices as pertinent in nurturing meaningful and enjoyable student learning, encouraging more equitable learning opportunities for students, and building students' curiosity and willingness to take up more learning opportunities.

The goal of this study was to investigate where secondary mathematics stand on this issue of promoting students' mathematical agency. Specifically, the study attended to how teachers think about students' mathematical agency, and how they support students' agency in their self-reported classroom practices. Understanding teachers' thinking and practices related to student agency is an important step toward supporting them in improving their practice.

Theoretical Framing

As mathematics teachers and students engage with each other during classroom learning, they participate in social practices wherein norms and obligations for what counts as mathematical competence are developed and negotiated (Cobb et. al., 2009; Langer-Osuna, 2016). Inherent in these norms and obligations are negotiations around the kinds of agency that students can bring to bear. Cobb and colleagues have delineated between two forms of agency (i.e., disciplinary and conceptual) that students can legitimately exercise in their classrooms. *Disciplinary agency* involves student performing mathematical practices that have already been established by the teacher, while *conceptual agency* involves students having the freedom to choose among mathematical practices and to make connections among mathematical concepts and principles beyond those taught directly by the teacher.

Students' mathematical agency is closely linked with the distribution of classroom authority (Cobb et. al., 2009). Classroom authority refers to the shifting dynamics around who has the legitimacy to lead classroom interactions (Amit & Fried, 2015; Pace & Hemmings, 2007) and determine the validity of mathematics concepts (Cobb et. al., 2009; Langer-Osuna, 2016). In Mathematics classroom, authority may be distributed solely to the teacher, with the teacher being the only one responsible for determining the correctness of mathematics concepts and deciding which ideas the class will engage with. Alternatively, authority may be distributed to the whole classroom community, with the teacher and students jointly determining the concepts to focus on and the validity of those math concepts. The types of agency that students exercise in math classrooms often depends on how authority is distributed in the classroom. That is, whether the teacher is the sole authority and students are obligated to just follow the teacher's lead, or there is shared authority between the teacher and students and hence students are invited to take lead in classroom learning (e.g., by devising solution strategies). This study draws on Cobb and colleagues' characterizations of student agency and classroom distribution of authority, and is guided by the question: *According to secondary mathematics teachers, what kinds of agency do students exercise in their mathematics classroom?*

Method

Study Context and Participants

This study is situated within an ongoing, exploratory study aimed at understanding and documenting how teachers view and support students' mathematical agency and authority. The participants are practicing secondary mathematics teachers working at various schools throughout the United States and who are all enrolled in a Mathematics Education master's program at a university in the midwestern United States. Practicing teachers were chosen as the target group because the goal is to investigate both the teachers' thinking and classroom actions related to students' mathematical agency and authority. Moreover, these teachers have taken at least two mathematics education courses within their program that urge them to reflect on various ways of meaningfully engaging their students mathematically, and hence this study would act as one avenue for this kind of reflection.

Data Sources and Analysis

For this study, the researcher focused on data from a teacher questionnaire. The questionnaire had a mixture of open-ended and closed-format questions related to students' mathematical agency. The analysis reported here is from 15 teacher responses. These teachers vary in their professional experience (i.e., from 1 year to 10 years of teaching), but are primarily white females (n=13) who are currently teaching Algebra and/or geometry to 9 - 12th grade students.

Data analysis was done in two phases. Phase one involved reading the teachers' responses and applying broad codes to any segments and response choices related to students' mathematical agency and classroom authority, alongside noting who the teachers highlighted as the authority in those instances. Phase two involved qualitative coding based on Cobb and colleagues' (2009) interpretive framework for student agency and classroom distribution of authority. These codes are described briefly below:

Table 1: Coding Categories

Category	Description
Authority figure	Identifies who takes the lead in classroom activities and determines the validity of math ideas. This could either be the teacher, students, or both.
Student agency	*Conceptual agency* – relates to if and how students choose solution methods, justify their work, assess their peers' work, and make mathematical connections.
	Disciplinary agency – relates to if and how students use established solution methods and follow teacher-led activities.

Findings

Most of the teachers expressed that they are the main authority in their math classrooms, with a few of them illustrating that they give their students minimal opportunities to share in the classroom authority. As a result of the teachers being the primary authority, students mostly exercise disciplinary agency, although there were some instances where teachers depicted that their students get opportunities to exercise some forms of conceptual agency.

Students' Agency when Teachers are the Main Classroom Authority

More than half of the teachers reported that the authority of leading the class in engaging with both new and known math concepts was distributed mainly to them. The teachers were responsible for delivering all lesson concepts and determining the methods for solving tasks. For example, one teacher said the following about her roles in a typical class:

> I review the assignment from the previous lesson then do notes over our new topic in a discussion style where I demonstrate to the students [how to solve] various examples.

This teacher, as well as several others saw themselves as the sole authority in reminding students how solve tasks from previous topics as well as "demonstrating" the solution methods for tasks from new topics. The teachers also took the lead in determining which tasks the classroom community would solve and in conveying the concepts required to solve new tasks. They reported that they usually "select tasks from the textbook, online materials or [personally] create them" without involving their students in task selection.

Teachers who saw themselves as the main classroom authority cited that their students exercise *disciplinary agency*. According to the teachers, students are obligated to "take notes" and "ask questions" during class to make sure they understand the concepts and can "apply [the concepts] correctly." Additionally, students are obligated to solve tasks using clearly established strategies. One teacher specified that she is currently teaching systems of equations and her students are expected to "solve these equations using the substitution, elimination, graphical and matrix methods." These student roles reveal a disciplinary agency of keenly following teacher-led activities and reproducing solution strategies as dictated by the teacher.

Students' Agency in Classrooms with Shared Authority

Although most of the teachers reported that they were the main classroom authority, a few of them highlighted that they provide opportunities for shared authority. For example, some teachers said that they engage students in critiquing and verbally assessing the validity of their peers' math solutions whenever the students share their work publicly during whole class discussions. One teacher added that when students publicly share their solutions, she invites all students to "recognize the different strategies … and make connections" between the strategies.

Other teachers expressed that even though their students treat them as the primary source/author of knowledge, they would like the students to view themselves and their peers as legitimate resources for learning mathematics. Although all the teachers reported that whenever their students encounter challenges while solving problems they always ask the teacher for help, a few teachers said that they wish that wasn't the norm. These teachers added that they preferred that students look at their notes and/or consult their peers. These teacher preferences show that they believe in their students' ability and authority to navigate math challenges without always relying on the teacher.

Teachers who stated that they share classroom authority with students expressed that their students exercise different types of mathematical agency. Some of the teachers highlighted that their students have the freedom to choose which strategy to use when solving math tasks. One teacher added that she encourages her students to "choose from multiple solution paths instead of having them pigeonholed into a specific strategy." These student choices relate to *conceptual agency* in the interpretive framework. Other teachers mentioned that their students exercise conceptual agency by performing classroom roles like explaining their thinking, "making connections between concepts" and assessing their peers' work (see above).

However, some of the teachers who expressed that they share classroom authority with students cited ideals related to *disciplinary agency* when describing what it means for a student to be good at math. For example, one of them said that a mathematically competent student should "be able to use the rules, procedures and proofs that I taught them efficiently in order to get the right answer." In this case, the teacher alluded that their standard for a mathematically competent student comprises a student who gets valid answers by recalling and reproducing the "rules" and "procedures" that the teacher taught.

Discussion and Conclusion

The goal for this study was to elicit how secondary mathematics teachers think about students' mathematical agency and the forms of agency their students exercise during classroom practices. Knowing the reality of what is happening in classes is helpful for scholars who want to work with teachers to improve their practice and student learning. Even though the teachers participating in this study were actively being taught (in their masters' program) the need to meaningfully engage their students by sharing classroom authority with them and positioning them as competent math learners, the teachers primarily employed teacher-centered practices in their classrooms. Consequently, their students exercise disciplinary agency, mostly appealing to teacher-led practices and reproducing teacher-taught solution strategies. There were however a few teachers who made attempts at sharing classroom authority and allowing their students to exercise the conceptual agency of selecting appropriate solution strategies, validating mathematical solutions, and making connections between multiple concepts. Interestingly, even these few teachers demonstrated a tendency to still expect their students to perform roles relating to disciplinary agency. These findings imply that teachers need more support beyond advanced university teaching for them productively engage their students in meaningful ways.

References

Amit, M., & Fried, M. N. (2005). Authority and authority relations in mathematics education: A view from an 8th-grade classroom. *Educational Studies in Mathematics, 58*(2), 145–168.

Bieda, K. N., & Staples, M. (2020). Justification as an equity practice. *Mathematics Teacher: Learning and Teaching PK-12, 113*(2), 102–108.

Bishop, J., Hicks, M.D., & Koehne, C. (2021). Mathematical authority in authoring, animating, & assessing mathematical ideas. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education Conference* (pp. 1520-1525). Philadelphia, PA: PME-NA.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for research in mathematics education, 40*(1), 40–68.

Langer-Osuna, J. M. (2016). The social construction of authority among peers and its implications for collaborative mathematics problem-solving. *Mathematical Thinking and Learning, 18*(2), 107–124.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Author.

Pace, J. L., & Hemmings, A. (2007). Understanding authority in classrooms: A review of theory, ideology, and research. *Review of Educational Research*, *77*, 4–27.

Ruef, J. (2021). How Ms. Mayen and her students co-constructed good-at-math. *Journal for Research in Mathematics Education, 5*2(2), 152–188.

Wilson, M. & Lloyd, G. M. (2000). Sharing mathematical authority with students: The challenge for high school teachers. *Journal of Curriculum and Supervision, 15*, 146–169.

MIDDLE SCHOOL STUDENTS' TYPES OF MATHEMATICAL PERSONIFICATIONS

Nicole M. Wessman-Enzinger
George Fox University
nenzinger@georgefox.edu

Clara Stilwell
George Fox University
cstilwell18@georgefox.edu

Unpacking middle school students' mathematical relationships is important as a step towards improving mathematical relationships. In this study, 500 middle school students drew personifications of mathematics. We examined these personifications of mathematics for insight into their relationships with mathematics. Using constant comparative methods, we present various ways the middle school students personified mathematics. Negative relationships were personified with terrible beasts, abusers, authoritarians, and pests/nuisances. Positive relationships were personified with best friends and nature. Some personifications supported both positive and negative relationships or were neutral relationships. Reflecting on these personifications point to components of positive relationships with mathematics that we should support and confronts ways we may be perpetuating negative relationships with mathematics.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education

All mathematical learners deserve the opportunity to see themselves as "good at math" (Ruef et al., 2020; Wickstrom & Gregson, 2017). Having a positive relationship with mathematics is a component of developing a positive mathematical identity. Students have both relationships *with* and *in* mathematics (Hunter & Anthony, 2011); these relationships influence identities (Black et al., 2009). Relationships *with mathematics* impact equity, access, and empowerment (Smith et al., 2017) in thinking and learning. Students enjoy mathematics more (Parks, 2020) and engage in the deep work of mathematics (Smith et al., 2017) if they have positive relationships with mathematics. Exploring what the elements of positive relationships with mathematics entail will provide insight into how to support students in healthy mathematical relationships.

Students enter middle schools where math is "tracked," which can have negative impacts on their identities (National Council of Teachers of Mathematics, 2018). Mathematics has the potential to be a challenging adventure that opens doors to their future (Stinson, 2004). But, students may not see themselves as mathematicians (Wickstrom & Gregson, 2017) and that impacts their mathematical relationships and identities. Although we suspect that some middle school students have negative relationships with mathematics, we need to understand the challenges in students' mathematical relationships in order to meet students where they are at. Insight into the challenges of mathematical relationships, can provide perspective into how we may transform negative mathematical relationships.

Purpose

Although teachers and researchers might inherently understand that middle schoolers have negative relationships with math, we do not fully understand the complexities that are included within those mathematical relationships with middle school students. We sought to understand middle school students' relationships with mathematics. The research question that guided our work is:

What types of mathematical personifications did our middle schoolers create?

Describing the nuances of the different types of personifications will provide insight into the mathematical relationships and experiences we need to change, enhance, or continue supporting.

Theoretical Perspective: Narrative Theory

Relationships, in general, are complex, and even positive relationships have a slew of emotions that come with that relationship. Having a positive relationship with mathematics does not mean that there will not be elements of frustration or struggle. In fact, we know that productive struggle is a critical attribute of doing mathematics deeply (Baker et al., 2020; National Council of Teachers of Mathematics, 2014). But, the relationship we hope students have with mathematics is not toxic or abusive. Yet, prior studies that have used personification tasks have revealed that university students describe their relationships with mathematics as toxic or abusive (Ruef, 2020; Zazkis 2015). Prior studies have also shown that personification tasks are better than "draw a mathematician" tasks alone (Picker & Berry, 2000; Zazkis, 2015). We extend this prior research further by examining the ways that middle school students personify mathematics. Through personifications, we have some insight into the narrative of their experiences (Langer-Osuna & Esmonde, 2017). Indeed, it is human nature to personify things. Three personifications present in the literature, created by preservice teachers, are: terrible beast, (former) best friend, and lover (Zazkis, 2015; Zazkis & Mamolo, 2016). The terrible beast relationship is equated to fear and repulsion. While the best friend, or even former friendship, or lover are relationships that can be translated to a level of comfortability and enjoyment, even if former, of mathematics. These types of personifications can provide insight into mathematical relationships and we entered our study expecting to see these personifications from middle schoolers as well.

Qualitative Methods

Participants and Data Collection

During Spring 2019, 505 middle schoolers (grades 6–8) volunteered to participate in the study and completed a personification of math task. At the time of data collection, the participants attended a middle school with students that scored in the 37th percentile in the state for mathematics.

The personification task included three different parts modified from (Ruef, 2020; Zazkis, 2015; Zazkis & Mamolo, 2016): (1) a picture of how they personify math and describe it; (2) a paragraph about who math is to them; and (3) a conversation with math. Of the 505 middle school participants, some of the responses were blank or illegible, leaving exactly 500 of the responses for analysis.

Data Analysis

The unit of analysis included the collective responses to the personification task—the drawing, description of drawing, and a written conversation with math. Sometimes students only had a drawing or a drawing and description, but no conversation with math. We used whatever they provided as the unit of analysis to help understand the personification.

The first pass of the data analysis entailed: (a) taking notes about each drawing and (b) coding the relationship with the Zazkis and Mamolo framework (2016): terrible beast, lover, (former) best friend. Using constant comparative methods (Merrriam, 1998), we went through each piece of data and negotiated what relationship was personified. We did about two passes of the data. First, we began with examining between 20-50 pieces of data at a time, iteratively meeting and comparing, and discussing if these themes aligned with our data. Second, we adjusted the themes—modifying the descriptions from Zazkis and Mamolo (2016) and adding categogies as needed. Third, we went through our data again with our new framework. This was a lengthy iterative process with the 500 pieces of data, but it spurred conversations about the themes and development of descriptions of our drawings.

In this research report, we present our modifications the Zazkis and Mamolo framework that fit the data for middle schoolers. For example, none of our participants personified their mathematical relationship with lovers as Zazkis and Mamolo described. Perhaps this is due to the age of our participants or their mathematical experiences. On the other hand, our participants emphasized authoritarians (e.g., drawings of mean teachers), pests, and nature which are were not highlighted in prior work. Therefore, for the results section of this research report, we describe the different personifications and how they relate to students' mathematical relationships.

Different Types of Mathematical Relationships

Negative Relationships

The middle schoolers personified negative relationships in mathematics with terrible beasts, authoritarians, abusers, and pest/nuisances.

The terrible beast. The terrible beast personifications included monsters, devils, or other terrifying animals (see Figure 1). Consistent with the Zazkis and Mamolo (2016) framework, the beast relationship showed fear or repulsion. When including beasts, students shared two dichotomous perspectives. One perspective with the beast included elements of consistent failure and defeat. The other perspective included elements of conquering math or beating "the beast." Figure 1 on the left shows a monster leaning over a student calling her stupid, dumb, and wrong. There is a sense of defeat in the student's face. However, in the second example in the middle of Figure 1, the student is trying to have some power and ability to fight back. The figure on the right, is a fetus that has a snake-like tongue, red eyes, and two sets of sharp teeth.

Figure 1: Terrible Beast Personifications

The authoritarian. The authoritarian personifications showed someone or something exerting power and control. This authority figure seems related to a beast relationship; but the authoritarian relationship differed from the terrible beast because the focus is on power and control and may not be a beast. Some drawings, for example, included stern or angry teachers. This is a theme that we added to the Zazkis and Mamolo (2016) framework. In Figure 2 math is illustrated as authoritarian that has all the control by putting the student in prison (math class). Freedom is outside of math class. In this example, the student wrote about how they desire to have more freedom within math class. In Figure 2 there is a student slumped and labeled with a depression; another student seems to be looking out of math class to freedom. The middle schooler wrote, "It's like a slap in the face every day. Math puts me in prison only [letting] me look at freedom."

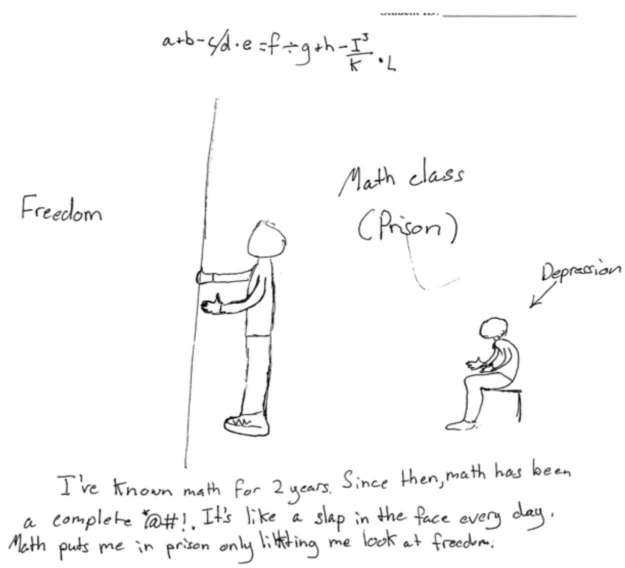


Figure 2: Authoritarian Personification

The abuser. With the abuser personification, the personifications included evidence of emotional or physical trauma. This is similar to the authoritarian relationship in that there is no power and control; however, the abuser relationship is an extreme version of an authoritarian with extreme emotions and/or trauma involved in this relationship. Although trauma and abuse in math is evidenced in the literature (e.g., Ruef et al., 2020), we made it a distinct category based on the amount of clear abuse that differed from a fantastical beast drawing (e.g., drawings of torture chambers). Within the abuser personifications, there are themes of hopelessness and cycles of abuse. Most of the abuser personifications in our study were incredibly graphic in ways that made us uncomfortable and reluctant to place in a proceeding. We selected Figure as a personification of abuser that "hit(s) kids with a ruler" as our example. Most of the personifications were more intense than this.

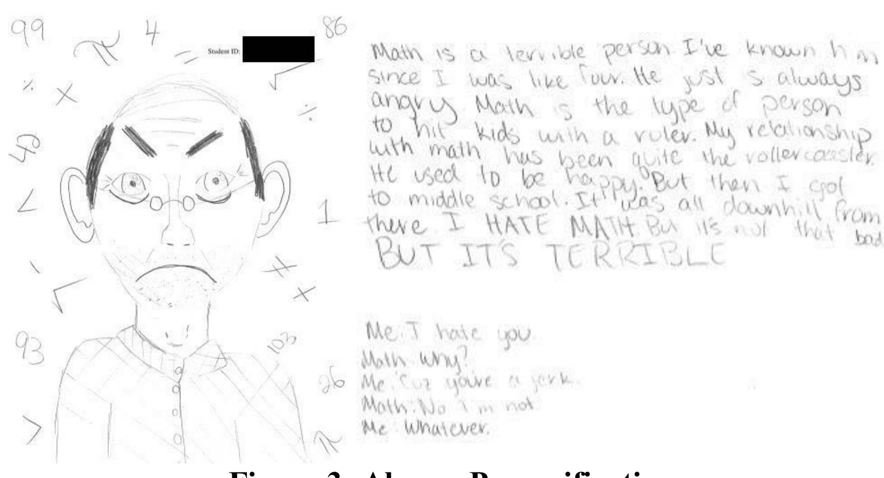


Figure 3: Abuser Personification

The pest/nuisance. Students that drew math as a pest often describe math as something that is intentionally annoying highlighted a pest/nuisance personification of mathematics. A pest is typically irritating or an inconvenience. These inconveniences can look like losing your keys or a skunk making you smell. It can also look like either doing math or smelling something gross. Figure 4 highlights a pesky macaw stealing keys form a student.

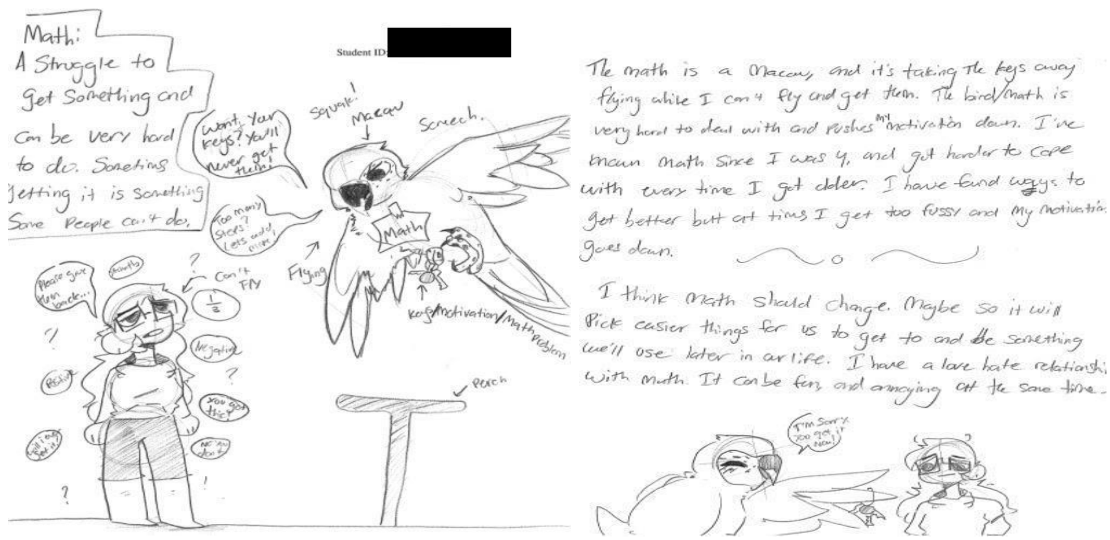


Figure 4: Pest/Nuisance Personification

This is a category that we added to the Zazkis and Mamolo (2016) framework of personifications.

Positive Relationships

The middle schoolers personified positive relationships in mathematics with best friends and nature.

The best friend. Students personified mathematics as a best friend. Zazkis and Mamolo (2016) described how math was personified as both a lover and also a (former) best friend. We did not see lovers as a theme; instead, there was an abundance of math personified as a best friend. The best friend personifications showed themes of comfortability and enjoyment of mathematics. The students described their friend as someone that you can do many things with, including but not limited to: playing games, going to the store with, and even being a helpful resource. This did not mean that the students did not think that mathematics had flaws. In Figure 5, the student described math as someone that is shy but kind. The student described her best friend as "She might be challenging to understand at first, you might have to tell her to slow down, and take a step back, but she will be with you through thick and thin." The student wants to be friends with math for as long as they can—It will be a useful asset all through life." This student highlighted that they are not a stranger to the challenges of mathematics, but it is worth the struggle.

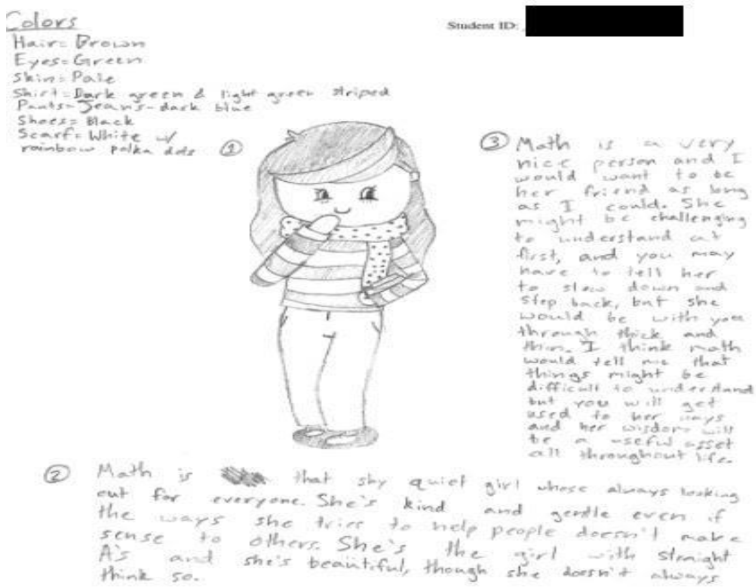


Figure 5: The Best Friend Personification

Nature. The students also personified mathematics with nature. These personifications described the push and pull of nature, how sometimes nature is beautiful and sometimes it is a challenge. Nature is beautiful and to be enjoyed, but also can be harmful and dangerous. We added this type of personification to the Zazkis and Mamolo (2016) framework. As the transcript below states, "math starts out looking like tangled videos with thorns like a rose bush, but after you get to know math and understand it, it can look like simple, harmless flowers."

Figure 6: Nature Personification

Both Positive/Negative Relationship or Neutral Relationships

The middle schoolers personified relationships in mathematics that could be considered both positive and negative or neutral with former friends and cartoons.

The former friend. Consistent with Zazkis and Mamolo's (2016), the students personified mathematics as a former friend. Figure 7 highlights a variety of emotions and types of relationships, both an abusive relationship and a former friend are illustrated. This is a reminder that sometimes relationships can be more than one thing. Emotions and relationships are complex in general and that does not exclude mathematics. This means that these classifications can overlap. Figure 7 shows some fingers that have various faces (e.g., smiles, frowns), indicating evidence of both positive and negative relationships. For example, in the transcript the relationship is described as "love/hate." The student described math as "fun," but math says things like "you ruin my life."

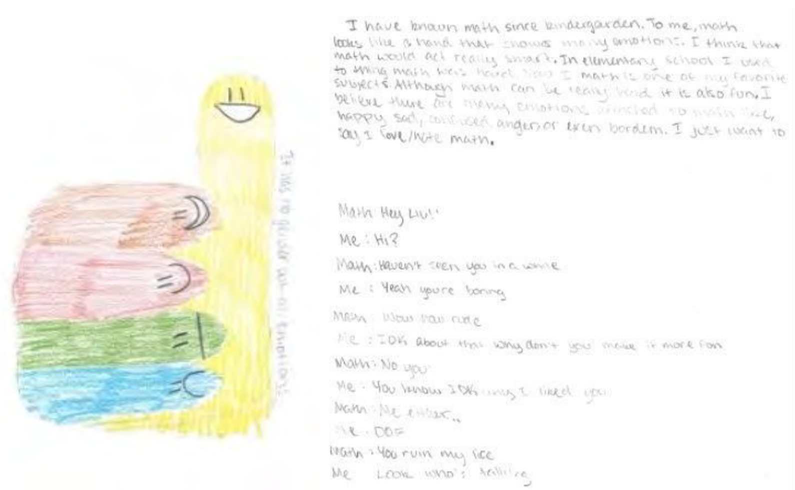


Figure 7: Former Friend Personification

The mathematical cartoons. We also added the mathematical cartoon personification to the Zazkis and Mamolo (2016) framework. Our students personified mathematics as cartoon mathematical symbols or objects, sometimes with no transcript or unemotional transcript. We struggled with determining what type of mathematical relationship was depicted. Although the drawing on the left in Figure 8 describes math as nice and funny, the cartoon depicted does not show much more emotion. The figure on the right says that "math likes to annoy kids," which is slightly negative, but the drawing itself is neutral. For the mathematical cartoon personifications, the drawings looked neutral, but if text was included, there was more insight the relationship with mathematics.

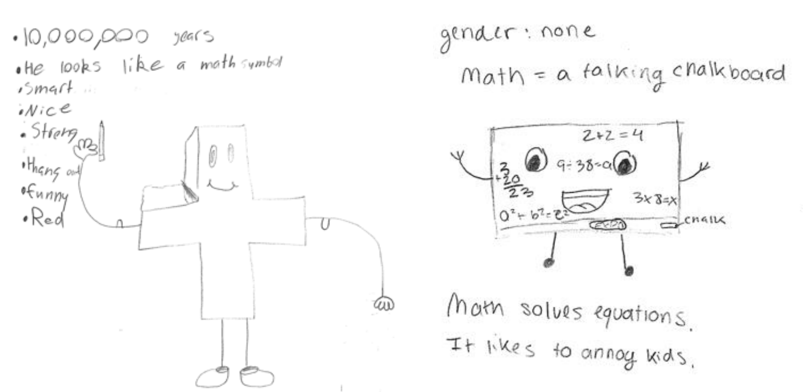


Figure 8: Cartoon/ Personification

Discussion

Unpacking middle schooler's personifications of mathematics is one aspect that makes this study significant. Prior studies on personifications of mathematics have been done with preservice teachers (Zazkis, 2015; Zazkis & Mamolo, 2016). This study extends this work to middle school students, which is important for understanding relationships with mathematics. Further, we describe additional personifications of mathematics (e.g., pests, nature).

Positive relationships in mathematics can lead to students pursuing careers in STEM. We want all students to have a positive relationship with mathematics and to see themselves as young mathematicians, especially those that have been traditionally underrepresented in STEM (Wickstrom & Gregson, 2017). Understanding and describing mathematical relationships helps researchers and educators support more positive relationships with mathematics. In this study, positive relationships were illustrated in personifications of mathematics as bests friends and in nature. The nature personification highlights the beauty and complexity of mathematics. The best friend personification illustrates someone that you spend extended, enjoyable time with. When we think about those personification, in what ways do we support our students in befriending mathematics or seeing the nature in mathematics?

Similarly, understanding the depth of negative relationships in the person is also important for supporting transformation in mathematical experiences of our students. If we understand what is negative about the relationship, then researchers and educators can work together towards specific changes. Negative relationships in this study were personified as mathematics as terrible beasts, authoritarians, abusers, and pest/nuisances. In what ways do we perpetuate these personifications in school mathematics?

Acknowledgments

We would like to thank the Paul K. Richter Memorial Fund and the Evalyn E.C. Richter Memorial Fund, distributed by the Bank of America, for funding this work.

References

Baker, K., Jessup, N. A., Jacobs, V. R., Empson, S. B., & Case, J. (2020). Productive struggle in action. *Mathematics Teacher: Learning & Teaching, PK–12*, 361–367.

Black, L., Mendick, H., & Solomon, Y. (2009). *Mathematical relationships in education: Identities and participation*. New York: Routledge.

Boaler, J. (2002). Paying the price for "sugar and spice": Shifting the analytical lens in equity research. *Mathematical Thinking and Learning, 4*(2–3), 127–144.

Braun, V., & Clarke, V. (2020). Thematic analysis. In H. Cooper (Ed.), *APA Handbook of Research Methods in Psychology* (Vol. 2., pp. 57–71). American Physiological Association.

Hunter, R., & Anthony, G. (2011). Forging mathematical relationships in inquiry-based classrooms with Pasifika students. *Journal of Urban Mathematics Education, 4*(1), 98–119.

Langer-Osuna, J. M., & Esmonde, I., (2017). Identify research in mathematics education. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 637–648).

Martin, D. B. (2000). *Mathematics success and failure among African-American youth: The roles of sociohistorical context, community forces, school influence, and individual agency.* Mahwah, NJ: Lawrence Erlbaum.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2018). *Catalyzing change in high school mathematics.* Reston, VA: Author.

National Academies of Sciences, Engineering, and Medicine. (2016). *Barriers and opportunities for 2-Year and 4-Year STEM degrees: Systemic change to support diverse student pathways.* Committee on Barriers and Opportunities in Completing 2-Year and 4-Year STEM Degrees. Washington, DC: The National Academies Press. doi: 10.17226/21739.

Parks, A. N. (2020). Creating joy in PK-Grade 2 mathematics classrooms. Mathematics Teacher: Learning and Teaching PK-12, 113(1), 61–65.

Picker, S. H., & Berry, J. S. (2000). Investigating pupils' images of mathematicians. Educational Studies in Mathematics, 43(1), 65–94.

Ruef, J. (2020). Visions of possible: Using drawings to elicit and support visions of teaching mathematics. Mathematics Teacher Educator, 8(2), 59–80.

Smith, M. S., Huinker, D., Bill, V. (2017). Taking action: Implementing effective mathematics teaching practices (K–5).Reston, VA: National Council of Teachers of Mathematics.

Steele, C. (2011). Whistling Vivaldi: How stereotypes affect us and what we can do. W. W. Norton & Company.

Stinson, D. W. (2004). Mathematics as a "gatekeeper" (?): Three theoretical perspectives that aim toward empowering all children with a key to the gate. The Mathematics Educator, 14(1), 8–18.

Wickstrom, M. H., & Gregson, S. (2017). Responding to inequities in mathematics education: Opening spaces for dialogue. Journal of Urban Mathematics Education, 10(1), 16–31.

Zazkis, D. (2015). Monsters, lovers, and former friends: Exploring relationships with mathematics via personification. *For the Learning of Mathematics, 35*(1), 33–38.

Zazkis, D., & Mamolo, A. (2016). Personification as a lens into relationships with mathematics. *Proceedings of the Conference for Research in Undergraduate Mathematics Education*. Pittsburgh, PA.

"I'M ONLY SIX.": EXPLORING YOUNG CHILDREN'S MATHEMATICS IDENTITY DEVELOPMENT

Mari Altshuler
University of Illinois Urbana-Champaign
mshuler@illinois.edu

Keywords: Affect, Emotions, Beliefs, and Attitudes, Early Childhood Education, Elementary School Education

When Reese and I had our first conversation over Zoom, she was just a few days from finishing Kindergarten, and she knew she had a lot of learning ahead of her. I asked if she expected to keep learning math as she got older, and she said, "I'm still learning math. I'm only six." Reese's reminder of her age was apropos because the literature exploring mathematics identity has focused on older students. Though previous research has revealed mechanisms that lead to negative relationships with mathematics for middle and high school students, including narrow pedagogies (e.g. Boaler & Greeno, 2000) and cultural and gender biases (e.g. Shah, 2017; Langer-Osuna, 2011), there is little related research with young children.

In this study, I looked at patterns in the literature that illustrate relationships between classroom features and students' mathematics identities, and I explored the applicability of these patterns with children in early elementary school. Specifically, I asked, what dimensions of young students' relationships with mathematics are relevant to understanding their emerging mathematics identities? And, what features of mathematics learning environments are supportive of young students forming positive relationships with the discipline of mathematics? The data for this study come from a larger project that took place between the spring and fall of 2020 with 30 young students and their caregivers from the Chicagoland area. The students began the study in either kindergarten, first, or second grade, and I conducted three interviews over Zoom with each student. Our conversations focused on students' experiences learning mathematics and their feelings towards mathematics within and outside of school, now and into the future.

Through qualitative analysis of the student interview transcripts, I compared this study's data to the mathematics identity development patterns identified in the work of Boaler & Greeno (2000), Cobb et al. (2009), and Nasir & Hand (2008). Together, their work explored mathematics identity from a variety of angles and highlighted a range of important and overlapping classroom features that impact students' developing mathematics identities including students' access to meaningful learning, the types of agency students experience, how authority in the classroom is allocated, and whether students have opportunities to express themselves. I found that whether students felt successful and whether students saw mathematics as a part of their futures were not suitable metrics for understanding young students' emerging mathematics identities. Rather, it was more meaningful to look closely at the combination of whether students enjoyed mathematics and whether they knowingly engaged in mathematics in their own time outside of school as a way to understand their relationships with the discipline. In addition, though experiencing classroom environments as collaborative and being granted access to mathematics as a meaningful and expansive domain have been documented to support older students' positive mathematics identities, for young learners those conditions were either less common or more difficult to perceive or articulate. In contrast, whether students felt that they had conceptual agency in their mathematics learning experiences and whether they perceived opportunities for self-expression appeared impactful across the Kindergarten through second grade participants in this study. This has implications for mathematics identity research and for classroom educators looking to ensure that more of their young students form positive relationships with mathematics.

References

Boaler, J. & Greeno, J. G. (2000). Identity, agency, and knowing. In J. Boaler (Ed.) *Multiple perspectives on mathematics teaching and learning,* (pp. 171-200). Ablex Publishing.

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students develop in mathematics classrooms. *Journal for Research in Mathematics Education, 40*(1), 40-68.

Langer-Osuna, J. M. (2011). How Briana became bossy and Kofi came out smart: Understanding the trajectories of identity and engagement for two group leaders in a project-based mathematics classroom. *Canadian Journal of Science, Mathematics and Technology Education, 11*(3), 207-225.

Nasir, N. I. S. & Hand, V. (2008). From the court to the classroom: Opportunities for engagement, learning, and identity in basketball and classroom mathematics. *The Journal of the Learning Sciences,* 17(2), 143-179.

Shah, N. (2017). Race, ideology, and academic ability: A relational analysis of racial narratives in mathematics. *Teachers College Record, 119*(7), 1-42.

IMPACT OF COVID INSTRUCTIONAL SHIFTS ON STUDENTS' PERCEPTION OF THEMSELVES AS MATHEMATICIANS

Emily Belford
West Chester University
eb9112900@wcupa.edu

Brian Bowen
West Chester University
bbowen@wcupa.edu

Carley Castura
West Chester University
cc902179@wcupa.edu

Amanda Griffith
West Chester University
ag913970@wcupa.edu

Briar Hines
West Chester University
bh940515@wcupa.edu

Ainsley Macrone
West Chester University
am918828@wcupa.edu

Paige Smith
West Chester University
ps943019@wcupa.edu

Christina Whalen
West Chester University
cw906124@wcupa.edu

Keywords: Elementary School Education, Middle School Education, Distance Education

There is no doubt that the shifts to virtual and hybrid models of instruction due to the pandemic have impacted learning for k-12 students. Many of the impacts of these changes are measurable through quantitative data such as reading levels or assessments scores (Dorn et. al, 2020). Other impacts are not as easily observed through traditional school measures. For example, the psychological impact of learning in isolation, may become evident in concerns related to mental and emotional health (Hertz & Barrios, 2021). Using similar periods of student separation from school such as summer break (Kuhfield & Tarasawa, 2020), research can begin to give some sense of how to best serve students post pandemic. However, there is no true comparison to draw from that accurately compares to what students are currently experiencing. To be best prepared to address student needs, we would be well served to better understand how students are currently reacting to their pandemic educational experience.

In this study we explore the way in which pandemic related educational changes have impacted how students perceive themselves as mathematicians. The lens used to describe approach to doing mathematics is observable actions that indicate either a positive, neutral, or negative perception of themselves as doers of mathematics. We are utilizing the observations of practicing veteran teachers as our primary lens of observation of student behavior. We interviewed 14 teachers, seven 3rd grade and seven 6th grades to be able to compare any similarities and differences observed between elementary and middle grades students. Results from the study will provide insight into the ways in which students perceptions may have been impacted from pandemic shifts in instruction, but also suggest ways in which teachers' expectations have shifted as well.

References

Dorn, E., Hancock, B., Sarakatsannis, J., & Viruleg, E. (2020). COVID-19 and student learning in the United States: The hurt could last a lifetime. *McKinsey & Company*, *1*.

Hertz, M. F., & Barrios, L. C. (2021). Adolescent mental health, COVID-19, and the value of school-community partnerships. *Injury Prevention*, *27*(1), 85-86.

Kuhfield, M., & Tarasawa, B. (2020). The COVID-19 Slide: What Summer Learning Loss Can Tell Us about the Potential Impact of School Closures on Student Academic Achievement. Brief. *NWEA*.

Legette, K. B., & Kurtz-Costes, B. (2021). Math track placement and reflected classroom appraisals are related to changes in early adolescents' math self-concept. *Educational Psychology*, *41*(5), 602-617.

LEVERAGING FAMILY STORIES TO SUPPORT THE MEANINGFUL LEARNING OF MATHEMATICS AND COMPUTATIONAL THINKING

Lynn Liao Hodge
The University of Tennessee
Lhodge4@utk.edu

Elizabeth Yoon
The University of Tennessee
elindqui@vols.utk.edu

Rebecca Layton
Knox County Schools
rebeccalayton@gmail.com

Nick Kim
The University of Tennessee
Nkim2@utk.edu

Shande King
Trevecca University
Nking2@trevecca.edu

Keywords: Integrated STEAM, Computational Thinking, Culturally Relevant Pedagogy

This poster focuses on a project that sought to build on family stories to support children's integrated learning of mathematics and computational thinking. The research conducted analyzed observational notes of the family workshop sessions, student presentations, and families' dioramas in order to identify and understand opportunities to learn about mathematics and computational learning that were co-constructed in the informal learning space. Analyses indicate that the family stories provided a purpose for learning how to code and for unpacking the code of others. Further, the family stories integrated with coding provided a hybrid space that supported community and voice while learning about mathematics and computational thinking.

Family Stories as Starting Points for Integrated Learning

In this project, we sought to answer the question of how to make mathematics and computational thinking meaningful to children and their families. We worked with families as part of a six-session coding workshop, initially conducted in a face-to-face format and then transitioning to a virtual format. Sessions 1 and 2 focused on an introduction to coding through the use of hands-on activities and picture that helped to unpack common coding terms (e.g. algorithm, loop, etc.). Sessions 3 and 4 introduced Scratch to families, and sessions 5 and 6 focused on families coding and adding robotics to their dioramas which featured a scene from their stories. Our approach was inspired by the *Techtales* curriculum. In this analysis, we draw on sociocultural perspectives that view learning as involving shifts in participation reflected in individuals' experiences and changes in trajectory (Nasir, 2012).

Analysis and Insights Gained

We drew on observational notes from each workshop session, the written artifacts, pictures of the dioramas, and children's presentations of their dioramas. We analyzed the notes for critical incidents in which both coding and families were discussed and examined common themes across these occurrences. Our findings indicate two themes in the critical incidents: First, when families discussed coding, they also included aspects of the story the majority of the time. Second, the families talked about the robotics features in relation to the purpose it served in telling the story. Third, the children were able to unpack and explain the code that was used. This research has implications for how we leverage family practices and identities in designing instruction.

Reference

Nasir, N. (2012). *Racialized identities: Race and achievement among African American youth*. Stanford University Press: Stanford, CA

INVESTIGATION OF MATHEMATICAL MINDSET, IDENTITY, AND SELF-EFFICACY OF COLLEGE STUDENTS

Yolanda G. Rush
Illinois State University
ygrush@ilstu.edu

Amanda L. Cullen
Illinois State University
almille@ilstu.edu

Keywords: Affect, Attitudes, and Beliefs.

Mathematical identity, mindset, and self-efficacy play a vital role in the success of students at the college level. These noncognitive components aid students in their ability to overcome setbacks (Center for Community College Student Engagement, CCCSE, 2019). The Center for Community College Student Engagement (2019) report recommends that community colleges should teach students the research behind growth mindset and improve student strategies on overcoming setbacks. Other studies have shown that students at 4-year institutions benefit from knowledge of productive mindsets especially minority students (Aronson et al., 2001; Hwang et al., 2019). Although some college students may benefit from a mindset intervention, how do we identify students who may benefit from this intervention? The purpose of the study was to determine if mathematical mindset, mathematical identity, or mathematical self-efficacy differ significantly among college students when looking at high school mathematics course history, race, gender, or 2- or 4-year institution enrollment, and which factors are helpful in identifying students who may benefit from a mindset intervention.

For this quantitative research study, a mass email was sent to students, through list serves from professors, instructors, college administrators or institution's marketing department. Over 500 college students participated in an online survey. The survey was comprised of demographic variables: gender, race, 2- or 4-year institution, and high school mathematics course history. There were three groups of questions that measured students' mathematical mindset, identity, and self-efficacy beliefs. Multiple independent sample *t*-tests were used to determine if there was a significant mean difference in student mathematical mindset score, mathematical identity score, and self-efficacy beliefs score across gender, race, and 2- or 4-year institution. A univariate ANOVA analysis was used to determine if mathematical mindset, identity, or self-efficacy differed significantly among college students when looking at high school mathematics course history (i.e., Algebra I, Geometry, and Algebra II) and grade level in which course was taken (e.g., before high school, Grades 9–12, does not apply).

The results showed: (1) significant differences in mathematical mindset, identity, and self-efficacy when considering grade level that Algebra I was taken; (2) significant differences in mathematical identity and self-efficacy when considering gender; (3) significant differences in mathematical mindset when considering race; and (4) significant differences among students' mathematical mindset when considering students who attended 2- vs. 4-year institution. Results confirm the CCCSE (2019) report recommendations that community colleges may need to teach their students the research behind growth mindset. Although the grade Algebra I was taken and current institution (i.e., 2 or 4-year) could be indicators, more research may be needed to determine additional factors important in identifying participants for a mindset intervention.

References

Aronson, J., Fried, C. B., & Good, C. (2001). Reducing the effects of stereotype threat on African American college students by shaping theories of intelligence. *Journal Experimental Social Psychology*, 1–14.

Center for Community College Student Engagement (2019). *A mind at work: Maximizing the relationship between mindset and student success*. Austin, TX: The University of Texas at Austin, College of Education, Department of Educational Leadership and Policy, Program in Higher Education Leadership.
Hwang, N., Reyes, M., & Eccles, J. S. (2019). Who holds a fixed mindset and whom does it harm in mathematics? *Youth & Society*, *51*(2), 247–267.

GLOBAL AND TASK-SPECIFIC MATHEMATICS SELF-EFFICACY AND PERFORMANCE: A QUALITATIVE STUDY IN INTERMEDIATE COLLEGE ALGEBRA

Bridgette E. Russell
Central Michigan University
buse1b@cmich.edu

Christine M. Phelps-Gregory
Central Michigan University
phelp1cm@cmich.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Algebra and Algebraic Thinking; Undergraduate Education

Mathematics self-efficacy (MSE) is defined as students' beliefs about their ability to learn and be successful in mathematics (Bandura, 1986). Self-efficacy can be measured globally (e.g., I am good at math) or locally for a specific task (e.g., I will be successful in solving this problem) (Pajares & Miller, 1995). MSE is important because it can predict future academic and career achievement, effort, and success on difficult tasks (Betz & Hackett, 1983; Hackett & Betz, 1989; Holenstein et al., 2021; Usher et al., 2018). However, sometimes there can be a mismatch between students' self-efficacy and their performance, which we will call *incongruous mathematics self-efficacy* (IMSE) (Schraw, 1995; Sheldrake, 2016). IMSE and the effect of IMSE on math performance and perseverance are not well understood.

Research Question, Design, and Analysis

This poster presents results from a pilot study in which students' global and task-specific MSE were compared to their performance on related mathematics exercises. This work adds to existing research about IMSE (Champion, 2010; Schraw, 1995; Sheldrake, 2016) by taking a qualitative look comparing students' efficacy and performance. Students (n = 24) in an intermediate algebra course were invited to take a global MSE survey. To allow for comparisons, five participants with a range of global MSE (low, average, and high) were then selected for additional interviews. As part of the interviews, participants were also asked about task-specific MSE and were asked to solve three exercises, two purely algebraic and one word problem, chosen directly from course material. Analysis focused on participants' global and task-specific MSE as well as comparing these to their performance on the exercises. A future full study will seek to build on these results by using a larger sample to examine students' global and task-specific self-efficacy in relation to their course performance.

Findings and Implications

Results show participants' global MSE was aligned with their performance on the three mathematical exercises. In particular, students with higher global MSE solved more problems successfully than students with lower global MSE. This aligns with previous research suggesting MSE can predict academic success on difficult tasks (Hackett & Betz, 1989; Holenstein et al., 2021). However, instances of incongruous task-specific MSE were identified for four of the five participants. For example, on the word problem, Gabrielle rated the problem as difficult and expressed low confidence in her work but was one of only two participants who solved the problem correctly. Also, interestingly, instances of task-specific under-confidence were identified in students with high global MSE and instances of task-specific over-confidence were identified in students with average or low MSE, building on the work of Champion (2010). Overall, these results, with richer qualitative data we will share in the poster, suggest that the relationship between MSE and performance is complex.

References

Bandura, A. (1986). Self-Efficacy. In *Social foundation of thought and action: A social-cognitive view* (pp. 290-453). *Englewood Cliffs*.

Betz, N. E., & Hackett, G. (1983). The relationship of mathematics self-efficacy expectations to the selection of science based college majors. *Journal of Vocational Behavior*, *23*(3), 329 – 345.

Champion, J. K. (2010). *The mathematics self-efficacy and calibration of students in a secondary mathematics teacher preparation program*. University of Northern Colorado.

Hackett, G., & Betz, N. E. (1989). An Exploration of the Mathematics Self-Efficacy/Mathematics Performance Correspondence. *Journal for Research in Mathematics Education*, *20*(3), 261 – 273.

Holenstein, M., Bruckmaier, G., & Grob, A. (2021). How do self-efficacy and self-concept impact mathematical achievement? The case of mathematical modelling. *British Journal of Educational Psychology*, e12443.

Pajares, F., & Miller, M. D. (1995). Mathematics self-efficacy and mathematics performances: The need for specificity of assessment. *Journal of counseling psychology*, *42*(2), 190 – 198. .

Schraw, G. (1995). Measures of Feeling-of-knowing Accuracy: A new look at an old problem. *Applied Cognitive Psychology*, 9, 321 – 332.

Sheldrake, R. (2016). Differential predictors of under-confidence and over-confidence for mathematics and science students in England. *Learning and Individual Differences*, *49*, 305 – 313.

Usher, E. L., Li, C. R., Butz, A. R., & Rojas, J. P. (2018). Perseverant grit and self-efficacy: Are both essential for children's academic success? *Journal of Educational Psychology*, *111*(5), 877 – 902.

"I AM PETRIFIED. EVERYONE SEEMS SO MUCH MORE KNOWLEDGEABLE THAN ME." – UNDERGRADUATE REASEARCH PERSPECTIVES

James C. Willingham
James Madison University
willinjc@jmu.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity, Inclusion, and Diversity; Informal Education; Undergraduate Education

The National Science Foundation's Research Experience for Undergraduates (REU) is a staple effort to increase the participation of underrepresented groups in advanced math, science, and engineering careers (Beninson, Koski, Villa, Faram, & O'Connor, 2011), expending close to $90 million yearly to support these efforts (Singer, 2019). However, many REU participants experience dissonance, as despite their high achievement and selection to competitive REUs, they lack an internal sense of success or belonging. This *imposter phenomenon* (Clance & Imes, 1978) is notoriously detrimental among women, and underrepresented racial, ethnic, and religious minorities (Chrousos & Mentis, 2020). With this background in mind, I consider how participants in a mathematics REU describe their experiences and the aspects of these programs that they find to best support their success.

The conceptual framing of this study is based on considering the value created by participants in the networks and communities (Wenger, Trayner, & de Laat, 2011) in which they experience situated learning (Lave & Wenger, 1991). In this context, a *network* is viewed as the relationships, interactions, and connections among the members of a *community* which is formed from a shared identity around a common set of challenges. As the primary recipients of value in this system are the participants, the emphasis of this framing is exploring the personal and shared narratives of this group in terms of their accounts of what has occurred and their common goals for the group.

To consider these narratives, I employed an embedded single-case design (Yin, 2014) to follow the experiences of five mathematics-REU participants across two eight-week research projects at a large mid-Atlantic university. I performed a simple time series analysis (Yin, 2014) across these eight weeks, codifying the participants experiences and aspirations through multiple interviews, weekly journals, and email exchanges.

The preliminary results of my analysis reveal a common anxiety among the participants, well characterized by one young woman's initial concern that, "Honestly, I am petrified. Everyone seems so much more knowledgeable than me." However, over the course of the project, she and other participants provide rich details of the specific activities that led them to individual and group success. This transition to more productive dispositions occurred through opportunities for sustained and substantial individual contributions to the groups' research, personal interactions with the faculty and other students involved in the project, and the recognition that, "Sometimes you don't know what exactly you are doing, you second guess yourself and don't feel confident. You have to be ok with asking for help and conversing with your peers so you understand."

References

Beninson, L.A., Koski, J., Villa, E., Faram, R., and O'Connor, S.E. (2011). Evaluation of the research experiences for undergraduates (REU) sites program. *Council on Undergraduate Research Quarterly 32*, 43-48.

Chrousos, G. P., & Mentis, A. F. A. (2020). Imposter syndrome threatens diversity. *Science*, *367*, 749-750.

Clance, P. R., & Imes, S. A. (1978). The imposter phenomenon in high achieving women: Dynamics and therapeutic intervention. *Psychotherapy: Theory, research & practice*, *15*(3), 241-247.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge, UK: Cambridge University Press.

Singer, S. R. (2019). Six decades of the National Science Foundation's commitment to undergraduate research. *Scholarship and Practice of Undergraduate Research*, *3*, 16-20.

Wenger, E., Trayner, B., & de Laat, M. (2011). Promoting and assessing value creation in communities and networks: A conceptual framework.

Yin, R. K. (2014). Case study research: Design and methods (5th ed.). Thousand Oaks, CA: Sage Publications.

Chapter 15:

Teaching Practice and Classroom Activity

THE ROLE OF A BOUNDARY OBJECT IN A STUDY OF MIDDLE GRADES MATHEMATICS INSTRUCTION

Angela R. Crawford
Boise State University
angelacrawford1@boisestate.edu

Michele Carney
Boise State University
michelecarney@boisestate.edu

Joe Champion
Boise State University
joechampion@boisestate.edu

Megan Schmidt
Boise State University
meganschmidt@boisestate.edu

This report is about how a group of U.S. teachers (N =98) and researchers used a boundary object in a collective study of grades 6-8 mathematics instruction. The focus is the teachers' engagement with a framework for effective instructional practices. Using qualitative content analysis of teachers' responses to use of the framework, we assessed its implementability and usefulness while supporting shared understandings of effective mathematics instruction. We found the framework supported shared understanding and implementation across varied contexts and was viewed as useful for meeting teachers' instructional and professional goals. Constraints were related to ambiguity around understanding and use of instructional practices related to student struggle. These findings demonstrate how a brief researcher-designed framework can serve as a bridge between teachers and researchers, meeting the professional goals of both.

Keywords: Middle School Education, Professional Development, Instructional Activities and Practices.

Our research is set in U.S. public middle schools, places where dissonance is a daily part of life (especially in band classrooms). In mathematics classrooms, students and teachers need to blend their often wildly different interests and goals to make the shared time together meaningful and mutually enriching. Nurturing harmony in these classrooms can be challenging, but it is a necessary condition for thriving mathematics education. Meanwhile, as researchers of effective mathematics instruction in the middle grades, we face an additional layer of dissonance. We come from a research culture and aim to build transferable knowledge of "what works", while our collaborating teachers are mostly focused on tuning their individual classroom practice to meet the needs of their students and schools. This study is about one way we have tried to orchestrate harmony in that context.

We are conducting a large-scale grant-funded study of sequences of mathematics instructional strategies within a teacher-researcher partnership (Kieran et al., 2012; Koichu & Pinto, 2018). Our project hinges on a guiding framework that effectively supports shared understandings, implementability, and motivation. While we aim to help teachers understand and use the framework, we take an explicitly asset-based approach to the work. As researchers, we know about effective instruction as described in the literature, and our collaborating teachers know about effectively working with students in their contexts. These ways of knowing about mathematics instruction introduce a "boundary" (Wenger, 1998), a useful analytic concept for studying teacher-researcher partnerships because it reflects the dissonances in our perspectives and goals while describing how learning may take place across the boundary (Akkerman & Bakker, 2011). The tool that has served as a boundary object (Star, 2010) for bridging our ways of knowing about mathematics instruction is a framework.

Theoretical Framework

EAC/SOS framework

We see our framework as the centerpiece of our work to learn alongside teachers about what works in their contexts. Based on decades of research addressing effective mathematics instruction, the Explicit Attention to Concepts (EAC) and Students Opportunity to Struggle (SOS) constructs were described by Hiebert and Grouws (2007) as primary clusters of instructional practices with robust evidence for supporting students' conceptual understanding of mathematics. Stein et al. (2017) subsequently operationalized the constructs, finding further evidence of the practices for supporting high student achievement on assessments of both procedural and conceptual knowledge in mathematics.

We developed a two-page practice guide for teachers to support the enactment of EAC and SOS in their classrooms (see Figure 1; Champion et al., 2020). The artifact presents three *features* for each construct. The features that characterize EAC instruction are (a) a focus on concepts, (b) making concepts explicit and public, and (c) emphasizing connections. The features that characterize SOS instruction are (a) a focus on sense-making, (b) application of sustained mental effort, and (c) engagement with important mathematics. The *constructs* and *features* are meant to broadly describe effective practice, but the guide also includes examples at a finer grain size. Each construct lists four instructional *strategies* with two accompanying *routines*, which more explicitly detail what enactment could look like in the classroom. Lastly, each strategy is annotated with potential instructional *tools* that teachers could use to support the enactment.

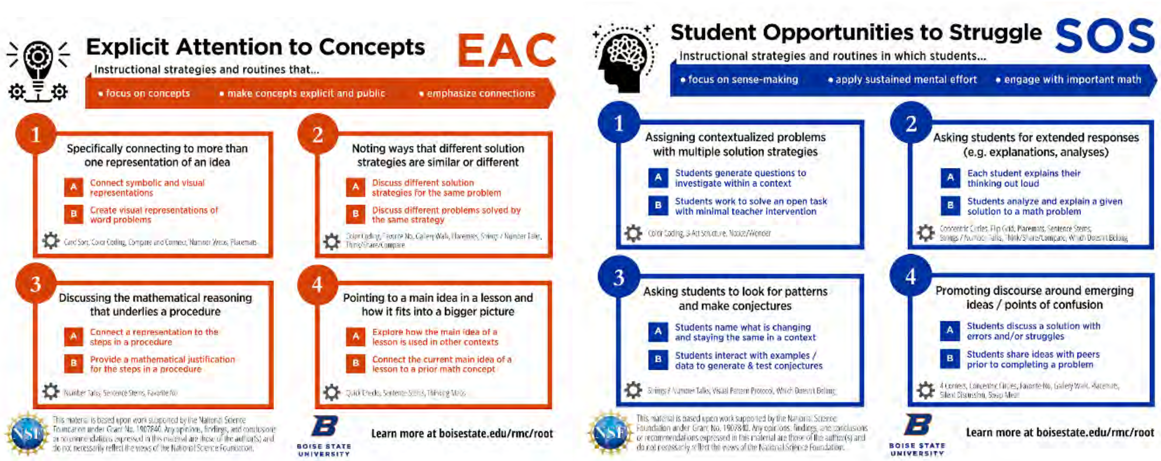


Figure 1: EAC/SOS Guide to Instructional Practices for Improving Math Achievement

EAC/SOS Framework as a Boundary Object

Our project is a boundary encounter between two communities of practice, mathematics education researchers and teachers, who are working toward a common goal of building useful knowledge for both communities (Wenger, 1998). Within boundary encounters, boundary objects are "things" (either physical or conceptual) that act as a bridge between communities (Star, 2010). As a bridge, a boundary object should be accessible to multiple perspectives, allowing shared meanings to be negotiated (Akkerman & Bakker, 2011; Star, 2010). In an investigation of a boundary object in teacher professional development, Edgington et al. (2016) found their framework was used by teachers in both expected and unexpected ways, suggesting opportunities for researcher learning through investigations of teachers' meanings around the boundary object. As researchers, our goal was to communicate research at a boundary while valuing collaborating teachers' expertise, so we identified teachers' interpretations of the boundary object as pivotal. Our guiding research question was, "To what extent does the EAC/SOS framework function well as a boundary object with middle school mathematics teachers, supporting shared understandings of effective mathematics instruction while being adaptable in teachers' diverse contexts?"

Method

Our study centers collaborating teachers' perspectives of the EAC/SOS framework. We used qualitative content analysis, which involves interpreting meaning from text, to understand teachers' feedback on their engagement with the framework across the first year of our partnership.

Participants

The participants in this study are Grades 6-8 mathematics teachers from 34 schools within 22 school districts in the western United States ($N = 98$). All teachers were working in public schools. Nearly all worked in brick-and-mortar schools, though one teacher worked for a virtual public charter school. Teachers' mathematics instruction often spanned multiple grades (49 taught Grade 6, 44 taught Grade 7, and 44 taught Grade 8) and courses (37 taught one course, 48 taught two courses, and 12 taught three or more courses). The teachers worked in a variety of school settings, both in terms of students' socio-economic status (mean eligibility for federal free or reduced school lunch was 58%, $SD = 21\%$) and locale type (31% rural, 69% suburban or small city). Teacher demographics indicated substantial variability in mathematics teaching experience (mean = 9.8 years, $SD = 7.4$, Range = 1 to 32), and they primarily self-identified as female (77%) and white (96%). Teachers' highest academic degree was typically a bachelor's degree (57%), though 40% held a master's degree, and 2% held an Ed.S.

EAC/SOS Guide & Professional Development Modules

The framework was enacted by inviting teachers to engage with the two-page guide and three supporting professional development modules. The first two modules were in-person and synchronous and, due to the COVID-19 pandemic, a third module was online and asynchronous. The focus of Module 1 was an orientation to the research project and the EAC/SOS framework. Modules 2 and 3 focused more deeply on EAC and SOS strategies, respectively. The in-person meetings gathered all teachers together for 4 hours on Saturday mornings. The asynchronous modules integrated online tools in the form of virtual bulletin boards, an interactive mathematics activity builder, and a video-based interaction platform.

Data Sources

We used extant data that was collected across the first year of the project to inform decisions about upcoming modules or engage teachers in exploring ideas (Table 1). These data were collected via graphic organizers and online forms. Questions asked of teachers varied each time, depending upon what information was desired to inform upcoming professional development sessions.

Data Analysis

We applied a three-level analysis (Simon, 2019). At the first level, we worked with the teachers' responses as raw data. We tabulated frequency counts for strategy and routine choices and coded textual data, guided by the research purpose and using constant comparison (Corbin & Strauss, 2008). To verify the trustworthiness of our coding at this level of analysis, we checked the comparability of coding within Data Source 3 (DS3, see Table 1). Two researchers applied open coding to half of the responses, and the team then discussed the codes and established a coding dictionary. To check for reliability of coding, two researchers used the coding dictionary and coded data in DS5. We compared the frequency with which codes were applied to responses; average exact agreement across codes was 68% and within 2 agreement was 91%. Disagreements were reconciled, and definitions were refined. Using this refined coding scheme, a single researcher conducted the remaining analyses and summarized each data source. These summaries described the context of the data (what the teachers were responding to and why), provided tables with frequencies of applied codes, and exemplar responses for frequent codes.

The second level of analysis identified themes across the summaries with reference to the original data as needed. Two researchers engaged in this process independently, and the full research team then compared the themes, discussing similarities and differences. A third researcher then compiled the results of the discussion into a final set of claims related to the research purpose. In the third level of analysis, we applied abductive reasoning to the second level claims to identify inferences which, if true, would account for the data (Simon, 2019). The goal is to consider the claims with the intention of contributing useful postulations that apply beyond the scope of this particular study. These inferences therefore inform our understanding of the success of the framework as a boundary object in a teacher-research partnership.

Table 1: Summary of Data Sources

Data Source ID	Module	Data Source	Content	# of Responses
DS1	1	Comfort-Frequency Notes	Compare comfort-frequency of strategy implementation to a peer	85
DS2	1	Implementation Cycle 1 Plans	Choose a strategy to try out in classroom and explain why it was chosen	94
DS3	1	Implementation Cycle 1 Reflection	Indicate what strategy was tried and what went well and what was difficult.	98
DS4	2	Implementation Cycle 2 Plans	Choose an EAC strategy to try out in classroom and explain why it was chosen	94
DS5	2	Implementation Cycle 2 Reflection	Indicate what EAC strategy was tried and reflect on how it went	93
DS6	3	SOS Impressions	Read a description of SOS, comment on using SOS in classrooms	89
DS7	3	SOS Strategy Identification	Watch a video, identify strategies and teacher moves to support struggle	97
DS8	3	Implementation Cycle 3 Plans	Choose and SOS strategy to try out in classroom and explain why it was chosen	83

Findings

This study investigated teachers' responses to determine the extent to which the framework supports a shared understanding of research-supported instructional practices and the extent to which teachers view the framework as implementable and useful.

Understandings about EAC and SOS

We looked for evidence of teachers' understandings of the framework across their reasons for selecting strategies. These data sources were coded for degree of alignment with our understandings as researchers. Coding categories were: aligned (described key features of the construct), partially aligned (described strategy without language related to key features), unaligned (confused the constructs or added unintended ideas), or unclear (e.g., "it fits my curriculum"). Figure 2 shows the respective percentage of responses in each category across the three modules. The majority of responses, 70% and above, were aligned or partially aligned. We believe this degree of alignment early in our project indicates success of the framework for supporting shared understandings. However, we note that the degree to which shared understandings were established was stronger for EAC than SOS. This discrepancy is also evident from DS7 in which we asked teachers to watch and respond to videos of a teacher implementing SOS strategies, from which only 39% of responses were coded as aligned, while 59% of responses were coded as partially aligned.

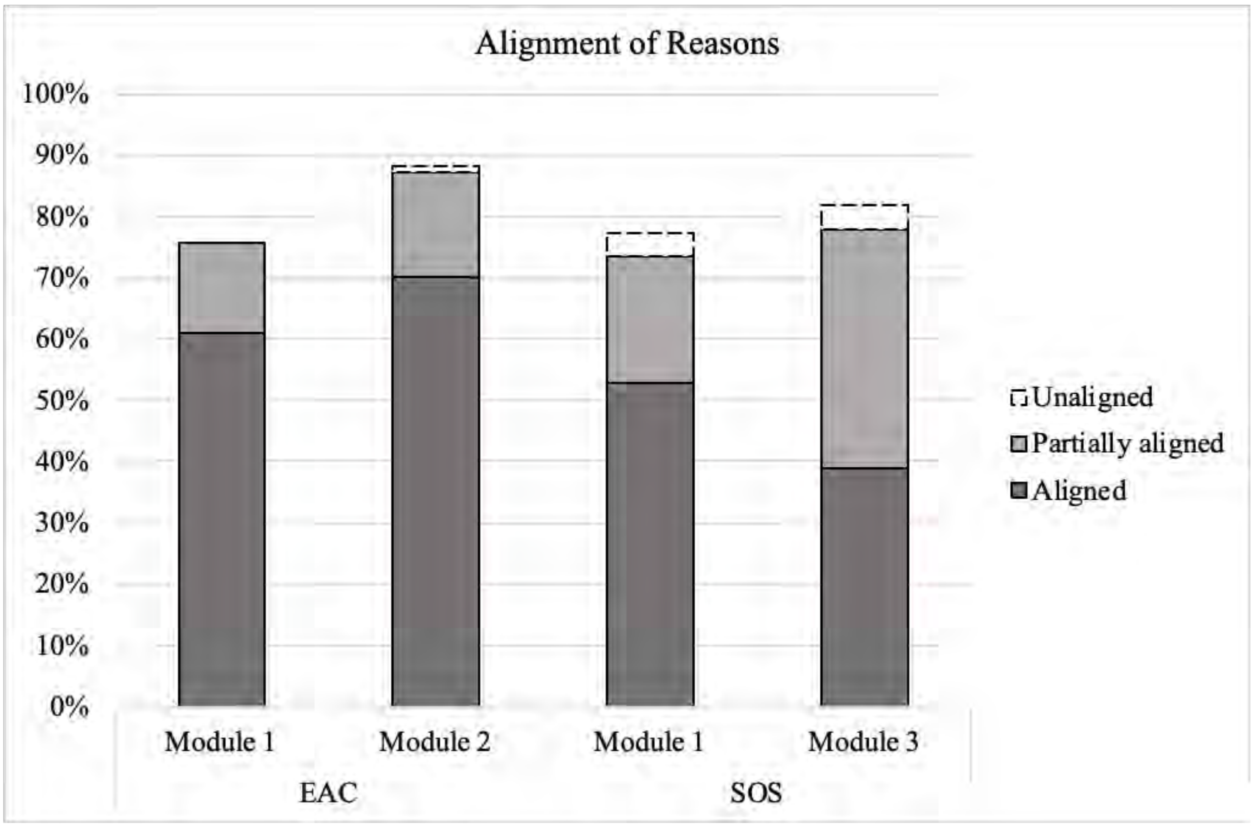


Figure 2: Alignment of Reasons for Strategy Choice to Researchers' Intended Meanings

Teaching Context

We wanted to determine to what extent the framework was viewed by teachers as implementable in their contexts. We found the framework offered options that were taken up by teachers across these contexts, and we found teachers preferred different strategies based on their experience, interests, and school or district context. Teachers' reasons for selecting strategies (Figure 3) indicated contextual factors such as the needs of their students, aligning with curriculum or learning goals, and teachers' experience or interests. When we asked teachers to rate EAC and SOS based on their comfort and frequency of use, and then comment on similarities and differences to the ratings of another teacher (DS1), 62% of comments related specifically to choices based on teaching context, including curriculum, working environment, years of experience, and teachers' individual personalities.

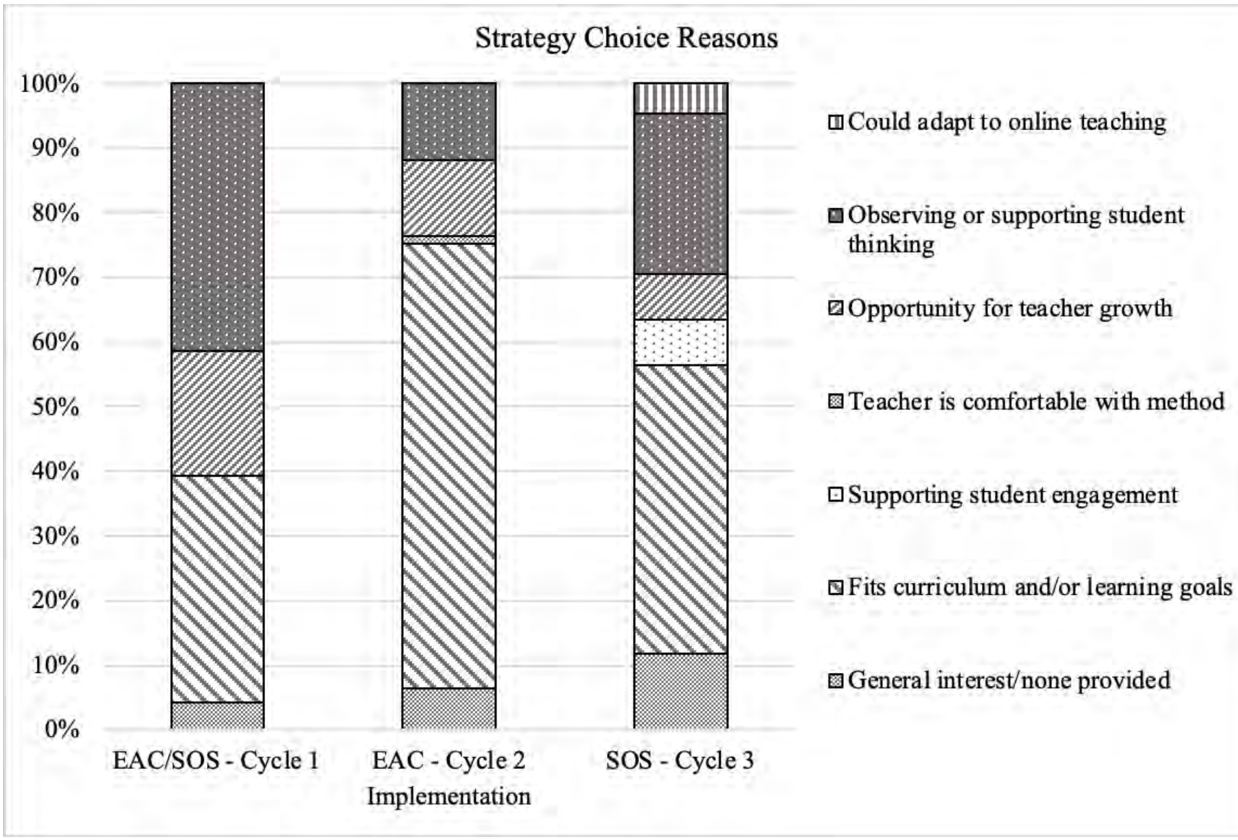


Figure 3: Themes Identified in Reasons for Strategy Choice

Despite evidence that teachers identified adaptable options in the framework, some data indicated a perception that SOS may not be successful with some students. After implementation cycle 1 (DS3), teachers reflected on the benefits and challenges of the experience. The primary difficulty related to SOS was "student engagement" which accounted for 61% of the SOS responses. A response that suggests SOS may not be successful with some students was, "It was very difficult to engage the students who struggle in math….A lot of scaffolding and accommodating was needed." Also, DS6 asked teachers to comment on a reading about the meaning of productive struggle, and 22% of the comments focused on the challenge of using SOS with some students. A characteristic response was, "With productive struggle it never really feels like they are learning anything, especially without teachers teaching step by step. Students definitely give up when they don't understand something, so how can they productively struggle if they keep giving up?"

Opportunities for Teacher and Student Learning

In support of implementation, we hoped teachers would interpret the framework as useful and motivating for innovating their professional practice. We found teachers described its usefulness in terms of offering opportunities for teacher and student learning. We identified the theme of opportunity for learning as reasons for selecting a strategy across all three cycles of implementation (see Figure 3). An example of a response that fell under this theme is a teacher's explanation for choosing an SOS strategy, "I believe that I guide my students too much to the answer, and give them too much information, which gives away the Ah Ha moments. I want to work on allowing the students to explore more of the problem without just giving them their answers" (DS2). In addition to identifying this theme in reasons for strategy choices, we identified this theme in 20% of teachers' reflections on implementation cycle 1 (DS3) and 17% during cycle 2 (DS5).

A number of teachers interpreted the strategies as opportunities to facilitate learning by focusing on student thinking. As seen in Figure 3, we identified the theme of "observing or supporting student thinking" in 41% of reasons for selecting a strategy in the first implementation cycle, 11% in the second, and 25% in the third. An example of a response that supports this claim is related to the SOS strategy of promoting discourse among students around emerging ideas, "Want to see if it will help with further understanding the concepts amongst the kids" (DS8). We also identified this theme in teachers' reflections. "Students learned" and "students were engaged" accounted for 88% of the descriptions of what went well after implementation cycle 1 (DS3) and 37% after implementation cycle 2 (DS5).

Discussion

This study aimed to understand the EAC/SOS framework as a boundary object bridging two communities of practice. We believe that this investigation of the boundary object can offer guidance to others involved in similar projects that involve communicating across such boundaries. We found the framework, by and large, supported shared understandings. We also found that teachers could identify appealing options and meaningful benefits from the framework, indicating they viewed it as implementable and useful. Despite the indications of shared understandings and positive reception of the framework, there are some challenges around the meaning and implementability of SOS. One potential contributing factor for this is the possibility that teachers who are accustomed to 'traditional' mathematics instruction may find EAC more accessible than the more student-centered 'reform-oriented' instruction associated with SOS. Researchers working with teachers to study SOS may need to provide additional experiences and resources for implementation. Teachers' context may be important, as well as teachers' beliefs about their students' abilities to learn through productive struggle.

Implications

Based on our findings, we offer some provisional recommendations for teacher-researcher partnerships to study mathematics instruction:

- Expect some aspects of the research lens you are using to be easily understood and translated by practitioners, while it may be more difficult to form shared understanding around others.
- Use cycles of data collection to monitor teachers' perceptions and implementation results, and adjust the planned meetings between each cycle.
- Give teachers an overview of the theory and research underpinning the project, and then ask teachers what information they need to translate this to their practice (e.g., video examples, curricular materials, rehearsals).
- Be mindful of teachers' context and beliefs. Provide multiple entry points, choices for teachers with different contexts, and be aware the assumptions and expectations of teachers may or may not align to those of the researchers.

Conclusion

To achieve harmony between our goals for learning “what works” in a broad sense and teachers’ goals of fine-tuning their individual classroom practice, we investigated the EAC/SOS framework as a boundary object. We centered teachers’ voices and found evidence in their responses that the framework supported shared understandings, strong implementability, and the usefulness of the framework across varying teaching contexts. These affordances are signs that the framework provides entry points for engaging teachers in professional learning around these constructs. However, we also found constraints of the framework, particularly around SOS. The constraints indicate specific ways to better support teachers’ understanding and implementation of SOS, and the study findings indicate several ways to scaffold teacher-researcher partnerships in order to build meaningful knowledge of effective mathematics instruction.

References

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169.

Champion, J., Crawford, A. R., & Carney, M. (2021). Articulating effective middle grades instructional practices in a teacher-researcher alliance. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020-292Corbin, J. & Strauss, A. (2008). *Basics of qualitative research* (3rd ed.). Sage.

Edgington, C., Wilson, P. H., Sztajn, P., & Webb, J. (2016). Translating learning trajectories into useable tools for teachers. Mathematics Teacher Educator, 5(1), 65–80.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students’ learning. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning*, (pp. 371–404). Information Age Publishing.

Kieran, C., Krainer, K., & Shaughnessy, J. M. (2012). Linking research to practice: Teachers as key stakeholders in mathematics education research. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 361–392). Springer. https://doi.org/10.1007/978-1-4614-4684-2_12

Koichu, B., & Pinto, A. (2018). Developing education research competencies in mathematics teachers through TRAIL: Teacher-Researcher Alliance for Investigating Learning. *Canadian Journal of Science, Mathematics and Technology Education*, *18*(1), 68–85. https://doi.org/10.1007/s42330-018-0006-3

Simon, M. A. (2019). Analyzing qualitative data in mathematics education. In K. Leatham (Ed.), *Designing, conducting, and publishing quality research in mathematics education* (pp. 111–122). Springer. https://doi.org/10.1007/978-3-030-23505-5_8

Star, S. L. (2010). This is not a boundary object: Reflections on the origin of a concept. Science, Technology, and Human Values, 35(5), 601–617. https://doi.org/10.1177/0162243910377624

Stein, M. K., Correnti, R., Moore, D., Russell, J. L., & Kelly, K. (2017). Using theory and measurement to sharpen conceptualizations of mathematics teaching in the common core era. *AERA Open*, *3*(1), 1–20. https://doi.org/10.1177/2332858416680566

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity.* Cambridge University Press.

LINGUISTIC TENSIONS IN GENERALIZING A MATHEMATICS EDUCATION FRAMEWORK FOR STEM EDUCATION

James Drimalla
University of Georgia
james.drimalla@uga.edu

AnnaMarie Conner
University of Georgia
acconer@uga.edu

Jenna Menke
University of Georgia
jennamenke@uga.edu

Anna Gillespie-Schneider
University of Georgia
agschnei@uga.edu

Timothy L. Foutz
University of Georgia
tfoutz@uga.edu

Lorraine Franco
University of Georgia
lorraine.franco@uga.edu

Aida Alibek
University of Georgia
aida.alibek@uga.edu

Barbara A. Crawford
University of Georgia
barbarac@uga.edu

Shaffiq Welji
University of Georgia
welji@uga.edu

Research processes are often messy and include tensions that are unnamed in the final products. In our attempt to update and generalize a framework used to examine teachers' support for collective argumentation in mathematics education classrooms to examining teachers' work in interdisciplinary STEM contexts, we have experienced significant linguistic tensions because of the context-dependent nature of language. We aim to acknowledge the difficulty of generalizing research beyond the mathematics education community, describe our attempts to resolve the problem we face, and discuss potential conclusions pertaining to the feasibility of generalizing frameworks beyond mathematics education.

Keywords: Classroom Discourse; Integrated STEM/STEAM; Research Methods

Background

In 2019, Eric Siy asked the mathematics education community to reveal their work to cultivate a healthier and more rigorous culture of research. Siy argued that one way to expose the messiness of research was to make our processes products. In response to his call, our team has decided to openly share a present, messy tension in our research process. By disclosing our present dilemma to the community for outside opinion and critique, we aim to practice academic humility and make our process a product.

The current tension emerged out of our attempt to update and extend the Teacher Support for Collective Argumentation (TSCA) framework (Conner et al., 2014) that was originally developed to understand and describe how secondary mathematics teachers support collective argumentation. We define collective argumentation as teachers and students working together to establish a claim and provide evidence to support it. The original framework described teachers' support of collective argumentation as (a) directly providing a claim, data, or a warrant, (b) prompting a student's contribution of argument components with a question, or (c) responding to a student's contribution. These three descriptions of teacher support for collective argumentation simplify to three categories: direct contributions, questions, and other supportive actions.

Given the utility of the original framework, we (a group of mathematics education, science education, and engineering education researchers) attempted to update and extend the framework to other STEM+C settings/classrooms. Our updates include specific kinds of questions within each broader category of question. For example, Table 1 shows a few of the kinds of questions originally conceptualized. In the updated framework, we want to potentially include additional kinds of questions as well as examples from multiple disciplines.

Table 1: Working Descriptions of Selected Types of Questions in New TSCA Framework

Category of Question	Description
Requesting a factual answer: Asks students to provide a mathematical fact	*Identification*: choose from a set of options, identify something *Observation*: describe what happened or is happening.
Requesting a method: Asks students to demonstrate or describe how they did something	*Describe a method*: state how they did, would, or could do something
Requesting an idea: Asks students to compare, coordinate, or generate mathematical ideas	*Conjecture*: put forward a prediction, suggestion, or other idea

Note. This table is adapted from the original TSCA framework (Conner et al., 2014, p. 419)

However, in our attempt to update and generalize the framework for use in other contexts, we experienced significant linguistic tensions. The complexity of meaning and the context-dependent nature of language pose severe challenges for crafting the framework for use beyond mathematics education. In this brief research report, we aim to acknowledge the difficulty of generalizing research beyond the mathematics education community, describe our attempts to resolve the problem we face, and discuss potential conclusions pertaining to the feasibility of generalizing frameworks beyond mathematics education.

Theoretical Framework

Ludwig Wittgenstein, in his *Philosophical Investigations* (1953/1958), sought to dismantle theories of language that were referential, representational, and ostensive at their core. A referential view of language is where words directly refer to objects and, consequently, a word has a referential meaning. Wittgenstein wanted readers to break free from the idea that there could be a unitary account of language (Grayling, 1988). Rather than an all-encompassing theory, Wittgenstein proposed the concept of language-games. Wittgenstein's concept of language-game was meant to emphasize that language is not independent from context nor is it referencing some outer reality. Rather, Wittgenstein (1953/1958) described language-games as "meant to bring into prominence the fact that the speaking of language is part of an activity, or of a form of life" (§23). Grayling (1988), expounding on Wittgenstein's phrase "form of life," wrote,

> Language is not something complete and autonomous which can be investigated independently of other considerations, for language is woven into all human activities and behaviour, and accordingly our many different uses of it are given content and significance by our practical affairs, our work, our dealings with one another and with the world we inhabit—a language, in short, is part of the fabric of an inclusive 'form of life.' (p. 67)

Through his exploration of concepts like "language-games" and "form of life," Wittgenstein sought to reinforce his main point: the meaning of language is largely found in its *use* in humans' embodied activities and social practices (i.e., in various language-games). Hence Wittgenstein wrote, "the meaning of a word is its use in the language" (§43). Given Wittgenstein's ideas about language and meaning, we should not be surprised that the same word is used across various

STEM+C disciplines with different meanings. Furthermore, a word's meaning may not be easily disentangled from its disciplinary use and the associated embodied practices and contexts.

Methods

The data for the larger project consist of many hours of video from elementary teachers' instruction in which they integrated coding with mathematics, science, literacy, and social studies lessons. To examine the teachers' use of argumentation and support of students' arguments across disciplines, episodes of argumentation were identified, verbatim transcripts were prepared, diagrams of arguments were constructed, and teachers' actions supporting the argumentation were identified according to the methods described in Conner et al. (2014).

Our research team met for multiple hours each week to analyze and discuss our coding of teachers' supportive actions with respect to the original framework. In the summer of 2021, a member of the research team highlighted a prominent linguistic tension about the use of codes within two categories (requesting a method and requesting an idea) in the framework. Subsequently, the use of two additional words was identified as problematic. Our team regularly debated how to use the framework in light of these tensions for the subsequent months. Multiple times, we attempted to settle the debate and move forward by adopting specific meanings for the words. However, further analysis caused additional complications of meanings, and more linguistic tensions were identified. In the most recent attempt to settle various meanings, all team members were asked to submit their own working definition of four contentious terms and to describe the types of practices and activities that are associated with the words within their home discipline.

Results

Table 2 shows four debated words and their meanings (derived from combining provided definitions) within each field. Each of these words was used as a kind of question in the evolving TSCA framework.

Table 2: The Meaning of Words in Different Fields

	Science	Engineering	Mathematics	Statistics
Conjecture	Guess, prediction, or tentative solution; cannot be positively justified; distinct from hypothesis	Idea or hypothesis that is tested out	Prediction, reasoned guess, or proposal of a method for a problem [a]; do not have a proof	Alternative hypothesis or claim related to a research question
Method	Systematic approach to answering a question; process or technique used to find a solution to a problem [a]	Process or technique to transform an idea into a physical reality	Abstract sequence of actions or steps used to solve a problem [a], prove something, or achieve something	Technique to collect data; statistical investigative cycle; statistical approach to solving a problem[a]
Identification	Categorizing or classifying an object into pre-	Pointing out attributes that define	Determine/select what thing (object, drawing, etc.) has a	Not applicable

	established categories based on observable characteristics	something such as a design problem[a] or a design solution	certain characteristic	
Observation	Method of data collection through one's senses and scientific tools; necessarily an interpretation in light of a theory	Assembling facts from what one can see or notice	Statement noting something's (object, drawing, etc.) characteristics	A single measurement of a variable; recognizing trends or patterns in data or a graph

[a] The word "problem" provides another layer of difficulty; the types of problems imagined by scientists, engineers, mathematicians, and statisticians are different.

Discussion and Continuing Dilemmas

Given these words have loaded meanings that are discipline-dependent, our team identified three potential options. First, we could concede and conclude that there is no opportunity to generalize this framework across disciplines—even in related STEM disciplines. Second, we could carefully choose words that are not loaded with meaning in multiple disciplines. This option poses two difficulties: (a) words that do not have discipline-related meanings can often have colloquial meanings and (b) by attempting to avoid discipline-related meanings, we may choose a watered-down word that does not sufficiently communicate the type of question the teacher used. Finally, we could use the words that we currently have in the framework and accept that they will cause friction for some of the disciplines.

As we explored these three options, our team was pushed to re-examine the original purposes of the project. Given our aim of helping STEM teachers support collective argumentation, we believe it is worth attempting to generalize the TSCA framework, and if possible, to use words that would be understandable across disciplines. In our attempt to pursue the second option, we concluded that, by replacing the word "Observation" with the word "Description," we could communicate our intended meaning and avoid discipline-specific meanings. However, we have not come to a consensus on "Conjecture," "Method," and "Identification."

This constructive work across disciplines required researchers to adapt and adopt new meanings for words—meanings that can cause friction with the discipline-specific meanings. Addressing this dilemma has required group members to intentionally learn more about the meanings and uses of words across disciplines, highlighting the epistemic and linguistic differences between related disciplines. This calls into question the ease with which teachers can engage in integrative STEM instruction and suggests the need for intentionality in building theory across disciplines.

Acknowledgments

This paper is based on work supported by the National Science Foundation under Grant No. 1741910. Opinions, findings, and conclusions in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Conner, A., Singletary, L.M., Smith, R.C., Wagner, P.A., & Francisco, R.T. (2014) Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401–429. https://doi.org/10.1007/s10649-014-9532-8

Grayling, A.C. (1988). *Wittgenstein*. Oxford University Press.

Siy, E. (2019). Creating a healthy and rigorous culture of research by revealing our work. In: Otten, S. Candela, A.G., de Araujo, Z., Haines, C., & Munter, C. (2019). *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 16–21). St Louis, MO: University of Missouri.

Wittgenstein, L. (1958). *Philosophical investigations* (G.E.M. Anscombe, Trans.; 2nd ed.). Basil Blackwell. (Original work published 1953)

USING PUBLIC RECORDS TO SUPPORT THE PRODUCTIVE USE OF STUDENT MATHEMATICAL THINKING

Ben Freeburn
Western Michigan University
byf5045@gmail.com

Keith R. Leatham
Brigham Young University
kleatham@mathed.byu.edu

Sini Graff
Brigham Young University
sinigraff21@gmail.com

Nitchada Kamlue
Western Michigan University
nitchada.kamlue@wmich.edu

Shari L. Stockero
Michigan Tech University
stockero@mtu.edu

Blake E. Peterson
Brigham Young University
peterson@mathed.byu.edu

Laura R. Van Zoest
Western Michigan University
laura.vanzoest@wmich.edu

The more researchers understand the subtleties of teaching practices that productively use student thinking, the better we can support teachers to develop these teaching practices. In this paper, we report the results of an exploration into how secondary mathematics teachers' use of public records appeared to support or inhibit their efforts to conduct a sense-making discussion around a particular student contribution. We use cognitive load theory to frame two broad ways teachers used public records—manipulating and referencing—to support establishing and maintaining students' thinking as objects in sense-making discussions.

Keywords: Classroom Discourse; Instructional Activities and Practices.

Recommendations for mathematics teaching address the importance of using student thinking during whole class discussions as well as propose teaching practices that one may use to prepare for and facilitate whole class discussions (e.g., National Council of Mathematics Teachers, 2014). In our work, we study a particular teaching practice—a practice that takes advantage of instances of high leverage student thinking (Leatham et al., 2015) by making those instances of student thinking the object of whole-class sense making discussions (Van Zoest et al., 2016). As we studied teachers leading class discussions surrounding high leverage instances, we noticed variation in their use of public records of the initial student's thinking and subsequent discussion. This variation suggested that purposeful use of a public record—a physical representation that holds some degree of permanence and is visually accessible to all participants simultaneously—had the potential to play an important role in supporting the desired practice. The use of a public record as a support aligns with the tenets of cognitive load theory (Swellers, 1988; Swellers et al., 2011), which suggest that if the load for remembering other students' contributions during sense making discussions is lightened, then students can focus more on the sense-making actions they are being asked to carry out. A written representation of students' contributions can thus lighten this load. In this paper we report the results of our initial investigation into *how* teachers create and use public records of an initial student's mathematical contribution and subsequent discussion. These results provide insight into the ways public records can support teachers leading class discussions surrounding high leverage instances of student thinking.

Teachers' use of publicly accessible media (e.g., blackboards, white boards, etc.) is a common and widely accepted practice for sharing student contributions as well as mathematical content (Villareal & Borba, 2010). Additionally, some research has found that teachers' effective use of public records of student mathematical thinking corresponded with an increased level of student mathematical activity during whole class discussions (Koehne et al., 2020). However, few studies have explicitly addressed how teachers use public records of student thinking during class mathematical discussions. Knowing *what* from a student's contribution gets captured and *how* it gets recorded is important since public records of student thinking provide permanence to student thinking, which can help maintain continuity during collaborative inquiry (Staples, 2007). Knowing more about how teachers use verbal and physical actions to reference and add to a public record can help researchers learn how teachers support students to engage with each other's ideas, a practice that has been linked to mathematical achievement (e.g., Webb et al., 2014). With respect to our research, knowing more about teachers' use of public records enables us to better understand how teachers effectively use student thinking during whole class sense making discussions. This understanding is important because there is evidence that teaching practices that use student thinking can be challenging to enact (e.g., Simpson & Haltiwanger, 2017; Peterson & Leatham, 2009). The more researchers understand the subtleties of teaching practices that productively use student thinking, such as creating and using public records, the better we can support teachers to develop these teaching practices.

Theoretical Framework

Our work focuses on understanding how mathematics teachers take advantage of teachable moments. We refer to these moments as **M**athematically significant pedagogical **O**pportunities to build on **S**tudent **T**hinking (MOSTs) (Leatham et al, 2015). Taking full advantage of MOSTs, what we refer to as the teaching practice of *building on a MOST* (or just *building*), is engaging the class in making sense of the MOST to better understand the mathematics of the MOST. As described elsewhere (Van Zoest et al., 2016), building consists of four elements:

- Establish: Make the MOST a clear object.
- Grapple Toss: Offer the MOST to the class with parameters that put them in a sense- making situation.
- Conduct: Conduct a whole-class discussion in which students collaboratively make sense of the MOST.
- Make Explicit: Facilitate the extraction and articulation of the mathematics of the MOST.

As can be seen from this description of building, the original student contribution—the MOST—needs to become a clear object and remain the object of consideration by the class throughout a sense-making discussion. Thus, in our work we focus on a *public record of a student's mathematical contribution and the class discussion of that contribution* (henceforth referred to as *public record*). Because of the central role of the student contribution to building, it seems reasonable to suppose that the way the public record of that contribution is created and used could help to facilitate (or possibly end up hindering) this discussion. One reason to suspect public records are an affordance when teachers are building comes from appealing to cognitive load theory.

Cognitive load theory (Swellers, 1988; Swellers et al., 2011) is a learning and instructional theory grounded on two types of memory —*long term* and *working*. Long term memory is the structure that serves as a person's permanent storage of information and is potentially unlimited in capacity. Working memory is the structure that processes incoming information in conjunction with information drawn from long-term memory. Working memory is very limited in capacity with regard to the amount of information processed and the duration that the information is held.

Given these limitations, learning and instruction impose *cognitive loads* of varying degrees on working memory. *Intrinsic load* is the load imposed by the basic structure of the information that is germane to learning. The other load is an *extraneous load* that is imposed by the instructional materials and activities in which the learner engages with the information. Since the extraneous load is not germane to learning and is typically under the control of a teacher, a goal of instructional design is to reduce the extraneous cognitive load so that a greater percentage of working memory resources can handle the intrinsic load. Extraneous load can occur when students try to hold information from one source while searching for and processing information from a separate source (Swellers et al., 2011). Teachers can help reduce the cognitive load by integrating the separate sources of information when the information needs to be considered simultaneously (Swellers et al., 2011). Additionally, researchers (e.g., Mousavi et al., 1995) have found that when a visual display is paired with speech during instruction, cognitive load can be reduced for students by drawing their attention to relevant pieces of the display that coordinate with referents in speech. Teacher use of public records has the potential to reduce extraneous load in such ways during whole-class sense making discussions, which would allow a greater percentage of working memory resources to be devoted to the intrinsic load imposed by the sense making activity.

In this paper, we address the following research question: How can teachers use public records of a MOST and of the subsequent discussion surrounding the MOST to support elements of the teaching practice of building on MOSTs?

Methods

This study is part of a larger project that included a data set of video-recordings of mathematics lessons from 6-12th grade teacher-researchers who endeavored to enact the teaching practice of building. The participants were selected because they expressed an interest in and a desire to know more about the productive use of student mathematical thinking. After receiving professional development related to building, the participants used mini-tasks designed to elicit anticipated MOSTs and then built on those MOSTs. For this study, we analyzed 27 lessons where a public record was used as part of the building practice.

We identified each time a public record was created in each lesson and then made note of whenever a change was made to the public record (e.g., an idea was added, circled, erased, etc). We refer to this creating and editing as *manipulating* the public record. After tracking how public records changed, we identified each instance of a teacher *referencing* the public record by identifying physical and verbal actions related to or directed at the public record. Physical actions include specific and general gestures that draw attention to parts of or the whole public record. Verbal actions include speech that draws attention to the public record in some way or implies the use or discussion of what was captured on the public record. Physical actions were captured using codes that determined whether gestures or looks were towards specific or general parts of the public record; codes for verbal actions included naming, pronoun, and repeat. Each enactment was coded individually by at least three researchers and then collaboratively reconciled.

Finally, we examined the collections of manipulating and referencing codes across enactments for each building practice element (Establish, Grapple Toss, Conduct, Make Explicit). Specifically, our analysis aimed at describing ways the teachers' use of public records appeared to support or inhibit the use of student thinking within each element of building. This study can thus best be characterized as theorizing grounded in data.

Results

These results are organized by building element and describe the primary ways that teachers' manipulating or referencing of public records support the practice of building on MOSTs.

Using a Public Record to Support the Establish Element of Building

During the Establish element of building a teacher establishes (a) the precision of the MOST, ensuring that the MOST is clear enough for the class to engage with it; (b) the MOST as an object that can be identifiable throughout the sense making discussion; and (c) that the students have an intellectual need to engage with the MOST. (See Van Zoest et al., 2022 for further details about Establish and the other elements of building.) As described below, manipulating the public record has substantial potential to support establishing the precision of the MOST.

Manipulating the public record. In order to build on a MOST, a teacher first needs to create a foundation for a sense making discussion; creating a public record that clearly and completely captures the MOST helps to create this foundation. A teacher's honing of a student's contribution is important for creating a clear and complete public record. One form of honing is the teacher's use of symbols to represent connections students are making among mathematical ideas, such as drawing arrows that indicate a relationship among two entities rather than a written description of the relationship. Another form of honing involves a teacher capturing the essence of a student contribution while not recording it word-for-word, leaving off extraneous information or extra verbiage. Honing may help decrease cognitive load for students, as they would not have to attend to a word-for-word re-presentation of the student's contribution and try to carry out this honing themselves. Additionally, the public record needs to be created on the board space in a location that is visible for all to see, clearly separated from other information with room for additions, and that will not need to be removed as the discussion progresses since we want the MOST to be the object of the discussion. In each case, these manipulating actions support the work of establishing the MOST in preparation for the next element of building—Grapple Toss.

Referencing the public record. While referencing was not a primary use of the public record during Establish, our analysis did reveal some ways referencing may support this element of building. Teachers verbally referencing the public record with a name or label (e.g., this claim, this argument) contributes to making the MOST an object, because the name or label gives the MOST an identity that can be referenced in the public record throughout the discussion. Providing the MOST such an identity during the Establish element reduces the extraneous load for students during later elements of building because they need only remember the name or label and the location of the MOST in the public record as opposed to remembering the entire MOST throughout the discussion. Also, teachers often pointed at specific pieces of the public record as they wrote in order to confirm that what they were writing aligned with a student's thinking.

Using a Public Record to Support the Grapple Toss Element of Building

There are two critical aspects to the Grapple Toss element of building: a clearly established object that is offered to the class (the established MOST), and an action (on that object) that puts the students into a sense-making situation. As described below, referencing the public record has the potential to help students attend to both of these aspects.

Manipulating the public record. A major goal of manipulating the public record during the Establish element is to allow teachers to avoid most manipulation during the Grapple Toss element. That said, a subtle way manipulating can help emphasize the MOST as an object during Grapple Toss is a teacher circling, drawing a box, or underlining the MOST in the public record

The teachers emphasizing the MOST in such a way can serve as a permanent gesture, similar to referencing.

Referencing the public record. A teacher's referencing of the public record during a Grapple Toss orients students to the details of the established object without the teacher having to repeat or revoice the MOST in its entirety. For example, a teacher could ask, "I want you to think about what Elisa has said. What do you find mathematically compelling or conflicting about this claim? *[pointing to a public record of the established MOST]*." Because the MOST is captured in the public record, the object can be referred to succinctly with "this claim" thus allowing the emphasis to be the sense making action. Referencing the public record in this way coordinates a visual mode of communication with the auditory mode of communication, thus potentially reducing the extraneous load of attending to the details of the MOST so that working memory resources can be devoted to the sense making action in the Grapple Toss question.

Using a Public Record to Support the Conduct Element of Building

After the established MOST has been tossed to the students, a teacher conducts a discussion that engages students in making sense of the MOST. As students offer additional contributions during this discussion, a teacher puts unrelated contributions aside, establishes contributions related to the MOST, and invites students to use related contributions to further their sense making of the MOST. As described below, both manipulating and referencing the public record can support these aspects of the Conduct element of building.

Manipulating the public record. As students offer their contributions during a discussion, the teacher may add other contributions related to the original contribution, the MOST, to the public record. When adding these related contributions, a teacher needs to attend to similar actions to those described in the Establish section above. In addition, teachers should consider the organization among contributions. This organization helps to support students in making connections between the new contribution and previous contributions. Parallelism, placement, and particularization are three important considerations for this public record organization. *Parallelism* (similar structuring) among the contributions may help the class make connections between those contributions. For example, the public record in Figure 1a has a variety of symbols and structure for the calculations that may lead to difficulties when comparing. By contrast, each student contribution in Figure 1b has been structured in the same way, potentially scaffolding student attention to their similarities and differences. The *placement* of additional contributions in relation to the original contribution can also support comparisons between contributions. For example, in Figure 1a it would likely take some effort for students to identify which of the contributions align with the MOST and which contributions do not. In Figure 1b, the contribution that agreed with the original MOST was vertically aligned below the MOST in a column on the left, while contradictory contributions were placed together in a column on the right. Teachers also may consider *particularizing* the contributions (e.g., drawing lines, using specific colors, assigning labels) to help distinguish the contributions. For example, in Figure 1a, it is difficult to identify when one contribution ends and another begins. However, in Figure 1b, each student's contribution is color coded differently, and each column of contributions is labeled with a "yes" or "no" so that students understand which contributions support or contradict the original. Also, within each contribution the initial and final prices are underlined to highlight the comparison each individual student's contribution is making. A well organized public record that addresses these considerations has the potential to reduce the extraneous cognitive load for students by integrating and aligning objects so that students' working memory resources can be allocated to processes germane to sense-making.

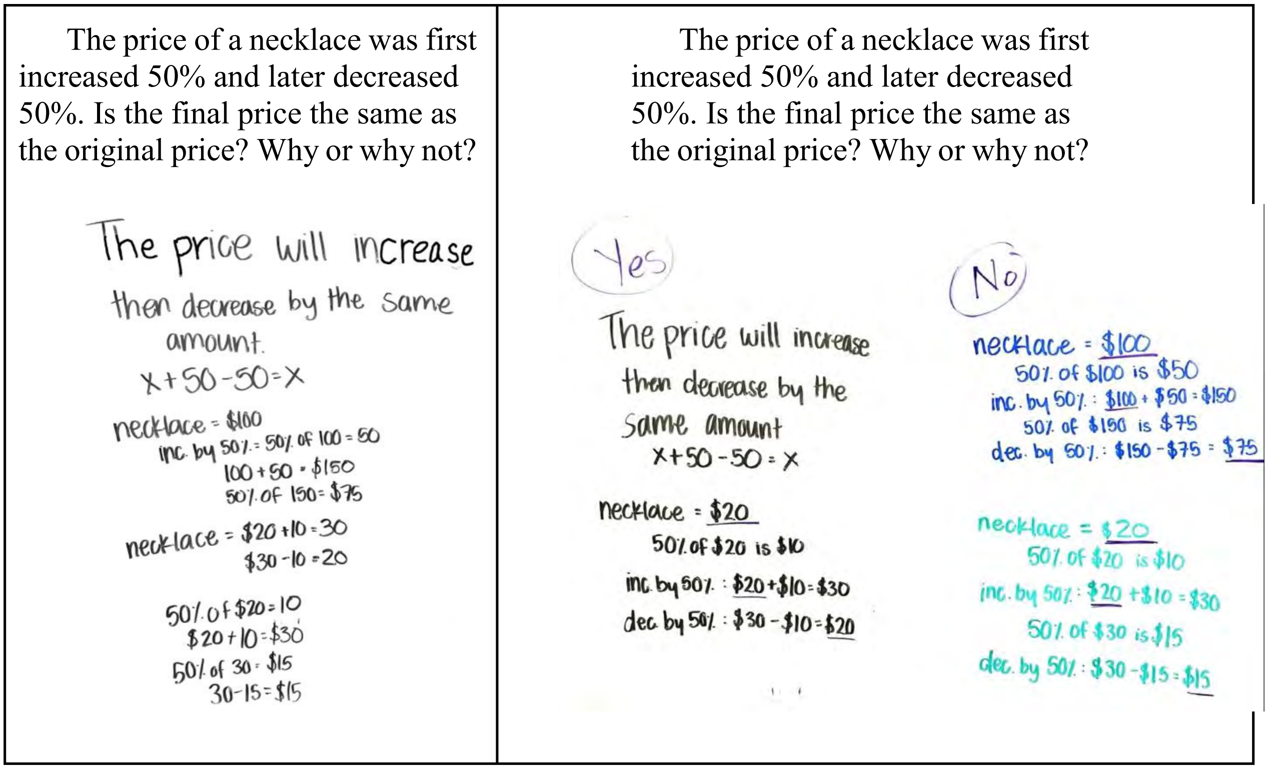


Figure 1: Public Record of MOST and Related Contributions During Conduct

Referencing the public record. One important part of conducting a sense-making discussion surrounding a MOST involves helping students to connect related student contributions (that have been established) to the MOST. Hence, a connecting move requires that both the MOST and a related student contribution be established as the object of the move. In addition, the move requires an action describing or requesting the nature of the connection that students are to make. Similar to referencing the public record during Grapple Toss, referencing to support connecting gives the action more prominence in the teacher's speech. However, because a teacher needs to reference two contributions in the public record, specific pointing gestures or verbal cues corresponding with the MOST and the related contribution become important, particularly because the public record may contain more information than these two contributions. For example, a teacher who asks, "How do you reconcile these two statements?" could use pointing gestures to make clear the objects in the public record that the class needs to reconcile. Differently, a teacher who asks, "How does this claim disprove Jaden's conjecture?" could use a pointing gesture to identify "this claim" in the public record, but the verbal naming "Jaden's conjecture" to identify the second contribution. This type of referencing reduces the cognitive load for students by clarifying the ambiguity of the referent for "this claim" and allowing the main focus to be on the requested reconcile action.

Another important part of using MOST related contributions is summarizing several established contributions so that they can be considered concurrently (i.e., synthesizing). A teacher's referencing supports the synthesis of student contributions to be connected by re-presenting the details of the contributions and helping students track those contributions in the public record. The public record provides permanence for student contributions so that teachers can use pointing gestures to support the class with attending to the details of multiple contributions. Supplementing speech with the use of the public record helps reduce the cognitive load of processing the sometimes lengthy details of contributions. Additionally, specific pointing gestures can clarify what piece of the public record corresponds with the teacher's speech. Such gestures reduce the cognitive load for students as they are directed to a specific piece of the public record and do not have to visually search for it on their own.

If a student shares a contribution during a discussion that is not related to the MOST, referencing the public record can help recenter the MOST as the focus of the discussion. In our research, we often found students responding to a Grapple Toss question with their own solutions to a task or their ways of thinking about an idea that was unrelated to or far from the focus of the MOST. Teachers can refocus students by gesturing at the MOST in the public record and telling students to focus on this idea, or referencing the MOST by its name while asking a student how their contribution is related to the MOST. Alternatively, teachers can use the same physical and verbal referencing of the MOST in the public record while reminding the class that the discussion is currently about making sense of the MOST.

Using the Public Record to Support the Make Explicit Element of Building

In the final element of building the teacher ensures that (a) the class agrees that the issue related to the MOST has been resolved, and (b) the mathematics of the MOST—the mathematical ideas that have emerged from making sense of the MOST—is explicitly articulated. We describe below how referencing can support resolving the MOST.

Manipulating the public record. In our data, teachers often did not manipulate the public record during Make Explicit since the ideas they wanted to discuss were already established, and connections and resolutions were made explicit throughout the discussion. However, we did see some manipulating that seemed to support Make Explicit. Editing the originally established MOST may further support students understanding the resolution of the MOST. Also, capturing the mathematics of the MOST succinctly may be important if the ideas will be used beyond the discussion of the MOST.

Referencing the public record. A teacher may reference the public record to emphasize the MOST as the object of a check-in question similar to the way referencing was used during Grapple Toss and Connect. For example, a teacher may ask, "Given our discussion for the past few minutes, how are you all thinking about our original argument? [gestures to the established MOST]." Another way a teacher may use referencing during Make Explicit is pointing to pieces of the public record that orient students to the details of ideas and connections said throughout the discussion that contribute to resolving the MOST.

Discussion and Conclusion

We reported the primary ways that a teacher's manipulating and referencing actions have the potential to support a teacher's productive use of student mathematical thinking when enacting the teaching practice of building on MOSTs. We begin the discussion by comparing and contrasting how manipulating and referencing actions play out across the four elements of building and conclude by reflecting on how these findings relate to the literature, particularly to cognitive load theory.

Although we described ways manipulating can support all elements of building, we see manipulating having a primary role during Establish and Conduct to provide permanence to student thinking shared in discussions. Manipulating that highlights pieces of the public record may occur during Grapple Toss or Make Explicit, but would be minimal as compared to altering contributions that would leave objects in the public record less clear for students. With respect to referencing, the combination and specificity of referencing actions often contributed to the extent to which these actions supported or hindered the building practice. Teachers clearly drew attention to particular contributions in the public record during Conduct with specific pointing gestures or using locator words (e.g., the upper right argument in blue). However, when teachers used vague pronouns (e.g., this calculation, that number) without pointing gestures to reference to a piece of a public record during conduct, the referent for the pronouns would likely be unknown for students.

There are at least two ways that teacher use of public records has the potential to reduce extraneous cognitive load imposed by split sources of information (Sweller et al., 2011) that inevitably occurs as a teacher builds on a MOST. First, sense making discussions require students to attend to a variety of information that is shared at different points in time and by different members of the class. Displaying (manipulating) the information in the public record can reduce the extraneous load imposed by having to remember the MOST and MOST-related contributions for the length of the discussion. Second, the public record may serve to reduce the extraneous load of attending to or searching for objects in the teacher's speech. During building, a teacher requests or positions students to engage in sense making actions with the MOST and MOST-related contributions. The teacher can reference these objects in the public record as opposed to revoicing or repeating them in their entirety, which has the potential to allow a greater percentage of working memory resources to be devoted to the load that is germane to the sense making of the mathematics of the MOST.

Our findings as to how manipulating and referencing can support elements of building extends the work of others investigating teacher use of student thinking to lead meaningful whole class discussions. Placement, parallelism, and particularization provide considerations for teachers manipulating and structuring public records that clearly and completely display student thinking during a sense-making discussion. These considerations offer insight as to how public records can be used to establish common ground, which is important for sustaining continuity of discussions on student thinking (Staples, 2007). During Grapple Toss, Conduct, and Make Explicit, referencing the public record can help support students' engagement with the details of each other's contributions. Referencing the public record in a Grapple Toss question or a request to connect ideas during Conduct can orient students to the specific objects with which they are to engage in sense making actions. This referencing provides insight into how teachers can use the public record as a resource to help students engage with the details of each other's thinking during whole class discussions, which can be key for students' mathematical learning (Webb et al., 2014).

Acknowledgments

This research report is based on work supported by the U.S. National Science Foundation (NSF) under Grant Nos. DRL-1720410, DRL-1720566, and DRL-1720613. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

References

Koehene, C., Thanheiser, E., Strickland, S., Pham, A., Heaton, R., & Melhuish, K. (2020). Profiling the use of public records of students' mathematical thinking in 4th-8th mathematics classrooms. In A. I. Sacrista´n, J. C. Corte´s-Zavala, & P. M. Ruiz-Arias (Eds.), Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 2017–2025). Cinvestav / AMIUTEM / PME-NA. https://doi.org/10.51272/pmena.42.2020-338

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. Journal for Research in Mathematics Education, 46(1), 88-124. https://doi.org/10.5951/jresematheduc.46.1.0088

Mousavi, S. Y., Low, R., & Sweller, J. (1995). Reducing cognitive load by mixing auditory and visual presentation modes. Journal of Educational Psychology, 87(2), 319-334. https://doi.org/10.1037/0022-0663.87.2.319

National Council of Teachers of Mathematics. (2014). Principles to actions: Ensuring mathematical success for all. Author.

Peterson, B. E., & Leatham, K. R. (2009). Learning to use students' mathematical thinking to orchestrate a class discussion. In L. Knott (Ed.), The role of mathematics discourse in producing leaders of discourse (pp. 99-128). Information Age Publishing.

Simpson, A., & Haltiwanger, L. (2017). "This is the first time I've done this": Exploring secondary preservice teachers' noticing of students' mathematical thinking. Journal of Mathematics Teacher Education, 20, 335–355. https://doi.org/10.1007/s10857-016-9352-0

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. Cognition and Instruction, 25(2-3), 161-217. https://doi.org/10.1080/07370000701301125

Sweller, J. (1988). Cognitive load during problem solving: Effects on learning. Cognitive science, 12(2), 257-285.

Sweller, J., Ayres, P., & Kalyuga, S. (2011). Cognitive load theory. Springer.

Van Zoest, L. R., Leatham, K. R., Stockero, S. L., Peterson, B. E. (2022). The teaching practice of building on MOSTs [Manuscript submitted for publication].

Van Zoest, L. R., Peterson, B. E., Leatham, K. R., Stockero, S. L. (2016). Conceptualizing the teaching practice of building on student mathematical thinking. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1281-1288). University of Arizona.

Villarreal, M. E., & Borba, M. C. (2010). Collectives of humans-with-media in mathematics education: notebooks, blackboards, calculators, computers and … notebooks throughout 100 years of ICMI. ZDM Mathematics Education, 42(1), 49-62. https://doi.org/10.1007/s11858-009-0207-3

Webb, N. M., Franke, M. L., Ing, M., Wong, J., Fernandez, C. H., Shin, N., & Turrou, A. C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning. International Journal of Educational Research, 63, 79-93. https://doi.org/10.1016/j.ijer.2013.02.001

COLLEGE INSTRUCTORS' PERCEPTIONS OF BARRIERS & DRIVERS THAT IMPACT THE IMPLEMENTATION OF ACTIVE LEARNING

Jessica Gehrtz	Stephen Lee	Priya V. Prasad
UT San Antonio	UT San Antonio	UT San Antonio
Jessica.Gehrtz@utsa.edu	Stephen.Lee@utsa.edu	Priya.Prasad@utsa.edu

College mathematics instruction that leverages evidence-based instructional practices, such as productive group work, can lead to many positive outcomes for students. In order to support instructors in adopting more evidence-based instructional practices, it is important to understand what barriers and drivers can impact their decision to implement such practices. In this study, we interviewed four introductory mathematics instructors teaching the same course in order to understand, in context, what aspects served as barriers and drivers. Transcripts were analyzed using thematic analysis. Initial results highlight how course coordination and weekly project meetings served as drivers, and the impact of the pandemic was seen as both a driver and a barrier to implementing evidence-based instructional practices.

Keywords: Instructional Activities & Practices, Undergraduate Education, Faculty Perceptions

Instruction that actively engages students in their learning has been shown to have many positive outcomes for students. Specifically, research at the college level has revealed that this type of instruction can increase student learning and students' conceptual understandings, as well as reduce achievement gaps (e.g., Freeman et al., 2014; Kogan & Laursen, 2014; Theobald et al., 2020). As such, there have been numerous calls to increase student engagement in college mathematics classrooms (CBMS, 2016; PCAST, 2012). Yet, didactic lecture still remains the most prominent form of instruction (Stains et al., 2018). To support college instructors in adopting more evidence-based instructional practices (EBIPs) that actively engage students, it is important to understand, both locally and globally, what barriers might be getting in the way of instructors' implementation of these strategies. It is also important to understand what drivers, aspects that mitigate barriers, foster instructional change and the adoption of EBIPs.

Previous research has examined various aspects that can impact an instructor's decision to implement EBIPs (e.g., Apkarian et al., 2021; Henderson & Dancy, 2007; Shadle et al., 2017). Although this work has helped to identify barriers, drivers, and individual or situational characteristics that may come into play, more work needs to be done to understand how institutions, departments, and individual mathematics educators can encourage the adoption of EBIPs by college mathematics instructors. Further, in order to foster instructional change locally, by mitigating barriers and leveraging drivers that have the most impact, it may be important to examine instructors' thinking about EBIPs at one's local institution (Henderson et al., 2011). As such, our work seeks to understand how college instructors at one institution think about the implementation of EBIPs in their classes. Specifically, we aim to answer the research question: What do college mathematics instructors consider to be barriers and drivers for implementing practices that engage students in active learning in their teaching?

Theoretical Framework Guiding Our Research

We examine our research question in the context of a larger project focused on transforming the teaching of introductory mathematics at one institution. The theoretical underpinning of this project is the Ethic of Practicality proposed by Doyle and Ponder (1977). This framework describes instructional change as a matter of how practical instructors find the proposed changes.

The Ethic of Practicality consists of three components: (1) *congruence* (compatibility with the instructor's classroom, setting, and instructional goals), (2) *instrumentality* (clearly articulated procedures for ease of implementation in the instructor's classroom), and (3) *cost* (potential benefits outweigh the effort and other costs of implementation). Motivated by this framing, we structured the project to maximize congruence and instrumentality and minimize cost for instructors. Specifically, instructor-participants were given a course release to make time for participation in the project, and the facilitators guided participants in implementing the Continuous Improvement (CI) model for instructional improvement (Berk & Hiebert, 2009), in a manner reminiscent of lesson study cycles (Dick et al., 2022). In CI, participants identify two or three different mathematical concepts and implemented the following cycle each semester: (1) design a task that targets a particular mathematical concept; (2) develop hypotheses about anticipated student responses; (3) collect data in the form of student work and classroom recordings, then analyze the data for evidence of the desired student learning outcomes; and (4) reflect on their own and their colleagues' instructional decisions in the implementation of the task, and revise the task as appropriate. In this way, instructors retain control of the congruence and instrumentality of the resources and materials that they themselves produce.

Methods

Participants were four fixed-term faculty teaching an introductory mathematics course at the same institution. Two participants, Shay and Nicholas (pseudonyms), were experienced instructors who had taught the course multiple times and served as course coordinators, and two participants, Alex and Ivy, were in their first semester teaching the course at this institution.

For this proposal, we focus on data collected in a 60-minute semi-structured interview with each of the instructors during the first semester of this larger project on instructional improvement. The interview protocol included questions to elicit details about each instructor's experience teaching, what a typical class period looked like, how their instruction has changed over time (if at all), and what factors supported or hindered them in implementing EBIPs (e.g., active learning activities). Interviews were transcribed verbatim. To answer our research question, we used thematic analysis of the transcripts to identify instructor reported barriers and drivers to implementing EBIPs (Braun & Clarke, 2006). We coded segments from the interviews, using the 18 barriers (e.g., Time Constraints) and 15 drivers (e.g., Aligns with existing resources) identified by Shadle et al. (2017) as a priori codes. We then created additional codes when instructors described barriers or drivers that were not represented in Shadle et al.'s (2017) findings.

Results

In this section, we describe two drivers, course coordination and weekly CI project meetings, that impacted all four participants decisions to implement EBIPs. We also discuss the impact of the COVID-19 pandemic, which was identified by participants as both a barrier and a driver to implementing EBIPs. In our talk, we will share additional barriers and drivers that arose in our data, including those that align with Shadle et al.'s (2017) findings.

Drivers to Implementing EBIPs: Course Coordination & CI Project Meetings

Course coordination played an important role in creating an environment that fostered the implementation of EBIPs. The course coordinators (Nicholas and Shay) created the course structure, outlining activities that would be done during class and building the homework sets and lecture videos that could be used to support students' learning outside of class. Ivy discussed how she appreciated having someone with departmental and course-specific knowledge in this role, especially when there was disagreement amongst the instructors. She said:

> So having somebody there that I know has the broader perspective … You've got to have somebody that's taking that leadership role and putting stuff into practice and saying this is the way it's going to be. But they have been extremely open with input.

Ivy also emphasized, in this quote and throughout the interview, that she appreciated how the coordinators took instructors' ideas as input when making decisions that impacted the entire instructional team.

All four instructors also emphasized that the resources that were created and made available as part of the course coordination contributed to their decision to implement more EBIPs in their teaching, which was also identified as a driver in Shadle et al.'s (2017) work (i.e., Aligns with Existing Resources). Ivy emphasized how these available resources prompted her to incorporate activities she had never quite been comfortable with implementing before. She said,

> I've got all these resources. My experience was lecture, and now I get to try the things I've always wanted to try. So I'm loving it. I love the activities. And I'll be honest, at the start of the semester I was like, 'Oh, we're going to do this now?' Because I didn't know how powerful it was. … It's just cool to see everything that we can do with [the activities], and it really helps the students understand.

These resources were initially created by the course coordinators, but as the semester progressed, other instructors began to add to this shared repository of course resources.

Weekly project meetings, which focused on developing materials as part of the CI cycle, also served as a driver for instructor implementation of EBIPs. These meetings created time and space for instructors to talk about best practices and pedagogy as they developed class activities. Notably, these meetings were separate from course coordination meetings (where the focus was primarily on logistics such as upcoming assessments, content, or due dates). All instructors shared how these weekly meetings were important for developing and implementing activities that centered student thinking, with the meetings also serving as a space for collaboration and as a form of accountability to ensure progress on shared objectives. Nicholas said,

> I think a lot of it's the kind of dedicated collaboration time. … It's kind of nice to just sit around for an hour and be like, 'What do you think about this? And how would you approach this? And how would you do that?'

Additionally, these weekly project meetings also created space for the facilitators to share related educational research as the instructors expressed an interest in learning more about EBIPs. As such, opportunities for professional development arose organically during these meetings.

COVID-19: Both a Barrier and Driver to Implementing EBIPs

Instructors discussed the impact of the COVID-19 pandemic both as a barrier and as a driver for implementing more EBIPs in their classes. On one hand, Shay described COVID as a driver because it challenged instructors to think carefully about how they were presenting material when the course modality shifted to online. Shay said,

> The idea that COVID kind of opened your [eyes] - you have to look for other ways to be able to present material to them. And so, oh! We'll make these videos. … Why can't we just make the video and they can watch it on their own time, and then come in and then do this

> [activity]? … I want them to - instead of just sitting there blankly looking at me while I'm talking - being able to interact.

This quote highlights the way that the pandemic enabled her to try new things in her class because she realized that she could create videos covering the content she would normally lecture on for students to watch on their own time. This freed up time during class for students to engage in activities and interact with one another.

Another instructor, Alex, pointed to how the COVID-19 pandemic was a barrier for him incorporating more EBIPs. He described how it was challenging to facilitate student discussions in breakout rooms on Zoom because he could not listen in on student discussions as easily. He felt as if there was not the same collaboration and excitement that happens in the classroom when one can hear other groups discussing and laughing together during student work time. Even when courses shifted back to being in-person, the pandemic continued to impact student interactions. Alex said:

> But on the very first day [of being back in person], some students are acting like everything's normal and sitting together, but others are sitting really far away, wearing their masks. And I'm not going to tell them [to get into a group]. … So that's brought me to where I'm a little bit less sure about how it can operate right now.

Discussion

In this study, instructors identified many of the same barriers to implementing EBIPs that were reported by Shadle et al. (2017). However, most of the drivers that were identified in our work differed from those raised in that study. This could be because Shadle et al. (2017) focused on faculty's perceptions of what "will help make change occur" (p. 10), whereas our study sought instructors' reflections as they were in the midst of a project designed to promote instructional change. Nonetheless, we do see some alignment between the drivers identified in both studies. Specifically, we see the drivers of course coordination and the weekly CI project meetings as aligning with the driver "Encourages collaboration and shared objectives" identified by Shadle et al. Critically, Shadle et al. describe this driver as an outcome of "increased emphasis on teaching and student success" (2017, p. 6), whereas in our work course coordination actually served to enable instructors in their adoption of EBIPs. Other recent research has also highlighted the important role course coordinators can play as change agents and drivers for instructional change (Williams et al., 2021). Two years before the start of this project, the mathematics department had embarked on a campaign for improved coordination and alignment across multi-section courses. This required instructors to meet and discuss their goals for and their teaching of the course, increasing collaboration. Then when course coordinators started experimenting with incorporating active learning in their own teaching, this led to the creation of a course structure that fostered the inclusion of more EBIPs.

As shown by the preliminary results of this study, the disruptions to university teaching that resulted from the COVID-19 pandemic complicated instructors' use of EBIPs in the expected ways (e.g. complicating the use of small-group work when social distancing was also encouraged during in-person classes), but they also provided instructors with opportunities to rethink their beliefs about their role in the classroom. As this project progresses, we anticipate that instructors will continue to interrogate their previous assumptions about mathematics teaching and learning.

Acknowledgements

Support for this work was provided by the National Science Foundation's (NSF) Improving Undergraduate STEM: Hispanic-Serving Institutions (IUSE: HSI) program under award #2116187. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

References

Apkarian, N., Henderson, C., Stains, M., Raker, J., Johnson, E., & Dancy, M. (2021). What really impacts the use of active learning in undergraduate STEM education? Results from a national survey of chemistry, mathematics, and physics instructors. *PloS one, 16*(2), 1-15.

Berk, D., & Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers, one lesson at a time. *Teachers and teaching: Theory and practice, 15*(3), 337-356.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative research in psychology, 3*(2), 77-101.

Conference Board of the Mathematical Sciences. (2016). *Active learning in post-secondary mathematics education.* Washington, DC: Conference Board of the Mathematical Sciences.

Dick, L. K., Appelgate, M. H., Gupta, D., & Soto, M. M. (2022). Continuous improvement lesson study: A model of MTE professional development. *Mathematics Teacher Educator, 10*(2), 111-128.

Doyle, W., & Ponder, G. A. (1977). The practicality ethic in teacher decision-making. *Interchange, 8*(3), 1-12.

Freeman, S., Eddy, S. L., McDonough, M., Smith, M. K., Okoroafor, N., Jordt, H., & Wenderoth, M. P. (2014). Active learning increases student performance in science, engineering, and mathematics. *Proceedings of the National Academy of Sciences, 111*(23), 8410-8415.

Henderson, C., Beach, A., & Finkelstein, N. (2011). Facilitating change in undergraduate STEM instructional practices: an analytic review of the literature. *Journal of Research in Science Teaching, 48*(8), 952–984.

Henderson, C., & Dancy, M. (2007). Barriers to the use of research-based instructional strategies: the influence of both individual and situational characteristics. *Physical Review Special Topics- Physics Education Research, 3*, 020102-1–020102-14.

Kogan, M., & Laursen, S. L. (2014). Assessing long-term effects of inquiry-based learning: A case study from college mathematics. *Innovative Higher Education, 39*(3), 183-199.

President's Council of Advisors on Science and Technology (2012). *Report to the president: engage to excel: Producing one million additional college graduates with degrees in science, technology, engineering, and mathematics.* Washington, DC: Executive Office of the President.

Shadle, S. E., Marker, A., & Earl, B. (2017). Faculty drivers and barriers: laying the groundwork for undergraduate STEM education reform in academic departments. *International Journal of STEM Education, 4*(1), 1-13.

Stains, M., Harshman, J., Barker, M. K., Chasteen, S. V., Cole, R., DeChenne-Peters, S. E., Eagan, M. K., Esson, J. M., Knight, J. K., Laski, F. A., Levis-Fitzgerald, M., Lee, C. J., Lo, S. M., McDonnell, L. M., McKay, T. A., Michelotti, N., Palmer, M. S., Plank, K. M., Rodela, T. M., … & Young, A. M. (2018). Anatomy of STEM teaching in North American universities. *Science, 359*(6383), 1468-1470.

Theobald, E. J., Hill, M. J., Tran, E., Agrawal, S., Arroyo, E. N., Behling, S., Chambwe, N., Cintrón, D. L., Cooper, J. D., Dunster, G., Grummer, J. A., Hennessey, K., Hsiao, J., Iranon, N., Jones II, L., Jordt, H., Keller, M., Lacey, M. E., Littlefield, C. E., … & Freeman, S. (2020). Active learning narrows achievement gaps for underrepresented students in undergraduate science, technology, engineering, and math. *Proceedings of the National Academy of Sciences, 117*(12), 6476-6483.

Williams, M., Apkarian, N., Uhing, K., Martinez, A. E., Rasmussen, C., & Smith, W. M. (2021). In the driver's seat: Course coordinators as change agents for active learning in university Precalculus to Calculus 2. *International Journal of Research in Undergraduate Mathematics Education*, 1-28.

RELATIONSHIPS BETWEEN TEACHER QUESTIONING AND STUDENT GENERALIZING

Allyson Hallman-Thrasher
Ohio University
hallman@ohio.edu

Jennifer Thompson
Ohio University
jh859394@ohio.edu

Kayla Heacock
Ohio University
kh426613@ohio.edu

Lizhen Chen
Ohio University
lizhen@ohio.edu

This study shares two frameworks for analyzing teacher actions that support students in generalizing and examines how those frameworks align with teacher questioning. One classroom teaching episode focused on the mathematical activity of generalizing is shared to illustrate effective generalizing promoting practices. We found several patterns of productive and unproductive generalizing promoting actions and questioning. Repeating generalizing promoting actions in succession were needed to produce student generalizations. Priming actions that set up for later generalizing promoting were helpful when students struggled to identify and state generalizations. Connection questions promoted generalizing, but justification and concept questions did not. Further research will explore the additional strategies to support teachers in fostering student-created generalizations.

Keywords: Instructional Activities and Practices, Classroom Discourse, Algebra and Algebraic Thinking

The mathematical practice of generalizing, identifying a relationship to describe multiple examples or instances of a phenomenon, is fundamental to learning in mathematics (Carraher & Schliemann, 2002; Kaput, 1999) and engages students in algebraic thinking (Blanton et al. 2011; Kieran et al. 2016), which requires students identify, investigate, and represent relationships. Understanding how to support students in developing, articulating, and refining generalizations is critical to mathematics teaching and learning. Teacher questioning plays a pivotal role in fostering students' generalizations, especially when students' reasoning does not lead to formal general statements (Radford, 2010). However, research on teacher questioning and generalizing remains distinct. This study is aimed at better understanding the relationship between teacher questioning and students generalizing. We describe our questioning framework and its relationship to two frameworks for analyzing actions to promote generalizing. We address how a high school mathematics teacher's questioning aligns with her actions to foster students generalizing and describe the patterns in student-teacher interactions that promoted student-created generalizations.

Literature

Generalizing skills contribute to algebraic understanding (Carpenter & Franke, 2001) and are identified as a key mathematical practice across all math content domains in the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Here, we adopt Kaput's (1999) definition of generalization as "lifting" and communicating reasoning to a level where the focus is no longer on a particular instance but rather on patterns and relationships of those instances. A formal generalization is the product of the mental activity of generalizing (Font & Contreras, 2008). In generalizing students must recognize quantities that vary and remain constant and represent these relationships using symbols or words. When students move to generalizing symbolically without having adequate time to understand and reason about quantities and their relationships in a variety of contexts beforehand, they can become dependent on procedures (Kieran, 2007). Teachers value, and thus place an instructional focus on, formal algebra such as symbols, notations, and procedures (Nathan & Koedinger, 2000). A focus on procedural approaches to algebra can obscure attention to engaging students in mathematical practices such as generalizing that build a conceptual understanding of mathematics. Given student difficulties in generalizing (e.g., Blanton & Kaput, 2002; Lannin, 2005; Lee & Wheeler, 1987; Stacey, 1989; Stacey & MacGregor, 1997), students' failure to justify generalizations (Breiteig & Grevholm 2006), and secondary math teachers' challenges in responding productively to student generalizations (Demonty et al., 2018), determining what instructional actions promote generalizing activities is warranted.

Ellis (2007) proposed an actor-oriented generalization taxonomy that consists of generalizing actions, which include students' activities while generalizing and their statements of generalization. To better understand the classroom interactions and discourse that promote generalizing actions, we use a modified version (Strachota, 2020) of Ellis' (2011) generalizing-promoting actions (GPAs) and Strachota's (2020) priming actions (PAs). Priming actions set the stage for more explicit attention to generalizing; they prepare students to build on an idea or refer to an idea later. Generalizing promoting actions, on the other hand, prompt immediate activities that have the potential to produce generalizations. Priming can include making the critical ideas of an individual public to the work of the class, making evident tools needed for generalizing, asking students to consider ideas or examine specific key examples, or setting up to extend an idea later. For example, a teacher who introduces *x* to represent a varying quantity in a pattern or who displays similar expressions for comparison is using a priming action, *reviewing a critical tool* and *constructing searchable and related situations*, respectively. Generalizing promoting requires students to identify a relationship, state a generalization, extend beyond cases available to them, or justify a general statement. For example, prompting students to describe a pattern algebraically is a generalizing promoting action that *encourages reflection*. Table 1 illustrate the codes we used for priming and generalizing promoting actions (Ellis, 2011; Strachota, 2020).

Table 1: Priming Actions and Generalizing-Promoting Actions

Priming Actions (PAs)[1]	
Naming a phenomenon, clarifying critical terms and tools	"Offering a common way to reference a phenomenon or emphasizing the meaning of a critical term or tool."
Constructing or encouraging constructing searchable and relatable situations	"Creating or identifying situations or objects that can be used for searching or relating. Situations that can be used for searching or relating involve particular instances or objects that students can identify as similar."
Constructing extendable situations	"Identifying situations or objects that can be used for extending. Extending involves applying a phenomenon to a larger range of cases than from which it originated."
Generalizing-Promoting Actions (GPAs)[2]	
Encouraging relating and searching	"Prompting the formation of an association between two or more entities; prompting the search for a pattern or stable relationship."

Encouraging Extending	"Prompting the expansion beyond the case at hand."
Encouraging Reflection	"Prompting the creation of a verbal or algebraic description of a pattern, rule, or phenomenon."
Encouraging Justification	"Encouraging a student to reflect more deeply on a generalization or an idea by requesting an explanation or a justification. Includes asking students to clarify a generalization, describe its origins, or explain why it makes sense."

[1]These categories and descriptions are from Strachota (2020, p. 7).
[2]These categories and descriptions are from Ellis (2011, p. 316). We adapted encouraging relating and searching, following Strachota (2020), by combining these into one category.

Teachers often struggle with asking 'good' questions that are cognitively demanding, involve higher-order thinking, and follow up on student input and explanations (Boaler & Brodie, 2004; Franke et al., 2009). In addition to asking good questions, teachers must be able to effectively interpret and make sense of students' questions during class to use student thinking to move the mathematics of a lesson forward (Hufferd-Ackles et al., 2004). The follow-up questions teachers ask after hearing student thinking are of critical importance. Franke et al. (2009) found that students benefited most when teachers asked a probing sequence of specific questions. This process of probing helped teachers better understand student thinking, helped the students who were responding to teacher questions to solidify their ideas and thinking, and helped other students connect ideas to their own thinking and address misconceptions. When a teacher asked only one question, they were often not able to obtain enough information to understand the student's thinking. Boaler and Brodie (2004) concluded that it is important for teachers to ask higher-order types of questions, so students have more opportunities to engage meaningfully with mathematics in ways that go beyond performing procedures. The finalized questioning framework used in this study is based on Boaler and Brodie (2004), Hallman-Thrasher & Spangler (2020), and Chen (2021) shown in Table 2 and described in the methods section.

Table 2: Question Types to Support Generalizing

Definition of Question Type	Example
Rhetorical	
Does not generate responses (teacher answers them or does not provide time for students to answer)	"Everyone else get that? Yeah? Ok."
Gathering Information	
Requires only a single short answer	"How many flowers are in step 5?"
Concept	
Attends students to underlying mathematical relationships and meanings	(No example from data)
Strategy	
Elicits descriptions of students' strategies, solutions, or procedures	"Do you want to come up here and show how you got that?"
Clarification	
Clarifies student input that has been shared or is known	"So, these would be the step numbers?"
Connection	
Seeks a connection across ideas, representations, or strategies	"Which part [of the picture] would be x?"

Justification	
Defends the appropriateness of a particular strategy, augmenting with connections that validate reasoning.	"Why does this [expression] not work [for other cases]?"
Extending Thinking	
Extends to examples beyond what is available or to examples where similar ideas could be used	"What about the 10th step?"

Adapted from Boaler and Brodie (2004) and Chen (2021).

Methods

The participant for this study was Ms. Patton, a teacher candidate enrolled in a one-year master's program with licensure for individuals with STEM degrees. She had earned an undergraduate degree in mathematics the previous year. At the time of data collection, she was in the Fall semester of a year-long teaching placement in an Algebra II classroom and enrolled in her only mathematics teaching methods course. In this lesson, Ms. Patton was supported by Mr. Dayton, her experienced mentor teacher. For this study, Ms. Patton planned, taught, and reflected on her video data from two episodes of teaching a pattern task with grades 9-10 students as part of an assignment for her mathematics teaching methods course. By pattern task we mean, a visual representation of objects that grows over instances of time (Figure 1). The data analyzed for this paper is the video recording of Ms. Patton's teaching where the most student-created generalizations were shared. In methods class, Ms. Patton first engaged in completing pattern tasks as a learner and analyzing videos of other teachers using pattern tasks. We carefully structured her planning for this task to attend to teacher questioning to elicit, understand, and make connections to student thinking.

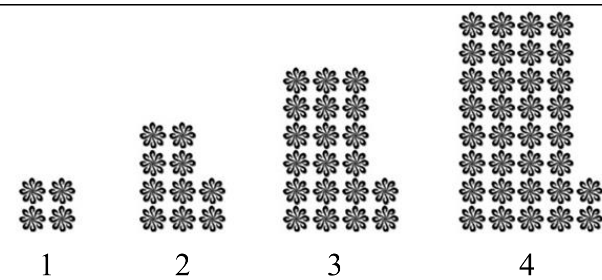


Figure 1. Ms. Patton's Pattern Task (Nguyen, 2020).

For our initial analysis, we coded the video data of teaching in 15-second segments using two established frameworks for generalizing (Table 1, Ellis, 2011; Strachota, 2020). Next, to capture teacher moves we modified Boaler and Brodie's (2004) question types. The question types (Boaler & Brodie, 2004) did not align well with the specific nature of supporting students in the creation of generalizations. Not all of the question types were applicable for this study and some were not sufficiently specific. None of the lessons we reviewed involved linking to concepts outside of mathematics so "linking and applying" and "establishing context" questions were not used. The "orienting and focusing" questions were not adequately specific, so we defined other categories that helped us classify the strategies a teacher would use to orient or focus (e.g., clarifying or justifying questions). "Probing" questions did not describe all the different ways a teacher might follow up on student thinking. We drew on Hallman-Thrasher & Spangler's (2020) broad categories of questions to develop a more comprehensive list which we compared against Chen (2021) to search for overlaps, gaps, and types needing more or less specificity. For example, Chen's (2021) "elicit thinking questions" were broken into strategy and concept questions in a new framework (Table 2). To establish a more descriptive framework specific to the evaluation of questioning that supports generalization, we also used thematic coding (Saldana, 2013) to identify questioning types not addressed by our existing frameworks. We defined a new question type "rhetorical" for questions that did not require a response or served as a means to garner attention to a thought, such as, "Does that make sense to everybody?" We also carefully considered what counted as a question: statements that functioned a question (e.g., "Find all the expressions you can for step 5") counted as questions, as did the questions that were not answered or were not intended to be answered (e.g., "You said that was *x*, right?"). To more closely examine the nuanced turns in conversation, we re-coded the data line-by-line for generalizing promoting actions, priming actions, student generalizations, and questioning with our revised framework for questioning. Interrater reliability was established by having all four researchers review and code all data. When disagreements arose, we discussed them using the frameworks to reach consensus (Syed & Nelson, 2015).

To better understand how the conversation developed over the course of the lesson, we divided the 45-minute lesson transcript into 13 blocks with each block representing a different instructional goal (e.g., launching the task, generating expressions for a particular step, applying a numeric expression for one step to a different step). We identified which blocks of conversation were productive for producing student generalizations. We wrote a description for each block and examined it in order to isolate characteristics of instruction that supported generalizing. We examined the coded data to identify patterns that signaled productive and unproductive questioning strategies and teacher moves to develop students' generalizations.

Results

We focus our results on Ms. Patton's most productive lesson; the one that included the most student-created generalizations. From our blocked classroom interactions, those producing at least one student generalization were identified as productive. Blocks that did not produce a student generalization were considered either missed opportunities for having had the potential to produce a student-created generalization or unrelated when the purpose of the interaction was not directly related to generalizing (e.g., clarifying directions, checking in with a small group). Students shared 13 generalizations spread over 7 productive blocks. Three blocks were unrelated to generalizing and three blocks were missed opportunities for generalizing. Table 3 shows which questioning types were used with priming actions (PAs) and generalizing promoting actions (GPAs). Ms. Patton relied primarily on connection, clarification, and extending thinking questions to develop generalizing actions. Two PAs did not involve questioning and eight GPAs did not involve questioning. Priming was associated with gathering information, clarification, strategy, and connection questions, whereas generalizing promoting incorporated all question types, relying most heavily on connection and extending thinking questions.

Particular types of questioning were more or less helpful for engaging students in conversations related to generalizing. Ms. Patton's use of connection, clarification, and rhetorical questions exceeded all other questioning types. She repeatedly used connection questions when encouraging students to relate between figural and algebraic representations of the pattern. When she asked, "Which part [of the picture] would be *x*?" her focus on a specific part of an expression prompted students to discover that the width of the large rectangle was the same as the step number. All of the connection questions she posed, were priming or generalizing promoting.

Ms. Patton used clarifying questions to have students further explain terminology or ideas and in doing so students revised an idea or stated it more precisely. For example, in prompting students to unpack an expression related to viewing the pattern as two equal-sized squares, Ms. Patton prompted, “Now what were you saying about perfect squares?” Clarifying questions often served to provide an opportunity for students to repeat an important point to which Ms. Patton wanted the class to attend. Clarifying questions, though important to understanding student thinking, did not consistently relate to generalizing; only 8 of her 17 clarifying questions functioned as priming or generalizing promoting actions.

Table 3: Alignment of Question Types, Priming, and Generalizing-Promoting Actions

Question Types	Priming Actions	Generalizing Promoting	Total Questions
Rhetorical	---	4	10
Gathering Information	1	1	5
Strategy	3	2	7
Clarification	4	4	17
Concept	---	---	---
Justification	---	3	3
Connection	4	13	17
Extending Thinking	---	5	6

Ms. Patton used rhetorical questions to state generalizations; the four teacher-stated generalizations were shared in the form of a rhetorical question. She did not, however, provide an opportunity for students to respond to these rhetorical questions. For example, she revoiced a generalization saying, “Does everyone see how he got that? For every picture you have this two right here [the constant two flowers on the rightmost column in each image] and then across [the width of the rectangle] is just the step number plus one.” This question had the potential to ensure that all students attended to and understood a key generalization shared earlier. However, by not providing students an opportunity to agree or disagree and justify their decisions she limited students’ opportunities to actively engage with another’s ideas.

Other questioning types, while not as prevalent as connection, clarification, and rhetorical, were more consistently associated with generalizing promoting. Extending thinking and justifying questions nearly always functioned as generalizing promoting actions. Extending thinking questions and justifying questions did not immediately lead to student generalizations. They always required follow-up supporting questions. Though justifying questions were not used often, they were effective at promoting generalizations. For example, Ms. Patton asked, “Why wouldn’t it work for say step number 3?” and followed with strategy, clarification, justification, and connection questions before a student correctly generalized that an expression for the pattern in step 4 would not work for any other step of the pattern. Justification questions, while present in this lesson, were less than we might have expected and may have established a tendency not to justify claims which could have limited opportunities for students to discover, refine, and state generalizations. We also noted that Ms. Patton did not use concept questions which may have contributed to the missed opportunities for students’ generalizing. By not consistently making underlying concepts and justifications evident, Ms. Patton may have focused more on what strategies were developed and less than the underlying structure that would have supported students in making their own generalizations.

Ms. Patton often used clarification, justification, and connection questions to make students’ generalizations accessible to the whole class. In clarifying to *encourage reflection*, she revoiced a student idea and pressed for detail that prompted a student to state a generalization more precisely; in connecting she *encouraged relating and searching*. Justifying questions such as "Why wouldn't that work for step 3?' served to *encourage justification* and in response a student produced a new general statement about how the formula could not extend to all instances of the pattern. To *encourage extending*, she repeatedly asked students to consider cases beyond those shared and to apply ideas developed from one step of the pattern to earlier or later steps.

Within each productive block we looked at the sequence and density of priming and generalizing promoting actions to identify patterns of actions that were productive towards producing student-created generalizations. One productive pattern for producing student generalizations was using repeated instances of generalization promoting over a short duration. The first such productive block included three GPAs (*encouraging extending*, *encouraging extending*, *encouraging justification*) over a one-minute time span. The questioning types that were used to accomplish these actions varied. She first gathered information and extended thinking to *encourage extending*, and then followed with a strategy question to *encourage justification*. By *encouraging extending* twice with two different question types, she first elicited the information she needed to build on in order to extend and justify. A student was then able to provide a general description of the pattern's structure. Ms. Patton also used repeated GPAs to shift students' attention to articulate a complete version of their generalizations.

A second related pattern, not consistently productive, but which Ms. Patton consistently employed, was to immediately follow every student generalization with another generalizing promoting action. For example, she followed the productive block described above with another GPA to build on that student's thinking by asking the class to translate the student's description into a generalized formula. These GPAs following a student generalization were productive in developing a new generalization when they took the form of a clarifying, connection, or justification question and functioned as *encouraging reflection*, *encouraging relating and searching*, or *encouraging justifying*. They were not productive when the teacher and students focused on different perspectives. Ms. Patton used a connection question, "How could we get it to be minus 6 in terms of x?" followed by a rhetorical question, "[x] Plus 2. Will that work?.. Distribute….Not quite, right?" as generalizing promoting actions to *encourage reflection* to generalize missing flowers in terms of step number. Yet, because students did not approach the problem from a 'what's missing' perspective, they did not readily generalize this strategy.

Within blocks that we labeled a missed opportunity, we identified a third pattern. This pattern involved a failure to use priming and generalizing promoting actions together to build towards a conclusion. When an initial PA or GPA failed to produce a student generalization, Ms. Patton abandoned the line of questioning. For example, Ms. Patton asked students to apply their numeric expressions created for step 4 to the first step of the pattern. Because the general structure is not evident in the first step of the pattern, students had trouble applying their formulas and because she failed to use generalizing promoting and priming together students could not meaningfully respond to this prompt.

A fourth pattern resulted from a missed opportunity where repeated GPAs produced no student generalizations. To scaffold students' thinking, Ms. Patton employed repeated priming before returning to GPAs to produce a student generalization. For example, over a two-minute span she used nine GPAs to try to help students develop a generalization for the number of missing flowers in the rightmost column. Students were able to recognize that she was looking for the missing flowers but were unable to make any general claims about the structure of the missing flowers relative to the full picture. Ms. Patton then used PAs to *construct and search for* *relatable situations.* Aware of the “tricky” part of representing missing flowers algebraically, she primed students by asking for clarification about the missing flowers. She continued priming by noting that there were 6, then 4, then 2 missing flowers (working backwards through the steps), and primed again to prompt students about the goal “We need to have something minus 6 and we need to end up with 2*n* squared plus 2, right? Cause right now we have…” [teacher draws attention to current representation prompting students to complete her sentence]. With this groundwork laid she used GPAs of *encouraging reflection and justification* which prompted students to state incorrect generalizations of the 6 missing flowers in step 4 and then to identify that those generalizations did not work for other steps of the pattern. A student generalized why the expression did not work and then Ms. Patton again primed to build on this idea. Finally, her mentor, Mr. Dayton used a connection question as a GPA to help students notice the connection between the missing flowers and the height of the rectangle. This lengthy exchange spanned three blocks and demonstrates how Ms. Patton’s and Mr. Dayton’s repeated questioning led to student generalizations for the blank spot of missing flowers.

Conclusion

We set out to determine what teacher moves were most productive in promoting students’ generalizing through a close examination of line-by-line student-teacher interactions situated within blocks of classroom dialogue with common purposes. It is not surprising that extending thinking questions and justifying questions aligned with generalizing promoting actions of *encouraging extending* and *encouraging justification,* respectively. However, it is surprising that these actions did not consistently produce student generalizations. Other questioning types (connection, clarification, and strategy) aligned with generalizing promoting and priming actions and, when used in particular patterns of promoting and priming, did lead to student-created generalizations. Our results confirmed the benefits of using connection questions to relate visual, numerical, and algebraic representations and to help students identify and articulate generalizations. Clarifying questions often provided an opportunity for students to repeat an important point critical to precisely state a generalization. Results also point to the difficulty of using justification and concept questions to promote generalizing even when teachers have expressly prepared to use these types of questions. Though justifying was used less than we expected, when employed it resulted in a generalizing promoting action each time.

There is not a one-to-one relationship between questioning and generalizing promoting or priming. A variety of questioning types served as priming and generalizing promoting actions. We expected to see a linear sequence beginning with teacher priming and generalizing promoting actions followed by student generalizations. But, in fact, generalizing promoting actions did not always lead to student generalizations; patterns leading to student generalizations were more complex. Repeated generalizing promoting actions and priming in conjunction with generalizing promoting were needed to produce a student generalization. When generalizing promoting actions did not produce student generalizations, priming followed by additional generalizing promoting was helpful. In this lesson, as with any lesson, questioning occurs in response to student contributions making the planning of all questioning challenging. Accepting this fact means that for questioning to support students in creating generalizations, teachers must possess the ability to respond in the moment with questioning and generalizing promoting moves that are likely to be productive. We suggest that this method of examining common pathways that lead students to generalize can be applied in other lessons to further develop suggestions for teachers as they work to support students as they develop their own generalizations.

Acknowledgments

The research reported here was supported in part by National Science Foundation Award #1758484. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Blanton, M., & Kaput, J. J. (2002). Design principles for tasks that support algebraic thinking in elementary school classrooms. In D. S. Newborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (Eds.), Proceedings of the 24th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 2-105). Eric Clearinghouse for Science, Mathematics, and Environmental Education.

Blanton, M., Levi, L., Crites, T. & Dougherty, B. (2011). Developing Essential Understanding of Algebraic Thinking for Teaching Mathematics in Grades 3-5. NCTM.

Boaler, J., & Brodie, K. (2004). The importance, nature, and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics in Education (pp. 774–782). OISE/UT.

Breiteig, T., & Grevholm, B. (2006). The transition from arithmetic to algebra: To reason, explain, argue, generalize and justify. In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez, A. (Eds.), Conference of the International Group for the Psychology of Mathematics Education: 17/07/2006-21/07/2006 (Vol. 2, pp. 225-232). PME, Charles University.

Carpenter, T. P., & Franke, M. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In Speiser, R., Maher, C. A., Walter, C. N.(Eds.), Proceedings of the 12th ICMI study conference. The future of the teaching and learning of algebra (pp. 155-162).

Carraher, D., & Schliemann, A. (2002). The transfer dilemma. Journal of the Learning Sciences, 11(1), 1–24. https://doi.org/10.1207/S15327809JLS1101_1

Chen, L. (2021). Preservice teachers' enactment of mathematical probes through speech and gestures [Unpublished doctoral dissertation], Perdue University.

Demonty, I., Vlassis, J., & Fagnant, A. (2018). Algebraic thinking, pattern activities and knowledge for teaching at the transition between primary and secondary school. Educational Studies in Mathematics, 99(1), 1–19. https://doi.org/10.1007/s10649-018-9820-9

Ellis, A. B. (2007). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. Journal of the Learning Sciences, 16(2), 221–262. https://doi.org/10.1080/10508400701193705

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. Journal for Research in Mathematics Education, 42(4), 308–345. https://doi.org/10.5951/jresematheduc.42.4.0308

Font, V., & Contreras, A. (2008). The problem of the particular and its relation to the general in mathematics education. Educational Studies in Mathematics, 69(1), 33–52.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. American Educational Research Journal, 38(3), 653–689. https://doi.org/10.3102/00028312038003653

Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. Journal of Teacher Education, 60(4), 380–392. https://doi.org/10.1177/0022487109339906

Hallman-Thrasher, A. & Spangler D. (2020). Posing purposeful questions with high-demand tasks. *Mathematics Teacher: Learning and Teaching PK-12, 113*, 446–459. https://doi.org/10.5951/MTLT.2019.0297

Hufferd-Ackles, K., Fuson, K. C., & Sherin, M. G. (2004). Describing levels and components of a math-talk learning community. Journal for Research in Mathematics Education, 35(2), 81–116. https://doi.org/10.2307/30034933

Kaput, James J. (1999). Teaching and learning a new algebra with understanding. In E. Fennema, & T. Romberg (Eds.), Mathematics classrooms that promote understanding (pp. 133-155). Erlbaum.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. J. Lester (Ed.), Second Handbook of Research on Mathematics Teaching and Learning (pp. 707–762). Information Age Pub.

Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). Early algebra: Research into its nature, its learning, its teaching (pp. 3–32). Springer.
Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. Mathematical Thinking and Learning: An International Journal, 7(3), 231–258.
Lee, L. & Wheeler D. (1987), Algebraic thinking in high school students: Their conceptions of generalisation and justification, Concordia University.
Nathan, M. J., & Koedinger, K. R. (2000). Teachers' and researchers' beliefs about the development of algebraic reasoning. Journal for Research in Mathematics Education, 31(2), 168–190. https://doi.org/10.2307/749750
National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). Common core state standards for mathematics. National Governors Association Center for Best Practices. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf
Nguyen F. (2020). https://www.visualpatterns.org/
Radford, L. (2010). Elementary forms of algebraic thinking in young students. In M. F. Pinto & T. F. Kawasaki (Eds.), Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 73–80). PME.
Saldaña, J. (2013). The coding manual for qualitative researchers (2nd ed). SAGE.
Stacey, K. (1989). Finding and using patterns in linear generalizing problems. Educational Studies in Mathematics, 20, 147-164.
Stacey, K., & MacGregor, M. (1997). Building foundations for algebra. Mathematics Teaching in the Middle School, 2(4), 252–260.
Strachota, S. (2020). Generalizing in the context of an early algebra intervention (La generalización en el contexto de una intervención algebraica temprana). Journal for the Study of Education and Development, 43, 347-394.
Syed, M., & Nelson, S. C. (2015). Guidelines for establishing reliability when coding narrative data. Emerging Adulthood, 3(6), 375–387. https://doi.org/10.1177/2167696815587648

TEACHERS' MOTIVATIONS FOR AND ENACTMENTS OF AMBITIOUS MATHEMATICS TEACHING: THE CASE OF "ROUGH DRAFT MATH"

Amanda Jansen
University of Delaware
jansen@udel.edu

Crystal Collier
University of Delaware
crystalc@udel.edu

The purpose of this study was to characterize variations in how teachers enacted an approach to ambitious mathematics teaching: "rough draft math." We also examined teachers' motivations for their enactments. Thirty-two teachers from five states in the U.S.A. were recruited to participate in interviews based on recommendations from leaders of book studies focusing on this teaching practice. All participants enacted "rough draft math" by intentionally building their classroom culture to welcome students' draft thinking to achieve the goal of promoting students' productive dispositions. However, additional variations in enactments drew attention to potential tensions between multiple goals of ambitious mathematics teaching (empowering students and learning through revising). Findings suggest insights for supporting teachers' learning to teach ambitiously; findings also contribute to a knowledge base for teaching.

Keywords: Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes

Ambitious mathematics teaching, according to Anthony and colleagues (2015) "involves skilled ways of eliciting and responding to each and every student in the class so that they learn worthwhile mathematics and come to view themselves as competent mathematicians" (p. 46). During ambitious mathematics instruction, students engage in discourse to learn mathematics and develop positive identities. Additionally, "the motivation to do things differently is as important as knowledge and skill to creating consistently ambitious practice" (Lampert et al., 2013, p. 227). To understand how to support mathematics teachers with learning to enact ambitious mathematics instruction, investigating variations in teachers' efforts to teach ambitiously, and their motivations to do so, can provide insight.

One approach to ambitious mathematics instruction involves inviting students to publicly share their in-progress or unfinished thinking, workshopping those ideas, and explicitly inviting students to revise their thinking – engaging students in *rough draft math* (Jansen, 2020; Jansen, Cooper, Vascellaro, & Wandless, 2016). Teaching mathematics through inviting drafts and revising is a complex activity. To do this work, teachers need facility with eliciting and responding to students' thinking (e.g, Fraivillig et al., 1999). It is not easy to listen to one another (teacher-student, student-to-student, student-teacher) in ways that lead to learning for students and the teacher during a mathematics lesson (Hintz et al., 2018). Building on students' thinking (Leatham et al., 2015) so that revising takes place is demanding work. It may be challenging in-the-moment to recognize strengths (Jilk, 2016) in students' draft thinking and manage classroom interactions to position students' contributions as valuable for their peers' learning (Wood, 2013).

The purpose of this study is to document variations of enactments of an ambitious teaching practice, "*rough draft math*," to contribute to building a knowledge base for teaching (c.f., Hiebert et al., 2002). Researchers can contribute toward building a knowledge base for teaching by documenting practitioners' work in a public, sharable manner so that more educators can learn from teaching practice. Given that incorporating rough drafts and revising in math class involves a range of practices, from creating a classroom culture where drafts are welcome to explicitly structuring revision experiences that move students' thinking forward (Jansen, 2020), there are many entry points into "rough draft math" that can be documented and understood.

Variations in Enactments and Teachers' Motivations

Ambitious teaching practices have interdependent but different goals. Consider scientific inquiry: this teaching practice has multiple goals, such as making sense of phenomena (constructing claims or explanations), articulating understandings (presenting arguments), and persuading others (critiquing, evaluating, and defending arguments) (Berland & Reiser, 2011). Similarly, "rough draft math" has multiple goals: making sense of mathematics, articulating understandings (drafting ideas), improving one's understanding (revising drafts), and promoting students' agency so that they see themselves as capable of knowing and doing mathematics. These multiple goals illustrate complexity and provide opportunities for tensions.

Berland and Reiser (2011) discuss that a complex and ambitious teaching practice, such as scientific inquiry, can vary in enactment in different ways, in part due to these multiple goals. Teachers could emphasize some goals more than others. Some goals might be adopted selectively, and other goals might be set aside. Efforts to achieve certain goals might take place more consistently. In scientific inquiry, this might look like one classroom of students learning to competitively persuade each other while another classroom of students learns to seek to understand one another's thinking (Berland, 2011). What do these variations of enactment look like in mathematics classrooms where teachers incorporate "rough draft math" and why might they occur?

Motivation can be understood through multiple perspectives, including goal-orientation theory and the role of interest. Goal orientation theory attempts to explain how and why people try to reach certain objectives (Kaplan & Maehr, 2006). Motivation can look like meaningful engagement with a practice, and such engagement develops, in part, through cultivating interest (Renninger & Hidi, 2015). Teachers with greater interest are likely to continue to enact a practice about which they have learned (Ginsberg & Wlodkowski, 2019), particularly if they see the practice as positively meeting the needs of their students (Appova & Arbaugh, 2017). However, their enactments could differ depending on the goals they seek to achieve.

In this study, we seek to understand the enactments of teachers who have expressed an interest in enacting "rough draft math," as well as the goals they sought to achieve when enacting it. This study addresses the following research questions: Among teachers with expressed interests in enacting "rough draft math," how do teachers define "rough draft math"? How do these teachers' self-reported enactments of "rough draft math" vary by their motivations?

Methods

Data Collection

We interviewed 32 mathematics teachers from five states in the U.S.A. Teachers were recruited for this study among those who had participated in book study groups to discuss *Rough Draft Math* (Jansen, 2020). Participating teachers were recommended by mathematics coaches or professional development leaders who had facilitated a book study. Additional participants were recruited through Twitter and invited to participate if they had read the book on their own and were attempting to enact relevant instructional practices in their classrooms. Teachers in this sample taught in grades K-12, from first grade through AP Calculus.

Prior to the interview, participating teachers were asked to send, via email, a digital artifact that provided an example of how they incorporated rough drafts and revising into their instruction. The artifact provided context for participants' descriptions of their enactments. Examples of artifacts included the following: web links to Desmos activities, slide decks used during lessons, handouts of mathematics tasks, sample student work that demonstrated students' initial drafts and revisions of their thinking, videos of classroom activities, assessments with directions, and photographs of students collaborating.

Interviews lasted between 35 and 60 minutes, and all interviews were conducted via video call, one-on-one. Interview questions included the following: If you were to tell a colleague about "rough draft math," how would you explain it to them? What can you tell me about this artifact? Tell me the story of this artifact. Why did you decide to do this in your classroom? What happened in your classroom? How is this artifact an example of "rough draft math"? Participants chose their own pseudonyms. All interviews were audio recorded and transcribed.

Data Analysis

Data analysis was conducted by both authors. In our first phase of analysis, we wrote analytic memos for each participant (Saldaña, 2013). In each memo, we described our conjectures for that teachers' thinking based on our initial listening to the interview and editing of the transcript. For the next phase of analysis, we used an exploratory process to inductively create descriptive codes (Saldaña, 2013) based on teachers' talk. We then created descriptive codes for two categories: (a) motivations to enact (or functions for) "rough draft math" and (b) enactments (forms) of "rough draft math." We identified the motivations that teachers described in relation to the artifact they shared. Although teachers described other ways of enacting "rough draft math" in their practice, in addition to the artifact, we assumed that the artifact represented an important part of their teaching practice, from their perspective. The second level of analysis involved frequency coding. We assessed whether, among the motivations reported for enacting instruction aligned with the shared artifact, any of the motivations were repeated throughout the interview and how often they were repeated. Repetition of an idea signals a level of importance to the speaker or an effort to emphasize an idea in their spoken talk (Tannen, 1989).

Results

Below, we illustrate three types of motivations that teachers in this sample expressed as functions for enacting "rough draft math," along with the enactments they described as forms for achieving these motivations. However, before we describe these motivations and enactments, we share teachers' definitions for "rough draft math." These definitions illustrate that teachers could think about rough drafting and revising in mathematics in a variety of ways, which then implies that they could have different motivations to enact "rough draft math."

Rough Draft Math: Teachers' Definitions

When teachers explained "rough draft math," they described components such as, (a) eliciting initial *drafts* of students' thinking, (b) providing students with opportunities to *revise* their thinking, often after some form of (c) *collaboration* with peers, and then (d) inviting students to *reflect* on their growth in their thinking and understanding. When asked to explain what "rough draft math" meant to a colleague, Ms. Dougherty said,

> …it's simply your first draft of thinking, if you're just initially starting out your thinking. You haven't really shared those thoughts with anybody. You haven't revised any of your thinking yet. It's just your first draft… then from there we build on that, and we use our rough draft to help build that understanding to make it stronger and understand where we came from along the way.

In Ms. Dougherty's explanation, she addressed all four components, highlighting the process of first drafts in students' thinking. She alluded to revising when she spoke about needing to "build" on the first draft "to make it stronger." Her explanation included an implicit reference to collective, collaborative work when she shifted to using a "we" pronoun. She suggested incorporating reflection on growth in thinking by saying, "understand where we came from along the way."

However, not every teacher mentioned all four components in their definitions, which suggests that, among some teachers, a subset of components of "rough draft math" resonated more than others. Ms. Briggs emphasized the process of iteratively drafting and collaborating in her explanation when she said, "I would say that it's thinking of math of less like a finished product and more like something you're continuously working on together."
Ms. Kakkar emphasized the process of revising in her explanation.

> For me, when I've heard about rough draft and when we had worked with [math coach] and talked about it, the first thought, to be honest, that was, yes, it should be in math. The rough draft particularly did not come into my mind at first, but since you know how in ELA teachers talk … this makes perfect sense in math, too… you do trials and errors and estimation and all, seeing the patterns, a couple of factors combined together to actually reach out to an answer… It has to be rough draft. It has to be multiple trials. It has to be multiple revising through your work rather than just, okay, here is my answer.

Mr. Vandelay highlighted reflecting on growth and humanizing mathematics learning in his explanation.

> I'd say it's a way of humanizing the students' mathematical experience. It gets away from a traditional feeling that math is about being good, getting right answers, and having your value either internally or affirmed from, without coming from a place of feeling like those are the things that matter. Whereas the true mathematical experience is always a work in progress. It's always about developing thinking and we learn so much from our mistakes. We learn so much from thought processes that lead the wrong direction. And so it's really a sense of honoring what all students should be having as an everyday experience learning mathematics.

In common across these teachers' descriptions, whether they expressed four components or not, was that it made sense to them to treat mathematics learning as a gradual, iterative process of drafting and revising ideas, and they expressed value for drafting and revising while learning and doing mathematics in school. These teachers' talk illustrates what drafting and revising meant to them, in their mathematics classrooms, and some teachers put relatively stronger emphases on different components of "rough draft math."

Motivations for and Enactments of "Rough Draft Math"

Teachers reported multiple motivations for enacting "rough draft math," as represented in their artifacts. The first motivation below, foster productive dispositions, was reported in some manner by every teacher. However, the other two motivations (cultivate learning through revising and reflecting; empower students) were not reported by every teacher. The table below illustrates differences in reported enactments that appeared to be aligned with specific motivations for enacting "rough draft math."

Table 1: Motivations for and Enactments of "Rough Draft Math"

Motivation	Enactment
Foster productive dispositions • Develop confidence in students	Intentionally build a classroom culture where students' ideas are valued at any stage • Labeling talk as "rough drafts"

• Increase students' participation and effort • Safety for intellectual risk taking	• Rights of Learners • Protocols for discussion • Anonymous student sharing • Multiple revision experiences in class
Cultivate learning through revising and reflecting • Normalize mistakes in math • Develop metacognition	Explicitly incorporate revising into mathematics instruction • Revision routines, such as test corrections • Reflection on revising
Empower students • Shift authority to students • Disrupt school structures that control students	Decenter teachers' authority • Student self-assessment • Peer collaboration to learn

Motivation: Foster Productive Dispositions. Every participant reported an overarching motivation to enact rough drafting in math class: to foster productive dispositions among students. The most frequently associated enactment with this motivation was teachers' efforts to intentionally build a classroom culture where students' ideas were welcome and valued at any stage of their thinking. Teachers often mentioned more than one motivation for enacting rough draft math, but this was the motivation and associated enactment shared by all participants. The function or motivation of fostering productive dispositions had a range of subdimensions: develop confidence in students, create safe experiences to encourage intellectual risk taking and collaboration, and increase students' participation and effort in math class.

One of the artifacts Ms. Dougherty shared was a set of pictures of her middle grades students collaborating – standing together at white boards on the wall working on math and sitting together on the floor working in a group, and when she described her motivations for collaborative work, Ms. Dougherty talked about building classroom culture. Ms. Dougherty described that she encouraged students to verbally label their thinking as being in a rough draft stage when they shared aloud, so that students could feel more confident to participate without being judged for imperfect or incorrect thinking.

> I definitely see the language that my students use. Um, they do a lot of, they'll use the language of rough draft math. So, very often they'll raise their hand or start their thinking saying, well, I'm not sure yet, but this is what I'm thinking right now. And they have a very comfortable space and, and I commend them. So, at the beginning of the year, I do a lot of like that [by saying], "Awesome, rough draft math talking, thank you, starting out your thinking that way. Now we all know that we're going to give you the space to share and revise if you need to." Um, I think they look for revision possibilities rather than thinking of their math as a final step. Um, so they're very aware of the idea that we can look at what others are thinking, um, and constantly revise. So I think that that's been really beneficial… I think I am making moves toward getting their confidence to be stronger in mathematics. I don't just mean confidence in ability as much as just confidence in sharing their thinking, you know, just, just get something out there. Like everybody can share something, you know?

Ms. Dougherty reported that welcoming drafts and normalizing revising in mathematics class supported students' development of their confidence so that more students would participate.

To demonstrate efforts to build culture, Ms. Briggs's artifact included a photograph of "rights of learners" (Kalinec-Craig, 2017) generated by her first-grade students, such as "I have the right to be cared for and care for others." She shared that her motivation for promoting these rights was to support students' safety and willingness to participate. According to Ms. Briggs, "…when students are talking, they have that space to work through some of their disequilibrium. … students feel psychologically safe to make those mistakes and to have confusion around a topic." She also said that it was important to her to be "really instilling that love of math and, um, making kids feel like that they were experts in math is going to have a long-term impact on them." She used "rights of learners" to build a classroom culture to support her students.

Ms. Kakkar shared an artifact of a sample of students' revised work, and she also said that she wanted to increase students' risk taking and promote participation. When describing her artifact, she said that she encouraged revising by using a protocol – learning to listen / listen to learn (Wolfe, 2021) – to teach students to hear and learn from each other. She also incorporated anonymous sharing of students' solutions during whole class discussion for workshopping and revising. Ms. Kakkar said that she enacted teaching in these ways so that students could "see their progression and how could they realize it, how much it is important to, to really build upon each other's ideas?" She also said that she wanted "to make students comfortable."

Ms. Burnett provided a short video of one of her mathematics lessons, edited into segments to illustrate an in-class, multiple revision experience, and she expressed that she wanted to encourage students to take risks, to build students' confidence, and increase their participation. She said that she wanted to share this video to illustrate what a "rough draft math" lesson looked like in her class: students worked in groups on a challenging task, she purposefully selected student strategies to be shared to the whole class while peers asked one another questions about the work, students would then revisit and revise their work in groups, and then share out again. Ms. Burnett said, "it takes the pressure off the kids" to welcome rough draft thinking. She said that, regarding a success that she has had,

> Definitely confidence building. And the students are not afraid to make mistakes… I would have [in the past] some kids just sit and do nothing. They would just hold their pencil. And I knew it was like a frozen fear of, I don't wanna make a mistake. It's [rough draft math] taken that away. Those kids are just like, let's do this.

These enactments were motivated by teachers' drives to foster productive dispositions among students and to support students with participating to learn mathematics through discourse.

Motivation: Cultivate Learning through Revising and Reflecting. Although every teacher described a motivation to promote productive dispositions, not every teacher described using rough drafting to explicitly cultivate learning through revising. Ms. Apple's artifact was an example of a revising routine and protocol that she used for students to correct their tests in AP Calculus AB, and she said that she wanted to promote growth in students' thinking and awareness of that growth.

> I like the metacognition that's involved in the protocol. So really thinking about their thinking and thinking about where their mistake, misunderstanding or, misconception or misunderstanding [was]. I think of all three of those things as different, different ways to think about the problem and where it was wrong in their thought process. Because sometimes they just make a silly mistake. Sometimes they don't understand the concept and they need to go back and revise the way they think about a problem like that.

Ms. Green also shared an activity for chapter test revisions and reflection activity and a "quicker rough draft" process for a routine for in the moment revisions.

> I do it on the test, the final draft thinking on the test revisions… the revision part is when I feel like a topic has been solidified… then the quicker rough draft would be for those moments where we're in a new learning. … I use it [rough draft math] in different ways, depending on what I'm working on. So are we working [on] solidifying an idea? Are we working on learning a new idea for the first time?

Both of Ms. Green's artifacts illustrated two revision experiences, but they had different goals: working toward solidifying thinking (test revisions) or developing new ideas (discussing and revising drafts). Teachers like Ms. Apple and Ms. Green spoke about reaching a "final draft."

Motivation: Empower Students. A subset of teachers described that they wanted to enact rough draft math in their teaching to promote equity through empowering students. Mr. Vandelay shared an artifact that documenting requirements for (and purpose of) a student portfolio for their self-assessment, which was his primary grading practice; he said that he incorporated self-assessment to transfer authority to students by dismantling traditional school grading practices. According to Mr. Vandelay,

> I personally have felt since I read *Rough Draft Math*, that one of the things that really does is honors the fact that we need to rethink the way we're evaluating students, so that what we value in terms of student thinking as a process isn't undermined by grading practices … especially with students who have low status or perceived low status or low mathematical identities, they may have come through in the high school setting case years of feeling like they're not good at math, they can't do it. They've never been successful… the greater idea is sort of building on identity. And the social sense is like, why do we believe the things about ourselves as humans, as math students as all those things, and where do those beliefs come from, whose voices have formed those beliefs and who's been neglected?

Mr. Vandelay's reflection on his assessment practices illustrate the value of empowering students to evaluate themselves and illustrate his questioning of evaluation systems in school.

In addition to photograph's Ms. Dougherty shared another artifact, which was a Desmos activity for engaging students in self-reflection of their learning and self-assessment to transfer autonomy and agency to her students.

> … I've been working toward a gradeless classroom, and I know that that's sort of a whole different topic, but, for me, the gradeless classroom is really part of rough draft math, because I really want the students to have that idea of judgment being removed from their thinking. So, each week I have the students go through a reflection... it's this constant balance between letting them know like, "Hey, yeah, I am in the room, and I am the teacher, but you have control of your learning. Own it, make it what it is."

Mr. Vandelay and Ms. Dougherty's enactments and motivations illustrate that assessments are a type of teaching practice that can be enacted in different ways for different reasons. If we contrast their assessment practice of student self-assessment with the activity of test corrections shared by Ms. Green and Ms. Apple, both sets of teachers enacted rough draft mathematics, but Ms. Green and Ms. Apple wanted students to reach perhaps an externally determined learning destination, while Mr. Vandelay and Ms. Dougherty wanted to empower their students.

Discussion

As Berland and Reiser (2011) discussed, variations in teachers' enactments of an ambitious teaching practice can be understood as variations in relative emphasis of different components of a practice, perhaps emphasizing different goals. One difference between enactments of "rough draft math" was indicated by participants' talk about assessments. Some teachers reported that they engaged their students in test revisions to help their students achieve a "final draft" understanding through revising errors. In contrast, other teachers said that they engaged their students in self-assessment to promote students' agency and autonomy. Tensions between goals are possible, such as a goal of empowering students and a goal of promoting learning through revising. Some teachers may intend for students to revise to achieve a final draft, or a pre-determined understanding, which could reflect either the discipline of mathematics or the teacher as an authority. Other teachers may want students to have a voice in not only whether they learn, but what and how much they learn, as a part of their efforts to empower students.

Future research could investigate how teachers can cultivate interest in and facility with enacting "rough draft math." We focused this study on teachers who reported a high level of interest in enacting "rough draft math." An analysis of the goals that teachers reported for enacting "rough draft math" can provide promise in generating greater uptake of the practices, as teachers could be encouraged to enact "rough draft math" if they gain awareness of the range of goals they can achieve. A contribution of this study is that understanding how these teachers enacted "rough draft math" provided insight for starting points teachers might take to learn to enact ambitious teaching; teachers could consider the goals they want to achieve and learn about possible enactments that could help them achieve these goals.

Additionally, future research could observe enactments of "rough draft math," in contrast to this study, which focused on teachers' self-reports. Observations of enactments would allow for characterizing collective practices of drafting and revising, which could provide insight into collective understandings generated and ways that students engage and participate. However, studying teachers' self-reports provides insights into teachers' goals through their voices as well as which enactments they value in pursuit of these goals.

A contribution of this study is the illustration of how ambitious mathematics teaching can vary in its enactments. Although all participating teachers reported a goal of promoting students' dispositions and enactments by encouraging a culture of drafting, they varied in their emphasis of goals to (a) learn through revising and reflecting or (b) to empower students. Additionally, results illustrated how varying goals for enacting ambitious mathematics teaching could potentially be in tension. These results contribute to building a knowledge base for how mathematics teachers can put "rough draft math" into practice.

References

Anthony, G., Hunter, R., Hunter, J., & Duncan, S. (2015). How ambitious is "ambitious mathematics teaching". *Set: Research Information for Teachers, 2, 45*, *52*(10.18296).

Appova, A., & Arbaugh, F. (2018). Teachers' motivation to learn: Implications for supporting professional growth. *Professional Development in Education*, *44*(1), 5-21.

Berland, L. K. (2011). Explaining variation in how classroom communities adapt the practice of scientific argumentation. *Journal of the Learning Sciences*, *20*(4), 625-664.

Berland, L. K., & Reiser, B. J. (2011). Classroom communities' adaptations of the practice of scientific argumentation. *Science Education*, *95*(2), 191-216.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education*, *30*(2), 148-170.

Ginsberg, M. B., & Wlodkowski, R. J. (2019). Intrinsic motivation as the foundation for culturally responsive social-emotional and academic learning in teacher education. *Teacher Education Quarterly*, *46*(4), 53-66.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one?. *Educational Researcher*, *31*(5), 3-15.
Hintz, A., Tyson, K., & English, A. R. (2018). Actualizing the rights of the learner: The role of pedagogical listening. *Democracy and Education*, *26*(2), 8.
Jansen, A. (2020). *Rough Draft Math: Revising to Learn*. Portsmouth, NH: Stenhouse Publishers.
Jansen, A., Cooper, B., Vascellaro, S., & Wandless, P. (2016). Rough draft talk in mathematics classrooms. *Mathematics Teaching in the Middle School, 22*(5), 304-307.
Jilk, L. M. (2016). Supporting teacher noticing of students' mathematical strengths. *Mathematics Teacher Educator*, *4*(2), 188-199.
Kalinec-Craig, C. A. (2017). The rights of the learner: A framework for promoting equity through formative assessment in mathematics education. *Democracy and Education*, *25*(2), 5.
Kaplan, A., & Maehr, M. L. (2007). The contributions and prospects of goal orientation theory. *Educational Psychology Review*, *19*(2), 141-184.
Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education*, *64*(3), 226-243.
Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education*, *46*(1), 88-124.
Renninger, K. A., & Hidi, S. E. (2015). *The power of interest for motivation and engagement*. Routledge.
Saldaña, J. (2013). *The coding manual for qualitative researchers* (2nd ed.). Sage Publishers.
Tannen, D. (1989). *Talking voices: Repetition, dialogue, and imagery in conversational discourse* (Vol. 6). Cambridge University Press.
Wolfe, J. A. (2021). Teaching is a journey: a journey in becoming. *Mathematics Teacher: Learning and Teaching PK-12*, *114*(3), 258-261.
Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education*, *44*(5), 775-808.

FUNCTIONAL CONCERNS THAT SHAPE TEACHERS' IN-THE-MOMENT DECISION-MAKING

Lara Jasien
CPM Educational Program
larajasien@cpm.org

Lisa Amick
University of Kentucky
lisa.amick@uky.edu

As a field, we have a limited understanding of what teachers do, what motivates them, and how they learn (Kennedy, 2016). Here, we develop a grounded framework of inquiry-based teaching in mathematics classrooms by beginning with primary evidence of teachers teaching. We seek to articulate an evidence-based account of teachers' functional (rather than aspirational) concerns so that those who support teachers — such as curriculum writers, professional learning specialists, coaches, and teacher educators — can ground their support in the realities of the work of teaching. Our research question is: "What are the functional concerns that shape teachers' decision-making in-the-moments of teaching?".

Keywords: Curriculum, Instructional Activities and Practices, Instructional Vision, Professional Development.

Vignette:

At the non-profit Mathematics Education Program *(*MEP*, pseudonym), the curriculum writing and professional learning teams frequently debate the question, "Where does the writing stop and the professional learning begin?" This is a problem of practice for them. When the writing team designs curricular features that are meant to support equitable outcomes for students, but the professional-learning team sees these curricular features being subverted in practice such that instruction does not actually change, then who is responsible for supporting teachers to enact the curriculum differently? How can these curriculum writing and professional learning teams bridge the ideals designed into the curriculum with supporting teachers to enact those ideals?*

The constraints teachers face in their local and institutional contexts of school as a workplace make answering these questions a perpetual challenge not just for MEP but for mathematics teacher educators, curriculum writers, professional learning specialists, coaches, and teachers everywhere. This is not surprising given the well-documented literature on intended versus enacted curriculum (e.g., Remillard 2005, 2014; Stein et al., 2007). Teachers have a lot to manage at multiple scales: over the span of a course, a unit, a lesson, and even within small group interactions (Ball & Cohen, 1996; Cohen, 2011; Ehrenfeld & Horn, 2020). The decisions that teachers make at each of these scales are consequential for students' learning. These decisions are big and small, conscious and unconscious. Surprisingly, these decisions often do not align with teachers' espoused beliefs about what good teaching looks like (Fang, 1996). This presents a challenge for those designing curricula and professional learning for teachers: Even if teachers know what equitable and ambitious instruction looks like and describe their practice as aligning with their beliefs about equitable and ambitious instruction, this alignment too often is not evidenced in observation. The example from MEP begins to make more sense. The problem of supporting teachers to shift their practices is not only a perpetual problem faced by MEP but is endemic to educational change.

To support teachers in making different decisions, researchers have developed reflection tools for use in professional learning and teacher education. For example, the Teaching for Robust Understanding (TRU) framework helps teachers think about dimensions of classroom activities that might serve as levers for change, helping them to be more reflective about their practice, thus more intentional, and, hopefully, more equitable in their decision-making (Schoenfeld, 2019). Other resources for teachers' decision-making are action-based, such as Smith and Stein's (2008) description of 5 Practices for orchestrating whole class mathematical discussions. Such tools are most helpful for instructional decisions made in the planning stages of instruction. In-the-moment decisions are much more difficult to influence because of the complexity and dynamism of classroom ecologies (Pfister et al., 2015). Likely because frameworks like TRU and the 5 Practices provide aspirational visions of mathematics instruction that focus on ideal models, their descriptions do not quite capture the complexity of decision-making in teaching mathematics. The simplicity of aspirational frameworks is useful for supporting teachers to try new things; yet, teachers still struggle with student-centered teaching.

Currently, our theories of and designs for teacher learning are based more on aspirational, ideal types than on nuanced, evidence-based understandings of what teachers do, what motivates them, and how they learn (Kennedy, 2016). Here, we develop a grounded framework of inquiry-based teaching in mathematics classrooms by beginning with primary evidence of teachers teaching. We seek to articulate an evidence-based account of teachers' functional—rather than aspirational—concerns so that those who support teachers, such as curriculum writers, professional learning specialists, coaches, and teacher educators can ground their support for teachers in the realities of the work of teaching. Our research question is, *What are the functional concerns that shape teachers' decision-making in-the-moments of teaching?.*

Conceptual Framework: Functional Concerns as Lived Pedagogical Responsibility

Recently, teachers' decision-making has been described through a tripartite model of pedagogical judgment (Horn, 2020), including dimensions of pedagogical action, pedagogical reasoning, and pedagogical responsibility. Pedagogical action describes the choices teachers make, whether intentional or not. Pedagogical reasoning is the logic behind these actions—the same action may be done for many different reasons. Pedagogical responsibility involves teachers' sense of obligations to ethical principles or to situational constraints such as mandates from administration. Together, these constitute the pedagogical judgment that informs teachers' decision-making.

Teachers' functional concerns involve a practical sense of pedagogical responsibility. Functional concerns are pragmatic in that they focus on elements of instruction that are within teachers' control. For example, teachers might feel an obligation to support learners' retention and transfer of content, but they focus their attention—their concern—in-the-moments of teaching on creating coherent and relevant mathematics experiences. In this way, learners' retention and transfer may be goals that emerge from teachers' sense of pedagogical responsibility, as distinguished from the phenomenon of interest here: teachers' concerns.

Teachers' functional concerns are pragmatic and grounded in teachers' lived experience of their work. There is no widely accepted theory or characterization of what teaching is or should be — teachers can view themselves as managers, mediators, actors, salespeople, role models, empowerers, and more (Kennedy, 2016). This is likely shaped by teachers' identity and experiences outside of teaching but is also likely shaped by local school and department cultures, as well as by the curriculum used.

As noted in the opening vignette and evidenced through mathematics education research (Remillard 2005, 2014), curriculum shapes but does not determine teacher's pedagogical activities. Depending on teachers' conceptual understanding of their work, it is quite possible to use a curriculum intended for inquiry in ways that reduce cognitive demand and short circuit opportunities for productive struggle. At the same time, curriculum likely does influence teachers' conceptual understanding of their work, as curriculum is one of the factors that shape school as a workplace (Kennedy, 2010; Jackson, 1968; Moore Johnson et al., 2012). For example, if the local curriculum is designed to get students to produce quick and correct answers on multiple-choice items, teachers' conceptual understanding of their work is likely shaped in ways that are unproductive for student learning.

Theoretical Framework: Conceptual Practices constitute Functional Concerns

Teachers' work is composed not just of a series of pedagogical actions but of conceptual practices — pedagogical actions and the meaning behind them. Examples of conceptual practices include building on student thinking, seeking correct answers, and offering mathematical choice.

From a situative perspective of learning, concepts are not ideas in a person's head, they are conceptual practices — "recurring patterns of purposeful activity that are distributed over people and technologies" (Hall & Horn, 2012, p. 241) in particular practices. Practices, simplistically explained, are activities and routines carried out for a particular taken-as-shared purpose with taken-as-shared meanings in a particular community (Lave & Wenger, 1991) and learning is "an active process of distributing cognition over people and things" (Hall & Horn, 2012, p. 214). Thus, conceptual practices are inextricably connected to the environments in which they take place. In teaching, conceptual practices are tied to curricula, local and national educational policies, departmental cultures, communities served, and more. Still, as mid-level concepts in teaching, teachers' conceptual practices are "more general than a particular set of actions taken in the specific context" with implications for learning that are "better instantiated (and thus more testable) than those of the more familiar Theories of education and psychology" (Lehrer & Schauble, 2004, p. 637-638).

Figure 1 offers a visualization of our conceptualization of conceptual practices as mid-level concepts in teaching. Pedagogical responsibility (part of Horn's tripartite of pedagogical judgment) is depicted as being composed of both functional and aspirational concerns, though the boundary between them is perhaps fuzzy and overlapping. Functional concerns — the topic of this manuscript — are composed of related, distinct, yet minimally overlapping conceptual practices. Each conceptual practice is composed of multiple teacher actions, represented by star shapes. The same teacher action may appear in multiple conceptual practices. Teacher actions contribute to particular conceptual practices based on the context in which the action occurs: It depends on the meaning behind the action.

Figure 1. Illustration of the relationship between pedagogical responsibility, functional concerns, conceptual practices, and teaching actions, with stars representing teacher actions.

Research Design

The mathematics curriculum used by teachers in this study was an inquiry-based support course from MEP that foregrounded student exploration, sensemaking, and relationship building. The course was ungraded and emphasized big ideas in algebra, leveraging technology to support student collaboration on complex mathematical ideas. Because inquiry-based support courses are the exception rather than the norm, the findings of this analysis should contribute to a better understanding of what it means to teach in such courses, and thus also support more nuanced understandings of how to support teacher learning.

Studying teachers' functional concerns in an inquiry-based support course is timely. In a 2019 ProQuest Social Sciences Premium Collection database search for "math*" AND "intervention" or "double dose" AND "inquiry" or "reform", only 177 peer reviewed articles in scholarly mathematics education journals were found. Of those, only 20 focused on middle and high school; and of those, only two contained research relevant to the current study on teachers' in-the-moment work of teaching (Johnson, 2001; Marshall et al., 2011).[1] In addition, because this is an inquiry course, the findings should also be relevant to teachers not in support courses but teaching heterogeneous (i.e., mixed-ability) classrooms.

Curriculum

The curriculum context for the teachers in this study was the *Big Ideas* (pseudonym) curriculum from MEP. It was designed based on a developing theory of action grounded in the assumptions of equitable, ambitious mathematics instruction: that all students can engage meaningfully in mathematics in ways that are personally and socially empowering, and that such meaningful engagement begins with enjoyable mathematical experiences in which students can exercise high levels of personal agency. Given this, the features of *Big Ideas* target four anticipated student outcomes: (1) increased engagement in and communication about mathematical sensemaking, (2) positive dispositions towards mathematics, (3) learning grade-level mathematics, and (4) stronger teacher-student relationships. Two lessons were observed in this study. Lesson 3.2 — WHEN WILL IT STOP? Representing Data with a Graph — required students to physically experiment with dropping a ball and recording their observations in Desmos. Lesson 4.11 — CAN YOU STAY ON THE PATH? Lines of Best Fit — required students to explore by plotting lines to stay in certain uncolored regions of a graph in Desmos.

Participants

We collected data from six teachers: Ms. Brewers, Ms. Ennings, Ms. Holder, Ms. Padshaw, Ms. Shea, and Ms. Walker (pseudonyms). These teachers taught in different regions across the United States in schools with different departmental cultures and serving different kinds of communities. The teachers' years of experience spanned from three years to twenty years. All teachers taught grade-level mathematics courses in addition to teaching *Big Ideas*.

Data Collection

Data collection aimed to capture teachers' talk, gestures, and movement through the classroom. Data include Swivl video-records of observations of six teachers from different locations across the United States teaching the same two lessons described above. Each lesson

[1] The other 18 articles returned in the search fell into categories of identifying the effectiveness of intervention (10) and the impact of policy (1), descriptions of interventions (2), the development of student thinking (1) and mathematical interest (1), historical overviews (1), the impact of course trajectories on career paths (1), and a discussion of methods of randomized trials (1).

was followed by a video-recorded debrief with a curriculum coach. These debriefs focused on teachers reflecting on how the lesson went, with coaches acting as thought partners rather than advisors. In addition, teacher interviews were conducted at the beginning and end of the year, but these data are not included in this analysis as interview data tended to illuminate teachers' aspirational concerns.

Data Analysis

To understand the work of teaching from teachers' perspectives, we used grounded theory's constant-comparative methods (Strauss & Corbin, 1990) and made close-to-data inferences about teachers' functional concerns in inquiry-based mathematics classrooms by looking at what was stable and variant across six teachers' enactments of and reflections on two of the same lessons.

Phase 1: Parsing the data. In the first round of analysis, we created time-indexed content logs (Derry et al., 2010) for classroom videos and teacher debriefs with their coaches. Then, we looked for recurring teacher actions — for example, circulating with and looking at the teacher's computer while talking to student groups, walking away from student groups when they have stated a correct answer (sometimes without saying anything), referencing a lesson plan, or reminding students of particular participation norms. We also attended to outcomes of these actions that seemed to satisfy teachers, allowing us to infer meaning behind the actions. We noted, for example, the ways in which teachers organized and set norms for participation structures, cared for relationships between themselves and students and between students, and held themselves and students accountable for pacing.

Phase 2: Narrating emergent themes. From the content logs of both debriefs and classroom videos, we generated descriptive, narrative memos (e.g., Cobb & Whitenack,1996; Powell et al., 2003) organized by themes for each lesson for each teacher. These themes (such as emphasis on individual students' math history and modeling curiosity) were not determined a priori, but rather were emergent from and closely tied to what became salient as we parsed each teacher's lessons and debriefs. Sometimes this process required going back to the video record to clarify or gain more nuance on a teacher's instruction.

Phase 3: Identifying conceptual practices. After identifying themes in each lesson for each teacher, the themes were compared and collapsed into broad descriptive categories (Strauss & Corbin, 1990). Collapsing qualitatively different kinds of pedagogical activities happening across the teachers allows their common conceptual practices of teaching that can be leveraged for more ecologically valid forms of professional development. We then looked through the content coded under each emergent conceptual practice and engaged in a process of defining each conceptual practice, selecting a spectrum of examples for each. This process allowed us to see how conceptual practices were both unique and related.

Collapsing conceptual practices into functional concerns. The Phase 3 examination of the distinctness of each conceptual practice resulted in four clusters of conceptual practices, with each cluster constituting a functional concern in teaching inquiry-based mathematics.

In our constant comparative analysis of six teachers teaching the same two lessons at different points in the year, we identified 15 conceptual practices. These 15 conceptual practices (codes) represent the daily work of teaching; they are inferred meanings of what teachers do and say (activities, not actions). While distinct, these conceptual practices are also related to each other, clustering together to support four overarching concepts/concerns in teaching inquiry-based mathematics.

Findings

The six teachers in this study came from different contexts but shared the same curriculum. We found that four functional concerns adequately described many of the different decisions teachers made across their classrooms. The overarching functional concerns are all mathematics teaching specific, but also differently emphasize concerns about mathematics or students. The four functional concerns were identified by examining clusters of codes that represented 15 emergently identified conceptual practices (Table 1). Most of these conceptual practices play into multiple functional concerns, including across more strongly mathematics- and student-centered concerns. The conceptual practices are located under the functional concern they most heavily constituted in the data, not the only functional concern they constituted.

Table 1. Teachers' functional concerns in the moments of teaching and the conceptual practices that most heavily constitute them

Functional Concerns	Key Conceptual Practices
Mathematical Correctness and Completeness *How can I make sure that student work is accurate, thorough, and finished?*	● Seeking final answers on a task or sub-task ● Navigating student uncertainty ● Assessing student progress ● Redirecting students' mathematical thinking ● Building on students' mathematical thinking
Mathematical Relevance and Coherence *How can I make sure that mathematics is meaningfully connected to students' prior knowledge and lives?*	● Generalizing mathematical strategies ● Connecting mathematics ● Connecting mathematics to everyday life ● Offering mathematical choices
Student Motivation *How can I make sure that students are interested enough to get started and to persevere through uncertainty and confusion?*	● Offering non-mathematical choices ● Attending to student confidence (high inference) ● Caring for relationships (high inference)
Student Access *How can I make sure that students can get started on the task and that struggle stays productive?*	● Unpacking the task ● Setting participation norms ● Organizing materials and student bodies

An abbreviated illustration of the functional concerns

Importantly, these functional concerns are neither positive nor negative. They look very different in their manifestations across the six teachers in our study. Table 2 provides example teaching profiles to illustrate how each functional concern looked different across the six teachers of this study. The table presents one conceptual practice for each teacher under each functional concern. Looking down each column provides a glimpse into the teachers' practice and looking across the rows provides a glimpse into the diversity with how conceptual practices manifest as teachers navigate their functional concerns.

Table 2. Teacher Profiles along the four functional concerns.

Brewer	Ennings*	Holder*	Padshaw	Shea	Walker*
Mathematical Correctness and Completeness					
Presses Ss** to build on their thinking by asking questions	Asks Ss questions but does not press for explanation	"Plays" a hot- and -cold game to get Ss to the correct answer	Prioritizes sensemaking over getting through the lesson and so rarely gets to closure	Accepts incorrect answers as correct and supplements with extra practice	Asks Ss to share strategies in a whole class but does not ask for connections
Mathematical Relevance and Coherence					
Asks Ss what they already know about the lesson's topic to connect to their prior learning	No evidence of connections across mathematics or daily life	References prior lessons to elicit transfer on discrete problem steps	Asks Ss what they already know about the lesson's topic to open up a pre-lesson	Creates spontaneous mini lessons to mathematize students' lives	Tells Ss about connections to prior lessons by mentioning the lessons
Student Motivation					
Expresses interest in Ss' mathematical intuitions	Expresses care for Ss through kind voice and personal greetings to students	Creates sense of urgency by using Desmos teacher dashboard during circulation	Highlights Ss' mathematical ideas and mistakes as interesting and valuable	Offers non-mathematical choices such as whether the lights should be on or off	Slows pacing to account for a perceived mismatch between ability and rigor
Student Access					
Explicitly reminds small groups of how to collaborate during every interaction	Gives students participation directions at the beginning of the task	Tells students to collaborate and addresses individual students	Pre-teaches mathematical ideas and explicitly reminds Ss of participation norms	Gives extensive directions for sometimes up to five minutes	Gives explicit participation directions but is unprepared with resources

*one observation only, **Ss = students

As is evidenced in the table, these functional concerns are not ideals but instead can look quite different in practice. Contrasting Ms. Brewer's and Ms. Shea's columns particularly highlights how differently teachers can enact these functional concerns. Being aware of the multiplicity of ways that teachers work to address these four functional concerns is critical for anyone striving to support mathematics teachers.

At the same time as the individual functional concerns are not ideals, these functional concerns as a set are not an ideal model. There is no particular emphasis or balance that everyone needs to follow; however, a simple thought experiment reveals that missing any one of these functional concerns may lead to inequitable teaching practice. For example, an overemphasis on student access and motivation and a lack of attention to mathematical correctness and completeness is often described as a characteristic of classrooms where teachers have low expectations of students' abilities. An overemphasis the other way around, for example, can describe classrooms with a "no excuses, zero tolerance" classroom culture. In such classrooms, little effort is made to meet students where they are. Productive teaching profiles can be described with all four functional concerns, although the distribution of the conceptual practices that teachers spend their time on can look very different.

Conclusion and Implications

The four functional concerns identified are not what we as educational researchers consider to be ideal. To harken back to the opening vignette, the writing team at MEP would not write curriculum to support teachers to focus on student correctness and completeness. While correctness and completeness are valued by MEP, MEP also understands that they are often overvalued in the classroom. Thus, the curriculum writers prioritize supporting teachers to focus on students' processes of collaborative problem-solving. Still, ignoring teachers' institutionally shaped functional concern for correctness and completeness presents a problem for teachers and the professional learning specialists who support them in the classroom. But, if teachers already focus on correctness and completeness, why do materials need to account for it rather than work around it? The evidence from this study suggests that even teachers using an ungraded inquiry-based curriculum and who have had a plethora of support from professional learning specialists still focus on these four functional concerns.

Because of this we argue that teachers' practical sense of pedagogical responsibility needs to be attended to in designs for learning. By attended to, we do not mean designed-out (e.g., preventing teachers from focusing on correctness), but instead, designed-in such that teachers can clearly see how these four functional concerns will be met. Since teachers will meet these functional concerns in many different ways when left to their own devices (often determined by apprenticeship of observation, Lortie, 1975), it is critical that our aspirational, idealized forms of instructional practice are designed-in through a lens of functional concerns. We wonder: Could designing curriculum, professional learning opportunities, and teacher education with these functional concerns and their conceptual practices in mind — in addition to aspirational concerns such as those articulated in the TRU framework and the 5 practices — help bridge the gulf between the ideals designed into curricula and the realities of teachers' ability to enact those ideals in the constrained contexts of school-as-a-workplace?

References

Cobb, P., & Whitenack, J. W. (1996). A method for conducting longitudinal analyses of classroom video recordings and transcripts. *Educational Studies in Mathematics, 30*(3), 213-228.

Cohen, D. K. (2011). Teaching and its predicaments. Harvard University Press.

Derry, S. J., Pea, R. D., Barron, B., Engle, R. A., Erickson, F., Goldman, R., ... & Sherin, B. L. (2010). Conducting video research in the learning sciences: Guidance on selection, analysis, technology, and ethics. *The Journal of the Learning Sciences, 19*(1), 3-53.

Ehrenfeld, & Horn, I. S. (2020). Initiation-entry-focus-exit and participation: a framework for understanding teacher

groupwork monitoring routines. *Educational Studies in Mathematics, 103*(3), 251–272. https://doi.org/10.1007/s10649-020-09939-2
Enyedy, N., Goldberg, J., & Welsh, K. M. (2006). Complex dilemmas of identity and practice. *Science Education, 90*(1), 68-93.
Fang, Z. (1996). A review of research on teacher beliefs and practices. Educational Research, 38(1), 47-65.
Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: theory and practice, 15*(2), 273-289.
Hall, R., & Horn, I. S. (2012). Talk and conceptual change at work: Adequate representation and epistemic stance in a comparative analysis of statistical consulting and teacher workgroups. *Mind, Culture, and Activity, 19*(3), 240-258.
Hall, R., & Jurow, A. S. (2015). Changing concepts in activity: Descriptive and design studies of consequential learning in conceptual practices. *Educational Psychologist, 50*(3), 173-189.
Horn, I.S. & Little, J.W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers' workplace interactions. *American Educational Research Journal, 47*(1), 181-217.
Horn, I. S. (2020). Supporting the development of pedagogical judgment. In G. M. Lloyd & O. Chapman (Eds.), *International handbook of mathematics teacher education, Volume 3: Participants in Mathematics Teacher Education* (2nd Edition, pp. 321-342). Brill Sense.
Horn, I. S., Garner, B., Buenrostro, P., & Marshall, S. (2022). Putting formative feedback into practice. In I. S. Horn & B. Garner, *Teacher learning of ambitious and equitable mathematics instruction: A sociocultural approach* (pp. 98-122). Routledge.
Jackson, P.W. (1968, 1990). The Daily Grind. Life in Classrooms. Teachers College Press.
Jackson, K., Gibbons, L., & Sharpe, C. (2017). Teachers' views of students' mathematical capabilities: Challenges and possibilities for ambitious reform. Teachers College Record, 119, 1-43. https://www.tcrecord.org/content.asp?contentid=21791
Johnson, I. (2001). Standards-Based Teaching: Alive and Well in Portugal. *Mathematics Teaching in the Middle School, 6*(9), 538-542. http://www.jstor.org/stable/41181024
Kennedy, M.M. (2010). Attribution error and the quest for teacher quality. *Educational Researcher, 39*(8), 591-598.
Kennedy, M. M. (2016). How does professional development improve teaching?. *Review of Educational Research, 86*(4), 945-980.
Lehrer, R., & Schauble, L. (2004). Modeling natural variation through distribution. *American Educational Research Journal, 41*(3), 635-679.
Lortie, D. (1975). School-teacher: A sociological study. University of Chicago Press.
Marshall, J. C., Smart, J., & Horton, R. M. (2011). Tracking perceived and observed growth of inquiry practice: A formative plan to improve professional development experiences. *Science Educator, 20*(1), 12-22.
Moore Johnson, S., Kraft, M.A., Papay, J.P. (2012). How context matters in high-need schools: The effects of teachers' working conditions on their professional satisfaction and their students' achievement. *Teachers College Record, 114* (10), p. –. http://www.tcrecord.org
Pfister, M., Opitz, E. M., & Pauli, C. (2015). Scaffolding for mathematics teaching in inclusive primary classrooms: A video study. *ZDM, 47*(7), 1079-1092.
Schoenfeld, A. H. (2019). Reframing teacher knowledge: a research and development agenda. *ZDM*, 1-18.
Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. Review of Educational Research, 75(2), 211-246.
Remillard, J. T., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM*, 46(5), 705-718.
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical thinking and learning, 10*(4), 313-340.
Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester's (Ed.), *Second handbook of research on mathematics teaching and learning, 1*(1), (pp. 319-370). National Council of Teachers of Mathematics.
Strauss, A., & Corbin, J. (1990). Basics of qualitative research. Grounded theory procedures and techniques. Newbury Park, CA: Sage.

RELATIONSHIP BETWEEN DISCOURSE AND STUDENTS' EVALUATION OF STRATEGIES TO SUBTRACT FRACTIONS WITH MIXED NUMBERS

Kristy Litster
Valdosta State University
klitster@valdsota.edu

Beth L. MacDonald
Illinois State University
blmacd1@ilstu.edu

This mixed methods study explored the relationship between two types of small-group mathematics discourse (Reflective and Exploratory) and the quantity and type of strategies discussed. The study focused on 97 fifth-grade students, organized into 34 small groups (17 Reflective, 17 Exploratory) and the strategies they used to solve two problems (4 ½ - 3 ¾; 4-3 ¾). Results indicated that both discourse types can encourage student evaluation of multiple computational strategies. However, several factors seemed to increase or decrease the strategies discussed. In exploratory discourse, the student in the group writing the answer may limit discussion by solving the problem themself. In reflective discourse, when students all agreed on the same answer they were less likely to discuss any strategies. For both discourse types, dissonance in answers or strategies often prompted the discussion of multiple strategies to determine accuracy.

Keywords: classroom discourse; number concepts and operations; elementary school education

Factions are one of the most difficult areas for elementary students, as well in-service elementary teachers, and prospective teachers, possibly due to limited abilities to mentally represent fractions and how these mental representations relate to computational procedures (Gabriel et al., 2013; Litster, MacDonald, & Shumway, 2020). Mathematics teacher educators (MTEs) can support a deep conceptual understanding of fractions through emphasizing multiple ways of solving fractional computations (AMTE, 2017; NCTM, 2014). Learning multiple computation strategies in isolation can increase student confusion, however comparing and contrasting fraction algorithms side-by-side for the same problems or problems that differ in small but import ways may help reduce misconceptions and deepen student understanding (Newton, Willar, & Teufel, 2014). Mathematics discourse is one strategy that MTEs can use to encourage their students to evaluate a variety of computational procedures for the same or similar problems (Wachira et al., 2013).

Theoretical Perspective

Discourse can impact student retention of learning and transfer of learning to novel situations in three ways (see Fig. 1). First, it has the potential to increase opportunities to learn by promoting higher levels of cognitive demand through reasoning and justification of solution strategies (Litster, 2019). Second, it has the potential to help build the automaticity of subskills by chunking fragmented schema in long term memory for easier retrieval (Sweller & Mayer, 2015). Third, it can help refocus the sophistication of student strategies from surface to structural similarities (Georgius, 2014; Schoenfeld, 2018).

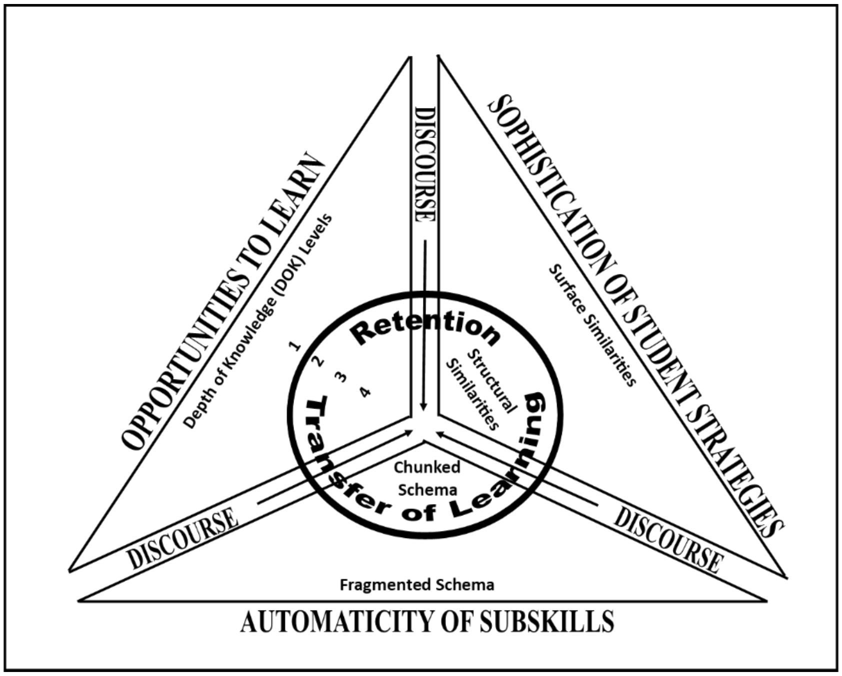


Figure 1: Influence of Discourse on Retention and Transfer of Learning

This study focuses on this third side of the triangle, sophistication of student strategies. When students develop a deeper conceptual understanding of fractional operation strategies, their retention is improved and they are more likely to transfer and apply this understanding in novel situations (Sidney & Alibali, 2015; Uttal et al., 2013). Siegler and Svetina (2006) explain that students are more likely to change their strategy only if the new strategy increases accuracy or efficiency. Novices may work backwards from a solution to identify their strategy whereas experts often work forward and tend to leave out important steps (Feldon, 2007). Discourse provides opportunities for students to see the value of new strategies as students share logical arguments around their strategy choices, which can not only focus students towards relational strategies, but also provide opportunities for chunked schemas. These discussions can also promote students to look at the structural similarities and differences between strategies.

MTEs can organize student-directed mathematics discourse in one of two ways. First, MTEs can group students into cooperative learning groups where students engage in exploratory discourse to collaboratively solve a mathematics task (Rojas-Drummond & Mercer, 2003). As students work together, they may share and evaluate a variety of strategies they may not have considered on their own to solve the problem or check their answer in order to quickly dismiss inaccurate or inefficient strategies. This co-construction of learning through explanations, redefinitions and interactions may promote a deeper understanding for multiple strategies (Wachira et al., 2013). However other researchers contest that an imbalance of power dynamics within a group engaging in exploratory discourse may limit student involvement which would decrease the strategies evaluated by the group (Walter, 2018).

Second, MTEs can provide time for students to engage with the task individually and then engage in reflective discourse using strategies such as the “Think-Pair-Show-Share” model to compare answers and strategies (Hunt et al., 2018; Walter, 2018). Time to think individually provides opportunities to evaluate potential strategies to solve the problem and write down ideas to bring to the group discussion, which can create more equitable opportunities to clearly explain their thinking for a variety of strategies (Kalamar, 2018). However other researchers contest that waiting until after the task is complete results in cumulative talk, meaning students agree with one idea and do not discuss other strategies (Rojas-Drummond & Mercer, 2003).

The purpose of this study was to evaluate the quantity and types of strategies discussed when fifth-grade students engaged in exploratory discourse (solving and discussion problems together) and reflective discourse (solving problems individually and then discussing solutions) for two related subtraction fractions problems with mixed numbers.

Methods

This mixed-methods study is part of a larger study where 97 fifth-grade students, organized into 34 small groups engaged with two real-world task-sets involving operations with fractions and decimals, for a total of 26 tasks (Litster, 2019). This study takes a closer look at student discourse relating to one of the tasks in the Harry Potter Task-Set (Task H5) to help answer the research question: What is the relationship between each discourse type (exploratory and reflection) and the quantity and types of subtraction fractional computation strategies discussed by fifth-grade students?

Participants and Procedures

Four fifth-grade classes participated in this study. The researcher organized discourse for the Harry Potter Task-Set so that half of the students (2 classes, 17 groups) engaged in exploratory discourse while the other half (2 classes, 17 groups) engaged in reflective discourse. Groups engaged in exploratory discourse were given all 13 problems in the Harry Potter task set which compares different aspects of success for the book and movie (e.g., ticket/book sales and ratings) and asked to work together to solve the problems. Groups engaged in reflective discourse were given the same problems in three separate chunks of blocked time, allowing students to work on one set of problems independently (e.g., H1-H4 on ticket/book sales) and then discussing that set of problems before working independently on the next set (e.g., H5-H9 on book/movie ratings).

Data Source and Analysis

The two main data sources in this study were students' written work and verbal discourse relating to problems 5b and 5c from the H5 task. In problems 5b and 5c (Fig. 1), students were asked to compare the difference between the average reviewer ratings for the Harry Potter and the Half-Blood Prince movie across multiple websites (i.e., Barnes & Nobles, IMDb, Amazon, iTunes, Rotten Tomatoes).

***Harry Potter & the Half Blood Prince* – Movie Adaptation**	
Source	**Average Rating (out of 5 stars)**
IMDb	$3\frac{3}{4}$
Rotten Tomatoes	$3\frac{3}{4}$
Amazon	$3\frac{3}{4}$
Barnes & Noble	$4\frac{1}{2}$
iTunes	4

5b. Compare the **difference** between the average Barnes & Nobles rating and the average IMDb rating.

5c. Compare the **difference** between the average Amazon rating and the average iTunes rating.

Figure 1: Task H5, Sub-questions 5b and 5c.

The researchers analyzed the data in three phases. First, they qualitatively coded video data to identify the types of strategies discussed by each of the 34 groups of students (Saldaña, 2015). One student in each group wore a Go-Pro camera on their chest which captured the audio and visuals for their group discussions. Students' written work (task sheets and scratch paper) was also used in this phase to verify the notations used in the discussed computational strategies. Second, the number of unique strategies were quantitatively counted to identify the number of strategies discussed by each group. Third, students' written work and verbal discussions were coded to identify any patterns that may contextualize the quantitative results.

Results

Qualitative results relating to the types of strategies identified six unique strategies discussed by students to solve Task H5. Figure 2 illustrates and explains five strategies students discussed relating to Sub-task 5b while Figure 3 illustrates and explains these same five strategies as well as a sixth unique strategy groups discussed relating to Sub-task 5c.

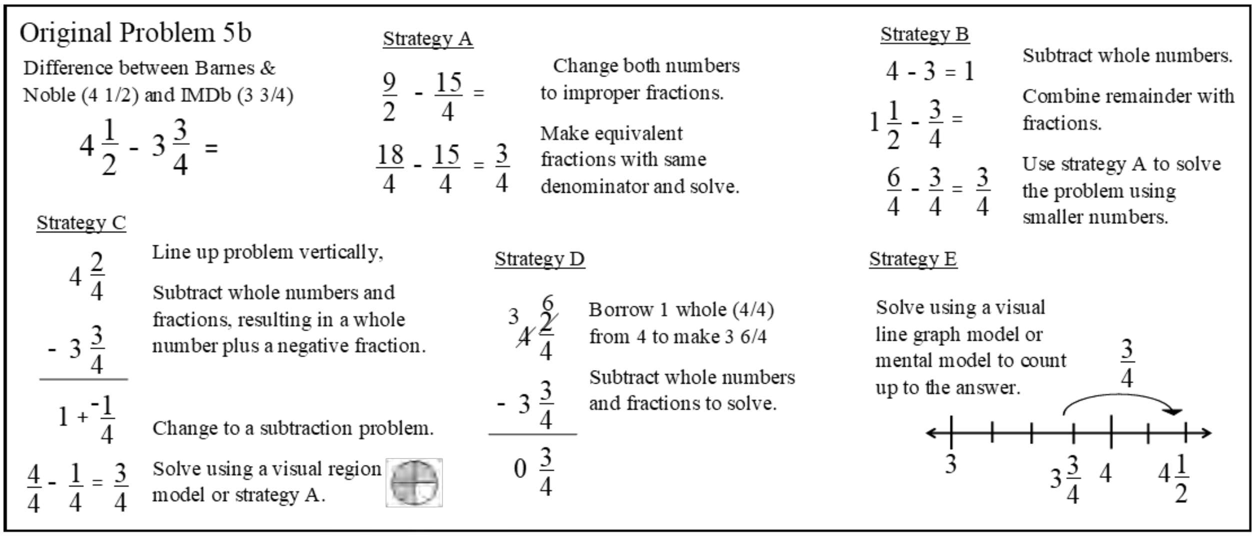


Figure 2: 5 Strategies Discussed Relating to Sub-task 5b.

As seen in Figure 2, Strategies A and B focus on procedures using equivalent fractions and the conversion of mixed numbers into improper fractions to subtract and solve the problem. Strategy C uses negative number and the relationship between addition and subtraction to solve the problem. Strategy D uses the subtraction procedure to "borrow" from whole number in order to subtract and solve the problem. Strategy E uses a number line or a mental model to count up to the answer.

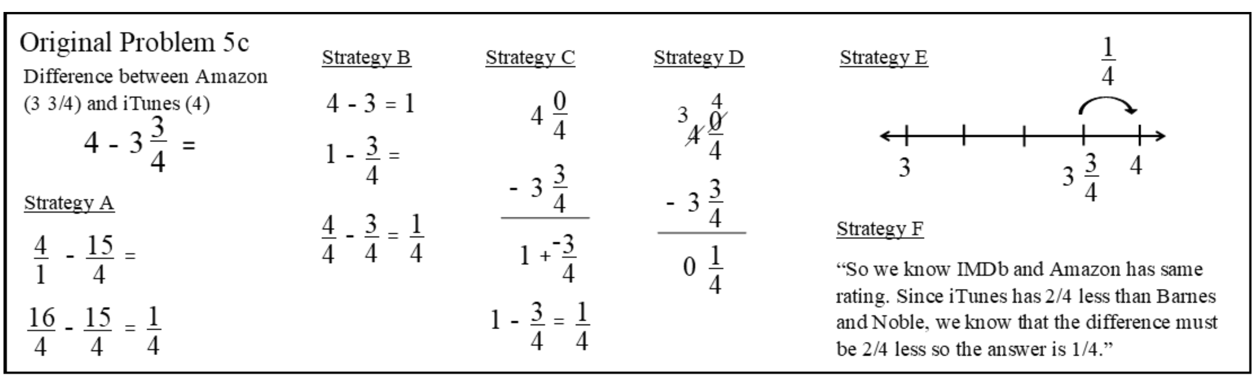


Figure 3: 6 Strategies Discussed Relating to Sub-task 5c.

As seen in Figure 2, the same five strategies discussed for sub-task 5b were also discussed for sub-task 5c. Additionally, some groups used a new strategy (F) to solve this sub-task, which involved using relational differences between the quantities in Problem 5b and 5c to mentally calculate the results.

Quantitative results identified several patterns between frequency and type of strategies discussed for each discourse type. Table 1 shows the frequency count of unique strategies discussed per group. A "0" for reflective indicates groups confirmed an answer only and did not discuss a specific strategy. "0" for exploratory indicates one student completed the task for the group independently, without discussing their strategy.

Table 1: Frequency Count of Unique Strategies Discussed per Group

	Reflective (N=17)		Exploratory (N=17)	
Strategy Count	5b	5c	5b	5c
0	4	7	1	2
1	5	2	10	8
2	5	6	3	5
3	3	1	1	2
4	0	1	2	0

As seen in Table 1, groups engaged in exploratory discourse were more likely to discuss at least one strategy; however reflective groups that did discuss strategies were more likely to discuss more than one strategy.

Table 2 shows the frequency count for the number of groups that discussed each strategy. In this table, columns do not equal the number of groups as the number of strategies discussed varied from group to group. N=Max possible number for each cell.

Table 1: Frequency Count of Unique Strategies Discussed per Group

	Reflective (N=17)		Exploratory (N=17)		Total (N=34)	
Strategy	5b	5c	5b	5c	5b	5c
A	6	6	10	11	16	17
B	3	3	7	9	10	12
C	1	1	2	1	3	2
D	1	1	3	0	4	1
E	3	5	3	5	6	10
F	0	2	0	1	0	3

As seen in Table 2, the three most common strategies across both groups were the two more traditional strategies, A (N=16/17) and B (N=10/12), as well as the visual strategy E (N=6/10). Additionally, similar strategies seemed to be discussed from one sub-question to the next.

The qualitative examination of student work showed that every strategy on the individual scratch paper for students in exploratory groups was discussed. However, only one reflective discourse group had every student in the group use the same strategy to solve problem 5b or 5c. This means that most reflective discourse groups had the potential to discuss 2-3 different fraction subtraction strategies. Unfortunately, an examination of verbal discourse showed that students were less likely to discuss the strategies used by students in the group when everyone agreed on the answer (e.g., "So I got 3/4." "Yep, me too and the next one is 1/4.") (see Table 1, strategy count 0). Only three reflective discourse groups where everyone agreed on the answer discussed all the different strategies they used to solve problems 5b and 5c. Instead, differences in answers for 5b and 5c prompted a discussion about the mathematics and student strategies. For example, when a reflective discourse group had different answers to a problem, their first action was usually a comparison of strategies to identify any errors within the procedures. Excerpt 1 below shows one example of students comparing strategies after having a difference in answers.

Matt: On this one [Q5b] I got 1 1/4
James: I got 3/4.
Matt: It's bigger cause 4 minus 3 is 1 and 3/4 minus 1/2 is 1/4. **[inaccurate Strategy B]**
James: You change that [pointing to 4 ½] to 4 2/4, which is 3 6/4 minus 3 ¾ [**Strategy D]**
Matt: Oh yea [changes answer to 3/4]
[Video 150H3, Time Stamp 5:21-5:37]

In Excerpt 2 below, a student in a reflective discourse group fixed their own mistake while sharing their strategy for the same problem.

Troy: I did 3/5
Laney: The ratings are a ¾ difference.
Adam: Yep. Did you change the denominators so [point 18/4 – 15/4 = ¾ on paper] [Strategy A]
Troy: [pointing to his paper, where he has 5 circles cut in fourths, with 4 1/2 shaded in and 3 ¾ crossed out, Strategy E] They are all cut up into different pieces and . . . but yea, it would be ¾
[Video 120H3, Time Stamp 7:46-9:23]

In a few cases, when students could not agree on an answer after sharing their own strategies, they turned to a new strategy to check their answer. Excerpt 3 is from an exploratory group that agreed on a general strategy (i.e., subtraction), worked on the problem individual, and then engaged in self-directed reflective discourse to discuss their answers to the same problem.

Anna: Well, look. Difference means subtract, right?
Tonya: Okay
Stacey: And the Barnes and Nobel were 4 ½ and IMBD were 3 ¾
[All 3 students calculating on their own paper]
Tonya: I got 1 3/4
Anna: I am thinking 1 ¼ cause 4 minus 3 is one. . . [Inaccurate Strategy B]
Stacey: [Points to 3 ¾ on her paper] I am usually wrong so . . .
Anna: But 3 ¾ you can't really do that.
Stacey: Yeah, but 3 ¾ we change this into a fraction and change this into 4 and times by 4 and make an 8 so . . . [Inaccurate Strategy A]
Tonya: But 4x4 is 16.
Anna: Oh yea, so even I got it wrong.
[All thinking]
Tonya: Look guys, 1, 2, 3 [pointing to number line] [Strategy E]
Stacey: Okay, so ¾ then.
Anna: So it would be ¾.
[Video 320H2, Time Stamp 3:22-12:22]

Five of the 17 Exploratory Discourse groups used this method to solve tasks H5b and H5c. The remaining exploratory discourse groups (n=12) worked through the problem together, sharing a single pen. The student holding the pen usually initiated which strategy their group used to solve the problem. Exploratory discourse groups were more likely to discuss multiple strategies when there was a disagreement on the validity or ease of use for a particular strategy. For example, in Excerpt 4 below, students are solving from H5c. Leah is holding the pen, but Karen has a disagreement on the validity of Leah's mental strategy.

Leah: [thinking, then writes ¼ on the paper]
Karen: But how did you get that, cause you couldn't get it that fast.
Leah: Um I . . [thinking] did it in my head?
Karen : You can't minus it in your head. That would equal [grabs scratch paper and writes 18/4 – 9/4 = 9/4, then converts to 2 ¼] 2 1/4 so how did you do that in your head. Because you have to change it into an improper fraction. This is the Barnes and Nobel one [18/4] and change it to same denominator and then improper. **[Inaccurate Strategy A]**
Bill: But this is the answer [points to 1/4.
Leah: If you take 1 off of it and spread it into 4/4 and then minus 3/4 is 1/4.
Bill: It's basically 4 0/4 – 3 ¾. Subtract wholes then the fractions.
Karen: You can't do 0 minus 3!
Bill: Yea, negative ¾ plus 1. **[Strategy C]**
Karen: But did you minus these? you have got to subtract these [points to 18/4 – 9/4]
Bill: Changing to same denominator would be 16/4 not 18. **[Correcting Error in Strategy A**, on Karen's paper to read "16/4-15/4=1/4"**]**
Karen: But I did the math.
Leah: So 4 minus 3 is 1 and then 4/4 minus 3/4 is 1/4. **[Strategy B]**
Bill: It's the same thing [pointing to the ¼ on scratch paper].
Karen: Yea, okay.
[Video 360H2, Time Stamp 3:27-13:20]

In Excerpt 5 below, Carla, Joy, and Tina used strategy A to complete problem 5b (Joy held pen), but struggled to use the same strategy to complete 5c (Tina held pen).

Joy: So first, find a common denominator [Attempting Strategy A]
Tina: Wait, then how do you turn 4 into a fraction?
Joy: So to keep it a whole number, you draw a line and put a 1 under it.
Carla: What if we subtract the wholes first, like 4 minus 3 is 1. [Strategy B]
Tina: So we would have 1 minus ¾?
Carla: And then times 4
Tina: So 4/4 and then take away ¾ . . . okay, ¼ [writes answer]
[Video 450H3-4, Time Stamp 1:00-3:49, 0:00-0:30]

In Excerpt 6 below, Amy (holding pen) is unsure about how to solve the problem and asks for help, which initiates a discussion on fraction strategies.

Amy: [Reads prompt for H5b out loud] I'm a little bit confuzzled [SIC].
Brit: Ok, so compare the difference, we are subtracting.
Chris: So, so, 4 1/2 minus 3 3/4 is you take the 3 away from the 4 and then you have one left over so 3/4 minus 4 1/2 is three left over **[Strategy B]**
Amy: Wait how is 3/4 minus 4 1/2, 3?

Brit: So, 4 minus 3 is 1 and 1 is 4/4 [draw circle and divides in 4 sections] and half [draws half circle and divided in 2 sections] then you take away 3/4 [shades in 3] and you are left with 3/4. **[Strategy E to Explain Strategy B]**

Amy: I'm a little bit confuzzled but not as much.

Chris: You try the next one [Indicating H5c]

Amy: Okay, one . . .[writes 4-3=1], two . . .[draws circle and cuts in fourths], three . . . [shades in three, writes 1/4 as answer], **[Strategy E]** I did it! Confuzzle over.

[Video 370H2, Time Stamp 5:13-9:58]

As we can see in each of these examples, dissonance created by differences in products (Excerpts 1-3) or processes (Excerpts 4-6) prompted students to move beyond simply confirming or computing answers to discussing the accuracy, reliability, or ease of use for different strategies.

Conclusions

The results in this study confirmed research that both types of discourse (reflective and exploratory) have the potential to encourage students to discuss multiple strategies (Kalamar, 2018; Wachira et al., 2013). They also confirmed fears that one student initiating the discourse in exploratory discourse may limit discussion to a single strategy (Walter, 2018) and fears that students engaging in reflective discourse may engage in cumulative talk (Rojas-Drummond & Mercer, 2003)

The results in this study also add to current research by highlighting factors that may increase or decrease the discussion of multiple strategies. For example, when students all agreed on an answer during reflective discourse, they were more likely to engage in cumulative talk; however, when there was dissonance, or differences in answers, they were more likely to discuss multiple strategies to determine accuracy. Dissonance also prompted the discussion of multiple strategies in exploratory discourse to determine accuracy.

In summary, teachers can use either discourse type to encourage students to discuss multiple ways of solving problems. Teachers may want to require exploratory discourse groups to identify two or more methods to verify their answers in order to encourage students to discuss multiple strategies to solve problems (Kazemi & Stipek, 2017). Teacher may also want to set up discourse procedures to encourage students engaging in reflective discourse to identify similarities and differences in strategies to promote discussions relating to how students solved the problems, beyond confirmation of the answer.

References

Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for Preparing Teachers of Mathematics*. Available online at amte.net/standards.

Boaler, J. (2006). Promoting respectful learning. *Educational Leadership, 63*(5), 74.

Feldon, D. F. (2007). Implications of research on expertise for curriculum and pedagogy. *Educational Psychology Review, 19*(2), 91-110.

Gabriel, F., Szucs, D., & Content, A. (2013). The mental representations of fractions: Adults' same-different judgements. *Frontiers in Psychology, 4*(385), 1-6.

Georgius, K. (2014). *Planning and enacting mathematical tasks of high cognitive demand in the primary classroom* (Doctoral dissertation, The University of Nebraska-Lincoln). Retrieved from http://digitalcommons.unl.edu/teachlearnstudent/41.

Hunt, J. H., MacDonald, B., Lambert, R., Sugita, T., & Silva, J. (2018). Think-Pair-Show-Share to Increase Classroom Discourse. *Teaching Children Mathematics, 25*(2), 78-84.

Kalamar, C. (2018). *Questioning techniques that increase student engagement during the mathematics lesson* (Master's thesis). Moravian College, Bethlehem, PA.

Kazemi, E., & Stipek, D. (2009). Promoting conceptual thinking in four upper-elementary mathematics classrooms.

Journal of education, 189(1-2), 123-137.
Litster, K. (2019). *The relationship between small-group discourse and student-enacted levels of cognitive demand when engaging with mathematics tasks at different depth of knowledge levels* (Doctoral dissertation). Utah State University, UT.
Litster, K., MacDonald, B. L., & Shumway, J. F. (2020). Experiencing active mathematics learning: Meeting the expectations for teaching and learning in mathematics classrooms. In A. Apposa, R. M. Welder, & Z. Feldman (Eds.) Special Issue: Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. *The Math Enthusiast 17*(1 & 2, June), 615-640.
National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to action: Ensuring mathematical success for all.* Reston, VA: Author.
Newton, K. J., Willard, C., & Teufel, C. (2014). An examination of the ways that students with learning disabilities solve fraction computation problems. *The Elementary School Journal 115*(1), 1-21.
Rojas-Drummond, S., & Mercer, N. (2003). Scaffolding the development of effective collaboration and learning. *International Journal of Educational Research, 39*(1-2), 99-111.
Schoenfeld, A. H. (2018). Video analyses for research and professional development: the teaching for robust understanding (TRU) framework. *ZDM*, 1-16.
Sidney, P. G., & Alibali, M. W. (2015). Making connections in math: Activating a prior knowledge analogue matters for learning. *Journal of Cognition and Development, 16*(1), 160-185.
Siegler, R. S., & Svetina, M. (2006). What leads children to adopt new strategies? A microgenetic/cross-sectional study of class inclusion. *Child development, 77*(4), 997-1015.
Sweller, J. & Mayer, R. (2015, November 1). *Virtual seminar – professor Richard Mayer talking with professor John Sweller*. Retrieved from https://www.youtube.com/watch?v=T88InIudNk0
Uttal, D. H., Amaya, M., del Rosario Maita, M., Hand, L. L., Cohen, C. A., O'Doherty, K., & DeLoache, J. S. (2013). It works both ways: Transfer difficulties between manipulatives and written subtraction solutions. *Child Development Research, Volume 2013*, 1-13.
Wachira, P., Pourdavood, R. G., & Skitzki, R. (2013). Mathematics teacher's role in promoting classroom discourse. *International Journal for Mathematics Teaching and Learning, 13(*1), 1-40.
Walter, H. A. (2018). Beyond turn and talk: Creating discourse. *Teaching Children Mathematics, 25*(3), 180-185.

UNDERSTANDING PRESERVICE TEACHERS' NOTICING OF ONLINE TEACHING

Jinqing Liu	Orly Buchbinder	Merav Weingarden
University of New Hampshire	University of New Hampshire	University of New Hampshire
jinqing.liu@unh.edu	orly.buchbinder@unh.edu	merav.weingarden@unh.edu

The rapid move to online teaching brought about by the global pandemic highlighted the need for the educational research community to develop new conceptual tools for characterizing these environments. In this paper, we propose a conceptual framework Instructional Technology Triangle (ITT) which extends the instructional triangle of teachers, students, and content to include technology as a mediating mechanism. We use the ITT framework to analyze noticing patterns in the written reflection of a prospective secondary teacher, Nancy, who, over the course of one semester taught online four lessons integrating reasoning and proof . The fluctuations in Nancy's noticing patterns, in particular, with respect to technology, shed light on her trajectory of learning to teach online and the role of reflective noticing in this process. We discuss implications for teacher preparation and professional development.

Keywords: Teacher Noticing, Preservice Teacher Education, Online and Distance Education.

Objectives

The pivotal role of technology in teaching mathematics has been widely recognized and thoroughly researched over the last decades (Ball et al., 2018; Bray & Tangney, 2017; Clark-Wilson et al., 2020; Hillmayr et al., 2020). These studies concerned teachers' classroom practice with technology, teachers' professional development (PD) for teaching mathematics with technology and impacts of technology on mathematics teaching and learning (see Clark-Wilson et al., 2020 for an extensive review). Most of these studies were conducted in a traditional face-to-face setting. Some teaching online has been addressed, but mainly in teacher PD and Massive Open Online Courses (MOOC) (e.g., Taranto et al., 2020). However, the complete and widespread shift to online teaching brought about by the global pandemic caught teachers, students and educational communities off guard (Seaton et al., 2022). In particular, the move to online teaching revealed its the uniqueness in the technology landscape. It also highlighted the need for the educational research community to develop new conceptual and analytical tools for characterizing teaching and learning that occurs in online environments.

In this paper, we propose a conceptual framework *Instructional Technology Triangle (ITT)* which extends the seminal notion of the instructional triangle, which conceptualizes instruction as interactions between teacher and students around particular content, situated in certain environments and time (Cohen et al., 2003; Lampert, 2001). In the context of online teaching and learning, technology becomes both the environment enabling the educational process to occur and the necessary mediator between teachers, students, and content. The ITT framework captures these relationships by placing technology within the instructional triangle.

We developed the ITT framework to capture and characterize noticing patterns of prospective secondary teachers (PSTs) as they reflected on video recordings of their own online teaching. The PSTs participated in the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* where they designed and taught lessons that integrated reasoning and proving within the regular mathematics curriculum (Buchbinder & McCrone, 2020). Due to the pandemic, in Fall 2020, the school teaching component moved online. The PSTs taught via *Zoom*, uploaded the recording to *Canvas* Learning Management System, where they watched and reflected on their teaching. The analysis focused on PSTs' noticing, which is an essential professional skill (Buchbinder et al., 2021; Mason, 2002; Kosko et al., 2021; Sherin et al., 2011). As we analyzed our PSTs' noticing, we were struck by the prevalence of technology-related aspects, their variety, and their change over time. To capture the richness of these data, we developed the ITT framework.

In this paper, we introduce the ITT framework and illustrate its use by analyzing one PSTs' (Nancy, a pseudonym) noticing patterns in four online lessons. We aim to *describe and characterize Nancy's noticing patterns as a way to understand how her learning to teach online evolved over time*. In addition, we triangulate data from multiple sources, such as Nancy's lesson plans, written essays and course materials, to understand possible reasons behind fluctuations in her noticing patterns, in particular with respect to the role of technology.

Conceptual Framework

Reflective Noticing

The concepts of noticing and reflection have been closely intertwined in educational literature dealing with teacher learning and professional growth (Seidel et al., 2011; Moore-Russo & Wilsey, 2014). Amongst many definitions of teacher noticing (Dindyal et al., 2021), in this study, we adopt Stockero's (2021) definition of noticing as comprised of *attending to* aspects of the classroom situation and *interpreting* them. In this study, we consider PSTs' noticing of their own teaching, as they reflect on their own classroom teaching, and use the term *reflective noticing* to describe a process that combines the tacit nature of attending and the goal-oriented nature of reflection (Buchbinder et al., 2021; Liu & Buchbinder, in press).

Teachers' reflective noticing happens with a specific context and is affected by many factors, such as individuals' knowledge, belief, experience (Schoenfeld, 2011), identity (Oyserman, 2009), and instructional practices (Liu et al., 2021; Sherin et al., 2009). Cross Francis et al. (2021) found that teachers' mathematical knowledge for teaching, efficacy, belief, emotions, and identity influenced teachers' noticing of and reflection on students' mathematics thinking.

Reflective noticing supports teachers' professional development because it engages teachers in regulating their attention resources to the teaching aspects that they perceive as important and allowing them to prioritize these aspects. Learning from reflection requires careful and critical deliberation on practice, connecting to theoretical ideas and contemplating takeaways for the future (Anderson, 2019; Wilson, 2008). Thus, it is not surprising that PSTs' reflection on the video-recordings of their own teaching was found to be beneficial to PSTs' professional learning (Buchbinder et al., 2021; Liu & Buchbinder, in press; Walshe & Driver, 2019).

The Instructional Technology Triangle Framework

The instructional triangle is a heuristic structure for describing and characterizing teacher-student-content relationships (Friesen & Osguthorpe, 2018). It highlights the nature of teaching as mutual interactions among teachers, students, and content in environments (Cohen et al., 2003; Herbst & Chazan, 2012; Lampert, 2001; Yeo & Webel, 2017). In a face-to-face setting where teachers or students interact with technology, technology can be understood as a tool or a part of the environment. In an online environment, it is impossible to avoid treating technology as a key factor in learning and instruction. Content is represented through technology, and teachers and students interact with technology and interact with each other through technology (Seaton et al., 2022). Technology is what mediates between teachers-students-content, making the educational process possible. We extend the instructional triangle framework by adding technology as an explicit component of instruction (Figure 1) to foreground the critical role of technology in an online teaching environment.

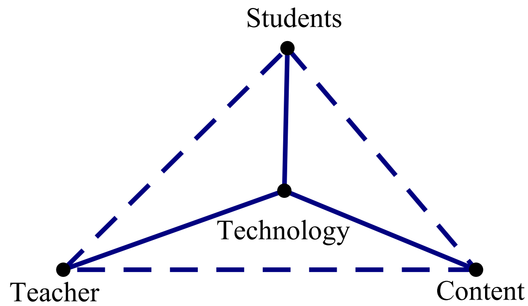


Figure 1: Instructional Technology Triangle Framework

The Instructional Technology Triangle (ITT) framework (Figure 1) considers four nodes: teacher, student, technology, and content and six sets of pair-wise relationships: student-content, student-technology, teacher-technology, teacher-student, teacher-content, content-technology. In the context of reflective noticing, we define the following ten categories. **Teacher -** considers teacher's reflection on their personality, behavioral characteristics like a voice pitch, thoughts, and self-impressions. **Students** - refers to noticing students' behavior and personalities, classroom participation and interactions with peers. **Content** refers to teacher reflection on the mathematics of the lesson and/or a rationale for including a certain content in the lesson. **Technology** involves reflecting on technology as a tool, without relating it to teaching moves or students' mathematical thinking. Noticing of **Teacher-Content** refers to teacher reflection on how they taught a content or made an instructional decision related to content. **Teachers-Students** noticing indicates attending to interactions between the teacher and the students. **Student-Content** describes reflecting on students' mathematical thinking while interacting with or responding to a mathematical question or task. **Content-Technology** refers to teacher's reflection on how technology is useful or not in representing a particular mathematical content; while **Teacher-Technology** describes teacher noticing of manage technology for effective teaching. **Student-Technology** captures the teacher's noticing of students' interaction with technology. We illustrate these categories in depth in the results section.

Methods

The Setting

This study is a part of the larger project that designed a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* and studied how PSTs' mathematical and pedagogical knowledge develops during the course (Buchbinder & McCrone, 2020). The four course modules focused on the following proof themes: (1) direct proof and argument evaluation; (2) conditional statements, (3) quantification and the role of examples in proving, and (4) indirect reasoning. In each module, the PSTs strengthened their subject matter knowledge of the proof themes, learned about students' proof-related (mis)conceptions and then designed and taught to small groups of students from local schools a 50-minute lesson that integrates a particular proof theme with an ongoing topic from the school curriculum. Then, the PSTs viewed the 360-video recording of their lesson and reflected on it (Buchbinder et al., 2021). In Fall 2020, due to the pandemic, the PSTs taught their four lessons online, via *Zoom*.

Nancy was a senior mathematics education major in a high school certification track. She had robust mathematical knowledge and educational orientation, as evidenced in her high GPA, and strong performance on pre-and post-course assessment on Mathematical Knowledge for Teaching Proof and Dispositions toward Proof survey (Buchbinder & McCrone, 2021). We chose to analyze Nancy's reflective noticing since it was elaborated and rich in detail.

Data Sources and Analytic Techniques

The main data source for this paper is Nancy's reflection reports she completed on *Canvas* Learning Management System 2-3 days after teaching the lesson. For this report, Nancy watched the video of her lesson, and used the commenting feature to write a reflective comment about every five minutes, about 8-9 comments per lesson. These comments provide information about Nancy's noticing in the moment of watching the video. We analyzed Nancy's reflective comments using the Instructional Technology Triangle coding scheme. Each comment was examined for the presence of a particular code; if needed, long comments were divided into shorter thematic units and assigned separate codes. The first two authors coded the data, discussed and resolved any discrepancies. Next, we examined the distribution of codes across the coding categories and four lessons (Table 1) in conjunction with data from supplemental sources: Nany's lesson plans, reflective essays submitted along with the *Canvas* comments, and a summative essay about the course. We also relied on the course syllabus and instructional materials (the second author was the course instructor) to construct narratives explaining Nancy's noticing patterns as she participated in the sociocultural contexts of the capstone course and of the online student-teaching embedded in it.

Results

Table 1 summarizes the distribution of codes of the ITT framework identified in Nancy's reflective comments across the four lessons. The total number of codes per lesson is specified in the heading; the modal response in each column is highlighted. We briefly describe each lesson and identify the key patterns in Nancy's noticing, followed by an interpretive analysis of these patterns.

Table 1: The Distribution of Nancy's Noticing Across ITT Coding Categories and Lessons

	Lesson 1 (*N* =14)	Lesson 2 (*N* =13)	Lesson 3 (*N*=12)	Lesson 4 (*N* =16)	Total (N)
Teacher-content	36%	15%	8%	31%	13
Teacher-Student	7%	23%	17%	25%	10
Teacher-Technology	14%	8%	17%	13%	7
Student-content	14%	0%	25%	0%	5
Technology	0%	15%	8%	6%	4
Teacher	21%	0%	0%	6%	4
Student-Technology	0%	8%	0%	13%	3
Content-Technology	0%	8%	0%	6%	2
Student	0%	8%	0%	0%	1
Content	0%	0%	8%	0%	1

Lesson 1. Direct Proof and Argument Evaluation: Supplementary and Vertical Angles.

Nancy started by engaging students in a discussion of "what makes a good two-column proof?" Next, she facilitated an exploration of the Vertical Angles Theorem (VAT) where she used *GeoGebra* to manipulate intersecting lines and had students observe the changing measures of vertical angles. Nancy guided students to notice that the vertical angles remained congruent.

In this lesson, Nancy was the only one engaging with GeoGebra while students observed and made conjectures verbally. Then, Nancy guided students through writing two-column proof of the VAT first with specific numeric values of the angles and then using variables.

Table 1 shows that Nancy mostly noticed the teacher-content relation (36%), and her own actions (teacher, 21%). This pattern is not surprising considering that this is the first time Nancy taught a lesson online and her first time watching her own teaching and reflecting on it. For example, Nancy noticed her speaking habits, e.g., "I am saying 'um' a lot," and her fast speaking pace and reflected on the need to be aware of and improve her speech: "I need to slow down and not rush through explanations. I'm naturally a fast talker and when I get nervous, I tend to talk even faster … I need to work on slowing it down in the future." Nancy seldom returned to these points in the following lessons, suggesting that the heightened focus on herself was due to the novelty of video reflecting experience.

Based on the prevalence of teacher-content codes (36%), Nancy's main concern in this lesson was making sure the lesson ran smoothly and achieved the teaching goals. These included engaging students in mathematical discourse to collectively construct a two-column proof of the VAT, develop criteria for a "good" two-column proof, and have students recognize there are multiple ways to prove a theorem. Nancy constantly monitored her progress towards these goals and justified her instructional decisions. For example, Nancy noticed her effective teaching practice of developing a list of ideas for proof before writing the two-column proof as a way to address a common student challenge of completing a proof. She wrote:

> As we worked through the proof, I asked the students questions and tried to guide their thinking. I brought their attention to our ideas that we developed earlier on, and they were able to see that we could use two of those expressions to help us with our proof. I think this is important because oftentimes when faced with a proof, it can be challenging to come up with the next step, but if you have already come up with ideas before writing the proof, it can make writing the proof a little easier.

Nancy also criticized some of her teaching choices like having a pre-labeled diagram which, in her words, "defeated the purpose" of having "the students realize that we needed to generalize the angles to prove all cases". Nancy also identified areas of improvement, like managing discussions in ways that increase student mathematical engagement.

> One thing that I noticed during this period was that I didn't really ask the students for their input about generalizing the proof. I just told them that we need to generalize the proof to make sure that we show all cases of vertical angles. After doing this lesson one of my revisions is to do less telling and have the students try coming to the conclusion on their own.

Fourteen percent of Nancy's comments concerned teacher-technology, which is not surprising given this was her first time teaching ever, and online. Nancy was familiar with *GeoGebra* software but struggled to show multiple screens on Zoom: "One teaching move that I found challenging occurred when I wanted to keep the problem up, but also wanted to show the list of ideas that the students were coming up with". Student interactions with technology did not show up in Nancy's noticing possibly because she was the only one manipulating GeoGebra in this lesson.

Lesson 2: Conditional Statements: Isosceles and Equilateral Triangles

Nancy used *Prezi* with multiple examples to introduce the concepts of conditional statement and its converse. Then, students used *GeoGebra* to explore three conditional statements about triangles. For each statement, students were to determine if it is true or false (if false, construct a counterexample), then write the converse and determine whether the converse is true or false.

After the first lesson Nancy seemed to gain comfort with teaching online and shifted the mode of student engagement by having them interact with *GeoGebra*. This is reflected in the shifts in Nancy's noticing patterns toward increased focus on interaction with students (23%). She wrote:

> While going over the conditional statement "if a number is divisible by 10, then it is divisible by 5" I asked the students if they thought the statement was true or false. One student responded that he thought it was false. Instead of saying that his answer was incorrect, I asked him why he thought that and as he started to explain, I realized he was talking about a different conditional statement. I'm glad that I asked him to clarify because otherwise I wouldn't have known that he was talking about a different statement.

By following up on students' answer, rather than discarding it as incorrect, Nancy inferred from the student explanation that he offered a correct response to a different question. Nancy then was able to steer this interaction towards the lesson goals.

An important trend of Nancy's noticing in lesson 2 is the presence of all technology-related codes. Nancy reflected how she transitioned from one technology to another, e.g., "I think the transition from the *Prezi* presentation to *GeoGebra* was pretty smooth and I think I did a nice job at explaining the key aspects of GeoGebra" (technology code, 15%); how the use of technology benefited her teaching, e.g., "I really like having exit tickets because it gives me really good feedback about what the students learned and how they felt with different parts of the lesson" (teacher- technology 8%). Nancy also noticed students' difficulty interacting with each other on *Zoom* (student-technology, 8%):

> When I told the students that they could work together, no one did. I think zoom makes this hard because it's not like you can turn to your neighbor and discuss. Instead, if you want to talk, you end up talking in front of everyone which can make people nervous. Also, as a student it can seem daunting to start up a discussion with your peers. This is something I need to keep in mind for the future.

The patterns of Nancy's noticing are consistent with her use of diverse technological tools (*Prezi, GeoGebra, Google forms*) and with having students interact with *GeoGebra* directly.

Lesson 3: Quantification and the Role of Examples: Triangle Similarity Theorems

Nancy first used *Google Slides* to introduce the concepts of universal and existential statements and the role of examples in proving/disproving them. She had students find counterexamples to three universal statements (e.g., All isosceles triangles are similar) and find examples proving two existential statements (e.g., There exist two right triangles that are similar). After briefly reviewing three similarity theorems as a group, the students spent most of the class time working on Side-Side-Side (SSS) similarity proof, each typing their work on a designated slide in a shared *Google Slides* document.

One unique feature of Nancy's noticing in lesson 3 is the focus on students' learning of content (25%). For example, she noted that "the students recognized that they needed to use SSS but were unable to connect/use proportions to relate the corresponding sides"; that "a lot of them [students] didn't understand how to include proportions into the proof," and that "the students were able to come up with some counterexamples." Nancy monitored students' progress when they typed their answers into *Google Slides,* but she did not have access to their *GeoGebra* screens. Nancy's reflective comments concerned her interactions with students (17%) and with technology (17%). For example, she reflected on the affordances of *Google Slides*: "I, again, liked the use of the google slides here, because I was able to see the students' progress through the similarity proof and get an understanding of what parts they found confusing."

Lesson 4: Indirect Reasoning: Coordinate Proofs

For this lesson, Nancy adapted her cooperating teacher's lesson plan about analytic geometry proofs by integrating indirect reasoning in it. The main activity was the game "The Quadrilateral Detective," where students determined the type of quadrilateral given the coordinates of its vertices. The students also created statements involving indirect reasoning, such as "The quadrilateral cannot be a kite because otherwise it would not have parallel sides". Nancy solved one task together with the students as an illustration. To avoid the complexity of students typing algebraic symbols, Nancy suggested students write their proofs on paper and post a picture of their work into the shared *Google Slides* document.

Despite Nancy's best intentions, not all parts of the lesson proceeded as planned. She seemed to manage better during the teacher-led parts of the lesson and reflected abundantly on her instructional decisions (teacher-content, 31%). For example, "One teaching move that I liked during this lesson was creating a theme for the lesson…" ; "One teaching move that I think was good for this portion was that I had already graphed and typed out the solution to case 1 in the presentation/desmos…"; "I think it was a successful teaching move to include some exposition about indirect reasoning at the end of the lesson instead of the beginning…"

Table 1 shows that from Lesson 1 to Lesson 3 Nancy's noticing shifted from more teacher-centered to more student-centered. But in lesson 4 Nancy mostly noticed her teaching of content (teacher-content, 31%), while also reflecting on her interactions with students (teacher-student, 25% – the largest percentage of this category across the four lessons). Specifically, Nancy reflected on her attempts to press students for explanations, e.g., "why he put his x in a certain spot"; on her calling on students to "get them to participate;" on how "open ended discussion allowed the students to express their answers and explain their reasonings" and on how "providing students enough time to work on their ideas leads to richer discussions."

Nancy also reflected on how she and her students used technology (teacher-technology, 13%; student-technology, 13%). One of the aspects she noticed was the difference between her expectations about students' use of technology and the reality. Nancy expected students to write their solutions to the Quadrilateral Detective Task on paper, scan, and post to the shared *Google Slides* document. Instead, the students created new slides within that document and started typing in their solutions. While Nancy sought to optimize student use of technology by providing space to post their work, the students chose an inefficient and time-consuming approach. Nancy noticed this and reflected on ways for making improvements for future practice.

Discussion and Conclusions

In this paper, we presented the Instructional-Technology Triangle (ITT) framework, which foregrounds technology as an essential element of online learning environments, and a primary mediator between teachers, students, and content. The ITT framework contributes to the literature advocating the need for extending the basic instructional triangle (Cohen et al., 2003) to represent additional elements of instruction in general (Herbst & Chazan, 2012) and with respect to technology (e.g., Yeo & Webel, 2017). The ITT framework was developed when teaching via online videoconferencing (e.g., Zoom, Google Meet, Microsoft Teams, VooV Meeting, etc.) became a norm during the pandemic in response to the seeming lack of theoretical and analytical tools for conceptualizing teaching and teacher noticing in the online context. In this context, teaching technology is not just another element of instruction, but a necessary medium that makes this instruction possible. The content is represented through technology, and both teachers and students interact with technology and through technology. This may include navigating *multiple* types of technology (e.g., video conferencing platform, digital interactive whiteboards, virtual manipulatives) simultaneously by teachers as well as students. Thus, technology started playing an essential role in determining the quality of instruction and the quality of student learning experience. The ITT framework represents our attempt to capture this unique role of technology in the online setting.

Our study also contributes to the body of knowledge on noticing and reflection by demonstrating the utility of the ITT framework in analyzing Nancy's reflective noticing and its development over time. Nancy's first teaching experience happened to be online, and she had to adjust to it quickly. Due to her strong mathematical content knowledge, Nancy invested most energy in designing interactive activities, choosing appropriate technological tools, and interacting with students. The novelty of the experience contributed to the heightened focus on herself and on her teaching in the first lesson (Table 1). As the semester progressed, Nancy seemed to become more comfortable and confident, as evidenced in her delegating responsibility to students by having them interact with technological tools like *GeoGebra, Desmos* and *Google Slides*. Accordingly, Nancy's noticing patterns shifted in lessons 2 and 3 toward increased focus on interactions between students, content, and technology (Table 1). These patterns shifted again in lesson 4 when Nancy enacted the cooperating teacher's modified lesson plan and encountered a mismatch between her expectations of student engagement and how the lesson unfolded. The ITT allowed us to create a nuanced representation of Nancy's reflective noticing and the changes in it. The observed patterns are consistent with the noticing literature (e.g., Buchbinder et al., 2021, Sherin & van Es, 2005; Stockero, 2021) on teachers' initial inclination to focus on their actions, followed by increased noticing on student mathematical thinking. Many researchers highlight the importance of noticing students' mathematical thinking by teachers (e.g., Barnhart & van Es, 2015; Cross Francis et al., 2021; Sherin & van Es, 2009). Despite its importance, it may be unwise to use it exclusively as an indicator of quality of teacher noticing and run the risk of overlooking other aspects. As Spangler (2019) advocates, "We as teacher educators need to demonstrate the curiosity and intellectual humility that allows us to understand how and why something a teacher did or said came from a place that made sense to them" (p. 2). In this paper, we attempted to interpret Nancy's noticing from a place that made sense to her, and to foreground her rationality in the context of her teaching practice (Herbst & Chazan, 2003).

Stepping back from our study, we believe that even as the schools begin to reopen, online education will retain a substantial presence in a variety of forms, such as teacher preparation, professional development, and situations where remote learning is a necessity for some reason. The ITT framework can be broadly applied for conceptualizing and analyzing teacher-student-content interactions in an online setting, responding to the need to attend to contexts in which the teaching and learning takes place, the experiences of the participants and to their individual voices (Liu et al., 2020; PME-NA 44 Conference Theme, 2022).

Acknowledgments

This research was supported by the National Science Foundation, Awards No. 1711163, 1941720. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Anderson, J. (2019). In search of reflection-in-action: An exploratory study of the interactive reflection of four experienced teachers. *Teaching and Teacher Education*, *86*, 102879.

Ball, L., Drijvers, P., Ladel, S., Siller, H. S., Tabach, M., & Vale, C. (2018). *Uses of technology in primary and secondary mathematics education*. Cham, Switzerland: Springer.

Barnhart, T., & van Es, E. (2015). Studying teacher noticing: Examining the relationship among pre-service science teachers' ability to attend, analyze and respond to student thinking. *Teaching and Teacher Education, 45*, 83-93.

Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research–A systematic review of recent trends. *Computers & Education*, *114,* 255-273.

Buchbinder, O., Brisard, S., Butler, R., & McCrone, S. (2021). Preservice secondary mathematics teachers' reflective noticing from 360-degree video recordings of their own teaching. *Journal of Technology and Teacher Education, 29*(3), 279-308.

Buchbinder, O., & McCrone, S. (2020). Preservice teachers learning to teach proof through classroom implementation: Successes and challenges. *The Journal of Mathematical Behavior*, 58, 100779.

Buchbinder, O., & McCrone, S. (2021, July 11-18). Characterizing mathematics teachers' proof-specific knowledge, dispositions and classroom practices[Eight-page paper]. 14th International Congress on Mathematical Education, Shanghai, China.

Clark-Wilson, A., Robutti, O., & Thomas, M. (2020). Teaching with digital technology. *ZDM – Mathematics Education 52*(7), 1223-1242. https://doi.org/10.1007/s11858-020-01196-0

Cohen, D., Raudenbush, S., & Ball, D. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119–142.

Cross Francis, D., Eker, A., Liu, J., Lloyd, K., & Bharaj, P. (2021). (Mis) alignment between noticing and instructional quality: the role of psychological and cognitive constructs. *Journal of Mathematics Teacher Education,* 1-34. https://doi.org/10.1007/s10857-021-09509-0

Dindyal, J., Schack, E. O., Choy, B. H., & Sherin, M. G. (2021). Exploring the terrains of mathematics teacher noticing. *ZDM–Mathematics Education, 53*(1), 1-16.

Friesen, N., & Osguthorpe, R. (2018). Tact and the pedagogical triangle: The authenticity of teachers in relation. *Teaching and Teacher Education, 100*(70), 255-264.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: The case of engaging students in proving. *For the Learning of Mathematics, 23*(1), 2–14.

Herbst, P., & Chazan, D. (2012). On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM - Mathematics Education, 44*(5), 601-612.

Hillmayr, D., Ziernwald, L., Reinhold, F., Hofer, S. I., & Reiss, K. M. (2020). The potential of digital tools to enhance mathematics and science learning in secondary schools: A context-specific meta-analysis. *Computers & Education, 153*, 103897. https://doi.org/10.1016/j.compedu.2020.103897

Kosko, K. W., Ferdig, R. E., & Zolfaghari, M. (2021). Preservice teachers' professional noticing when viewing standard and 360 video. *Journal of Teacher Education, 72*(3), 284-297.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* New Haven, CT: Yale University Press

Liu, J., & Buchbinder, O. (in press). Learning to Teach Reasoning and Proof in an Online Setting: The Case of Nancy. To appear in the *Proceedings of the 24th Annual Conference on Research in Undergraduate Mathematics Education, Special Interest Group of the Mathematical Association of America*, Boston, MA.

Liu, J., Cross Francis, D., Eker, A., Lloyd, K. R., & Bharaj, P. K. (2021) (Mis)Alignment Between Teachers' In-the-Moment Noticing and Post-Instruction Noticing. In Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp.744-748). Philadelphia, PA.

Liu, J., Galindo, E., Yoder, G. B., & Bharaj, P. K. (2020). The Design, Implementation, and Effectiveness of a Collaborative Responsive Professional Development (CRPD) Model. In Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp.1897-1901). Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.

Moore-Russo, D. A., & Wilsey, J. N. (2014). Delving into the meaning of productive reflection: A study of future teachers' reflections on representations of teaching. *Teaching and Teacher Education*, *37*, 76-90.

Oyserman, D. (2009). Identity-based motivation: Implications for action-readiness, procedural-readiness, and consumer behavior. *Journal of Consumer Psychology, 19*(3), 250–260.

PME-NA 44 Conference Theme. Retrieved February 15, 2022, from http://www.pmena.org/pmena44/2022-theme-en/

Schoenfeld, A. H. (2011). Toward professional development for teachers grounded in a theory of decision making. *ZDM- Mathematics Education, 43*(4), 457-469.
Seaton, K., Loch, B., & Lugosi, E. (2022). Takeaways from teaching through a global pandemic–practical examples of lasting value in tertiary mathematics education. *International Journal of Mathematical Education in Science and Technology*, *53*(3), 559-565. https://doi.org/10.1080/0020739X.2022.2008551
Seidel, T., Stürmer, K., Blomberg, G., Kobarg, M., & Schwindt, K. (2011). Teacher learning from analysis of videotaped classroom situations: Does it make a difference whether teachers observe their own teaching or that of others?. *Teaching and Teacher Education, 27*(2), 259-267. https://doi.org/10.1016/j.tate.2010.08.009
Sherin, M. G., Jacobs, V. R., & Philipp, R. A. (2011). Situating the study of teacher noticing. *In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers eyes* (pp. 3–14). London: Routledge.
Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education*, *60*(3), 213-230.
Sherin, M. G., & van Es, E. A. (2009). Effects of Video Club Participation on Teachers' Professional Vision. *Journal of Teacher Education, 60*(1), 20-37.
Sherin, M., & van Es, E. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*(3), 475-491.
Spangler, D. A. (2019, Feb 20, p.2) Fundamental Commitments of My Work as a Mathematics Teacher Educator. Retrieved from https://amte.net/sites/default/files/Spangler-2019-JudithJacobsLecture.pdf
Stockero, S. L. (2021). Transferability of teacher noticing. *ZDM–Mathematics Education*, *53*(1), 73-84.
Taranto, E., Robutti, O., & Arzarello, F. (2020). Learning within MOOCs for mathematics teacher education. *ZDM - Mathematics Education, 52,* 1439–1453. https://doi.org/10.1007/s11858-020-01178-2.
Walshe, N., & Driver, P. (2019). Developing reflective trainee teacher practice with 360-degree video. *Teaching and Teacher Education*, *78*, 97-105.
Wilson, J. P. (2008). Reflecting-on-the-future: A chronological consideration of reflective practice. *Reflective Practice*, *9*(2), 177-184.
Yeo, S., & Webel, C. (2017). Extension of interactions based on technology: bridging elementary mathematics classrooms in Korea. In E. Galindo & J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 122-1225). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

NUMBER TALKS: PRELIMINARY RELATIONSHIPS BETWEEN TEACHERS' USE OF QUESTIONS AND STUDENTS' AGENCY

Beth L. MacDonald
Illinois State University
blmacd1@ilstu.edu

Hilary Tanck
High Point University
htanck@highpoint.edu

Number Talks are widely used across the U.S., as they uniquely place student reasoning at the center of mathematics instruction while allowing for few resources and time throughout enactment. The purpose of this brief paper is to describe how relations between teachers' use of questions and students' engagement in a Number Talk associate with students' number sense. Findings suggest students' "student-to-student dialogue" was significantly associated with teachers' "generating discussion" questions and students' number sense development.

Keywords: Number Concepts and Operations, Classroom Discourse, Instructional Activities and Practices

Number sense fundamentally predicts mathematics outcomes in future schooling experiences (e.g., Jordan et al., 2009). One set of resources designed to support students' number sense development are Number Talks (Humphreys & Parker, 2015). Number Talks are widely used across the U.S., as they uniquely place student reasoning at the center of mathematics instruction, while allowing for few resources and time throughout enactment. Parrish and Dominick (2016) describe Number Talks as a "five- to fifteen-minute classroom conversation around purposefully crafted problems that are solved mentally" (p. 13). As Matney et al. (2020) explicate, few blind reviewed papers exist examining Number Talks. The purpose of this brief paper is to describe *how relations between teachers' use of questions and students' engagement in a Number Talk associate with students' number sense*.

Theoretical Framework

Our framing draws from Cultural Historical Activity Theory (CHAT – Engeström, 1999). CHAT frames activity systems, which can be used to make visible development within a system of activity. When examining Number Talks, we recognize teachers may have activity related to their individual goals. Classrooms of students may have collective goals associated with their collective activity. Engeström explains CHAT draws from the interaction between an object and subject, which are mediated by community, rules, division of labor, and instruments. In Figure 1, the subject (teachers' use of questions) interacts with the object (students' number sense) and is mediated by students' agentic behaviors (Cobb et al., 2009). There are three less visible mediators, described here as socio-mathematical norms, Number Talk parameters, and discourse patterns (see Figure 1). Existing tension within the students' development and reorganization of their number sense, leverages change in the system (evidenced through teachers' use of questioning and mediated by students' agentic behaviors). The following sections better explain (1) teachers' use of questions, (2) students' agency, and (3) students' number sense development.

Teachers' Use of Questions

When teachers engage in Number Talks, Humphreys (2016) found teachers used eight question types: *funneling, evaluating, probing, restating, revoicing, generating discussion*, *gathering information*, and *double-checking*. These are similar to other Number Talk questioning frameworks (e.g., Conner et al., 2022, Joswick et al., 2020).

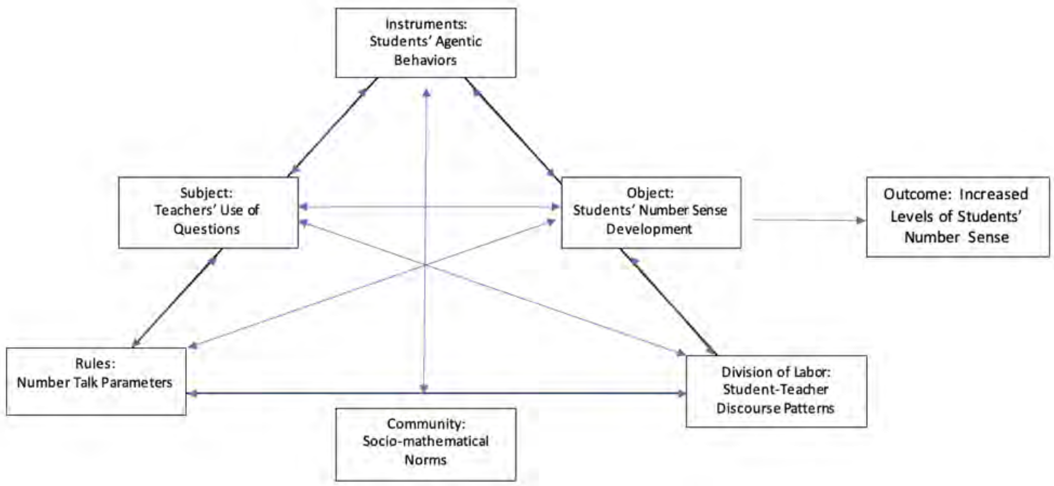


Figure 1: Cultural Historical Activity Theory (CHAT) theoretical framework

We used Humphrey's (2016) questioning framework due to the direct relationships these were found to have with students' agency during Number Talks. For instance, when teachers assessed students' reasoning by asking probing and clarifying types of questions. When one teacher asked restating questions or revoicing questions, Humphreys (2016) found students evidenced more agentic behaviors. By asking these questions, a teacher would ask a student to further explain other students' thinking. When teachers used generating discussion questions, they often asked students to contribute by adding on to other students' reasoning. Conner et al. (2022) described gathering information as questions that press students to "connect" their mathematical ideas and "engage with others' ideas." For example, when asking gathering information questions, teachers often asked students "Okay, where did you go from there?". When describing double-checking types of questions, teachers were simply confirming students' reasoning by restating their strategy in their own words. These five categories framed our structural coding and through open coding we established the additional three categories.

Students' Agency

We framed student engagement with agency and measured it with agentic behaviors. *Agency* is described as an individual's choice of how to engage in a classroom community (Cobb et al., 2009). Cobb and colleagues describe agency in relation to authority, thus, to measure students' agency, we attempted to capture students' goals, as navigated by authority in the classroom.

During Number Talks, Humpreys (2016) found students evidenced six types of agentic behaviors: *appropriates authority for sensemaking, injects an unsolicited comment, asks an unsolicited question, student-to-student dialogue, public risk taking, and shares "in process" thinking.* This study utilized these agentic for analysis.

When students engaged with the mathematics in an unprompted manner, students appropriated authority for sensemaking, injecting unsolicited comments, and/or asking unsolicited questions. When students engaged with each others' mathematical ideas directly, we categorized this as student-to-student dialogue. By evidencing vulnerability in their sense-making students were described as taking risks. Students were noted as evidencing "in-process" thinking when stating how their methods developed. Students' agency is considered as embedded in socio-mathematical norms, discourse patterns, and Number Talk parameters.

Students' Number Sense Development

To examine productive activity wherein students' construction of number was at the center, we collected data evidencing students' number sense development. To examine students' number sense development, we worked closely with David Woodward, CEO of a non-profit organization, Forefront Education, which develops number sense screeners from the research literature (e.g., Wright & Ellemor-Collins, 2018). Each tool was given to teachers to assess their students' number sense development three times throughout the academic year (fall, mid-year, spring). Scores were structured with a 3-point rubric (1, 2, or 3) for each item. The authors aggregated these scores along a 5-point scale to both capture overall number sense performance per student and variance in their number sense.

Methods

This brief paper explains baseline data from a larger Convergent Mixed Methods study (Creswell & Plano-Clark, 2011), focusing on explaining students' number sense development over time, while in Number Talk lessons. For this brief paper, we explain the results after one semester of enactment of Number Talks. Moreover, we draw from three of the five main constructs of these data to explain how teachers' use of questions related to students' number sense development.

Participants and Setting

This study initially included 20 practicing teacher participants assigned to teach grades 4-6 in a school district located in the mountain west region of the United States. Of these 20 participants, we collected complete data from 14 participants. Of these 14 participants, we purposefully chose 11 participants from a cross-reference analysis of an efficacy survey (Enochs et al., 2000) with a two-by-two factor analysis with two levels of two types of efficacy and beliefs. Participants' classroom teaching experience ranged from X to X and from X to X with the particular grade they were assigned to teach. To gather data, researchers scheduled Zoom observations and emailed number sense assessment materials.

Procedures

After analyzing survey data, we scheduled the observations. Throughout the observations, we used the following protocol: (1) open Zoom and ensure video and audio operate successfully, (2) reposition the camera and/or device to be sure students and teacher can be heard, (3) remind teacher to set up a device to audio-record the observation, (4) once a Number Talk began, both authors noted students' agentic behaviors (more in analysis section), (5) once the Number Talk finished, the authors reminded the teacher to upload the audio file in a secure box folder, and (6) closed Zoom. Audio files were transcribed and organized by teacher. The participants completed the number sense screener and uploaded the de-identified results in a secure box folder.

Analysis

Data were first qualitatively analyzed with open and structural coding schemes. These schemes were framed by the aforementioned theoretical and conceptual frameworks. Once descriptive codes were gathered from open coding, the data were revisited for structural coding and then tallied with a frequency count. Quantitative data of these frequency counts and number sense scores were analyzed with a Pearson's Correlation test. Mixed analysis took the form of convergent analysis of the frequency of questions and agency and significant correlations.

Results

Qualitative results suggest that on average teachers asked the most types of questions with the goal to "gather information" (m = 29.9, SD = 15.4). Here, teachers asked about students' solutions, such as, “And then what was the next step after that?”. On average, teachers asked “funneling” types of questions the least number times (m = 2.9, SD = 2.5). Examples of these were when teachers asked students to focus on procedural and/or solution accuracy, such as, “Were you subtracting too much or too little right here?” (see Table 1).

Table 1: Descriptive Statistics of Teachers’ Questions

Frequency of Teachers’ Questions M(SD)							
Funnel	Evaluate	Probe	Gen. Dis.	Repeats	Revoices	Gather Info.	Double Check
2.9(2.5)	3.4(3.2)	6.6(5.0)	4.7(5.3)	14.1(11.9)	6.1(2.8)	29.9(15.4)	9.3(7.6)

Qualitative results also suggest that students on average shared most often their “in-process thinking” (m = 8.0, SD = 3.1). This was often observed when students explained the processes they used to solve a computational problem. Moreover, students, on average, asked “unsolicited questions” the least number of times (m = 0.5, SD = 0.8). When students asked questions in this way, students often asked classmates to explain their solutions (see Table 2).

Table 2: Descriptive Statistics of Students’ Agency

Frequency of Students’ Agency M(SD)					
Authority	Unsolicited Comments	Unsolicited Questions	Student-to-Student	In-Process	Risk Taking
1.9(1.7)	2.4(1.8)	0.5(0.8)	2.9(3.8)	8.0(3.1)	0.9(1.3)

Quantitative results revealed significant positive correlations between “student-to-student dialogue” and students’ number sense scores (0.235, $p = 0.011$), and teachers “generating discussion” questions and “student-to-student dialogue” (0.832, $p < 0.001$). Teachers’ use of questions were not significantly correlated with students’ number sense, suggesting student-to-student dialogue acted as a mediator between number sense and teachers generating discussions.

Divergent themes suggest that when teachers asked gathering information types of questions, they did so on average about 30 times per Number Talk. Given this, it is striking that did not have any significant associations with students’ agency and students’ number sense development. However, convergent themes found that students’ “student-to-student dialogue” was significantly associated with teachers’ “generating discussion” questions. The qualitative data found teachers only asked generating discussion types of questions on average 4 to 5 times during a Number Talk, the sixth most asked question. Students engaged with student-to-student dialogue on average about 3 times throughout a Number Talk, the second most agentic behavior. Finally, it seems students’ agency acted as a mediator between teachers’ use of questions and students’ number sense development during a Number Talk.

Conclusion

Number Talks are rarely examined in a rigorous manner and with comprehensive foci (Matney et al., 2020). Findings suggest teachers can increase their generating discussion questions to leverage opportunities for student-to-student dialogue. Future work should consider how teachers’ questions, students’ agency, and number sense develops over time.

References

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). An interpretive scheme for analyzing the identities that students

develop in mathematics classrooms. *Journal for research in mathematics education*, *40*(1), 40-68.
Conner, K., Joswick, C. & McMillian., B. (2022, February). Elementary and secondary teachers' questioning patterns during Number Talks. *Individual session presented during Number Talks at the annual meeting of the Association of Mathematics Teacher Educators, Las Vegas, Nevada.*
Creswell, J. W., & Clark, V. P. (2011). Designing and conducting mixed methods research. Sage Publications. *Thousand Oaks: USA*.
Engeström, Y. (1999). Activity theory and individual and social transformation. *Perspectives on activity theory*, *19*(38), 19-30.
Enochs, L. G., Smith, P. L., & Huinker, D. (2000). Establishing factorial validity of the mathematics teaching efficacy beliefs instrument. *School Science and mathematics*, *100*(4), 194-202.
Humphreys, C., & Parker, R. (2015). *Making number talks matter: Developing mathematical practices and deepening understanding, grades 4-10*. Stenhouse Publishers.
Humphreys, C. (2016). *Letting Go: Cultivating Agency and Authority Through Number Talks in the Secondary Mathematics Classroom* (Doctoral dissertation, Stanford University).
Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: kindergarten number competence and later mathematics outcomes. *Developmental psychology*, *45*(3), 850.
Joswick, C., Meador, A., Fletcher, N., Conner, K., McMillian, B. (2020, December). Responding to the current field experience challenges with the virtualization of Number Talks. *AMTE Connections,* Retrieved from https://www.amte.net/connections/winter-2020.
Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black holes of research on instructional practice: the case of number talks. *Investigations in Mathematics Learning*, *12*(4), 246-260.
Parrish, S., & Dominik, A. (2016). *Number Talks: Fractions, Decimals, and Percentages: A Multimedia Professional Learning Resource*. Math Solutions.
Wright, R. J., & Ellemor-Collins, D. (2018). *The Learning Framework in Number: Pedagogical tools for assessment and instruction*. Sage.

ENGLISH LANGUAGE LEARNER STUDENTS' DEVELOPMENT OF THE MATHEMATICS REGISTER DURING A PROBLEM-SOLVING LESSON

Karla Matute
University of Wyoming
kmatute@uwyo.edu

Libni Catsellón
University of Wyoming
Lcastel1@uwyo.edu

Richard Kitchen
University of Wyoming
rkitchen@uwyo.edu

This paper describes how fifth-grade English Learner students (ELs) in an urban school district develop the mathematics register during a problem-solving lesson. It provides examples of students' work to illustrate how they use the mathematics register to communicate their mathematical ideas orally and in writing. The teacher implemented teaching practices such as mathematics discourse to facilitate their students' development of the mathematics register during the problem-solving lesson. Students were engaged in a problem-solving task that involved fractions. Findings provide insights into EL students' challenges when learning the mathematics register and inform instruction about the importance of incorporating teaching practices such as paraphrasing and assessing others' reasoning to support students in learning the mathematics register through problem-solving.

Keywords: Classroom discourse, Mathematics register, Problem Solving

English Learners (ELs) population is significantly growing in U.S. schools (Abedi & Gándara, 2006; Campbell et al., 2007; de Araujo et al., 2018). Almost 10% of students enrolled in public schools in the United States are classified as ELs (NCES, 2020). As the population of ELs increase, meeting their academic needs deserve an urgent national response. ELs often have limited access to a challenging education in mathematics, adequate resources, and qualified teachers (Borjian, 2008; Dong, 2016). Research highlights several issues that influence the performance gap between ELs and monolingual speakers (see, e.g., Campbell et al., 2007; Chval & Chávez, 2012; de Araujo et al., 2018). Among these issues, language is viewed as a source of difficulty in learning mathematics, particularly for ELs required to learn the content of mathematics and English language skills simultaneously (de Araujo et al., 2018; Martiniello, 2008; Moschkovich, 2007). This paper describes how two fifth-grade mathematics classes with a high percentage of ELs (30% or more) developed the mathematics register during a problem-solving lesson. It also shows how ELs use the mathematics register when engaging in a problem-solving task.

In spring 2021, the first author worked with "Ms. Ware," a 5^{th}-grade teacher, and her students in an urban school district to conduct a case study as part of her dissertation project. One of the research goals was to examine how Ms. Ware supported her ELs in developing the mathematics register during problem-solving lessons. The research questions that we explore here are: How do ELs develop the mathematics register while engaging in a problem-solving lesson? How does Ms. Ware support ELs developing the mathematics register during a problem-solving lesson?

The Mathematics Register

The mathematics register was initially introduced into mathematics education by Halliday in 1978. He defined the mathematics register as a set of "meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes" (Halliday,1978, p.195). The literature also refers to the mathematics register as language variations constructed by individual interactions (Schleppegrell, 2004). These variations are characterized in terms of *field* (area or subject), *tenor* (relationship between the individuals), and *mode* (organization of text or conversation) (Halliday, 1978).

The mathematics register includes semiotic resources, such as symbols, visual displays, graphs, diagrams, and language (O'Halloran, 2015). Similar to other disciplines, mathematics has its own language, and learning this discipline implies learning its language. Researchers focused on academic languages have highlighted several issues that emerge when ELs are required to use the mathematics language in informal and formal contexts (see, e.g., Barwell, 2005; Ernst-Slavit & Mason, 2011; Lucero, 2012). For example, Barwell (2005) points out that ELs often face challenges when communicating their mathematical ideas in English. He highlights that solving and writing word problems can support students in integrating mathematics content and language. Ernst-Slavit & Mason (2011) argues that ELs have limited access to mathematics classrooms in which they can consistently hear the mathematics register. Also, teachers constantly use slang, homophones, colloquialism, and idiomatic expressions that can hinder ELs' understanding (Ernst-Slavit & Mason, 2011). Lucero (2012) argues that ELs need repeated opportunities to engage in problem-solving tasks to communicate their ideas to others, define mathematics terms, and compare strategies.

Communicating mathematical ideas in oral and writing forms require students to know the knowledge of mathematics and the knowledge of the language (Wilkinson, 2018). Students need to combine everyday language, symbols, visual displays, and non-linguistic representations to communicate effectively in mathematics. Teachers should incorporate instructional strategies to help their students combine these linguistics features when communicating their mathematics ideas, and consequently, they learn the mathematics register (Moschkovich, 2014)

The mathematics register also involves linguistics features that define the communication styles (Schleppegrell, 2004; Wilkinson, 2018). For example, in an academic context, the mathematics register used can vary when the communication is oral or written (Wilkinson, 2018) Mathematics includes words borrowed from everyday English (e.g., degree, factor, relation, power, radical, product, mean, real, imaginary, rational, and natural) that can have a different meaning in other contexts than mathematics. It also involves some grammatical constructions that differ in their qualitative and quantitative meaning (e.g., *four fours*, the first *four* is an adjective while the second one illustrates a nominal status) (Pimm, 1987).

The mathematics register has two features: multiple semiotic systems and grammatical patterns (Schleppegrell, 2007). The multiple semiotic systems refer to symbol notation, oral and writing language, and visual displays to construct mathematical meaning. The grammatical patterns refer to technical vocabulary, dense noun phrases, nominalization, verbs, conjunction, and logical connectors to communicate mathematics ideas. These features work together to develop the mathematics register, and they are often used to provide mathematical arguments, justify mathematical reasoning, and build mathematical ideas and meanings (Schleppegrell, 2007).

Research Methodology

During the 2021 spring semester, I collaborated with Ms. Ware, a fifth-grade teacher, and her two 5th fifth-grade mathematics classes on three occasions. The study's goal was to examine how Ms. Ware implemented an instructional protocol called the "Discursive Mathematics Protocol" to help her ELs develop the mathematics register and how her students used the mathematics register during problem-solving lessons (see Kitchen et al., 2020, for additional information on the DMP). The DMP was built based on Polya's (1945/1986) problem-solving framework and incorporated research-based strategies and essential teaching practices. These research-based strategies are based on theories of academic language development (Barwell, 2005; Ernst-Slavit & Mason, 2011; Moschkovich, 2015) and the essential teaching practices proposed in *Principles to Actions* (NCTM, 2014) that guide teachers in facilitating their students' development of the mathematics register ad mathematical reasoning, respectively. For the three problem-solving lessons, Ms. Ware and I worked together to plan the lesson, then Ms. Ware delivered the lesson as planned, and finally, we had a debriefing session. For the planning session, we met on Zoom to discuss how we hoped to develop the problem-solving lesson, identified questions, discussed strategies and challenges students could face, discussed the mathematics register involved in the task, and how she would support her ELs to develop it. Ms. Ware and her students met in *Google Meet* for 90 minutes each group during class. I observed the lessons and, on a few occasions, interacted with Ms. Ware and her students, particularly in group discussions. In the debriefing session, Ms. Ware and I met in Zoom for 30-minutes after the problem-solving lesson to discuss what went well in the lesson, what needed improvements for the next lesson, and what adjustments were necessary to incorporate to help ELs to develop their mathematics register.

For this study, 27 students provided consent across the two classes (15 ELs and 12 non-ELs). These classrooms had a high percentage of ELs (30% or more). Most of them speak Spanish as their first language. Ms. Ware taught one group in person and another group entirely online. Each group attended mathematics classes for 90 minutes in two blocks of 45 minutes each. Ms. Ware had Due to the COVID-19, Ms. Ware and her students used a technology tool called *Pear Deck* and *Google Meet* to participate in the class. *Pear Deck* is an application to make interactive presentations in virtual meetings, and *Google Meet* is a virtual tool that allowed Ms. Ware to teach mathematics remotely. The in-person and online groups were connected through *Google Meet* in mathematics classes.

This paper describes how students used the mathematics register while engaging in a problem-solving task. We also highlight the strategies Ms. Ware's used during the lesson to help her ELs develop the mathematics register. Data collection for this study included sources such as video recording of the planning session, problem-solving lessons in *Google Meet,* the debriefing session, ELs' work samples, and class observations. The videotapes and students' work samples were analyzed inductively using interpretive methods (Creswell & Poth, 2018). Each videotape was transcribed, and they were viewed as a whole, followed by open coding to reflect on and clarify teaching strategies to deliver the problem-solving lessons and procedures to address the mathematics register inherent in the task. The analysis process went through multiple iterations to check the consistency of the findings and look at commonalities and differences across the solutions that students provided.

Problem-Solving Task implemented

In the problem-solving lesson, we used a task called 'connecting area model to context," referred to as the multiplication story task. The multiplication story task is a performed task that is publicly accessible through the Illustrative Mathematics (IM) project. This task is considered a cognitively and linguistically demanding task. Students were required to write a multiplication story problem and give a diagram with two types of shading. They needed to establish connections among a procedure (multiplication), a diagram, and a context. Also, they needed to recognize the concept of fractions as operators (Charalambous & Pitta-Pantazi, 2007) inherent in the double shading shown in the diagram. The complexity of the language of this problem appeared when students were required to create a story with a context that related to the notion of fractions as operators represented in the diagram and translate symbols into words (3/4 of 1/5).

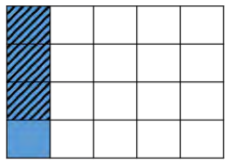
The diagram below represents one whole. 	1. Write down an equation that represents the doubled shaded area on the diagram. 2. Write a multiplication story that could be solved using the diagram with its two types of shading. Explain how your story context relates to the diagram provided.

Figure 2: Task implemented with the DMP

Findings

We now present an example of a problem-solving lesson to illustrate how students in Ms. Ware's classroom used the mathematics register when they solved the multiplication story task. Ms. Ware followed three primary teaching practices to help her students develop the mathematic register inherent in the task. These practices included: (1) implementing socio-mathematical norms and activating previous knowledge, (2) introducing the task and reviewing vocabulary, and (3) facilitating meaningful mathematical discourse as a medium to support students in developing the mathematics register.

Ms. Ware started the problem-solving lesson by establishing some mathematical norms and directions students needed to follow during the virtual meeting. She also invited her students to participate and communicate their ideas in oral and writing form through *Pear Deck* during the lesson. Students were expected to collaborate with each other, share their reasoning, engage in mathematical discussion, negotiate the meaning of mathematical terms, and reach agreements about their mathematical ideas. Prior to the problem-solving task, Ms. Ware activated her students' prior knowledge about fractions. She had students work with model diagrams and identify unit fractions (e.g., 1/3,1/5) in both columns and rows of a rectangular diagram. Students were also asked to identify expressions represented in the area model diagram that could be derived by multiplying fractions (1/4 of 1/3). The goal of the review was to emphasize the notion of fractions as operators, which would help students work on the multiplication story task.

Following the first stage, Pólya's heuristic *Understanding the Problem* (stage1 of the DMP), Ms. ware introduced the multiplication story task. She began asking questions such as: *What is the task asking for? What do you think it means to write a multiplication story? What do you guys think we're doing?* Students discussed in small groups what the problem asked them to do. Ms. Ware allowed students to use their vocabulary and familiar language to explain what they understood about the problem. She also helped her students to understand the vocabulary used in the task, such as "story," "multiplication story," and "two types of shading." Some students provided their reasoning about the meaning of these terms. For example, Marcia, an EL, explained that: a multiplication story is "like a story that has a multiplication in it," Max, a non-EL student, said, "it is like using a story, but like using math." Similarly, students defined "two types of shading" as "overlapping" of "two types of colors." Moreover, Once Ms. Ware was satisfied that most of her students understood the task and vocabulary found in the task, she continued with the activities planned and moved to stage two of Polya's heuristic, *Create a Plan* (stage 2 of the DMP)

As Ms. Ware progressed through the lesson, she engaged her students in mathematics discourse on several occasions by allowing them to share their ideas, paraphrasing others' reasoning, providing justifications, and comparing their strategies while solving the task. Throughout the second stage of the DMP, Ms. Ware asked her students to create a plan to solve the task. She asked questions such *How could we solve this problem*? *What is our plan to solve it? What do we need to solve this problem? What two things are we looking for?* In stage 3 of Polya's heuristic, *Carry out the plan* (stage 3 of the DMP), students were required to work individually on *Pear Deck* to write an equation and a multiplication story. Then, students shared their solutions with their partners in small group discussions and justified whether the equation was related to the diagram.

During this stage, Ms. Ware engaged her students with others' reasoning. For example, Ana, an EL, wrote the numerical expression 3/4 x 1/5 to represent the area double shaded in the diagram, and she wrote the following multiplication story: "Mis. Ware ate 1/5 of a cookie; later the day, she ate ¾ of the cookie; how much did Ms. Ware eat?" Ana was able to translate "two types of shading" in the diagram into an expression (3/4 x 1/5), in which they demonstrated the ability to interpret the diagram by identifying the whole and the two factors of the multiplication. These elements are features of the mathematics register because Ana could identify the multiplication of fractions (1/5 x 3/4) shown in the diagram and use two different representations (e.g., the diagram and an equation). However, Ana’s story problem demonstrated a misconception of finding fractional pieces of the original whole rather than finding a fraction of a fraction. Ana did not find a part of a part; instead, she found a part of a whole twice. Her stories did not indicate the multiplication of the two fractions represented in the diagram. Ms. Ware used Ana’s story to generate a discussion and help her students reflect on their solutions. She asked her students to think and explain how the story could be fixed to match the diagram and the equation. She encouraged her students to use the mathematics register by assessing another student's strategy. Andrew, a non-EL student, gave his reasoning about Ana’s story:

> I think this multiplication problem isn't correct because this was like an adding problem…if you like to write in a better way to do so, it would be like: Miss Ware split and took away 1/5 of a cookie, and later in the day, she took ¾ of the piece or something like that to represent the diagram. Also, it has a lot more sense because it shows the 1/5 that she took away and then the ¾ that you ate from the 1/5.

Ms. Ware asked another student to paraphrase what Andrew said to ensure that her students understood the issue in Ana’s story. Through paraphrasing, students had the opportunity to show their understanding and use the mathematics register involved in the task. Antonio, an EL, said:

> What Aiden explained is that it [referring to Ana's problem] was an addition. So, I agree with Andrew because whenever it is multiplication, you basically have to multiply that, no matter what the denominators are, but when adding, you need to find a common denominator.

Ms. Ware revoiced Andrew’s reasoning to emphasize that the notion of a part of a part did not appear in the story. She said:

> I want to point out how she [referring to Ana’s story] has 1/5 of a cookie and then ¾ of a cookie. Right? And so, the ¾ is not referring to the 1/5. It is referring to a whole cookie. Whereas, as Andrew said, she [Ana] took 1/5 of something, and she ate ¾ of the 1/5 later. Do you guys hear the difference? Is it just a multiplication problem, or is this an addition problem?

Ms. Ware was aware that her students needed support in the mathematics register required to write their stories. Specifically, she helped them learn grammatical patterns (the second feature of the mathematics register) needed to write their stories. She took advantage of Ana's story and Andrews' ideas to highlight why their stories were not related to the multiplication expression ¾ x 1/5 and why the language used implied part of a whole twice in the story. Ms. Ware explained that because the unit is the same cookie used in the problem (1/5 of a cookie and then 3/4 of the cookie), the problem referred to the same "whole." Ana used the notion "part of a whole" when she wrote the phrase "1/5 of a cookie" and "¾ of the cookie," but the idea "part of a part" was not used in her story. Moreover, Ana's question, "how much did Miss Ware eat?" led students to think about adding the fractions to find the solution. Ms. Ware provided feedback on Ana's story. She suggested a sentence structure that she could use to enhance her problem "1/5 of something and then she ate ¾ of the 1/5 later." In this sentence, Ms. Ware provided her students a template they could use to write their story problem.". This sentence could help other students to use the notion of "part of a part" in their story problem. After engaging her students in the previous discussion, Ms. Ware asked them to return to Pear Deck to revise and re-write their story problems when needed.

As another example, Ms. Ware intentionally chose another student's solution while monitoring their work on *Pear Deck*. In her multiplication story, Alejandra, an EL, demonstrated a strong understanding of fractions as operators and used the mathematics register in her answer. Ms. Ware engaged her students to assess Alejandra's problem, and Ms. Ware and I asked Alejandra to clarify the language she used in her story.

Alejandra: My problem was Ms. Ware's daughter had a birthday party. She invited six of her friends. Ms. Ware had to get cupcakes for the party. One packet had 1/5 vanilla cupcakes, and 3/4 of those cupcakes had strawberry sprinkles on them. How many cupcakes in total had sprinkles?
Ms. Ware: ok, what makes this work?
Andrew: So, Alejandra's problem she explained that 1/5 of all of the cupcakes are vanilla and ¾ of the 1/5 has sprinkles on them, which is what the diagram shows because it shows 1/5 of the vanilla cupcakes, and then ¾ of the 1/5 is shaded, which shows that ¾ of the 1/5 have sprinkles on them.
Researcher: When you said "three-fourths of those cupcakes," what do you mean? Can you say more?
Alejandra: What I mean is that the 1/5 is the total cupcakes in the package and ¾ of those cupcakes had sprinkles on them.
Andrew: I think Alejandra…you mean by of those cupcakes is the vanilla cupcakes, because I feel like that could be a little bit confusing to some people because like… if you are trying to be very specific in making it very clear, I'd add that ¾ of the vanilla cupcakes had sprinkles on them because somebody could ask anything that when you say of those cupcakes…they might think ¾ of the chocolate cupcakes.

The excerpt above illustrates an example of how a student combined familiar language, symbols, and diagrams to write her multiplication story. In her written solution, Alejandra also used features of the mathematics register (e.g.,1/5 of all of the cupcakes are vanilla, and ¾ of those cupcakes had sprinkles on them). For Alejandra, the phrase "of those" means 3/4 of the 1/5 of vanilla cupcakes. She used undefined references (of those cupcakes) to indicate a specific antecedent in the sentence (the vanilla cupcakes). Using this word in a sentence to translate a mathematics expression into words can lead students, especially ELs, to miss data in the problem and get misconceptions about the whole used. In this sense, I asked Alejandra to clarify what she meant when she said "of those cupcakes" to help other students understand her story. During this discussion, Andrew helped to explain Alejandra's story. He highlighted the meaning of the phrase "3/4 of those cupcakes." Both Alejandra and Andrew demonstrated a high level of

English language proficiency and understanding of the mathematics register used in the story. However, it was essential to clarify the language used in the story to help other students identify the notion of "part of a part."

During the lesson, Ms. Ware devoted significant time engaging her students in meaningful mathematical discourse to help them understand the problem, engage with their peers in mathematical reasoning, revoice and paraphrase others' ideas, and communicate their mathematical ideas. She also had her students work individually on their stories and communicate their reasoning in writing. During the last stage of Polya's heuristic, *Looking Back* (Stage 4 of the DMP), Ms. Ware asked her students to return to *Pear Deck* and revise their stories. She encouraged her students to explain how their stories related to the diagram and the equation $\frac{3}{4} \times 1/5 = 3/20$ using the mathematics vocabulary learned in the class. Some students could refine their solutions and re-write their stories in this stage.

Discussion

This paper described how fifth-grade ELs in Ms. Ware's mathematics classes used and developed the mathematics register while engaging in a problem-solving lesson. Regarding the first research question, students had consistent opportunities to develop the mathematics register during the problem-solving lesson. Ms. Ware engaged her students in meaningful mathematics discourse. Multiple linguistic recourses such as everyday language, mathematics vocabulary, and visual displays were integrated to help students learn the mathematics register. Students had multiple opportunities to communicate their ideas, assess their peers' reasoning, negotiate meanings of mathematical terms, and paraphrase others' reasoning. These strategies helped students enhance and advance their mathematics register (Moschkovich, 2015; Schleppegrell, 2007). For example, students had opportunities to learn the mathematics register inherent in the task by negotiating the mathematical meaning of terms used in the problem (e.g., multiplication story, equation), using their everyday language to explain their problems (e.g., Alejandra's story). Furthermore, students assessed Ana's problem and reflected on the language used in her story and how the phrases used (e.g., "1/5 of a cookie and ¾ of the cookie") indicated part of a whole twice instead of part of a part. Moreover, ELs had the opportunity to revise their stories and incorporate the language learned to write a multiplication story that reflected the notion of the fraction of a fraction. Alejandra's problem illustrated how students used the mathematics register to write their stories and justify the language employed in their stories (e.g., ¾ of those cupcakes").

Regarding the second research question, the multiplication story task requires students to have English language skills and abilities to translate mathematics expressions into words. Ana's story illustrates that learning the mathematics register can pose challenges for ELs, specifically when students need to use the English language to translate a mathematics expression into text (O'Halloran, 2015). In this sense, Ms. Ware implemented instructional strategies to help her students develop the mathematics register. These strategies included facilitating meaningful mathematics discourse in which students had multiple opportunities to paraphrase others' ideas, assess others' strategies, and engage with their peer's reasoning (Herbel-Eisenmann et al., 2013; Moschkovich, 2015). Ms. Ware intentionally selected specific students' stories (e.g., Ana and Alejandra's stories) to help them reflects on how they could use the mathematics register to fix their stories and represent the notion of "part of a part."

The findings in this paper provide insights into the importance of supporting ELs to develop the mathematics register through problem-solving. In Ms. Ware's mathematics classes, ELs had the opportunity to apply their knowledge about fractions and their knowledge of language to write multiplication stories. Ms. Ware asked purposeful questions to help her students understand specific phrases that they could use to address the language complexity found in rich tasks such as the multiplication story task.

References

Abedi, J., & Gándara, P. (2006). Performance of English language learners as a subgroup in large-scale assessment: Interaction of research and policy. *Educational Measurement: Issues and Practice*, *25*(4), 36–46. https://doi.org/10.1111/j.1745-3992.2006.0007710.1111

Barwell, R. (2005). Integrating language and content: Issues from the mathematics classroom. *Linguistics and Education*, *16*(2), 205–218. https://doi.org/10.1016/j.linged.2006.01.002

Borjian, A. (2008). A new approach to meeting the needs of Latinos in American schools: Students' views on attending a redesigned small high school. *International Journal of Learning*, *15*(2), 277–285. https://doi.org/10.18848/1447-9494/CGP/v15i02/45642

Campbell, A. E., Davis, G. E., & Adams, V. M. (2007). Cognitive demands and second-language learners: A framework for analyzing mathematics instructional contexts. *Mathematical Thinking and Learning*, *9*(1), 3–30. https://doi.org/10.1080/10986060709336603

Charalambous, C., & Pitta-Pantazi, D. (2007, March). Drawing on a theoretical model to study students' understandings of fractions. *Educational Studies in Mathematics*, *64*(3), 293- 316.

Chval, K. B., & Chávez, Ó. (2012). Designing math lessons for English Language Learners. *Mathematics Teaching in the Middle School*, *17*(5), 261–265. https://doi.org/10.5951/mathteacmiddscho.17.5.0261

Creswell, J. W., & Poth, C. N. (2018). *Qualitative inquiry and research design choosing among five approaches.* (4th Edition). SAGE Publications, Inc., Thousand Oaks.

de Araujo, Z., Roberts, S. A., Willey, C., & Zahner, W. (2018). English Learners in K–12 mathematics education: A review of the literature. *Review of Educational Research*, *88*(6), 879–919. https://doi.org/10.3102/0034654318798093

Dong, Y. R. (2016). Create a responsive learning community for ELLs. *The Mathematics Teacher*, *109*(7), 534–540. JSTOR. https://doi.org/10.5951/mathteacher.109.7.0534

Ernst-Slavit, G., & Mason, M. R. (2011). "Words that hold us up:" Teacher talk and academic language in five upper elementary classrooms. *Linguistics and Education*, *22*(4), 430–440. https://doi.org/10.1016/j.linged.2011.04.004

Halliday, M (1978). *Language as a social semiotic: The social interpretation of language and meaning*. Baltimore, MD: University Press.

Herbel-Eisenmann, B., Johnson, K. R., Otten, S., Cirillo, M., & Steele, M. D. (2015). Mapping talk about the mathematics register in a secondary mathematics teacher study group. *The Journal of Mathematical Behavior*, *40*, 29–42. https://doi.org/10.1016/j.jmathb.2014.09.003

Herbel-Eisenmann, Steele, M. D., & Cirillo, M. (2013). (Developing) Teacher discourse moves A framework for professional development. Mathematics Teacher Educator, 1(2), 181–196. https://doi.org/10.5951/mathteaceduc.1.2.0181

Kitchen, R., Castellón, L. B., & Matute, K. (2020). Supporting English learner students' development of the mathematics register through an instructional intervention. In A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), Mathematics Education Across Cultures: Proceedings of the 42nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education Conference, México (pp. 1993-2000). Cinvestav/AMIUTEM/PME-NA. https://doi.org/10.51272/pmena.42.2020

Lucero, A. (2012). Demands and opportunities: Analyzing academic language in a first-grade dual language program. Linguistics and Education, 23(3), 277–288. https://doi.org/10.1016/j.linged.2012.05.004

Martiniello, M. (2008). Language and the performance of English-Language Learners in math word problems. *Harvard Educational Review; Cambridge*, *78*(2), 333-368,429. https://search.proquest.com/docview/212283604/abstract/A87B58C0AC984034PQ/1

Moschkovich, J. (2007). Bilingual mathematics learners. How views of language, bilingual learners' mathematics communication effect instruction. In N.S. Nasir and P. Cobb (eds.). *Improving access to mathematics: Diversity and equity in the classroom* (pp. 89 -104). New York, NY: Teachers College Press.

Moschkovich, J. (2014). Building on student language resources during classroom discussions. In. M. Civil & E. Turner, (Eds.), The common core state standards in mathematics for English language learners: Grade K-8 (pp.127-138). Alejandra, VA: TESOL.

Moschkovich, J. (2015). Academic literacy in mathematics for English Learners. *The Journal of Mathematical Behavior*, *40*, 43–62. https://doi.org/10.1016/j.jmathb.2015.01.005

National Center for Education Statistics (NCES) (2021). Preprimary, Elementary, and Secondary Education Section: Elementary and Secondary Enrollment. U.S. Department of Education. Retrieved from https://nces.ed.gov/programs/coe/indicator/cgf

National Council of Teachers of Mathematics (NCTM) (2014). *Principles to Actions: Ensuring mathematics success for all*. Reston, VA: National Council of Teachers of Mathematics.
O'Halloran, K. L. (2015). The language of learning mathematics: A multimodal perspective. *The Journal of Mathematical Behavior*, *40*, 63–74. https://doi.org/10.1016/j.jmathb.2014.09.002
Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. New York, NY: Routledge.
Pólya, G. (1986). *How to solve it: A new aspect of mathematical method*. Princeton University Press. (Original work published in 1945)
Schleppegrell. (2004). *The language of schooling: A functional linguistics perspective*. Lawrence Erlbaum Associates.
Schleppegrell. (2007). The linguistic challenges of mathematics teaching and learning: A research review. *Reading & Writing Quarterly*, *23*(2), 139–159. https://doi.org/10.1080/10573560601158461
Wilkinson, L. C. (2018). Teaching the language of mathematics: What the research tells us teachers need to know and do. The Journal of Mathematical Behavior, 51, 167–174. https://doi.org/10.1016/j.jmathb.2018.05.001
Wisconsin Center for Education Research (WCER). (2014). 2012 amplification of the English language development standards: Kindergarten – grade 12. WCER: Board of Regents of the University of Wisconsin.

EXPLORING THE RELATIONSHIP BETWEEN QUALITATIVE LESSON SCORES AND QUANTITATIVE QUALITIES OF INDIVIDUAL CODES

Kathleen Melhuish
Texas State University
melhuish@txstate.edu

Alexander White
Texas State University
whiteale@txstate.edu

Sharon K. Strickland
Texas State University
strickland@txstate.edu

Elizabeth Wrightsman
Texas State University
ewrightsman@txstate.edu

Describing and measuring instructional quality of mathematics lessons is a common goal amongst mathematics education researchers. Such work takes several forms such as classifying and coding instructional moves and student activity or providing high-level rubric-based scores in relation to categories. In this work, we share an innovative mixed methods approach to analyzing lesson data that includes both a time-based classification of instruction and an overall scoring component. Using the Math Habits framework, our project team analyzed a set of 97 fourth-eighth grade mathematics lessons including overall scores. From this qualitative analysis, we developed quantitative models to predict overall scores and better understand the ways that individual codes do or do not contribute to overall lesson score characterizations.

Keywords: Research Methods, Instructional Activities and Practices

In this report, we share recent work aiming to further both our approach to classroom observation tool measures and our understanding of which elements of a classroom are salient in a coding process. This work is situated in a larger validation study focused on the Math Habits Tool (Melhuish, et al., 2020). The Math Habits Tool decomposes a mathematics classroom into four types of codeable activities: teaching routines, catalytic teaching habits, student habits of mind, and student habits of interaction (each of which will be expanded in the next section.) The categories capture teacher and student activity that characterize student-centered, conceptually-oriented classrooms. As in many instruments (e.g., Mathematical Quality of Instruction, Hill, 2014), qualitative coders analyze the lesson at two levels: during the lesson and holistically at the end of the lesson. While coding during the lesson involves identifying time-stamped, individual occurrences, the holistic codes use a rubric-based approached to make a subjective judgement call as to the quality of the teacher and student activity.

We frame our contribution as two-fold. First, we make a methodological contribution -- the development of a quantitative models to estimate overall lesson scores after a qualitative coding process. Notably, we go beyond just using occurrence counts for codes to characterize a class, but also introduce a measure of spread (the degree these occurrences are found at different times in the lesson). We conjectured that although spread was not an explicit portion of the coder's rubrics, it was likely to inform the qualitative evaluations at the lesson level. For example, consider this extreme version. Suppose a classroom has ten rich student contributions, but all occurred within the first five-minute interval. Then the remainder of the class was a lecture. Contrast this situation with a class where student and teachers are interacting, and ten rich student contributions occur throughout the lesson. A frequency-based approach would characterize these two classrooms in the same manner; however, it is unlikely that we would want such classes to be equivalent.

Second, in order to estimate overall lesson scores, we confronted issues of which codes "mattered" and in what ways. Specifically, whether a code occurred spread throughout a lesson sometimes mattered more than how often (and vice versa, along with other combinations). These findings have implications for researchers interested in teaching practices and students' classroom activity.

Background and Framing

Broadly, we take a social cultural approach to the mathematics classroom focusing on social interactions between people in the classroom. Knowledge is co-constructed in these interactions between students and between teacher and students. While we largely assume that individual cognition and social interactions are interrelated (as in Cobb & Yackel, 1996) where individual understanding is developed via social interactions, we focus on the observable social side. Further, we specifically attend to components of classroom interactions that may promote sense-making and mathematical argumentation inclusive of justifying and generalizing. Justifying and generalizing can support the co-construction of mathematical meaning (Brown & Renshaw, 2000; Simon & Blume, 1996) and students' development of conceptual understanding (Staples et al., 2012). Instruction aligned with such goals reflects a standards-based instructional approach (defined in Rubel, 2017 and reflected in standards documents such as the *Common Core*, National Governors Association, 2010)

We use the instructional triangle (Hawkins, 2002) to situate our analytic framing focusing on relationships between teachers, students, and content. We incorporate both Lampert (2001) and Cohen et al.'s (2003) expansion to capture the mediating role a teacher plays in the student-content relationship and the relationships between the students themselves. Figure 1 reflects the components of the BI Framework overlayed on the instructional triangle.

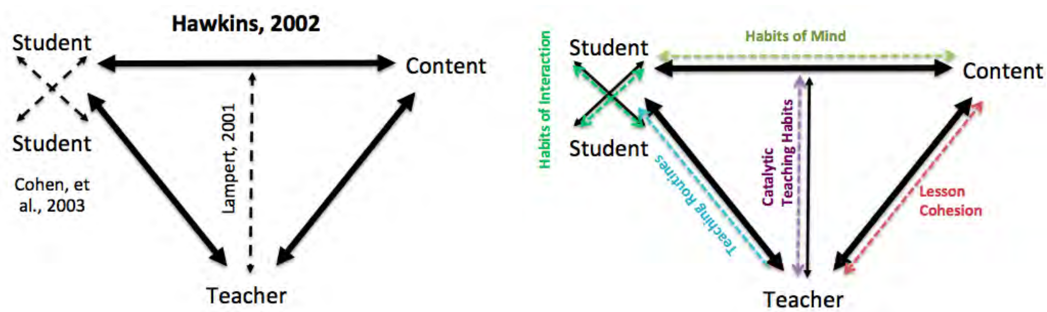


Figure 1: Instructional Triangle and the BI Framework

The blinded framework was developed to operationalize specific instructional routines, moves, and student activity that can be observed within the classroom setting. The coding categories include: Student Habits of Mind which reflect productive ways students engage in mathematics, Habits of Interactions which reflect ways students engage with each other around the mathematics, Catalytic Teaching Habits which capture specific teaching moves that may engender students in engaging with mathematics and each other's mathematical ideas, and Mathematical Productive Teaching Routines. The teaching routines are "recurring, patterned sequences of interaction teachers and students jointly enact to organize opportunities for student learning in classrooms" (DeBarger et al., 2011, p. 244). Unlike the other categories, teaching routines are not identified as instances, but rather over time intervals when they occur. Table 1 includes the categories and subcodes. Each of these categories and subcategories are rooted in the literature on promoting student-centered instruction and mathematical argumentation (e.g., Kazemi, 1998; Staples, 2007; Stein, et al., 2008; Thanheiser & Melhuish, 2022).

Table 1: BI Framework with Variable Names in Parentheticals

Student Habits of Mind	
Mathematical (MathHoM)	Representations; Connections; Mathematical structure
Reflection (ReflectHoM)	Metacognition; Reasoning with mistakes; Making meaning
Capstone (JGHoM)	Justifying; Generalizing
Student Habits of Interaction	
Private Reasoning Time (PRHoI)	Private Reasoning Time
Explaining (ExplainHoI)	Explaining my reasoning; Exploring multiple pathways
Engage with Peer (PeerHoI)	Revoicing; Comparing logic and ideas; Critiquing and Debating
Question (QuestionHoI)	Asking genuine questions
Catalytic Teaching Habits	
Private Reasoning Prompt (ThinkCTH)	Private Reasoning Time Prompt
Sharing Thinking Prompts (ShareCTH)	Prompt to share meaning; Prompt to share thinking; Prompt to share why; Prompt to share representation
Peer Prompts (PeerCTH)	Prompt to analyze strategy; Prompt to analyze mistake; Prompt to compare or connect across strategies; Prompt to revoice or make sense of strategy
Capstone Habit Prompts (JGCTH)	Prompt to justify; Prompt to notice, wonder, or conjecture
Teaching Routines	
Access (AccessTR)	Making meaning of tasks, contexts, and/or language
Public Records (RecordsTR)	Working with selected & sequenced student math ideas Teacher; Working with public records of students' mathematical thinking
Discussion (DiscussionTR)	Orchestrating productive whole class discussions
Groupwork (GroupworkTR)	Structuring mathematically worthwhile student talk; Conferring to understand students' mathematical thinking & reasoning

Methods

Data

This study draws on 96 video-recorded lessons (taken near the end of the school year) from 3 school districts stemming from diverse projects. The samples include 33 lessons from District 1 (Melhuish, et al., 2022), 31 lessons from District 2 (Sorto, et al., 2018), and 33 lessons from District 3 (Kane, et al., 2016). Data on each district can be found in Table 2.

Table 2: Demographic Information on Data Set Districts

District	Race/Ethnicity	Socio-Economic Status	Language

District 1 (grades 4 and 5)	56% White 19% Black/African American, 11% Latino/Hispanic 9% Asian	55% eligible for free and reduced lunch	6% Transitional Bilingual
District 2 (grades 6-8)	99% Latino/Hispanic	95% "economically disadvantaged"	33% Limited English Proficiency
District 3 (grades 4 and 5)	51% Black/African American, 30% White, 13% Latino/Hispanic, 4% Asian	73% eligible for free and reduced lunch	23% Limited English Proficiency

All of these lessons had previously been analyzed with the Mathematical Quality of Instruction (MQI; Hill, 2014) instrument. For the larger databases, we selected a random subset within MQI strata. For District 2, we included all middle school teachers who had opted into recording. We sampled in this manner to assure a range of instructional contexts and practices.

Qualitative Analysis

Each lesson was then coded independently by two researchers according to the BI framework. Any discrepancies were resolved via discussion. After an initial round of coding, the coded lessons were then reviewed by a third member of the research team to identify any coding drift or inconsistences across the coded lessons. Additionally, discrepancies were identified and resolved via discussion. Besides coding using the framework, each coder also assigned an overall rubric-based score for student and teacher activity. Krippendorff's α=0.79 and α=0.57 for overall student and teacher, respectively. The levels for overall teaching score are as follows: (1) No evidence of use of Teaching Routines or an attempted teaching routine (but without Catalytic Teaching Habits embedded.) (2) Use of more than one Teaching Routines; some evidence of Catalytic Habits; OR Use of only one Teaching Routine; but many (variety) of Catalytic Habits. (3) Multiple Teaching Routines; Catalytic Habits embedded; (4) Multiple Teaching Routines; Catalytic Habits embedded with pushes towards justifying and/or generalizing. The levels for overall student codes are as follows: (1) Students engaged in at most a Habit of Interaction or two and maybe a Habit of Mind; (2) Students engaged in some Habits of Mind and/or Habits of Interactions (3) Students engaged in multiple Habits of Mind and Habits of Interaction; (4) Students engaged in multiple Habit of Interaction or two and maybe a Habit of Mind with justifying and/or generalizing. As in many rubrics, the overall levels provide some guidance, but also rely on subjective judgements made by the coders. For this reason, consensus was reached through discussion.

Quantitative Analysis

In order to examine how individual codes are associated with the overall Teacher and Student codes, two summary statistics were computed for each lesson and code type: the count and the spread. The count is simply the total number of occurrences of the code during a lesson. To compute the spread, the lessons were partitioned into 10 equal intervals. The spread is the number of intervals in which a code occurred at least once. The two statistics capture the difference between number of times the behavior is observed and the consistency with which it is observed throughout the lesson.

Least Absolute Shrinkage and Solution Operator (lasso) models were used to investigate the relationship between the individual and overall codes. Lasso (James, et al., 2013, pp. 219 - 227) models use l1 regularization to prevent overfitting, reduce the variance of the coefficient estimates of a linear model, and perform variable selection. Unlike stepwise techniques used with standard least squares regression, variable selection in Lasso models does not rely on normality assumptions. The lasso coefficients minimize the quantity

$$\sum_{i=1}^{n}(y_i - \beta_0 - \sum_{i=1}^{p} \beta_j x_{ij})^2 + \lambda \sum_{i=1}^{p} |\beta_j|$$

where the y_i is the overall code for the i^{th} lesson, and the x_{ij} are the corresponding scaled versions of the count and spread summaries for individual codes. The regularization constant, λ is determined separately for the Teacher and Student models via cross-validation.

Results

In this results section, we share estimates from our models and interpretation; however, the majority of the theorizing and contextualizing of these results can be found in the Discussion section. Recall, our goal is to create a model that can take the human coders' individual timestamped BI codes to predict the human coders' overall student and teacher codes. That is, can we use the detailed BI coding with a model to generate the overall codes? To this end, we needed to examine which (if any) of the individual BI codes were having more or less impact on overall codes. In this section we present what we learned about the role of BI codes in relation to overall codes. We begin with Overall Student codes.

Table 3: Coefficients from the LASSO model for Student Codes

	Model Coefficients	
Code	**Count**	**Spread**
MathHoM	-0.208	0.443
ReflectHoM	0.143	0
JGHoM	0	0.273
PRHoI	0	0.045
ExplainHoI	0.089	0.249
PeerHoI	0	0.060
QuestionHoI	0.120	0

The coefficients for the lasso model for Overall Student code are shown in Table 3. Recall that these habits reflect observable ways that students engaged with the mathematics and with each other mathematically. The zero coefficients for the count variables for JGHoM (capstone habits of justifying and generalizing) and PRHoI (private reasoning time), and the spread variable for QuestionHoI (asking genuine questions) indicate that lasso dropped those respective count or spread variables from the model. We use *z*-scores to interpret the coefficients of the remaining predictors. For example, for a lesson with one standard deviation more ReflectHoM (reflection habits of mind) than the average lesson, the predicted overall code increases by 0.143. We can unpack the slightly more complex case of the mathematical habits of mind (MathHoM) coefficients where we see a negative relationship. Consider two lessons where the count for MathHoM differs by one. If the additional code occurs in an "empty" interval, the count and the spread both increase and the predicted overall code increases. On the other hand, if the additional code occurs in an interval where MathHoM has already been observed, the predicted overall code decreases. That is, observing the code throughout the lesson is more beneficial than simply counting a total. On the other hand, some codes only mattered in terms of count. For example, students asking a genuine question (QuestionHoI) is associated with an increase in the overall code, no matter where it occurs. That is all to say, that some codes matter where they occur in a lesson (indicated by spread) and some only matter how often they occur (indicated by count).

Table 4: Coefficients from the LASSO model for Teacher Codes

	Model Coefficients	
Code	**Count**	**Spread**
GroupworkTR	0.280	0.096
RecordsTR	0	0.241
DiscussionTR	0	0
AccessTR	0.253	0
ThinkCTH	0	0.058
ShareCTH	0	0.104
JGCTH	0.144	0
ReflectCTH	0.139	0
PeerCTH	0	0
RevoiceCTH	0.056	0.097

Table 4 provides the coefficients for the lasso model for the Overall Teacher codes. Again, we can notice that different types of activities are differently related to the overall codes. The teaching routines related to groupwork (GroupworkTR) matter both in terms of how frequently they occur (counts) and how they spread throughout the lesson (spread). In contrast, the teaching routines related to public records (RecordsTR) were only significant in terms of how spread they were throughout the lesson and the meaning making teaching routine (AccessTR) was significant only in terms of frequency, not spread. If we turn to individual teaching moves (the catalytic teaching habits), we can see spread is significant for prompts related to private think time and to share thinking (ThinkCTH; ShareCTH), but overall frequency, but not spread, is significant for reflection prompts (ReflectCTH). For teacher revoicing (RevoiceCTH), both frequency and spread were small, but positive predictors of overall score. Finally, we note that the orchestrating discussion routine (DiscussionTR) and the prompts to engage with peers' reasoning (PeerCTH) did not contribute to predicting overall teacher scores.

We also briefly share results about how closely these models fit our overall scores. Table 4 and Table 5 present a crosstabulation comparing the true and predicted codes. In grey, we have emphasized the lessons where the coder overall code matched the predicted code. For the students, the model correctly predicts 71.1% of our lessons. For the overall teacher score, the model correctly predicts 76.77% of our lessons. Further, only one lesson in each case is predicted more than one level off. We can calculate Krippendorf's α, as we would when comparing coders. For the overall teacher scores, $\alpha=0.879$ and for the overall student scores $\alpha = 0.813$. Both numbers are over 0.80 indicating substantial agreement. That is, our models are doing a relatively good job predicting the overall scores arrived at by coders.

Table 5: Crosstabulation of Overall Student Codes: True vs. Predicted

		Predicted Overall Code				
		1	2	3	4	total
Coder Overall Code	1	16	10	0	0	26
	2	2	24	6	0	32
	3	0	6	13	0	19
	4	0	1	4	15	20
Total		18	41	23	15	97

Table 6: Crosstabulation of Overall Teacher Codes: True vs. Predicted

		Predicted Overall Code				
		1	2	3	4	total
Coder Overall Code	1	24	7	0	0	31
	2	1	26	3	0	30
	3	0	4	12	1	17
	4	0	1	6	12	19
Total		25	38	21	13	97

Discussion

When we code data, we often make (explicit or implicit) inferences based on quantitative characteristics such as frequency of a code. However, when we consider a complex setting like a mathematics classroom, we often rely on subjective judgement calls about other features such as the degree it feels like a particular activity characterizes a lesson. In fact, this sort of expert judgement is why qualitative coding can be powerful. Yet, we have found that it can be quite challenging to come to agreement on lesson level scores because, by nature, coders are not noticing or perhaps not weighting elements of instruction in the same way. Our goal with this study was to develop quantitative means to estimate overall scores. We focused on the measure of spread in addition to count to avoid collapsing some of the dimensionality in a classroom.

If we consider our results, we can see that in some cases spread was more important, some cases count, and in yet others, they played out in more complex ways. We return to a couple of examples to conjecture what might account for these differences. If we turn to the overall teacher scores, we can see that both orchestrating discussion and prompts to engage with peers' ideas did not contribute to predicting overall scores. First, these are types of codes that are theoretically related. In order to orchestrate discussion, we required that multiple students are engaged with each other's ideas in some way. This typically occurs when teachers make related prompts for engagement. From a simplistic view, the rubric would reward both types of activities with higher overall scores. However, if we examine the data, we can note two features that may account for this result. First, the DiscussionTR and the PeerCTHs were only meaningfully different for teachers with an overall high score. For example, the mean spread for the DiscussionTR for overall high teachers was 3.1 (meaning, on average, discussion happened in 3 of 10 intervals) and mean count was 6.9 (meaning on average discussion happened 7 times per lesson). In contrast, spread was less than one for all other levels of overall score and less than two for mean count. Rather than gradual increases, this routine served to discriminate between high level lessons from other levels. This leads to the second point, the high-level lessons all had higher spreads and counts in other categories. Theoretically this makes sense. It is likely that in a class where a teacher orchestrates discussion, there is overlap with other teaching routines like using public records of students thinking – in many cases the discussion is about such a record. Thus, these codes are not contributing new information about the overall teaching in the lesson.

Now let's contrast two teaching routines that both were significant but in different ways. Working with public records of student thinking and selecting and sequencing (RecordTR) mattered in terms of spread whereas Making meaning of task and terms (AccessTR) mattered in terms of counts. In this case, we conjecture the way these routines operate in the classroom may account for the difference. Engaging students in making meaning around tasks and terms is a routine that comes up when students encounter an idea, task, or piece of language in which they may be unfamiliar. This is likely to occur at specific points in the lesson such as when a task is launched. In contrast, working with records of students' ideas may be threaded throughout. Thus, the number of occurrences of meaning making may be more salient than the spread of meaning making; while the spread of record use is likely a more salient feature of overall teaching.

If we turn to the student codes, we see the relationship between observed math habits of mind (including capstone habits) and overall scores are more linked to spread. These habits include things like reasoning with representations, structure, connections, and justifying and generalizing. Frequencies alone may paint a misleading picture because a single student contribution may embed many of these habits (e.g., justifying a result by use of a pattern within a table). A short span of time with high counts is not as meaningful as occurrences spaced throughout a lesson (reflected in spread), thus theoretically spread is likely to be more salient. A second explanation may be that at a certain threshold, frequency might not contribute new information. That is, a class with 20 instances and a class with 30 instances of habits of mind are both likely at a high level and the difference of 10 instances does not contribute something new. In contrast, the spread is capped at the interval number and any difference has the potential to provide meaningful information that characterizes a lesson across time.

For space limitations, we will not unpack all of the differences in how the codes are operating but will spend a brief amount of time comparing the difference in the ExplainHoI (students explaining their thinking) and the MathHoMs. Explaining mattered in terms of both count and spread, although with a relatively small coefficient for count. The threshold to explain one's thinking is much lower than the threshold for that thinking to include math habits of mind (which reflect higher level reasoning). Explaining was by far the most frequently observed student activity at all levels of overall student code. The mean number of occurrences of explain was 6.4 for the Overall Student =1 classes and 39.8 for the Overall Student=4 classes. However, for lessons that received the lowest overall score, the mean spread was less than 3 intervals whereas in highest lessons, it was nearly 9 intervals. Both frequency and spread appear to provide important information to characterize a lesson.

This leaves several questions open for future research. First, are these relationships a consequence of the coders and rubrics or a consequence of how these activities unfold in the classroom? This is work that could be further addressed with additional qualitative analysis of the classrooms as well as attention to classrooms where overall codes as assigned by the model diverged from overall code as determined by the qualitative coders. Second, how might we develop more accurate models and measures when considering coding at this grainsize? Spread and count provided a start, but other measures such as interrelated (different types of codes within a timespan), existence (binary), or alternative models (such as non-linear models) could lead to additional insights.

Acknowledgments

This research was supported by the National Science Foundation grant DRL-1814114.

References

Brown, R., & Renshaw, P. (2000). Collective argumentation: A sociocultural approach to reframing classroom teaching and learning. In H., Teoksessa Cowie & G. van der Aalswoort (Eds.), *Social interaction in learning and instruction. The meaning of discourse for the construction of knowledge*. Amsterdam: Bergamon Press.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, 31(3-4), 175–190.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). *Resources, instruction, and research. Educational evaluation and policy analysis,* 25(2), 119–142.

DeBarger, A. H., Penuel, W. R., Harris, C. J., & Schank, P. (2011). Teaching routines to enhance collaboration using classroom network technology. In *Techniques for fostering collaboration in online learning communities: Theoretical and practical perspectives* (pp. 224-244). IGI Global.

James, G., Witten, D., Hastie, T., & Tibshirani, R. (2013). An introduction to statistical learning. Springer.

Hawkins, D. (2002). I, thou, and it. In The informed vision: Essays on learning and human nature (pp. 51–64). Algora Publishing.

Hill, H. (2010). Mathematical Quality of Instruction (MQI). Coding tool. Lansing, MI: University of Michigan.

Kazemi, E. (1998). Research into Practice: Discourse That Promotes Conceptual Understanding. Teaching Children Mathematics, 4(7), 410-414.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* Yale University.

Landis, J.R.; Koch, G.G. (1977). The measurement of observer agreement for categorical data. Biometrics. 33 (1): 159-174.

Melhuish, K., Thanheiser, E., Heaton, R., Sorto, A. Strickland, S., & Sugimoto, A. (2020). The
Math Habits Tool - Research Version [Measurement instrument] Retrieved from http://mathhabits.wp.txstate.edu

Melhuish, K., Thanheiser, E., White, A., Rosencrans, B., Foreman, L., Shaughnessy, J. M., Riffel, A. & Guyot, L. (2022). The Efficacy of a "Mathematics for All" Professional Development. *Journal for Research in Mathematics Education*.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Washington, DC: Authors.

Rubel, L. H. (2017). Equity-directed instructional practices: Beyond the dominant perspective. *Journal of Urban Mathematics Education,* 10(2).

Simon, M. A., & Blume, G. W. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. The Journal of Mathematical Behavior, 15(1), 3–31.

Sorto, M. A., Wilson, A. T., & White, A. (2018). Teacher knowledge and teaching practices in linguistically diverse classrooms. In *Language and communication in mathematics education* (pp. 219-231). Springer, Cham

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and instruction*, 25(2-3), 161-217.

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *The Journal of Mathematical Behavior,* 31(4), 447-462.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. Mathematical thinking and learning, 10(4), 313-340.

Thanheiser, E., & Melhuish, K. (2022). Teaching routines and student-centered instruction: a case of divergent lessons. [Manuscript submitted for publication].

CONCEPTIONS OF HONDURAN SECONDARY SCHOOL TEACHERS ON THE NUMBER LINE IN SYMBOLIC MANAGEMENT TASKS

Sharon Samantha Membreño Estrada
Cinvestav - IPN
sharon.membreno@cinvestav.mx

Claudia Margarita Acuña Soto
Cinvestav - IPN
claudiamargarita_as@hotmail.com

The number line is a model that is used to measure, count, order and even operate, which requires a symbolic interpretation. Then, we investigate the conceptions of 72 in service secondary teachers, when they manage the model of the number line associated with order, spatial location and the relative position between numbers and marks. In a workshop carried out through Internet, we found that the participants consider that the usefulness of the model is in the manipulation, and they do not see the need to modify it, despite the requirements of the problem, taking for granted its usefulness and the control that resources could give them, such as the change of the numerical scale. They don't even realize the symbolic properties to interpret the numerability.

Keywords: Number line, Teacher conceptions, Artifact management

The number line can be a model for teaching, it serves for measuring, counting, ordering, or operating. Its several presentations even allow it to be used as a kind of note (Heirdsfield, 2011), where at least there is no need for homogeneous units between consecutive numbers or the properties of a homogeneous representation (Nemirovsky, 2003).

We consider that the model of the number line is more than a metaphor (Doritou & Gray, 2007) because it allows us to approach the numerical properties through a symbolic representation, and its use must consider properties such as the homogeneous representation that gives spatial meaning to the numbers according to their position. As well as to the marks and the intervals between them, these symbolic activities are the basis for numerability.

Therefore, it is desirable that the teacher takes into account that signs are regulated in a way that gives it mathematical meaning, so we investigate the conceptions of Honduran in service secondary school teachers, about their willingness to make changes to the representation according to the needs, as well as to the symbolic treatment considering the order, the spatial location and the relative position of numbers and marks on the number line to development a symbolic treatment in class.

Background

At this time, it has been established the importance of teachers acquiring three types of knowledge to develop their teaching effectively, which requires: 1) Common knowledge of the content; 2) Specialized knowledge of the content; and 3) Knowledge of content and students, as stated by Hill, et al. (2008).

In this research, we will refer to the specialized knowledge of the content, associated with the symbolic interpretation, use and management of the number line in numerability sense where the properties of homogeneity and order are highlighted, that is, the use of a model where the position of the signs are relevant both for their value, as for their relative positions associated with order and inequality to perform model functions for numerability.

The so-called number line is capable for giving body to different arithmetic operations at the basic level we can find it in the work that refers to the relations of numerical inequalities and linear inequalities (Membreño and Acuña, 2021), which requires interpreting the order, relative position and spatial location of numbers and marks since an index to symbols in an intentional way.

In general, teachers use the number line as a pedagogical tool that it is useful to the student, since it gives body with certain relationships, however, they do not always realize that it is also an abstract structure, which is a symbolic model with some restrictions (Doritou & Gray, 2010), that accepts a treatment as a representation of the numbering system and also gives body to the metaphor of numbering (Doritou & Gray, 2007).

From our point of view, although it is true that the number line gives body to the metaphor of numerability, it itself is formed by a series of signs with mathematical properties that are of two types: explicit and implicit, which must be considered for their teaching.

In the case of explicit signs, we have those that will clearly be detected and used for interpretation as in the case of numbers and the marks associated with them, the numbers have numerical value, so they acquire an ordinal position, but they are also the numerical labels for the associated marks, which allows the order of the objects represented. However, there are other implicit signs such as segments that, although relevant to interpretation, often go unnoticed, for example, when they represent the norm of the homogeneous unit of numbering or in the case of representing the solution sets of numerical inequalities and linear inequalities, (Membreño and Acuña, 2021).

Objective

We investigate the conceptions and justifications of the secondary school in service teachers when they are in a position to manage the type of representation of the model of the number line that would be used under changing conditions, which implies accepting a management on the representation, as well as the use of the signs involved in the model either marks or numbers based on the order, the spatial location and relative position of this representation when it is homogeneous.

Theoretical framework

Using models as symbolic tools

The number line is a symbolic representation, made up of signs that must be interpreted mathematically to use it as a model for teaching, which includes detecting the role of implicit and explicit signs as symbolic mediators.

The Theory of Objectification suggests that learning is linked to the use of instruments, signs or artifacts that are not aids to learning, but they are a semiotic media whose presence and use print a distinctive stamp on the constructed, becomes part of it through a semiotic mediation that allows to develop a reflective praxis, which is understood as: “what we know and how we come to know it is circumscribed by ontological positions and processes of meaning production that shape a certain kind of rationality that allows us to pose certain problems” (Radford, 2004, p. 13)

If mathematics is essentially symbolic, it is important to observe the role of the signs that allow individual to mediate knowledge and it is through them that the actions and reflections of this are oriented, so it is suitable to know the conceptions of the teachers who would be guiding their use on the signs about the representation of the number line.

Radford states that " The sign has meaning when it is related to other signs" (Radford 2006b, p. 8), so meaning would be given to the representation when the explicit signs are being related to those implicit on the number line model.

Regarding the treatment of signs, we are interested to know the use made of them either as index or symbols, Pierce (2005) which will be relevant for an adequate symbolic treatment, throughout his teaching activity by the treatment of the number line and in the use of the properties of order and homogeneity.

For the purposes of this research, on the use of signs we have incorporated the restriction of homogeneity, Nemirovsky (2003) in the number line in which the only property of the objects represented is their position in order to investigate the inclination of teachers to modify or not, the spatial conditions of it under changing conditions, as well as its interpretation of the order and spatial location of the numbers and their marks on the signs present in the use of the model.

In the present research, the symbolic management of the line is present when teachers adjust the representation of it, depending on the needs of the problems posed and the semiotic treatment is characterized by recognizing the mathematical norms of order, spatial location and relative position of the numbers and marks represented.

Methodology

About the sample

The present research explored the conceptions of 72 secondary school teachers in service from Honduras, who teach students between 12 and 15 years old, research that was developed through an online workshop as part of the Congress of Educational Mathematics (COME) in Honduras in 2021, aimed at teachers from all over of the country, the open call was freely attended by those interested in the improvement of mathematics teaching in that country.

Method

The workshop was carried out through Zoom supported using the GeoGebra classroom platform, through a questionnaire where teachers could manipulate graphic representations and interact with the researcher either in front of all participants, in a particular and instantaneous way or even leave written comments to have subsequent communication. This was useful to analyze more broadly their opinions and conceptions to management on the model of the number line.

The questionnaire was split into eight tasks associated with the management of the number line to know the conceptions of the teachers in front of tasks that require managing the number line, specifically about order and homogeneity.

In general, graphs of number lines were presented in which the position of relevant marks such as zero was asked, as well as the choice of lines where zero is not in the center and maybe they have chosen a more appropriate numerical representation. We also ask for their opinion about the treatment in class of the relevance of a representation of extreme numbers by its position. In addition, we proposed representations that require considering the relative position of given numbers, according to the available information, both in correct and incorrect cases. The reason for this being that these tasks would show us the conceptions of the teachers and the recurring schemes that their students learn in math classes.

For the analysis of the data, we have considered the choice of certain graphs as a disposition towards management. At the same time, we observe verbal indicators in the use of the terms: scale, direction and measure that are adequate to manage the representation institutionally leaving aside the symbolic interpretation.

Results and Discussion

In the first task, teachers were presented with a number line and asked to place zero on it, 62.5% placed it in the center, also 11% chose on the left most mark because they were considering only positive numbers, which could reflect their usual practice.

In the second task we asked: *"Of the number lines shown below, which one would you choose to propose to your students if you want to place the numbers 120 and – 3 on it? Explain"*

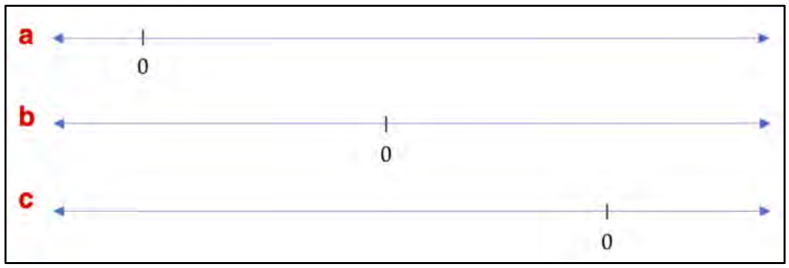


Figure 1: Task 2. Choosing the right line.

In the first part of the question, most of the teachers (59.7%) chose **a**. In the second part, the justifications were: 1) Because there is more space (40%); 2) Scales can be made to locate the points (2%); 3) A larger amount is required for positive numbers than for negative ones (6%) and they did not answer (32%).

Of the 59.7% who chose option **a.**, 58.1%. that is, 34.7% of the total, justified that there was more "space" or "distance" between the numbers, which makes us see the use and type of favorite number line that is intended to be taught in the class, as well as the type of examples that will be worked mostly are positive.

The third task asked: "*How would you explain to your students that the following number line can be used to place the numbers -200 and 5 on it?*"

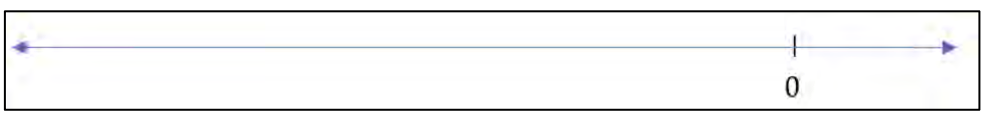


Figure 2: Task 3. Explanation of the type of line used.

Teachers who included the terms: scale, space and direction are 45.9%, who, apparently, considered change in representation, showing a form of management over representation that consists of using segments of 20 in 20 units, for example.

With the purpose to deepening the conceptions and justifications of teachers when faced with this type of task, below, we show one of their answers to this case.

> P2: I would explain to they that -200 is farther from zero, therefore a larger scale is needed to represent it, on the contrary, 5 is closer to zero, so you don't need much space.

This justification shows us that he was thinking of possibly using a scale on one side of the number line and having less space on the other side.

Other justifications for the choice of this type of line and that are associated with the scale, are the following:

P7: We can make divisions of 5 cm that represent 40 units with which to the left of the zero we would have 5 divisions and to the right side that we should divide it into 8 equal parts, where in the first division we would represent the 5.

P60: This line can be used to locate -200 because it has enough space from the zero point and can be located from 5 to 5, 10 to 10, or 20 to 20 for the negative side, and for the positive side it takes up less space because there are only 5 positives (sic).

Understanding both cases, the scale as a form to adopt the numerability in which it is intended to avoid placing many marks and their respective labels.

On the other hand, 45.8% did not answer, which possibly means that they had not had contact with this situation.

Regarding the order, spatial location, and relative position of numbers on the line, we request in the fourth task: to place the missing numbers in the marks that appear on the next number line.

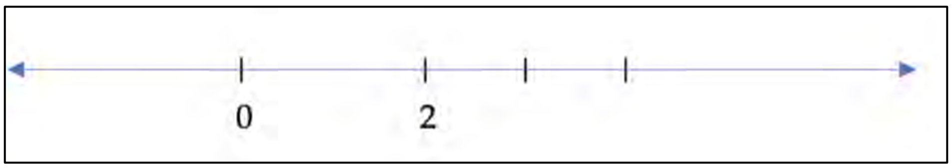


Figure 3: Task 4. Location of missing numbers.

Most teachers properly used the unit segments and correctly located the missing numbers, which would be showing that they went through the process of identifying the magnitude of the unit, only the answers of four teachers showed that they have not considered it, taking the distance from zero to 2 like the unit, hence the error. We also observed that in the treatment of the magnitude in these last answers the Gestalt relationship between background and form was present in the representation and it was not well resolved.

Insisting on the order and interpretation of the space between the numbers, we proposed the next line warning that it is wrong, then we asked to reorder it, as well as the drawing of a suitable number line.

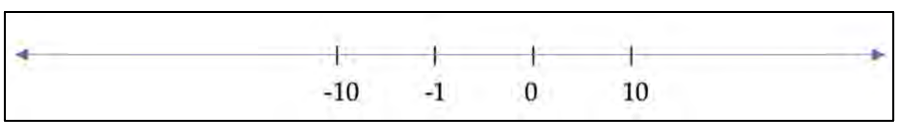


Figure 4: Task 5. Interpretation of the space between numbers.

The results tell us that 40.7% consider the unit segments, 51.4% did not answer the question, while we have 7% of teachers who either do not take distances between numbers or consider that the representation is not incorrect to no consider the relative position of the numbers.

Regarding the use of signs to place numerical values on the number line we asked to place the numbers -5, -3, -1, 4 on the next number line

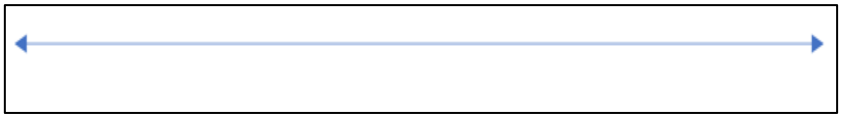


Figure 5: Task 6. Location of numbers.

In this task, the teachers (33%) used small vertical marks or points to place the requested numbers, as well as extra marks that served as a guide for the position taking care that the distance between it was relatively the same, although they were not accompanied by the respective numbers having an approximation that we can call *local position*.

There are also a few (26%) who only place the marks on the requested numbers, visually estimating their position, which suggests that they use a *global position* in this task.

In task 7, that requested to reorder an incorrect representation, from all the teachers who answered, 50% ordered the numbers properly, which shows that the order between them was not a conflict. However, in task 8 the zero was also missing, so the number line had to be rebuilt assigning the necessary space, 37.5% of the teachers resolved correctly, however, 8.3% simply added the mark and number in the middle of 1 and -1 without considering the location and the unit space.

Conclusions

In the development of the tasks, it is important to note that only half of the teachers answered consistently the tasks of the questionnaire shown in GeoGebra, possibly for two reasons: the first may be the need of the management of virtual tools, and the second because they had never faced similar situations in which they had to manage their own representation by their own.

We detected in the results a certain tendency to use some favorite representations such as placing the 0 in the center or the inclination to structures linking to the positives numbers that suggest a dominant presence of their needs to teach in the classroom. These conceptions show us a strong and utilitarian use of the model. In addition to that, when teachers agree to modify the representation, their suggestions are linked to the idea and control by mean of scale that they can rule, that apparently solves the problems of representation and interpretation in their teaching work.

The order is not a problem for them or even adding the missing numbers and the spatial location is respected in cases of need, however, it is not identified as an obligatory condition or as a symbolic need of the number line, but as a convention that must be respected, in certain cases.

The teachers' conceptions of the requirements of symbolic representation are that it is ordered and that the position has a homogeneous basis, but that it can be modified according to the scale used. This suggests that there are modifications to the representation for utilitarian purposes, but the symbolic properties that make it function as a mathematical model are not detected.

References

Doritou, M., & Gray, E. (2009). Teachers' subject knowledge: the number line representation. Paper presented at Sixth Conference of the European society for Research in Mathematics Education (CERME 6). Lyon: France.

Heirdsfield, A. (2011) Teaching mental computation strategies in early mathematics. YC Young Children, 66(2), pp. 96-102Doritou, M., & Gray, E. (2007). The number line as metaphor of the number system: A case study of a primary school. In D. Pitta-Pantazi, & J. Philippou (Eds.), Proceedings CERME 5 (pp. 111-120).

Hill, H. C., Ball, D. L. y Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. Journal for Research in Mathematics Education, 39, 372-400.

Membreño, S., & Acuña, C. (2021) The number line model as a homogeneous representation space. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. (pp. 297 – 301)*

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education (Vol. 1, pp. 103-135). Honolulu, Hawai‘I: PME.

Peirce, C. (2005). El ícono, el índice y el símbolo. Traducción castellana de Sara Barrena. Fuente textual en CP 2.274-308.

Radford, L. (2004). Semiótica cultural y cognición. Conferencia plenaria Décima octava Reunión latinoamericana de Matemática Educativa. Universidad Autónoma de Chiapas.

Radford, L. (2006b). Introducción Semiótica y Educación Matemática. Revista Latinoamericana de Investigación En Matemática Educativa. Número especial, 7–21.

CONCEPCIONES DE PROFESORES HONDUREÑOS DE SECUNDARIA SOBRE LA RECTA NUMÉRICA EN TAREAS DE GESTIÓN SIMBÓLICA

Sharon Samantha Membreño Estrada
Cinvestav - IPN
sharon.membreno@cinvestav.mx

Claudia Margarita Acuña Soto
Cinvestav - IPN
claudiamargarita_as@hotmail.com

La recta numérica es un modelo que en la enseñanza es usada para medir, contar, ordenar e incluso operar, lo que requiere de una interpretación simbólica, luego indagamos las concepciones de 72 profesores cuando gestionan el modelo de la recta numérica asociada al orden, la ubicación espacial y la posición relativa entre números y marcas. En un taller por internet, encontramos que, los participantes consideran que la utilidad del modelo está en la manipulación y no ven la necesidad de modificarlo, pese a los requisitos del problema, dando por sentado su utilidad y el control que les puede dar recursos como el cambio de la escala numérica. Tampoco se percatan de las propiedades simbólicas para la interpretación de la numerabilidad y la operatividad.

Palabras clave: recta numérica, concepciones de profesores, gestión de artefactos

La recta numérica funciona como un modelo para la enseñanza, ya que sirve para medir, contar, ordenar u operar. Sus diversas presentaciones permiten incluso usarla como una especie de apunte (Heirdsfield, 2011), donde, en este último caso no hay necesidad de unidades homogéneas entre números consecutivos ni de las propiedades de una representación homogénea (Nemirovsky, 2003).

Consideramos que el modelo de la recta numérica es más que una metáfora (Doritou & Gray, 2007) porque nos permite abordar las propiedades numéricas mediante una representación simbólica y su uso debe considerar propiedades como la de la representación homogénea que da sentido espacial a los números de acuerdo con su posición, así como a las marcas y a los intervalos entre ellos, estas actividades simbólicas que son la base para la numerabilidad.

Por ello, es deseable que el profesor tenga en cuenta que estos signos están regulados de manera que le dan sentido matemático, por lo que indagamos las concepciones de profesores hondureños de secundaria en activo, sobre su disposición para hacer cambios a la representación según las necesidades, así como al tratamiento simbólico considerando el orden, la ubicación espacial y la posición relativa de números y marcas sobre la recta numérica para desarrollar un tratamiento simbólico en la clase.

Antecedentes

Actualmente, se ha establecido la importancia de que los profesores adquieran tres tipos de conocimientos para desarrollar su docencia con efectividad, lo que requiere de: 1) Conocimiento común del contenido; 2) Conocimiento especializado del contenido; y 3) Conocimiento del contenido y de los estudiantes, tal como lo afirma Hill, Ball, et al. (2008).

En esta investigación nos referiremos al conocimiento especializado del contenido, asociado a la interpretación simbólica, al uso y la gestión de la recta numérica en tareas de numerabilidad donde se destacan las propiedades de la homogeneidad y el orden, esto es, el uso de un modelo donde la posición de los signos es relevante tanto por su valor, como por sus posiciones relativas asociadas al orden y a la desigualdad para desempeñar funciones de modelo para la numerabilidad.

La llamada recta numérica es susceptible de dar cuerpo a distintas operaciones aritméticas en los niveles básicos, en particular, la podemos encontrar en el trabajo que se refiere a las relaciones de desigualdades numéricas y de las inecuaciones lineales (Membreño y Acuña, 2021), lo cual requiere de interpretar el orden, la posición relativa y la ubicación espacial de números y marcas desde índices a símbolos de una manera intencional.

En general, los profesores usan la recta numérica como una herramienta pedagógica que sirve como ayuda para el estudiante, ya que da cuerpo a ciertas relaciones, sin embargo, no siempre se percatan de que también es una estructura abstracta, que es un modelo simbólico con algunas restricciones (Doritou & Gray, 2010), que acepta un tratamiento como representación del sistema numérico y que además da cuerpo a la metáfora de la numeración (Doritou & Gray, 2007).

Desde nuestro punto de vista, si bien es cierto que la recta numérica da cuerpo a la metáfora de la numerabilidad, ella misma está formada por una serie de signos con propiedades matemáticas que son de dos tipos: explícitas e implícitas, las que deben ser consideradas para su enseñanza.

En el caso los signos explícitos tenemos aquellos que claramente serán detectados y utilizados para la interpretación como en el caso de los números y las marcas asociadas a ellos, los números tienen valor numérico por lo que adquieren una posición ordinal, pero también son las etiquetas numéricas para las marcas asociadas, lo que permite el orden de los objetos representados. Sin embargo, hay otros signos implícitos como los segmentos que, aun siendo relevantes para la interpretación, pasan frecuentemente desapercibidos, por ejemplo, cuando representan la norma de la unidad homogénea de la numeración o en el caso de representar los conjuntos solución de desigualdades numéricas e inecuaciones lineales, (Membreño y Acuña, 2021).

Objetivo

Pretendemos investigar las concepciones y las justificaciones del profesor de secundaria cuando está en condiciones de gestionar el tipo de representación del modelo de la recta numérica que se usaría bajo condiciones cambiantes, lo que implica aceptar una gestión sobre la representación, así como el uso de los signos involucrados en el modelo ya sean marcas o números con base en el orden, la ubicación espacial y la posición relativa de esta representación cuando es homogénea.

Marco teórico

Uso de modelos como herramientas simbólicas

La recta numérica es una representación simbólica, formada por signos que deben ser interpretados matemáticamente para usarlo como un modelo para la enseñanza, lo que incluye detectar el papel de los signos implícitos y explícitos como mediadores simbólicos.

La Teoría de la Objetivación sugiere que, el aprendizaje está ligado al uso de instrumentos, signos o artefactos que no son ayudas para el aprendizaje, sino medios semióticos cuya presencia y uso imprimen un sello distintivo sobre lo construido, llega a ser parte de éste a través de una mediación semiótica que permite desarrollar una praxis reflexiva, la cual se entiende como: “(lo) que conocemos y la manera en que llegamos a conocerlo esta circunscrito por posiciones ontológicas y procesos de producción de significados que dan forma a cierta clase de racionalidad que permite plantear ciertos problemas" (Radford, 2004, p. 13)

Si la matemática es esencialmente simbólica, es importante observar el papel de los signos que permiten al individuo mediar el conocimiento y es a través de ellos que se orientan las acciones y reflexiones de este, por ello es conveniente conocer las concepciones de los profesores que estarían orientando su uso sobre los signos presentes en la representación de la recta numérica.

En particular, Radford afirma que "El signo tiene significado cuando está relacionado con otros signos" (Radford 2006b, p. 8), por lo que se le daría significado a la representación cuando se están relacionando los signos explícitos con los implícitos sobre el modelo de la recta numérica.

Respecto al tratamiento de los signos, nos interesa detectar el uso que se hace de ellos ya sea como índices o símbolos, Pierce (2005) lo que será relevante para un adecuado tratamiento simbólico, a lo largo de su actividad docente, en el tratamiento de la recta numérica y en el uso de las propiedades de orden y homogeneidad.

Para los fines de esta investigación sobre el uso de los signos hemos incorporado la restricción de la homogeneidad, Nemirovsky (2003) de la recta numérica en la que la única propiedad de los objetos representados es su posición con el fin de para indagar la inclinación de los profesores para modificar o no, las condiciones espaciales de la recta numérica bajo condiciones cambiantes, así como de su interpretación del orden y la ubicación espacial de los números y de sus marcas sobre los signos presentes en el uso del modelo.

En la presente investigación, la gestión simbólica de la recta está presente cuando los profesores ajustan la representación de la recta numérica, dependiendo de las necesidades de los problemas planteados y el tratamiento semiótico se caracteriza por reconocer las normas matemáticas de orden, ubicación espacial y posición relativa de los números y marcas representadas.

Metodología

Sobre la muestra

La presente investigación exploró las concepciones de 72 profesores de secundaria de Honduras, que atienden estudiantes de entre 12 y 15 años, investigación que se desarrolló mediante un taller en línea como parte del Congreso de Matemática Educativa (COME) en Honduras en el año 2021, dirigido a profesores de todo el país, la convocatoria abierta fue atendida libremente por los interesados para la mejora de la enseñanza de las matemáticas en dicho país.

Método

El taller se desarrolló vía Zoom apoyado en el uso de la plataforma GeoGebra classroom, a través de un cuestionario donde los profesores podían manipular representaciones gráficas e interactuar con la investigadora ya sea frente a todos los participantes, de manera particular e instantánea o incluso dejar comentarios por escrito para tener comunicación posterior, esto fue útil para analizar más ampliamente sus opiniones y concepciones de su gestión sobre el modelo de la recta numérica.

El cuestionario estaba dividido en ocho tareas asociadas a la gestión de la recta numérica con el fin de conocer las concepciones de los profesores frente a tareas que requieran de gestionar la recta numérica, específicamente sobre el orden y a la homogeneidad.

De manera general, se presentaron gráficas de rectas numéricas en las que se preguntaba por la posición de marcas relevantes como el cero, así como la elección de rectas en donde el cero no está en el centro y puede ser una más adecuada representación numérica. Solicitamos también su opinión sobre el tratamiento en clase de la pertinencia de una representación de números extremos por su posición. Además, les propusimos representaciones que requieren considerar la posición relativa de números dados, según la información disponible, tanto en casos correctos como incorrectos. Debido a que estas tareas nos mostrarían las concepciones de los profesores y los esquemas que aprenden sus estudiantes en las clases de matemáticas.

Para el análisis de los datos, hemos considerado la elección de ciertas gráficas como una disposición hacia la gestión, al mismo tiempo observamos indicadores verbales en el uso de los términos: escala, dirección y medida que son adecuados para gestionar la representación institucionalizada, dejando de lado la interpretación simbólica.

Resultados y Discusión

En la primera tarea, se les presentó a los profesores una recta numérica y se les pidió ubicar al cero en ella, 62.5% lo colocan en el centro, también el 11% eligió en el extremo izquierdo posiblemente porque estaban considerando los números positivos, lo que podría ser reflejo de su práctica usual.

En la segunda tarea pedimos: *"De las rectas numéricas que se muestran abajo ¿Cuál elegiría para proponérsela a sus estudiantes si desea colocar sobre ella los números 120 y – 3? Explique"*

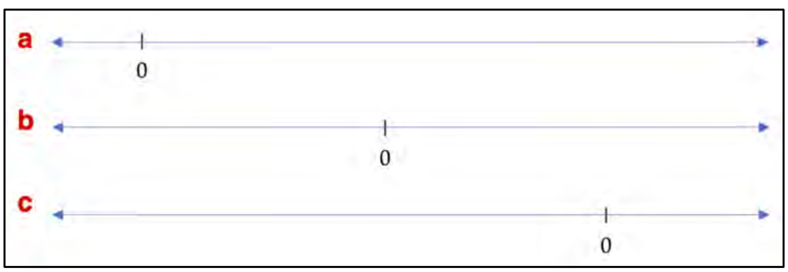


Figura 1: Tarea 2. Elección de la recta adecuada.

En la primera parte de la pregunta la mayoría de los profesores (59.7%) elije **a**. En la segunda, algunas de las justificaciones son: 1) Porque hay mayor espacio (40%); 2) Se puede hacer escalas para ubicar los puntos (2%); 3) Se requiere una cantidad mayor para los números positivos que para los negativos (6) y no contestaron (32%).

Del 59.7% que eligieron la opción **a.**, el 58.1%. es decir, el 34.7% del total justificaban que había más "espacio" o "distancia" entre los números, lo que nos hace ver que el uso y el tipo de recta favorito están pensadas para impartir la clase, así como el tipo de ejemplos que serán mayormente positivos.

En la tercera tarea se solicita: "*¿Cómo les explicaría a sus alumnos que se puede utilizar la siguiente recta numérica para colocar sobre ella los números -200 y 5?"*

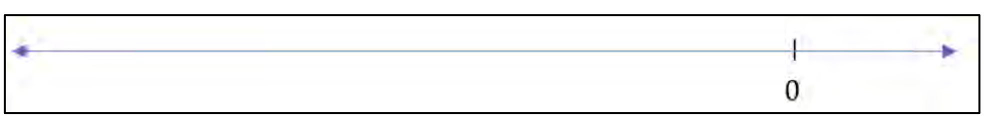


Figura 2: Tarea 3. Explicación del tipo de recta usada.

Los profesores que incluyen los términos: escala, espacio y dirección son el 45.9%, quienes, al parecer, consideran algún tipo de cambió en la representación, mostrando una forma de gestión sobre la representación que consiste en usar segmentos de 20 en 20 unidades, por ejemplo.

Con el fin de profundizar en las concepciones y justificaciones de los profesores cuando se enfrentan a este tipo de tareas, a continuación, mostramos una de sus respuestas para el caso.

P2: Les explicaría que el -200 está más alejado del cero, por lo tanto, se necesita una escala más grande para representarlo, por el contrario, el 5 está más cerca del cero, así que no es necesario mucho espacio.

Justificación que nos muestra que posiblemente está pensando en utilizar una escala en un lado de la recta numérica y en el hecho de contar con menos espacio al otro lado.

Otras justificaciones para la elección de este tipo de recta y que están asociadas con la escala, son las siguientes:

P7: Podemos hacer divisiones de 5 cm que representen 40 unidades con lo cual a la izquierda del cero tendríamos 5 divisiones y a la derecha una sola que deberíamos dividirla en 8 partes iguales donde en la primera división representaríamos el 5.

P60: Esta recta se puede utilizar para ubicar -200 porque tiene el espacio suficiente desde el punto cero y se puede ir ubicando de 5 en 5, de 10 en 10, o de 20 en 20 para los lados negativos, y para los lados positivos ocupa menos espacio porque solo son 5 positivos.

Entendiendo en ambos casos, la escala como una forma de numerabilidad en la que se pretende evitar colocar muchas marcas y su respectiva etiqueta.

Por otro lado, el 45.8% no contestó, lo que posiblemente significa que no habían tenido contacto con esta situación.

Respecto al orden, ubicación espacial y la posición relativa de números sobre la recta, solicitamos en la cuarta tarea: colocar los números faltantes en las marcas que aparecen en la siguiente recta numérica.

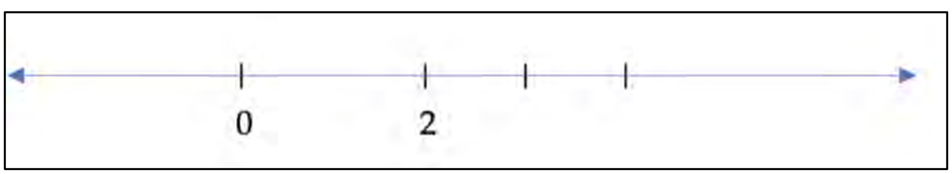


Figura 3: Tarea 4. Ubicación de números faltantes.

La mayoría de los profesores utilizan adecuadamente los segmentos unitarios y ubican correctamente los números faltantes, lo que estaría mostrando que pasan por el proceso de identificación de la magnitud de la unidad, sólo las respuestas de cuatro profesores muestran no haberla considerado, tomando la distancia del cero al 2 la unidad de ahí el error. Observamos también que en el tratamiento de la magnitud en estas últimas respuestas no se resuelve bien la relación Gestalt entre fondo y forma presentes en la representación.

Insistiendo sobre el orden y la interpretación del espacio entre los números, propusimos la siguiente recta advirtiendo que está incorrecta, pedimos reordenarla, así como el dibujo de una recta adecuada.

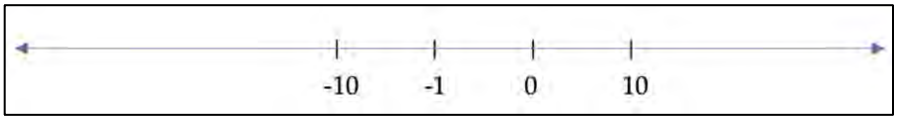


Figura 4: Tarea 5. Interpretación del espacio entre los números.

Los resultados nos dicen que el 40.7% respeta los segmentos unitarios, que el 51,4% no contestan la pregunta, mientras que tenemos 7% de profesores que o bien no respetan distancias entre números o consideran que no la representación no es incorrecta dejando de considerar la posición relativa de los números.

En lo relativo al uso de signos para colocar valores numéricos sobre la recta numérica pedimos colocar los números -5, -3, -1, 4 en la siguiente recta numérica

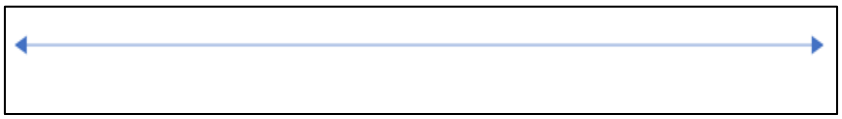


Figura 5: Tarea 6. Ubicación de números.

En esta tarea, los profesores (33%) usaron pequeñas marcas verticales o puntos para colocar los números solicitados, así como marcas extras que le sirven de guía para la posición cuidando que las distancia entre ella sea relativamente la misma, aunque no estén posición local.

También hay unos pocos (26%) que sólo colocan las marcas sobre los números solicitados, estimando la visualmente su posición, lo que nos sugiere que usan una estimación global en esta tarea.

En la tarea 7 que solicitaba reordenar una representación incorrecta todos los profesores que contestaron (50%) ordenaron adecuadamente lo que muestra que el orden entre ellos no es un conflicto, sin embargo, en la tarea 8 faltaba además el cero, por ello se debía rehacer la recta asignando el espacio necesario, el 37.5% de los profesores resolvió correctamente, sin embargo, el 8.3% simplemente agregó la marca y el número en medio del 1 y el -1 sin considerar la ubicación y el espacio unitario.

Conclusiones

En el desarrollo de las tareas es importante notar que solamente la mitad de la cantidad total de profesores respondió consistentemente las tareas del cuestionario mostrado en GeoGebra, posiblemente por dos razones: la primera se puede deber al manejo de las herramientas virtuales y la segunda, porque nunca se habían enfrentado a situaciones similares en las que tenían que gestionar la propia representación por su cuenta.

Detectamos en los resultados cierta tendencia del uso de algunas representaciones favoritas como la de colocar el 0 en el centro o la inclinación por estructuras ligadas a los positivos que sugieren una presencia dominante de sus necesidades en el salón de clase, estas concepciones muestran un uso fuerte y utilitario del modelo. Además, cuando los profesores aceptan modificar la representación ésta está ligada a la idea y al control que les da la escala que pueden dominar, lo que al parecer resuelve los problemas de representación e interpretación en su labor docente.

El orden no es un problema para ellos o incluso agregar los números faltantes y la ubicación espacial es respetada en los casos de necesidad, sin embargo, no se identifica como una condición obligada ni como una necesidad simbólica de la recta numérica, sino como una convención que debe ser respetada, en ciertos casos.

Las concepciones de los profesores respecto a los requisitos de la representación simbólica son que esta es ordenada y que la posición tiene una base homogénea, pero que puede ser modificada según la escala que se use. Lo anterior sugiere que hay modificaciones a la representación con fines utilitarios, pero no se detectan las propiedades simbólicas que lo hacen funcionar como un modelo matemático.

Referencias

Doritou, M., & Gray, E. (2009). Teachers' subject knowledge: the number line representation. Paper presented at Sixth Conference of the European society for Research in Mathematics Education (CERME 6). Lyon: France.

Heirdsfield, A. (2011) Teaching mental computation strategies in early mathematics. YC Young Children, 66(2), pp. 96-102Doritou, M., & Gray, E. (2007). The number line as metaphor of the number system: A case study of a primary school. In D. Pitta-Pantazi, & J. Philippou (Eds.), Proceedings CERME 5 (pp. 111-120).

Hill, H. C., Ball, D. L. y Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. Journal for Research in Mathematics Education, 39, 372-400.

Membreño, S., & Acuña, C. (2021) The number line model as a homogeneous representation space. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.). Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. (pp. 297 – 301)

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education (Vol. 1, pp. 103-135). Honolulu, Hawai'I: PME.

Peirce, C. (2005). El ícono, el índice y el símbolo. Traducción castellana de Sara Barrena. Fuente textual en CP 2.274-308.

Radford, L. (2004). Semiótica cultural y cognición. Conferencia plenaria Décima octava Reunión latinoamericana de Matemática Educativa. Universidad Autónoma de Chiapas.

Radford, L. (2006b). Introducción Semiótica y Educación Matemática. Revista Latinoamericana de Investigación En Matemática Educativa. Número especial, 7–21.

EXPANDING PROFESSIONAL NOTICING TO EXAMINE TEACHERS' DECISION-MAKING DURING INTERVENTION IN SMALL GROUPS

Byungeun Pak
Utah Tech University
byungeun.pak@dixie.edu

This study explores what affects teachers' decision-making when they intervene in small groups. I draw specifically on professional noticing to examine teachers' decision-making. Working with two novice teachers, I observed their mathematics lessons in which both teachers used small groups as an instructional structure to teach mathematical concepts and conducted follow-up interviews to have them elaborate on their intervention. The analysis of both teachers' elaboration on their specific intervention indicates that two teachers decided to intervene in small groups based on interpretations of certain actions as patterns, meaning what teachers have constructed working with students over weeks or months, not only what happened at the moment. This finding suggests that conceptualization of profession noticing needs to be expanded. Implications for research and teacher educators are offered.

Keywords: Elementary School Education; Teacher Education; Teacher Noticing; Group Work

Leslie, a fourth-grade teacher, has students work together with each other in small groups to solve a mathematical task. The teacher notices one student with head down and two students talking quietly to each other in a group of three students. The teacher approaches the group and decides to intervene in the small group by encouraging the student who has his head down to participate actively in the group work and the two students to include the other student in their group work. A question arises in terms of the teacher's decision-making: What affects the teacher's decision-making of how to intervene? This teacher's decision might be based on what she noticed in the immediate moment and on the understanding the teacher has constructed while working with students over weeks or months. In this paper, I try to answer this question by examining how two beginning teachers explained their decision-making related to how to intervene in small groups in mathematics instruction. I draw particularly on the framework of professional noticing (Jacobs, Lamb, & Philips, 2010) to investigate the two beginning teachers' decision-making.

Perspectives

There are different ways for researchers to conceptualize teacher noticing (See Jacobs and Spangler (2018) and Sherin, Russ, and Colestock (2011) for details). I draw specifically on professional noticing (Jacobs et al., 2010) in this paper because it helps us understand what informs teachers' decision-making. In general, professional noticing consists of three interrelated cognitive components- attention (what teachers attend to in classroom activity), interpretation (how they make sense of what they attend to) and decisions of how to respond (how they plan to respond). Even though these three components "are tied together conceptually and temporally" (Sherin et al., 2011, p. 81), I view interpretation as an important component that can inform what affects teachers' decision-making during teachers' intervention in small groups.

The current dominant assumption of professional noticing as a practice is that it can inform "teachers' in-the-moment decision-making" (Jacobs & Spangler, 2018, p. 772). Viewing professional noticing as part of in-the-moment decision-making indicates that teachers make decisions about how to respond to what happens because they interpret what happens as something to respond to immediately and urgently. For example, Leslie in the scenario might have attended to the head-down student, interpreted his action as being off-task, and decided to encourage them to work together because she felt an immediacy of correcting his off-task action. Alternatively, she could attend to the head-down student, interpreted his action as a sign of boredom, and decided to have the two students include the head-down student.

There has been an argument in which the notion of professional noticing should be expanded. Louie (2018) pointed to "the limitations of conceptualizing teacher noticing in terms of individual teachers' skill at internal mental processes" (p. 68). She argued that the current conceptualization does not explain that noticing is socially and culturally constructed. It needs to be expanded to recognize impacts of socially and culturally constructed ideas, such as deficit views, related to students and their behavior on what teachers notice, which affects teacher's decision-making. Her argument suggests that teachers' decision-making might be not simply based on what they interpret as immediate and urgent but also certain ideas constructed in the long term. Building on this idea, I argue for considering *patterns*, meaning what teachers have constructed working with students over weeks or months, to be a part of what affects their decision-making.

Teachers might make decisions of how to respond based on *patterns* they have observed of students in terms of actions and what happened as a result of the actions, not simply their in-the-moment noticings. *Patterns* teachers have observed over weeks or months shape teachers' perceptions. When teachers attend to certain actions, they may see them as *patterns* and interpret such actions based on *patterns*. As such, interpreting certain behaviors as *patterns* is not likely to be the same as noticing something immediate and urgent in that it is shaped by teachers' perceptions of students built up over weeks or months.

In terms of intervention in small groups, noticing certain *patterns* might affect teachers' decisions of how to intervene in small groups. Leslie in the scenario earlier might interpret the student's action (of his head down) as typical behavior based on what she had observed repeatedly over months and decide to respond to such action. Literature on intervention in small groups knows little of what affects teachers' decision-making drawing on teacher noticing and professional noticing.

In this paper, I argue that teachers might make decisions of how to respond to students and their behaviors in the context of small groups not only based on the current dominant conceptualization of noticing (what happens at the moment) but also on the expanded conceptualization (*patterns*). As such, the purpose of this paper is to understand what affects teachers' decision-making when they intervene in small groups in mathematics instruction drawing on both the current and expanded conceptualizations of teacher noticing.

Methods

To collect data, I worked with two beginning teachers, Leslie and Marva (pseudonyms). They were novice teachers in the sense that they were in their second or third year in teaching careers in 2018-2019. Leslie taught fourth grade students in a city area. Marva taught first grade students in an urban area. I recruited them because they reported they implemented small groups regularly as an instructional structure to teach mathematical concepts and decided to participate in the study. To collect the data, I conducted three stimulated recall interviews per teacher. Before each stimulated recall interview, I first video-recorded the entire length of a mathematics lesson. As a result, I obtained video recordings of six different mathematics lessons, three of which came from each teacher. Watching each video recording on my own, I identified four or five short moments of intervention in each video where I could see teachers intervene in and interact with students in a productive way (e.g., encouraging students to participate actively in group work or asking students to explain their strategies). During each stimulated recall interview, I had the teacher watch each intervention I had identified prior to the interview. After watching each intervention, I asked the teachers questions building on three components of professional noticing to elaborate on what they noticed, what they thought was happening in the small group, and why they intervened in the small group the way they did. The interview data were audio-recorded and fully transcribed for analysis. The data sources were 24 intervention excerpts, four of which were obtained from each of the six interview transcripts. Each intervention excerpt consisted of two parts- 1) a transcript of intervention and 2) teachers' elaboration on their intervention. There were three steps I took to analyze the data. First, I read through an intervention transcript and the teacher's elaboration to make sense of attention, interpretation, and decisions of how to intervene in small groups. Second, I examined how the teachers explained their intervention in each intervention excerpt in terms of interpretation of what happened in small groups, whether it was based on immediacy or *patterns* over time. Third, by comparing each interpretation, I identified certain regularities related to interpretation across intervention excerpts.

Finding(s)

As a result of the data analysis, I found that two teachers decided to intervene in small groups based on interpretations of certain actions as *patterns*, not only what happened at the moment. To illustrate this finding, I present a part of an excerpt of interaction between Leslie and a group of students. This excerpt comes from Leslie's first lesson. In this lesson, the students were asked to work together on choosing ingredients to make the meal from the list on the worksheet. They had to figure out the cost of dinner by adding the cost of ingredients that they chose to make the meal. This intervention happened when, as she was monitoring the classroom, Leslie looked at the numbers on the worksheet that a group of four students (Alice, Jacob, Fred and Bob) had worked on. The teacher began to initiate conversation with the small group by asking questions.

1. Leslie: (Points to some numbers on the worksheet expecting to hear from anyone in the small group) So what's going on over here? What's this?
2. Fred: This is the number of- (interrupted by Leslie who asked another student questions).
3. Leslie: Alice, what's this? What are all the numbers over here?
4. Alice: This is, uh, addition problem ... for this.
5. Leslie: What's two plus six? What's that?
6. Alice: Eight (answers quietly.)
7. Leslie: But where did you get those numbers from? I don't see two and six.
8. Alice: (Points numbers the group chose on the worksheet) From the dollars... [inaudible] … two numbers.
9. Leslie: Okay. So you're taking just the dollars and adding them up?
10. Alice: Yeah.
11. Leslie: Okay. Jacob, do you think that's the best way of doing this?

This excerpt showed Leslie's decision-making based on what she thought happened at the moment. According to the teacher's elaboration in relation to this intervention, when she looked at the numbers on the group's worksheet, she was wondering "if they were estimating" or "if they were just taking the number, the dollar values, or what they were going on." Leslie attended to what she saw on the worksheet and interpreted what the students did as something she needed to respond immediately. Based on her sense of its immediacy, she decided to ask the small group questions to make sure they were doing right.

Leslie's elaboration on her intervention in relation to her intervention in this excerpt also showed how the teacher's decision-making was based on *patterns*, not only on what happened at the moment. In Line 2, Fred tried to answer Leslie's question in Line 1. She interrupted his initiation to talk. She chose to call on Alice first to ask questions. The teacher was asking Alice to explain the group's strategy to solve an addition problem they created. From Line 3 to Line 10, she intervened in the small group by asking students to explain their mathematical thinking. After listening to Alice's explanation, she shifted her attention from Alice to Jacob to ask him to evaluate Alice's explanation (Line 11). Leslie's elaboration on this specific intervention shows that she interpreted Fred's initial response (Line 2) as *patterns* she has known of what might happen when she allows Fred to talk. She interpreted their actions as *patterns* she has constructed of how Fred, Alice, and Jacob might do in group work without her intervention. Fred "could have just taken over and been writing the answers" and Alice and Jacob "could have been sitting there the whole time." Otherwise, it would have been possible for her to not interrupt Fred so that he could continue to provide his explanation. Instead, however, she decided to call on Alice first to ask questions. Like shown in the example above, both teachers' decision-making during intervention was affected by *patterns*, not only what they interpreted as something immediate and urgent.

Discussions and Implications

I demonstrated how *patterns* (what teachers have constructed working with students over weeks or months) could affect teachers' intervention in small groups. There are two points to discuss. First, this paper contributes to expanding the current dominant conceptualization of professional noticing based in-the-moment noticings. Like shown in Louie's (2018) argument, the notion of professional noticing should be expanded, in this paper, to include patterns as part of what affects teachers' decision-making during intervention. The expanded conceptualization of professional noticing could be applied to teachers' decision-making in other teaching contexts, such as whole class or independent work. Second, this paper provides a more detailed understanding of what affects teachers' decision-making during intervention in small groups. Literature on intervention in small groups has not attended to teachers' decision-making using teacher noticing and professional noticing. Based on this study, researchers with an interest in intervention in small groups may examine how *patterns* affect teachers' intervention in small groups in a productive way. A few questions to explore include: what types of patterns do teachers notice? How would patterns be related to socially and culturally constructed ideas (e.g., deficit views towards minority students)? In terms of teacher education, teacher educators may need to support novice teachers to be aware of what affects their decision-making during intervention in small groups in mathematics instruction. Teacher educators may design an instructional activity in which novice teachers analyze their elaboration on their interaction with students in small groups to examine how *patterns* affect their decision-making.

References

Jacobs, V. R., & Spangler, D. A. (2018). Research on core practices in K–12 mathematics teaching. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 766–792). National Council of Teachers of Mathematics.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for research in mathematics education, 169-202.

Louie, N. L. (2018). Culture and ideology in mathematics teacher noticing. *Educational Studies in Mathematics*, 97(1), 55-69.

Sherin, M. G., Russ, R. S., & Colestock, A. A. (2011). Assessing mathematics teachers' in-the-moment assessing. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), Mathematics teacher noticing: Seeing through teachers' eyes (pp. 80–94). New York: Routledge.

CONDUCTING A WHOLE CLASS DISCUSSION ABOUT AN INSTANCE OF STUDENT MATHEMATICAL THINKING

Shari L. Stockero
Michigan Technological University
stockero@mtu.edu

Blake E. Peterson
Brigham Young University
peterson@mathed.byu.edu

Keith R. Leatham
Brigham Young University
kleatham@mathed.byu.edu

Laura R. Van Zoest
Western Michigan University
laura.vanzoest@wmich.edu

Productive use of student mathematical thinking is a critical aspect of effective teaching that is not yet fully understood. We have previously conceptualized the teaching practice of building on student mathematical thinking and the four elements that comprise it. In this paper we begin to unpack this complex practice by looking closely at its third element, Conduct. Based on an analysis of secondary mathematics teachers' enactments of building, we describe the critical aspects of conducting a whole-class discussion that is focused on making sense of a high-leverage student contribution.

Keywords: Classroom Discourse, Communication, Instructional Activities and Practices.

The mathematics education community has advocated using students' mathematical thinking as a critical component of whole-class instruction (e.g., Association of Mathematics Teacher Educators, 2017; National Council of Teachers of Mathematics, 2014). Elsewhere, we have argued that some instances of in-the-moment student thinking are particularly high-leverage and, if taken advantage of productively, can be used to engage students in a whole-class discussion focused on making sense of the important mathematical ideas that underlie the student thinking (Leatham et al., 2015). Taking advantage of such instances, however, requires that a teacher coordinate a collection of teaching actions into a coherent practice. Furthermore, certain aspects of this practice do not occur naturally in whole-class instruction (Stockero et al., 2020). Thus, taking advantage of such instances is a complex practice.

Grossman and colleagues (2009) have described the decomposition of teaching practices into their "constituent parts" (p. 2069) for the purpose of better understanding and improving teachers' ability to engage in complex practices. We have previously conceptualized the teaching practice of *building on student mathematical thinking* and decomposed it into four elements that comprise the practice: "(1) *Establish* the student mathematics of the MOST so that the object to be discussed is clear; (2) *Grapple Toss* that object in a way that positions the class to make sense of it; (3) *Conduct* a whole-class discussion that supports the students in making sense of the student mathematics of the MOST; and (4) *Make Explicit* the important mathematical idea from the discussion" (Leatham et al., 2021, p. 1393). In this paper we further decompose the practice of building on student thinking by looking closely at its third element, Conduct. Specifically, we examine this research question: What are critical aspects of the Conduct element of building as revealed through teachers' attempts to enact the practice?

Literature Review

Conducting a whole-class discussion that engages students in making sense of an instance of student mathematical thinking is a complex teaching practice that requires deliberate actions by the teacher. Research on teacher moves during whole-class discussion sheds some light on a range of teacher actions that can support student sense making. Some actions prompt students to make sense of an individual student contribution by, for example, pressing for justification (e.g. Drageset, 2014; Ellis et al., 2019), asking probing questions that encourage other students to engage with specific details of the contribution (e.g., Webb et al., 2019), requesting that students evaluate the correctness of an idea (e.g., Drageset, 2014; Bishop et al., 2016), or asking students to reflect on an idea to advance their mathematical understanding (e.g., Ellis et al., 2019). Other teacher actions support students in making sense of how ideas are related, such as positioning one student contribution relative to another (Webb et al., 2019) and requesting that students make connections among two or more contributions (Lineback, 2015; Bishop et al., 2016). Still other actions keep students focused on the contribution that they are making sense of, such as putting aside contributions that are unrelated to the idea at hand (Dragset, 2014), and redirecting students' attention (Lineback, 2015) to the idea that is the focus of the sense-making discussion.

Although all of these individual moves have potential, in general, to support student sense making, we have come to understand that conducting a whole-class discussion focused on making sense of a particular student contribution requires a coordinated collection of teacher moves. Smith and Stein (2018) articulated one such coordinated collection of moves for orchestrating a whole-class discussion around a high-cognitive demand task. Their 5 Practices have been widely used to support teachers in facilitating mathematical discussions when the teacher has an opportunity to monitor students' work, and then intentionally select and sequence the solutions they wish to incorporate in the follow-up discussion. When a high-leverage instance of student thinking emerges in the moment during a whole-class discussion, however, a different collection of actions is needed to conduct a whole-class discussion around that student contribution. We define this collection of actions as the teaching practice of building on student mathematical thinking (Leatham et al., 2021; Van Zoest et al., 2016).

Theoretical Framework

When we discuss the teaching practice of building, we are focused on building on a high-leverage instance of student mathematical thinking that emerges during whole-class instruction. More specifically, we focus on building on MOSTs (**M**athematically **O**pportunities in **S**tudent **T**hinking), which are described in Leatham et al. (2015). MOSTs are instances of student mathematical thinking that are related to significant mathematics and that create an opportunity in that moment they are shared to engage the class in joint sense making of the mathematics embedded in the instance.

When we say *building on a MOST* we mean the teaching practice that takes advantage of the opportunity that a MOST provides (Van Zoest et al., 2016). We define *building on a MOST* (hereafter referred to as *building*) as making a MOST "the object of consideration by the class in order to engage the class in making sense of that thinking to better understand an important mathematical idea" (Van Zoest et al., 2017, p. 36). As mentioned earlier, *building* is comprised of four elements: (1) *Establish*; (2) *Grapple Toss*; (3) *Conduct*; and (4) *Make Explicit* (Leatham et al., 2021). This paper focuses on the third element, Conduct, but we frame our discussion of this element by first sharing some detail about the first two elements. Two important aspects of Establish are establishing precision by taking steps to assure that the MOST is "clear, complete, and concise" enough for the students to engage in making sense of it (Leatham et al., 2021, p. 1395) and then establishing the precise MOST as an object to be discussed. Once the MOST is established, it is then grapple tossed—offered to the class in a way that positions them to make sense of it. The key aspects of a Grapple Toss are an explicitly established object and an action that students are to apply to that object—actions such as justify or make sense of.

Methods

Twelve practicing secondary school mathematics teachers worked with us to study the theorized practice of building. These teacher researchers used four mini tasks (see Figure 1) in their classrooms that were designed to elicit particular MOSTs; this allowed them to enact the practice of building in a context where there would be a predictable MOST. We compared the 27 resulting videotaped enactments to our theorized conceptualization of building by coding the enactments for teacher actions that seemed to either facilitate or hinder the overall practice of building. Our analysis included identifying critical aspects of each element, as well as subtleties of each. In this paper we report what we learned about the Conduct element of building, specifically the aspects of this element and actions teachers might take to increase the likelihood that students will productively engage in a collaborative discussion focused on making sense of a peer's contribution.

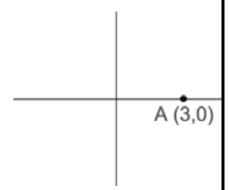
(a) Percent Discount The price of a necklace was first increased 50% and later decreased 50%. Is the final price the same as the original price? Why or why not?	**(c) Points on a Line** Is it possible to select a point B on the y-axis so that the line $x + y = 6$ goes through both points A and B? Explain why or why not. 
(b) Variables Which is larger, x or $x + x$? Explain your reasoning.	**(d) Bike Ride** On Blake's morning bike ride, he averaged 3 miles per hour (mph) riding a trail up a hill and 15 mph returning back down that same trail. What was his average speed for his whole ride?

Figure 1: The Four Mini Tasks Used to Create Instantiations of Building

Context of the Conduct Element

Once a MOST has emerged during a class discussion, been established as a precise object, and grapple tossed to the class, the teacher enacts the Conduct element— perhaps the most complex element of building. The overarching goal of Conduct is to keep the whole-class discussion focused on the MOST as the teacher facilitates the class's movement toward making sense of the mathematics of the MOST. In essence, making sense of the MOST and the mathematics associated with it becomes a new mini-discussion that the class becomes temporarily engaged with before returning to the larger discussion. Thus, conducting a discussion about a MOST requires a different collection of coordinated actions than those that a teacher might engage in other circumstances (such as after they have students work on a task and then collect ideas about that task). Once a teacher decides to build on a MOST, we have found that it is unproductive to continue to collect ideas about a broader task or anything other than ideas related to the MOST. In fact, our data suggest that doing so is detrimental to taking advantage of the opportunity the MOST provided to make sense of important mathematics.

Before discussing the specifics of the Conduct element, we begin by providing a "big picture" overview of the teacher actions that take place while effectively carrying out this element of the building practice. Note that this overview is not meant to oversimplify the complex activity of conducting a discussion about a MOST, but to help the reader think about how specific teacher actions might coordinate to carry out the nuances of the work. When a student contributes an idea during Conduct, the teacher first determines whether the mathematics underlying that contribution will help students make sense of the MOST–whether it is a MOST-related contribution (MRC). The subsequent sequence of teacher actions depends on that determination. If a teacher determines that the contribution is not MOST-related (a non-MRC), we have found that a productive teacher action is to gracefully put aside the idea (Drageset, 2014) and recenter the MOST as the focus of the discussion. If the teacher determines that the idea is an MRC, we have found a productive sequence of actions is to first establish the MRC and then make a move that invites students to connect the MRC to the MOST itself—to make sense of how the new idea helps to make sense of the MOST. In essence, the flow of the Conduct element is a repetition of this pattern, since in doing the work of responding to either a MRC or non-MRC, the teacher sets students up to continue to contribute their ideas to make sense of the MOST. We elaborate on the details of these teacher actions in the following sections.

Is the Student Contribution MOST-related?

When conducting a discussion around a MOST, it would be ideal if all of the contributions that students add to the discussion helped to make sense of the MOST. Students do not always do or say what a teacher might want or expect, however. In order to keep the discussion focused on making sense of the MOST, each time a new contribution is shared during the Conduct element, the teacher needs to begin their work by determining whether the contribution is actually a MOST-related contribution (MRC). This determination informs the teacher's subsequent actions.

Determining whether a contribution is an MRC may require the teacher to clarify what the student has said or expand the contribution to get a sense of the student's reasoning. After the teacher has sufficient information to infer the mathematics of the contribution, they then must ask themself, "Is the mathematics of this contribution directly related to the MOST?" (rather than, for example, a contribution that shares another idea about a broader task that students have worked on) and "Will it help students make sense of the MOST?" If both answers are yes, the contribution is an MRC. Otherwise, it is not. In general, for a contribution to be MOST-related, it needs to have the same underlying mathematics as the MOST.

In the following sections, we elaborate on the work that our data suggest a teacher needs to engage in after they determine whether a student contribution is an MRC. We first discuss actions around contributions that are not MRCs, and then actions that support the class's sense-making around an MRC. As we discuss these actions, we will share examples from our data based on whole- class discussions of the tasks designed to elicit MOSTs in Figure 1.

The Student Contribution is Not an MRC

If the teacher determines that a student contribution is not an MRC, our data suggests that it is best to redirect the conversation back to the MOST by *putting aside* the unrelated contribution and recentering the MOST as the focus of the conversation. The action of *putting aside* is adapted from the work of Drageset (2014) who stated that "redirecting actions might be used to put aside suggestions without too much discussion to keep the class concentrated and in order not to lose the line of thought" (p. 300). Recentering the MOST draws on the work of Lineback (2015) who talks about a type of "focus redirection" in which the teacher "refocus[es] the students on an earlier discussion topic" (p. 432). In the case of building, the MOST that is under consideration would be the earlier discussion topic. In this section we elaborate on the ways that the putting aside and recentering might occur.

One way we saw teachers put aside a non-MRC and recenter the MOST was by simply reminding students that they are to be making sense of the MOST. For example, in the Points on a Line task, a common shared solution was "Yes. Point B is $(0,3)$ because you get $3 + 3 = 6$," including the explanation that they took the *x*-value for the equation $x + y = 6$ from point A and the *y*-value from point B. Despite being asked to make sense of this particular student contribution, in the ensuing discussion students often offered other approaches to solve the task itself, such as “If you write the equation in slope-intercept form, the y-intercept is (0, 6).” This contribution is a non-MRC because the underlying mathematics related to slope-intercept form is different from the mathematics of the MOST (related to what it means for a point to be a solution to an equation). In this situation, the teachers said things like, “That’s another interesting approach, but right now we want to be making sense of the claim that Point B is $(0,3)$ because $3 + 3 = 6$.” This statement validates the contribution but immediately puts it aside and recenters the MOST as the focus of the discussion. In our data, we have found that it is essential to not only put aside a non-MRC, but to also redirect the conversation by recentering the MOST; this recentering provides guidance to focus students’ subsequent contributions. Other teacher actions that supported recentering included gesturing to the public record of the MOST while reminding the class that the MOST is what they are currently discussing, and re-grapple tossing the MOST to re-engage students in sense-making around that contribution. These actions for handling a non-MRC are seen on the left side of Figure 2.

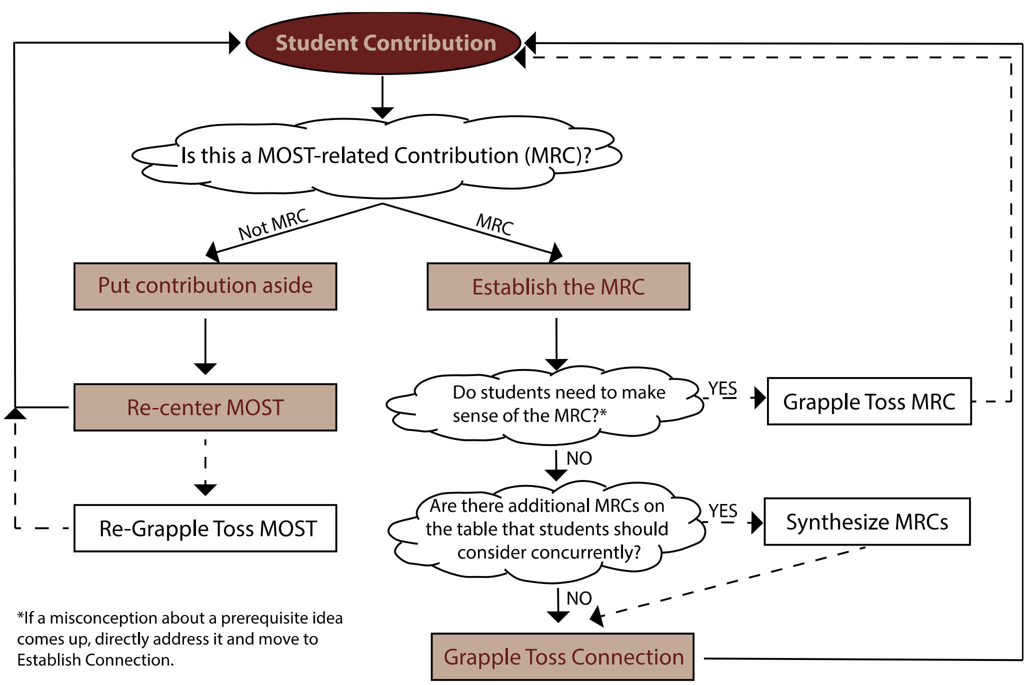


Figure 2: The Conduct Element of Building

The Student Contribution is an MRC

If it is determined that a student contribution is an MRC, the teacher’s focus should be on weaving this MRC into the conversation about the MOST. Prior to doing this weaving, we have found that the teacher needs to ensure that everyone in the class understands the essence of the MRC they will consider. Our data indicate that this involves two essential actions—establishing the MRC, and positioning students to connect the MRC to the MOST in a way that allows them to use the MRC to help make sense of the MOST. We first talk about these two essential actions when working with an MRC (the tan boxes on the right side of Figure 2) and then discuss other considerations when working with an MRC (the white boxes on the right side of Figure 2).

Establishing the MRC. Our data suggests that prior to weaving an MRC into the discussion, similar to establishing the MOST, the teacher should first establish the MRC as clear, complete, and concise (Leatham et al., 2021). For example, recall that in the Points on a Line task, the

MOST that emerged was that the point B (0, 3) was the point that satisfied the conditions of the problem because $3 + 3 = 6$. In the discussion of the MOST, Joelle (all names are pseudonyms) said, "Well, I mean, it doesn't ever, I don't know, I just feel like there's no, it doesn't say that the x and y are coming from different points, I just don't know how that would work." With so much hedging, it was necessary for the teacher to hone in on the key aspect of what Joelle said to make her statement concise, and clarify her statement in the process. He did so by saying "Okay so you're describing this [the teacher points at the MOST written on the board, 'Yes, use (0,3) because 3+3=6'] as taking x and y from different points." The teacher made this statement with a tone indicating that he was seeking confirmation from Joelle that he had interpreted her statement correctly. His gesture to the board also clarified that Joelle's contribution was describing the approach used in the MOST. Teacher moves to establish the MRC have the impact of making the MRC clear, complete and concise for the rest of the students in the class.

Grapple Toss Connection. Asking students to consider how an MRC helps them make sense of the MOST is an important aspect of the Conduct element. When there is an MRC on the table, the teacher needs to make a move to position not only the MOST, but also the MRC, as the object of the discussion. The *grapple toss connection* tells students the action they are to take on the MRC and the MOST—to connect the MRC to the MOST in a way that positions the students to use the MRC to help make sense of the MOST. Because a connection necessarily implies tying two things together, we have found that an important aspect of the grapple toss connection is clearly identifying the two ends of the connection, the established MRC and the MOST. Without clearly defining both ends of the connection, some students may not know what object they are to consider. In addition, we have found that the teacher needs to clearly define the action that students are to take on that object. Failing to do so often leaves students wondering what they are to think about, rather than putting them in a sense-making situation. The reason this is called a grapple toss connection is because the action-defining move resembles the Grapple Toss of the MOST, but the object that is tossed is the connection between the MRC and the MOST.

We have seen some variations in how a teacher connects an MRC to the MOST. One of the most productive connecting actions we have seen in terms of helping students use the MRC to make sense of the MOST—the purpose of the Conduct element—is explicitly asking students to relate the new information (the MRC) to the MOST. For example, in the Percent Discount Task, the common MOST is the response that the original and final price are the same because you are adding and subtracting 50%. In the discussion of the MOST, Kaleb contributed the following MRC using a sample necklace price of $32, "If you add the half of 32, which is 16, it would equal 48. [*teacher writes "If you add ½ of the 32, 32+16 = 48" on the whiteboard*.] But, if you subtract the half of 48, it would give you 24 which is not the original price." The teacher responded to this MRC by saying, "Okay, so, if you add half of the original, 16, that would give you 48 and then if you subtract half of the 48, it would be 24? So what does that have to do with the original claim here? How does that prove or disprove it?" Here we see the teacher identify both ends of the connection by restating Kaleb's contribution and asking how it relates to "the original claim"— the MOST. The fact that the original claim is recorded on the whiteboard strengthens the connection because students can refer to the public record to recall exactly what the original claim was. Not only does the teacher clearly identify both ends of the connection but they also identify an action, "prove or disprove", that the students are expected to take as they consider the connection between the MRC and the MOST. This grapple toss connection clearly positions students to make sense of the connection between the MRC and the MOST.

Another way we have seen teachers connect the MRC and the MOST is to ask students to consider conflicting information. In this case, the teacher typically highlights that there is in fact a conflict, and then asks students to make sense of that conflict. For example, after an MRC surfaced in the discussion of the Percent Discount task, the teacher said, "So in Bruce's argument, [the initial and final price are] not the same but Jaden was saying it would be the same cause you would just add 5 and minus 5. So how do we know which one we're supposed to do?" In this response, we see the teacher connecting the MRC (Bruce's argument) to Jaden's claim (the MOST) by highlighting the contradiction that is now on the table and asking students the action-defining question, figuring out "which one we're supposed to do." Pointing to Jaden's claim on the board, restating a portion of the claim, and explicitly referencing Bruce's argument when contrasting it with Jaden's all support having a clear object of consideration.

In summary, when integrating MRCs into the discussion of a MOST, we have found that explicitly defining both ends of the connection puts students in a position where they are able to use the new information provided in the MRC to make sense of the MOST. A second important aspect of the grapple toss connection is to define some type of mental action the students are to carry out when considering how the MRC helps them make sense of the MOST.

Other Considerations. After an MRC is established, but prior to the grapple toss connection, we have found that additional teacher actions are sometimes needed. These moves–grapple tossing the MRC itself and synthesizing the ideas that are on the table–only seem necessary when it is clear that they will better position students for the grapple toss connection.

Grapple Tossing the MRC. Before they are asked to make sense of the connection between the established MRC and the MOST, we have found that sometimes students need to spend some time making sense of the MRC itself. This is important when the students do not yet appear to understand the MRC. If the teacher determines that this is the case, our data suggest that the best move is to Grapple Toss the MRC itself to the class by asking a question that positions students to make sense of that contribution (seen on the right side of the flowchart in Figure 2).

In the discussion of why the simple average of 9 mph (the MOST) is not correct in the Bike Ride task, Corbin contributed the MRC, "You're going 3 miles per hour for the same length of time as you're going 15 miles." Recognizing the potential value of Corbin's thinking to make sense of the MOST, the teacher responded, "Do you guys believe that Blake rode the bike for the same amount of time going 3 miles per hour and the same amount of time at 15 miles an hour?" In this case, Corbin's thinking was a productive contribution to the discussion because knowing the lengths of time at each speed would help students make sense of the MOST, but it wouldn't be productive to connect it to the MOST until students have had a chance to make sense of the validity of this claim. In this instance, grapple tossing just the MRC was a productive move prior to asking students to connect the MRC to the MOST because it put students in a better position to think about how the MRC might help them make sense of the MOST.

Synthesizing. After an MRC has been established but before the grapple toss connection, a teacher needs to ask themselves whether there are prior MRCs on the table that, if considered in conjunction with the new MRC, would better position students to make sense of the MOST. If so, then some synthesis needs to take place that highlights the salient aspects of the MRCs that students will be asked to collectively consider. In this case, the grapple toss connection might position the class to consider the relationship among multiple MRCs and the MOST.

For the Variables task, the common MOST is for students to say that $x + x$ is greater than x because $x + x$ is $2x$ and that is twice as big as x. After several MRCs had emerged, the teacher synthesized with the following response: "So here we have Andre's thinking that x + x is gonna be larger than x because x + x is double so it makes everything larger, right? Somebody else said 5 plus 5, would be greater than just 5, right? And Briana, now you're saying that if x were -9, then -9 is greater than -9 plus -9." In this synthesis, the teacher has tied the concrete example of x = 5 to the statement of "$x+x$ is larger than x" which is written on the board and then highlights the MRC where x = -9 by writing "-9 > -9 + -9". Note, however, that they stop short of writing or saying $x > x + x$ so the students are left to make that connection and see the contradiction. In this example, the teacher highlights the salient aspects of two MRCs and also ties one MRC to the MOST. With the appropriate grapple toss connection, the students are in a position to make sense of the MOST with a new perspective provided by the MRCs.

Discussion and Conclusion

Engaging students in whole-class discussion focused on making sense of the mathematics in an instance of student thinking is a complex teaching practice. Although past research has identified individual teacher actions that support student sense making, our work suggests that conducting such a discussion requires a coordinated collection of teacher decisions and actions. Each time a new student contribution emerges, the teacher needs to assess whether the contribution will support the joint sense making, or take students in an unproductive direction. This critical initial decision determines the teachers' subsequent decision points and actions.

There are two additional insights about the Conduct element of building that we wish to share. First is the importance of keeping the discussion focused on making sense of the object of discussion, the MOST. In our work, we have seen teachers pursue every student idea that surfaces, resulting in a discussion that meanders among ideas. Although it seems that teachers do this with good intent as a means of honoring students and their ideas, we argue that putting some ideas aside better honors the student who contributed the MOST by maintaining a focus on the important mathematics that they initially brought to the discussion.

Second, we have come to realize the importance of ensuring that students have a clear understanding of the object they are to focus on and how they are to engage with that object at any point in a discussion. During the practice of building, the discussion begins by positioning the MOST as the object that students are to make sense of. In the Conduct element, for each MRC that is contributed, students first need to understand the MRC itself, and then need to shift their attention to considering a connected object—the MRC and the MOST—and how the MRC helps to make sense of the MOST. Without a clear sense of these shifts in focus, students are unlikely to engage in a focused sense-making discussion. Thus, an important part of the teacher's work during Conduct is explicitly helping students productively focus their attention.

Decomposing complex teaching practices is an important first step in supporting teacher learning of such practices (Grossman et al., 2009). Unpacking the Conduct element of building has allowed us to better understand the complexity of the practice of building on MOSTs and positions us to move to the next stage of our work in supporting teachers to develop their abilities to productively use student mathematical thinking.

Acknowledgements

This research report is based on work supported by the U.S. National Science Foundation (NSF) under Grant Nos. DRL-1720410, DRL-1720566, and DRL-1720613. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF. We thank the teacher-researchers for opening up their practice to help us learn about building, and the undergraduate and graduate students who have helped us to collect, process, and analyze building enactments.

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. Retrieved from amte.net/standards

Bishop, J., Hardison, H., & Przybyla-Kuchek, J. (2016). Profiles of responsiveness in middle grades mathematics classrooms. In M. B.Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics education* (pp. 1173–1180). The University of Arizona.

Drageset, O. G. (2014). Redirecting, progressing, and focusing actions—a framework for describing how teachers use students' comments to work with mathematics. *Educational Studies in Mathematics*, *85*(2), 281-304.

Ellis, A., Özgür, Z., & Reiten, L. (2019). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal*, 31(2), 107–132. https://doi.org/10.1007/s13394-018-0246-6

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, 111(9), 2055-2100.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88-124. https://doi.org/10.5951/jresematheduc.46.1.0088

Leatham, K. R., Van Zoest, L. R., Freeburn, B., Peterson, B. E., Stockero, S. L. (2021). Establishing Student Mathematical Thinking as an Object of Class Discussion. In D. Olanoff, K. Johnson, & S. Spitzer, (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1392-1400). Philadelphia, PA.

Lineback, J. E. (2015). The redirection: An indicator of how teachers respond to student thinking. *Journal of the Learning Sciences*, 24(3), 419–460. https://doi.org/10.1080/10508406.2014.930707

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

Smith, M. S.,&Stein, M. K. (2018). *Five practices for orchestrating productive mathematics discussions*. National Council of Teachers of Mathematics.

Stockero, S. L., Van Zoest, L. R., Freeburn, B., Peterson, B. E., & Leatham, K. R. (2020). Teachers' responses to instances of student mathematical thinking with varied potential to support student learning. *Mathematics Education Research Journal*, 1-23. https://doi.org/10.1007/s13394-020-00334-x

Van Zoest, L. R., Peterson, B. E., Leatham, K. R., Stockero, S. L. (2016). Conceptualizing the teaching practice of building on student mathematical thinking. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1281-1288). University of Arizona.

Van Zoest, L. R., Stockero, S. L., Leatham, K. R., Peterson, B. E., Atanga, N. A., & Ochieng, M. A. (2017). Attributes of instances of student mathematical thinking that are worth building on in whole-class discussion. *Mathematical Thinking and Learning*, *19*(1), 33-54. https://doi.org/10.1080/10986065.2017.1259786

Webb, N. M., Franke, M. L., Ing, M., Turrou, A. C., Johnson, N. C., & Zimmerman, J. (2019). Teacher practices that promote productive dialogue and learning in mathematics classrooms. *International Journal of Educational Research*, *97*, 176-186. http://dx.doi.org/10.1016/j.ijer.2017.07.009

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

AESTHETIC DIMENSIONS OF STUDENT MATHEMATICAL CREATIVITY

Meghan Riling
Vanderbilt University
meghan.riling@vanderbilt.edu

There is a popular myth that mathematical creativity is a trait possessed by a small group of people. This is perpetuated by the lack of understanding of what influences mathematical creativity, especially in the classroom. In particular, professional mathematicians often name aesthetic experiences as a critical aspect of their mathematical creativity. Therefore, this study examines aesthetic dimensions of episodes of student creativity from the high school classroom. Several types of aesthetic experiences arose across the episodes. Students were motivated to take creative action by experiences of mystery or discomfort. Furthermore, in some instances, taking a creative action was accompanied and propelled by a sense of craziness or satisfaction.

Keywords: Affect, Emotion, Beliefs, and Attitudes; High School Education

Students must have the opportunity to be creative during mathematics class. Otherwise, they may learn that creating mathematics is for others to do; their only role is to reproduce. This is problematic because creating mathematical ideas is often necessary for practical purposes in life (e.g., crafting a personal finance plan) and can be a source of pleasure and meaning (e.g., making music or noticing patterns) (Su & Jackson, 2020). If students learn that their role with respect to mathematics is to use mathematical ideas or tools made by others, they may not be empowered to take ownership of aspects of their life with prominent mathematical components. This can be dehumanizing and could have devastating impacts (Gutiérrez, 2018).

And yet, it often seems that many students are not capable of being creative during math instruction. Some previous studies of student mathematical creativity have even been unable to detect any student creativity within the context of classroom learning (Clack, 2017; Mhlolo, 2016). I argue, though, that being mathematically creative is not an inborn trait possessed by a limited group of people, but instead a type of action that is influenced by personal, social, and cultural factors. Factors such as these have been explored in creativity in other fields (Csikszentmihalyi, 2014; Hanson, 2015), but are understudied in the case of mathematical creativity. This perpetuates the harmful myth that there are some people who can create new mathematical ideas, and others who cannot.

In mathematics, one factor that has been named by some professionals in the field as impactful in their creative process has been aesthetic experiences (Burton, 1998; Sriraman, 2004). What about students? In this paper, I examine aesthetic experiences and their role in student mathematical creativity by analyzing enacted lessons in high school mathematics classrooms. What roles can aesthetic experiences play in student mathematical creativity?

Conceptual Frameworks

In this study, I assert that mathematics is human-made and fallible (Ernest, 1998; Lakatos, 1976). Thus, human action can create new ways of doing mathematics. Creativity is the act of creating ways of doing or thinking about mathematics that were not previously possible (Riling, 2020). An action with creative mathematical potential is one that an individual takes of their own volition, rather than because of following an instruction or norm (Pickering, 1995; Sinclair, 2005). The action realizes its creative potential if it goes on to enable members of the mathematical community in which it occurs to think about or do mathematics in new ways.

Whether or not an action fulfills its creative potential is not solely based on the merits of the action, but various personal, social, and cultural forces that govern the community of mathematicians.

Because mathematics is a human endeavor, the relationship between a person and mathematics is a crucial aspect governing how mathematics develops, which I conceptualize here through *aesthetics*. Aesthetic experiences can function in multiple ways in the doing of mathematics. They may motivate a person to take some action, or they may occur as a result of a person taking an action (Brinkmann & Sriraman, 2009; Sinclair, 2004). Here I describe three forms of aesthetic experience that may play a role in student mathematical creativity.

First, aesthetic experiences can motivate the exploration of mathematics. Sinclair (2004) explains that as there are few practical reasons to pursue a great deal of the field of advanced mathematics, much of the motivation for doing so is derived from aesthetic aspects of the field. Mathematicians may become "hooked on" (p. 275) the order that can arise from completing a mathematical investigation or proof, or become intrigued by a paradox or moment of surprise. As these types of experiences have been shown to motivate mathematical activity in general, they may specially motivate mathematical action with creative potential. Thus, for the purposes of this study, a *motivating aesthetic experience* is one that prompts students to try to create new mathematical possibilities.

Sinclair (2004) also describes how an aesthetic experience can play a *generative* role in mathematics, in that people may be struck by a sense that mathematical objects or events that arise in the process of doing mathematics that they sense will be valuable or provide insight, compelling them to investigate further. It is possible that while taking an action that has the potential to create mathematics, a student might experience pleasure or excitement that generates further curiosity.

Finally, Sinclair (2004) describes the role of an aesthetic experience in considering a mathematical object that has been produced already; an *evaluative aesthetic experience*. Beauty is a particularly important form of evaluative aesthetic for many professional mathematicians (Brinkmann & Sriraman, 2009; Sinclair, 2004). This type of aesthetic experience may happen as a student regards their own work, or the work of another, and may play a part in what happens after a student has taken an action with creative potential.

Methods

To learn about the role of aesthetic experiences in student mathematical creativity, I examined enactments of secondary mathematics lessons. The lessons were designed to be aesthetically captivating for students. The lessons were analyzed by identifying instances of student action with creative potential and student aesthetic experiences, and by crafting narrative episodes of these actions and experiences, which were then examined for common themes.

Participants and Lesson Selection

The lessons in this study were enacted in high school classrooms in the northeastern region of the US, taught by teachers with 4-6 years of experience in the classroom. The teachers taught at three schools that varied in size, school type, and racial demographics (see Table 1). Each teacher taught three specially-designed lessons. These lessons were designed by small groups, including the classroom teacher, another colleague participating in the study, and university-based researchers. The topics of these lessons fell within the curricula of each course. These lessons were unique in that they were designed by teachers and researchers to provide the opportunity for aesthetic experiences. This was done by way of conceptualizing lessons as mathematical stories (Dietiker, 2015), meaning that the teachers thought of the mathematical context as a setting for the development or action of mathematical characters (e.g., numbers or geometric objects). Some teachers aimed to design stories with compelling narrative devices, such as a surprising plot twist or an air of suspense. This data set was useful for studying aesthetic dimensions of student creativity because a set of lessons designed without attention to students' aesthetic experience could fail to provide any detectable displays of student aesthetic experiences, or any variety in student aesthetic experiences.

Table 1: Participating High Schools and Students

High School	*Cooperating teachers*	*Number of participating students*	*Students' racial identification (categories >15%)*	*Mean disposition to math (1-4)*
Forte Charter Secondary (Forte)	Mr. Davis and Ms. Bacheldor	85	79% - Latinx**	2.54
Gladbury High School (GHS)	Mr. Anderson and Ms. Evans	56	58% - White**	2.73
Motion High School (MHS)	Ms. Curran and Ms. Fontaine	70	50% - Asian** 16% - Latinx**	2.81

**Based on student responses to survey items.*
***Based on demographic data reported by the state; demographics below 15% not represented in this table.*

Each specially-designed lesson was observed and audio- and video-recorded from multiple perspectives in the classroom. Some were enacted and observed for multiple classes of students, if a teacher taught multiple sections of the same course. After the lessons, all participating students completed a Lesson Experience Survey (LES). This survey included a prompt to select three descriptors to describe their experience of the lesson, out of a bank of sixteen terms. Some terms were positive (e.g., “intriguing”), some were neutral (e.g., “fine”), and others were negative (e.g., “boring”). In addition, two or three students were selected to be interviewed after each lesson. Students were typically approached about being interviewed because of some strong affective display during the lesson. For every specially-designed lesson that was filmed, the research team also observed and recorded a typical lesson in the same class of students within approximately one week, following the same observation protocols.

In order to select a group of lessons that were more likely to contain student actions with creative potential, I identified the lessons that were more frequently described on the LES with a group of target descriptors that I theorized could be selected by students to describe experiences that involved creativity (i.e., *thought-provoking*, *amazing*, *fascinating*, *fun*, *surprising*, *intriguing*, and *enjoyable*). I identified each teacher's lessons for which the highest percentage of students selected the target descriptors. Upon beginning to analyze the lessons, I began with one lesson per teacher. I iteratively added additional lessons, aiming to maintain variety in teachers and courses, until I had identified a robust set of student actions with creative potential. All lessons that were included were ones that the teachers had designed to be aesthetically engaging (see Table 2). Each lesson was transcribed, including full-group discussions and the discussions that occurred in a small group that had been filmed from an additional, close-up camera.

Table 2: Lessons Analyzed

School	*Teacher*	*Course*	*Lesson*	*High frequency descriptors*
Forte	Ms. Bacheldor	Honors Algebra 2	Extraneous solutions	Thought-provoking
Forte	Mr. Davis	AP Calculus	Derivatives of exponential functions	Intriguing, Enjoyable
Forte	Mr. Davis	AP Calculus	Solids of revolution	Fun, Enjoyable
GHS	Mr. Anderson	Algebra 2	Imaginary numbers	Thought-provoking, Surprising
GHS	Ms. Evans	Algebra 2	Introduction to inverses	Amazing, Fascinating
GHS	Ms. Evans	Algebra 2	Introduction to inverses	Thought-provoking, Enjoyable
MHS	Ms. Curran	Integrated Math 1	Geometric transformations	Amazing, Enjoyable
MHS	Ms. Curran	Integrated Math 1	Linear functions	Surprising, Fun, Enjoyable
MHS	Ms. Fontaine	Honors Integrated Math 3	Rational root theorem	Fascinating

Identifying and Categorizing Creative Actions

I next scanned the selected lessons for actions with creative potential. By creative potential, I mean actions that *could* lead to the creation of new mathematics. These are those actions that students have taken using their own agency, rather than following disciplinary or classroom norms (Pickering, 1995). In order to broaden my perspective, and thereby increase my ability to detect student creative potential, I consulted first-hand accounts of professional creators. To avoid reinforcing the narrative of mathematics' white, male history, I curated a set of professional creators who varied in medium, gender, country of origin, race and ethnicity, and age. The set of creators included individuals such as Chinese architect and conceptual artist Ai Weiwei, Black American film-maker Ava DuVernay, French-Cuban musicians Lisa-Kaindé Diaz and Naomi Diaz, of Ibeyi, and white French mathematician Sophie Germain. After learning about their creative work, I revisited the classroom data and indeed was able to identify additional student actions with creative potential.

I categorized each action with creative potential based on how the actions opened new possibility for students in the lessons (Riling, 2022). For example, *setting out* refers to instances in which students decide to embark on some kind of creative work. *Imagining* refers to the act of mentally projecting a new version of the mathematical reality. *Recognizing* involves reinterpreting some existing mathematical object in a new way. *Manifesting* is the act of taking a action that immediately changes the current mathematical context.

Identifying and Explaining Aesthetic Experiences

In order to identify any aesthetic experiences that could have had a role in students' actions with creative potential and how they developed in the classroom, I asked the following questions of the lesson data: (1) What emotions do individuals display in reference to their mathematical context? (2) Do the individuals explicitly relate their response to mathematical ideas, or do emotional displays coincide with changes in their understanding of the mathematical context? (3) What kind of aesthetic experience do the individuals appear to be having? I marked displays of emotion such as changes in vocal pitch or tone (e.g., singing or shouting), use of exclamatory language (e.g., "oh my god" or "boom"), use of explicitly aesthetic or emotional language (e.g., "mystery" or "hate"), and changes in physical behavior (e.g., smiling or dancing). I recorded my initial understanding of the type of aesthetic experience that students appeared to be having (e.g., surprise or satisfaction).

Next, I worked to establish the roles of the aesthetic factors (i.e., motivating, generative, or evaluative). I did this in part by considering when the individuals displayed evidence of having an aesthetic experience, in comparison to when the action with creative potential was taken. That is, a motivating aesthetic experience would happen before the action, a generative one would be concurrent, and an evaluative one after. In addition, in order for an aesthetic experience to be considered to have motivated an action with creative potential, there would have to be evidence that the action taken was related to the context that provoked the aesthetic response. For generative aesthetic experiences, students either would explicitly connect their affect to a mathematical object or event, or express emotion at the same time as taking an action, in the absence of any other change that could explain their expression. To be considered to be evaluative, an aesthetic expression would have to occur as the expressing student attended to a mathematical object related to the action with creative potential under consideration.

In addition to the audio and video recordings, in order to ascertain the type and role of students' aesthetic experiences, I consulted post-lesson interviews. For eight of the fourteen episodes, the student who took the action with creative potential was interviewed after the lesson. In some cases, they spoke explicitly about the action and the feelings they had related to it. For example, one student shared during their interview, "when I saw like the coefficients, with the pattern of the roots, I thought that maybe like, it's kinda crazy but maybe the coefficients are related to the polynomial." Here, the student explains how his feeling of something being "crazy" related to his actions.

I identified the episodes that seemed to contain similar dimensions within students' aesthetic experiences (e.g., all episodes that contained mysterious elements) and used memo-ing to explore the similarities in how those experiences functioned in different episodes. For example, I asked myself, "What is the role of satisfaction across these examples?" and "When does satisfaction happen in relation to creative actions?" After answering these questions for individual episodes, I constructed a model of how that type of aesthetic experience functioned with respect to actions with creative potential and to other types of aesthetic experiences. This enabled me to identify and define common dimensions in students' aesthetic experiences.

In the process of analyzing student language in relation to aesthetic experiences, I noticed that students used terms that evoked the existence of some alternative to their current accepted reality (e.g., conspiracy theories, drug use). Therefore, I examined the ways in which students described aesthetic experiences with mathematics related to the existence of a new mathematical world of some kind. This interpretation was overwhelmingly consistent with students' descriptions. Therefore, I defined the common dimensions of aesthetic experiences that I found in the episodes in terms of how they related to students' perception of the existing and/or a new mathematical world.

Findings

Within the data, when students took actions with creative potential, they sometimes seemed to be motivated by two different types of aesthetic experiences: some kind of mysterious uncertainty, or discomfort. Additionally, some actions with creative potential were accompanied by a break in mathematical reality that students referred to as "crazy," or a sense of satisfaction, which played a generative role as it spurred continued investigation, or an evaluative role as students concluded their work exploring an action with creative potential. In this section, I define *mystery*, *craziness*, *discomfort*, and *satisfaction* in terms of students' aesthetic experience with mathematics and describe how students' actions with creative potential were influenced or marked by aesthetic experiences.

Mystery

One dimension of an aesthetic experience remarked upon by students in the data was *mystery*. Mystery appeared in some instances of *imagining*, one type of action with creative potential. In several cases, students indicated feeling that their previous assumptions about the mathematical context were incorrect or incomplete in some way. This happened when students encountered something that did not make sense, such as finding that they need to take the square root of negative numbers in order to solve a problem, when they had previously been told they could not do so. After encountering such a clue, students began to doubt the general premises of the mathematical realm in which they were operating. A sense of uncertainty is inherent in mystery. For example, in the case of a group of students in a scenario that suggested that they should take the square root of negative numbers, when their teacher asked if they believed that was feasible, several students responded "I don't know." In other instances in which students sensed a mystery, they used phrases like "maybe" and "I think," conveying this lack of certainty.

Another common feature of students' references to mystery is that they referred to it broadly. For example, in one lesson, a student said "we have a situation," indicating that, in general, something about the mathematical context was amiss. She did not, however, specify exactly what the potential problem was. Another student was even less detailed, intoning "dun dun dun." This suggests that a sense of mystery might be something that students do not affix to a specific object in their mathematical context, but to the mathematical context more broadly.

Discomfort

In the lessons, students experienced various forms of discomfort as a result of the current mathematical task. This discomfort often seemed to motivate students to take actions with creative potential. Although students also can have negative emotions about other things during mathematics class, such as peer relationships or grades, I am not referring to those experiences here. One form of discomfort in the data was frustration. Frustration often occurred when students were unhappy that their attempts to solve a problem were unsuccessful. It should be noted that other students could be excited to have a problem that they could not solve quickly. To qualify as discomfort, students would have to not enjoy these experiences. Students typically demonstrated discomfort by using qualitatively negative language about mathematics. For example, "I hate *e*," or "Forty-six? Yikes." At other times, they might not verbally refer to a mathematical concept, but instead have a negative outburst (e.g., "Jesus Christ!" or "This shit's confusing") while working on a mathematical task. Another form of discomfort in the data was tedium, in which students were bored by work that they perceived as too repetitive.

In several cases, students seemed to engage in manifesting (a type of action with creative potential) in an attempt to remove discomfort. This bears something in common to students attempting to imagine their way out of a mystery. However, by manifesting, students did not return to the status quo, as the students attempted to do through imagining, but instead took an action that created a new mathematical reality. Consider that in several of these instances, the students verbally wrestled with whether or not to manifest when experiencing a state of discomfort. For example, one student who was struggling to solve an equation expressed that he was uncertain about whether he could square the equation. He asked, "Why can't we?" and then finally rejecting this felt external pressure to not square, stating, "I don't care" as he squared the equation. The evidence that students often manifested in spite of not being sure that it was allowed suggests that these students were aware that they were changing something about the mathematical context. In this way, they acted more like the students who moved toward a new mathematical reality when they experienced something crazy that they chose to explore further.

Craziness

Craziness is related to mystery in that it also indicates that students have come to doubt their current realm of known mathematics. However, it differs in that rather than only sensing that an alternate mathematical reality might exist, students are in the act of crossing over into the new realm. They are compelled to continue doing so, meaning that this aesthetic experience is generative. For example, consider a student in a lesson in which a teacher asked students to group cards that contained representations of linear functions. This student and his groupmates had begun to form pairs of cards that they believed contained representations of the same functions. When the student found out that he could make groups of more than two linear functions, he exclaimed, "more! That's crazy! There's more!" Since students who sense that something is crazy are crossing into the new reality, they can typically point to a specific mathematical object or event that provoked the feeling.

When students described a crazy aesthetic experience, they often spoke loudly in excited tones. They sometimes used language related to conspiracy theories ("it's Illuminati confirmed, bro") or drug use ("that's trippy"). These are both experiences in which an alternate version of reality is suddenly revealed, by way of learning secret information or by consuming drugs. However, the most common vocabulary that students used when expressing that they are experiencing this aesthetic is the word "crazy." It is because of the extreme frequency of this word choice that I use "crazy" to label this type of aesthetic experiences, even though it is a word I typically avoid due to its ableist connotations. To date, I have not encountered an appropriate synonym that expresses both the perceived disruption in students' understanding of the mathematical context, and the sudden onset of feelings that accompanies this shift.

Several students who took the action with creative potential of setting out seemed to be motivated by a sense that something crazy had occurred. For example, the student who found out that he could make groups of more than two linear functions then set out to learn more about individual functions in order to be able to form groups. Additionally, the student in the quadrilateral midpoint lesson set out to learn about parallelograms only after finding a parallelogram in a place that they did not expect it to appear. The feeling of craziness prompted these students to want to learn more about the possible new mathematical context. This is different from the way in which mystery functioned in the cases of *imagining*. In those cases, students responded to a prevailing sense of mystery by attempting to avoid the mystery. That is, they did not want to 'uncover the truth' and potentially create a new reality; they wanted to maintain the status quo.

Satisfaction

Evidence of the sensation of *satisfaction* also appeared in the data, accompanying several instances of recognizing. When students recognized something, be it a pattern or a familiar mathematical structure, they sometimes gasped or exclaimed a phrase such as “oh my god,” seemingly involuntarily. In some instances, the students also loudly called out to their classmates. Gaining a new perspective through which to understand the current mathematical context seemed to cause pleasure for these students. These experiences seemed to be generative, in that they compelled students to continue to investigate the implications of the action.

In some ways, satisfaction bears resemblance to the craziness that some students expressed. Both types of aesthetic experiences involved students suddenly recognizing shifts in what they knew to be the current mathematical reality. Students even used some similar language, such as “oh my god.” The difference is that, whereas the students who experienced things as being crazy were suddenly aware that a different mathematical world existed, and that the one they knew previously might even cease to exist, students who experienced satisfaction instead found evidence of being in a familiar context. For example, consider a student who recognized that an equation was in a form could be factored. Upon sensing a familiar structure, the student expressed happiness through gasps and singing. A further difference is that, whereas satisfaction is virtually always a positive feeling, the sensation of being jolted into a new reality by something that seems crazy might be enjoyable for some students, but not all, in the same way that some people enjoy the thrill of roller coasters and others do not.

Discussion

Aesthetic experiences played prominent roles in students’ creative episodes in these lesson enactments, suggesting a need for lessons to be designed with dynamic aesthetic experiences. In many cases, students seemed to be deeply impacted by their aesthetic experiences, as indicated by behaviors ranging from gasping to singing. Maybe this is why it often seems that many individuals are not capable of being creative; they have not been able to experience mathematics in a way that is conducive to mathematical creativity. I suspect that many mathematics lessons differ drastically from the dynamic enactments analyzed here, considering that mathematics is so often described as dull or boring by those who do not become professional academic mathematicians. How many more students could be moved to take action with creative potential if they were to experience suspense or mystery during their mathematics lessons?

Of course, lessons with more varied aesthetic experiences and opportunities for student creativity may make new demands of teachers, in terms of managing both behavior and mathematical ideas. The student behavior described in this study does not match the image of quiet students working diligently at their desks. It involved sudden outbursts, minor references to drugs, even the use of curse words. What new skills would teachers need to facilitate lessons in which these behaviors were not only allowed, but maybe even encouraged? This adds difficulty to the increased demands of teaching for student creativity, in which students have more control over the mathematical content of class.

Finally, these aesthetic experiences were studied within a fairly unique set of lessons, in that they were designed to be aesthetically captivating, but the lessons still aligned with standard secondary mathematics curriculum in the US. This raises the question: What other kinds of creativity might students engage in if they were to have even more diverse aesthetic experiences, such as those that integrate expressive arts or social justice topics into mathematics instruction? Future research is needed to learn what other mathematical creativity students may be capable of.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1652513.

References

Burton, L. (1998). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, *37*(2), 121. https://doi.org/10.1023/A:1003697329618

Clack, J. (2017). Distinguishing between 'macro' and 'micro' possibility thinking: Seen and unseen creativity. *Thinking Skills and Creativity*, *26*, 60–70. https://doi.org/10.1016/j.tsc.2017.06.003

Csikszentmihalyi, M. (2014). Society, culture, and person: A systems view of creativity. In M. Csikszentmihalyi (Ed.), *The Systems Model of Creativity: The Collected Works of Mihaly Csikszentmihalyi* (pp. 47–61). Springer Netherlands. https://doi.org/10.1007/978-94-017-9085-7_4

Dietiker, L. (2015). Shaping mathematics into compelling stories: A curriculum design heuristic. *Journal of the International Society for Design and Development in Education*. https://open.bu.edu/handle/2144/25609

Ernest, P. (1998). *Social constructivism as a philosophy of mathematics* (1st edition). SUNY Press. https://epdf.tips/social-constructivism-as-a-philosophy-of-mathematics.html

Gutiérrez, R. (2018). The need to rehumanize mathematics. In *Rehumanizing Mathematics for Black, Indigenous, and Latinx Students* (pp. 1–10). National Council of Teachers of Mathematics.

Hanson, M. H. (2015). *Worldmaking: Psychology and the ideology of creativity*. Palgrave Macmillan Limited. http://ebookcentral.proquest.com/lib/bu/detail.action?docID=4082226

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge University Press.

Mhlolo, M. K. (2016). Regular classroom teachers' recognition and support of the creative potential of mildly gifted mathematics learners. *ZDM*, *49*(1), 81–94. https://doi.org/10.1007/s11858-016-0824-6

Pickering, A. (1995). *The mangle of practice: Time, agency, and science*. University of Chicago Press. http://ebookcentral.proquest.com/lib/bu/detail.action?docID=648147

Riling, M. (2020). Recognizing mathematics students as creative: Mathematical creativity as community-based and possibility-expanding. *Journal of Humanistic Mathematics*, *10*(2), 6–39. https://doi.org/10.5642/jhummath.202002.04

Riling, M. (2022). *Creating new mathematical possibility: Types of student actions with creative potential*. 2022 AERA Annual Meeting.

Sinclair, N. (2005). Chorus, colour, and contrariness in school mathematics. *THEN: Journal*, *1*(1). http://thenjournal.org/feature/80/

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator*, *14*(1), 19–34.

Su, F., & Jackson, C. (2020). *Mathematics for human flourishing* (Illustrated edition). Yale University Press.

SELF-EFFICACY, INSTRUCTIONAL BELIEFS AND THE USE OF THE STANDARDS FOR MATHEMATICAL PRACTICE

Amy E. Roehrig
Illinois State University
aeroehr@ilstu.edu

Tami S. Martin
Illinois State University
tsmartin@ilstu.edu

Teachers' beliefs, knowledge, and decisions can affect the way teachers teach, and, consequently, what students learn. Self-efficacy beliefs may also interact with beliefs about the most appropriate and effective teaching and the selection of instructional practices to implement. We examined the relationships among teachers' mathematics self-efficacy, mathematics teaching self-efficacy, instructional beliefs, and use of the Standards for Mathematical Practices (SMP; NGACBP & CCSSO, 2010) for teachers who had been identified as effective teachers. We found that although the teachers scored similarly on beliefs surveys, there were some differences in their use of the SMP (NGACBP & CCSSO, 2010).

Keywords: Standards, Teacher Beliefs, Instructional Activities and Practices

Affective factors such as self-efficacy, motivation, anxiety, and confidence can be related to learning and teaching processes. Self-efficacy beliefs—beliefs about one's capacity to be successful in various situations—are situation-specific. Teachers have self-efficacy beliefs about mathematics, teaching, and mathematics teaching. These beliefs are important because they may interact with beliefs about the effective teaching practices (McLeod, 1987; Opera & Stonewater, 1987; Raymond, 1997) and the implementation of instructional practices (Peterson et al., 1989).

Mathematics teaching practices are often categorized as either student-centered (e.g., reform oriented, learner-centered) or teacher-centered (e.g., traditional). The use of student-centered teaching practices has been shown to relate positively to student learning in mathematics, as measured by increased post-test scores on a curriculum-based, researcher-constructed classroom test (Jong et al., 2010). Student-centered practices may also decrease negative affective factors, such as mathematics anxiety, in students (Alsup, 2004). There is evidence that teachers' beliefs concerning the use of student-centered practices and teacher self-efficacies influence teachers' decisions about what teaching practices to enact when teaching mathematics (Hadley & Dorward, 2011; Peterson et al., 1989). However, this evidence is largely based on self-reported teaching practices. Thus, it is not clear how self-efficacies may relate to the enacted teaching practices of teachers, as verified by an outside observer. Merriam and Tisdell (2016) noted that observation in the form of "firsthand encounters with the phenomenon" is more reliable than "a secondhand account" by teachers, who are sharing an interpretation of their own teaching practices (p. 137).

Further, the role that teachers' instructional beliefs play in mediating self-efficacies as part of teachers' decision-making processes about which instructional strategies to implement is not clearly understood (Allinder, 1994; Bandura & Wood, 1989; Klassen & Tze, 2014). And, there has been no exploration of the extent to which teachers who have been identified as effective may vary with respect to their beliefs and practices. Teachers, even effective teachers, are not monolithic. Knowing more about the breadth of ways to be effective would add nuance to our understanding of what it means to be effective. For these reasons, we explored this research question: How do effective teachers vary with respect to their instructional beliefs, mathematics self-efficacy, mathematics teaching self-efficacy, and classroom-observed uses of the Standards for Mathematical Practice (SMP; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010)?

Theoretical Perspectives

Ernest (1994) posited that an individual's learning cannot be separated from their social environment. In conversation with others and themselves (Ernest, 1994), students negotiate ideas in a way that expands their prior knowledge. Marchitello & Wilhelm (2014) reported that the mathematics component of the Common Core was based on cognitive principles and, in particular, the authors assumed that students learn by building on prior knowledge and working collaboratively with peers. In this way, the SMP are aligned with social constructivist perspectives on learning. Within a constructivist learning environment, teachers and students share knowledge, responsibility, and authority (Chung, 1991). A teacher's role is as a facilitator, rather than an expert, who creates and sustains a collaborative learning environment (Tam, 2000). Thus, social constructivism undergirds the student-centered teaching practices that we examined.

The theory of self-efficacy (Bandura, 1977), also aligned with social constructivism (Bandura, 2001), is a lens through which we interpreted how teachers' self-efficacies mediated their instructional practices and beliefs. A firm understanding of self-efficacy, not only as an affective factor of teachers, but also as a theory, aided in the interpretation of teachers' instructional beliefs and practices. According to Bandura (1997), an individual has the capacity to control their actions based on how confidently they exercise that control. Bandura (1997) described a phenomenon in which individuals make choices based on how successfully they believe they can perform a particular task. Thus, mathematics self-efficacy may influence the decision to use challenging tasks and mathematics teaching self-efficacy may influence a teacher's decision to engage in student-centered teaching practices that require coping with uncertainty about how students will respond when given the freedom to explore.

Literature Review

Those who have examined how the interplay between beliefs and practices can affect student learning (e.g., Allinder, 1994; Polly et al., 2013) have found that teachers' beliefs about teaching practices may be tied to other affective factors, including self-efficacy beliefs.

Self-Efficacy Beliefs of Teachers

Self-efficacy beliefs have the potential to influence teachers' choices about which instructional practices to use (Bates et al., 2013; Perera & John, 2020; Swars, 2005). In addition, self-efficacy beliefs potentially mediate teachers' instructional beliefs as they choose whether to use student- or teacher-centered practices (Wilkins, 2008). Such self-efficacy beliefs are based on a person's interpretation of past experiences (e.g., Bandura, 1986, 1997; Tschannen-Moran et al., 1998; Wilkins, 2008), are constructed from four sources (i.e., verbal persuasion, vicarious experiences, physiological arousal, and mastery experiences; see Bandura, 1986), and are situation specific (Pajares & Miller, 1994). People with high self-efficacy beliefs are more likely to: (a) attempt tasks they might find challenging; (b) expend more initial effort on successful completion of the task; and (c) persist when a task becomes difficult (Bandura, 1986).

Mathematics self-efficacy. Kahle (2008) defined mathematics self-efficacy—a situation-specific self-efficacy—as one's beliefs in their capability to successfully carry out a mathematical task. Using the Mathematics Confidence Scale (Dowling, 1978), Pajares and Miller (1984) found significant and positive correlations among gender, high school mathematics experience, college mathematics experience, and mathematics self-efficacy. Thus, they concluded, mathematics self-efficacy affects mathematics performance directly rather than through mediated variables (e.g., gender, prior experiences).

Mathematics teaching self-efficacy. Another type of self-efficacy, mathematics teaching self-efficacy, has been defined as the belief in one's capability to teach others mathematics (Enochs et al., 2000; Kahle, 2008; Swars, 2005). Teaching self-efficacy influences choices teachers make in teaching practices, curriculum delivery, and task choice (e.g., Gulistan et al., 2017). More specifically, high teaching self-efficacy has been linked to teachers' (a) openness to student responses and inquiry, including student engagement (e.g., Toropova et al., 2019); (b) positive responses to academic coaching (Ross, 1992); (c) high-quality instruction (Perera & John, 2020; Wilkins, 2008); and (d) positive effects on student attitudes toward mathematics (Nurlu, 2015).

Teaching Practices in Mathematics

Skemp (1978) distinguished between two types of student learning goals for teachers. Relational understanding is "knowing both what to do and why" (p. 9) whereas instrumental understanding refers to using "rules without reason" (p. 9). Student-centered practices align with relational thinking because student-centered practices are specifically designed to help students construct knowledge by having them grapple with both what procedures to use and why they work. Students who have experienced mathematics through student-centered practices viewed mathematics as creative and flexible, leading them to see a future in which they would use mathematics in their everyday lives (Boaler, 1997)

Teacher-centered practices align with instrumental understanding because the teacher is viewed as one who possesses knowledge and who should impart that knowledge to passive learners. These learners, in turn, are responsible for emulating the procedures that the teacher has demonstrated without necessarily reflecting on the reasons undergirding those procedures. For this study, because effectively implemented student-centered practices are aligned with social constructivist assumptions about learning and regarded as appropriate for the teaching and learning of mathematics (National Council of Teachers of Mathematics [NCTM], 2000, 2014; NGACBP & CCSSO, 2010; NRC, 2001; Smith & Stein, 2018), we equated effective student-centered practices with effective mathematical teaching practices.

Teachers' Mathematics Instructional Beliefs

Wilkins (2008) posited that instructional beliefs are mediated by teachers' attitudes about mathematics and mathematics teaching, and those instructional beliefs, in turn, influence instructional practices. O'Hanlon et al. (2015) described teachers' instructional beliefs—views teachers hold about the best teaching practices—as a complex system that integrates teachers' beliefs concerning the nature of mathematics with their beliefs about the relationship between the teaching and learning of mathematics. Woolley et al. (2004) classified teachers' instructional beliefs as aligned with either constructivist or traditionalist (i.e., behaviorist) approaches to learning. Yet researchers have found that such instructional beliefs sometimes conflict with enacted teaching practices (Peterson et al., 1989; Raymond, 1997; Yurekli et al., 2020).

We examined the interplay of these factors: teachers' self-efficacy beliefs, teachers' mathematics instructional beliefs, and teachers' implementations of the SMP.

Methods

Participants

For this study, we focused on two mathematics teachers who were identified as effective by both mathematics teacher educators and their school principals. Kathy and Frances (pseudonyms) both taught at the same suburban school in the midwestern United States. Kathy taught kindergarten and Frances taught Grade 5/6 mathematics.

Data Collection

To determine levels of self-efficacies, instructional beliefs, and use of effective teaching practices, we used the instruments and procedures described in this section.

Self-efficacy beliefs. To measure both mathematics and mathematics teaching self-efficacy, we used Kahle's (2008) Mathematics Teaching and Mathematics Self-Efficacy (MTMSE) survey. The MTMSE has a measured reliability of $\alpha = .942$ and produced positive results for both face and content validity. The MTMSE originally was intended for teachers who taught grade 3–6, but Kahle (personal communication, June 8, 2020) verified that the survey is suitable for elementary and middle school teachers.

We assessed mathematics teaching self-efficacy beliefs with Part 2 and Part 4 of the MTMSE survey. In Part 2 of the MTMSE teachers rated, using a Likert scale of 1 (strongly disagree) to 6 (strongly agree), their agreement with statements such as "I will generally teach mathematics ineffectively" or "I will typically be able to answer students' questions." In Part 4 of the survey, teachers rated (on a scale of 1–6, from low to high) how confident they felt about teaching specific content (e.g., fractions, decimals, shapes).

We measured mathematics self-efficacy using Part 1 and Part 3 of the MTMSE survey. These components focus on teachers' beliefs about their capabilities regarding a variety of mathematical tasks. In the survey, teachers are asked to rate their level of confidence, on a scale from 1–6, in their own ability to complete certain tasks, though they are not required to complete the tasks. One of the tasks was: "On a map, 7/8 inch represents 200 miles. How far apart are two towns whose distance apart on the map is 3 1/2 inches?"

During pre-lesson interviews, the first author asked teachers to rate their confidence about the content and the teaching of the content.

Instructional beliefs. We used an instructional beliefs survey that incorporated items from O'Hanlon et al.'s (2015) Teaching and Learning Mathematics Beliefs survey to determine whether teachers' instructional beliefs were primarily aligned with student-centered or teacher-centered instructional practices. The beliefs survey is applicable to all the teachers of Grades K–12 as the items were designed to elicit teachers' instructional beliefs about the content they teach at their own grade level. In developing the survey, O'Hanlon et al. designed items to reflect NCTM's (2000) process standards recommendations, which are aligned with the values described in the CCSS-M standards (NGA & CCSSO, 2010). O'Hanlon et al.'s items represented three perspectives: personal learning, student learning, and teaching. For this study, we used items that focused on teachers' perspectives about student learning and teaching.

Use of practice standards. To assess teachers' use of practice standards, the first author observed 5 or 6 lessons for each teacher and collected all lesson plans and materials used in those lessons. The first author also conducted short pre- and post-observation interviews to discuss the lesson plan.

To obtain a detailed description of each teacher, the first author conducted five in-person observations with the kindergarten teacher and six with the 5th and 6th grade mathematics teacher during the spring of 2021. The teacher selected the lessons for each observation. We video and audio recorded each observation. To assess the frequency with which teachers engaged students in the Standards of Mathematical Practice (SMP; NGACBP & CCSSO, 2010), we used the Mathematics Classroom Observation Protocol for Practices (MCOP^2; Gleason et al., 2015) during the observation and subsequent viewing of video data. The MCOP^2 was developed to examine aspects of teaching facilitation (TF) and student engagement (SE) for the purpose of teaching mathematics for conceptual understanding and was grounded in the Instruction as Interaction framework (Cohen et al., 2003). Gleason et al. (2017) found the MCOP^2 to have an interrater reliability for both TF ($IRR = 0.616$) and SE ($IRR = 0.669$) subscales. Teacher facilitation refers to the role of the teacher to provide lesson structure and guidance through problem solving and mathematical discourse. Student engagement refers to students fulfilling their roles as active learners within the classroom environment (Gleason et al., 2015). The MCOP^2 includes both TF and SE because an observer cannot assess whether teacher actions are effective without observing whether those teacher actions produce meaningful engagement from students. The MCOP^2 observation guide aligns with teacher implementation of the SMP. In Table 1, from Gleason et al.'s (2017) publication, we note the alignment between the MCOP^2 and the SMP. Gleason et al. (2015) recommended that multiple observations occur over a period of time so that a comprehensive view of teachers' practices can be established.

	Standards for Mathematical Practice							
MCOP^2 item	Make sense of problems and persevere in solving them	Reason abstractly and quantitatively	Construct viable arguments and critique the reasoning of others	Model with mathematics	Use appropriate tools strategically	Attend to precision	Look for and make use of structure	Look for and express regularity in repeated reasoning
1	X						X	X
2	X				X			
3	X							
4	X		X					X
5	X	X	X		X			
6							X	X
7		X		X				
8							X	X
9	X							
10			X			X		
11	X							
12			X					
13			X					
14	X							
15			X					
16	X							

Table 1. Relationship between the MCOP^2 and the SMP (NGACBP & CCSSO, 2010)

Data Analysis

We did not perform statistical tests on survey data because of the small sample size. We used the quantitative tools only to estimate levels for each teacher's self-efficacies as part of a description of that teacher's beliefs and practices.

Self-efficacy beliefs. We used analysis results from the MTMSE survey for the purpose of establishing whether teachers' mathematics and mathematics teaching self-efficacy beliefs were high or low. Survey questions were worded both positively and negatively. We reordered ratings so that all statements were on a consistent scale with high scores representing more self-efficacious views. Unlike Kahle (2008), who based analysis on the sums of survey ratings, we tallied the responses for each question so that we could consider each type of self-efficacy separately. For mathematics self-efficacy, we classified teachers as having a high mathematics self-efficacy if they provided at least 17 responses (out of 31 questions) that were fours, fives, or sixes on a Likert scale. For mathematics teaching self-efficacy, we classified teachers as having a high mathematics teaching self-efficacy if they recorded at least 15 responses (out of 26 questions) that were fours, fives, or sixes.

Instructional beliefs. The survey questions were constructed both as positively and negatively worded statements. For the analysis, we again reordered ratings so that all statements were on a consistent scale with high scores representing more student-centered views. Rather than looking at the mean rating of survey items, we examined the frequencies of each level, which was more appropriate for our purpose. Following O'Hanlon et al. (2015), we classified teachers whose responses were primarily in the top half of the Likert scale (scores of 4, 5, or 6) as having instructional beliefs that aligned with more student-centered practices. In contrast, teachers for whom a majority of their responses were in the bottom half of the Likert scale were classified as having primarily teacher-centered instructional beliefs (O'Hanlon et al., 2015).

Use of practice standards. We classified teachers' use of the SMP by their scores in the relevant section of the MCOP^2. The MCOP^2 rates teachers' performance on a scale of 0–3 for each of the 16 items, with a 0 indicating the non-use or incorrect use of the item description and a 3 indicating the use of the item description. For each teacher, we reported the median scores across all observations for individual component of the scale (see Table 1). We used a median because the data were ordinal. We focused our analysis on the frequency of attaining medians at the top two rating levels (i.e., 2 and 3) within each of the SMP because those levels indicate at least an intentional effort to used student-centered practice.

Results

The purpose of this research was to determine the relationships among effective teachers' instructional beliefs, mathematics and mathematics teaching self-efficacies, and their use of the SMP. Using the collapsed rating method described above, both teachers scored similarly with respect to their instructional and self-efficacy beliefs. However, there were differences in their MCOP^2 results. In the following sections we discuss the specific results to each of the surveys and the MCOP^2 observations.

Self-Efficacy and Instructional Beliefs

We found that both teachers scored identically on the sections measuring mathematics teaching self-efficacy, meaning that both teachers felt as though they were capable of teaching mathematics in a student-centered approach. The teachers' mathematics self-efficacy survey results showed a slight difference between the two participants, Kathy scored higher than Frances (see Figure 1). However, both teachers achieved scores that we interpret as indicating a high mathematics self-efficacy, because both scores were above the cutoff score of 17.

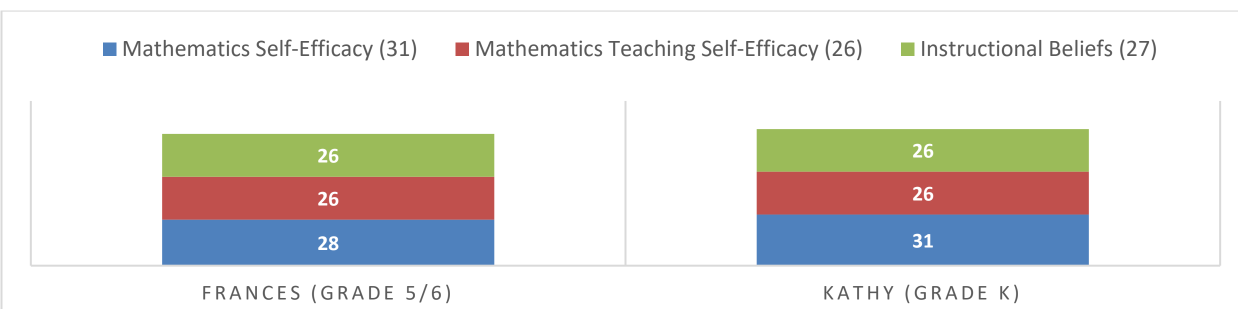


Figure 1. Beliefs Survey Results

Both teachers seemed to score similarly on instructional beliefs indicating that they both believed in a student-centered approach to teaching. Of the 26 items, each teacher selected a single 3-level rating, and their remaining 25 ratings were within the 4–6 range. However, when we examined the distribution of ratings, Frances' rated all 25 statements with a 4, whereas Kathy's ratings were all 5s and 6s (see Figure 2). Although both teachers had student-centered instructional beliefs, Kathy's beliefs may be more robustly student-centered than Frances' beliefs. We found further evidence of a possible disparity during the pre- and post-observation interviews, as Kathy spoke more frequently—without prompting—about her student-centered instructional beliefs. Frances often required prompting to discuss her instructional beliefs.

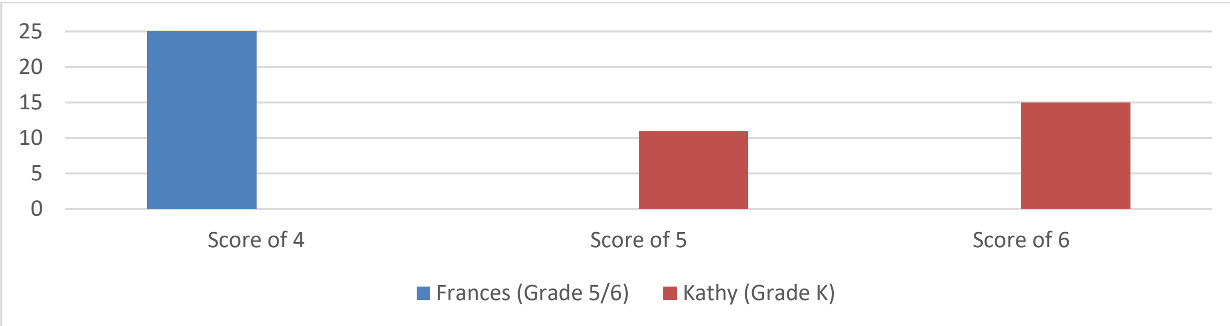


Figure 2. Specific Score Categories for Instructional Beliefs

Use of Standards of Mathematical Practice

Using the MCOP^2 observation tool, Frances received more medians of 3 than Kathy, and Frances' medians were always equal to or greater than Kathy's (see Table 2). For teacher facilitation items across all observations, Frances' median score was 3 and Kathy's was 2. For student facilitation items, Frances' median score was 2.5, whereas Kathy's median score was 2.

Table 2. Medians of Scores on the MCOP2 Across All Items and Observations

	Make sense of problems and persevere in solving them		Reason abstractly and quantitatively		Construct viable arguments and critique the reasoning of others		Model with mathematics		Use appropriate tools strategically		Attend to precision		Look for and make use of structure		Look for and express regularity in repeated reasoning	
MCOP^2 Items	K	5/6	K	5/6	K	5/6	K	5/6	K	5/6	K	5/6	K	5/6	K	5/6
1	3	3											3	3	3	3
2	1	2							1	2						
3	3	3														
4	2	3			2	3									2	3
5	2	2	2	2	2	2			2	2						
6													2	3	2	3
7			2	2			2	2								
8													3	3	3	3
9	1	3														
10					2	3					2	3				
11	2	2														
12					2	2										
13					2	3										
14	2	3														
15					2	3										
16	1	3														

Each teacher consistently facilitated the SMP in their classroom. Although Kathy received a median of 2 more often than a 3 on the MCOP^2, she regularly provided her students with opportunities to look for and make use of structure and look for and express regularity in repeated reasoning. Frances—having a median of 3 more often—provided opportunities for her students to make sense of problems and persevere in solving them, attend to precision, look for and make sense of structure, and look for and express regularity in repeated reasoning.

Relationships Among Beliefs and Practices

Although teachers had only minor variation in the robustness of their beliefs, based on their survey responses, their MCOP[2] showed somewhat greater variation in their use of the SMP in their lessons. For example, Kathy had more robust student-centered beliefs, as measured by the beliefs survey (see Figure 2); but then Kathy employed the SMP less consistently throughout the observations. Figure 3, showing the medians of level 2 and level 3 implementations of the SMP, shows that Kathy had a greater median of level 2 implementations, although her beliefs were more robustly student centered. In contrast, although Frances' student-centered beliefs were less robust, she scored a 3 across more categories.

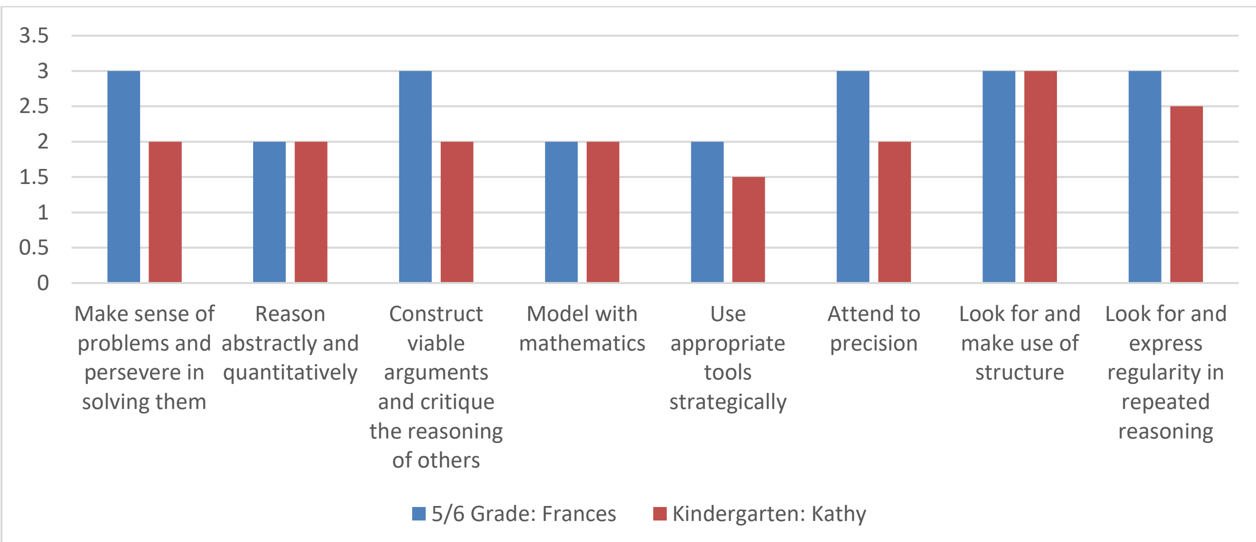


Figure 3. Median SMP Scores Across All Observations

Discussion

Echoing prior research, we noted a connection between the teachers' beliefs and their use of instructional practices (Hadley & Dorward, 2011; Peterson et al., 1989). However, there may be some differences with respect to which practices teachers use and how often they use those practices. We conjecture that our observed differences between the teachers were not simply due to grade-level differences but instead were due to the complexity of teaching and teacher beliefs. Building on prior work (e.g., Yurekli et al., 2020) by using classroom observations, we conclude that teachers with more-robust student-centered beliefs may not enact those beliefs in practice.

This study was limited by the number of participants and the pandemic context in which we observed them[1]. More research, using in-depth interviews, is needed to explore the extent to which beliefs, experiences, and other factors influence teachers' use of the SMP. Even so, this study adds to the growing body of research on the beliefs of teachers.

1 Because of the pandemic, we had limited access teachers who were teaching in person and there were many constraints to the classroom environment. Thus, the practices we observed may not have reflected teachers' typical or desired approaches.

References

Allinder, R. M. (1994). The relationship between efficacy and the instructional practices of special education teachers and consultants. *Teacher Education and Special Education, 17*(2), 86–95. https://doi.org/10.1177/088840649401700203

Alsup, J. (2004). A comparison of constructivist and traditional instruction in mathematics. *Educational Research Quarterly, 28*(4), 3–17. https://eric.ed.gov/?id=EJ718122

Bandura, A. (1977). Self-efficacy: Toward a unifying theory of behavioral change. *Psychological Review*, 84(2), 191–215. https://doi.apa.org/doi/10.1037/0033-295X.84.2.191

Bandura, A. (1986). *Social foundations of thought and action.* Prentice-Hall.

Bandura, A. (1997). *Self-efficacy: The exercise of control.* Freeman.

Bandura, A. (2001). Social cognitive theory: An agentic perspective. *Annual Review of Psychology, 51*(1), 1–26. https://www.annualreviews.org/doi/full/10.1146/annurev.psych.52.1.1#_i4

Bandura, A., & Wood, R. (1989). Effect of perceived controllability and performance standards on self-regulation of complex decision making. *Journal of Personality and Social Psychology, 56*(5), 805–814. https://doi.org/10.1037/0022-3514.56.5.805

Bates, A. B., Latham, N. I., & Kim, J. (2013, August). Do I have to teach math? Early childhood pre-service teachers' fears of teaching mathematics. *Issues in the Undergraduate Mathematics Preparation of School Teachers, 5*, 1–10. https://files.eric.ed.gov/fulltext/EJ1061105.pdf

Boaler, J. (1997). When even when the winners are losers: Evaluating the experiences of "top set" students. *Journal of Curriculum Studies, 29*(2), 165–182. https://doi.org/10.1080/002202797184116

Chung, J. (1991). Collaborative learning strategies: The design of instructional environments for the emerging new school. *Educational Technology,* 31(12), 15–22. https://www.jstor.org/stable/44427555

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119–142. https://doi.org/10.3102/01623737025002119

Dowling, D. M. (1978). The development of a mathematics confidence scale and its application in the study of confidence in women college students [Unpublished doctoral dissertation]. Ohio State University.

Enochs, L. G., Smith, P. L., & Huinker, D. (2000). Establishing factorial validity of the Mathematics Teaching Efficacy Beliefs Instrument. *School Science and Mathematics, 100*(4), 194–202. https://doi.org/10.1111/j.1949-8594.2000.tb17256.x

Ernest, P. (1994). Social constructivism and the psychology of mathematics education. In P. Ernest (Ed.) *Constructing mathematical knowledge: Epistemology and mathematical education* (pp. 62–72). Routledge. https://doi.org/10.4324/9780203454206

Gleason, J., Livers, S., & Zelkowski, J. (2015). *Mathematics Classroom Observation Protocol for Practices (MCOP^2): Descriptors manual.* University of Alabama. http://jgleason.people.ua.edu/mcop2.html

Gleason, J., Livers, S., & Zelkowski, J. (2017). Mathematics Classroom Observation Protocol for Practices (MCOP^2): A validation study. *Investigations in Mathematics Learning, 9*(3), 1–20. http://doi.org/10.1080/19477503.2017.1308697

Gulistan, M., Hussain, M. A., & Mushtaq, M. (2017). Relationship between mathematics teacher's self-efficacy and students' academic achievement at secondary level. *Bulletin of Education and Research, 39*(3), 171–182. http://pu.edu.pk/images/journal/ier/PDF-FILES/11_39_3_17.pdf

Hadley, K. M., & Dorward, J. (2011). The relationship among elementary teachers' mathematics anxiety, mathematics instructional practices, and student mathematics achievement. *Journal of Curriculum and Instruction, 5*(2), 27–44. https://web.archive.org/web/20161117222213id_/http://www.joci.ecu.edu:80/index.php/JoCI/article/viewFile/100/pdf

Jong, C., Pedulla, J. J., Reagan, E. M., Salomon-Fernandez, Y., & Cochran-Smith, M. (2010). Exploring the link between reformed teaching practices and pupil learning in elementary school mathematics. *School Science and Mathematics, 110*(6), 309–326. https://doi.org/10.1111/j.1949-8594.2010.00039.x

Kahle, D. K. B. (2008). How elementary school teachers' mathematical self-efficacy and mathematics teaching self-efficacy relate to conceptually and procedurally oriented teaching practices (Doctoral dissertation, Ohio State University). Retrieved from https://etd.ohiolink.edu/apexprod/rws_etd/send_file/send?accession

=osu1211122861&disposition=inline
Klassen, R. M., & Tze., V. M. C. (2014, June). Teachers' self-efficacy, personality, and teaching effectiveness: A meta-analysis. *Educational Research Review, 12*, 59–76. http://doi.org/10.1016/j.edurev.2014.06.001
Marchitello, M., & Wilhelm, M. (2014, September). The cognitive science behind the Common Core. Center for American Progress. https://www.americanprogress.org/article/the-cognitive-science-behind-the-common-core/
McLeod, D. B. (1987, July19–25). *Beliefs, attitudes, and emotions: Affective factors in mathematics learning* [Paper presentation]. Proceedings of the International Conference on the Psychology of Mathematics Education, Montreal Canada.
Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Author.
National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Author.
National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common Core State Standards for mathematics*. https://www.isbe.net/Documents/math-standards.pdf
National Research Council. (2001). *Adding it up: Helping children learn mathematics.* J. Kilpatrick, J. Swafford, and B. Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. National Academy Press.
Nurlu, Ö. (2015). Investigations of teachers' mathematics teaching self-efficacy. *International Electronic Journal of Elementary Education, 8*(1), 21–40. https://www.iejee.com/index.php/IEJEE/article/view/95
O'Hanlon, W. A., Barker, D. D., Langrall, C. W., Dossey, J. A., McCrone, S. M., and El-Zanati, S. A. (2015). Mathematics research experiences for preservice teachers: Investigating the impact of their beliefs. In J. M. Dewar & C. D. Bennett (Eds.), *Doing the scholarship of teaching and learning in mathematics* (pp. 171–179). Mathematical Association of America.
Opera, J. M., & Stonewater, J. (1987, July 19–25). *Mathematics teachers' belief systems and teaching styles: Influences on curriculum reform*. [Paper presentation]. Proceedings of the 11th International Conference on the Psychology of Mathematics Education, Montreal, Canada.
Pajares, F., & Miller, M. D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: A path analysis. *Journal of Educational Psychology, 86*(2), 193–203. https://doi.org/10.1037/0022-0663.86.2.193
Perera, H. N., & John, J. E. (2020). Teachers' self-efficacy beliefs for teaching math: Relations with teacher and student outcomes. *Contemporary Educational Psychology.* Advance online publication. https://doi.org/10.1016/j.cedpsych.2020.101842
Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teacher's pedagogical content beliefs in mathematics. *Cognition and Instruction, 6*(1), 1–40. https://doi.org/10.1207/s1532690xci0601_1
Polly, D., McGee, J. R., Wang, C., Lambert, R. G., Pugalee, D. K., & Johnson, S. (2013). The association between teachers' beliefs, enacted practices, and student learning in mathematics. *The Mathematics Educator, 22*(2), 11–30. https://files.eric.ed.gov/fulltext/EJ1013943.pdf
Raymond, A. M. (1997). Inconsistency between a beginning Elementary school teachers' mathematics beliefs and teaching practice. *Journal for Research in Mathematics Education, 28*(5), 550–576. https://doi.org/10.2307/749691
Ross, J. A. (1992). Teacher efficacy and the effects of coaching on student achievement. *Canadian Journal of Education, 17*(1), 51–65. https://doi.org/10.2307/1495395
Skemp, R. R. (1978). Relational understanding and instrumental understanding. *The Arithmetic Teacher, 26*(3), 9–15. http://math.coe.uga.edu/olive/EMAT3500f08/instrumental-relational.pdf
Smith, M. S., & Stein, M. K. (2018). *5 practices for orchestrating productive mathematics discussions* (2nd ed.). National Council of Teachers of Mathematics.
Swars, S. L. (2005). Examining perceptions of mathematics teaching effectiveness among Elementary preservice teachers with differing levels of mathematics teacher efficacy. *Journal of Instructional Psychology, 32*(2), 139–147. https://www.thefreelibrary.com/Examining+perceptions+of+mathematics+teaching+effectiveness+among...-a0134381062
Tam, M. (2000). Constructivism, instructional design, and technology: Implications for transforming distance learning. *Educational Technology & Society,* 3(2), 50–60. https://www.jstor.org/stable/jeductechsoci.3.2.50
Toropova, A., Johansson, S., & Myrberg, E. (2019). The role of teacher characteristics for student achievement in mathematics and student perceptions of instructional quality. *Education Inquiry*, 10(4), 275–299. https://doi.org/10.1080/20004508.2019.1591844

Tschannen-Moran, M., Hoy, A. W., & Hoy, W. K. (1998). Teacher efficacy: Its meaning and measure. *Review of Educational Research, 68*(2), 202–248. https://doi.org/10.3102/00346543068002202

Wilkins, J. L. M. (2008). The relationship among elementary teachers' content knowledge, attitudes, beliefs, and practices. *Journal of Mathematics Teacher Educator, 11*(2), 139–164. https://doi.org/10.1007/s10857-007-9068-2

Woolley, S. L., Benjamin, W-J. J., & Woolley, A. W. (2004). Construct validity of self-report measure of teacher beliefs related to constructivist and traditional approaches to teaching and learning. *Educational and Psychological Measurement, 64*(2), 319–331. https://doi.org/10.1177/0013164403261189

Yurekli, B., Stein, M. K., Correnti, R., & Kisa, Z. (2020). Teaching mathematics for conceptual understanding: Teachers' beliefs and practices and the role of constraints. *Journal for Research in Mathematics Education, 51*(2), 234–247. https://doi.org/10.5951/jresematheduc-2020-0021

A CLASSROOM OBSERVATION TOOL FOR EQUITY-ORIENTED TEACHING OF MATHEMATICAL MODELING IN THE ELEMENTARY GRADES

Erin Turner
University of Arizona
eturner@arizona.edu

Mary Alice Carlson
Montana State University
marycarlson5@montana.edu

Elizabeth Fulton
Montana State University
elizabeth.fulton@montana.edu

Jennifer Suh
George Mason University
jsuh@gmu.edu

Julia Aguirre
University of Washington Tacoma
jaguirre@uw.edu

Mathematical modeling can be a lever for equity in the elementary math classroom, as it empowers teachers to build on the knowledge and cultural resources that children bring to the classroom and empowers students to draw on their experiences and identities to inform their mathematical work. To better support this transformative synergy between mathematical modeling and equity-oriented practices, we need a tool to deepen our understanding of variations and potential trajectories of teacher practice. In this report, we briefly describe our process for developing an equity-oriented mathematical modeling classroom observation protocol. We then discuss two sample dimensions from our tool to illustrate our integrated attention to equity-focused and mathematical modeling-specific teaching practices.

Keywords: Modeling; Elementary School Education; Instructional Activities and Practices; Equity, Inclusion and Diversity

In elementary mathematics education, classroom observation tools can support teacher learning by making visible key features of instructional practice and outlining potential trajectories for developing new practices (Bostic et al., 2021; Boston et al., 2015). Observation tools also serve research by generating evidence of teachers' enactment of specific practices which can inform the design of professional learning experiences, or evaluate the effectiveness of interventions. While there are numerous observation tools focused on high-quality math teaching (e.g., MQI; Learning Mathematics for Teaching Project, 2011; M-SCAN, Walkowiak et al., 2014), tools focused specifically on teaching mathematical modeling are limited.

Mathematical modeling is an iterative process involving problem posing, and testing, validation, and revision of mathematical models to inform decision-making (Lesh & Zawojewski, 2007; Pollak, 2012). Observation tools that attend to modeling often include modeling as a single dimension (e.g., Bostic, et al., 2019; Gleason et al., 2017). Although researchers have begun to conceptualize modeling-specific protocols (e.g., Hwang, 2020), validating tools for use in elementary grades is still needed. Additionally, while there is growing recognition that mathematical modeling can be a lever for equity (Aguirre et al., 2019; Anhalt et al., 2018; Carlson et al., 2016; Cirillo et al., 2016; Suh et al., 2018; Turner et al., 2022), tools that integrate an explicit focus on modeling and equity-oriented practices are lacking. Equity-oriented tools often lack a content focus (e.g., CLASS, Pianta & Hamre, 2009; Salazar, 2018), or focus generally on standards-based mathematics instruction, but not modeling (e.g. Nava, et al., 2019).

The lack of a validated classroom observation protocol that attends, in substantive ways, to both mathematical modeling and equity-oriented, culturally responsive instruction is problematic because of the synergy between mathematical modeling and advancing equity. To address this need, our project is building on prior exploratory work that involved adapting dimensions of an existing classroom observation protocol for elementary mathematics (the M-Scan, Walkowiak et al., 2014) to focus on culturally responsive mathematical modeling in grades 3-5. In this report, we describe our development process and discuss tool dimensions that illustrate our integrated attention to equity-focused and modeling-specific teaching practices. We end with a discussion of potential uses of the classroom observation tool, and next steps in our validation process.

Equity-Focused Teaching Practices for Mathematical Modeling

Teaching mathematical modeling is challenging because modeling includes processes like posing problems, making assumptions, and testing and revising models, that are not typical in mathematics classrooms (Niss, Blum, & Galbraith, 2007). Modeling tasks are more open and less predictable than those in most lessons (Cai et al., 2018) and require teachers to know about the real-world contexts that motivate modeling problems, potential mathematical solutions, and ways to maintain rigor and support for students as they develop, refine, and communicate their models (Suh et al., 2021; Turner et al., 2021). However, there is synergy between equity-focused mathematics teaching and mathematical modeling. Modeling empowers teachers to elicit and build on the knowledge and cultural resources that students bring to the classroom and empowers students to draw on their identities and experiences to inform mathematical work and take action (Aguirre et al., 2019; Turner & Bustillos, 2017). Classroom modeling also encourages diverse student contributions and gives teachers opportunities to "recognize and reward a broader range of mathematical abilities than those traditionally emphasized" (Lesh & Doerr, 2003, p. 23). To better support teachers' learning and practices related to the transformative integration of equity-focused teaching and mathematical modeling, we need a tool to sharpen our vision and deepen our understanding of variations and potential trajectories of teacher practice.

Overview of Classroom Observation Tool Development

To develop our classroom observation tool, we followed a multi-stage process for protocol validation outlined by Bostic and colleagues (2019). *Stage 1* involved a review of observation protocols for mathematical modeling and equitable teaching practices, and research on effective teaching practices for mathematical modeling, especially in grades K-2. *Stage 2* focused on synthesizing key outcomes from the literature, and using these ideas to draft an initial version of the tool. This version included 6 dimensions, each of which focused on equity-oriented teaching practices for a specific phase of the modeling process (e.g., making sense of the context and posing problems; identifying important quantities and making assumptions). Several of the dimensions were adapted from similar dimensions in existing, exploratory tools (Foote, Aguirre, Turner & Roth McDuffie, 2020. In *Stage 3*, we assembled an expert review panel consisting of 15 scholars with expertise in mathematical modeling, equitable teaching practices, and classroom observation tools. Scholars reviewed the draft protocol and provided feedback on its alignment with the construct (i.e., to what extent do the dimensions capture key features of mathematical modeling across the elementary grades?); the range of practice captured (i.e., do the indicators capture teacher moves and supports that are appropriate for K-5 students from diverse cultural, racial, linguistic, and geographic backgrounds?); and the clarity and usability of each dimension to describe and inform practice. In *Stage 4,* we revised the tool based on our review of the literature and feedback from our expert panel. *Stage 5,* our current stage, is focused on testing the revised observation tool with video of modeling lessons from different grade levels, and from teachers with various levels of experience teaching modeling.

Dimension 1: Connections to Students' Experiences and Cultural/Community Contexts

Dimension 1 (Table 1) captures teaching practices to support connections between students' experiences and the modeling process. Unlike some of the tool's dimensions, which focus on a particular modeling phase, (see Table 2), Dimension 1 should be present throughout the cycle. Its descriptors address the degree to which the teacher sustains connections to students' experiences and cultural and community contexts. Key Terms and Ideas, listed below the descriptors, clarify the nature of the connections to be made and teacher moves that could support students' work.

Table 1: Connections to Students' Experiences and Cultural/Community Contexts

Not Present (0)	Emerging (1)	Proficient (2)	Advanced (3)
The teacher does not engage the class in making connections between the modeling context and their lives.	In one phase of the modeling process, teachers support multiple students to make connections to their experiences (lives, communities, cultures) to inform their work.	In two or more phases of the modeling process, teachers support multiple students to make connections to their experiences to inform their work.	In two or more phases of the modeling process, teachers support multiple students to make connections to their experiences to inform their work.
Key Terms and Ideas **Connections** include references to experiences related to the context or scenario, or references to understandings about a specific context, setting, scenario or activity. Connections can include connections to students' or teachers' experiences outside of school, and students'/teachers' shared experiences as members of the school community. **Teacher Support** includes teacher moves such as asking students to recall experiences related to the context to help them identify key quantities; reminding students to use what they know about a situation to help them evaluate their solution; asking students to draw on their experiences with the situation to propose revisions to their models; reminding students of a shared experience that might inform their work. Teacher support must involve prompting, probes and/or follow up and go beyond a single question, hook or statement, and beyond a teacher-provided connection that does not invite student input or response.			

Dimension 1 makes practices that advance equity while teaching mathematical modeling explicit and offers a trajectory for supporting students to connect to their experiences and cultural and community contexts. At the *emerging* level, teachers support connections in one phase of the modeling cycle, likely through a single prompt or statement. As teachers grow in their ability to support classroom modeling, they sustain connections to students' experiences throughout the cycle - perhaps by prompting students to draw on their experiences to identify important quantities, propose revisions to their model, or interpret their solution. The tool also makes clear that teachers support multiple students to connect to their backgrounds and experiences, increasing the likelihood that contributions of students from diverse backgrounds are honored.

Dimension 2: Posing Mathematical Problems

Dimension 2 (Table 2) captures practices that support students learning to pose mathematical problems. Its descriptors focus on the degree to which the teacher involves students in posing the question that drives the modeling task. Unlike typical mathematics lessons, where students must understand the problems posed by the teacher or the textbook, when students model they learn to focus on specific features of real-world problems and then translate those features into mathematical questions. Thus, supporting students in learning to pose mathematical problems requires teaching practices that are unique to mathematical modeling.

Table 2: Posing Mathematical Problems

Not Present (0)	Emerging (1)	Proficient (2)	Advanced (3)
The teacher poses the modeling problem. The problem is pre-determined by the teacher. There is no evidence of actions by the teacher to connect students' questions or wonders about the context to the problem posed.	The teacher poses the modeling problem, but the teacher makes connections between the problem posed and the questions/ wonders/ observations/ experiences that students posed about the context.	The teacher invites students to shape some components of the problem posed. Includes: supporting students to recognize math questions, inviting students to connect the problem with their wonders or to share insights about the problem posed.	The teacher consistently invites and builds on student ideas to pose and refine the modeling problem. The teacher may facilitate the discussion, but student ideas drive problem-posing.
Key Terms and Ideas **Posing problems** refers to formulating a mathematical question that can be invested through the modeling process. Problems may be descriptive, predictive, or evaluative in nature. **Teacher Connects** student ideas or wonders to the problem posed (by the teacher). The teacher might say, "You all asked about ……., that is very similar to the problem we are going to work on today. Here is our problem…" Or, the teacher may invite students to share an observation or experience related to the problem posed (by the teacher).			

Dimension 2 captures equity-oriented instruction by focusing on teaching practices that empower students to shape the mathematical questions that are asked. Teacher development in this trajectory moves from *emergent* practices that are limited to teachers making connections between students' ideas a pre-determined problem to *advanced* practices that consistently invite student contributions and allow student ideas to drive problem posing. When teachers support students in this way, they extend authority for what is legitimized as important and worthwhile mathematical work to students. Not only does this disrupt power structures present in most classrooms, it supports students in developing a critical mathematical modeling competency.

Discussion and Implications

Observation tools that combine equity-oriented teaching and a content focus advance the field by offering specific descriptions of practice and trajectories for developing effective, equity-focused mathematics teachers. Our next step is to test our tool on a larger set of modeling lessons from diverse grade levels and classroom contexts, and to conduct psychometric analysis of data generated from use of the tool (i.e., reliability analyses). We anticipate that mathematics teacher educators can use our tool to design mathematical modeling professional development experiences that center equity, and that researchers can use the tool to study potential shifts or growth in teacher practice of mathematical modeling over time. Teachers can use selected dimensions of our tool to reflect on their practice and to consider strengths and areas for growth.

Acknowledgments

The work reported here is supported by a grant from the National Science Foundation (Grants 2010269, 2008997, 2010202, 2010178).

.

References

Aguirre, J.M., Anhalt, C., Cortez, R., Turner, E.E., & Simi-Muller, K., (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice. *Mathematics Teacher Educator*. 7(2) 7-26.

Anhalt, C., Cortez, R. & Been Bennett, A. (2018). The emergence of mathematical modeling competencies: An investigation of prospective secondary mathematics teachers. *Mathematical Thinking and Learning International Journal,* 20(3) 1-20. DOI10.1080/10986065.2018.1474532

Bostic, J., Lesseig, K., Sherman, M. & Boston, M. (2021). Classroom observation and mathematics education research. *Journal of Mathematics Teacher Education, 24*, 5-31.

Bostic, J., Matney, G. & Sondergeld, T. (2019). A validation process for observation protocols: Using the Revised SMPs Look-for Protocol as a lens on teachers' promotion of the standards. *Investigations in Mathematical Learning, 11*(1), 69-82.

Boston, M., Bostic, J., Lesseig, K. & Sherman, M. (2015). A comparison of mathematics classroom observation protocols. *Mathematics Teacher Educator, 3*(2), 154-175.

Cai, J., Cirillo, M., Pelesko, J. A., Borromeo Ferri, R., Borba, M., Geiger, V., . . . Kwon, O. N. (2014). Mathematical modeling in school education: Mathematical, cognitive, curricular, instructional and teacher education perspectives. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA* 36 (pp. 145–172). Vancouver, Canada: PME.

Carlson, M.A., Wickstrom, M.H., Burroughs, B. & Fulton, E. (2016). A Case for Mathematical Modeling in the Elementary School Classroom. *Annual Perspectives in Mathematics Education (APME) 2016: Mathematical Modeling and Modeling Mathematics*, ed., Christian R. Hirsch, pp. 121–29. Reston, Va.: National Council of Teachers of Mathematics.

Cirillo, M., Bartell, T. G., & Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education: Mathematical modeling and modeling with mathematics* (pp. 87-96). National Council of Teachers of Mathematics.

Foote, M.Q., Aguirre, J.M., Turner, E.E., & Roth McDuffie, A. (2020). M2C3-Scan: Mathematical Modeling with Cultural and Community Contexts Observation protocol. Unpublished tool.

Gleason, J., Livers, S. & Zelkowski, J. (2017). Mathematics Classroom Observation Protocol for Practices (MCOP2): A validation study. *Investigations in Mathematics Learning, 9*(2), 111-129.

Hwang, J. (2020). Instructional alignment observation protocol (IAOP) for implementing the CCSSM: Focus on the practice standard, "Model with mathematics". *Journal of Korean Society of Mathematics Education,* 23(3), 149-164.

Learning Mathematics for Teaching Project. (2011). Measuring the mathematical quality of instruction. *Journal of Mathematics Teacher Education, 14*, 25–47.

Lesh, R. A., & Doerr, H. M. (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*. New York: Routledge.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. K. Lester & National Council of Teachers of Mathematics (Eds.), *Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics* (pp. 763–804). Charlotte, NC: Information Age.

Nava, I., Park, J., Dockterman, D., Kawasaki, J., Schweig, J., Quartz, K. H. & Martinez, J. F. (2019). Measuring teacher quality of secondary mathematics and science residents: A classroom observation framework. *Journal of Teacher Education, 70*(2), 139-154.

Niss, M., Blum, W., & Galbraith, P. L. (2007) Introduction. In W. Blum, P. L. Galbraith, H. W. Henn, & M. Niss (Eds.) *Modelling and Applications in Mathematics Education: The 14th ICMI Study* (pp. 3-32). New York: Springer.

Pianta, R. C., & Hamre, B. K. (2009). Conceptualization, measurement, and improvement of classroom processes: Standardized observation can leverage capacity. *Educational Researcher, 38*(2), 109–119.

Pollak, H. (2012). Introduction. In H. Gould, D. R. Murray, & Sanfratello (Eds.), *Mathematical modeling handbook* (pp. viii–xi). Bedford, MA: Consortium for Mathematics and Its Applications.

Salazar, M.d.C. (2018). Interrogating teacher evaluation: Unveiling Whitenessa as the normative center and moving the margins. *Journal of Teacher Education, 69*(5), 463-476.

Suh, J., Matson, K., Seshaiyer, P., Jamieson, S., & Tate, H. (2021). Mathematical Modeling as a Catalyst for Equitable Mathematics Instruction: Preparing Teachers and Young Learners with 21st Century Skills. Mathematics, 9(2), 162. MDPI AG. Retrieved from http://dx.doi.org/10.3390/math9020162

Suh et al (2018): Every penny counts: Promoting community engagement to engage students in mathematical modeling. In Goffney & Gutiérrez (Eds) *Rehumanizing mathematics for Black, Indigenous, and Latinx students*.

pp. 63-76. Reston, VA: NCTM
Turner, E. E., Bennett, A. B., Granillo, M., Ponnuru, N., Roth Mcduffie, A., Foote, M. Q., Aguirre, J.M. & McVicar, E. (2022). Authenticity of elementary teacher designed and implemented mathematical modeling tasks. *Mathematical Thinking and Learning*, 1-24.
Turner, E., Roth McDuffie, A., Aguirre, J., Foote, M. Q., Chapelle, C., Bennett, A., Granillo, M. & Ponnuru, N. (2021). Upcycling Plastic Bags to Make Jump Ropes: Elementary students leverage experiences and funds of knowledge as they engage in a relevant, community-oriented mathematical modeling task (pp. 235-266). In J. Suh, M. Wickstram, L. English (Eds). *Exploring Mathematical Modeling with Young Learners*. The Netherlands: Springer. DOI:10.1007/978-3-030-63900-6_11
Turner, E. & Bustillos, L. (2017). ¿Qué observamos aquí? ¿Qué preguntas tienen? Problem posing in Ms. Bustillos's Second-Grade Bilingual Classroom. (pp. 45-62.) In S. Celedón-Pattichis, D. White and M. Civil (Eds.) *Access and Equity: Promoting High Quality Mathematics in Grades PreK-2*! Reston, VA: National Council of Teachers of Mathematics.
Walkowiak, T., Berry, R., Meyer, J., Rimm-Kaufman, S., & Ottmar, E. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*, 109–128.

SCORING WITH CLASSROOM OBSERVATIONAL RUBRICS: A LONGITUDINAL EXAMINATION OF RATERS' RESPONSES AND PERSPECTIVES

Temple A. Walkowiak
North Carolina State University
tawalkow@ncsu.edu

Jonee Wilson
North Carolina State University
jwilson9@ncsu.edu

Elizabeth Adams
Southern Methodist University
eladams@smu.edu

Anne Garrison Wilhelm
Southern Methodist University
awilhelm@smu.edu

This study examines the utilization of cognitive interviews longitudinally over a one-year period to collectively trace raters' response processes as they interpreted and scored with observational rubrics designed to measure teaching practices that promote equity and access in elementary and middle school mathematics classrooms. We draw on four rounds of cognitive interviews (totaling 14 interviews) that involved four raters at purposeful time points spread over the year. Findings reported in this study focus on raters' responses about one rubric, positioning students as competent. The findings point to the complexities of utilizing observational rubrics and the need to track response processes longitudinally at multiple time points during data collection in order to attend to rater calibration and the reliability and validity of resulting rubric scores.

Keywords: instructional activities and practices; research methods; measurement

In the field of education, researchers and evaluators regularly develop rubrics to assess or measure a particular construct with the intent for trained raters to apply the rubrics and assign reliable scores so that valid inferences can be drawn from the resulting data. One such example is classroom observational rubrics designed to measure mathematics teaching practices (e.g., Boston, 2012; Walkowiak et al., 2014). However, we know that using classroom observational rubrics is a complex and intense endeavor, due to the many nuances that exist in classroom interactions and instruction. Therefore, attending to how raters interpret the rubrics and apply their interpretations to assign scores is a critical type of validity evidence. That is, the response processes of raters can be utilized to evaluate if raters interpret the rubrics and apply scores as intended. In this study, we attend to raters' response processes over time using rubrics designed to measure teaching practices that promote equity and access in mathematics classrooms. The significance of this study is twofold: (1) it illustrates the complexity of one component of an interpretation/use argument (IUA) (Kane, 2016), but with longitudinal data; and (2) it draws the field's attention to the importance of iteratively examining raters' interpretations of rubric language and levels. It is critical that classroom observational rubrics generate reliable scores from which we can make valid inferences.

Background

The EAR-MI (Equity and Access Rubrics for Mathematics Instruction) is a set of classroom observation rubrics designed to focus on specific practices that support more equitable participation and access in mathematics classrooms. The EAR-MI began as a theoretically-derived and empirically validated set of instructional practices (Wilson et al., 2019). For this study, we focus on raters' interpretation of one of the instructional practices and its accompanying rubric, positioning students as competent.

When teachers position students as competent, they explicitly and publicly value, identify, and acknowledge the brilliance of their students, framing their actions and statements as intellectually valuable (Bartell, 2011). The positioning rubric emphasizes the extent to which teachers specify what students do that is productive *and* the extent to which they provide rationales as to why what was done was considered productive. Figure 1 illustrates how positioning may be elevated in a classroom, corresponding to the levels on the rubric.

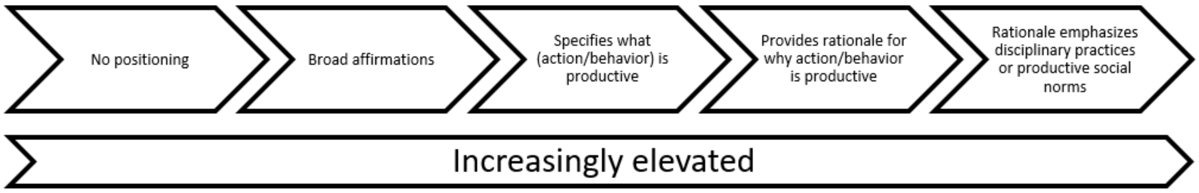


Figure 1: Positioning students as competent, increasingly elevated.

Our overarching goal in further developing the EAR-MI is to utilize Kane's (2016) "argument-based approach to validity" to systematically build an argument for validity. Kane describes the process of developing an interpretation/use argument (IUA) with claims to be evaluated with evidence. One such source of evidence is cognitive interviews to evaluate if the scoring is being applied accurately and consistently (Groves et al., 2009; Willis, 2005) and more specifically, to illuminate factors that could be impacting the scoring process. Cognitive interviews provide insights into four components of the raters' response processes: comprehension, retrieval, estimation, and scoring (Tourangeau, Rips, & Rasinski, 2000). Comprehension is the rater's process of understanding terms within a rubric and their combined meaning to interpret the rubric as intended. Retrieval refers to the rater recalling the necessary information or evidence in the video-recorded lesson. Estimation is the process of the rater judging the quality of the retrieved information for completeness and integrating information from their notes and memories to estimate a score. Scoring is the process of mapping the estimated response to the rubric's scale. We utilized these four components in the context of the current study to longitudinally examine raters' interpretations of the positioning rubric.

Methods

Data Collection

Participants in this study were four raters on the scoring team for the EAR-MI. Each rater completed three or four cognitive interviews, conducted one-on-one with an interviewer. Before each cognitive interview, each rater watched a video-recorded mathematics lesson (the same lesson for all participants at the given time point). During the cognitive interview, the rater talked aloud about each rubric in light of the mathematics lesson, providing justifications and evidence for the scores they assigned.

Cognitive interviews occurred at four purposeful time points (TPs) across the course of one year. Four raters participated in the *first* interview, which took place in January 2021 (TP1), after a live, four-day rubric training, before raters entered a phase called training reliability. During the training reliability phase, raters scored lessons and subsequently met with an expert rater to discuss their scores. Two raters participated in the *second* interview, which occurred in April 2021 (TP2), mid-way through the training reliability phase. At this point, raters had scored 11 lessons and participated in an additional live training, what we refer to as "construct jams". During the construct jams, raters refined and adjusted their understanding of the rubric constructs. Four raters participated in the *third* interview, which occurred in June 2021 (TP3), after the training reliability phase and before raters scored lessons for a generalizability study. Following the training reliability phase, raters' scores indicated acceptable agreement with expert scores; 83% of raters' scores matched with expert scores exactly on the last five videos scored. At this point, raters had scored 10 additional lessons post- construct jams. Four raters participated in the *fourth* interview, which occurred in November 2021 (TP4), after the generalizability study and before raters started scoring lessons as a part of the larger sample of lessons to be included in continued examination of the validity of rubric scores. All interviews were transcribed. Having cognitive interview data at four time points allows for the tracing of interpretations of the rubric over time. It is important to note that in this study, we center raters' thoughts and perspectives in an effort to understand what is working and what needs improvement in scoring procedures as we aim to produce reliable rubric scores.

Data Analysis

We focused on raters' collective interpretation of the focal rubric over the course of the four TPs. We chose the positioning rubric for two reasons. First, we are able to look across time at raters' interpretations of the rubric without changes in rubric language. This is possible because the generalizability study did not indicate that rubric changes were necessary. Second, the positioning rubric was functioning fairly well. For example, exact-match agreement rates between rater and expert scores exceeded 80%. We wanted to dive deeper with this rubric to understand the nuances that supported or hindered use from a rater perspective, particularly when everything looks like it is going well based on agreement statistics.

We reduced the 14 interview transcripts to only include when raters talked about scores for the positioning rubric. We conducted a line-by-line qualitative analysis of the raters' responses. After an initial read, we read through their responses using an open coding approach, tagging codes to the four components of the response process framework (Tourangeau et al., 2000). Codes were collapsed or fine-tuned based on a third reading of the data. We then identified themes for each component of the response process framework, focusing on the group of raters collectively, not an individual rater's development over time.

Findings

Comprehension refers to raters' interpretations of the rubric's terms and their combined meaning. Across the first three time points, raters consistently grappled with the definition of a "rationale" and identifying when a rationale was present. As displayed in Figure 1, positioning is elevated when a teacher includes a rationale for why the student's action or idea is considered productive. In some instances, raters were "on the fence of whether or not [a teacher's comment] was a rationale" (TP3). Sometimes, the raters pondered the level of clarity and/or explicitness of the rationale (e.g., "I think it could be debatable because it's not 100% explicit, but this [teacher action] demonstrates the ability to provide rationales, but they are not necessarily clear" [TP3]). At TP4, the raters were much more decisive and did not grapple with the term as evidenced by "I'm going to stick with [my score] because I don't see a rationale."

Retrieval is the process of recalling the necessary information in relation to the rubric. Raters scored the lesson immediately after watching the lesson. They also utilized structured notetaking and applied "soft scoring" approximately every 20 minutes. "Soft scoring" means pausing and recording a score that represents what has happened so far in the lesson. The processes of structured notetaking and soft scoring were implemented between TP2 and TP3. Raters described how soft scoring "helps me calibrate and make sure that I know what I have strong evidence for" (TP3). Between TP1 and TP2, the positioning rubric changed from a rubric based on preponderance of evidence to highest evidence because the intent is to acknowledge a teacher's potential for implementing the equitable teaching practice of positioning students as competent. A direct implication of this change was evident in raters' ability to retrieve information for scoring efficiently, with less emphasis on finding every instance of positioning within a lesson: "I don't really have to search too far [in my notes] because I felt the earlier instance was stronger" (TP3).

Estimation occurs when the rater estimates a score based on notes and recollection of the video. With the exception of TP4, raters tended to ponder the distinctions between the levels on the rubrics. One rater described "the distinction between a two and a three is still kind of blurry….it would be hard for me to teach someone else what counts as a three. I'm not sure if I could articulate it clearly" (TP3). Here, the rater grappled with the levels on the rubric; this grappling corresponds to the uncertainty described earlier about the term, "rationale".

Scoring is the actual application of a score on the rubric. Across the four TPs, the raters became increasingly more confident, particularly at TP4 ("I'm always confident now"), when they spent less time grappling with the rubric and its terms, did not verbalize or demonstrate indecisiveness, and moved more quickly to assigning the score.

Discussion

While our work is situated within the context of the EAR-MI, we present our discussion as three broader recommendations for researchers, both for those who are designing rubrics to measure a target construct and for those who utilize rubrics. Both groups should critically and carefully consider raters' interpretations and scoring applications. First, the cognitive interviews shed light on reasons for our raters' misinterpretations. If we did not have the cognitive interview data, our awareness of issues with the term, "rationale", would be less robust. Implementing cognitive interviews, whether in the context of the development of new rubrics or in the application of existing rubrics, is critical for understanding the nuances of raters' interpretations. Cognitive interviews also center the voices of raters and their perspectives about the important work they are doing. Second, we implemented procedures that resulted in better and more efficient retrieval of relevant information for scoring. The structured notetaking and soft scoring turned out to be fruitful as raters became more proficient with the rubric. When utilizing rubrics to score data, we recommend researchers attend to the process, not just the resulting scores. The systematic examination of these processes and their influence on raters seems to be an important component of collecting and evaluating evidence of validity. Finally, based on our read of the literature in mathematics education and beyond, cognitive interviews are typically implemented at one (or maybe two) TPs, and often only in the context of the initial development of a rubric. Our data suggests that iterative, longitudinal implementation of cognitive interviews is significant in improving raters' scoring procedures and thought processes. We were able to iteratively provide training to raters (as in the example of the "construct jams") throughout the year of these four rounds of cognitive interviews. In summary, the use of classroom observational rubrics to document and measure teaching practices is messy, but exciting work. Cognitive interviews at multiple time points serve as a mechanism for attending to the production of reliable scores such that valid inferences can be made when utilizing rubric data.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant #1908481. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of NSF.

References

Bartell, T. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. *Journal of Urban Mathematics Education, 4*(1), 50-74.

Boston, M. (2012). Assessing instructional quality in mathematics. *Elementary School Journal, 113*(1), 76-104.

Groves, R. M., Fowler, F. J., Coupter, M. P., Lepkowski, J. M., Singer, E., & Tourangeau, R. (2009). *Survey methodology* (2nd ed.). Hoboken, NJ: Wiley & Sons.

Kane, M. T. (2016). Explicating validity. *Assessment in Education: Principles, Policy & Practice, 23*(2), 198-211.

Tourangeau, R., Rips, L. J., & Rasinski, K. (2000). *The psychology of survey response.* Cambridge, United Kingdom: Cambridge University Press.

Walkowiak, T. A., Berry, R. Q., Meyer, J. P., Rimm-Kaufman, S. E., & McCracken, E. R. (2014). Introducing an observational measure of standards-based mathematics teaching practices: Evidence of validity and score reliability. *Educational Studies in Mathematics, 85*(1), 109-128.

Willis, G. B. (2004). *Cognitive interviewing: A tool for improving questionnaire design.* Thousand Oaks, CA: SAGE.

Wilson, J., Nazemi, M., Jackson, K., & Wilhelm, A. G. (2019). Investigating teaching in conceptually-oriented mathematics classrooms characterized by African American student success. *Journal for Research in Mathematics Education, 50*(4), 362-400. https://doi.org/10.5951/jresematheduc.50.4.0362

EXAMINING ELEMENTARY AND MIDDLE SCHOOL MATHEMATICS INSTRUCTION: ARE WE PROMOTING EQUITY AND ACCESS?

Anne Garrison Wilhelm
Southern Methodist University
awilhelm@smu.edu

Elizabeth Adams
Southern Methodist University
eladams@smu.edu

Jonee Wilson
North Carolina State University
Jwilson9@ncsu.edu

Temple Walkowiak
North Carolina State University
tawalkow@ncsu.edu

This descriptive study attended to the extent to which we see evidence of the presence of four practices that promote equity and access in 141 grades 3-8 mathematics lessons in the United States. We found that lessons generally showed evidence of some incorporation of the practices but often not at the highest level. Teachers in this sample engaged in social coaching at a relatively high level, across elementary and middle school classrooms. Teachers tended to do less with respect to supporting connection and engagement between student context and the math learning environment. We also found statistically significant differences between elementary and middle school lessons in positioning students as competent and supporting a nurturing environment by proactively building relationships and productive classroom culture. We offer possible interpretations and a few brief implications of these findings.

Keywords: Equity, Inclusion, and Diversity; Elementary School Education; Middle School Education; Instructional Activities and Practices

Instruction that meets national standards for student learning (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) is considered "ambitious" because it is more demanding and requires more of teachers than the prior focus on procedural competence (Lampert et al., 2010; Lampert & Graziani, 2009). However, ambitious mathematics instruction remains uncommon and opportunities for students to develop the understandings outlined in the *Standards* are not distributed equally (Banilower et al., 2018; Boston & Wilhelm, 2017). This is particularly true for students of color and students for whom English is not their first language (e.g., Nasir & Cobb, 2002; Gutstein & Peterson, 2005). Recently, there has been progress towards identifying specific instructional practices that support historically marginalized groups of students, particularly as they participate in more rigorous mathematics. In order for teachers to develop and implement the identified practices, they need support in understanding distinctions between strong and weak examples of the practices (Goodwin, 1994; Grossman et al., 2013; Little 2003).

The Equity and Access Rubrics for Mathematics Instruction

Gutiérrez (2012) specified four dimensions to attend to the extent to which learning environments support historically marginalized groups of students and might be characterized as aiming for equity: access, achievement, identity, and power. Access and achievement comprise the "*dominant axis*" of equity, while identity and power make up what Gutiérrez called the "*critical axis*" of equity. The EAR-MI (Equity and Access Rubrics for Mathematics Instruction) is a set of classroom observation rubrics developed to capture seven practices that support marginalized students particularly along the dominant axis in gaining access to and more equitably participating in rigorous mathematics activity (Wilson et al., 2019). The EAR-MI was designed by carefully observing the practices of middle school teachers in classrooms characterized by "*Standards*-based" instruction where historically marginalized students have been successful (Wilson, accepted). In this paper, we examine the classroom instruction of elementary and middle grades teachers to investigate the extent to which we see evidence of the presence of practices from four of the rubrics that produced reliable results and remained unchanged across two generalizability studies of the EAR-MI. Specifically, we ask the following question: To what extent are grades 3-8 teachers using some of the practices that support equity and access outlined in the EAR-MI?

Building an Argument for Validity

One intended use of the EAR-MI is to support researchers to assess teachers' progress in enacting mathematics teaching practices that support equity and access. Since the initial empirical study that described the identified classroom practices (Wilson et al., 2019), a set of rubrics were developed and have undergone numerous revisions. The rubrics were revised based on feedback from experts in ambitious and equitable mathematics teaching as well as experts in rubric development. Additional revisions were made based on multiple rounds of pilot coding and feedback from raters who were trained to use the rubrics. We then engaged in a generalizability study to understand how the rubrics were functioning. Based on the results of the initial generalizability study, we removed several rubrics due to a lack of variation in score distribution and we made revisions to several other rubrics to reduce rater variance or to improve the score distribution. We then engaged in a second generalizability study with the revised set of rubrics. The following four practices have rubrics that remained the same through both generalizability studies, and, hence, are the focus of this analysis: positioning students as competent; social coaching; supporting connection and engagement between student context and the math learning environment; and supporting a nurturing environment by proactively building relationships and productive classroom culture.

Four Focal Practices

Based on initial evidence of validity from the first generalizability study, the rubrics that correspond with the four focal practices had scores that demonstrated variability across lessons, and variability was mostly attributed to the lesson and not to differences across raters.

Positioning students as competent. Positioning students as competent is about teachers explicitly and publicly valuing, identifying, and acknowledging the brilliance of their students and framing their actions and statements as intellectually valuable (Bartell, 2011). Note, this is not "appointing" or "giving" students competence. Whether or not a teacher recognizes it, all students already have the capability and know-how to do important and brilliant things. The rubric that attends to this practice emphasizes the extent to which teachers specify what students do that is "productive" as well as the extent to which they provide rationales that support listening students as well as those being positioned in understanding why what was done was considered productive.

Social coaching. Coaching is one way teachers can support students in negotiating productive ways of participating and meeting expectations without decreasing the rigor of the task at-hand. Specifically, social coaching is about teachers deliberately intervening, scaffolding, or providing additional supports to help as students engage with one another (e.g., as they work in cooperative groups or present their thinking to one another). The rubric that attends to this practice focuses on the extent to which the teacher provides concrete suggestions in support of social participation. In addition, this rubric attends to how often the teacher provides rationales for their suggestions.

Supporting connection and engagement between student context and the mathematics learning environment. This practice is about connecting students' lives to discussions and interactions that take place in mathematics classrooms by making the most of connections between the mathematics discussed in class and the everyday lives of students. In particular, teachers may attend to aspects of students' lives and incorporate them into the curriculum (Ladson-Billings, 1995; Banks & McGee, 2001; Gay, 2002), or they may provide learning opportunities that make the mathematics problems discussed in class feel experientially real for students (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013).The rubric that attends to this practice focuses on the extent to which there are connections between mathematics and students' contexts, and whether the connections involve students through dialogue.

Supporting a nurturing environment by proactively building relationships and productive classroom culture. This practice is about establishing personal relationships and developing a sense of community in the classroom (Timmons-Brown & Warner, 2016). This practice often involves the teacher building rapport with students and reinforcing "classroom values." The rubric that attends to this practice emphasizes the extent to which the teacher connects with students in ways that are substantial or that are reciprocated. It also attends to the extent to which the teacher highlights or reiterates classroom values.

Method

Sample

This research project draws on extant classroom video data from two prior research projects, the Responsive Classroom Efficacy Study (RCES) study (Rimm-Kaufman et al., 2014) and the Middle-School Mathematics and the Institutional Setting of Teaching (MIST) study (Cobb et al., 2018). The RCES lessons included in this study were collected in upper elementary classrooms (grades 3-5) during the 2008–09, 2009-10, and 2010-11 school years, and the MIST lessons were collected in middle school classrooms (grades 6-8) during the 2009-10 and 2010-11 school years. The districts and schools within each district varied in their student demographics (see Table 1).

Table 1: District Student Demographics (Rounded)

District	Number of Students	% White	% Black	% Hispanic	% ELL	% Free/ reduced price lunch
RCES	175,000	45	10	20	15	20
MIST-A	35,000	30	40	15	10	65
MIST-B	80,000	15	25	60	30	70
MIST-C	160,000	15	30	65	35	85
MIST-D	95,000	55	35	5	5	55

We hired and extensively trained 5 raters to use the EAR-MI rubrics. At the conclusion of several months of training, rater reliability was assessed with multiple measures including percent exact agreement with an expert score across the last five lessons rated, as well as across 21 lessons scored as a part of training. We consider the five most recently scored lessons because we expected agreement between the raters and expert to improve over time due to ongoing learning. Table 2 shows that exact agreement based on the last five relative to overall was higher for Positioning and Proactive, was similar for Social Coaching, and was slightly lower for Context.

Examining raters' scores relative to the expert scores and relative to each other across the 21 lessons included in training, we calculated Cohen's kappa, Fleiss's kappa, and Krippendorff's alpha (Table 2). Rater consistency with the expert tended to be higher than relative to one another. Generally speaking, kappa and alpha statistics above .20 indicate fair agreement and above .40 indicate moderate agreement (Klein, 2018). Agreement rates with the expert were generally fair or moderate. The Intraclass Correlation Coefficients (ICC) allow us to measure consistency in raters' scores relative to each other. Higher ICCs indicate that scores are trending in similar directions. The ICCs observed in our training data were all above .50, except social coaching which was .40.

For this analysis, we drew on data coded as part of two consecutive generalizability studies, and selected the four focal practices because the related rubrics remained the same and produced reliable results across the two generalizability studies. We analyzed a sample of 141 lessons, representing 65 teachers. This resulted in 83 upper elementary lessons, and 58 middle school lessons. For the purpose of this analysis, a set of scores was generated for each of these lessons through one of two different methods: 1) Expert scores, or 2) Averaging scores across raters. The expert scores resulted from lessons used for bi-weekly drift checks, and the other lessons were scored by three or more raters, and those scores were averaged to create a unique score for each rubric for each lesson. Rubrics include 5 discrete score points. By taking the average across raters, resulting scores could take on any value between 0 and 4.

Table 2: Measures of Rater Agreement

	Positioning	Social Coaching	Context	Proactive
Training				
% Exact Agreement (Last 5 Videos)	83.3%	53.3%	63.3%	73.3%
% Exact Agreement (Overall)	71.4%	53.2%	66.7%	55.6%
Relative to the Expert				
Cohen's Kappa	.53	.29	.53	.37
Fleiss's Kappa	.53	.29	.52	.35
Krippendorff's Alpha	.53	.29	.52	.35
Relative to Other Raters				
Cohen's Kappa	.39	.20	.41	.28
Fleiss's Kappa	.39	.19	.41	.28
Krippendorff's Alpha	.39	.19	.41	.28
Intraclass Correlation	.54	.40	.71	.51

Analysis

This study reports on a descriptive analysis of scores assigned by raters, based on four rubrics designed to assess the four focal practices: positioning students as competent (POSITIONING), social coaching (SOCIAL COACHING), supporting connection and engagement between student context and the mathematics learning environment (CONTEXT), and supporting a nurturing environment by proactively building relationships and productive classroom culture (PROACTIVE). With the unique set of scores for each lesson, we examined the score distributions for the four practices as well as compared score distributions for the middle and elementary school sub-samples. We utilized two-sample t-tests to determine whether perceived differences between the elementary and middle school sample means were statistically significant. A limitation of this analysis is that it does not take into account the nested nature of the lessons within teachers or the order of the lessons for teachers. Future analyses will investigate the influence of these factors on the results included in this report.

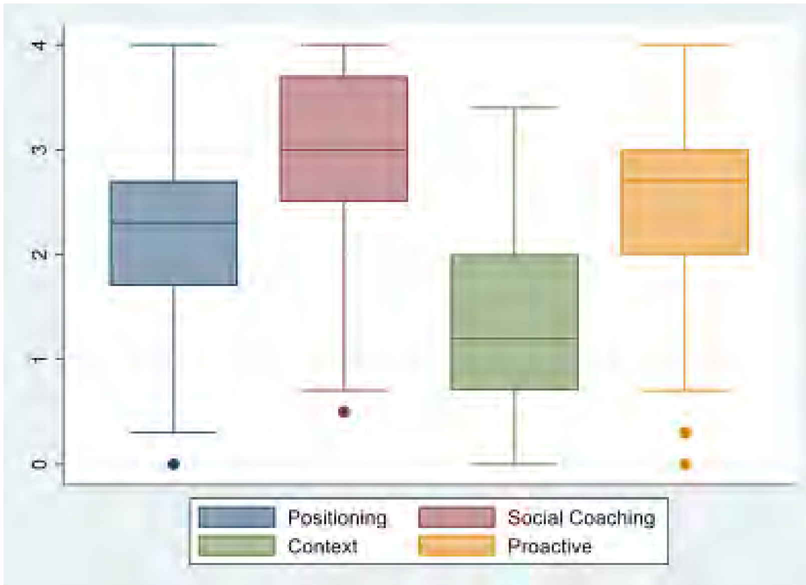


Figure 1: Box Plots Demonstrating Score Distributions for Four Focal Practices

Table 3: Descriptive Statistics for Scores Related to Four Focal Practices

	Overall (n=141) Mean (SD)	Middle (n=58) Mean (SD)	Elementary (n=83) Mean (SD)	Grade band T-test P-value
Positioning	2.20 (0.71)	2.05 (0.61)	2.31 (0.75)	p=.033*
Social Coaching	3.01 (0.75)	3.02 (0.80)	3.00 (0.72)	p>.05
Context	1.30 (0.87)	1.38 (0.92)	1.24 (0.84)	p>.05
Proactive	2.46 (0.67)	2.60 (0.43)	2.36 (0.79)	p=.035*

Note: * denotes statistically significant at the p<.05 level.

Results

Overall, the practices vary significantly in their score distributions (see Figure 1). Of the four focal practices, social coaching had the highest mean (m=3.01, see Table 3), and the score distribution is shifted considerably higher than for the other practices (see Figure 1). A mean of about 3 for social coaching can be interpreted as lessons in which the teacher provided concrete suggestions for social participation with occasional rationales for following those suggestions. The practice with the second-highest mean (m=2.46) was proactive (i.e., supporting a nurturing environment by proactively building relationships and productive classroom culture), which is between levels 2 and 3. For a level 2, the teacher made just one substantial attempt to connect with students (e.g., the teacher sharing personal information about their life). For a level 3, the teacher made more than one substantial attempt to connect with students. The practice with the next highest mean was positioning students as competent (m=2.20), which corresponds between levels 2 and 3. At a level 2, we see lessons where the teacher positions at least one student as competent by specifying what the student did that was productive, but does not provide a rationale for why it was productive (e.g., “Great strategy”). At a level 3, the teacher provides rationales that may not be clear or that may not be focused on disciplinary practices of mathematics like generalizing, justifying, and making connections among multiple representations (e.g., “Using the lines on your paper is a great strategy because your work will be neat and structured and it will be easier to find your answer”). The focal practice with the smallest mean (m=1.30), but also the largest standard deviation (SD=0.87) was context (i.e., supporting connection and engagement between student context and the mathematics learning environment). At a level 1, either the connections to students’ contexts are superficially related to the math task or the students do not participate meaningfully in the discussion of the context (e.g., with a math problem about the perimeter of a lake, the teacher might say, “Raise your hand if you have seen a lake before”). At levels 2 and above, the connections to students’ contexts are substantially related to the math task, with different levels of student participation for levels 2-4. At a level 2, students participate using brief or one-word responses, and at a level 4, multiple students participate in developing a shared understanding of the connections to the context.

Dividing the sample into middle school and elementary school lessons revealed additional nuance with respect to some of the practices (see Table 3). First, there were no statistically significant differences between the elementary and middle school lesson sample means for context and social coaching. There were statistically significant differences for positioning and proactive, and they were in opposite directions. In particular, middle school lessons received significantly lower scores with respect to positioning students as competent ($p<.05$). A middle school mean at a level 2 indicates that, on average, middle school teachers did not provide rationales for their statements that positioned students as competent, whereas a mean of 2.31 for elementary lessons suggests that more elementary teachers provided some sort of rationale in their lessons. The significant difference between means was in the opposite direction for proactive. In other words, middle school lessons received higher scores with respect to supporting a nurturing environment by proactively building relationships and productive classroom culture. This means that there were more teachers in middle school lessons (when compared with elementary school lessons) who made more than one substantial attempt to connect with students. These grade band differences are interesting and warrant further investigation. In the discussion we offer several possible interpretations as well as implications for researchers and teacher educators.

Discussion

Our analysis of four practices that support equity and access in mathematics lessons highlighted interesting differences between the average middle school lesson and the average elementary school lesson with respect to positioning students as competent and proactively building relationships. On average, in elementary lessons compared with middle school lessons, more teachers tended to provide rationales when specifying what the students did that was productive while positioning them as competent. The rubric attends specifically to the explicit ways that teachers position students. With this in mind, it could be that middle school teachers position students as competent in ways that are mostly implicit and thus would not be documented as outlined in the rubric (e.g., some teachers position students by calling them up to the board and asking them to demonstrate their mathematical strategies in the front of the class). It is important to note, that we are not saying that implicit positioning cannot be useful.

However, one theme that we have found across the practices (and especially with the practice of positioning) is that the most supportive implementations of the practices usually reveal the often invisible “rules of the game” being played in mathematics classrooms that may not be apparent to students, particularly students who historically have been minoritized and marginalized in these contexts. In other words, what we have found is that the more transparent and explicit teachers can be in supporting their students the better. These transparent and explicit moves could support students in more directly accessing what is going on and what they are being asked to do, which may empower them in finding their *own* individual ways of “doing math”.

On the other hand, in our sample, the average middle school lesson was rated significantly higher than the average elementary school lesson with respect to proactively building relationships. It could be that teachers in elementary classrooms connect with students in more superficial ways. Alternatively, we note that the distinctions within this rubric attend to the extent to which there are reciprocal personal connections and bonds being built between teachers and students while working on mathematics problems. Knowing that teachers of elementary-aged students tend to teach all subject areas and are usually the instructor for their students throughout the whole school day, it is possible that elementary teachers compartmentalize and make these types of substantial connections at other times during the school day (e.g., some teachers facilitate discussions about their own lives and inquire about their students’ lives during “Calendar Time” or “Circle Time on the Carpet”). However, we are finding that these rich interpersonal connections are particularly important when teachers and students are working on mathematics as it is one way to support students in "showing up" completely and as their whole selves. These interactions also support students in viewing their teachers as approachable, which we have seen improve student participation both in terms of who participates and how they participate. In general, we find that these reciprocal interpersonal connections support the development of a space in which students, particularly those whose voices and natural ways of being are typically pushed to the margins, are likely to feel seen and to be comfortable being their authentic selves. These interactions also support relationships that help students feel safe and secure to take the necessary risks involved in “doing math” (e.g., knowing that disagreeing with shared ideas is common and can be non-threatening or knowing that they are free to make rough draft or not fully formed conjectures while discussing mathematics in class).

As we work to support mathematics teachers to go beyond high-quality mathematics instruction and specifically attend to equity and access in mathematics classrooms, we need more guidance about concrete practices that teachers can engage in (Grossman et al., 2013). This analysis is part of a larger effort to both specify those practices and develop research tools that can be used to assess teachers’ progress as they work to provide instruction that aims towards equity in their mathematics classrooms. At this stage of the validation process, the attention in the EAR-MI rubrics is on whether the teacher engages in particular practices and not with which of the students the practices are enacted. Once we have established meaningful differences between scores on the rubrics, a possible extension would be to combine the EAR-MI rubrics with a participation-focused approach like the EQUIP (Reinholz & Shah, 2018) to attend to whether there are patterns in with whom the teacher enacts particular practices. By attending to individual students in the classroom, we could more intentionally address aspects of the critical axis, specifically highlighting how individual students experience specific aspects of instruction in a mathematics classroom.

The field of mathematics education needs additional research specifically focused on tools for researchers and practitioners that attend to the extent to which mathematics learning environments support historically marginalized groups of students. With the evidence that teachers are not consistently enacting practices that support historically marginalized students, it is clear that teachers would benefit from learning more about these practices and how to enact them in mathematics classrooms. Teacher educators and professional development providers can use these practices and related rubrics to help teachers understand the practices and important distinctions in how they get enacted. By discussing important principles that guide work with students (e.g., the importance of building relationships with students), and pairing those with specific practices described by the rubrics, teachers can begin to envision how to enact those principles with students (e.g., see Pruitt-Britton et al., 2022).

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant #1908481. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Banilower, E. R., Smith, P. S., Malzahn, K. A., Plumley, C. L., Gordon, E. M., & Hayes, M. L. (2018). *Report of the 2018 NSSME+.* Chapel Hill, NC: Horizon Research, Inc.

Banks, J. A., & McGee, C. A. (2001). Families and teachers working together for school improvement. *Multicultural education: Issues and perspectives, 4*, 402-420.

Bartell, T. (2011). Caring, race, culture, and power: A research synthesis toward supporting mathematics teachers in caring with awareness. *Journal of Urban Mathematics Education, 4*(1), 50-74.

Gay, G. (2002). Preparing for culturally responsive teaching. *Journal of Teacher Education, 53*(2), 106-116.

Goodwin, C. (1994), Professional Vision. *American Anthropologist, 96*, 606-633.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In *Equity in discourse for mathematics education* (pp. 17-33). Springer, Dordrecht.

Gutstein, E., & Peterson, B. (Eds.). (2005). *Rethinking mathematics: Teaching social justice by the numbers*. Rethinking Schools.

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction *Journal for Research in Mathematics Education, 44*(4), 646-682.

Klein, D. (2018). Implementing a general framework for assessing inter-rater agreement in Stata. *The Stata Journal: Promoting Communications on Statistics and Stata.* https://doi.org/10.1177/1536867X1801800408

Ladson-Billings, G. (1995). But that's just good teaching! The case for culturally relevant pedagogy. *Theory into Practice, 34*(3), 159-166.

Lampert, M., Beasley, H., Ghousseini, H., Kazemi, E., & Franke, M. (2010). Using designed instructional activities to enable novices to manage ambitious mathematics teaching. In *Instructional explanations in the disciplines* (pp. 129-141). Springer, Boston, MA.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal*, *109*(5), 491-509.

Little, J. (2003). Inside teacher community: Representations of classroom practice. *Teachers college record*, *105*(6), 913-945.

Nasir, N. I. S., & Cobb, P. (2002). Diversity, equity, and mathematical learning. *Mathematical thinking and learning*, *4*(2-3), 91-102.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*. Washington, DC: Authors. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Pruitt-Britton, T., Wilhelm, A. G., & Wilson, J. (2022). Nurturing students through social interactions. *Phi Delta Kappan*, *103*(5), 18-23.

Reinholz, D., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns

in mathematics classroom discourse. *Journal for Research in Mathematics Education, 49*(2), 140-177.
Rimm-Kaufman, S. E., Larsen, R. A. A., Baroody, A. E., Curby, T. W., Ko, M., Thomas, J. B., Merritt, E. G., Abry, T., & DeCoster, J. (2014). Efficacy of the Responsive Classroom Approach: Results from a 3-Year, Longitudinal Randomized Controlled Trial. *American Educational Research Journal, 51*(3), 567-603.
Timmons-Brown, S., & Warner, C. (2016). Using a Conference Workshop Setting to Engage Mathematics Teachers in Culturally Relevant Pedagogy. *Journal of Urban Mathematics Education*, *9*(1), 19-47.
Wilson, J. (Accepted). Initial steps in developing classroom observation rubrics designed around instructional practices that support equity and access in mathematics learning environments with potential for "success". *Teachers College Record: Special Issue*.
Wilson, J., Nazemi, M., Jackson, K., & Wilhelm, A. G. (2019). Investigating Teaching in Conceptually-Oriented Mathematics Classrooms Characterized by African American Student Success. *Journal for Research in Mathematics Education, 50*(4). 362-400. https://doi.org/10.5951/jresematheduc.50.4.0362

CULTURALLY RELEVANT COMPUTING TASKS: EVIDENCE OF SYNERGIES BETWEEN STUDENTS' MATHEMATICAL AND COMPUTATIONAL THINKING

Fernando Alegre
Louisiana State University
falegre@lsu.edu

Zuhal Yilmaz
Louisiana State University
zyilmaz@lsu.edu

Juana Moreno
Louisiana State University
moreno@lsu.edu

Rose Kendrick
Louisiana State University
rkendr3@lsu.edu

In this study, we explore the synergies between students' mathematical and computational thinking as they engage in an open-ended computing "Quilts" project. The activity aims to create a culturally relevant learning environment in which students have opportunities to code personalized quilts that demonstrate their culture and values. An open-coding process was used to analyze team projects. The results of the study suggested that the computational medium gave students flexibility to devise multiple design plans using the power of mathematics and opportunities to test their chosen approaches. Some of the plans included defining functions, using geometric transformations and estimating coordinates. The code provided insights into the students' familiarity with mathematical concepts, level of abstraction and precision in their calculations. Each team created unique quilts that demonstrate multiple values such as freedom, equity and personal life experiences such as music, family, and homeland.

Keywords: Computational Thinking, Mathematical Thinking, Culturally Relevant Pedagogy

Culturally relevant pedagogy (CRP) (Ladson-Billings, 1995) has been used to create engaging and personally relevant learning opportunities, in particular for traditionally underserved and marginalized students (Enyedy & Mukhopadhyay, 2007). CRP not only focuses on students' achievement in a learning environment. It also supports students' development of critical consciousness about inequities while acknowledging and accepting their cultural identity (Ladson-Billings, 1995). In this study, we explore the synergies between students' mathematical and computational thinking as they engage in an open-ended computing "Quilts" project. In the project, students express their values and thoughts by designing their own quilts as a team, using the flexibility of a computing medium and the power of mathematical thinking.

A growing number of studies is exploring the connection between computational thinking and mathematics learning (Barcelos et al., 2018). The majority of these studies mainly focused on teaching programming skills with a limited or no explicit connection to key mathematical ideas and concepts (Hickmott et al. 2018). A limited number of recent studies (e.g. Alegre et al., 2020; Schanzer et al. 2015) explore the connection between mathematical and computing concepts. However, few studies use culturally relevant computing tasks. Aligned with Li et al. (2020)'s perspective, we believe that computational thinking is more about thinking than computing. This study will present examples of how students used various mathematical strategies as they code personalized quilts that reflect their own values, lives and cultures.

Method

We analyzed the data gathered from the end of year Cultural Quilts project of the year-long Introduction to Computational Thinking (ICT) course that is designed to foster 9th graders' understanding of mathematics and teach programming. The quilts activity aims to create a culturally relevant learning environment in which students have opportunities to code personalized quilts. This activity also enables them to demonstrate their programming skills and use mathematical knowledge such as geometric transformations, and functions.

The activity has 3 phases. In the first phase, students research about the history of quilts and reflect on the relevance of quilts in American history (e.g. quilts have been used to carry over traditions across generations). In the second phase, students work in teams to design a quilt that has a common theme but is made by patching several patterns together. The theme represents values that are important for the team. In the third phase, each student needs to create a geometric pattern and decorate it with symbols related to the values chosen by the team. Finally, all the decorated patterns made by each team member are put together in a single quilt design, and the students write a reflection on how the final quilt is related to the chosen values.

To design quilts, students used CodeWorld, which is an integrated coding platform designed for young students, where the code is based on constructing basic shapes (polygons, circles) and performing geometric transformations (translations, rotations, scalings) on them. These simple instructions can be combined to create very elaborate designs.

Data Sources and Analysis

We collected the data from 352 projects from students in 14 different schools. The data set included code of 1) selected symbols and geometric shapes 2) each team members' pattern (a patch of the whole quilt formed of geometric shapes and symbols) and 3) team's final quilt. It also included the theme of the final quilt and a written description of the values, culture and life demonstrated in the quilt. Two researchers engaged in an open-coding process (randomly selected half of the projects) to identify the mathematical strategies being used in the code and the major themes in students' descriptions.

Findings

In the initial phase of dynamic quilts design, the majority of the students used mathematical estimation to decide how to partition the output window, where to place their geometric shapes and symbols to create their individual patterned patch. A few students used precise mathematical calculations in this process. Figure 1 shows an example for both approaches.

```
--elephant--
midForeHead = solidPolygon[ (0,8), (-1.4,7), (0,5.5), (1.4
rightForeHead = painted(solidPolygon[ (1.4,7), (0,5.5), (1
rightLowerForeHead = painted(solidPolygon[ (1.9,7.3), (1.4
rightUpperForeHead = painted(solidPolygon [ (0,8), (1.4,7)
```

```
p = painted(solidPolygon(points), "brown")
points = [(-3*unit,1*unit),(-1*unit,3*unit),
unit = 10/3
```

Figure 1: Mathematical estimation (on the left) and precise calculation (on the right)

As they created their quilts, students used various mathematical strategies such as 1) define function(s) to construct repetitive patterns (24%), 2) use geometric transformations to set positions relative to other object(s) (37%) 3) create each object systematically (e.g. by defining its coordinates considering their size, output size etc.) (22%) 4) create objects unsystematically (e.g. by defining its coordinates without considering the locations of the other objects) (17%). In some codes, they used multiple strategies. The elegance of their code varied depending on the mathematical approach students used to create objects and patterns. Figure 2 shows an example of how one team (3 students) used the first strategy while creating an object (a pattern formed by geometric shapes) for their quilt.

```
s71 = combined[s(0,"red"),s(1,"green"),s(2,"yellow"),s(3, "blue")]
  where
  s(r,c) = painted(rotated(translated(rotated(solidRectangle(8,8), 45),
                                      (u62,u62)), r*90), c)
  u62 = 67/24
```

Figure 2: Creating an object by defining functions

To create the object in figure 2, students defined a function, s, that takes a rotation index (r) and a color (c). In the definition of s, the construction starts with an 8x8 square, which is rotated 45 degrees, then translated, then rotated again by a multiple of 90 degrees given by the argument r, and finally painted in the given color c. The squares are created in the order given in the first line of the code (red, green, yellow, blue) and then combined to make a single object out of them. Note that the order in the list also determines which squares partially cover other squares, as the first one is on top, the second one is right below it, and so on. As seen in figure 3, to complete the pattern, the students created a small 4x4 blue square and placed it on top of the red one. Alternatively, to avoid the problem with the overlap, they could have constructed an L-shaped object and rotated it 4 times.

```
s7 = combined[painted(translated(rotated(solidRectangle(4,4), 45),
                                  (u62+0.1,-0.025)), "blue"),
              s71]
```

Figure 3: Pattern completion

Each team member created different objects and combined all objects to create the team's final quilt (See Figure 4). The final code is composed of 258 lines.

Figure 4: Unified Quilt

The students explained how their quilt represents their values, as follows:

> The values of generosity, respect, etc. contribute to the sense of unity which is what this quilt is all about. Every quilt comes together to make one single quilt that is better than any of them alone. This represents humans in a lot of ways.

Other students also represented various themes in their projects. For instance, 47.5% of the quilts represented the importance of family and friends in their lives, 21% represented life needs such as shelter, food and safety, 17% of the quilts represented the importance of freedom and peace, 12.5% represented southern culture, 8.75% represented students' homelands. Some quilts represented more than one aspect. 59% of the quilts are combined in harmony, each patch represents an aspect of the overall theme (See figure 5) and 41% are not combined in harmony

(See figure 4), each patch may not represent an aspect of an overall theme or the quilt lacks an overall theme.

In another quilt named “Black teens” students described their final quilt (See Figure 5) as:

> Morals, hobbies, and heritage are important because it makes up a person's beliefs and can decide the actions that they choose to take. K. made symbols about Kongo, S. made symbols about cotton, and I made symbols with music notes and paints.

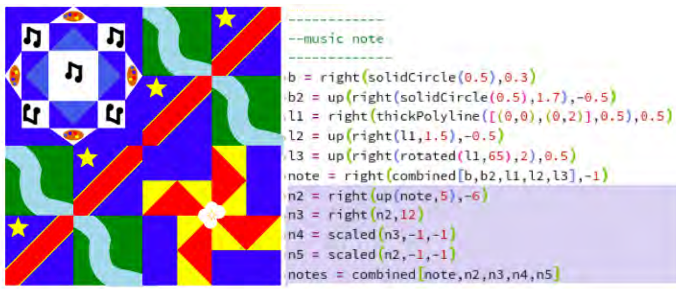


Figure 5: Black Teens Quilt

In the top-left patch in figure 5, a student built a note at the center and then made four copies at different corners. The code not highlighted shows how the student built a single note by combining circles and a segment [thickPolyline]. The highlighted code shows how the student translated the initial note to the two upper corners (n2 and n3), and then they used a reflection in each coordinate direction [scaled(...,-1,-1)] to get to the lower two notes (n4 and n5).

Results and Discussion

The results of this study suggest that students’ codes gave insights about how they used mathematics to design their quilt. The code gave detailed information about students’ mathematical strategies, which mathematical concepts they used correctly and the level of abstraction in their mathematical thinking. In addition, the computational medium gave students flexibility to devise multiple design plans using the power of mathematics (Barcelos et al., 2018) and opportunities to test their chosen approaches. The open-ended nature of the quilt task enabled students to engage in higher order thinking. To design their unique quilts, students had to figure out what procedures to use and why. This dynamic formative feedback had a potential to foster an understanding of the conceptual underpinnings of mathematical procedures (Guzdial & Shreiner 2021). In addition to synergies between mathematics and computation, this activity created a learning environment that was personally relevant to the students. As diSessa (2018) suggested “the freedom to take … ideas and put them to personal use [in this case expressing their values and beliefs tied to their cultural heritage]” (p. 14) makes the tasks personally relevant to the students and creates personal variations in the designed quilts.

Acknowledgments

This research project is partly funded by the NSF award CNS-1923573 and the U.S. Dep. of Education award U411CI90287

References

Alegre, F., Underwoood, J., Moreno, J. & Alegre, M. (2020). Introduction to Computational Thinking: A New High School Curriculum using CodeWorld. In *Proceedings of the 51st ACM Technical Symposium on Computer Science Education,* 992-998.

Barcelos, T. S., Muñoz-Soto, R., Villarroel, R., Merino, E., & Silveira, I. F. (2018). Mathematics Learning through Computational Thinking Activities: A Systematic Literature Review. *J. Univers. Comput. Sci.*, *24*(7), 815-845.

DiSessa, A. A. (2018). Computational literacy and "the big picture" concerning computers in mathematics education. *Mathematical thinking and learning*, *20*(1), 3-31.

Enyedy, N., & Mukhopadhyay, S. (2007). They don't show nothing I didn't know: Emergent tensions between culturally relevant pedagogy and mathematics pedagogy. *The Journal of the Learning Sciences*, *16*(2), 139-174.

Guzdial, M., & Shreiner, T. (2021). Integrating computing through task-specific programming for disciplinary relevance: Considerations and examples. *Computational Thinking in Compulsory Education: A Pedagogical Perspective,, Aman Yadav and Ulf Dalvad Berthelsen (Eds.). Routledge Taylor & Francis Group*.

Hickmott, D., Prieto-Rodriguez, E., & Holmes, K. (2018). A scoping review of studies on computational thinking in K–12 mathematics classrooms. *Digital Experiences in Mathematics Education*, *4*(1), 48-69.

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American educational research journal*, *32*(3), 465-491.

Li, Y., Schoenfeld, A. H., diSessa, A. A., Graesser, A. C., Benson, L. C., English, L. D., & Duschl, R. A. (2020). On computational thinking and STEM education. *Journal for STEM Education Research*, *3*(2), 147-166.

Schanzer, E., Fisler, K., Krishnamurthi, S., & Felleisen, M. (2015, February). Transferring skills at solving word problems from computing to algebra through Bootstrap. In *Proceedings of the 46th ACM Technical symposium on computer science education* (pp. 616-621).

RETHINKING CLASSROOM DATA COLLECTION

Amber G. Candela
University of Missouri – St. Louis
candelaa@umsl.edu

F. Paul Wonsavage
University of Florida
wonsavagef@ufl.edu

Zandra de Araujo
University of Florida
zdearaujo@coe.ufl.edu

Samuel Otten
University of Missouri
ottens@missouri.edu

Keywords: Research Methods, Technology

Gathering evidence of teachers' classroom instruction has been a hallmark of data collection in mathematics education research. In-person data collection, though often preferable, has become increasingly challenging in recent years, partly due to COVID-19, but also because of logistical challenges (e.g., cost, travel, time) associated with collecting classroom observation data (Namey et al., 2020). Given that research typically occurs near higher education institutions, voices outside these areas are often underrepresented. These challenges have implications around geographic equity, researchers' resources, and time. We explore the possibilities of remote data collection as a means of capturing key aspects of teachers' mathematics instruction.

Previous research on observation protocols that used video versus live coding primarily utilized quantitative methodologies (e.g., Casabianca et al., 2013; Curby et al., 2016) where researchers looked for significant differences using statistical tests. Gridley and colleagues (2018) found no significant differences between live and video coding for their observation protocol, but suggested there might be qualitative differences that were not captured. For example, the informal conversations between the teacher and researcher, the richness of data from being immersed in a bustling classroom, the ability to pause and rewind video, or having an entire research team together to watch video. Understanding aspects of protocols best suited for video or live observations is needed to develop and refine instruments for remote data collection, in order to ensure coding video captures key aspects of instruction, as if the researchers were in the classroom.

Through mixed methods (Creswell & Plano Clark, 2018) we will identify differences in live and video recorded classroom observations with our goal to revise an existing observation protocol to use with video-recorded lessons so we can diversify participants and decrease logistical challenges. Data collection includes live observations of classroom instruction using the *Flipped Mathematics Instruction Observation Protocol* (Otten et al., 2018; Otten et al., 2021) while simultaneously using a 360° video camera. Two researchers will complete the protocol during a live observation and two different researchers will complete it using the video recording. Then the researchers will come together and identify possible differences in coding between the live and video recordings in order to refine the protocol for video recorded observations. Our ultimate goal is to mitigate the dissonance of in-person versus live recordings and work towards a more harmonious approach for collecting observation data that attends to geographic equity, resources, and time.

Acknowledgments

This work was supported by the National Science Foundation (award #2206774) though any opinions, findings, and conclusions expressed here are those of the authors and do not necessarily reflect the views of the NSF.

References

Casabianca, J. M., McCaffrey, D. F., Gitomer, D. H., Bell, C. A., Hamre, B. K., & Pianta, R. C. (2013). Effect of observation mode on measures of secondary mathematics teaching. *Educational and Psychological Measurement*, *73*(5), 757-783.

Creswell, J. W., & Plano Clark, V. L. (2018). *Designing and conducting mixed methods research* (3rd ed.). Thousand Oaks, CA: Sage.

Curby, T. W., Johnson, P., Mashburn, A. J., & Carlis, L. (2016). Live versus video observations: Comparing the reliability and validity of two methods of assessing classroom quality. *Journal of Psychoeducational Assessment*, *34*(8), 765-781.

Gridley, N., Bywater, T. J., & Hutchings, J. M. (2018). Comparing live and video observation to assess early parent-child interactions in the home. *Journal of Child and Family Studies*, *27*(6), 1818-1829.

Namey, E., Guest, G., O'Regan, A., Godwin, C. L., Taylor, J., & Martinez, A. (2020). How does mode of qualitative data collection affect data and cost? Findings from a quasi-experimental study. *Field Methods*, *32*(1), 58-74.

Otten, S., de Araujo, Z., Sherman, M., & Birişçi, S. (2021). A framework for capturing structural variation in flipped mathematics instruction. *International Journal of Mathematical Education in Science and Technology*, 1-32.

Otten, S., de Araujo, Z., & Sherman, M. (2018). Capturing variability in flipped mathematics instruction. In *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

.

DEVELOPING EXPERTISE IN TEACHING THAT IS RESPONSIVE TO CHILDREN'S MATHEMATICAL THINKING: A CASE STUDY

Amy Dunning
University of Illinois Springfield
ahewi2@uis.edu

Victoria R. Jacobs
University of North Carolina at Greensboro
vrjacobs@uncg.edu

This poster presents the 3-year story of Ms. Brown, showcasing her development of expertise in teaching that is responsive to children's mathematical thinking. In the first two years, Ms. Brown was a fifth-grade teacher, and in the final year, she served as a mathematics coach. Throughout the study, she was engaged in a 3-year professional development (PD) that was part of a larger project, *Responsive Teaching in Elementary Mathematics* (Empson & Jacobs, 2021). The PD was designed to help teachers develop expertise in teaching that is responsive to children's thinking, with an emphasis on teaching fractions. Research-based frameworks of both children's thinking and instructional practices were central to this work (Jacobs et al., 2019).

We identified growth storylines for Ms. Brown in relation to three instructional practices linked to teaching that is responsive to children's mathematical thinking: (a) anticipating children's mathematical thinking (Stein et al., 2008); (b) noticing children's mathematical thinking (Jacobs et al., 2010); and (c) questioning children's mathematical thinking (Jacobs & Empson, 2016). Data for this case study included 7 classroom observations (2–3 per year), 19 individual interviews (6–7 per year), 4 focus group conversations (1–2 per year), and two written assessments, one each for anticipating and noticing children's mathematical thinking.

Using constant comparative analysis, we identified growth storylines for Ms. Brown's anticipating, noticing, and questioning of children's mathematical thinking. Overall, children's thinking became increasingly centered and addressed in more depth with each instructional practice. Further, Ms. Brown became increasingly confident in her use of children's thinking in these practices. For example, Ms. Brown began by struggling with noticing children's thinking in written work. In an interview early in the first year, she shared: "I've really been trying to take [written work] home and look through the work, and I don't know fully what I'm looking at yet." During the second year, her confidence and ability to notice children's thinking increased, and she highlighted this growth in an interview: "That's been the biggest growing for me over the past 2 years, how to look at your students' work." During the third year, Ms. Brown demonstrated further growth not only in her own confidence but also in her ability and desire to help others notice children's thinking in written work. In an interview about her mathematics coaching, Ms. Brown stated how she encouraged her mentee to work on noticing: "For the first 2 weeks, I just want you to notice. I want you to be able to come to me and say here are some things I notice about my kids' work." These examples provide a brief glimpse of Ms. Brown's growth storyline for the instructional practice of noticing children's mathematical thinking.

This case study contributes to the literature on teaching that is responsive to children's mathematical thinking (see, e.g., Robertson et al., 2016). We provide rich examples of the three instructional practices and their development over time. Implications for ways to foster that development are also presented.

Acknowledgements

This research was supported in part by the National Science Foundation (DRL 1316653 and 1712560). The opinions expressed do not necessarily reflect the position, policy, or endorsement of the supporting agency. We thank Paige Brown for her collaboration and dedication to children's thinking.

References

Empson, S. B., & Jacobs, V. R. (2021). Exploring teachers' responsiveness to children's fraction thinking and relationships to fraction achievement. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1449–1458). Philadelphia, PA.

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. ZDM–The International Journal on Mathematics Education, 48(1–2), 185–197.

Jacobs, V. R., Empson, S. B., Pynes, D., Hewitt, A., Jessup, N., & Krause, G. (2019). The responsive teaching in elementary mathematics project. In P. Sztajn & P. H. Wilson (Eds.), Designing professional development for mathematics learning trajectories (pp. 75–103). Teachers College Press.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169–202.

Robertson, A. D., Scherr, R. E., & Hammer, D. (2016). Responsive teaching in science and mathematics. Routledge.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. Mathematical Thinking and Learning, 10(4), 313–340.

CREATIVITY IN PROVING: IMPLEMENTING A GROUP REFLECTION ACTIVITY

Amanda Lake Heath
Middle Tennessee State
Alh2ei@mtmail.mtsu.edu

Sarah K. Bleiler-Baxter
Middle Tennessee State
sarah.bleiler@mtsu.edu

Jordan E. Kirby
Middle Tennessee State
jek3b@mtmail.mtsu.edu

Keywords: Reasoning and Proof, Instructional Activities and Practices, Undergraduate Education, Metacognition

Proof and creativity are foundational to the work of mathematicians (Karakok et al., 2015; Sriraman, 2004). For this reason, it is essential that creativity be encouraged in proof-based university mathematics courses (Mann, 2006; Schumacher & Seigel, 2015) and that students have opportunities to explicitly reflect upon their creativity when proving (Savic et al., 2017). Given the current emphasis on collaborative learning in such courses, we explore one possible way to promote explicit reflection on creativity among undergraduate students as they work together to co-construct proofs (i.e., the implementation of a modified version of the Creativity-in-Progress Reflection on Proving (CPR; Savic et al., 2017). We adopt a theoretical framing of intersubjectivity (Matusov, 1996; Sawyer, 2019) and social metacognition (Chiu & Kuo, 2009) to answer the research question, how do undergraduate students in an Introduction-to-Proof (ITP) course use the CPR to reflect upon their experiences in collaborative proving? Providing a picture of how students use a tool such as the CPR will provide a foundation for future research aimed at understanding the effectiveness of the CPR in promoting explicit reflection upon creativity in collaborative proving and supporting students' development as creative provers.

Context and Methods

This study was conducted in an undergraduate ITP course at a large public university in the United States. This course facilitated an inquiry-based learning environment centered around small-group work. We implemented the CPR group reflection activity (Heath et al., in press) following collaborative group proving activities that occurred during different course modules (a) Logic and Grammar of Mathematics (early in semester), (b) Direct and Indirect Proof Methods (mid semester), and (c) Set Theory (late semester). For each implementation, we analyzed video data of student groups engaging in group reflection using the CPR.

Results

Each implementation followed a similar trajectory where students (a) worked in small-groups to collaboratively prove a given conjecture, (b) used the CPR to reflect on their proving process as a group, and (c) used the CPR to reflect on their individual contributions to the group's proof, and (d) reconvened as a small-group to debrief and discuss their overall observations on creativity in proving. As an example, during the Logic module, small groups were tasked with, "Let A, B, and C be logical statements and prove the following: $(A \wedge B) \vee C$ is logically equivalent to $(A \vee C) \wedge (B \vee C)$." Analysis of video data of group reflection following this task revealed two central themes. First, there was a general difference in approach between groups with respect to how they reached a consensus during their discussion of the CPR. Second, the CPR helped facilitate the recollection of group members' contributions to their group proving that were previously undervalued or unnoticed. This poster will, for each implementation, describe the process of implementing the CPR and report the themes observed in the groups' reflections. These processes and themes will inform the continued refinement of the CPR and provide future directions for our research regarding how to implement of the CPR in a collaborative setting.

References

Heath, A. L., Bleiler-Baxter, S. K., & Kirby, J. E. (in press). Group reflection on mathematical creativity in proving. *Proceedings of the 24rd annual conference on research in undergraduate mathematics education*.

Chiu, M., & Kuo, S. (2009). From metacognition to social metacognition: Similarities, differences, and learning. *Journal of Education Research*, 3(4), 1–19. https://www.researchgate.net/publication/291046856

Karakok, G., Savic, M., Tang, G., & El Turkey, H. (2015). Mathematicians' views on undergraduate students' creativity. In K. Krainer & N. Vondrová (Eds.), *Ninth congress of the European society for research in mathematics education* (pp. 1003–1009). ERME. https://hal.archives-ouvertes.fr/hal-01287301

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, *30*(2), 236–260. https://doi.org/10.4219/jeg-2006-264

Matusov, E. (1996). Intersubjectivity without agreement. *Mind, Culture, and Activity*, *3*(1), 25–45. https://doi.org/10.1207/s15327884mca0301_4

Savic, M., Karakok, G., Tang, G., el Turkey, H., & Naccarato, E. (2017). Formative assessment of creativity in undergraduate mathematics: Using a creativity-in progress rubric (CPR) on proving. In R. Leikin & B. Sriraman (Eds.), *Creativity and giftedness* (pp. 23–46). Springer. https://doi.org/10.1007/978-3-319-38840-3

Sawyer, R. K. (2019). Individual and group creativity. In J. C. Kaufman & R. J. Sternberg (Eds.), *The Cambridge handbook of creativity* (Second, pp. 366–380). Cambridge University Press. https://doi.org/10.1017/CBO9780511763205.023

Schumacher, C. S., & Siegel, M. J. (2015). *2015 CUPM curriculum guide to majors in the mathematical sciences.* Mathematical Association of America.

Sriraman, B. (2004). The characteristics of mathematical creativity. *The Mathematics Educator*, *14*(1), 19–34. https://doi.org/10.1007/s11858-008-0114-z

A SURVEY ON ELEMENTARY TEACHERS' USES OF MANIPULATIVES

UNE ENQUÊTE SUR LES PRATIQUES D'UTILISATION DU MATÉRIEL DE MANIPULATION AU PRIMAIRE

Doris Jeannotte	Claudia Corriveau	Anne-Sophie Charest
UQAM	Université Laval	Université Laval
jeannotte.doris@uqam.ca	Claudia.corriveau@fse.ulaval.ca	anne-sophie.charest@mat.ulaval.ca

Simon Bilodeau-Carrier	Xavier Serra	Catherine Verret
Université Laval	Université Laval	Université Laval
simon.bilodeau-carrier.1@ulaval.ca	xavier.serra.1@ulaval.ca	catherine.verret.1@ulaval.ca

Keywords: Elementary school education, Instructional Activities and Practices, Teacher Beliefs, Teacher Knowledge

More than 50 years ago, Kieren (1971) asked: "For whom, for which topics, and with what materials are manipulative and play-like activities valuable?" (p. 232). Since then, most curricula around the world have been promoting or even prescribing the use of manipulatives to teach mathematics at the elementary level. However, the questions asked by Kieren are still relevant as recent research continues to show mixed results when it comes to the efficiency of manipulatives (Carbonneau, Marley & Selig, 2013). The ways in which manipulatives are used by teachers should be considered to better understand how this use can contribute to mathematical learning. In addition, Puchner, Taylor, O'Donnell, & Fick (2008) underlined that "[...] teachers need support making decisions regarding manipulative use, including when and how to use manipulatives to help them and their students think about mathematical ideas more closely" (p. 323). According to Hatfield (1994), Swan & Marshall (2008), and Johnson, O'Meara & Leavy (2020), the use of manipulatives decreases across grades as some teachers tend to believe that manipulatives can be childish or that their use could be a hindrance to the development of abstraction. Those studies also report some factors identified by teachers as discouraging the use of manipulatives (e.g., availability of material, cost, time, distraction) or encouraging it (e.g., visual aid, hands-on learning, building a better understanding). However, we know little about how teachers choose a specific manipulative to use, when and why they use them.

In this project, we aim to better understand teachers' practices around manipulatives and to help them develop meaningful ways of using them. Our first step was to develop a survey around the use of manipulative that will be distributed during spring 2022 to a random sample of elementary teachers from Québec, Canada. Building from previous surveys found in math education literature, we modified and enriched the different questions based on our collaboration with teachers (Jeannotte & Corriveau, 2020; Corriveau & Jeannotte, 2015). The first version was administered to 5 teachers who provide feedback to clarify questions through interviews. The last version is composed of 3 sections. The first section is about demographic information. The second section deals specifically with teachers' use of manipulatives. The third is specific to the pandemic situation. In this poster presentation, we will present the rationale behind some questions from section 2 and some preliminary results.

Acknowledgments

This research is funded by SSHRC, #435-2021-0862.

References

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*(2), 380-400. https://doi.org/10.1037/a0031084

Corriveau, C. & Jeannotte, D. (2015). L'utilisation du matériel de mathématique en classe de mathématiques au primaire : quelques réflexions sur les apports possibles. *Bulletin AMQ, LV*(3), 32-49.

Hatfield, M. M. (1994). Use of Manipulatives Devices: Elementary School Cooperating Teachers Self-Report. *School Science and Mathematics, 94*(6), 303-309.

Jeannotte, D. et Corriveau, C. (2020). Interactions Between Pupils' Actions and Manipulative Characteristics when Solving an Arithmetical Task. *Proceedings of the 11th Congress of the European Society for Research in Mathematics Education.* Utrecht, NL.

Johnson, P., O'Meara, N., & Leavy, A. (2020). Factors supporting and inhibiting teachers' use of manipulatives around the primary to post-primary education transition. *International Journal of Mathematical Education in Science and Technology*, 1-23. https://doi.org/10.1080/0020739X.2020.1736348

Kieren, T. E. (1971). Manipulative Activity in Mathematics Learning. *Journal for Research in Mathematics Education*, 2(3), 228. https://doi.org/10.2307/749048

Marshall, L., & Swan, P. (2008). Exploring the Use of Mathematics Manipulative Materials: Is It What We Think It Is? *Sustainability in Higher Education: Directions for Change,* 338-350. https://ro.ecu.edu.au/ceducom/33

THE BELIEFS AND PRACTICES OF GRADUATE STUDENT TUTORS ENGAGED IN ONLINE TUTORING DURING THE PANDEMIC

Taylor Mule'
The University of Memphis
tnmule@memphis.edu

Scotty Houston
The University of Memphis
sghoustn@memphis.edu

Justine Piontek
The University of Memphis
jpiontek@memphis.edu

Jessica Webb
The University of Memphis
jessicajohnsonwebb@gmail.com

Leigh Harrell-Williams
The University of Memphis
leigh.williams@memphis.edu

Keywords: Affect, Emotions, Beliefs, and Attitudes; Informal Education; Instructional Activities and Practices; Undergraduate Education

Many higher education institutions in the United States provide mathematics tutoring services for undergraduate students. These informal learning experiences generally result in increased final course grades (Byerly & Rickard, 2018; Rickard & Mills, 2018; Xu et al., 2014) and improved student attitudes toward mathematics (Bressoud et al., 2015). In recent years, research has explored the beliefs and practices of undergraduate and, sometimes graduate, peer tutors, both prior to (Bjorkman, 2018; Johns, 2019; Pilgrim et al., 2020) and during the COVID-19 pandemic (Gyampoh et al., 2020; Mullen et al., 2021; Van Maaren et al., 2021). Additionally, Burks and James (2019) proposed a framework for Mathematical Knowledge for Tutoring Undergraduate Mathematics adapted from Ball et al. (2008) Mathematical Knowledge for Teaching, highlighting the distinction between tutor and teacher. The current study builds on this body of work on tutors' beliefs by focusing on mathematical sciences graduate teaching assistants (GTAs) who tutored in an online setting during the 2020-2021 academic year due to the COVID-19 pandemic. Specifically, this study addresses the following research question: What were the mathematical teaching beliefs and practices of graduate student tutors participating in online tutoring sessions through the mathematics learning center (MLC) during the COVID-19 pandemic?

The graduate students in this study were part of a multi-institution implementation of a multiple component graduate teaching training program, Promoting Success in Undergraduate Mathematics through Graduate Teaching Assistant Training (PSUM-GTT; Harrell-Williams et al., 2020), which was developed with the intent to strengthen the teaching skills of mathematical sciences GTAs to help the undergraduate students they currently serve as GTAs and to position them to become more effective instructors if they become faculty. GTAs at three institutions participated in a teaching seminar offered by their department and a Critical Issues in Undergraduate STEM Education seminar and received peer mentoring from more advanced graduate students and support from a peer TA Coach.

During the Spring 2021 semester, ten GTAs who had tutoring assignments instead of teaching assignments participated in structured interviews about their beliefs and practices. These GTAs had completed the teaching seminar and MLC-specific training. The interview protocol used modified items from the Pilgrim et al. (2020) tutor-focused survey based on The Teacher Beliefs Interview (TBI; Luft & Roehrig, 2007). The interview data are being coded using the modified coding scheme from Pilgrim et al. (2020), which involves using categories describing the responses on a continuum from more tutor-focused to more student-focused experiences. Results regarding graduate student tutors' beliefs will be summarized and presented in this poster.

Acknowledgments

The PSUM-GTT project is funded by the National Science Foundation under Award Numbers DUE #1821454, 1821460, and 1821619 to the University of Colorado Denver, Auburn University, and the University of Memphis. The opinions, findings, and conclusions or recommendations are those of the authors, and do not necessarily reflect the views of the funding agency.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59(5), 389-407.

Bjorkman, K. (2018). The undergraduate mathematics learning center: Perspectives of peer tutors. In Hodges, T.E., Roy, G. J., & Tyminski, A. M. (Eds.). (2018). Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Greenville, SC: University of South Carolina & Clemson University.

Bressoud, D. M., Mesa, V., & Rasmussen, C. L. (Eds.). (2015). Insights and recommendations from the MAA national study of college calculus. MAA Press.

Burks, L., & James, C. (2019). Mathematical Knowledge for Tutoring Undergraduate Mathematics. Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education. Oklahoma City, Oklahoma.

Byerley, C. & Rickard, B. (2018). Evaluation of impact of Calculus Center on student achievement. Proceedings of the 21st Annual Conference on Research in Undergraduate Mathematics Education. San Diego, CA.

Gyampoh, A. O., Ayitey, H. K., Fosu-Ayarkwah, C., Ntow, S. A., Akossah, J., Gavor, M., & Vlachopoulos, D. (2020). Tutor perception on personal and institutional preparedness for online teaching-learning during the COVID-19 crisis: The case of Ghanaian Colleges of Education. African Educational Research Journal, 8(3), 511-518.

Harrell-Williams, L.M., Olson, G., Webb, J., Houston, S., & Gomez, J. (2020, February). How different is different? Examining institutional differences prior to scaling up a graduate teacher training program to improve undergraduate mathematics outcomes. Proceedings of 23rd Annual Conference on Research in Undergraduate Mathematics Education Conference (RUME), Boston, MA.

Johns, C. (2019). Tutor Behaviors in Undergraduate Mathematics Drop-In Tutoring. [Doctoral dissertation, Ohio State University]. ProQuest Dissertations & Theses Global.

Luft, J. A., & Roehrig, G. H. (2007). Capturing science teachers' epistemological beliefs: The development of the teacher beliefs interview. Electronic Journal of Science Education, 11(2), 38-63.

Mullen, C., Pettigrew, J., Cronin, A., Rylands, L., & Shearman, D. (2021). Mathematics is different: student and tutor perspectives from Ireland and Australia on online support during COVID. Teaching Mathematics and its Applications, 40(4) 332-355.

Pilgrim, M., Miller, E., Hill-Lindsay, S., & Segal, R. (2020). Analyzing the beliefs and practices of graduate and undergraduate mathematics tutors. Proceedings of the 23rd Annual Conference on Research in Undergraduate Mathematics Education. Boston, MA.

Rickard, B. & Mills, M. (2018). The effect of attending tutoring on course grades in Calculus I. International Journal of Mathematical Education in Science and Technology, 49(3), 341-354.

Van Maaren, J., Jensen, M., & Foster, A. (2021). Tutoring in higher education during COVID-19: Lessons from a private university's transition to remote learning. Journal of College Reading and Learning, 1-20.

Xu, Y., Hartman, S., Uribe, G. & Mencke, R. (2011). The effects of peer tutoring on undergraduate examination scores in mathematics. Journal of College Reading and Learning, 32(1), 22-31.

HAVE WE CUT OURSELVES OFF AT THE NECK? CENTERING RELATIONALITY AND HUMANITY IN OUR RESEARCH

Charlotte Sharpe
Syracuse University
cjsharpe@syr.edu

Nicole Fonger
Syracuse University
nfonger@syr.edu

Keywords: Research methods, Design Experiments, Teacher Educators

"Humans – feeling, living, breathing, thinking humans – do research. When we try to cut ourselves off at the neck and pretend an objectivity that does not exist in the human world, we become dangerous, to ourselves first, and then to the people around us (Hampton, 1995, p. 52).

This poster summarizes the developing praxis of two mathematics education researchers who consider themselves community-engaged scholars, design researchers, and community partners with public school educators in our city. Recently we have both reflected on our tendency to quietly append mentions of *human connection* to our research manuscripts in humble but brief "closing thoughts" or acknowledgements. We reduce to a sentence or two the months or years we've spent building relationships with students, teachers, and communities, and the patience and kindness they have shown us by inviting us into their spaces. Here we assert that relegating 'connectedness' to the hidden corners of manuscripts is a symptom of personal or academic fear: fear of looking too emotional, too feminine, or too soft in the bid for legitimacy among the bench scientists. Or perhaps we are afraid that making and talking about connections with practitioners or research subjects makes us accountable to them, and vulnerable (Howard & Hammond, 2019; McDevitt, 2021). *What if I care too much about this research*? We join with other critical scholars who challenge the notion of rational objectivity as a fiction of the colonial worldview (Wilson, 2003; Hampton, 1995). By allying ourselves through genuine connections with the students, teachers, school leaders, parents, and other community members in our research, we trade the illusion of objectivity for the vulnerability of feeling (Hampton, 1995; Halle-Erby, 2020). Examining our own research using Indigenous theories and methodologies of *relationality*, we argue in this poster that human connections are not a fringe benefit, a lubricant, or even a foundation of "real research." Instead, we contend that human connection and relationship but should be a goal of mathematics educational research.

This poster represents our first steps adopting the Indigenous theoretical and methodological framework of *relationality* (Hampton, 1995; Wilson, 2003; Steinhauer, 2002) for use in writing and research in mathematics education. To demonstrate the utility of this framework for the PME-NA community, we present a *relationality* analysis of our own past and current writing projects, discussing the affordances and vulnerabilities in privileging human connections as a research pursuit. For example, in examining a manuscript for which she has recently received a "revise and resubmit," Author 1 identifies a problematic absence of discussion of the way friendship, trust, fear, excitement, and disappointment emerged in a sequence of mediated field experiences for pre-service teachers in an elementary classroom. Using the relationality framework, the authors will use this text, as well as Author 2's developing book proposal, as concrete examples of how we might rehumanize mathematics education scholarship and research design by centering connections and relationality. We include implications for how we might incorporate relationality in studies of children's mathematical thinking as well, giving examples from recent PME-NA plenaries that honor relationality in timely and critical ways (Frank, 2021; Marchant & Jones, 2021; Kalinec-Craig, 2021).

References

Frank, T. (2021). From practice to theory: listening to and learning with Black mathematics teachers. *Proceedings of the 43rd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Philadelphia, PA.

Gomez Marchant, C. N., Jones, S. (2021). Let Us Be the Healing of the Wound/Seamos la Curación de la Herida. *Proceedings of the 43rd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Philadelphia, PA.

Halle-Erby, K. (2022). “Relationships are reality”: centering relationality to investigate land, indigeneity, blackness, and futurity. International Journal of Qualitative Studies in Education, 1-18.

Hampton, E. (1995). Memory comes before knowledge: Research may improve if researchers remember their motives. *Canadian Journal of Native Education*, *21*, 46-54.

Kalinec-Craig, C. (2021). Pandemics, scholarship, and rethinking what counts. *Proceedings of the 43rd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Philadelphia, PA.

McDevitt, S. E. (2021). “Don’t Be Afraid”: Exploring Methodological Relationships in (Re) searching the Experiences of Immigrant Teachers of Color. *Multicultural Perspectives*, *23*(1), 40-47.

Steinhauer, E. (2002). Thoughts on an Indigenous research methodology. *Canadian Journal of Native Education*, *26*(2), 69.Howard & Hammond, 2019;

Wilson, S. (2003). Progressing toward an Indigenous research paradigm in Canada and Australia. *Canadian Journal of Native Education*, *27*(2), 161.

Chapter 16:

Technology and Learning Environment Design

LEVERAGING TECHNOLOGY IN A SOCIAL JUSTICE MATHEMATICS LESSON: MODELING HEALTHY FOOD PRIORITY AREAS WITH SCRATCH

David Johnson	Jean Ciscell	Kimberly Corum
Towson University	Towson University	Towson University
djohns77@students.towson.edu	jcisce1@students.towson.edu	kcorum@towson.edu

Keywords: Social Justice, Computing and Coding, Modeling

Using mathematics to understand and address social issues can empower students to act as agents of change in their communities (Gewertz, 2020). Despite calls from education researchers to integrate social justice issues into K-12 mathematics (NCSM & TODOS, 2016), teachers rarely engage in this type of instruction (Bartell, 2013). In our task, students decide where to place a grocery store in a local neighborhood that has been classified as a healthy food priority area (HFPA or food desert) and write a program in Scratch that models different ways of traveling to the grocery store. By using Scratch in this task, students are exposed to coding which can increase their fluency in mathematics and digital literacy as well as promote future careers in STEM (Amador & Soule, 2015). Integrating coding into the mathematics classroom can also support students' ability to visualize abstract concepts, develop authentic problem-solving skills, and explore real-world applications of mathematics (Germia & Panorkou, 2020).

We implemented the *Food from Scratch* task with two groups of students. The first group consisted of 16 university students enrolled in a required course for secondary math and science education majors. This implementation consisted of one 75-minute session that took place in November 2021. The second group consisted of 18 elementary and middle school aged students who participated in math enrichment tasks offered through a university-community partnership program. This implementation consisted of two 60-minute sessions that took place on consecutive Saturdays in December 2021. The activity sequence for the *Food from Scratch* task consisted of five parts: (1) analyzing a neighborhood map to record what students notice and wonder, (2) determining the unit rate for different methods of travel (i.e., walking, driving), (3) selecting a grocery store location, (4) modeling travel from different locations to their grocery store using Scratch, and (5) engaging in a mathematical analysis of their grocery store placement (e.g., distance to their grocery store, average travel times for walkers versus drivers, accessibility of their grocery store based on data collected from Scratch).

Analysis of students' written work and Scratch programs revealed key themes across both implementations: accessibility, prior knowledge of the coordinate plane, and understanding of unit rates. Students were constantly aware of how their decision making affected the people in the community and wanted to ensure an equitable grocery store location for residents. In doing so, students used their knowledge of a coordinate plane to construct various routes of travel and utilized rich mathematical reasoning to justify their grocery store placements. For example, students from the first implementation calculated the distance between different points while students from the second implementation considered the difference in unit rates between walkers and drivers when reflecting on their grocery store placement. Students were able to apply their knowledge of the coordinate plane and unit rates to justify their ideal location of a grocery store in a HFPA in their community. By addressing a social justice issue that is facially neutral (all people should have access to healthy food options) and utilizing students' prior mathematical knowledge, our findings from *Food from Scratch* task could prove helpful for educators wanting to utilize similar tasks that integrate social justice issues and modeling with technology.

References

Amador, J. M., & Soule, T. (2015). Girls Build Excitement for Math from Scratch, Mathematics Teaching in the Middle School MTMS, 20(7), 408-415. Retrieved Jan 27, 2022, from https://pubs.nctm.org/view/journals/mtms/20/7/article-p408.xml

Bartell, T. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. Journal for Research in Mathematics Education, 44, 129-163.

Germia, E., & Panorkou, N. (2020). Using Scratch Programming to Explore Coordinates, Mathematics Teacher: Learning and Teaching PK-12, 113(4), 293-300. Retrieved Jan 27, 2022, from https://pubs.nctm.org/view/journals/mtlt/113/4/article-p293.xml

Gewertz, C. (2020). Teaching Math Through A Social Justice Lens. Education Week, 40(15), 14–16. Retrieved February 13, 2022.

National Council of Supervisors of Mathematics and TODOS: Mathematics for ALL. (2016). Joint position statement: Mathematics education through the lens of social justice: Acknowledgement, actions, and accountability, 1-8. Retrieved form totos-math.org.

IDENTITY AND POSITIONING DURING A TECHNOLOGY-ENHANCED MATHEMATICS TASK: WHO TAKES THE STAGE?

Samantha Fletcher
Middle Tennessee State University
scf3a@mtmail.mtsu.edu

Kristen Fye
University of North Carolina at Charlotte
krfye@uncc.edu

It is important that mathematics teachers cultivate the positive identities of their students, as identity is an essential dimension for equity in mathematics education (AMTE, 2017; Gutiérrez, 2012). Consequently, an important role of the mathematics teacher is to foster students' positive mathematics identities to create an equitable mathematics learning environment. Identities can be developed through positioning, meaning recognizing discursive actions as social acts within storylines familiar to participants in discourse (Harré & van Langenhove, 1991). The way students are positioned has been shown to positively and negatively affect their learning as they develop mathematics identities (Bishop, 2012; Tait-McCutcheon & Loveridge, 2016; Wood, 2013), suggesting it is a crucial component of identity development. From a perspective of storylines, mathematics identity can be thought of as "the repetition of 'performances' in mathematics learning contexts that generates our recognition of ourselves in certain ways as learners of mathematics" (Darragh, 2015, p. 85). As a student performs their mathematics identity, those they are interacting with are audience members that recognize and interpret the performance in different ways. In the context of a technology-enhanced mathematics task, students participate in the mathematics actively through their actions with the technology, so technology use can be thought of as part of their performance. The purpose of this study was to investigate how two students' mathematics identities were cultivated through being positioned as the mathematical expert – by themselves, each other, and the teacher – when engaging in a technology-enhanced mathematics task. We address the research question: How are mathematics identities performed and leveraged by students during a technology-enhanced mathematics task?

In this pilot case study, two ninth-grade students (age 14) attended an in-person after school session to engage with a task designed to leverage the power of sliders in Desmos to enhance and empower their learning around key characteristics of the sine function (i.e., amplitude, midline, and period). As they engaged in the activity, we collected screen capture recordings of their work with Desmos. We drew from Darragh (2015) to develop an analytical framework for examining their mathematical discourse, which, over time, constitutes their mathematical identities. Each instance of a student being the performer (which then made the teachers and the other student the audience) served as our unit of analysis, which Darragh referred to as "the stage." New stages presented themselves when the performer of mathematical expertise shifted as a result of social acts during the discourse. As a research team, we coded the performer/mathematical expert and the action which caused the shift in each stage. Preliminary findings indicate the teachers positioned the students as the performer/mathematical expert through questioning, and one student de-positioned herself as the performer/mathematical expert when struggling with the mathematical concepts. Additionally, both students were simultaneously positioned as the performer/mathematical expert when they grappled with or agreed on mathematical ideas. Although performer/mathematical expert was not always the person driving the technology, the actions with the technology were leveraged in their discourse. Implications for fostering positive identity development with technology-enhanced tasks will be shared.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at https://amte.net/standards

Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education*, *43*(1), 34-74. https://doi.org/10.5951/jresematheduc.43.1.0034

Darragh, L. (2015). Recognizing 'good at mathematics': Using a performative lens for identity. *Mathematics Education Research Journal, 27*(1), 83-102. https://doi.org/10.1007/s13394-014-0120-0

Harré, R. & Langenhove, L. V. (1991). Varieties of positioning. *Journal for the Theory of Social Behavior, 21*(4), 393-407. https://doi.org/10.1111/j.1468-5914.1991.tb00203.x

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education?. In B. Herbel-Eisenmann, J. Choppin, & D. Wagner (Eds.) *Equity in discourse for mathematics education* (pp. 17-33). Springer. https://doi.org/10.1007/978-94-007-2813-4_2

Tait-McCutcheon, S. L. & Loveridge, J. (2016). Examining equity of opportunities for learning mathematics through positioning theory. *Mathematics Education Research Journal, 28*, 327-348. https://doi.org/10.1007/s13394-016-0169-z

Wood, M. B. (2013). Mathematical micro-identities: Moment-to-moment positioning and learning in a fourth-grade classroom. *Journal for Research in Mathematics Education, 44*(5), 775-808. https://doi.org/10.5951/jresematheduc.44.5.0775

EXAMINING CONTENT AND STRUCTURE OF FLIPPED ALGEBRA I VIDEOS

Jaepil Han
University of Missouri
jaepilhan@mail.missouri.edu

Zandra de Araujo
University of Florida
zdearaujo@coe.ufl.edu

Courtney Vahle
University of Missouri
courtneyvahle@mail.missouri.edu

Samuel Otten
University of Missouri
ottensa@missouri.edu

Keywords: Online and Distance Education, High School Education, Technology

Flipped instruction is an instructional model in which content is delivered via video lessons students typically watch as homework. The increase in teachers' implementation of flipped instruction in conjunction with the rise of virtual schooling due to the COVID-19 pandemic has heightened attention and interest in the videos used to deliver content in mathematics classrooms. Despite the proliferation of such videos, little attention has been given to aspects of lecture videos in mathematics, such as what constitutes flipped lecture videos or how each lecture video is structured. To this end, the purpose of this study was to answer the question, What is the content and structure of lecture videos used in flipped Algebra I classrooms?

As part of a larger research project studying flipped Algebra I instruction, we analyzed the video structure (i.e., how each video introduced, explored, and summarized the mathematics content) of 59 lecture videos from 13 Algebra I teachers employing flipped instruction came from school districts of varying sizes throughout the state. Based on the shift of the images on the screen, the team identified the types, lengths, and sequences of video segments and summarized the purpose of each segment in addition to the basic instructional features and quality (e.g., focus, rationale, unmitigated errors) and video production (e.g., creator, interactive features) adapted from Authors (Han et al., 2019; Otten et al., 2021).

The teachers typically assigned just one video per lesson and each video ranged from 41 seconds to 23 minutes and 18 seconds with an average of 8 minutes and 8 seconds. All videos functioned to deliver content, so-called lecture videos. The videos were primarily composed of definitions, worked examples, and problem solutions. The most common sequence of video segments aligned with the typical model of direct instruction, rule-example-practice (Okeeffe, 2013). Given the videos' alignment with direct instruction, it is not surprising the segments comprising the teachers' videos largely centered on procedures and the enactment of those procedures. Interestingly, *Introduction*, *Worked Example*, and *Definition* were present in nearly every video (72.88%, 93.22%, and 62.71%, respectively) and constituted the majority of video duration (3.12%, 68.59%, and 17%, respectively). Other segment types (e.g., *Try Example*, *Solution*, *Homework*, *Conclusion*) only took up less than 6% of video duration in total.

Across the videos, there were no differences in terms of the content and structure of the videos and they were largely modeled after the textbook. Thus, as we have argued elsewhere (de Araujo et al., 2017), the videos are a form of digital textbook delivered by the teacher. Our findings shed light on the content and structure of the videos used in flipped classrooms. In addition, it provides a better understanding of current implementation of flipped instruction in Algebra 1 and a snapshot of how teachers aim to help students learn outside of the classroom environment, which potentially inform us of supports that we may offer teachers to improve the segments that they are using or promote alternative purposes and structures for the videos.

Acknowledgments

This study is funded by the National Science Foundation (#1721025). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

de Araujo, Z., Otten, S., & Birisci, S. (2017). Teacher-created videos in a flipped mathematics class: digital curriculum materials or lesson enactments?. *ZDM*, *49*(5), 687-699.

Han, J., Hirt, S., Kamuru, J., de Araujo, Z., Otten, S., & Zhao, W. (2019). Features of video homework in flipped Algebra instruction. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 141). St Louis, MO: University of Missouri.

Okeeffe, L. (2013). A framework for textbook analysis. *International Review of Contemporary Learning Research*, *2*(1), 1-13.

Otten, S., de Araujo, Z., Sherman, M., & Birişçi, S. (2021). A framework for capturing structural variation in flipped mathematics instruction. *International Journal of Mathematical Education in Science and Technology*, 1-32.

COMPARING EXPERT AND NOVICE TEACHERS' NOTICING WITH EYE-TRACKING IN 360 VIDEO

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Chris Lenart
Kent State University
clenart4@kent.edu

Qiang Guan
Kent State University
qguan@kent.edu

Keywords: Teacher Noticing; Technology

Teacher noticing involves attending to pedagogically significant events amongst an array of occurrences in the classroom, and using one's professional resources to reason about those events (Jacobs et al., 2010; van Es et al., 2017). Initially, more novice teachers may focus on the teacher or generic classroom behaviors, while more knowledgeable teachers focus on students and those students' mathematical reasoning (Huang & Li, 2012; Teuscher et al., 2017). Scholars studying teachers' eye-tracking data with standard videos have found preservice teachers (PSTs) attempt to focus on more students, with less time per student, and usually those proximally closer to the camera, whereas inservice teachers (ISTs) are more focused (Huang et al., 2021). In a similar manner, analysis of PSTs viewing of 360 video has found that those attending to students' mathematics with more specificity tend to focus on fewer students for longer durations (Kosko et al., 2021) and position students more in the center of their field of view (FOV) (Kosko et al., 2022). This scholarship provides evidence that where teachers gaze with their eyes (Huang et al., 2021) and turn their body/head (Kosko et al., 2022) are related to what and how they attend to in the classroom. The present study represents an effort to combine both technological approaches by examining ISTs (n=4) and PSTs' (n=10) professional noticing in relation to their eye-tracking data within a 360 video on the Commutative Property of Multiplication. Analysis of written noticing suggest that ISTs attended to students' reasoning about the property while PSTs focused on students' procedural knowledge. Results from eye-tracking data suggest ISTs (42.46%) and PSTs (47.23%) gazed at children for similar proportions of time. However, ISTs showed evidence of looking at different students and their work, as conveyed with larger average gaze distances from students (U=34.00, p=.048).

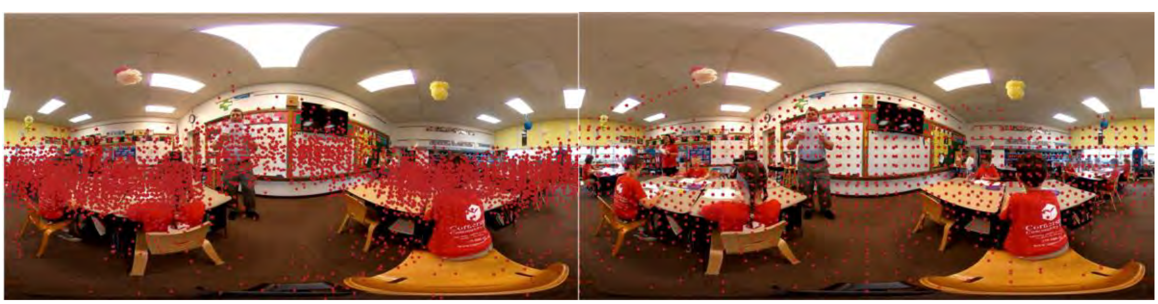


Figure 1: Cumulative Raw Gaze Data for Experts (left) and Novice (right) Teachers.

Acknowledgments

Research reported here received support from the National Science Foundation (NSF) through DRK-12 Grant #1908159. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

References

Huang, R. & Li, Y. (2012). What matters most: A comparison of expert and novice teachers' noticing of mathematics classroom events. *School Science and Mathematics, 112*(7), 420–432. https://doi.org/10.1111/j.1949-8594.2012.00161.x

Huang, Y., Miller, K. F., Cortina, K. S., & Richter, D. (2021a). Teachers' professional vision in action. *Zeitschrift für Pädagogische Psychologie*. https://doi.org/10.1024/1010-0652/a000313

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional Noticing of Children's Mathematical Thinking, *Journal for Research in Mathematics Education*, 41 (2), 169-202. https://www.jstor.org/stable/20720130

Kosko, K. W., Ferdig, R. E., & Zolfaghari, M. (2021). Preservice teachers' professional noticing when viewing standard and 360 video. *Journal of Teacher Education, 72*(3), 824–841. https://doi.org/10.1177/0022487120939544

Kosko, K. W., Heisler, J., & Gandolfi, E. (2022). Using 360-degree video to explore teachers' professional noticing. *Computers & Education, 180*. https://doi.org/10.1016/j.compedu.2022.104443

Teuscher, D., Leatham, K. R., & Peterson, B. E. (2017). From a framework to a lens: Learning to notice student mathematical thinking. In E. O. Schack, M. H. Fischer, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 31-48). Springer. https://doi.org/10.1007/978-3-319-46753-5_3

van Es, E. A., Cashen, M., Barnhart, T., & Auger, A. (2017). Learning to notice mathematics instruction: Using video to develop preservice teachers' vision of ambitious pedagogy. *Cognition and Instruction, 35*(3), 165-187. https://doi.org/10.1080/07370008.2017.1317125

INTEGRATIONS OF TECHNOLOGY AND CULTURALLY SUSTAINING MATHEMATICS PEDAGOGY FOR PRESERVICE TEACHERS

Idrissa Presley-Peterson
University of Florida
ipresleypeterson@ufl.edu

Hyunyi Jung
University of Florida
hyunyi.jung@coe.ufl.edu

Keywords: Technology, Culturally Sustaining Pedagogy, Elementary Preservice Mathematics Teachers

The objective of this study is to examine how elementary preservice teachers (PSTs) design and reflect on culturally sustaining mathematics lesson plans by utilizing instructional technology. Culturally sustaining pedagogy (CSP) values the development of student cultural competency, emphasizing the ever-changing states of student cultures and sociopolitical contexts surrounding students' lives (Paris, 2012). Technology is a tool that can support CSP and accessible STEM education, due to adaptability, engagement, and opportunities for student representation (Galbraith et al., 2001). The Association of Mathematics Teacher Educators [AMTE] (2017) emphasized the importance of preparing future teachers to deeply understand ways to increase equity and access for every student's mathematical success and the roles of technology in enhancing access to mathematics learning (AMTE, 2017). As teacher educators of elementary PSTs, we aimed to support their capacity to develop culturally sustaining mathematics lessons that incorporate instructional technology.

The participants of this study were PSTs enrolled in two sections of a mathematics education course for an inclusive elementary education program at a public Southeastern university in the U.S. PSTs interacted with weekly lessons on math-based technology, featuring instructional design and collaborative media, and learned about ways to teach mathematics to culturally and linguistically diverse students. All technology components were selected for potential integration into culturally sustaining mathematics teaching. Examples were game-based applications, augmented or virtual reality, real-time collaborative software, and digital note taking tools. Data include PSTs pre/post-surveys, weekly written reflections on the use of technology in mathematics classrooms and CSP, culturally sustaining mathematics lesson plans with technology integration, and microteaching of the lessons. Our preliminary data analysis on PSTs' work indicates that they gained an understanding of diverse technological tools, integration into mathematics teaching, and systemic barriers they were facing. For example, PSTs reflected upon ways to utilize students' funds of knowledge and instructional technology, to design and implement mathematics lesson plans that cultivate equitable learning experiences for all students. They discussed ways to honor students' cultures and identity by using technology in mathematics curricula while acknowledging the systemic barriers to implementing ideal lessons in an educational setting. In our poster session, we will showcase the utilization of technology to promote culturally sustaining mathematics teaching to address equity and universally accessible curriculum design. The specific ways PSTs created technology-based, culturally sustaining lesson plans to promote students' understanding of mathematics will provide insights into curricular designs for a similar course and PSTs' perspectives on these learning opportunities. We will create a space for discussion in which teacher educators share their experiences of incorporating technology and/or culturally sustaining pedagogy into programs. We hope to build future collaborations to support the learning of PSTs in multiple teacher education programs.

References

Association of Mathematics Teacher Educators. (2017). *Standards for Preparing Teachers of Mathematics.* Available online at amte.net/standards.

Garbraith, P., Goos, M., Renshaw, P., & Geiger, V. (2001). Integrating technology in mathematics learning: What some students say. *Numeracy and beyond*, 223-230.

Paris, D. (2012). Culturally sustaining pedagogy: A needed change in stance, terminology, and practice. *Educational researcher*, *41*(3), 93-97.

CONTENT KNOWLEDGE AND TEACHER NOTICING: THE CASE OF VERTICAL ASYMPTOTES

Demet Yalman Ozen
Middle Tennessee State University
dy2i@mtmail.mtsu.edu

Nina G. Bailey
University of North Carolina Charlotte
nbaile15@uncc.edu

The practice of teacher noticing (TN, i.e., attending, interpreting, and deciding how to respond; Jacobs et al., 2010) when focused on students' mathematical thinking in a technology-mediated learning environment is a core component of technological pedagogical content knowledge (TPACK; Niess, 2005) and is essential for mathematics instruction (Jacobs & Spangler, 2017). Some researchers conceptualize content knowledge as the foundation of TPACK (Lee & Hollebrands, 2011). This conceptualization, coupled with research that has found that content knowledge influences teachers' interpretations of students' understanding on paper and pencil tasks (e.g., Dick, 2017), suggests that it is important to study this relationship in the context of technology-mediated learning environments. Thus, we investigated the relationship between preservice secondary mathematics teachers' (PSMTs') content knowledge and their practice of TN of students' mathematical thinking in a technology-mediated learning environment. The content focused on the key characteristics of rational functions—specifically the existence and location of vertical asymptotes.

This study was situated in a course focused on teaching secondary mathematics with technology in Spring 2021. The course had an emphasis on supporting PSMTs in developing TPACK in general and TN very explicitly. All PSMTs were math majors and had successfully completed at least through Calculus 2. PSMTs completed two assessments, one focused on content and one on the practice of TN. The content assessment asked PSMTs how they would explain to a student why the vertical asymptote of a rational function is at a given value(s). The TN assessment presented PSMTs with a video clip of a pair of secondary students working on a technology-mediated task in which they used sliders to explore the parameters of rational functions to reason about the existence and locations of vertical asymptotes. The content assessment was scored based on PSMTs' reasoning (e.g., correct and complete, correct but incomplete, incorrect). The TN assessment was analyzed using a coding rubric which focused on PSMTs' skills of attention to and interpretation of students' spoken and written mathematical thinking and technology engagement. To develop the TN assessment coding rubric, we identified the mathematically significant details of the students' exploration in the video clip and included three levels (i.e., lacking, limited, and robust) on each of the four components similar to the coding scheme used by Jacobs et al. (2010). We then compared the TN evidence with the PSMTs reasoning about the existence and location of vertical asymptotes to assess any trends. Findings include that PSMTs who provided correct and complete responses to the content assessment provided either limited or robust evidence for their interpretations on the TN assessment. In comparison, PSMTs who responded with incorrect components on the content assessment provided lacking interpretation evidence on the TN assessment. Thus, preliminary findings suggest that although there may be a connection between PSMTs' knowledge of vertical asymptotes and the TN skill of attending, there is likely a stronger connection between PSMTs' knowledge of vertical asymptotes and the skill of interpreting in a technology-mediated learning environment. A full analysis will be presented along with implications for supporting PSMTs as they develop both content knowledge and TN.

References

Dick, L. K. (2017). Investigating the relationship between professional noticing and specialized content knowledge. In E. O. Schack, M. H. Fisher, & J. A. Wilhelm (Eds.), *Teacher noticing: Bridging and broadening perspectives, contexts, and frameworks* (pp. 339–358). Springer.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202. https://doi.org/10.5951/jresematheduc.41.2.0169

Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K-12 mathematics teaching. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 766–792). National Council of Teachers of Mathematics.

Lee, H. S., & Hollebrands, K. F. (2011). Characterising and developing teachers' knowledge for teaching statistics with technology. In C. Batanero, G. Burrill, & C. Reading (Eds.), *Teaching statistics in school mathematics-Challenges for teaching and teacher education* (pp. 359–369). Springer, Dordrecht.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, *21*(5), 509–523. https://doi.org/10.1016/j.tate.2005.03.006

DIGITALIZED INTERACTIVE ITEM COMPONENTS IN COMPUTER-BASED-ASSESSMENT IN MATHEMATICS FOR K12 STUDENTS: A RESEARCH SYNTHESIS

Moosa A. A. Alhadi, EdM
Rutgers University, NJ, USA
maa405@scarletmail.rutgers.edu

Dake Zhang, PhD.
Rutgers University, NJ, USA
dake.zhang@gse.rutgers.edu

Ting Wang, PhD.
ETS, NJ, USA
twang001@ets.org

Carolyn A. Maher, Ed.D.
Rutgers University, NJ, USA
carolyn.maher@gse.rutgers.edu

This research synthesizes studies that used a Digitalized Interactive Component (DIC) to assess K-12 student mathematics performance during Computer-based-Assessments (CBAs) in mathematics. A systematic search identified ten studies that categorized existing DICs according to the tools that provided language assistance to students and tools that supported students problem solving. We report on the one study that involved students with learning disabilities and three studies involved English Language Learners. One study focused on assessing geometry content and four studies targeted on number and operations understanding. For other studies included a mixture of mathematics domains. Mixed results were reported as to the effectiveness of the availability of DICs. The research suggests that older children were more likely to benefit from availability of the DIC than younger children, and that DICs have greater impact on students with special needs.

Keywords: Computerized mathematics tests, mathematics assessment, Interactive digitalized components

While there are multiple literature reviews evaluating the effectiveness of Computer-based-Assessment (CBA), no systematic review was found in evaluating the effects of Digitalized Interactive Components (DICs) that are used in CBAs. Computer- based-assessments (CBA) have been widely used in national and international large-scale tests as part of recent reform efforts stressing teaching, learning and testing in digital contexts. Important CBAs include high-stake state standard assessments (e.g., the Partnership for Assessment of Readiness for College and Careers, PARCC, the Smarter Balanced Assessment), the national report card (National Assessment of Educational Progress, NAEP), and the international comparison assessments (e.g., Program for International Student Assessment, PISA). The shift towards implementing CBAs for assessment can be an important improvement during testing, potentially impacting students' engagement, interpretations, and responses to test items (Jerrim, 2016).

CBAs are created to optimize learning goals and assessment techniques through high-quality tests (Smoline, 2008), and to improve the participation and performance of students with disabilities (Flowers et al., 2011). Computers provide the opportunity to offer tasks that display animated or dynamic changes in different aspects over time: Interactive games, simulations, and microworlds, that can be made by computers, enable exploring problems, discovering rules, forming relationships, and developing successful strategies. Computers also can help students deal with complicated data sets (Ridgway & McCusker, 2003). In a study to compare CBAs with pen-and-paper assessments (PPAs) for students with a read-aloud accommodation, Flowers et al. (2011) found that students prefer CBAs and report that they perform better when CBAs are available. According to Hoogland and Tout (2018), CBAs have advantages over traditional paper and pencil assessments (PPAs) in: Supporting the assessment of high-level mathematical thinking process, illustrating authentic problems from our real-life settings to implement mathematical concepts, and building tests and analyzing results through complicated psychometric processes with new techniques in scoring and reporting.

Digital-Interactive-Components

As a key element of CBA, DICs are part of the item stem with which students can interact and employ to answer the question. For example, an interactive ruler is an on-screen ruler that students can use to measure the length of objects in the screen. DICs commonly used in the current CBAs include, but are not limited to, drag-and-drop, annotations, changing the color of a shape upon clicking, etc. The NAEP items that have access to the touch-screen tablets mode include DICs such as using a read-aloud tool, making digital notes, adjusting screen themes, highlighting texts, dynamic texts, interactive maps, embedded digital calculators, interactive graphs, and virtual simulations (National Center for Educational Statistics, NCES, n.d.).

DICs enable practitioners to measure students' knowledge and skills in ways more authentic to a CBA environment (Bergner & von Davie, 2019; Lissitz & Jiao, 2012; Masters & Gushta, 2018), and to improve the measurement of student learning by targeting constructs that are otherwise difficult to capture with the PPA mode (Bennett et al., 2008; Chen & Perie, 2018; Jerrim, 2016). DICs are expected to offer benefits to make CBAs over traditional PPAs by engaging students in simulated real-world tasks and measuring aspects of a construct that cannot be measured with PPA items (Huff & Sireci, 2001). Some interactive tools require students to work with them in the process of producing products/responses, a more authentic form of measurement (Archbald & Newmann, 1988; Bennett, 1999; Harlen & Crick, 2003; Huff & Sireci, 2001; Masters & Gushta, 2018; McFarlane, et al., 2000). Interactive features of DICs have the potential to improve assessment accessibility for some disadvantaged groups, such as students with disabilities and English Learners (ELs). Further, technologies involving the use of DIC tools can enable the collection of real-time process data that could offer diagnostic assessment information for educators to support improved teaching and student learning (Jiang et al., 2021). Despite the potential benefits of DIC inclusion in CBAs and the emerging trend of their inclusion in testing, implementations of DICs in testing have been challenging.

A first challenge in current assessment practices is the lower percent of DICs in all CBA items. Despite the fast-growing research on educational technology (Lindner, 2020), many of the studies in the literature focus on technology as a tool for instruction, whereas sparse research has been conducted on technology as a tool for assessment (Cheng & Basu, 2009). Moreover, even with knowledge from the current CBA literature, computerized test items often apply traditional formatting similar to PPAs, such as multiple choice, true-and-false, and fill-in-the-blanks (Cheng & Basu, 2009), whereas the use of interactive features in rarely reported (Dindar et al., 2015).

A second challenge in the current assessment implementation of DICs lies in the poor design of existing DICs. Invalidated DICs might introduce construct-irrelevant variance even if students have mastered the content domain, thus diminishing validity and utility of scores for students of certain subgroups. Russel and Moncaleano (2019) examined the use and construct fidelity of technology-enhanced items employed in international K12 large-scale assessment programs, and reported that only 40% of the technology-enhanced items has good fidelity in measuring the aimed construct. In particular, a sizable percentage of drag-and-drop items has a low level of fidelity. Likewise, student cognitive interviews in NAEP 2016 indicated that students experienced tremendous confusions when interacting with DICs. More than half of the students encountered various difficulties using DICs, even though more than half of the students had experiences with similar tools. Their confusion centered around figuring out how to use the tools to respond to the assessment items rather than applying the tools in their problem solving.

Even more alarmingly, poorly designed DIC tools can become a source of bias towards students from under-represented groups. Studies show that students need to be trained to use computerized items that involve DICs since the absence of learning to use the tools properly can negatively impact potential benefits of DICs (Olson, 2005). Students with less access to learn digital skills are more vulnerable to benefit with potentially negative impacts on their performance using poorly designed DIC features (Jerrim, 2016; Schatz, et al., 2010; Stickney et al., 2012). The extent to which a disparity might be present for some students is still largely unexplored (Wang et al., 2021). According to Buzick, (2021), with the exception of embedded calculators, students with disabilities showed an even lower frequency of using any DICs than their peers without disabilities, even though all embedded DICs were rarely used in all students. It still remains unclear what roles specific types and features of DICs play in measuring students' knowledge and skills, and how these features might impact the performance of student subgroups. For example, research to date has yet to examine the degree to which items with DIC yield construct irrelevant variance for different subgroups of students with different socioeconomic status (SES) and gender.

Although there have been systematical reviews pertinent to comparing the different effects of PPA and CBA for K12 students on mathematics related measures, there is an absence of research reviews specifically on the availability and effects of existing DICs. To this end, we offer a research synthesis with the goal of exploring the use of interactive tools from research studies investigating students' mathematics performance. Questions that guide our review are:

1. What existing DICs are available for use in assessing student learning mathematics in the literature of CBA in mathematics? Specifically, what are the main features and purpose of the tools? How effective are they in capturing student performance? What student characteristics, such as age, gender, ability level, influence the effectiveness of these interactive tools?
2. What mathematics domains are involved in the reviewed DIC studies? Are the DICs more effective within certain mathematical content domains? Are certain item features more or less relevant?

Method

Search Procedures

For this literature review, we followed guidelines from the Preferred Reporting Items for Systematic reviews and Meta-Analyses (PRISMA; Moher et al. 2009) and three approaches were completed to accurately search for peer-reviewed journal articles. We conducted (a) an electronic search of databases with key words related to computer-based assessment, (b) another round of electronic search of databases with key words of specific DIC types, and (c) an ancestral search to identify any additional relevant articles.

First, we searched five databases: Academic Search Premier, ERIC, ProQuest Education Database, Teacher Reference Center, and APA PsycInfo (including PsycArticles). Within the abstracts of English studies, we searched using a combination of the following terms: "Computer based, assessment, mathematic*" and "Computer based, Test*, mathematic*" The total of the resulted studies was 1589, which were screened to 39 potentially viable studies for inclusion, based on titles and abstracts. Full-text review of each study led to an inclusion of eight articles for this synthesis.

Secondly, we also searched additional key words emphasizing interactive features with the above databases, with combined terms of "interactive technology", "computer* feature", "digit* feature", "interact* computer*", as well as specific DICs from the studies identified from the first round of search, including "calculator" "glossary" "drag drop" "animat*" and "digit* ruler." Every single phrase was attached using AND condition with the two terms "assessment" and "mathematic*." The total of the resulted studies was 444 and 2 additional studies were identified.

Thirdly, we examined the included studies reviewed by related meta-analysis or review synthesis. Specifically, we reviewed mathematics studies included in the three prior meta-analyses that was conducted by Kingston (2009), Wang et al. (2007), and by Kingston (2009) that reviewed research comparing PPA vs. CBA effects. No additional studies were identified beside these three meta-analysis studies.

Inclusion and Exclusion Criteria

In the research synthesis we only included quantitative studies that investigated the effectiveness of using DICs in assessing mathematics performance for K-12 students. Studies included in the review were required to meet all the three criteria: (a) inclusion of quantitative data of students' mathematics assessment results in CBA with specific DICs; (b) all participants were K-12 students; and (c) studies were published in peer-reviewed journals in English. Ten studies were identified, including six studies comparing a DIC condition with a non-DIC condition, and four studies exploring specific features of DIC conditions.

The following types of studies were excluded: (a) those that provided feedback or corrections in an assessment (e.g., Hippisley et al., 2005); (b) those based on computer-based programs with DICs as interventions rather than assessments (e.g., Crawford et al., 2016; Gambari et al., 2014; Ninness et al., 1998); (c) unpublished dissertations, conference presentations, technical reports (e.g., Bridgeman & Potenza, 1998; Christensen et al., 2014; Price et al., 2014); (d) qualitative studies (Pantzare, 2012); (e) focusing on non-mathematics subjects (e.g., Koong & Wu, 2010 on social studies; Cheng & Basu, 2009 on science); (f) those with participants not in K-12 (e.g., Arslan et al., 2020; Gallagher et al., 2002; Gulley et al., 2017; Mavridis & Tsiatsos, 2017), or (g) no specific DICs described (e.g., Logan, 2015).

Coding Procedure

The first and second author developed a coding scheme that included the following variables: (a) author(s) and the year of publication, (b) number of participants, (c) \DIC description, (d) participants' age or school grade level, (d) participants' other characteristics such as ability level, SES, El, etc., (e) mathematics content involved, (f) research design, (g) independent and dependent variable(s) and their means (M) and standard deviations (SD), (h) effect size (ES), and (i) narrative research findings. Coding reliability was found by calculating the percentage of agreement between two independent raters on 30% of the studies found in the first stage of searching and screening. The inter-rater reliability was 90.91%.

Results

Types of Existing DICs

To address our first research question regarding the availability of existing interactive tools and their effectiveness that were reported in the review literature of (K-12) mathematics CBAs, based on the functions of the DICs reviewed in this study, we categorized the existing DICs into two major types: (a) DICs that provided language or vocabulary assistance, and (b) DICs that enabled students to construct solutions. Four studies were conducted to investigate the impact of implementing interactive tools that provided language or vocabulary glossaries (Calhoon et al., 2000; Cohen, et al., 2017; Cohen et al., 2020; Roohr & Sireci, 2017). Six studies were found under the second category of effectiveness (Adesina et al., 2014; Applegate, 1993; Kong et al., 2018; Jiang et al., 2021; Ninaus et al., 2017; Threlfall et al., 2007).

Language tools. For the effectiveness category, four studies addressed the use of pop-glossary in mathematics assessment for students who have language or reading problems. Roohr and Sireci (2017) compared the effects of two DICs: A pop-up glossary tools and sticker paraphrasing tool for English language learners (EL s). Findings reported that the pop-up glossary did not provide definitions for all the words or phrases in the test, but only for those that were underlined. For example, when a student clicks on the underlined words or phrases, a window appears with a clarification using definitions, pictures, synonyms, or animations. The sticker paraphrasing tool is intended to provide less difficult paraphrasing after clicking on an icon. More than 2000 students participated in the study. The results show that students used the sticker paraphrasing tool more frequently than the pop-up glossary tool (d = -0.34). Regarding the item-level response time in seconds, results showed that all students took longer on the items that provided either of the two DICs compared with the items that did not provide DICs (d = 0.17). Unfortunately, this study did not report student scores, so we do not know if students scored higher or lower on items with DICs than items without DICs, or if students scored higher on items with one DIC than items with the other DIC.

Two other studies specifically examined the effects of the glossary pop-up tool. Cohen et al. (2017) investigated the impact of using pop-up English glossaries with audio through randomized controlled trials. More than 32,000 third- and-seventh-grade students participated in the study. The items were selected based on experts' judgements of the likelihood that EL s may encounter language difficulties when solving the problems. Math items from a field test were randomly selected to be provided with a pop-up English glossary, whereas the other items were not provided with the pop-up glossary. The results showed that the pop-up glossary slightly inhibited students' performance on mathematics assessments. In the subsequent study about the pop-glossaries, Cohen et al. (2020) scaled up the previous study and over 60000 students from 3rd-grade to 11th-grade participated. Results show that the pop-up glossary accommodation overall was effective for mathematics assessment and did not degrade the measured construct. However, ES could not be obtained for these two studies.

The reading-aloud tool is another DIC that falls into the first category of language assistance tools. Calhoon, et al. (2000) investigated the differences between using a teacher-read and a computer-read tool with and without video. They found that teacher-read tools, computer-read, and computer-read tools with video increased students’ scores on mathematics performance assessment in comparison to the traditional PPA without any reading accommodations. No significant difference was found among these three conditions.

Tools for constructing problem solutions. The second category of DICs identified from the reviewed papers deals with enabling or assisting test takers to construct responses or solutions by interacting with prompts from the computer screen. Adesina et al. (2014) investigated a computerized tool that enabled students to construct responses with an interactive digital device. Two panes were provided to students: the problem pane and the answer pane. The problem pane showed the worded problem and the answer pane showed students’ responses. The problem pane allows the exam taker to drag numeric values using one or both hands. The user could touch and hold two numeric values simultaneously. The touch-and-drag feature activated a menu that contained the main four symbols of the arithmetic operations (i.e., +, -, O, ÷). After selecting one of these symbols a numeric keypad and a text box were displayed. The study investigated the usability and applicability of the tool and compared study students' performance using PPAs, with 39 fifth grade participants. Results reported no significant difference between mathematics scores using the DIC and their corresponding scores in the traditional PPA condition, suggesting access to the DIC did not influence students' scores.

Applegate (1993) investigated the impact of a response-construction tool to test children's analogical reasoning. He compared student performance with a DIC condition and a PPA condition on geometry items and involved 24 kindergarteners. In the DIC condition, a joystick was used to enter data in the response portion. The examinee needed to put the cursor anywhere on the top graphic shapes and press the fire button. The examinee could choose a shape or color. If the correct object was chosen, then it was displayed in an empty box. If the incorrect color was chosen, then all objects were shaded with that color. The study revealed that the DIC condition is significantly more difficult than the PPA condition.

Threlfall et al. (2007), Kong et al., (2008) and Jiang (2021) examined the effects of a pull-and-drag tool. The tool in Threlfall et al., (2007) allowed students to select number cards and put them in boxes to perform an arithmetic operation). This study found that changing from PPA to CBA made little difference in testing validity and legitimacy, and also indicated that some PPA items showed poorer validity than the corresponding equivalent CBA items with DICs, possibly because an equivalent CBA version was less cognitive-demanding and made the problem solving process easier, for example, the pull-and-drag DIC might decrease the likelihood of making mistakes such as using numbers that are not mentioned in the question. Kong et al. (2018) compared the response time differences between computers and tablets on items either with a DIC (e.g., drag-and-drop) or without a DIC (e.g., filling- in-the blank, multiple choice). However, this study did not compare either students' accuracy or response time between items with the "drag-and-drop" DIC and items without a DIC. Jiang et al. (2021) studied students' performance on drag-and-drop items in a large-scale assessment (i.e., National Assessment of Educational Progress, NAEP) in the fourth and eighth graders. The results also revealed that students who answered the item correctly spent shorter time to respond compared to their peers who responded incorrectly.

Effects of tools. Mixed results of the DIC effectiveness were reported in two DIC categories. In all studies that compared differences in students' scores between a DIC condition and a non-DIC condition, only one study (Applegate, 1993) used a between-subject design and provided needed data to calculate an ES, whereas all other studies with a within-subject design did not provide needed data to calculate an ES. Consequently, we were unable to conduct a meta-analysis to estimate an overall effect size for each DIC category. Instead, we summarized the mixed findings, which seem to favor a positive effect of language tools -- two studies (Calhoon et al., 1997; Cohen et al., 2020) reported positive effects, one (Cohen et al., 2017) reported negative effects, and one did not report student accuracy scores. There also seemed to be a negative effect of the response construction category DIC, with two non-significant effects (Adesina et al., 2014; Threlfall et al., 2007), and one negative effects (Applegate, 1993).

Roles of Student Characteristics

To explain the varying effectiveness reported in research studies on the same type of DIC, we further examined whether and how student characteristics (e.g., age, grade level, ability level, EL status) and item features (e.g., mathematics subjects) were related to the effectiveness of DICs. Although we could not calculate weighted effect sizes or perform a meta-regression to detect any significant moderating effects, we identified some patterns, including the following: (a) DICs may inhibit young' children' performance whereas support older children's performance, and the positive effects of DICs increase with age or grade level, and (b) DICs seemed to have different effects in different subgroups, and generally speaking, it appeared to produce greater influences, either positive or negative, on students who were in greater need of help. For example, the language DICs had greater influence on ELs than non-EL students (Cohen et al., 2017; Cohen et al., 2020) and EL students also tended to use the language DICs more frequently (Roohr & Sireci, 2017); and across all 6 studies comparing non-DICs with DICs, students with disabilities benefited from the DIC (Calhoon et al. 2000) and general-education students did not show any changes (Adesina et al., 2014; Threlfall et al., 2007). However, the sample with gifted students was inhibited in a DIC condition (Applegate, 1993). Due to the limited number of studies available, we interpreted these patterns as a hypothesis for future research to verify, rather than as validated conclusions from this synthesis. Well-designed DICs could provide accommodations to address the special needs of students with disabilities and Els, and hence offer promise to make the assessments more assessable and equitable for these special student populations. With poorly-designed DICs, disadvantages can be exacerbated with special student populations by creating extra barriers and demanding greater cognitive load.

Effects of Mathematics Domains

Regarding the influence of mathematics domains and item features, the current findings reported mixed effects in both geometry items and numerical items. It is reasonable to expect that students might obtain greater benefits on geometry items than numerical items because geometry assessment items rely heavily on visual presentations for which DIC features may play a more effective role by enabling students' interactions with diagrams. However, three studies (Applegate, 1993; Cohen et al., 2020; Trelfall et al., 2017) that involved a form of geometry assessment suggested mixed findings. This may be partially attributed to the nature of the items intended to be measured, confounded by other variables such as text or perhaps the less-than-satisfactory geometry DIC effects to the very few available DICs. In the reviewed studies no DICs that might be particularly useful for geometry problem solving, such as drawing supplemental lines, rotations etc., were included, suggesting the need for further research in order to gain richer understanding of the potential benefits of well-designed geometry DICs.

Discussion

This study synthesized prior studies that examined the effects of specific Digitalized-Interactive-components (DICs) in computer-based assessment (CBA). Research questions included: What DICs were available in the literature of CBA in mathematics; what were the main features of these DICs, and how effective were they? How did students' age, gender, and ability level influence performance with the interactive tools? And what mathematical content domains were involved, and whether the DIC effectiveness was influenced by mathematics domains or item features? We systematically searched the literature and reviewed 10 studies that met our inclusionary criteria. The small number of studies identified suggested a need for more research in this important field. While CBA has already been widely implemented in high-stakes, large-scale assessments and DIC is an essential component to ensure the effectiveness of CBA, very little empirical data is available as to the effectiveness of DICs. Although in educational practice, there have been greater use of a variety of DICs that have been developed and used in school practice, results of our systematic search revealed that these DICs that are widely used in school-assessment practice were never validated, suggesting that the research base is lagging behind educational practice. We also discovered that 7 out of the 10 studies were published after 2014, including six studies published in 2017 –2021, suggesting that this is a topic gaining increased attention by the research community.

We categorized DICs that were used as a language tool, and DICs that were used to enable students to construct solutions. Both categories of DICs do not reflect the numerous DIC types in actual practice. For example, NAEP assessments, with the touch-screen tablets mode, have made available DICs such as the read-aloud tool, making digital notes, adjusting screen themes, highlighting texts, dynamic texts, interactive maps, embedded digital calculators, interactive mathematical graphs, and virtual simulations (NCES, n.d). However, most of these DICs are not included in any of the reviewed articles. Also, lacking were some sophisticated DICs that can be found in some commercial online assessment programs such as rotating/flipping/transforming images, digital rulers, annotations. Such a mismatch between the limited types of DICs in research papers and the rich resources of DIC options in practice certainly reflects the gap between the research and the rapid development of technology in the assessment industry. Ideally, educational practice should be guided by education theories and validated by evidence from rigorous educational research; however, in actual school practice it seems that the assessment industry has moved much faster than the tools included by researchers in the studies. Earlier literature (Kim, 1998; Logan, 2015; Smolinsky et al, 2020; Wang et al., 2007) shows that some DICs in practice inhibited, rather than supported students' mathematical problem solving during a CBA. Also, there remains a concern that a CBA with poorly designed DICs could create greater assessment biases against disadvantaged students (Pan, 2016). More research is warranted to ensure the development and validation of science-based assessment tools based on educational theories and cognitive principles.

A limitation of this synthesis is the sparsity of research available - only 10 studies found in peer-reviewed journals. Because access to technical reports produced by some testing companies or organizations (e.g., Educational Testing Service, or American Institute of Research) was not available, a decision was not to include these non-peer-reviewed reports. Moreover, insufficient information was available in the research publications, making us unable to calculate an ES for within-subject design studies. Therefore, the DIC effects and possible influences of participant characteristics and item features were presented as an overall summary of the findings of the related reviewed studies, rather than as a result from a rigorous meta-regression. Therefore, cautions need to be applied to the findings, such as DICs may inhibit young' children' performance whereas support older children's performance, and the positive effects of DICs increase with age or grade level, and DICs seemed to have different effects in different subgroups, and generally speaking, it appeared to produce greater influences, either positive or negative, on students who were in greater need of help.

Conclusions

In the field of mathematics assessment, numerous studies were conducted to investigate the impact of implementing computerized tools, but few studies have examined the effectiveness of implementing DICs. We identified 10 studies that implemented a DIC in a mathematics assessment. The DICs were categorized into two groups: Interactive tools that provided language or vocabulary assistance, and interactive tools that enabled students to construct solutions and ideas. DICs may have differential effects on children of varying ages, and appeared to produce greater influences, either positive or negative, on students who were in greater need of help. Further research studies should be conducted to investigate the effectiveness of implementing DICs in assessing mathematics performance of K-12 students of different ages and with varying special needs. The need for continued research that has the potential to gain greater insight into what knowledge can be gained when struggling students have access to appropriate tools is clear.

References

*Adesina, A., Stone, R., Batmaz, F., & Jones, I. (2014). Touch arithmetic: A process-based computer-aided assessment approach for capture of problem-solving steps in the context of elementary mathematics. *Computers & Education, 78*, 333–343. https://doi.org/10.1016/j.compedu.2014.06.015

* Applegate, B. (1993). Construction of geometric analogy problems by young children in a computer-based test. *Journal of Educational Computing Research, 9*, 61-77. https://doi.org/10.2190/F77B-PPYA-V1EX-LWDY

Aspiranti, K. B., Henze, E. E. C., & Reynolds, J. L. (2020). Comparing paper and tablet modalities of math assessment for multiplication and addition. *School Psychology Review, 49*(4), 453–465. https://doi.org/10.1080/2372966X.2020.1844548

Archbald, & Newmann, F. A. (1988). *Beyond standardized testing : assessing authentic academic achievement in the secondary school.* National Association of Secondary School Principals.

Arslan, B., Jiang, Y., Keehner, M., Gong, T., Katz, I. R., & Yan, F. (2020). The effect of drag-and-drop item features on test-taker performance and response strategies. *Educational Measurement: Issues & Practice, 39*(2), 96–106. https://doi.org/10.1111/emip.12326

Ashton, H. S., Beevers, C. E., Korabinski, A. A., & Youngson, M. A. (2006). Incorporating partial credit in computer-aided assessment of mathematics in secondary education. *British Journal of Educational Technology, 37*(1), 93–119. https://doi.org/10.1111/j.1467-8535.2005.00512.x

Bennett, R. E. (1999). Computer-based testing for examinees with disabilities: On the road to generalized accommodation. *Assessment in higher education: Issues of access, quality, student development, and public policy*, 181-191.

Bennett, S., Maton, K., & Kervin, L. (2008). The digital natives debate: A critical review of the evidence. *British Journal of Educational Technology, 39*(5), 775–786. https://doi.org/10.1111/j.1467-8535.2007.00793.x

Bridgeman, B., Lennon, M. L., & Jackenthal, A. (2003). Effects of screen size, screen resolution, and display rate on computer-based test performance. *Applied Measurement in Education, 16*(3), 191–205. https://doi.org/10.1207/S15324818AME1603_2

Bergner, Y., & von Davier, A. A. (2019). Process Data in NAEP: Past, Present, and Future. *Journal of Educational and Behavioral Statistics, 44*(6), 706–732. https://doi.org/10.3102/1076998618784700

Bergstrom, B. A. (1992). *Ability Measure Equivalence of Computer Adaptive and Pencil and Paper Tests: A Research Synthesis.* Paper presented at the Annual Meeting of the American Educational Research Association (Sa Francisco, CA, April 20-24, 1992).

Bridgeman, B., & Potenza, M. (1998, April 12-16). *Effects on performance on the computerized SAT I: Reasoning test in mathematics*. The annual meeting of the national council on measurement in education, San Diego.

Buzick (2021, June). Exploring item level use of accessibility supports [Paper Presentation]. Virtual Conference of Educational Accommodations: Future directions.

* Calhoon, M., Fuchs, L. & Hamlett, C. (2000). Effects of computer-based test accommodations on mathematics performance assessments for secondary students with learning disabilities. *Learning Disability Quarterly, 23*, 271-282. https://doi.org/10.2307/1511349

Cheng, I., & Basu, A. (2009). Interactive graphics for computer adaptive testing. *Computer Graphics Forum, 28*(8), 2033–2045. https://doi.org/10.1111/j.1467-8659.2009.01427.x

Chen, J., & Perie, M. (2018). Comparability within Computer-Based Assessment: Does Screen Size Matter? *Computers in the Schools, 35*(4), 268–283. https://doi.org/10.1080/07380569.2018.1531599

Christensen, L. L., Shyyan, V., Rogers, C., & Kincaid, A. (2014). *Audio support guidelines for accessible assessments: Insights from cognitive lab*s. Minneapolis, MN: University of Minnesota, Enhanced Assessment Grant (#S368A120006), U.S. Department of Education.

* Cohen, D. J., Ballman, A., Rijmen, F., & Cohen, J. (2020). On the reliable identification and effectiveness of computer-based, pop-up glossaries in large-scale assessments. *Applied Measurement in Education, 33*(4), 378–389. https://doi.org/10.1080/08957347.2020.1789137

* Cohen, D., Tracy, R., & Cohen, J. (2017). On the effectiveness of pop-up English language glossary accommodations for EL students in large-scale assessments. *Applied Measurement in Education, 30*(4), 259–272. https://doi.org/10.1080/08957347.2017.1353986

Crawford, L., Higgins, K. N., Huscroft-d'angelo, J.,N., & Hall, L. (2016). Students' use of electronic support tools in mathematics. *Educational Technology, Research and Development, 64*(6), 1163-1182. https://doi.org/10.1007/s11423-016-9452-7

Dindar, M., Kabakçı Yurdakul, I., & Dönmez, F. (2015). Measuring cognitive load in test items: static graphics versus animated graphics. *Journal of Computer Assisted Learning, 31*(2), 148–161. https://doi.org/10.1111/jcal.12086

Flowers, C., Kim, D., Lewis, P., & Davis, V. C. (2011). A comparison of computer-based testing and pencil-and-paper testing for students with a read-aloud accommodation. *Journal of Special Education Technology, 26*(1), 1-12. https://doi.org/10.1177%2F016264341102600102

Gallagher, A., Bennett, R., Cahalan, C., & Rock, D. (2002). Validity and fairness in technology- based assessment: Detecting construct- irrelevant variance in an open-ended, computerized mathematics task. *Educational Assessment, 8*(1), 27–41. https://doi.org/10.1207/S15326977EA0801_02

Gambari, I. A., Ezenwa, V. I., & Anyanwu, R. C. (2014). Comparative effects of two modes of computer-assisted instructional package on solid geometry achievement. *Contemporary Educational Technology, 5*(2), 110–120. https://doi.org/10.30935/cedtech/6119

Gulley, Ann P, Smith, Luke A, Price, Jordan A, Prickett, Logan C & Ragland, Matthew F. (2017). Process-driven math: An auditory method of mathematics instruction and assessment for students who are blind or have low vision. *Journal of Visual Impairment & Blindness, 111*, 465-471. https://doi.org/10.1177/0145482X1711100507

Harlen, W., & Deakin Crick, R. (2003). Testing and Motivation for Learning. *Assessment in Education : Principles, Policy & Practice, 10*(2), 169–207. https://doi.org/10.1080/0969594032000121270

Hassler Hallstedt, & Ghaderi, A. (2018). Tablets instead of paper-based tests for young children? Comparability between paper and tablet versions of the mathematical Heidelberger Rechen Test 1-4. *Educational Assessment, 23*(3), 195–210. https://doi.org/10.1080/10627197.2018.1488587

Hensley, K. K. (2015). *Examining the effects of paper-based and computer-based modes of assessment on mathematics curriculum-based measurement*. ProQuest Dissertations Publishing

Hippisley, J., Douglas, G. & Houghton, S. (2005). A cross-cultural comparison of numeracy skills using a written and an interactive arithmetic test. *Educational Research, 47*, 205-215. https://doi.org/10.1080/00131880500104325

Hoogland, K., Tout, D. (2018). Computer-based assessment of mathematics into the twenty-first century: pressures and tensions. ZDM : *The International Journal on Mathematics Education, 50*(4), 675–686. https://doi.org/10.1007/s11858-018-0944-2

Huff, K. L., & Sireci, S. G. (2001). Validity Issues in Computer-Based Testing. *Educational Measurement, Issues and Practice, 20*(3), 16–25. https://doi.org/10.1111/j.1745-3992.2001.tb00066.x

Jerrim, J. (2016). PISA 2012: how do results for the paper and computer tests compare? *Assessment in Education: Principles, Policy & Practice, 23*(4), 495-518. http://dx.doi.org/10.1080/0969594X.2016.1147420

* Jiang, Y., Gong, T., Saldivia, L., Cayton-Hodges, G., & Agard, C. (2021). Using process data to understand problem-solving strategies and processes for drag-and-drop items in a large-scale mathematics assessment. *Large-Scale Assessments in Education, 9*(1), 1–31. https://doi.org/10.1186/s40536-021-00095-4

Kim, J.-P. (1999, October 13-16). *Meta-analysis of equivalence of computerized and P&P tests on ability measures* [Conference Session]. The annual meeting of the Mid-Western Educational Research Association, Chicago, IL.

Kingston, N. M. (2009). Comparability of computer- and paper-administered multiple-choice tests for K-12 populations: A synthesis. *Applied Measurement in Education, 22*(1), 22–37. https://doi.org/10.1080/08957340802558326

Koong, C., & Wu, C. (2010). An interactive item sharing website for creating and conducting on-line testing. *Computers and Education, 55*(1), 131–144. https://doi.org/10.1016/j.compedu.2009.12.010

* Kong, X., Davis, L., McBride, Y., & Morrison, K. (2018). Response time differences between computers and tablets. *Applied Measurement in Education, 31*(1), 17–29. https://doi.org/10.1080/08957347.2017.1391261

Lee, G., & Weerakoon, P. (2001). The role of computer-aided assessment in health professional education: a comparison of student performance in computer-based and paper-and-pen multiple-choice tests. *Medical Teacher, 23*(2), 152–157. https://doi.org/10.1080/01421590020031066

Lissitz, R. W., & Jiao, H. (Eds.). (2012). Computers and their impact on state assessments: Recent history and predictions for the future. *IAP*

Logan, T. (2015). The influence of test mode and visuospatial ability on mathematics assessment performance. *Mathematics Education Research Journal, 27*(4), 423-441. https://doi.org/10.1007/s13394-015-0143-1

Masters, J., & Gushta, M. (2018). *Using Technology-Enhanced Items to Measure Fourth Grade Geometry Knowledge*.

McFarlane, A., Williams, J., & Bonnett, M. (2000). Assessment and multimedia authoring - a tool for externalising understanding: Assessment and multimedia authoring. *Journal of Computer Assisted Learning, 16*(3), 201–212. https://doi.org/10.1046/j.1365-2729.2000.00133.x
Moher, D., Liberati, A., Tetzlaff, J., & Altman, D. G. (2009). Preferred reporting items for systematic reviews and meta-analyses: the PRISMA statement. *BMJ, 339*(7716), e78–336. https://doi.org/10.1136/bmj.b2535
Mavridis, A., & Tsiatsos, T. (2017). Game-based assessment: investigating the impact on test anxiety and exam performance. *Journal of Computer Assisted Learning, 33*(2), 137-150. https://doi.org/10.1111/jcal.12170
National Center for Educational Statistics (NCES). (n.d.). *Digitally based assessment – NAEP*. https://nces.ed.gov/nationsreportcard/dba/
* Ninaus, M., Kiili, K., McMullen, J., & Moeller, K. (2017). Assessing fraction knowledge by a digital game. *Computers in Human Behavior, 70*, 197–206. https://psycnet.apa.org/doi/10.1016/j.chb.2017.01.004
Ninness, H. A. Chris, Ninness, Sharon K, Sherman, Sandra & Schotta, Chuck. (1998). Augmenting computer-interactive self-assessment with and without feedback. *The Psychological Record, 48*(4), 601-616.
Olson, L. (2005). Impact of paper-and-pencil, online testing is compared. *Education Week, 25*(1), 14.
Pantzare. (2012). Students' use of CAS calculators--effects on the trustworthiness and fairness of mathematics assessments. *International Journal of Mathematical Education in Science and Technology, 43*(7), 843–861.
Price, B., Steinle, V., Stacey, K., Gvozdenko, E. (2014). *Using Percentages to Describe and Calculate Change* [Conference Session]. The Annual Meeting of the Mathematics Education Research Group of Australasia (MERGA), Sydney, New South Wales, Australia.
Ridgway, J., & McCusker, S. (2003). Using computers to assess new educational goals. *Assessment in Education: Principles, Policy & Practice, 10*(3), 309–328. https://psycnet.apa.org/doi/10.1080/0969594032000148163
* Roohr, K. C., & Sireci, S. G. (2017). Evaluating computer-based test accommodations for English learners. *Educational Assessment, 22*(1), 35–53.
Russell, M., & Moncaleano, S. (2019). Examining the use and construct fidelity of technology-enhanced items employed by K-12 testing programs. *Educational Assessment, 24*(4), 286–304. https://doi.org/10.1080/10627197.2019.1670055
Schatz, P., Neidzwski, K., Moser, R. S., & Karpf, R. (2010). Relationship Between Subjective Test Feedback Provided by High-School Athletes During Computer-Based Assessment of Baseline Cognitive Functioning and Self-Reported Symptoms. *Archives of Clinical Neuropsychology, 25*(4), 285–292. https://doi.org/10.1093/arclin/acq022
Smoline, D. V. (2008). Some problems of computer-aided testing and "interview-like tests". *Computers & Education, 51*(2), 743–756. https://doi.org/10.1016/j.compedu.2007.07.008
Smolinsky, L., Marx, B. D., Olafsson, G., & Ma, Y. A. (2020). Computer-based and paper-and-pencil tests: A study in calculus for STEM majors. *Journal of Educational Computing Research, 58*(7), 1256–1278. https://doi.org/10.1177%2F0735633120930235
Steedle, J., Pashley, P., Cho, Y., & ACT, I. (2020). Three studies of comparability between paper-based and computer-based testing for the ACT. *ACT Research & Policy*. https://www.act.org/content/dam/act/unsecured/documents/R1847-three-comparability-studies-2020-12.pdf
Stickney, E. M., Sharp, L. B., & Kenyon, A. S. (2012). Technology-Enhanced Assessment of Math Fact Automaticity: Patterns of Performance for Low- and Typically Achieving Students. *Assessment for Effective Intervention, 37*(2), 84–94. https://doi.org/10.1177/1534508411430321
* Threlfall, J., Pool, P., Homer, M., & Swinnerton, B. (2007). Implicit aspects of paper and pencil mathematics assessment that come to light through the use of the computer. *Educational Studies in Mathematics, 66*(3), 335–348. https://doi.org/10.1007/s10649-006-9078-5
Wang, S., Jiao, H., Young, M. J., Brooks, T., & Olson, J. (2007). A meta-analysis of testing mode effects in grade K-12 mathematics tests. *Educational and Psychological Measurement, 67*(2), 219–238. https://doi.org/10.1177/0013164406288166
Wang, T.-H, Kao, C.-H., & Chen, H.-C. (2021). Factors Associated with the Equivalence of the Scores of Computer-Based Test and Paper-and-Pencil Test: Presentation Type, Item Difficulty and Administration Order. *Sustainability, 13*(17), 9548–. https://doi.org/10.3390/su13179548
Yao, D. (2020). A comparative study of test takers' performance on computer-based test and paper-based test across different CEFR levels. *English Language Teaching, 13*(1), 124–133. http://dx.doi.org/10.5539/elt.v13n1p124

INTERACCIÓN CON REPRESENTACIONES DINÁMICAS PARA ARGUMENTAR SOBRE LA VALIDEZ DE UNA CONSTRUCCIÓN GEOMÉTRICA

INTERACTION WITH DYNAMIC REPRESENTATIONS TO ARGUE ABOUT THE VALIDITY OF A GEOMETRY CONSTRUCTION

María Alejandra Calderón González
CINVESTAV
alejandra.calderon@cinvestav.mx

Luis Enrique Moreno Armella
CINVESTAV
lmorenoa@cinvestav.mx

Desde la matemática educativa, gran variedad de estudios se ha enfocado en analizar la presencia de la tecnología digital desde diferentes puntos de vista. Quienes abordan la prueba y la argumentación se han resistido a que se presenten resultados con un componente tecnológico, debido a la idea de formalidad asociada estos términos, lo que ha aumentado la tensión existente entre la naturaleza teórica y empírica de la matemática. Teniendo en cuenta que con la presencia de entornos como DGS el estudiante puede acceder a representaciones dinámicas, nosotros buscando entender el impacto de la interacción con estas representaciones en los argumentos que proponen los estudiantes. Para esto realizamos un estudio de tipo exploratorio y descriptivo cuyos resultados muestran que la interacción con representaciones dinámicas brinda a los estudiantes herramientas de validación diferentes a las del papel.

Palabras clave: Razonamiento y prueba.

Introducción y antecedentes

Argumentar y validar proposiciones son habilidades que el ser humano necesita para desenvolverse en sociedad. Desde muy temprano, uno de los objetivos educativos consiste en promover la participación de los estudiantes para la comprensión y desarrollo de argumentos. Por ejemplo, la NCTM (2000) propone que "Al final de la escuela secundaria, los alumnos deberían estar capacitados para comprender y elaborar demostraciones matemáticas, es decir, argumentos que consisten en deducciones o conclusiones lógicamente rigurosas a partir de hipótesis" (p. 59). Todo esto es analizado desde la perspectiva del profesor (Bleiler et al, 2014; Lesseig et al, 2019; Dickerson y Doerr, 2014), desde las producciones del estudiante (Buchbinder y Zaslavsky, 2011), y desde la incorporación de la tecnología digital (Leung y Or, 2007; Olivero y Robutti, 2001; Sinclair y Robutti, 2012).

En el salón de clases el profesor es quien usualmente promueve el desarrollo de argumentos y pruebas, por lo que se han desarrollado investigaciones en las que se exploran las ideas que tienen los profesores sobre este tema (Bleiler et al, 2014; Lesseig et al, 2019). Un hallazgo en este sentido es que los profesores consideran la formalidad y la lógica como factores relevantes para determinar la validez de un argumento y más aún cuando son profesores con pocos años de experiencia (Dickerson y Doerr, 2014). Sin embargo, para algunos autores (Wittmann, 2021; de Villiers, 2004) este énfasis formalista sobre la prueba inhibe la importancia de la comprensión, por lo que se propone que para la educación matemática el objetivo sea que los estudiantes desarrollen distintos tipos de *razonamientos* que les permitan comprender y argumentar por qué una proposición es válida (Hanna, 2014).

La geometría es una de las áreas de las matemáticas en la que el desarrollo de argumentos y pruebas se puede dar de manera natural, ya que "los métodos geométricos son, hasta cierto punto, una combinación de ver y razonar, ya que, allí, la razón comprueba los desarrollos lógicos, y los guía sobre lo que los ojos ven en la figura" (Northrop, 1968, p.131). En particular, el razonamiento se desarrolla sobre una figura, de modo que la figura es parte constitutiva del proceso argumentativo. Esta misma idea es compartida por Netz (1988) quien afirma que, en el caso de la geometría griega clásica, el diagrama es la esencia de la prueba.

En clase de geometría se busca que los estudiantes expresen sus razonamientos para ilustrar la generalidad de un hecho a partir del diagrama, sin embargo, esto no siempre es posible debido a que generalizan a partir de un solo caso (Stylianides y Stylianides, 2017). Por ejemplo, piensan que la mediana y la mediatriz de un triángulo son el mismo objeto porque al trazarlas con regla y compás estás coinciden para el triángulo construido (isósceles). Este tipo de situaciones han hecho que la incertidumbre sobre la generalidad de un hecho sea una pieza clave para incentivar en los estudiantes la necesidad de producir pruebas deductivas (Buchbinder y Zaslavsky, 2011). Sin embargo, cuando las representaciones son dinámicas (e.g., presentadas en GeoGebra) se puede generar un alto nivel de certeza mediante el arrastre de puntos de una construcción y la observación de que las propiedades se mantienen a través de las distintas figuras resultantes. Esta percepción de la permanencia de una propiedad (que ya no sería accidental), es lo que lleva a proponer otras estrategias y funciones de la prueba diferentes a la tradicional (de Villiers, 2004). Además, el acceso a una pantalla digital ha promovido el estudio del impacto de las tecnologías digitales en el cambio sufrido en los razonamientos y las pruebas que proponen los estudiantes.

Existe un gran número de investigaciones en las que se analiza desde diferentes perspectivas el uso de geometría dinámica en procesos de argumentación y prueba. Algunas se han enfocado en tipificar los argumentos que proponen los estudiantes, concluyendo que la mayoría son de tipo empírico (Morales et al.,2021); otros estudios han buscado analizar cómo la interacción con los objetos de la geometría dinámica puede servir como trabajo previo para proponer y construir pruebas deductivas/teóricas (Leung y Or, 2007; Olivero y Robutti, 2001; Dello-Iacono, 2021). Sin embargo, también se ha reconocido que la interacción con representaciones dinámicas puede llevar a nuevas formas de razonamiento genuinas (Moreno-Armella y Brady, 2018) y por lo tanto a establecer nuevas ideas sobre lo que significa una prueba cuando se trabaja con representaciones digitales. El estudio que aquí se reporta, investigó ¿cómo es la interacción entre el estudiante y las representaciones dinámicas al momento de realizar construcciones y argumentar sobre su validez?

Teniendo en cuenta la importancia de la interacción entre el estudiante y las representaciones dinámicas, a continuación, presentamos el constructo teórico de la *co-acción*, el cual extiende algunas ideas de génesis instrumental. Además, presentaremos una visión general de la argumentación que no expresa la necesidad de un nivel de formalidad.

Perspectiva Teórica

Cada vez más las estructuras curriculares están siendo habitadas por la tecnología digital mediante los sistemas de representación que reaccionan de inmediato a las acciones del estudiante, permitiendo, por una parte, que algunas actividades que se pueden adelantar con lápiz y papel se desarrollen de una forma más eficiente y se eviten errores de cálculo —la herramienta como *amplificadora*—. Sin embargo, la ejecutabilidad de estas representaciones hacen posible que el estudiante pueda transformar sus conocimientos y acceder a ideas nuevas —la herramienta como *reorganizadora conceptual*—. Esto es particularmente tangible en los entornos de geometría dinámica (Moreno-Armella y Sriraman, 2005).

La acción de *arrastrar* un vértice de un triángulo, por ejemplo, genera una respuesta del medio digital que gradualmente va develando la naturaleza estructural de la construcción que se haya realizado. Hay allí un espacio de exploración nuevo para el estudiante, pues la pantalla digital es un espacio que *dialoga* con el estudiante a través de las representaciones digitales. La ejecutabilidad de las representaciones se traduce en la capacidad de respuesta del medio durante una exploración y contribuye al desarrollo de nuevas maneras de pensar. Termina generándose una dialéctica que ha sido estudiada como la génesis instrumental (Rabardel y Beguin, 2005), pero en un salón de clases se aprende *de, con y a través* de los otros. Hay formas de articular el pensamiento propio con el pensamiento de los demás incluidas sus experiencias con el medio digital presente. Esta *co-acción cognitiva* (Hegedus y Moreno-Armella, 2010) es un rasgo central de la mediación instrumental.

Si bien las posibilidades abiertas por los medios digitales de cómputo numérico y visualización han cuestionado la demostración tradicional sobre papel y lápiz como único criterio de validación matemático, ubicados en el terreno de la educación matemática y viendo la *construcción del sentido* como un objetivo central del aprendizaje, consideramos que hay, aparte del tradicional, otros caminos para construir la justificación de una proposición (digamos geométrica) en el salón de clases. La exploración de una situación geométrica (nuestro interés aquí) encuentra una respuesta *teóricamente controlada* del lado del mediador digital—por ejemplo, Geogebra. Al adelantar una exploración, el estudiante encuentra en la pantalla una respuesta a sus acciones a través de la plasticidad de las representaciones. De esa manera su entendimiento matemático va encontrando y desechando rutas de aproximación a una respuesta plausible a un problema o a una demanda de justificación. Por lo tanto, los argumentos matemáticos pueden ser abordados desde una perspectiva general como una línea de *razonamiento* que pretende mostrar y/o convencer de que un resultado (una declaración general sobre un objeto, una solución a un problema, un cálculo) es correcto (Sriraman y Umland, 2020; Hanna, 2020). Nuestro trabajo responde a las líneas aquí trazadas.

Metodología

Situamos nuestra investigación dentro de un enfoque cualitativo centrado en comprender, profundizar y describir fenómenos a través de la perspectiva de los participantes (Hernández et al., 2010). En este caso buscamos comprender la interacción con las representaciones dinámicas al momento de argumentar. Seguimos la línea de las investigaciones exploratorias y descriptivas (Steffe y Thompson, 2000) ya que buscamos familiarizarnos con los modos y formas de operar de los participantes.

Los participantes fueron 6 profesores de matemáticas que estaban llevando un curso de geometría como parte de un programa de maestría. Para este reporte, describimos el caso de Ana quien manifestó no estar familiarizada con los contenidos de geometría (ya que nunca ha enseñado esta materia y solo llevó un curso sobre este tema durante su licenciatura) y quién demostró gran destreza con el manejo de GeoGebra.

El cuestionario propuesto a los participantes consistía en seis construcciones que debían realizar y argumentar por qué su solución era válida. Posteriormente se realizó una entrevista semi estructurada a cada participante con el fin de profundizar en sus argumentos/ razonamientos y en la interacción con el medio digital.

Para el análisis de los datos se extrajeron los fragmentos de la entrevista en los que se evidenciaba un proceso donde: los estudiantes realizaban acciones sobre el medio digital, interpretaban la respuesta del medio (la información de la pantalla) y esta información los llevaba a realizar nuevas acciones. En general, nosotros analizamos cómo este proceso contribuyó a que los estudiantes pudieran argumentar sobre la validez de su construcción.

Resultados

En esta sección analizamos las interacciones de los estudiantes con el entorno digital, en particular, la naturaleza de los argumentos que se producen gracias a dicha interacción. Encontramos tres usos que describimos a continuación.

GeoGebra como amplificador

El entorno es, en un momento inicial, un amplificador que le permite a los estudiantes hacer explícitas sus ideas sobre las circunferencias, elipses, mediatrices y demás objetos geométricos. En ese momento sus ideas aún no han sido afectadas por el nuevo sistema de representación. Lo ilustramos primero con el trabajo de Ana, quien aborda el problema: dado un ángulo y una recta transversal a los lados del ángulo hallar un punto sobre la transversal equidistante de los lados del ángulo. En su respuesta Ana afirma que el punto buscado está en la intersección de la bisectriz y la transversal, por lo que durante la entrevista se le solicita que realice la construcción, dando paso al siguiente dialogo.

[1]E: ¿Qué trazaste ahí?
[2]A: Una bisectriz.
[3]E: ¿Por qué la bisectriz?
[4]A: Porque la bisectriz me va a dar todos los puntos que equidistan de (…) de ambos lados.

La construcción de Ana está guiada por su conocimiento teórico, por lo que el medio digital (GeoGebra) se usa como una herramienta que le permite construir la bisectriz del ángulo dado de manera casi idéntica a cómo lo haría con lápiz y papel.

En cuanto al argumento, Ana *concluye* que el punto de intersección entre la bisectriz y la transversal equidista de los lados del ángulo porque existe una *regla general* vinculada a la bisectriz que le permite afirmar dicha equidistancia (línea 4, diálogo anterior).

GeoGebra como reorganizador conceptual

El medio se ha convertido en un reorganizador conceptual cuando el razonamiento del estudiante trae indeleble la presencia del mediador digital, de modo que el estudiante reconfigura sus argumentos incluyendo la *respuesta* del mediador. Esto se ilustra cuando Ana debe probar que los puntos de la bisectriz equidistan de los lados de un ángulo, hecho que usualmente se prueba determinando dos triángulos congruentes, sin embargo, Ana usa otra estrategia (ver Figura 1) y dice:

[5]A: ¿Por qué equidista? Porque, pues (…) puedo dibujar esta circunferencia con centro aquí (señala H) que es tangente a este lado y a este (señala los dos lados del ángulo).
[6]E: Ok. Y, ¿el poder dibujar esa circunferencia me permite verificar la equidistancia?
[7]A: Aja (mueve constantemente el punto B por la pantalla).
[8]E: ¿Por qué? ¿qué característica da (…)? ¿por qué no una elipse, o un cuadrado?
[9]A: Ah, pues porque la distancia del centro a cualquier punto, pues es la misma, desde aquí que desde aquí (señala dos puntos de la circunferencia).

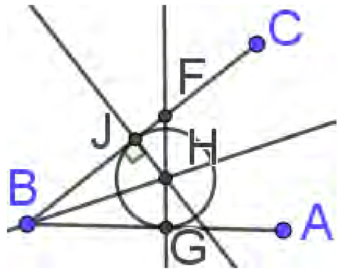


Figura 1: Construcción de Ana para probar una propiedad de la bisectriz

Para probar que los puntos de la bisectriz equidistan de los lados del ángulo, Ana guiada por sus *conocimientos* sobre las propiedades de las circunferencias (línea 9) *construye* una circunferencia con centro en H que es tangente a uno de los lados del ángulo (BC), anticipando que la circunferencia resultará tangente al otro lado. La *respuesta* dada por el medio gracias a la exactitud de los trazos y la geometría *incrustada* en ese medio permite que Ana configure su argumento basado en lo que ve en la pantalla. En este mismo sentido, Ana usa la función de arrastre para ilustrar la generalidad del hecho cuando *arrastra* uno de los puntos de la construcción y la *respuesta* que le da el medio es que la circunferencia permanece tangente a ambos lados del ángulo.

Sobre el argumento de Ana, ella *concluye* que los puntos de la bisectriz equidistan de los lados del ángulo porque puede dibujar una circunferencia que es tangente a los dos lados del ángulo (línea 5). En este caso, la garantía del argumento es una regla que establece como general gracias a sus conocimientos sobre las propiedades de la circunferencia y la co-acción con la representación dinámica. La presencia del medio dinámico en el argumento de Ana se presenta en dos momentos: primero con la exactitud de los trazos, ya que si estuviera trabajando en el papel podría no darse la tangencia o sería *su voluntad* la que haría que la circunferencia resultara tangente. El segundo momento se presenta cuando ella mueve un punto de la construcción, en este caso el vértice del ángulo, característica que diferencia el trabajo en el medio digital y el papel y lápiz.

GeoGebra como guía

El medio digital es "experto en geometría" de modo que puede impactar el razonamiento del estudiante a través de un proceso en donde se plantean hipótesis o conjeturas. Esto puede darse con la observación persistente de un hecho que modifique las ideas geométricas del estudiante, lo que ocurriría al evaluar la respuesta del medio digital. Este proceso dialéctico lo observamos cuando se le propone a Ana construir la bisectriz de un ángulo sin utilizar la herramienta que viene preconstruida en el medio digital —GeoGebra en este caso— y después que pruebe que efectivamente es la bisectriz.

[10]A: Déjame lo pruebo.
[11]E: ¿Qué cosa vas a probar?
[12]A: Si (…) si trazando alguna de estas rec(…) sí, alguna de éstas (señala a HJ y HG en la figura 1) me da este punto de aquí (señala a H), y ese punto, pues va a ser parte de la bisectriz.

Ana construye un ángulo NML y un punto O sobre MN, después construye una recta perpendicular a MN por O e intenta construir una recta perpendicular a ML, pero solo mueve el apuntador sobre el segmento ML (ver Figura 2) y se da cuenta que el punto no puede ser ubicado arbitrariamente cuando dice:

[13]A: “No puede ser cualquiera”.

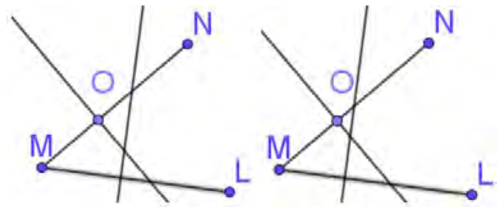


Figura 2: Arrastre de Ana para construir la perpendicular deseada

En este caso, la construcción de Ana está guiada por su intuición y la *respuesta* del medio digital se puede interpretar como la posible ubicación de la recta perpendicular a ML y el punto de intersección de las dos rectas perpendiculares. La respuesta del medio es interpretada por las ideas previas que tiene Ana gracias a la interacción con su representación anterior (línea 12), las cuales le permiten afirmar que el punto sobre ML debe tener alguna característica particular que desconoce (línea 13).

Continuando con su exploración, Ana empieza a usar GeoGebra como un medio que *guía* su razonamiento, ya que construye un punto P en el segmento LM y la recta perpendicular a dicho segmento por P, después determina el punto Q como la intersección de las dos rectas perpendiculares y construye la circunferencia con centro en Q y radio QO, por último, mueve P hasta que parece que la circunferencia con centro en Q es tangente a ML por P.

La construcción de la circunferencia por parte de Ana muestra la presencia de un hecho geométrico que ha sido adquirido gracias a la interacción con el medio digital, en el que una recta es bisectriz si y solo sí al construir una circunferencia con centro en la bisectriz y tangente a un lado del ángulo resulta tangente al otro lado (línea 5). Esta idea se ha incorporado a la construcción del objeto bisectriz que tiene Ana, y ahora hace parte de sus herramientas conceptuales que le permiten probar si un punto es un punto de la bisectriz o no.

Continuando con la interacción con el medio digital, Ana conjetura sobre la ubicación de P (por ejemplo, que es punto medio de ML) y realiza las construcciones respectivas, pero gracias las respuestas del medio digital a través del arrastre, ella comprende que sus hipótesis no son correctas. Mueve el punto P sobre el segmento ML y observa que el punto de intersección entre la circunferencia y la recta perpendicular a ML se mueve cuando P se mueve, por lo tanto, determina dicho punto de intersección como R y decide usar la herramienta “Rastro”. Esta última le muestra el lugar geométrico de R cuando P se mueve (ver Figura 3), pero al ver el resultado Ana afirma que necesitaría conocer P y decide cambiar su estrategia.

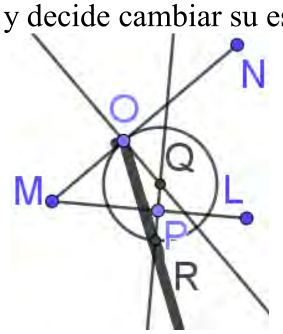


Figura 3: Uso de la herramienta “Rastro”

Aquí subrayamos dos tipos de interacción con el medio digital, en la primera, Ana busca dentro de sus conocimientos geométricos ideas que le permitan plantear hipótesis sobre la ubicación de P y en este caso la *respuesta* del medio digital es la ubicación del punto solicitado por Ana, como el punto medio de ML y esta respuesta es contrastada con las ideas previas de Ana. En la segunda interacción, ella no parte de ideas geométricas, sino del comportamiento de la representación digital de los objetos, de modo que la *respuesta* del medio muestra el comportamiento del punto R. Entonces, Ana busca asociar esta respuesta con su conocimiento previo y al no poder hacerlo, decide usar otra estrategia.

Ana decide volver a explorar la configuración inicial (la que se presenta en la Figura 1) y después de mover los punto A, B y C decide construir una circunferencia con centro en M y radio MO.

[14]: Y, ¿esa circunferencia por qué se te ocurrió?

[15]: (mueve el punto P hasta que la circunferencia con centro en Q resulta tangente a los dos lados del ángulo) ¡Ah, sí es!

Al ver que su nueva hipótesis resulta ser correcta Ana decide eliminar el punto P que había construido y ahora lo construye como punto de intersección de la circunferencia con centro en M con el segmento ML. Pero, esta vez no construye Q como punto de intersección de las dos perpendiculares, sino que lo determina con punto medio de OP y construye la recta MQ como bisectriz. Para probar que la recta es la bisectriz ella construye la circunferencia con centro en Q y radio QO, pero como Q no es el punto de intersección de las dos rectas perpendiculares, entonces la circunferencia no resulta ser tangente a los lados del ángulo (ver Figura 4). Ana mueve los puntos de su construcción y duda que la recta construida sea la bisectriz, entonces decide usar la herramienta "bisectriz" que la construye a partir de tres puntos y nota que la recta bisectriz coincide con MQ, por lo que reevalúa la construcción de la circunferencia con centro en Q. Por último, Ana construye la circunferencia con centro en la intersección de las dos perpendiculares y así comprueba que la recta construida es la bisectriz del ángulo LMN.

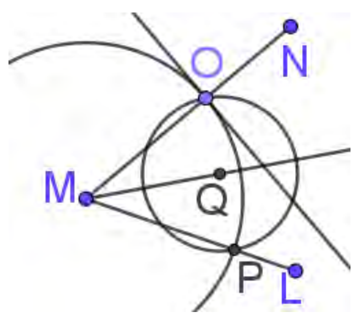


Figura 4: Construcción de circunferencia que no es tangente a ambos lados del ángulo

En general, la construcción de Ana está guiada por las observaciones sobre la representación *ejecutable* y el constante diálogo con el medio digital. Este medio se usa como un recurso que le permite *explorar* partiendo de hipótesis (como la idea que se ilustra en la Figura 2) o buscando hipótesis gracias a la interacción con la representación dinámica—estrategia presentada en Figura 3 y cuando explora nuevamente la Figura 1.

Sobre el argumento de Ana, ella *concluye* que la recta construida es la bisectriz del ángulo porque puede construir una circunferencia que es tangente a los dos lados del ángulo y permanece tangente ante el arrastre. En este caso, la garantía del argumento es una *regla de* *validación digital* que ha mostrado tener incorporada a sus conocimientos (línea 5). Afirmamos que esta es una regla de validación digital ya que cuando el medio responde con una circunferencia que no es tangente (Figura 4), ella piensa que la recta construida no es bisectriz, porque una idea sobre este objeto que forma parte de sus conocimientos previos es que: una recta es bisectriz si y sólo si se puede trazar la circunferencia tangente.

Un hecho que resulta interesante del proceso de Ana es que cuando traza la recta bisectriz y el medio responde haciendo coincidir dicha recta con la recta ya construida a saber, MQ (Figura 4), ella no usa dicha coincidencia para argumentar que la recta construida sea la bisectriz del ángulo. Solo cuando construye la circunferencia tangente considera que ha probado que su construcción es correcta. Las reglas de validación están vinculadas al proceso continuo de formación y adquisición de conceptos.

Palabras finales

Este trabajo ha permitido observar diferentes interacciones cuando se recurre a la mediación de las representaciones dinámicas durante las tareas de construcciones geométricas y cómo dicha mediación afecta los procesos de validación. Reportamos tres usos del medio digital: i) como un socio constructivo en donde la respuesta del medio no hace parte del argumento, ii) como un medio de prueba en donde la interacción con el medio desempeña un rol importante en el argumento, y iii) El medio digital como un guía que corrige y da indicios del camino a seguir, al igual que en el caso anterior la interacción con el medio es crucial en el argumento.

Las diferentes interacciones que analizamos permitieron observar formas híbridas de interacción, es decir, presencia de ideas geométricas previamente desarrolladas en un medio de lápiz y papel que se manifiestan en medio de un razonamiento adelantado en el contexto de la geometría dinámica. Los razonamientos de Ana se auxilian de los recursos que ofrece el medio digital para establecer que la circunferencia siempre será tangente al otro lado del ángulo. En efecto, Ana razona sobre una representación que le brinda información diferente a la que tenía en un medio estático. Pero ¿se podría afirmar que la prueba de Ana es deductiva? Este cuestionamiento nos lleva a plantearnos una pregunta que es todavía más amplia: *¿qué caracteriza una prueba deductiva en el medio digital?*

Ofrecemos aquí un punto de partida hacia la caracterización de una *prueba digital*.

Referencias

Bleiler, S. K., Thompson, D. R., & Krajčevski, M. (2014). Providing written feedback on students' mathematical arguments: Proof validations of prospective secondary mathematics teachers. Journal of Mathematics Teacher Education, 17(2), 105-127.

Buchbinder, O., & Zaslavsky, O. (2011). Is this a coincidence? The role of examples in fostering a need for proof. ZDM, 43(2), 269-281.

Dello Iacono, U. (2021). From Argumentation to Proof in Geometry Within a Collaborative Computer-Based Environment. Digital Experiences in Mathematics Education, 7(3), 395–426.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. International Journal of Mathematical Education in Science and Technology, 35(5), 703–725.

Dickerson, D. S., & Doerr, H. M. (2014). High school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics. Mathematics Education Research Journal, 26(4), 711-733.

Hanna, G. (2014). The width of a proof. PNA, 9(1), 29-39.

Hanna, G. (2020). Mathematical Proof, Argumentation, and Reasoning. In S. Lerman, (Ed.), Encyclopedia of Mathematics Education (pp. 561–566). Springer, Dordrecht.

Hegedus, S. J., & Moreno-Armella, L. (2010). Accommodating the instrumental genesis framework within dynamic technological environments. For the Learning of Mathematics, 31(1), 26-31.

Hernández, R., Fernández, C., & Baptista, P. (2010). Metodología de la investigación. (5ta ed.) McGraw-Hill.

Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. Mathematics Education Research Journal, 31(4), 393-418

Leung, A., & Or, C. M. (2007). From construction to proof: Explanations in dynamic geometry environments. In J. H. Woo., H. C. Lew., K. S. Park, & D. Y. Seo (Eds.), Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education (Vol. 3, pp. 177–184). Seoul, Korea: PME

Morales, G., Rubio, N., & Larios, V. (2021). Tipificación de argumentos producidos por las prácticas matemáticas de alumnos del nivel medio en ambientes de geometría dinámica. Bolema, 35(70), 664-689.

Moreno-Armella, L., & Brady, C. (2018). Technological Supports for Mathematical Thinking and Learning: Co-action and Designing to Democratize Access to Powerful Ideas. In L. Ball., P. Drijvers., S. Ladel., H. S. Siller., M. Tabach., & C. Vale. (Eds.). Uses of Technology in Primary and Secondary Mathematics Education (pp. 339-350). Springer, Cham.

Moreno-Armella, L., & Sriraman, B. (2005). Structural stability and dynamic geometry: Some ideas on situated proofs. ZDM, 37(3), 130–139.

National Council of Teachers of Mathematics. (2000). Principios y Estándares para la educación matemática (M. F. Reyes, Trans). Sociedad Andaluza de Educación Matemática Thales. (Obra original publicada en 2000).

Netz, R. (1998). Greek Mathematical Diagrams: Their Use and Their Meaning. For the Learning of Mathematics, 18(3), 33–39.

Northrop, E. (1991). Paradojas Matemáticas (R. Ortiz Vasquez, Trans), México: Unión tipográfica editorial hispanoamericana. (Obra original publicada en 1944).

Olivero, F. & Robutti, O. (2001). Measures in Cabri as a bridge between perception and theory. In M. van den Heuvel-Panhuizen (Ed.), Proceedings of the 25th PME International Conference, (Vol. 4, pp. 9-16). Utrecht.

Rabardel, P. & Beguin, P. (2005). Instrument mediated activity: from subject development to anthropocentric design. Theoretical Issues in Ergonomics Science, 6(5), 429-461.

Sinclair, N., & Robutti, O. (2012). Technology and the Role of Proof: The Case of Dynamic Geometry. In: Clements M., Bishop A., Keitel C., Kilpatrick J., & Leung F. (Eds.), Third International Handbook of Mathematics Education. Springer International Handbooks of Education (Vol. 27, pp. 571-596). Springer, New York.

Sriraman, B., & Umland, K. (2020). Argumentation in Mathematics Education. In S. Lerman, (Ed.), Encyclopedia of Mathematics Education (pp. 63–66). Springer, Dordrecht.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), Research design in mathematics and science education (pp. 267–307). Hillsdale, NJ: Erlbaum

Stylianides, G. J., & Stylianides, A. J. (2017). Research - based interventions in the area of proof: the past, the present, and the future. Educational Studies in Mathematics, 96(2), 119-127.

Wittmann, E. C. (2021). When Is a Proof a Proof? In E.C. Wittmann. (Ed.), Connecting Mathematics and Mathematics Education (pp. 61–76). Springer, Cham.

INTERACTION WITH DYNAMIC REPRESENTATIONS TO ARGUE ABOUT THE VALIDITY OF A GEOMETRY CONSTRUCTION

INTERACCIÓN CON REPRESENTACIONES DINÁMICAS PARA ARGUMENTAR SOBRE LA VALIDEZ DE UNA CONSTRUCCIÓN GEOMÉTRICA

María Alejandra Calderón González
CINVESTAV
alejandra.calderon@cinvestav.mx

Luis Enrique Moreno Armella
CINVESTAV
lmorenoa@cinvestav.mx

From mathematics education, a great variety of studies have focused on analyzing the presence of digital technology from different points of view. Those who work with proof and argumentation have resisted presenting results with a technological component, due to the idea of formality associated with those terms, which has increased the existing tension between the theoretical and empirical nature of mathematics. Taking into consideration that with the presence of environments such as dynamic geometry software (DGS) the student can access to dynamic representations, we seek to understand the impact of the interaction with these representations on the arguments proposed by the students. For this, we carried out an exploratory and descriptive study whose results show that the interaction with dynamic representations provides students with validation tools different from those on paper.

Keywords: Reasoning and Proof.

Introduction and background

To argue and to validate propositions are skills that human beings need to function in society. From very early on, one of the educational objectives is to promote the participation of students for the understanding and development of arguments. For example, the NCTM (2000) proposes that "By the end of secondary school, students should be able to understand and produce mathematical proofs —arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments" (p. 56). All this has been analyzed from the perspective of the teacher (Bleiler et al, 2014; Lesseig et al, 2019; Dickerson and Doerr, 2014), from the student's productions (Buchbinder and Zaslavsky, 2011), and from the incorporation of digital technology. (Leung and Or, 2007; Olivero and Robutti, 2001; Sinclair and Robutti, 2012).

In the classroom, the teacher is the one who promotes the development of arguments and proofs, consequently, research has been developed to explore the teachers' ideas on this subject. (Bleiler et al, 2014; Lesseig et al, 2019). A finding is that teachers with few years of experience consider formality and logic to be relevant factors that determine the validity of an argument (Dickerson & Doerr, 2014). However, some authors (Wittmann, 2021; de Villiers, 2004) consider that this formalist emphasis on proof inhibits the importance of the comprehension process. Under this last perspective, it is concluded that the objective, in mathematics education, should be for students to develop different types of reasoning that allow them to understand and argue about the validity of a proposition (Hanna, 2014).

Geometry is one of the areas of mathematics in which the development of arguments and proofs can occur, because geometric methods are, to a certain extent, a combination of seeing and reasoning, since reason verifies logical developments and guides them about what the eyes see in the figure (Northrop, 1968). The reasoning, in geometry, is developed on a figure, so that the figure is a constituent part of the argumentative process. This same idea is shared by Netz (1988) who states that, in the case of classical Greek geometry, the diagram is the essence of the proof.

In geometry class, students are expected to express their reasoning to illustrate the generality of a fact from the diagram, however, this is not always possible because they usually generalize from a single case (Stylianides and Stylianides, 2017). For example, they think that the median and perpendicular bisector of a triangle are the same object because when they are drawn with a ruler and compass, they coincide for the constructed triangle (isosceles). These types of situations have made uncertainty about the generality of an important fact in promoting the need to produce deductive proofs in students (Buchbinder & Zaslavsky, 2011). However, when the representations are dynamic (e.g., presented in GeoGebra) a high level of certainty can be generated by dragging points of a construction and observing that the properties are maintained across the different resulting figures. This perception of the permanence of a property (which would no longer be accidental), is what leads to proposing strategies and functions of the proof that are different from the traditional one (de Villiers, 2004). In addition, access to a digital screen has promoted the study of the impact of digital technologies on the change suffered in the reasoning and proofs proposed by students.

There are many investigations in which the use of dynamic geometry in argumentation and proof processes is analyzed from different perspectives. Some have focused on typifying the arguments proposed by students, concluding that most of them are empirical (Morales et al., 2021); Other studies have analyzed how interaction with dynamic geometry objects can serve as preliminary work to propose and build deductive/theoretical proof (Leung and Or, 2007; Olivero and Robutti, 2001; Dello-Iacono, 2021). However, it has also been recognized that the interaction with dynamic representations can produce new forms of genuine reasoning (Moreno-Armella and Brady, 2018) and therefore establish new ideas about what a proof means when working with digital representations. The study reported here investigated how is the interaction between the student and the dynamic representations when making constructions and arguing about their validity?

Considering the importance of the interaction between the student and the dynamic representations, we present the theoretical construct of co-action, extending some ideas of instrumental genesis. We will present an overview of the argumentation that does not express the need for a level of formality.

Theoretical perspective

More and more curricular structures are being inhabited by digital technology through representation systems that react immediately to student actions, allowing, on the one hand, that some activities that can be developed with pencil and paper, are carried out in a more efficient way, and avoid calculation errors —the tool as an amplifier—. However, the executable nature of these representations makes it possible for the student to transform their knowledge and access new ideas —the tool as a conceptual reorganizer—. This is particularly tangible in dynamic geometry environments (Moreno-Armella and Sriraman, 2005).

The action of dragging a vertex of a triangle, for example, generates a response from the digital environment that gradually reveals the structural nature of the construction that has been made. There is a new exploration space for the student since the digital screen is a space that dialogues with the student through digital representations. The performance of the representations translates into the responsiveness of the medium during an exploration and contributes to the development of new ways of thinking. It ends up generating a dialectic that has been studied as instrumental genesis (Rabardel and Beguin, 2005), but in a classroom you learn from, with and through others. There are ways to articulate one's own thinking with the thinking of others, including their experiences with the present digital medium. This cognitive co-action (Hegedus and Moreno-Armella, 2010) is a central feature of instrumental mediation.

Although the possibilities opened by the digital means of numerical computation and visualization have questioned the traditional proof on paper and pencil as the only mathematical validation criterion, located in the field of mathematics education and seeing the construction of meaning as a central objective of learning, we consider that there are, apart from the traditional one, other ways to build the justification of a proposition (let's say geometric), in the classroom. The exploration of a geometric situation (our interest here) finds a theoretically controlled response from the side of the digital mediator—for example, GeoGebra. By advancing an exploration, the student finds on the screen a response to her actions through the plasticity of the representations. In this way, the mathematical understanding of it finds and discards routes of approximation to a plausible answer to a problem or to a demand for justification. Therefore, mathematical arguments can be approached from a general perspective as a line of reasoning that aims to show and/or convince that a result (a general statement about an object, a solution to a problem, a calculation) is correct (Sriraman and Umland, 2020; Hanna, 2020). Our work responds to the lines described before.

Methodology

We place our research within a qualitative approach, which is focused on understanding, deepening, and describing phenomena through the perspective of the participants (Hernández et al., 2010). In this case we seek to understand the interaction with the dynamic representations when the student proposes arguments. We follow the line of exploratory and descriptive research (Steffe and Thompson, 2000) as we seek to familiarize ourselves with the ways and means of operating of the participants.

The participants were 6 mathematics teachers who were taking a geometry course as part of a master's program. For this report, we describe the case of Ana who said she wasn't familiar with the contents of geometry (because she has never taught it and only took one course during her undergraduate degree) and who showed great skills in using GeoGebra.

The questionnaire proposed to the participants consisted of six constructions that they had to make and argue why their solution was valid. Subsequently, a semi-structured interview was conducted with each participant to deepen their arguments/reasoning and their interaction with the digital medium.

For the analysis of the data, the fragments of the interview were extracted in which a process was evidenced where the students carried out actions on the digital medium, they interpreted the response of the medium (the information on the screen) and this information led them to carry out new actions. In general, we analyze how this process contributed to the students being able to argue about the validity of the construction.

Results

In this section we analyze the interactions of students with the digital environment, particularly the nature of the arguments that are produced thanks to said interaction.

GeoGebra as an amplifier

The environment is, initially, an amplifier that allows students to make explicit their ideas about circles, ellipses, perpendicular bisectors, and other geometric objects. At that time her ideas have not yet been affected by the new system of representation. We illustrate this idea with the work of Ana, who solves the following problem: given an angle and a line transversal to the sides of the angle, find a point on the transversal equidistant from the sides of the angle. In her written solution proposal, Ana states that the point sought is at the intersection of the bisector and the transversal, so during the interview she performs the construction, which leads to the following dialogue:

[1]E: What did you draw there?
[2]A: A bisector.
[3]E: Why the bisector?
[4]A: Because the bisector will give me all the points that are equidistant from (...) both sides.

Ana's construction is guided by her theoretical knowledge, so the digital medium (GeoGebra) is used as a tool that allows her to construct the bisector of the given angle almost identically to how she would do it with pencil and paper.

Regarding the argument, Ana concludes that the point of intersection between the bisector and the transversal is equidistant from the sides of the angle because there is a general rule linked to the bisector that allows her to affirm said equidistance (line 4, previous dialogue).

GeoGebra as a conceptual reorganizer

The digital environment has become a conceptual reorganizer when the student's reasoning is transformed by the presence of the digital mediator, so that the student reconfigures her arguments including the mediator's response. This is illustrated when Ana must prove that the points of the bisector are equidistant from the sides of an angle, a fact that is usually proved by determining two congruent triangles, however, Ana uses another strategy (see Figure 1) and says:

[5]A: Why equidistant? Because, well (...) I can draw this circumference with center here (points to H) that is tangent to this side and to this (points to the two sides of the angle).
[6]E: Ok. And does being able to draw a circle allow me to check the equidistance?
[7]A: Ujum (constantly drag point B around the screen).
[8]E: Why? what characteristic does it give? why not an ellipse, or a square?
[9]A: Ah. Well, because the distance from the center to any point is the same, from here as from here (indicates two points on the circumference).

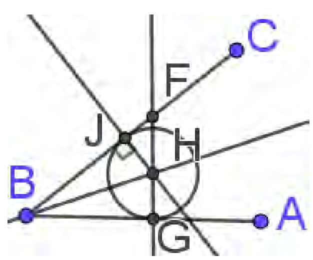


Figure 1: Construction of Ana to prove a bisector property

To prove that the points of the bisector are equidistant from the sides of the angle, Ana, guided by her knowledge of the properties of circles (line 9), constructs a circle with center at H that is tangent to one of the sides of the angle (BC), anticipating that the circumference will be tangent to the other side. The response given by the representation, thanks to the accuracy of the lines and the geometry embedded in that environment, allows Ana to configure her argument based on what she sees on the screen. In this same sense, Ana uses the drag function to illustrate the generality of the fact when she drags one of the points of the construction and the response given by the environment is that the circumference remains tangent to both sides of the angle.

Regarding Ana's argument, she concludes that the points of the angle bisector are equidistant from the sides of the angle because she can draw a circle that is tangent to both sides of the angle (line 5). In this case, the guarantee of the argument is a rule that she establishes as general thanks to her knowledge of the properties of the circle and the co-action with the dynamic representation. The presence of the dynamic environment in Ana's argument is evidenced in two moments: first with the accuracy of the lines, since if she were working on paper the tangency might not occur or it would be her will that would make the circumference tangent. The second moment occurs when she drags a point of the construction (in this case the vertex of the angle), since it is an action that is allowed by the digital environment and differentiates it from working with paper and pencil.

GeoGebra as a guide

The digital environment is "expert in geometry" so that it can impact the student's reasoning through a process where hypotheses or conjectures are raised. This can occur from the persistent observation of a fact that modifies the geometric ideas of the student. This would occur when evaluating the response of the digital medium. We observe this dialectical process when Ana is proposed to construct the bisector of an angle without using the tool that is pre-built in the digital environment – GeoGebra in this case – and then she proves that it is indeed the bisector.

[10]A: Let me try it.
[11]E: What are you going to try?
[12]A: If (...) if tracing any of these (...) if any of these (points to HJ and HG in Figure 1) it gives me this point here (points to H), and that point, well, it will be part of the bisector.

Ana constructs an angle NML and a point O on MN, then constructs a line perpendicular to MN through O and tries to construct a line perpendicular to ML, but just moves the pointer over segment ML (see Figure 2) and realizes that the point cannot be arbitrarily located when it says:

[13]A: Can't be anyone.

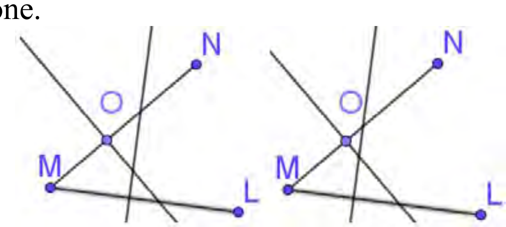


Figure 2: Ana dragging to build the desired perpendicular line

In this case, Ana's construction is guided by her intuition and the response from the digital environment can be interpreted as the possible locations of the perpendicular line to ML and the possible points of intersection of the two perpendicular lines. The response of the environment is interpreted by the previous ideas that Ana had thanks to her interaction with her previous representation (line 12), which allow her to affirm that the point on ML must have some characteristic that she does not know (line 13).

Continuing her exploration, Ana begins to use GeoGebra as an environment to guide her reasoning, as she constructs a point P on segment LM and the line perpendicular to that segment through P, then determines point Q as the intersection of the two lines perpendiculars and constructs the circle with center at Q and radius QO. She finally moves P until it seems that the circle with center at Q is tangent to ML through P.

The construction of the circumference by Ana shows the presence of a geometric fact that has been acquired thanks to the interaction with the digital environment, in which a line bisects an angle if and only if when a circle is constructed with center on the bisector and tangent to one side of the angle becomes tangent to the other side (line 5). This idea has been incorporated into Ana's construction of the bisector object, and now it is part of her conceptual tools that allow her to state whether a point is a bisector point or not.

Continuing with the interaction with the digital environment, Ana conjectures about the location of P (for example, that it is the midpoint of ML) and performs the respective constructions, but thanks to the responses of the digital environment through the drag, she understands that her hypotheses are not correct. She moves the point P on the segment ML and observes that the point of intersection between the circumference and the perpendicular line to ML moves when P moves, therefore, she names R as point of intersection and decides to use the "Trace" tool. The latter shows her the locus of R when P moves (see Figure 3). Seeing the result, Ana affirms that she would need to determine P and decides to change her strategy.

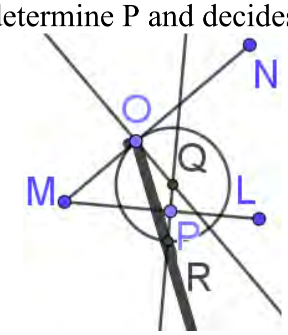


Figure 3: Using the "Trace" tool

Here we underline two types of interaction with the digital environment. In the first, Ana searches within her geometric knowledge for ideas that allow her to hypothesize about the location of P. In this case, the response of the digital environment is the location of the point requested by Ana, as the midpoint of ML, and this is contrasted with Ana's previous ideas. In the second interaction, she does not start from geometric ideas, but from the behavior of the digital representation of objects, so that the answer that is observed on the screen is the behavior of point R. So, Ana tries to associate this response with her prior knowledge and, unable to do so, decides to use another strategy.

Ana decides to re-explore the initial configuration (the one shown in Figure 1) and after moving points A, B and C, she decides to build a circle with center M and radius MO.

[14]: And why did you come up with that circumference?

[15]: (drag point P until the circle centered at Q is tangent to both sides of the angle) Ah, yes, it is!

Seeing that her new hypothesis turns out to be correct, Ana decides to eliminate the point P that she had constructed and now constructs it as the point of intersection of the circle with center in M with the segment ML. But this time she does not construct Q as the point of intersection of the two perpendiculars, but rather she determines it with the midpoint of OP and constructs the line MQ as a bisector. To prove that the line is the bisector, she constructs the circle with center at Q and radius QO, but since Q is not the point of intersection of the two perpendicular lines, then the circle does not turn out to be tangent to the sides of the angle (see Figure 4). Ana moves the points of her construction and doubts that the constructed line is the bisector, so she decides to use the "bisector" tool that constructs it from three points and notices that the bisector line coincides with MQ, so she reevaluates the construction of the circle with center at Q. Finally, Ana constructs the circle with center at the intersection of the two perpendiculars and thus checks that the constructed line is the bisector of the angle LMN.

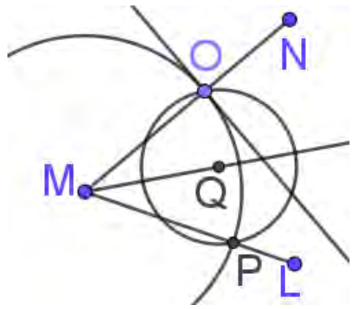


Figure 4: Construction of a circle that is not tangent to both sides of the angle

In general, the construction of Ana is guided by the observations about the executable representation and the constant dialogue with the digital environment. This environment is used as a resource that allows you to explore starting from hypotheses (such as the idea illustrated in Figure 2) or searching for hypotheses thanks to the interaction with the dynamic representation—strategy presented in Figure 3 and when exploring Figure 1 again one—.

On Ana's argument, she concludes that the constructed line is the bisector of the angle because she can construct a circumference that is tangent to the two sides of the angle and remains tangent during the drag. In this case, the argument guarantee is a digital validation rule that she has built into her knowledge (line 5). We affirm that this is a digital validation rule because when the environment responds with a circumference that is not tangent (Figure 4), she thinks that the constructed line is not a bisector, because an idea about this object that is part of her previous knowledge is that a line is a bisector if and only if the tangent circle can be drawn.

An interesting fact about Ana's process is that when she draws the bisecting line and the environment responds by matching said line with the already constructed line, namely MQ (Figure 4), she does not use said coincidence to argue that the constructed line is the bisector of the angle. Only when she constructs the tangent circle does she consider that she has proven that her construction is correct. Validation rules are linked to the continuous process of concept formation and acquisition.

Final words

This work has allowed us to observe different interactions when the mediation of dynamic representations is used during geometric construction tasks and how this mediation affects the validation processes. We report three uses of the digital environment: i) as a constructive partner where the response of the environment is not part of the plot, ii) as a test environment where the interaction with the environment plays an important role in the plot, and iii) The digital environment as a guide that allows contrasting and finding solution strategies. As in the previous case, interaction with the environment is crucial in the argument.

The different interactions that we analyzed allowed us to observe hybrid forms of interaction, that is, the presence of geometric ideas previously developed in pencil and paper that manifest themselves during advanced reasoning in the context of dynamic geometry. Ana's reasoning is aided by the resources offered by the digital medium to establish that the circumference will always be tangent to the other side of the angle. Indeed, Ana reasons about a representation that gives her different information from what she had in a static medium. But could it be said that Ana's proof is deductive? This questioning leads us to ask ourselves a question that is even broader: what characterizes a deductive proof in the digital medium?

We offer here a starting point towards the characterization of a digital evidence.

References

Bleiler, S. K., Thompson, D. R., & Krajčevski, M. (2014). Providing written feedback on students' mathematical arguments: Proof validations of prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education*, 17(2), 105-127.

Buchbinder, O., & Zaslavsky, O. (2011). Is this a coincidence? The role of examples in fostering a need for proof. *ZDM*, 43(2), 269-281.

Dello Iacono, U. (2021). From Argumentation to Proof in Geometry Within a Collaborative Computer-Based Environment. *Digital Experiences in Mathematics Education, 7*(3), 395–426.

De Villiers, M. (2004). Using dynamic geometry to expand mathematics teachers' understanding of proof. *International Journal of Mathematical Education in Science and Technology, 35*(5), 703–725.

Dickerson, D. S., & Doerr, H. M. (2014). High school mathematics teachers' perspectives on the purposes of mathematical proof in school mathematics. *Mathematics Education Research Journal*, *26*(4), 711-733.

Hanna, G. (2014). The width of a proof. *PNA, 9*(1), 29-39.

Hanna, G. (2020). Mathematical Proof, Argumentation, and Reasoning. In S. Lerman, (Ed.), *Encyclopedia of Mathematics Education* (pp. 561–566). Springer, Dordrecht.

Hegedus, S. J., & Moreno-Armella, L. (2010). Accommodating the instrumental genesis framework within dynamic technological environments. *For the Learning of Mathematics*, *31*(1), 26-31.

Hernández, R., Fernández, C., & Baptista, P. (2010). *Metodología de la investigación*. (5ta ed.) McGraw-Hill.

Lesseig, K., Hine, G., Na, G. S., & Boardman, K. (2019). Perceptions on proof and the teaching of proof: a comparison across preservice secondary teachers in Australia, USA and Korea. *Mathematics Education Research Journal*, *31*(4), 393-418

Leung, A., & Or, C. M. (2007). From construction to proof: Explanations in dynamic geometry environments. In J. H. Woo., H. C. Lew., K. S. Park, & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 177–184). Seoul, Korea: PME

Morales, G., Rubio, N., & Larios, V. (2021). Tipificación de argumentos producidos por las prácticas matemáticas de alumnos del nivel medio en ambientes de geometría dinámica. *Bolema*, *35*(70), 664-689.

Moreno-Armella, L., & Brady, C. (2018). Technological Supports for Mathematical Thinking and Learning: Co-action and Designing to Democratize Access to Powerful Ideas. In L. Ball., P. Drijvers., S. Ladel., H. S. Siller., M. Tabach., & C. Vale. (Eds.). *Uses of Technology in Primary and Secondary Mathematics Education* (pp. 339-350). Springer, Cham.

Moreno-Armella, L., & Sriraman, B. (2005). Structural stability and dynamic geometry: Some ideas on situated proofs. *ZDM, 37*(3)*,* 130–139.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Netz, R. (1998). Greek Mathematical Diagrams: Their Use and Their Meaning. *For the Learning of Mathematics*, *18*(3), 33–39.

Northrop, E. (1991). *Paradojas Matemáticas* (R. Ortiz Vasquez, Trans), México: Unión tipográfica editorial hispanoamericana. (Obra original publicada en 1944).

Olivero, F. & Robutti, O. (2001). Measures in Cabri as a bridge between perception and theory. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th PME International Conference,* (Vol. 4, pp. 9-16). Utrecht.

Rabardel, P. & Beguin, P. (2005). Instrument mediated activity: from subject development to anthropocentric design. *Theoretical Issues in Ergonomics Science, 6*(5), 429-461.

Sinclair, N., & Robutti, O. (2012). Technology and the Role of Proof: The Case of Dynamic Geometry. In: Clements M., Bishop A., Keitel C., Kilpatrick J., & Leung F. (Eds.), *Third International Handbook of Mathematics Education. Springer International Handbooks of Education* (Vol. 27, pp. 571-596). Springer, New York.

Sriraman, B., & Umland, K. (2020). Argumentation in Mathematics Education. In S. Lerman, (Ed.), *Encyclopedia of Mathematics Education* (pp. 63–66). Springer, Dordrecht.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267–307). Hillsdale, NJ: Erlbaum

Stylianides, G. J., & Stylianides, A. J. (2017). Research - based interventions in the area of proof: the past, the present, and the future. *Educational Studies in Mathematics*, *96*(2), 119-127.

Wittmann, E. C. (2021). When Is a Proof a Proof? In E.C. Wittmann. (Ed.), *Connecting Mathematics and Mathematics Education* (pp. 61–76). Springer, Cham.

THE ADAPTED IGS FRAMEWORK: DESIGNING DESMOS TASKS

Kayla Chandler
East Carolina University
chandlerk@ecu.edu

Charity Cayton
East Carolina University
caytonc@ecu.edu

Research has identified specific frameworks that focus on the design and evaluation of mathematics technology tools and tasks (McCulloch et al., 2021a). One of those identified, the IGS Framework (Sherman & Cayton, 2015), has proven to be a useful tool to assist teachers in designing technology tasks that use the affordances of a specific math tool as a reorganizer (Pea, 1985) to support students' high-level thinking (Sherman et al., 2017; Cayton & Chandler, Under Review). This study adapted the IGS Framework to examine in-service teachers' design of Desmos tasks in an asynchronous, paid professional development. Findings indicate when teachers designed a task that used Desmos as a reorganizer along an explicitly stated goal for student thinking, the result was a task of high potential cognitive demand.

Keywords: Technology, Professional Development, Instructional Activities and Practices

Setting mathematical goals and selecting tasks that promote reasoning and sense making serve a central role in guiding instruction (National Council of Teachers of Mathematics, 2014). When adding the use of technology to the scenario, a teacher must consider the mathematical goals and how to leverage the affordances of the technology tool to support student thinking (Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences, 2012).

When considering tasks that use technology, research is limited regarding the cognitive demand of tasks teachers use or to what features of a tool a teacher attends when considering a task (Sherman et al., 2017; Bray & Tangney, 2017; Sherman, 2014). Rather, most studies about technology use in mathematics classrooms focus on the type of technology used (e.g., Goos & Bennison, 2006; Pierce & Ball, 2010), barriers to incorporating technology (e.g., Ertmer et al., 1999; Washira & Kenngwe, 2011), positioning of the technology (e.g., Goos et al., 2003) and the reasons teachers decided whether or not to use technology (e.g., Goos, 2005; Zbiek & Hollebrands, 2008). Additionally, in a nationwide survey regarding the frameworks mathematics teacher educators use in their work of preparing secondary mathematics teachers to teach with technology, only five of 17 identified frameworks address the design and evaluation of technology tools and tasks (McCulloch et al., 2021a). Altogether, these findings indicate there are few tools available to teachers to support them in evaluating, revising, and designing cognitively demanding tasks that use technology to support students' mathematical learning.

Framework

The IGS Framework (Sherman & Cayton, 2015) identifies goals for students' mathematical thinking based on Sinclair's (2003) design principles along each row (i.e., dimension) and each column contains descriptions to help the teacher distinguish whether technology is used as an amplifier or as a reorganizer (Pea, 1985) with respect to a particular goal. While these goals are described separately within the IGS Framework, it is important to note that they are not mutually exclusive. It is also important to note that, while potentially related, these are different from a teacher's mathematical goals for a task. Research based on the IGS Framework found it to be a productive tool for both pre-service (Sherman et al., 2017) and in-service teachers (Cayton &

Chandler, under review) in creating technology tasks that are of high cognitive demand, as well as a reorganizer in relation to student thinking.

The Adapted IGS Framework (Figure 1) was created by adapting the IGS Framework to move away from language specific to dynamic geometry software (e.g., sketch) to be more inclusive of math action technologies more broadly (e.g., task). Additionally, emphasis was added to words like static and dynamic to further draw attention to how the technology is being used. The purpose of this qualitative study was to consider the ways in which in-service mathematics teachers utilized the Adapted IGS Framework to design tasks that use Desmos.

		Technology Used As	
		Amplifier Students could achieve the same goal without the technology.	**Reorganizer** The mathematical goal of the task would be difficult to achieve without the technology.
Goals	**Questions to Ask Yourself**	**Use of Design Principles**	
Make mathematically meaningful observations; look for invariant relationships	Do the task and prompts use the *dynamic* aspects of the technology in a way that would be difficult or impossible to replicate without its use?	Students create multiple *static* examples, either by construction or dragging, and reason from those static examples. For example, students are prompted to make observations or generalizations based on a table or static measurements without reference to the technology.	The task allows for continuous dragging, and students are guided to examine measurements or relationships *dynamically*. Students are required to make or explain observations or generalizations based on their dynamic use of the technology.
Mathematical exploration; use appropriate tools strategically	How does technology support mathematical exploration?	The task and prompts guide students to investigate the same example or a set of examples to explore mathematical connections or invariances. Freedom with respect to technology use does not provide alternative paths if students are all investigating the same example(s).	The task and prompts allow students to explore their individual observations of mathematical concepts, connections, or invariances based on their *dynamic* use of the technology. The task supports students' mathematical exploration by providing alternative paths.
Make and test conjectures; modify thinking; foster curiosity	Does the technology provide feedback? Do the prompts encourage or require students to use feedback?	The task is limited by restrictive construction or does not provide feedback to allow students to explore their conjectures. Prompts do not explicitly guide students to test conjectures.	The technology provides feedback or allows students to test and refine conjectures. Prompts explicitly guide students to use the technology to test conjectures.

Figure 1: The Adapted X Framework

Methods

This study included 13 in-service teachers in an online, paid professional development (PD), representing a range of teaching experience, from one to over 30 years. Most teachers had between 11-20 years of experience, with a group average of 15.6 years. Self-reported comfortability with Desmos included three teachers with minimal experience, eight that used Desmos once or twice a week, and two reported using Desmos almost daily. The four-week, asynchronous PD focused on utilizing Desmos for teaching and learning mathematics and consisted of seven self-paced modules. The first four modules provided an overview of tools within Desmos, while the remaining three modules focused on pedagogical considerations when using Desmos. During the last two modules, participants were 1) introduced to the Adapted IGS Framework, 2) analyzed Desmos tasks using the Adapted IGS Framework, and 3) revised a task so that technology would be used as a reorganizer for one of the goals for students' thinking. In the culminating activity participants selected one goal for students' thinking (i.e., target dimension) to design a task that used Desmos as a reorganizer with respect to that goal. Written rationales articulating design choices related to their target dimension were also required.

Data Sources and Analysis

The data sources for this study consisted of the teachers' designed tasks and their written rationales. The unit of analysis was the identified goal for student thinking. Data was collected, blinded, and analyzed following the completion of the PD. Tasks were independently coded as using technology as an amplifier or a reorganizer along all three dimensions of the Adapted IGS Framework. Interrater reliability for coding tasks along each goal for student thinking was 87%. Discrepancies were resolved through synchronous re-analysis and discussion. This analysis was done without knowledge of the participants' identified goal for their task design.

Analysis of cognitive demand was also independently coded using the Potential of the Task Rubric within the Instructional Quality Assessment Toolkit for Mathematics (Matsumura et al., 2008; Boston, 2012). Interrater reliability for level of cognitive demand was 100%. For the third level of coding, written rationales were examined to discern the identified goal for student thinking and to determine if participants successfully attained their target goal.

Findings

As noted above, each of the 13 tasks were coded as amplifier or reorganizer along each dimension of the Adapted IGS Framework prior to considering participants' identified goal for student thinking (e.g., row/dimension of the framework). Recall, these identified goals were for students' thinking, not the mathematical goals. However, interpreting this analysis proved to be more difficult than previous iterations due to variation in how well participants clearly identified their goal for student thinking. Of the 13 participants, only six clearly identified a single goal for student thinking, as directed in the assignment. Five did not explicitly identify their goal for student thinking, and two participants stated they were trying to design a task to be a reorganizer along all three dimensions of the Adapted IGS Framework. Since five participants did not identify their goal for student thinking, we could not ascertain if they met their design intentions.

Thus, we decided the unit of analysis should be each goal that was explicitly identified by a teacher. Table 1 shows a summary of these results with respect to each chosen target dimension. There were seven instances where participants successfully designed a task that was a reorganizer for their specified goal (58%, met). Results for the potential level of cognitive demand analysis revealed that for every task that used technology as a reorganizer, the result was a task of high potential cognitive demand.

Table 1: Summary of Results

Target Dimension	Teacher Identifier	Met Goal	Cognitive Demand
Row 1	T5, T9, T12	Yes	High
	T3	No	Low
Row 2	T12	Yes	High
	T9	No	High
	T4, T10	No	Low
Row 3	T6, T9, T12	Yes	High
	T1	No	High
Did not explicitly identify	T8, T13	-	High
	T2, T7, T11	-	Low

Discussion

We echo the sentiments many have expressed lately with challenges related to conducting research during Covid (e.g., Kalinec-Craig, 2021). This study was not without its challenges. Interpretation of the results proved more difficult than anticipated because teachers did not complete the task design assignment as instructed. The assignment asked teachers to identify one goal for students' thinking from the Adapted IGS Framework; however, some teachers identified more than one goal for students' thinking from the framework and others neglected to explicitly state any goal for students' thinking. Due to the asynchronous nature of the PD, we were unable to ascertain why this was the case.

Despite the challenges presented in this asynchronous PD, one promising result was if a teacher designed a task that was a reorganizer along the identified goal for student thinking, then the task was also of high cognitive demand. This is a result that has been consistent across three iterations of this work, including a variety of contexts (e.g., in-service/pre-service teachers, face-to-face courses/hybrid courses, undergraduate/graduate) and two different technologies (e.g., dynamic geometry software and Desmos) (Cayton & Chandler, under review (n=14); Sherman et al., 2017 (n = 15); as well as this study (n = 7)). In other words, when teachers use the IGS Framework or the Adapted IGS Framework to design a task that leverages the affordances of the math action tool as a reorganizer, the consistent result has been a high cognitive demand task.

When teachers implement high cognitive demand tasks, learners are provided more equitable learning opportunities to engage in high level thinking (Aguirre et al., 2013). Thus, we propose that the Adapted IGS Framework offers the potential to support teachers in creating tasks that improve learning conditions for all learners. Yet, there is still more work needed. As noted in McCulloch et al. (2021a), the frameworks we are using in our preparation of teachers do not address the issue of equity. As such, we are considering how we might incorporate the recommendations from McCulloch et al. (2021b) to position all learners as explorers of mathematics when using technology into revisions of the Adapted IGS Framework. Our hope is that by updating the framework, we can help support teachers in understanding practical ways they can make use of math action technologies to engage students in deeper mathematical learning.

References

Aguirre, J. M., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K-8 mathematics; Rethinking equity-based practices*. National Council of Teachers of Mathematics.

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics.*

Boston, M. D. (2012). Assessing instructional quality in mathematics. *The Elementary School Journal, 113*, 76-104.

Bray, A., & Tangney, B. (2017). Technology usage in mathematics education research – a systematic review of recent trends. *Computers & Education, 114*, 255–273.

Cayton, C., & Chandler, K. (Under Review). A learning sequence for developing technology tasks that support students' mathematical thinking: A replication study. *International Journal of Mathematical Education in Science and Technology.*

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. American Mathematical Society & Mathematical Association of America.

Ertmer, P. A., Addison, P., Lane, M., Ross, E., & Woods, D. (1999). Examining teachers' beliefs about the role of technology in the elementary classroom. *Journal of Research on Computing in Education, 32*(1), 54–72.

Goos, M. (2005). A sociocultural analysis of the development of pre-service and beginning teachers' pedagogical identities as users of technology. *Journal of Mathematics Teacher Education, 8*(1), 35–59.

Goos, M. & Bennison, A. (2006). Technology use in secondary mathematics classrooms: A survey of Queensland teachers. *Proceedings of the AARE International Education Research Conference 2006*, 1-15. Adelaide, Australia.

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2003) Perspectives on technology mediated learning in secondary school mathematics classrooms. *Journal of Mathematical Behavior, 22*, 73–89.

Kalinec-Craig, C. (2021). Pandemics, scholarship, and rethinking what counts. *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Philadelphia, PA.

Matsumura, L. C., Garnier, H., Slater, S. C., & Boston, M. D. (2008). Toward measuring instructional interactions "at-scale." *Educational Assessment*, *13*, 267–300.

McCulloch, A. W., Leatham, K. R., Bailey, N. G., Cayton, C., Fye, K., & Lovett, J. N. (2021a). Theoretically framing the pedagogy of learning to teach mathematics with technology. *Contemporary Issues in Technology and Teacher Education, 21*(2).

McCulloch, A. W., Lovett, J. N., Cayton, C., & Dick, L. K. (2021b). Positioning students to explore math with technology. *Mathematics Teacher: Learning and Teaching PK-12, 114*(10), 738-749.

National Council Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*.

Pea, R. D. (1985). Beyond amplification: Using the computer to reorganize mental functioning. *Educational Psychologist, 20*(4), 167–182.

Pierce, R., & Ball, L. (2010). Secondary teachers' use of technology for teaching mathematics. In *Making a difference: Proceedings of the 2010 Conference of the Australian Association for Research in Education*. Melbourne, Australia.

Sherman, M. F. (2014). The role of technology in supporting students' mathematical thinking: Extending the metaphors of amplifier and reorganizer. *Contemporary Issues in Technology and Teacher Education, 14*(3), 220-246.

Sherman, M. F., & Cayton, C. (2015). Using appropriate tools strategically for instruction. *Mathematics Teacher, 109*(4), 306-310.

Sherman, M. F., Cayton, C., & Chandler, K. (2017). Supporting teachers using appropriate tools strategically: A learning sequence for creating technology tasks that support student thinking. *Mathematics Teacher Educator, 5*(2), 122-134.

Sinclair, M. P. (2003). Some implications of the results of a case study for the design of pre-constructed, dynamic geometry sketches and accompanying materials. *Educational Studies in Mathematics, 52*(3), 289–317.

Washira, P., & Keengwe, J. (2011). Technology integration barriers: Urban school mathematics teachers' perspectives. *Journal of Science Education and Technology, 20*, 17–25.

Zbiek, R., & Hollebrands, K. (2008). A research-informed view of the process of incorporating mathematics technology into classroom practice by inservice and prospective teachers. In M. K. Heid, & G. Blume (Eds.). *Handbook of research on technology in the learning and teaching of mathematics: Syntheses and perspectives* (pp. 287–344). Information Age.

STUDENTS' IMPRESSIONS OF EQUATION-SOLVING WORKED EXAMPLES IN AN ONLINE ENVIRONMENT

Avery H. Closser	Hannah Smith	Jenny Yun-Chen Chan
Worcester Polytechnic Institute	Worcester Polytechnic Institute	Worcester Polytechnic Institute
ahclosser@wpi.edu	hcsmith2@wpi.edu	jchan2@wpi.edu

Worked examples are an effective form of instructional support. As online tools are increasingly used for instruction, designers should optimize how worked examples are presented to support students. We present a small exploratory study on 92 algebra students who rated and evaluated formats of worked examples on equation-solving as part of a larger experiment. We found that students' ratings of the worked examples aligned with their posttest scores. Further, across conditions, explanations of students' ratings revealed themes of worked example speed, equation-solving process, and content review. These findings provide considerations for researchers and teachers creating worked examples for online algebra learning environments.

Keywords: Algebra and Algebraic Thinking, Online and Distance Education, Metacognition, Technology.

Worked examples effectively help students learn math procedures by displaying the step-by-step problem-solving process (e.g., Booth et al., 2015; Carroll, 1994; Foster et al., 2018). However, worked examples are typically presented as static images with limited research on the impacts of alternative formats of worked examples. As online tools become more prevalent in classrooms, worked examples should be intentionally designed to optimize learning in online environments. Further, since prior work has shown a positive association between students' actual and perceived learning gains from online instructional materials (Whitehill et al., 2019), students' perceptions of worked examples should inform their design. Here, we explore students' perceptions of worked examples presented in *static*, *sequential*, or *dynamic* formats.

We consider two competing theoretical perspectives. First, perceptual scaffolding (e.g., visual cues that direct students' attention to important information in instructional materials) can support students' cognitive processes and increase learning (Harrison et al., 2020; Gibson, 1969; Kirshner, 1989). Online tools present opportunities to design worked examples which use dynamic transformations and display the connections between steps. These worked examples may help guide students' attention to make connections about the problem-solving process. Second, worked examples that minimize extraneous complexity can decrease students' cognitive load and improve learning (Schwartz et al., 2016; Sweller et al., 2019). By this logic, students may find that worked examples with dynamic transformations take more effort to study: these examples might increase students' cognitive load rather than decrease it.

We have found comparable learning gains among students who viewed different worked example formats (Smith et al., 2022). Here, we report an exploratory analysis of how those students perceived the worked examples and why they found them to be helpful or unhelpful. We explore: 1) *Are students' posttest scores correlated with their rating of the worked examples?* 2) *Which themes emerge in students' explanations for their ratings?* 3) *How do the themes vary across students who viewed worked examples with different degrees of dynamic presentation?*

Methods

A total of 230 grade-school algebra students from North and Central America completed a larger online experiment investigating how the format of worked examples impacted learning gains in an online tutoring system (Smith et al., 2022). Of those students, 92 rated the helpfulness of the worked examples and provided a rationale, thus were included in the analyses.

We recruited teachers who already used the tutoring system with their classes. At their discretion, teachers assigned this one-hour study to students as an online problem set during class or as homework. Students worked at their own pace to complete the assignment. First, they completed an eight-item pretest on algebraic equation solving (e.g., solve for x in $8(2x + 9) = 56$; reliability: KR-20 = 0.86). Next, they were randomly assigned to one of six experimental conditions varying in the worked example presentation. Within each condition, students completed six problem pairs consisting of one worked example and one practice problem. Next, students completed the posttest (reliability: KR-20 = .89) matching the structure of the pretest. Half of the pretest and posttest problems directly mirrored the equation structure in the worked examples whereas the other half used similar but different equation structures. Last, students were asked how much they agree with the statement: "*The worked examples were helpful for learning how to solve equations*". Students rated the statement (1 = Strongly Disagree; 6 = Strongly Agree) and were prompted to explain their rating.

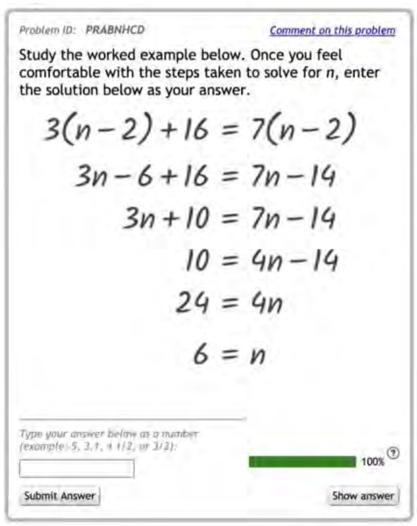


Figure 1: A Static Worked Example

The six worked example conditions of the larger study varied in their *extensiveness* (i.e., displaying major problem steps or every transformation) and degree of *dynamic* presentation (i.e., static, sequential, or dynamic). Here, we focus on how students perceive worked examples displayed with different degrees of dynamic presentation as supported by online tools so we do not further describe or discuss the variations in the extensiveness of worked examples. First, students in the *static* conditions viewed images of worked examples as they might be displayed in a textbook (Figure 1). Second, students in the *sequential* conditions viewed worked examples as looping GIF videos that presented the worked examples line-by-line in approximately 3-second intervals, creating a history of the derivation over time. Third, students in the *dynamic* conditions viewed looping videos that demonstrated the transformation process through a screen recording. For example, to simplify $2(x - 3) = 8$, students would watch as the 2 is dragged over the parentheses to create the transformed expression, $2x - 6 = 8$ (Figure 2).

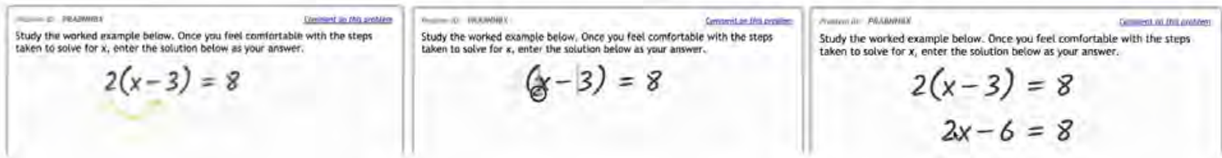


Figure 2: Dynamic Presentation of Worked Examples with Fluid Transformations

Results

We first conducted a partial correlation between students' perceived helpfulness and their posttest performance while controlling for pretest performance (M = .38, SD = .37). Students who scored higher on the posttest also rated the worked examples as more helpful, $r(90)$ =.24, p = .02. A scatter plot also illustrated a positive, albeit weak, association between students' posttest performance and perceived helpfulness of the worked examples (Figure 3).

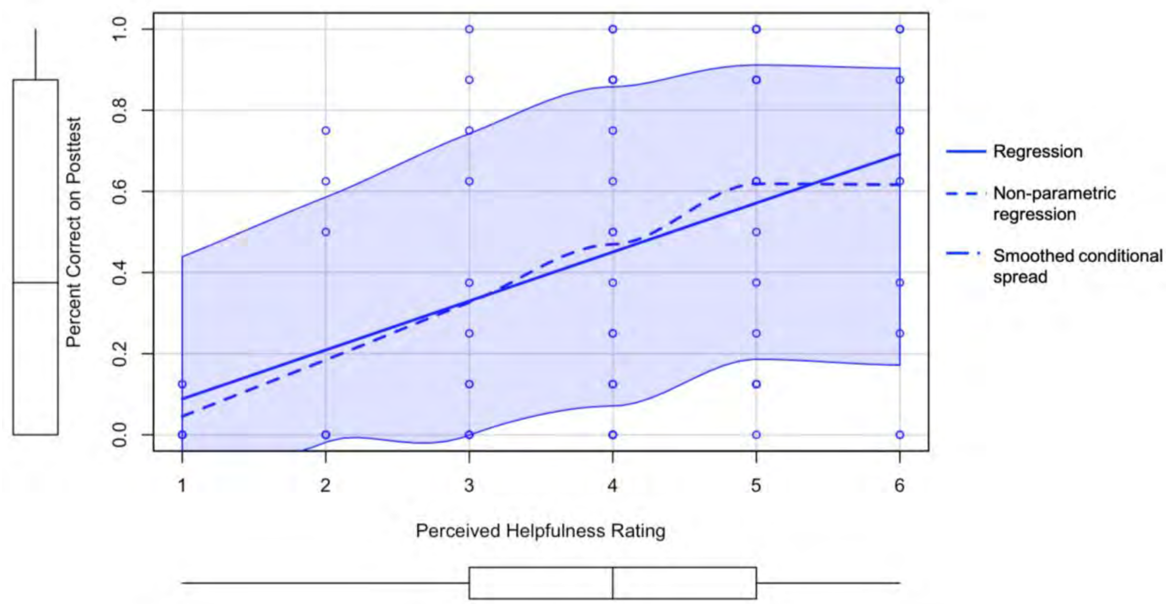


Figure 3: Scatter Plot of Students' Perceived Helpfulness Rating and Posttest Performance

Next, we identified themes in students' explanations indicating why they found the worked examples to be helpful or not. Three themes emerged: the *speed* of the worked examples, the *process* within the worked examples, and the worked examples as *review* material. After identifying and defining these themes, two researchers individually coded responses that included one of these themes and compared how these themes emerged across students who viewed static (n=29), sequential (n=38), or dynamic (n=25) worked examples.

First, four students (three in dynamic conditions; one in sequential) commented on the *speed* of the worked examples, expressing that the worked examples were displayed too quickly. One student in a dynamic condition noted that the worked examples "*were helpful but they went too fast*" and another student expressed that having a pause button (rather than the looping video) might change the way they study the example.

Second, nine students noted that the worked examples demonstrated the *process* for equation solving. For example, one student in a dynamic condition wrote, "*The examples were helpful because it showed the full procedures on how they got the answer*". Of the nine students, five were in dynamic, two in sequential, and two in static conditions. A Chi-square test comparing the number of students noting vs. not noting that the process revealed no significant difference across the three types of conditions, $\chi^2(2, 92)=4.11$, $p=0.13$.

Third, ten students noted that the worked examples or the assignment as a whole served as a *review*. For example, one student in the dynamic condition expressed, "*Doing all these refreshed my mind on how to do equations with variables. It was very helpful and I believe it helped improve my skill*". Three of those students were in dynamic, four in sequential, and three in static conditions. A Chi-square test comparing the number of students noting vs. not noting about review revealed no significant difference across the condition types, $\chi^2(2, 92)=0.05$, $p=0.98$.

Discussion

The results align with existing research and provide considerations for designing online algebra worked examples. First, these results support prior work on students' ability to gauge their learning (Whitehill et al., 2019) by showing that their perceived helpfulness of worked examples was positively associated with their posttest scores. Second, these results reveal features that students notice when studying worked examples: the speed of the worked example, the equation-solving process, and connecting examples with course content. Third, these themes did not emerge more often in one condition than another, potentially suggesting that students may attend to these elements when evaluating any worked examples.

Multiple students in the sequential and dynamic conditions commented that the *speed* of the worked example videos was too fast, indicating that the worked examples in the GIF video format may have increased the cognitive demand on students rather than serving as a helpful tool. Next, students across conditions noted that the worked examples demonstrated the problem-solving *process*, suggesting that all three formats may have provided visual cues that help with connection-making as a form of perceptual scaffolding. Finally, multiple students commented on the content as a *review*; studying the worked examples in a brief experimental context may have cued their prior knowledge to improve equation solving (Sidney, 2020).

Importantly, our sample size limited the analyses conducted and conclusions we were able to draw. Further, not every response included one of the themes identified; many students simply wrote, "*I don't know*" or "*it was good*". This narrowed our sample for comparisons across conditions. For example, since only four students across two of the conditions noted the speed of the worked examples, we did not conduct a Chi-square test for statistical comparisons. However, the students' comments provide suggestions for future iterations of sequential and dynamic worked examples. When designing these worked examples, it would be worthwhile to consider a slower video speed. Alternatively, since research has shown that undergraduate students can effectively self-regulate their use of worked examples for learning (Foster & Dunlosky, 2018), it may be worthwhile to explore how algebra students learn from worked examples with features that provide more autonomy (e.g., speed setting or pause).

Looking ahead, it would be worthwhile to conduct a larger study using refined iterations of these worked example formats and more targeted questions for students. For instance, asking, "What would you change about this worked example and why?" and "Which worked example format do you prefer and why?" may provide further insights about which features of worked examples students notice. Future studies should also consider long-term impacts from worked example practice with alternative presentations. Ultimately, we aim to offer suggestions that guide algebra teachers in designing online worked examples to support student learning.

References

Booth, J. L., Oyer, M. H., Paré-Blagoev, E. J., Elliot, A. J., Barbieri, C., Augustine, A., & Koedinger, K. R. (2015). Learning algebra by example in real-world classrooms. *Journal of Research on Educational Effectiveness*, *8*(4), 530-551.

Carroll, W. M. (1994). Using worked examples as an instructional support in the algebra classroom. *Journal of Educational Psychology*, *86*(3), 360.

Foster, N. L., Rawson, K. A., & Dunlosky, J. (2018). Self-regulated learning of principle-based concepts: Do students prefer worked examples, faded examples, or problem solving? *Learning and Instruction*, 55, 124– 138.

Gibson, E. J. (1969). *Principles of perceptual learning and development*. Englewood Cliffs, NJ: Prentice Hall.

Harrison, A., Smith, H., Hulse, T., & Ottmar, E. (2020). Spacing out!: Manipulating spatial features in mathematical expressions affects performance. *Journal of Numerical Cognition, 6*(2), 186-203. https://doi.org/10.5964/jnc.v6i2.243

Kalyuga, S., Chandler, P., Tuovinen, J., & Sweller, J. (2001). When problem solving is superior to studying worked examples. *Journal of Educational Psychology*, *93*(3), 579.

Kirshner, D. (1989). The visual syntax of algebra. *Journal for Research in Mathematics Education*, *20*(3), 274. https://doi.org/10.2307/749516

Lusk, M. M., & Atkinson, R. K. (2007). Animated pedagogical agents: Does their degree of embodiment impact learning from static or animated worked examples?. *Applied Cognitive Psychology: The Official Journal of the Society for Applied Research in Memory and Cognition, 21*(6), 747-764.

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge? An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561.

Schwartz, D. L., Tsang, J. M., & Blair, K. P. (2016). *The ABCs of how we learn: 26 scientifically proven approaches, how they work, and when to use them.* W W Norton & Co.

Schworm, S., & Renkl, A. (2006). Computer-supported example-based learning: When instructional explanations reduce self-explanations. *Computers & Education, 46*(4), 426-445.

Sidney, P. G. (2020). Children’s learning from implicit analogies during instruction: Evidence from fraction division. *Cognitive Development, 56*(August 2019), 100956. https://doi.org/10.1016/j.cogdev.2020.100956

Smith, H., Closser, A. H., Ottmar, E. R., & Chan, J. Y. C. (2022). The impact of algebra worked example presentations on student learning. *Applied Cognitive Psychology, 36*(2), 363-377. https://doi.org/10.1002/acp.3925

Sweller, J., van Merriënboer, J. J., & Paas, F. (2019). Cognitive architecture and instructional design: 20 years later. *Educational Psychology Review*, *31*(2), 261-292.

Whitehill, J., Aguerrebere, C., & Hylak, B. (2019). Do learners know what's good for them? Crowdsourcing subjective ratings of OERs to predict learning gains. In the *Proceedings of the 12th International Conference on Educational Data Mining (2019),* 462-467.

THE TEACHER'S ROLE IN SUSTAINING COGNITIVE DEMAND WITH DESMOS

Victoria Delaney
Stanford University
vdoch@stanford.edu

Teachers play a crucial role in influencing students' cognitive demand during mathematics tasks. This study investigates the interaction between teachers and cognitive demand when digital technologies are integrated during students' enactment of such tasks. The research includes a video analysis of two eighth grade teachers' task launches and enactments of six student small groups using Desmos to solve a high-demand task. The findings indicate that the teachers' launches had less overall impact on students' cognitive demand and Desmos use than their ensuing small-group interactions, including how they related Desmos to the task's goals.

Keywords: Curriculum, Middle School Education, Technology, Teacher Knowledge

As digital technologies become more commonplace in classrooms, there is increasing demand for teachers to instruct students with such technologies to better prepare for career and college readiness (National Governors Association, 2010). Yet despite general agreement toward incorporating these tools, there are few standardized practices that instruct teachers on how and when to use digital technologies for learning (Greenhow, Robelli, & Hughes, 2009). As a result, the ways in which teachers frame the role of such technologies to students in a math task's launch and following enactment of the task may differ, impacting how students view digital technologies in relation to solving rigorous problems. Taking two middle school math teachers' classrooms as contrasting case studies, I ask: how do teachers' framing of digital technology during a high-demand mathematics task impact the cognitive demand of the task (a) during the task launch and (b) during students' enactment of the task?

Theoretical Framework

This study examines the changes in cognitive demand throughout a rigorous mathematics task and the interactions between teachers, students, and digital technology during its enactment. To situate this work, I use Stein and colleagues' definition of a *mathematical task* as the arrangement of mathematics within the task, combined with the resources provided to solve it. *Cognitive demand,* the primary variable of interest, refers to the cognitive processes required of students to navigate a task, identify key information, and activate prior knowledge that is useful for constructing solutions (Doyle, 1983). The current study focuses on a *high cognitive demand* task (Stein & Smith 1998), which requires students to make conceptual connections, use multiple representations and consider multiple solution strategies. Because cognitive demand is adapted and negotiated by teachers and students throughout enactment (Remillard, 2005), a key goal of this research is to examine its changes when digital technologies are present and when teachers play a central role in encouraging students' uses of such technologies.

This study takes an in-depth examination of students' uses of Desmos, a *dynamic representational technology* (Vahey et al., 2020), which can automate graphing, give instantaneous feedback, and display multiple modes of representation. Such technologies may potentially alter the cognitive demand of the task and influence how students learn mathematics (Sherman, 2014). However, technology's potential to mediate task outcomes may be contingent on how teachers frame them to students and how students take up such tools. Because teachers' framing of digital technologies in relation to the task's outcomes varies, there is currently mixed evidence of digital technology's efficacy in relation to students' learning. Some studies point toward positive learning outcomes when students express aptitude for using technology to solve problems (e.g., Hollebrands, 2003; Gökçe, & Güner, 2022) although there is not yet clear consensus on how the role of the teacher impacts student outcomes.

Methods

Broader Context

This study is one component of a larger research-practice partnership conducted between a West Coast research university and Urban Unified School District (UUSD), where many students are from non-dominant cultural and linguistic communities (UUSD, 2020). UUSD strives to provide a high-quality mathematics education for its students, and as such, developed a task-based curriculum that aligns with the Common Core State Standards (Borko et al., 2017). As part of its long-term vision for implementing the curriculum, UUSD identified and recruited Teacher Leaders (TLs) in 6th-8th grades to participate in professional development workshops six times per year. The goal of each workshop was for TLs to deeply understand the curriculum and facilitate professional development around it to their own mathematics departments.

Task Selection

The Washing Machine Problem is a high demand, *Doing Mathematics* (Boston & Wolf, 2006) task taken from UUSD's 8th grade curriculum (see Figure 1). It requires students to analyze various costs mathematically in order to give advice to "Elena" toward purchasing a washer and dryer or continuing to visit her local laundromat. The task's design purposefully omits critical information, such as how often Elena does laundry, so that students can grapple with the task's inherent ambiguity and practice voicing their assumptions explicitly. As such, the task does not have a single "correct" solution. Rather, the challenge of solving the task comes from how students draw upon multiple representations and solution strategies when solving it.

Figure 1: The Washing Machine Task, as Written in the District Curriculum

"Elena doesn't have a washing machine or dryer. She is considering buying one so that she doesn't have to go to the Laundromat. She wants to know if it makes sense financially to buy a washer and dryer. Each load of laundry at the Laundromat costs $1.25 for the washing machine and $1.50 for the dryer. A top-loading washer costs $250, and each load costs $0.26 for water and energy. A front-loading washer costs $400, and each load costs $0.09 for water and energy. A dryer costs $300, and each load costs $0.35 for energy. **What should Elena do? Give her some advice.**"

Teacher Participants

Two teachers from UUSD were selected based on their integration of Desmos during instruction of *The Washing Machine Problem* in Spring 2017. Annie, the first teacher, was involved in the research-practice partnership as a TL during the time of the study. Tia, the second teacher, was involved in a different professional learning program supported by UUSD. Although the district maintained an initiative to support technology use in the classroom, the extent to which Tia and Annie used Desmos with students prior to the task is unknown.

Data Sources and Analysis

Data for this project were obtained from video recordings from both teachers' classrooms while instructing *The Washing Machine Problem*. Recordings contained both the launch of the task and the enactment by students in small groups. Three recordings were taken from each classroom and were subjected to additional screening for visual and audio quality. Each teacher's launch was additionally captured in a separate recording from start to finish.

The research team rated and coded units of mathematical conversation ("conversation units" e.g., sequences of talk where students discussed mathematical ideas) according to the Instructional Quality of Assessment (IQA) Rubric (Boston & Wolf, 2006). We identified (1) changes in students' cognitive demand throughout the task and (2) instances where Desmos was used and the teacher participated in student interactions. Both researchers rated and coded independently, reaching 87.4% agreement, then met to resolve remaining disagreements.

Findings

Teachers' Launches

Annie announced that students would solve a system of equations and give advice. She presented the task as a series of steps: first, students would create three equations to model the two different washing machines and laundromat options. Then, students would graph their equations in Desmos, where Desmos was mandated as part of the second step of solving. Finally, students would give advice for the best washer-dryer combination. We rated the cognitive demand of Annie's launch as a 2: *Procedures without Connections.* This launch focused on a sequence of procedures for students to follow, which removed ambiguity and reduced opportunities for multiple solution strategies. Although Annie informed students to produce multiple representations, there was little question on how to do so. The focus of the launch seemed more suited toward completion of a sequence of steps than on non-algorithmic thinking.

Tia opened the task by situating it in the experiences of her students: "Who here has ever been to a laundromat?" She gave an overview of the three models, including how to build them. Then, she encouraged multiple solutions and representations of the three models, and presented Desmos as a strategic, optional tool. Tia ended the launch by directing students to construct a poster of their results. We rated the cognitive demand of Tia's launch as a 3: *Procedures with Connections*. Tia's explanation of how to build the linear models slightly lowered the cognitive demand from the task as written. However, unlike Annie, Tia left open and encouraged multiple solutions and representations, which preserved the most challenging aspects of the task.

Student Enactments

Although Annie's launch initially lowered cognitive demand, her three student groups spent substantially more time in high demand activity (see Table 1). Compared to Tia's students, Annie's used Desmos earlier in the task, which allowed them to analyze their models' mathematical features and compare multiple representations for a longer duration. Because they spent more time critically comparing their models, two of Annie's three groups realized that the task did not contain a single correct answer; rather, their perception of the "best" model depended on their interpretations of unknown variables, such as how often Elena did laundry.

Annie's small-group interactions were imperative to students' successes. When she approached each group, Annie routinely asked students to revoice their ideas aloud, then asked probing questions (e.g., "What assumptions are you making?" to focus students' thinking on mathematically relevant information provided by Desmos. By directing students to Desmos, she helped students connect affordances of the technology with the goal of the task. Her interactions routinely raised cognitive demand or maintained it at a level 3 or 4 (see Table 2).

Table 1: Student groups' cognitive demand during enactment, by teacher

	Annie's Students		Tia's Students	
IQA Level	Minutes	Conversation Units	Minutes	Conversation Units
Level 1	0.3	1	5.0	7
Level 2	17.2	**34**	**37.8**	**86**
Level 3	19.7	27	13.5	24
Level 4	**22.4**	17	6.9	10

Table 2: Student groups' shifts in cognitive demand during small-group interactions

	Increase	No change	Decrease		Increase	No change	Decrease
Annie	6	10	0	**Tia**	2	12	5

Tia's students spent a majority of the task in level 2 (see Table 1) despite her initial preservation of the task's rigorous aspects. Compared to Annie's students, Tia's small groups took longer to reach the analytic portion of the task, expressing more confusion on how to create their models. Two of the three groups did not know how to use Desmos, requiring an additional tutorial from Tia before they could use it for graphing. These groups did not finish the task or engage in any interpretive comparisons of the models before enactment ended. The sole group that was familiar with Desmos used it immediately and was responsible for all of the level 4 activity in Table 1, successfully solving the task and recognizing the ambiguity inherent in it.

Tia's interactions with students often maintained low demand or lowered demand, although there were a few instances where she assisted students in becoming unstuck and entering high-demand activity. When she visited small groups, Tia encouraged students to build off of one another's ideas. However, students' ideas were not always high-demand or mathematically correct. Furthermore, Tia desired that all students work together and at the same pace, which encouraged frequent low-demand tutoring conversations where the tutor tended to focus on the reproduction of their own work. Finally, when the focus was not on equitable groupwork, Tia aided students by directly telling them what to do (e.g., "You need to add the costs together to find the slope.") Demand was kept low because Tia often prioritized correctness without connecting it back to the goals of the task or highlighting the affordances of Desmos.

Discussion and Conclusion

The degree of divergence between Tia and Annie's approaches to launching and facilitating *The Washing Machine Problem* had varying impact on their students. Tia's focus on equitable groupwork and procedural precision, as well as her choice to frame Desmos as an optional tool, kept cognitive demand low and discouraged students from using Desmos until it was absolutely necessary. In contrast, Annie foregrounded demanding analytic components of the task, anticipated the necessity of Desmos in allowing students to access those components, and bridged students' Desmos use to the task's goals during small group interactions. This work suggests that, although digital technologies are becoming more commonplace in the classroom, teachers must first establish congruence between the learning goals, the features of the task and technology that are highlighted in the launch, and their interactions with students. Furthermore, they must understand their role in maintaining cognitive demand throughout the launch (Jackson et al., 2013) and task enactment (Stein, Engle, Smith, & Hughes, 2008).

References

Borko, H., Carlson, J., Mangram, C., Anderson, R., Fong, A., Million, S., ... & Villa, A. M. (2017). The role of video-based discussion in model for preparing professional development leaders. *International journal of STEM education*, *4*(1), 1-15.

Boston, M., & Wolf, M. K. (2006). Assessing Academic Rigor in Mathematics Instruction: The Development of the Instructional Quality Assessment Toolkit. CSE Technical Report 672. *National Center for Research on Evaluation, Standards, and Student Testing (CRESST)*.

Doyle, W. (1983). Academic work. *Review of educational research*, *53*(2), 159-199.

Gökçe, S., & Güner, P. (2022). Dynamics of GeoGebra ecosystem in mathematics education. *Education and Information Technologies*, 1-23.

Greenhow, C., Robelia, B., & Hughes, J. E. (2009). Learning, teaching, and scholarship in a digital age: Web 2.0 and classroom research: What path should we take now?. *Educational researcher, 38(*4), 246-259.

Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *The Journal of Mathematical Behavior*, *22*(1), 55-72.

Jackson, K., Garrison, A., Wilson, J., Gibbons, L., & Shahan, E. (2013). Exploring relationships between setting up complex tasks and opportunities to learn in concluding whole-class discussions in middle-grades mathematics instruction. *Journal for Research in Mathematics Education*, *44*(4), 646-682.

National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). Common Core State Standards Key Considerations. Washington, DC: Author.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of educational research*, *75*(2), 211-246.

Sherman, M. (2014). The role of technology in supporting students' mathematical thinking: Extending the metaphors of amplifier and reorganizer. *Contemporary Issues in Technology and Teacher Education*, *14*(3), 220-246.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical thinking and learning*, *10*(4), 313-340.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics teaching in the middle school*, *3*(4), 268-275.

[Urban Unified School District]. [UUSD] *Facts at a glance.* [District website]. [Urban Unified School District]. (June 2014). *[UUSD] Vision 2015: Reimagining education in [City] for a new generation.*

ELEMENTARY STUDENTS' EMERGENT USE OF PARENTHESES WHEN WRITING EXPRESSIONS IN A SANDBOX DIGITAL LEARNING GAME

Christopher Engledowl
New Mexico State University
chriseng@nmsu.edu

Ted Stanford
New Mexico State University
stanford@nmsu.edu

Developing competency with writing numeric expressions that include grouping symbols is an important goal of early algebra (Stephens et al., 2017). In this study, we draw on numeric expressions written by 473 fourth and fifth grade students from 10 schools and 29 teachers as they played a digital learning game in order to examine their use of grouping symbols. We describe use of parentheses as being of three types: Type 1 are trivial, Type 2 indicate the order things should be done in, but are unnecessary, and Type 3 are necessary. Preliminary results show that regardless of whether a student played the game for much longer, the use of Type 2 and Type 3 parentheses increased as students became more experienced players. The sand box style of the game design appears to have played a role in students feeling free to experiment with different types of parentheses. Future research is needed on the sophistication development.

Keywords: Algebra and Algebraic Thinking, Technology, Number Concepts and Operations

The field of *early algebra* (Stephens et al., 2017) has established itself as a promising developmental pathway to support students' later learning of formal algebra (Blanton et al., 2015; Venenciano et al., 2020), which has a long-standing role as a gatekeeper to higher level mathematics and college entrance (Slaught, 1908; Moses & Cobb, 2001; Knuth et al., 2006; Martin et al., 2010). One of the foundational big ideas of early algebra involves learning about expressions (Blanton et al., 2015). Expressions have not only had a longstanding place in the elementary school curriculum (NCTM, 2000; CCSSM, 2010), but also in the MET II and the latest AMTE (2017) standards for preservice teachers.

Despite the fact that the topic of order of operations has been, and continues to be, a point of hot debate amongst the general population—resulting even in popular press articles discussing it, such as the New York Times article about mathematicians' sentiments on the social media firestorm (Chang, 2019)—few studies have documented students' struggles and successes with using grouping symbols (e.g., Papadopoulos & Gunnarsson, 2018). Moreover, most studies have largely focused on how students perform when they are presented with different types of structures of expressions (Gunnarsson et al., 2016). Perhaps this combination of lack of research and widespread misunderstanding is why there have been recent suggestions for using a different model for order of operations (Taff, 2017). In this study, we present preliminary results documenting how 4th and 5th grade students used grouping symbols while beginning to learn how to write numeric expressions in the context of a digital mathematics learning game.

Background and Framing

Blanton et al. (2015) described the following five big ideas of early algebra: a) Equivalence, Expressions, Equations, and Inequalities, b) Generalized Arithmetic, c) Functional Thinking, d) Variable, and e) Proportional Reasoning. In this study, the project key idea we are focusing on is *Write and Interpret Numeric Expressions*, which overlaps with Blanton et al.'s first big idea of early algebra. In the current study, students wrote dozens to hundreds of numeric expressions while playing a digital learning game. When writing the expressions, based on the need, sometimes it was not necessary to write them with any specific reason in mind. However, the longer students played the game, and in order to be more intentional about what they were creating in the game, students had to write expressions with intent. For instance, writing an expression that would evaluate to the value of 4 and also include two different operations and one set of parentheses. Such thinking is in line with the expectations that Blanton et al. had for the third graders in their study, such as having students be able to "identify different ways to write an expression" (p. 48).

Earlier research has distinguished *necessary* from *unnecessary* parentheses, sometimes referred to as *superfluous* (Gunnarsson et al., 2016). We have found it helpful in this study to refine this distinction into three types. Type 1 parentheses are trivial in the sense that they give no instructions about the order of operations. Examples of expressions with Type 1 parentheses are (2) + 3, (2 + 3), and the extra set of parentheses in ((2 + 1)) × 3. Note that Type 1 parentheses are not mathematically incorrect, in that they are not ambiguous and are evaluated correctly under standard procedures and by machine calculators. Moreover, Type 1 parentheses may naturally occur as intermediate steps in simplifying an expression, or in substituting one expression for an equivalent one. Type 2 parentheses are those which do give instructions as to the order that operations should be evaluated in, but the instructions are superfluous, either because of the standard order of operations, or because of the mathematical properties of operations. Examples of expressions with Type 2 parentheses are 4 + (3 × 1) and 2 + (3 + 1). Type 3 parentheses are those which are necessary, in that the expression would evaluate to a different outcome without them. An example with Type 3 parentheses is 2 × (3 +1).

Method

Participants and Setting

This study stems from a larger project involving an early algebra intervention incorporating two games and an interactive tool, as well as associated lesson plans. Some of the information shared in this method section has been similarly described elsewhere (Engledowl, 2020; Engledowl et al., 2021). This study focuses on the expressions that 473 students, from 10 schools and 29 teachers' classrooms, wrote while playing a digital mathematics learning game called *Agrinautica* (available at mathsnacks.org).

Digital Learning Game

The digital mathematics learning game, *Agrinautica*, is a sandbox-style game that was designed to align with the larger project's key idea of *Writing and Interpreting Numeric Expressions*. This key idea also aligns with Blanton et al.'s (2015) big idea of *Equivalence, Expressions, Equations, and Inequalities*. In this game, players first select a world and then set out to populate it with plants, animals, and artifacts. To create them, players write numeric expressions. Depending on the number of operations, the number of sets of parentheses, and the value the expression evaluates to (an integer from 1–9), different species of plants, animals, and artifacts are created and players can place them in their world. For instance, to create a plant called *Vinus duoclus* (see Figure 1), a player must write an expression that evaluates to the value 2, includes two types of operations, and has no parentheses.

Figure 1: Screenshots from *Agrinautica* of field guide (left) and how plants are created

Data Sources and Analysis

From February through May of 2019, game telemetry data was collected from all participants—whether they played at school or at home. This data included every expression they wrote and subsequent plant that was made, every button they clicked on—such as if they viewed the field guide that shows every possible item with a description of what is necessary to include in the expression to create each item—as well as associated date and time stamps. Both the time spent and the number of total actions taken varied widely across the study group, and we are using here the number of total actions taken as a measure of the amount of game play for a given student. We divided the students into two equal-sized groups, according to the number of total actions they took. For each student, we recorded all unique expressions, in the order that they were first created. Some students entered many repeated, identical expressions, but these were not counted for the purpose of this analysis. Expressions such as 1+1 and (1+1) were counted as different, even though the game would consider them the same expression for the purpose of creating a plant. Then, all the expressions within the group were combined, maintaining the order in which they were created, and now with many duplicates possible since different students often made the same expression. This list was divided into quarters in order to create the tables below. This division into quarters keeps all students' first expressions together, and their second, and so on, and since the number of unique expressions created varied widely from one student to another, students who made more unique expressions are represented more heavily in the later quarters than students who made fewer. This is one reason we analyzed the two groups separately, to be sure we were capturing a pattern of increasing sophistication as game play progressed, and not just an overall difference between students. After students were grouped, we totaled the number of expressions that were categorized as Type 1, Type 2, or Type 3.

Findings

Preliminary results can be seen in Tables 1 and 2. Table 1 shows the number of times each type of use of parentheses was used for students who took between 20–399 total game actions. Table 2 shows this same information for students who took between 400 and 3,844 actions. From these two tables, there is evidence that student use of Level 1 parentheses remained relatively constant over time—that is, the total number being used didn't change as students gained experience writing more expressions. *Agrinautica* does not reward students for using these kinds of parentheses, but it also does not penalize their use—they are simply ignored. Thus, it is unclear why they continued to be used. Student use of Level 2 parentheses was found to increase as students wrote more expressions. Similarly, student use of Level 3 parentheses also increased as students wrote more expressions. Both of these are expected results because *Agrinautica* allows their use in creating plants, animals, and artifacts. However, *Agrinautica* does not specify that parentheses must be used in one of these two ways to create any particular plant, and does not prioritize Level 3 over Level 2. And yet, we observed student use of both types of parentheses increasing over time.

Table 1: Number of expressions created with certain types of parentheses for students who completed 20–399 total actions

Time Created	Type 1	Type 2	Type 3
1st Quarter	134	147	68
2nd Quarter	117	163	75
3rd Quarter	114	329	108
4th Quarter	104	519	192
Total	469	1158	443

Table 2: Number of expressions created with certain types of parentheses for students who completed 400–3844 total actions

Time Created	Type 1	Type 2	Type 3
1st Quarter	394	887	395
2nd Quarter	439	2217	1047
3rd Quarter	372	2598	1326
4th Quarter	363	3204	1511
Total	1568	8906	4279

Discussion and Implications

Preliminary results indicate that as students played longer, they tended to engage more in in using Type 2 and Type 3 parentheses—that is, they began to increase their use of parentheses in meaningful ways. For instance, when attempting to create items in the field guide listed under *Artifacts*, the item called “Unknown alien device 9” requires three types of operations and two sets of parentheses. To create this item, it is perhaps the most likely situation in which a student would be faced with creating an expression in which the parentheses would fall into Type 3—the use of parentheses changes the value the expression evaluates to. This is because it is more likely that student would use a subtraction symbol or multiplication symbol between two sets of parentheses, thus requiring use of the distributive property with the second set of parentheses. However, it is still possible that a student could have written such an expression without Type 3 parentheses. Thus, it is interesting to see an increase in their use for both groups of students as they became more experienced with playing the game. In contrast to prior studies (e.g., Gunnarsson et al., 2016), by providing an environment in which students’ expression writing could occur in a risk free environment, there appears to be evidence of broad success in encouraging the increase of sophistication in expression writing over time. Further research is needed to understand the nuances of the patterns in their development.

Acknowledgements

This research was supported from the National Science Foundation (Award # 1503507).

References

Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*.

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J.-S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39–87.

Chang, K. (2019, August). Why mathematicians hate that viral equation: It's formatted to confuse people, and there are no interesting underlying concepts. *The New York Times*. https://www.nytimes.com/2019/08/06/science/math-equation-pemdas.html

Engledowl, C. (2020). Constructing and validating an early algebra assessment. In J. Cribbs & H. Marchionda (Eds.), *Proceedings for the 47th Annual Meeting of the Research Council on Mathematics Learning* (pp. 51–58). https://www.rcml-math.org/assets/Proceedings/RCML%202020%20Proceedings%202-19-20.pdf#page=57

Engledowl, C., Al-younes, M., Chamberlin, B., & Stanford, T. (2021). Learning acquired from a computer game-based early algebra intervention. *School Science and Mathematics*, *121*, 495–508. https://doi.org/10.1111/ssm.12500

Gunnarsson, R., Sönnerhed, W. W., & Hernell, B. (2016). Does it help to use mathematically superfluous brackets when teaching the rules for the order of operations? *Educational Studies in Mathematics*, *92*(1), 91–105. https://doi.org/10.1007/s10649-015-9667-2

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*(4), 297–312.

Martin, D., Gholson, M., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education*, *3*(2), 12–24.

Moses, R. P., & Cobb, C. E. (2001). Algebra and civil rights? In *Radical equations: Math literacy and civil rights* (pp. 3–22). Beacon Press.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. NCTM.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*.

Papadopoulos, I., & Gunnarsson, R. (2018). The use of "mental" brackets when calculating arithmetic expressions. *42nd Conference on the Psychology of Mathematics Education*, *3*, 451–458.

Slaught, H. E. (1908). What should be emphasized and what omitted in the high-school course in algebra? *The School Review*, *16*(8), 503–516.

Stephens, A. C., Ellis, A. B., Blanton, M., & Brizuela, B. M. (2017). Algebraic thinking in the elementary and middle grades. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 386–420). National Council of Teachers of Mathematics, Inc.

Taff, J. (2017). Rethinking the order of operations. *Mathematics Teacher*, *111*(2), 126–132.

Venenciano, L. C. H., Yagi, S. L., Zenigami, F. K., & Dougherty, B. J. (2020). Supporting the development of early algebraic thinking, an alternative approach to number. *Investigations in Mathematics Learning*, *12*(1), 38–52. https://doi.org/10.1080/19477503.2019.1614386

A BAKHTINIAN LENS ON THE EFFICACY OF DIALOGIC INSTRUCTIONAL VIDEOS

Michael Foster
San Diego State University
Mfoster8969@sdsu.edu

Inquiry into the role of dialogic instructional videos is still in its infancy. There is a need to better understand if dialogic instructional videos are viable and how to best understand the impact they have on students. To date, the limited findings paint the use of dialogic videos in a positive light, but the results are at times inconsistent. Within this project I attempt to leverage the theoretical writings of Mikhail Bakhtin to explain these inconsistencies and to shed a new explanatory light on why dialogic instructional videos can be effective.

Keywords: Technology, Online and Distance Education, Learning Theory

Introduction

Instructional videos have continued to grow in popularity, and for good reason. Videos allow students to engage with content at their own pace and repeatedly (Lin & Michko, 2010; Vidergor & Ben-Amram, 2020), they broaden who has access to information (Parslow, 2012), and videos present an alternative to traditional instruction (e.g., flipped classrooms, Fyfield et al., 2019). The more options the better, but some forms of instructional videos are better than others – particularly for conceptually deep subject material like mathematics.

The most common and widely used form of instructional videos within mathematics education are those that follow the model set forth by Khan Academy (Bowers, Passentino, & Conners, 2012). This form of instructional video contains a "talking hand", or a voice dictating with a virtual pen, describing a procedure that is presented on a virtual whiteboard. Under the goal of providing open access to content, the "talking hand" model of instructional videos has been extremely successful with millions of worldwide users (Kelly & Rutherford, 2017; Noer, 2012).

The main issue with this form of video – particularly for mathematical content – is its emphasis on procedures (Bowers, Passentino, & Connors, 2012). With videos typically lasting less than 10 minutes, it is difficult for deep conceptual meaning to emerge for the presented mathematical content (Danielson & Goldenberg, 2012). Within that time viewers can be reminded of, or learn new, procedures (e.g., how to find a derivative), but establishing conceptual understanding takes time (e.g., that a derivative is a function for instantaneous rate of change). Instead of challenging the status quo through a new medium of technology, the "talking hand" style of instructional videos serves as reinforcement for the conception of mathematics as a set of facts to be memorized and enacted given the proper context (Bowers, Passentino, & Connors, 2012).

Another issue for "talking hand" videos, and other forms of online instructional videos (e.g., asynchronous lectures), is the lack of student voices (Lobato, Walker, & Walters, 2017). These models of video emphasize the exposition of content by a knowledgeable teacher, as opposed to the inclusion of inexperienced student voices. Student voices are important because they can serve as voices that resonate with the thoughts and difficulties of the viewer – voices of fellow learners (Chi, Kang, & Yaghmourian, 2017). As online instructional videos and distance learning, more generally, grows in popularity there are less opportunities for students to engage with and experience fellow students' voices (McKendree et al., 1998). Instructional videos could fill that void, but exposition focused videos do not.

An alternative to the dominant approaches to instructional videos gaining traction in math education are dialogic videos (e.g., https://calcvids.org/, Kolikant & Broza, 2011, Lobato, Walker, & Walters, 2017). Foremost, dialogic videos contain people engaged in a dialogue. This means that the videos contain conversations and attempts at creating new meaning or new ways of experiencing (Alrø & Skovsmose, 2004). Fundamentally, this form of instructional videos can emphasize students' voices. There are typically two people within these videos – one student and a teacher (e.g., Muldner, Lam, & Chi, 2014), or two students (e.g., Lobato, Walker, & Walters, 2017). Importantly, the voices of the students within these videos are engaged in a process of meaning making, and the meaning making process may produce deeper conceptual understanding for the viewer than simply viewing a set of procedures.

To date, there is a limited body of empirical research into the experiences of the viewers of dialogic videos. The central focus within this literature is on the learning outcomes of dialogic videos in comparison to those of monologic videos (e.g., Chi, Roy, & Hausmann, 2008; Craig, Driscoll, & Gholson, 2004; Muldner, Lam, & Chi, 2014; Muller, Sharma, & Reimann, 2008). Taking a quantitative approach, these studies have reported mixed findings (Gholson). The trend in the research suggests that dialogic videos lead to significant differences in learning outcomes, but the differences in findings across studies is not easily explained.

Within this theoretical paper I forward a lens into these contradictory findings based on the writings of Bakhtin that I believe can begin to explain the supposed contradictions and can be applied to future inquiry into students' experiences with dialogic instruction videos. This project is guided by the following research question: Does a Bakhtinian framework illuminate reasons for the inconsistency in learning outcomes of dialogic versus monologic quantitative studies and can Bakhtinian framework be applied to a wider range of literature on dialogic instruction videos.

Theoretical Framework

The following section presents a brief introduction to the philosophical writings of Bakhtin. This will include a definition of dialogue that will frame the Bakhtinian concept of voice and the subsequent introduction to the constructs of internally persuasive and authoritative voices.

Bakhtinian Dialogue

For Bakhtin, dialogue is an omnipresent interaction between an individual and the other (Morson, 2004; Mustova, 2007). Bakhtin (1981) defines the other as everyone external to an individual. While interacting with the other, words enter the individual's mind and begin to interact with the web of meaning emanating from that individual's experience with the word. As Bakhtin (1981) famously wrote:

> …language, for individual consciousness, lies on the border between oneself and the other. The word in language is half someone else's. It becomes 'one's own' only when the speaker populates it with his own intentions, his own accent, when he appropriates the word, adapting it into his own semantic and expressive intention. Prior to this moment of appropriation, the word does not exist in a neutral and impersonal language (it is not, after all, out of a dictionary that the speaker gets his words!), but rather it exists in other people's mouths, in other people's concrete contexts, serving other people's intentions: it is from there that one must take the word, and make it one's own. (p. 293)

As an individual engages with the use of a word they are simultaneously engaging with the history of that word and the intended meaning behind the other's use of the word. Through dialogue, an individual can begin to appropriate the word through a personal interpretation and negotiation of historical and local meanings. In the colloquial sense, a conversation with another person can be seen as a dialogue, but a Bakhtinian dialogue is a subjective internal process that does not need another person's physical presence. Dialogue only requires words and meanings.

Bakhtin, working in the early 20th century, was a literary scholar (Bakhtin, 1981). Critiquing genres that are unchanging and containing fixed features (e.g., every epic contains a hero), Bakhtin appreciated themes and characters that grew and changed as the story, and the genre as a whole, progressed. For Bakhtin (1981), the epitome of literature was the genre of the novel because the novel contained evolving dialogue between characters, between author and character, and between author and reader. Clearly, in the case of a novel, author and reader are not directly interacting, but meaning is still being negotiated. Even though there is only one voice, that of the author, in a novel there can still be a dialogue. Bakhtin's primary example of this comes from the writings of Dostoevsky. Dostoevsky is famous for his use of polyphony (i.e., multiple voices) within his writings. Through this style of writing, Bakhtin suggests, multiple perspectives can emerge, and dialogue can take place between and within characters. This dialogic experience can then be experienced by the reader.

One affordance of a Bakhtinian definition of dialogue, is an expansion of what can be considered a dialogue. A dialogue can occur between two people, between a book and a reader, or between an individual and their thoughts. Importantly, Bakhtin's dialogue emphasizes the subjective nature of constructing meaning from previous experiences, and what matters most is how the voices are experienced.

Voices

A voice is an utterance or an action and the associated meaning (Kolikant & Pollack, 2015; Silserth, 2012). For Bakhtin, voices are either experienced as internally persuasive svoice (IPV) or as authoritative voices (AV). An internally persuasive voice is voice that is experienced as open for, and enters into a process of, negotiation (Bakhtin, 1981). Authoritative voices, on the other hand, are the voices that are not negotiated. This can happen because the voice is experienced as not open for negotiation or because an attempted negotiation fails.

For Bakhtin (1986), the process of negotiation with a voice is where deep knowledge and understanding is constructed. Therefore, fostering IPV should be a goal of educators (Morson, 2004; Mustova, 2007), but IPV are subjective experiences (Bakhtin, 1981). One individual listening to the same voice as another will experience what is being said or done differently. Included in this difference in experience is a difference in negotiability of what is being said. For example, an experienced rock climber may hear the beta (jargon for suggested method of climbing a route) for a difficult climb and understand that this advice is a suggestion that is not one size fits all. Climbers of different sizes have different strengths; thus, this beta may not be suitable for a taller climber and the beta needs to be adjusted accordingly – the beta needs to be negotiated with. On the opposite side, a newer climber may hear the beta and fixate on trying to reproduce the moves exactly as it appears in the beta. The experienced climber is experiencing the voice of the beta as an IPV because they are open to and able to negotiate with it. The newer climber, on the other hand, is then experiencing the beta as an AV.

AV are antithetical to dialogue because meaning is not being negotiated. If an individual is experiencing the other's voice as authoritative, then they are not actively engaged in the dialogue. Instead, the individual takes on a more passive role in the dialogue as an information receptacle. Internally persuasive voices, on the other hand, trigger the process of appropriating the other's words and negotiating its meaning to fit one's own purposes. Bakhtin (1986) states that "Active agreement/disagreement (if not dogmatically predetermined) stimulates and deepens understanding (p. 142)". Agreement/disagreement that is predetermined are not open for negotiation and fall into the category of AV. When an active process is involved, on the other hand, the agreement/disagreement is produced by a negotiation process and would, thus, fall into the category of IPV. Together, this means that learning is best fostered by voices that are IPV. AV can produce learning, but when learning is "dogmatic" it is not as deep as the learning produced by an active negotiation process.

Considering this definition of dialogue, and the two kinds of voices, I forward a lens into the literature on student's use of dialogic videos that focuses on experiences of IPV. When greater learning is fostered, I will argue that more voices are being experienced as internally persuasive.

Methods

To find the papers reviewed within this theoretical paper a systemic review of the field was conducted. This included the use of the database ERIC-EBSCO. Allowable dates included in this search accounted for anything up to the date of the search (October 2020). Additionally, this search was limited to those with electronic access in the above database. Finally, the search terms were limited to "vicarious learning", "video", and "education" contained anywhere in the title or body of the text. This search resulted in 12 articles. Further limiting the scope of this search, the article abstracts were read and articles not focused on student learning were omitted. This process of omission was used to weed out articles focused on nursing education. Given the infancy of the field's inquiry into vicarious learning, the remaining 5 articles were read and an iterative bibliographic exploration was used to find an addition 9 articles. These 14 articles form the basis of this theoretical analysis.

Results

In the following section the limited literature on dialogic instructional videos is reviewed and then reinterpreted from a Bakhtinian perspective. This literature can be broken into two sets of findings: learning outcomes and learning practices.

Learning Outcomes

A number of research projects have sought to establish the effectiveness of learning from dialogic videos via quantitative comparative studies. Within these empirical studies, a commonality was the comparison of a treatment that used recordings (e.g., videos, audio tapes, etc.) that contains a single speaker (i.e., monologic) to resources that contained multiple speakers (i.e., dialogic). The trend within these findings suggests that dialogic videos are more favorable for learning outcomes than monologic, but there are several conflicting results.

Findings. The majority of studies found supporting evidence for the claim that dialogic videos are better than monologic (e.g., Chi, Kang, and Yaghmourian, 2017; Chi, Roy, & Hausmann, 2008; Cox, et al., 1999; Craig, Chi, & Vanlehn, 2009; Driscoll et al., 2003; Gholson & Craig, 2006; Muldner, Lam, & Chi, 2014; Muller et al., 2007; Muller, Sharma, & Reimann, 2008). For example, in one set of studies Chi and colleagues position their dialogic videos as tutoring sessions with one speaker being the tutor and all other speakers being tutees. They then compared the learning gains from viewers of the dialogic videos to viewers of a single tutor presenting the same set of information (Chi, Kang, & Yaghmourian, 2017; Chi, Roy, & Hausmann, 2008; Craig, Chi, & Vanlehn, 2009; Muldner, Lam, & Chi, 2014). These studies have found statistically significant differences in favor of the pre/post-test learning gains of the treatments who engaged with resources that contained multiple speakers, suggesting that two speakers are better than one

Some projects have found conflicting results, with either no statistical difference between viewers of monologic and dialogic videos or a difference in favor of the learning gains of viewers who engaged with monologic videos (e.g., Cooper et al., 2018; Monaghan & Stenning, 1998; Muller, Bewes, et al., 2008). For example, Cooper et al., (2018) found a statistically significant difference favoring the students who viewed monologic videos of their professor. The literature, thus, suggests that two speakers are better than one, but two speakers are not a sufficient condition for improving learning outcomes of the viewers.

Bakhtinian Lens. For Bakhtin, more learning is indicative of the presence of more IPV. The findings that support the use of dialogic videos thus suggest that dialogues foster the experiences of IPV. Multiple speakers (e.g., dialogic videos) may foster IPV if the multiple speakers are engaged in a negotiation process, and if the indirect participation with voices that are already being negotiated makes it easier for IPV to develop. The presence of negotiation within dialogic videos can establish as a norm that the voices present within the videos are open for negotiation. This could be particularly important for students who are from traditional lecture classrooms that typically contain voices that are not open for negotiation (Morson, 2004).

At times, when multiple speakers were not a sufficient condition for fostering IPV, it is possible that other features were. For example, Cooper et al (2018) found learning outcomes were improved for viewers of monologic videos, but the students viewed monologic videos that contained their professor. Compared to Chi and colleagues, whose monologue videos contain an unfamiliar tutor and learning outcomes favored dialogue viewers, it’s possible that the familiarity with the voice of the professor in Cooper and colleague’s study influenced the learning outcomes of their quantitative findings. In other words, because the students were familiar with their professor, and perhaps because of norms established in their class, the voice of the teacher was positioned as open for negotiation.

In another study, Muller, Bewes, et al. (2008) found that the presence of alternative conceptions and resolutions was more important for learning outcomes than whether the viewers engaged with monologic or dialogic videos. This suggests that the number of voices was less important than the content of the voices. Particularly, the presence of alternative conceptions appears to foster IPV. If multiple conceptions and productive struggles are presented, then the viewers would disadvantage themselves if they simply accepted information without negotiation. In the case of multiple conceptions, accepting without negotiation would lead the viewer to take on conceptions that are not accurate. When that information is eventually corrected, within the video, the viewer may be left more confused. This confusion is not evidenced by the learning gains of students who watch and are exposed to alternative conceptions (e.g., Chi, Kang, and Yaghmourian, 2017; Chi, Roy, & Hausmann, 2008; Cox, et al., 1999; Craig, Chi, & Vanlehn, 2009; Driscoll et al., 2003; Gholson & Craig, 2006; Muldner, Lam, & Chi, 2014; Muller et al., 2007; Muller, Sharma, & Reimann, 2008). Further, I believe that the presence of multiple conceptions and struggle can again produce a norm of negotiating with what is being presented. With conceptions present that are understood by the viewer to be merely possibilities, then that conception is ripe for negotiation. The viewer can determine for themselves whether or not they hold that same conception through negotiation. Additionally, when a final conception is realized within the video, and other conceptions are dispelled, the reasoning that led to the concluding conception can serve as a model for viewer’s future negotiation processes.

Practices

A small body of work has applied an analytical lens to the actions of observers by probing the effect that dialogic videos have on the behaviors of the viewers. The findings from this literature suggests that the individuals contained within the videos serve as a model for the viewers. The findings are limited, but the behavior modeled ranges from content specific practices like the construction grammar trees to broader practices like posing deep questions (Bandura, 1965; Braaksma, Rijlaarsdam, Van den Bergh, & van Hout-Wolters, 2004; Craig, Gholson, Ventura, & Graesser, 2000; Chi, Kang, & Yaghmouian, 2017; Kuhn & Modrek, 2021; Rummel & Spada 2005; Schunk Hanson & Cox, 1989). This suggests that viewers not only learn the content of dialogic videos, but that they can also learn behaviors that can benefit them as learners (e.g., productive problem-solving techniques).

From a Bakhtinian lens, these findings support the claim that dialogic videos can foster IPV. With voices defined as utterances or actions and their associated meaning, the reported modeled behaviors demonstrate the role that actions have on the viewers. Specifically, if the dialogue of the dialogic videos contains a negotiation process, then any modeled negotiation would be evidence that the viewer is experience the videos as containing IPVs.

Discussion

From a Bakhtinian perspective, dialogue is a subject experience. Whether a video contains a single expert explaining the process of Riemann approximation, or two students describing the difference between right-hand and left-hand approximations, both may be a dialogue, but one may be internally persuasive while the other is authoritative. What matters is the content of the voices and the subjective experience of those voices. What will foster more learning is when voices within the video are experienced as IPVs. The empirical literature comparing different feature of instructional videos, namely the comparison between treatments using monologic or dialogic instructional videos, are then useful in determining what can foster negotiation of voices.

While the analysis presented here is preliminary, the theoretical framework forwarded offers a starting point for future inquiry into student's use of dialogic videos. With a lack of empirical work assessing the mechanisms by which viewers of instructional videos learn, future work can extend the use of Bakhtin's voices to analyze and assess possible negotiations occurring between the viewer and the viewed.

References

Alrø, H., & Skovsmose, O. (2004). Dialogue and learning in mathematics education: Intention, reflection, critique (Vol. 29). Springer Science & Business Media.

Bakhtin, M. M. (1981). The dialogic imagination: Four essays by M.M. Bakhtin (C. Emerson & M. Holquist, Trans.). Austin, TX: University of Texas Press.

Bakhtin, M. M. (1986). The Bildungsroman and its Significance in the History of Realism. Speech genres and other late essays, 10, 21.

Bowers, J., Passentino, G., & Connors, C. (2012). What is the complement to a procedural video?. Journal of Computers in Mathematics and Science Teaching, 31(3), 213-248.

Braaksma, M. A., Rijlaarsdam, G., & Van den Bergh, H. (2002). Observational learning and the effects of model-observer similarity. Journal of Educational Psychology, 94(2), 405–415.

Chi, M. T., Kang, S., & Yaghmourian, D. L. (2017). Why students learn more from dialogue-than monologue-videos: Analyses of peer interactions. Journal of the Learning Sciences, 26(1), 10-50.

Chi, M. T., Roy, M., & Hausmann, R. G. (2008). Observing tutorial dialogues collaboratively: Insights about human tutoring effectiveness from vicarious learning. Cognitive science, 32(2), 301-341.

Craig, S. D., Chi, M. T., & VanLehn, K. (2009). Improving classroom learning by collaboratively observing human tutoring videos while problem solving. Journal of Educational Psychology, 101(4), 779.

Craig, S., Gholson, B., Ventura, M., & Graesser, A. (2000). the Tutoring Research Group: Overhearing dialogues and monologues in virtual tutoring sessions. Intl Journal of Artificial Intelligence in Education, 11, 242-253.

Danielson, C., & Goldenberg, M. P. (2012). How well does khan academy teach. The Washington Post.

Fyfield, M., Henderson, M., Heinrich, E., & Redmond, P. (2019). Videos in higher education: Making the most of a good thing. Australasian Journal of Educational Technology, 35(5), 1-7.

Gholson, B., & Craig, S. D. (2006). Promoting constructive activities that support vicarious learning during computer-based instruction. Educational Psychology Review, 18(2), 119-139.

Kelly, D. P., & Rutherford, T. (2017). Khan Academy as supplemental instruction: A controlled study of a computer-based mathematics intervention. The International Review of Research in Open and Distributed Learning, 18(4).

Khan, S. (2012). The one world schoolhouse: Education reimagined. Twelve.

Lin, G., & Michko, G. (2010, May). Beyond YouTube: Repurposing online video for education. In Global Learn (pp. 257-267). Association for the Advancement of Computing in Education (AACE).

Lobato, J., Walker, C., & Walters, C. D. (2017). Designing and Investigating Dialogue-Intensive Online Math Videos.

Matusov, E. (2007). Applying Bakhtin scholarship on discourse in education: A critical review essay. Educational Theory, 57(2), 215-237.

McKendree, J., Stenning, K., Mayes, T., Lee, J., & Cox, R. (1998). Why observing a dialogue may benefit learning. Journal of Computer Assisted Learning, 14(2), 110-119.

Morson, G. S. (2004). The process of ideological becoming. Bakhtinian Perspectives on Language, Literacy, and Learning, 317-331.

Muldner, K., Lam, R., & Chi, M. T. (2014). Comparing learning from observing and from human tutoring. Journal of Educational Psychology, 106(1), 69.

Muller, D. A., Bewes, J., Sharma, M. D., & Reimann, P. (2008). Saying the wrong thing: Improving learning with multimedia by including misconceptions. Journal of Computer Assisted Learning, 24(2), 144-155.

Muller, D. A., Sharma, M. D., Eklund, J., & Reimann, P. (2007). Conceptual change through vicarious learning in an authentic physics setting. Instructional Science, 35(6), 519-533.

Muller, D. A., Sharma, M. D., & Reimann, P. (2008). Raising cognitive load with linear multimedia to promote conceptual change. Science Education, 92(2), 278-296.

Noer, M. (2012). One man, one computer, 10 million students: How Khan Academy is reinventing education. Forbes (www. forbes. com/sites/michaelnoer/2012/11/02/one-man-one-computer-10-million-students-how-khan-academy-is-reinventing-education).

Parslow, G. R. (2012). Commentary: The Khan academy and the day-night flipped classroom. Biochemistry and Molecular Biology Education, 40(5), 337-338.

Vidergor, H. E., & Ben-Amram, P. (2020). Khan academy effectiveness: The case of math secondary students' perceptions. Computers & Education, 157, 103985.

THINKING OUTSIDE THE BOX: PREPARING ELEMENTARY TEACHERS INTEGRATE COMPUTATIONAL THINKING AND MATHEMATICS

Terrie M. Galanti
University of North Florida
terrie.galanti@unf.edu

In this report I investigate how elementary teachers in an online graduate-level computational thinking course make sense of mathematical relationships using block-based programming. Teachers were provided with a Scratch program that structures the dynamic relationship between changing height and changing volume in the context of rich mathematics task. Analysis of teachers' mathematical representations of this task revealed how teachers can construct conceptual understandings by modifying parameters within a block-based program. Implications for preparing teachers to leverage the computational thinking concepts of abstraction, dynamic modeling, and automation in teaching mathematics are discussed.

Keywords: Computational Thinking, Technology, Computing and Coding, Professional Development

With the increasing emphasis on computational thinking as a critical skill in K-12 teaching and learning (Committee on STEM Education, 2018), the mathematics education community has an exciting opportunity to broaden its conceptualization of mathematics as a school subject. Computer science, particularly at the elementary level, is often framed as a discipline that is distinct from other subjects. However, there is growing research attention on the natural overlap between computational thinking and mathematics in K-12 education with respect to structure, abstraction, and modeling in problem solving (Israel & Lash, 2019; Silva et al., 2020; Strickland et al., 2021; Weintrop et al., 2016).

Mathematics continues to be taught in many K-12 schools through a siloed approach instead of as an authentic integration of reasoning and sense-making (Li et al., 2019). Yet computational thinking, particularly in relation to programming, has the unique potential to change the way that students think about the rigor and relevance of mathematics. Li and colleagues (2020) recently described computational thinking as "searching for ways of processing information that are always incrementally improvable in their efficiency, correctness, and elegance" (p. 4). Computational thinking and mathematics are naturally and historically connected (Gadanidis et al., 2017) because of their shared emphasis on modeling and exploring quantitative relationships.

The power of computer programming as a sense-making tool in the context of K-12 mathematics activities is only beginning to be explored in elementary teacher preparation and professional development programs (e.g., Gadanidis et al., 2017; Gleasman & Kim, 2020). Preparing teachers to leverage computational thinking opportunities through mathematical representations can enrich students' exploration of patterns, structure, and sequence. When teachers provide opportunities for students to explore mathematical relationships by changing parameters within code, they can foster conceptual understandings of mathematics. There is a need to understand how teachers reason with code to design professional development experiences that build teachers' abilities to make this overlap explicit (Kotsopoulos et al., 2019).

This study focuses on how elementary teachers made sense of one rich mathematics task using a computational thinking perspective. The following research question guided this investigation of teachers' reasoning about varying measurement quantities: How do elementary teachers enrolled in a graduate STEM education course use and describe computational thinking in their mathematical exploration of the volume of an open box?

Theoretical Background

Seymour Papert (1980) first introduced computational thinking as a constructionist approach to learning mathematics with tangible making experiences using the LOGO programming language. However, it was Wing's (2006) essay on computational thinking as a fundamental skill along with reading, writing, and arithmetic that inspired the growing call to bring more computational thinking opportunities to all children. While there is no scholarly consensus on the definition of computational thinking (Cansu & Cansu, 2019), the International Society for Technology in Education (ISTE) in conjunction with the Computer Science Teachers Association (CSTA) offer a robust definition of computational thinking in relation to mathematics. It is a problem-solving process with specific characteristics related to problem formulation; organization and analysis of data; use of abstractions such as models and simulations; efficiency of algorithmic solutions; and generalizability of solutions (ISTE & CSTA, 2011). Within the field of mathematics teacher education, we need research on developing this implicit synergy between computational thinking and mathematics, especially within professional development experiences that elementary teachers need to realize its advantages (Bull et al., 2020; English, 2018).

It is important to emphasize that programming is both a process and product of computational thinking. As such, programming within a mathematical learning context can help students solve problems and use data effectively (Weintrop et al., 2016; Wolfram, 2020). Gadanidis and colleagues (2017) explained the possibilities of "what might be" with computational thinking as they used the Scratch programming language to integrate advanced grade level geometry and probability content in a Grade 1 mathematics classroom (p. 78). They proposed seven pedagogical affordances associated with integrating computational thinking and mathematics teaching and learning. These affordances are as follows: 1*) low floor, high ceiling;* 2) *abstraction and automation;* 3) *dynamic modeling;* 4) *tangible feel;* 5) *conceptual surprise;* 6) *wide walls; and* 7) *agency*. These affordances also constitute an analytic framework that I used examine the potential of computational thinking to position teachers as both producers and consumers of mathematical knowledge (Gadanidis et al., 2017, p. 93).

Methods

As the instructor in a synchronous online computational thinking course for elementary teachers who were enrolled in a STEM+C graduate certificate program, I encouraged teachers to explore the natural relationship between computational thinking and mathematics learning in a series of mathematical tasks. Before they engaged in problem solving, the teachers analyzed computational thinking concepts and practices (Grover & Pea, 2018) and the Standards for Mathematical Practice (NGA & CCSSO, 2010) for conceptual overlap. Second, the teachers experienced three collaborative problem-solving opportunities with high cognitive demand mathematical tasks. As they built their solutions, they were encouraged but not required to explore dynamic modeling opportunities using starter code in the Scratch programming language. These open exploration tasks were centered on number sense and geometry.

Their mathematical goal in the "Thinking Outside the Box" task was to construct a box with the largest possible volume (Dodge & Viktora, 2002) by cutting congruent squares from each corner of an 8.5 x 11-inch rectangular paper. The task was aligned with Grade 5 geometric measurement standards and prompted the use of patterns to develop a solution. Teachers were provided with hyperlinks to a Google spreadsheet for recording box dimensions and volume calculations and Scratch starter code to automate the calculations of a user-specified sequence of heights (see Figure 1). The Scratch starter code created a list of 6 boxes with evenly spaced integer heights ranging from 0 to 5 and associated volumes. Teachers were able to modify the range of heights and the differences between the heights by adjusting the parameters in the code. Teachers could choose to represent their mathematical thinking as outputs of the Scratch starter code. They also reflected how they experiences the computational thinking concepts of pattern seeking, abstraction, and automation from their perspectives as learners in this task.

The unit of analysis for this study was teacher collaborative sense-making as evidenced in Google Slides artifacts (for five groups of 3-4 teachers) and instructor notes from breakout room conversations. I used the Gadanidis et al. (2017) pedagogical affordances as apriori codes to analyze how teachers interacted with the Scratch to think deeply about mathematics.

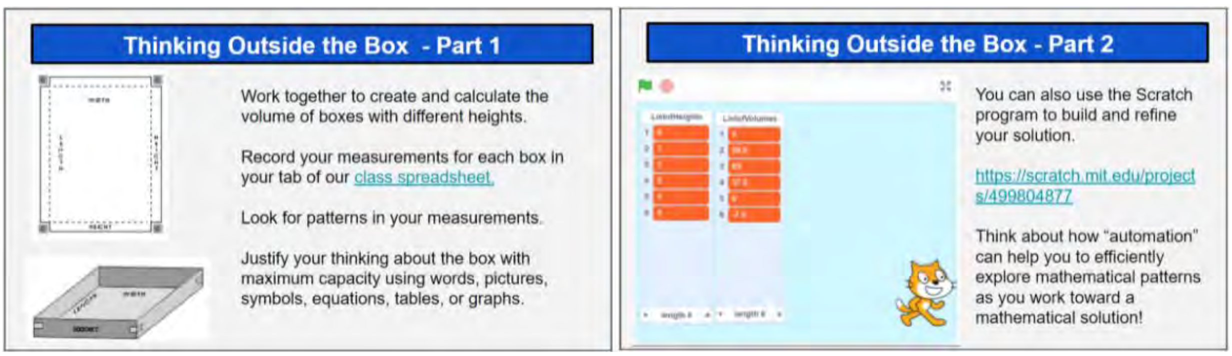


Figure 1: Task Directions – Finding the Maximum Volume of an Open-Top Box

Findings

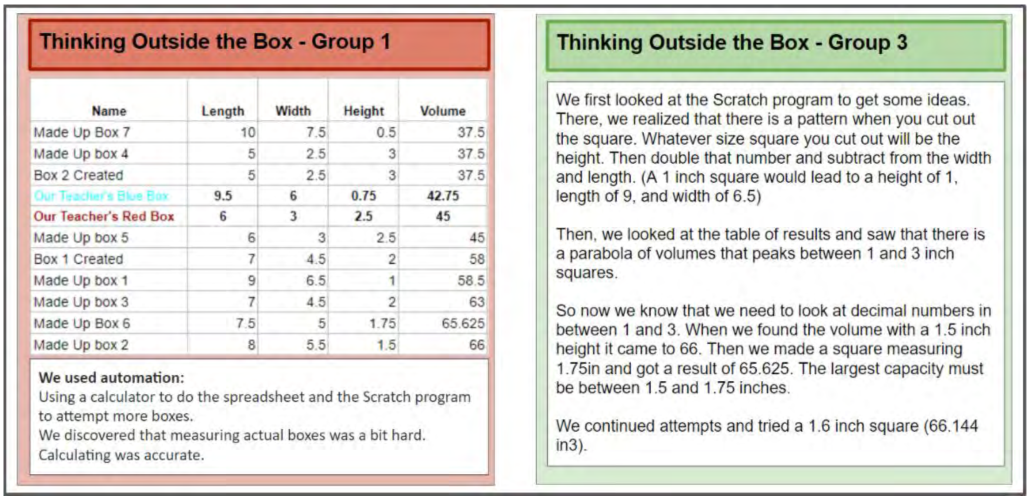
Three of the five groups chose to use of the Scratch program to examine structure and to generalize with differing outcomes. In breakout room conversations, two groups expressed *conceptual surprise* as they noticed the negative value for volume with a height of 5 inches in the Scratch program output. I encouraged these groups to think about the length and width dimensions that they were not seeing on the list of values and to tinker with the program further to explain their noticings. They further expressed *conceptual surprise* as they discovered they might not have been able to find one correct answer based on the limited precision of the code.

As Group 5 observed the *dynamic modeling* within Scratch, they generalized that as the height of the box increased, the volume decreased. However, they did not comment on the increasing and decreasing patterns with the initial height values. They "noticed that all of the heights were whole numbers and decided to experiment with decimals". The affordance of *abstraction/automation* enabled them to adjust the height parameters to increase by 0.25 inch with each iteration. They concluded that a 1.5-inch height would yield the greatest volume.

Group 1 (see Figure 2, left) used the Scratch program to extend the sense making that they had recorded in Google Sheets using box measurements and formula calculations. Because they did not order their spreadsheet rows, they did not offer evidence that they were examining patterns to see how volume was changing. Instead, they seemed to be looking for the largest volume by trial and error. Their comment that using the Scratch improved the accuracy of their calculations suggests that they needed additional support to see how the *dynamic modeling* of the program could deepen opportunities for mathematical reasoning *automating* the formula.

Group 3 (see Figure 2, right) used the Scratch-generated list structure to look for patterns in changing heights and generalized that volume was increasing and then decreasing between heights of 1 and 3 inches. They also leveraged *abstraction/automation* to change the parameters with increasingly narrow intervals of 0.5 and 0.1 inches as they concluded that the largest volume would be 1.6 inches. Their mathematical explanation offered contextual evidence of tinkering with code as a pedagogical experience in programming (Kotsopoulos, 2017).

Teachers' use of Scratch varied from a *low floor* of automating the calculation of a variety of individual volumes (Group 1) to a *high ceiling* of modifying code parameters and examining the resulting patterns (Group 3). These artifacts of teacher sense-making show how teachers can use starter code can build the reciprocal relationship between computational thinking and mathematical exploration (Weintrop et al., 2016). The Scratch program was used as a *dynamic model* of the covarying relationship between height and volume. The list structure was used as an *automated abstraction* of the volume function increased the opportunities for rich mathematical conversations about covariation, pattern seeking, and reasonable mathematical answers.



Thinking Outside the Box - Group 1

Name	Length	Width	Height	Volume
Made Up Box 7	10	7.5	0.5	37.5
Made Up box 4	5	2.5	3	37.5
Box 2 Created	5	2.5	3	37.5
Our Teacher's Blue Box	9.5	6	0.75	42.75
Our Teacher's Red Box	6	3	2.5	45
Made Up box 5	6	3	2.5	45
Box 1 Created	7	4.5	2	58
Made Up box 1	9	6.5	1	58.5
Made Up box 3	7	4.5	2	63
Made Up Box 6	7.5	5	1.75	65.625
Made Up box 2	8	5.5	1.5	66

We used automation:
Using a calculator to do the spreadsheet and the Scratch program to attempt more boxes.
We discovered that measuring actual boxes was a bit hard.
Calculating was accurate.

Thinking Outside the Box - Group 3

We first looked at the Scratch program to get some ideas. There, we realized that there is a pattern when you cut out the square. Whatever size square you cut out will be the height. Then double that number and subtract from the width and length. (A 1 inch square would lead to a height of 1, length of 9, and width of 6.5)

Then, we looked at the table of results and saw that there is a parabola of volumes that peaks between 1 and 3 inch squares.

So now we know that we need to look at decimal numbers in between 1 and 3. When we found the volume with a 1.5 inch height it came to 66. Then we made a square measuring 1.75in and got a result of 65.625. The largest capacity must be between 1.5 and 1.75 inches.

We continued attempts and tried a 1.6 inch square (66.144 in3).

Figure 2: Excerpts from Teacher Work - Using Scratch for Mathematical Sense-Making

Discussion and Implications

Analysis of the mathematical representations in this "Thinking Outside the Box" task reveal the potential for mathematics teacher educators to facilitate a productive interplay between computational thinking and mathematical sensemaking. Only two of five groups in this study modified parameters for the purpose of pattern seeking and generalization. This finding raises important questions about how professional development facilitators can scaffold deep mathematical reasoning through code. Facilitating this sense making is crucial in building elementary teacher confidence that mathematical learning can happen through computational exploration. This study shows that teachers may need further support to see code as a tool for reasoning about mathematical relationships beyond efficient calculation of answers.

There is a need for more research to conceptualize how programming "about and for the sake of" mathematics can support learning (Goldenberg et al., 2021, p. 49). Opportunities to program in Scratch in elementary schools are often framed as expressive activities outside of curriculum. Yet this study shows that carefully designed starter code can empower teachers to leverage abstraction, automation, and dynamic modeling as students investigate mathematical relationships. This knowledge can position teachers to provide all students with tools enjoy the computational nature of mathematics as a discipline.

References

Bull, G., Garofalo, J., & Hguyen, N. R. (2020). Thinking about computational thinking: Origins of computational thinking in educational computing. *Journal of Digital Learning in Teacher Education, 36*(1), 6-18. https://doi.org/10.1080/21532974.2019.1694381

Cansu, S. K., & Cansu, F. K. (2019). An overview of computational thinking. *International Journal of Computer Science Education in Schools,* 3(1), 1-11. https://doi.org/10.21585/ijcses.v3i1.53

Committee on STEM Education. (2018). *Charting a course for success: America's Strategy for STEM education.* National Science and Technology Council. https://eric.ed.gov/?id=ED590474

Dodge, W., & Viktora, S. (2002). Thinking out of the box…problem. *The Mathematics Teacher, 95*(8), 568-574. https://www.jstor.org/stable/i20871154

English, L. D. (2016). On MTL's second milestone: Exploring computational thinking and mathematics learning. *Mathematical Thinking and Learning, 20*(1), 1–2. https://doi.org/10.1080/10986065.2018.1405615

Gadanidis, G., Hughes, J. M., Minniti, L., & White, B. J. (2017). Computational thinking, grade 1 students and the binomial theorem. *Digital Experiences in Mathematics Education, 3*(2), 77-96. https://doi.org/10.1007/s40751-016-0019-3

Gleasman, C., & Kim, C. (2020). Pre-service teacher's use of block-based programming and computational thinking to teach elementary mathematics. *Digital Experiences in Mathematics Education, 6*(1), 52-90.https://doi.org/10.1007/s40751-019-00056-1

Goldenberg, E. P., Carter, C. J., Mark, J., Reed, K., Spencer, D., & Coleman, K. (2021). Programming as language and manipulative for second-grade mathematics. *Digital Experiences in Mathematics Education, 7*(1), 48-65. https://doi.org/10.1007/s40751-020-00083-3

International Society for Technology in Education and Computer Science Teachers Association (2011). Operational definition of computational thinking for K–12 education. http://www.iste.org/docs/ct-documents/computational-thinking-operational-definition-flyer.pdf

Israel, M., & Lash, T. (2020) From classroom lessons to exploratory learning progressions: mathematics + computational thinking. Interactive Learning Environments, 28(3), 362 – 382. https://doi.org/10.1080/10494820.2019.1674879

Kotsopoulos, D., Floyd, L., Khan, S., Namukasa, I. K., Somanath, S., Weber, J., & Yiu, C. (2017). A pedagogical framework for computational thinking. *Digital Experiences in Mathematics Education, 3*(2), 154-171. https://doi.org/ 10.1007/s40751-017-0031-2

Li, Y., & Schoenfeld, A. H. (2019). Problematizing teaching and learning mathematics as "given" in STEM education. *International Journal of STEM Education, 6*, 44. https://doi.org/10.1186/s40594-019-0197-9.

Li, Y., Schoenfeld, A. H., diSessa, A. A., Graesser, A. C., Benson, L. C., English, L. D., & Duschl, R. A. (2020). Computational thinking is more about thinking than computing. *Journal for STEM Education Research, 3*(1), 1-18. https://doi.org/ 10.1007/s41979-020-00030-2

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). Common core state standards for mathematics. Authors.

Papert, S. (1980). *Mindstorms, children, computers, and powerful ideas. Basic* Books.

Silva, R. S. R. D., Gadanidis, G., Hughes, J., & Namukasa, I. K. (2020). Computational thinking as a heuristic endeavour: students' solutions of coding problems. *Pro-Posições,* 31. https://doi.org/10.1590/1980-6248-2018-0034

Strickland, C., Rich, K. M., Eatinger, D., Lash, T., Isaacs, A., Israel, M., & Franklin, D. (2021, March). Action fractions: The design and pilot of an integrated math + CS elementary curriculum based on learning trajectories. In *Proceedings of the 52nd ACM Technical Symposium on Computer Science Education* (pp. 1149-1155).

Weintrop, D., Beheshti, E., Horn, M., Orton, K., Jona, K., Trouille, L., & Wilensky, U. (2016). Defining computational thinking for mathematics and science classrooms. *Journal of Science Education and Technology, 25(*1), 127-147. https://doi.org/10.1007/s10956-015-9581-5

Wolfram, C. (2020). *The maths fix: an education blueprint for the AI age.* Wolfram Media, Inc.

HOW TRANSITIONS BETWEEN RELATED ARTIFACTS SUPPORT STUDENTS' COVARIATIONAL REASONING

Erell Germia	Toni York	Nicole Panorkou
Montclair State University	Montclair State University	Montclair State University
germiae1@montclair.edu	yorka1@montclair.edu	panorkoun@montclair.edu

Many studies use instructional designs that include two or more artifacts (digital manipulatives, tables, graphs) to support students' development of reasoning about covarying quantities. While students' forms of covariational reasoning and the designs are often the focus of these studies, the way students' interactions and transitions between artifacts shape their actions and thinking is often neglected. By examining the transitions that students make between artifacts as they construct and reorganize their reasoning, our study aimed to justify claims made by various studies about the nature of the synergy of artifacts. In this paper, we present data from a design experiment with a pair of sixth-grade students to discuss how their transitions between artifacts provided a constructive space for them to reason about covarying quantities in graphs.

Keywords: Instructional Activities and Practices; Mathematical Representations; Design Experiments.

Over the past three decades, mathematics educators have characterized students' reasoning about the simultaneous change of two quantities in various ways (e.g., Confrey & Smith, 1994; 1995; Saldanha & Thompson, 1998). Some proposed frameworks to describe students' progression of covariational reasoning over time (Carlson et al., 2002; Thompson & Carlson, 2017) and a handful of studies used these to examine students' reasoning of various mathematical ideas, such as the rate of change (e.g., Johnson, 2012), and scientific phenomena, such as gravity (e.g., Panorkou & Germia, 2021). To examine *what* forms of covariational reasoning students exhibit, these studies often engage students with various artifacts such as dynamic manipulatives, tables, or graphs (e.g., Castillo-Garsow, 2012; Ellis, 2011).

Multiple studies point to the role of particular artifacts or combinations of artifacts used during the learning process to explain *why* this development of students' reasoning happens or does not happen (e.g., Ellis et al., 2018). The role of technology has been to advance students' reasoning because it can be utilized to illustrate change in progress (Castillo-Garsow, 2012) as well as the ability to reverse change. For example, Johnson et al. (2016) used a Ferris wheel animation on Geometer's Sketchpad (GSP) and asked the students to click and drag the car to control the motion and graph the relationship between the car's height from the ground and its distance traveled within one revolution. Examining relationships between quantities using dynamic manipulatives before graphing has shown to advance students' conceptions of graphs of functions as a representation of coordinated change (e.g., Ellis et al., 2018). While some asked students to transition straight from the digital tool to the graph (e.g., Ellis et al., 2018; Stevens & Moore, 2016), others asked students to create a table as an intermediate artifact before transitioning to the graph (e.g., Ellis et al., 2015).

Although each study uses different technologies (GSP, GeoGebra, Desmos), often their goal is for students to connect these dynamic representations of relationships with the graphing of those relationships. One of the prevailing difficulties that students have with graphing is that they focus on the shape of the graph and ignore the relationship between the two covarying quantities that it illustrates. To explain this, Moore and Thompson (2015) distinguished between *static* or *emergent* shape thinking. Students who think statically operate on a graph as an object, such as a piece of wire. In contrast, students with emergent shape thinking illustrate an understanding of a graph as a trace (or a snapshot of a trace) in progress that depicts the relationship between two covarying quantities. To examine students' reasoning across multiple representations, we expanded the ideas of static and emergent thinking in this study to also include applying them to reasoning with simulations and tables as well as graphs. To reason emergently in this way, a student must construct a more abstracted structure of covarying quantities that is operative in each of the representational contexts, rather than being inherently tied to any one of them.

While research studying *what* forms of reasoning students develop during the transition between artifacts and *why* this might happen prevails, an examination of *how* this process happens is rarely the focus of the investigation. In this paper we argue that the *how* lens can show us specific methods and conditions in which students' actions and reasoning are influenced during the interactions with the artifacts. This can make a contribution for informing both the design of artifacts, tasks, and questioning around these ideas and also the analysis of students' reasoning during those transitions. Considering the above, this study aims to examine *how* students make connections between different representations of the same covarying quantities as they transition between artifacts, and to explore the conditions that made those transitions productive for their learning. Specifically, we aim to examine the following research question: *How may students' transitions between artifacts provide a constructive space for them to reason emergently with a graph?*

Theoretical Framework: Examining the Transitions Between Artifacts

In this paper we use the term *artifact* to refer to a tool made by humans that is used for performing a specific task (Verillon & Rabardel, 1995; Trouche, 2004). As the student interacts with an artifact, the artifact imposes on the user some affordances and constraints, which shape both the actions and the ideas emerging from the activity (Artigue, 2002; Noss & Hoyles 1996). Through their interaction with an artifact, students build an *instrument* (Verillon & Rabardel, 1995; Lagrange et al., 2001) which is a mental construction that consists in part of the artifact and in part of cognitive schemes. *Instrumental genesis,* or the process by which an artifact becomes an instrument, is therefore influenced by both the artifact's characteristics and the user's prior knowledge and experience (Rabardel, 2000).

The new instruments formed from the instrumental genesis do not develop in isolation but instead become part of a system that consists of instruments that students developed earlier (Rabardel & Bourmaud, 2003). Mariotti and Montone (2020) discussed the nature of a *synergy* between two artifacts as "an implicit or explicit reference to both artifacts [that] creates a relationship between meanings emerging from their use" (p. 113). To support students in relating these different meanings emerging from the different uses of artifacts, Soury-Lavergne (2021) proposed three characteristics of design that the group of artifacts needs to have. Specifically, she talked about making the use of each artifact necessary, what she refers to as *complementarity* between the two artifacts. To make their relation visible, she argued that there must be some form of *redundancy* of some characteristics of one artifact in the other. To her "redundancy in the system of instruments produces robustness and adaptability of the system" (p. 6). Finally, she stated that each artifact should have different constraints that would lead to students to adapt, by challenging and reorganizing their initial system of instruments. She referred to this characteristic as an *antagonism* between artifacts.

We may view each artifact as becoming a transitional instrument for the continuous (re)organization of a system of instruments. We refer to *reorganizations* (Piaget, 2001) of reasoning as the inferences we make about students' projections and reflections of particular forms of reasoning and their connections as these are (re)structured into a more coherent whole. While designs may imply a sequential instructional process from one artifact to the next, our experience shows that students' transitions do not follow a linear process but rather shift back and forth in unrestrained ways between different artifacts. Consequently, in this paper we use the term *transitions* to refer to the dynamic, continuous, and "messy" shifts (physical and cognitive) that the individual makes between artifacts as they (re)structure their system of instruments.

Methods

The findings we report here are part of a design experiment (Cobb et al., 2003) in which we iteratively developed and tested theories about both the process of learning and the nature of the synergy of artifacts that supported that learning. In this section we discuss the design and initial conjectures explored in this study as well as our methods of data collection and analysis.

Design and Initial Conjectures

Since our previous work (e.g., Panorkou & Germia, 2021) showed that students as young as sixth grade can engage in sophisticated reasoning about covarying quantities when these are presented in meaningful contexts such as scientific phenomena, we focused on the scientific phenomenon of climate in this study. We thus designed a set of artifacts based on this phenomenon, specifically involving the exploration of the covarying quantities of temperature and latitude in the earth's climate.

The first of these artifacts is the Climatic Zones simulation (Figure 1), with which students can explore how temperature and latitude covary in the earth's polar, temperate, and tropical zones. The temperature in these zones is largely determined by the distance away from the earth's equator, or latitude. To explore the simulation, students control the location of the arrow on the right side of the screen by moving their mouse into the different zones. The readouts above and to the left allow the student to observe resulting changes in the quantities. Similar to other studies (e.g., Ellis et al., 2018), our conjecture was that students' interactions with this simulation would support them in constructing their understanding of graphs as records of covariation, thereby engaging in emergent shape thinking (Moore & Thompson, 2015).

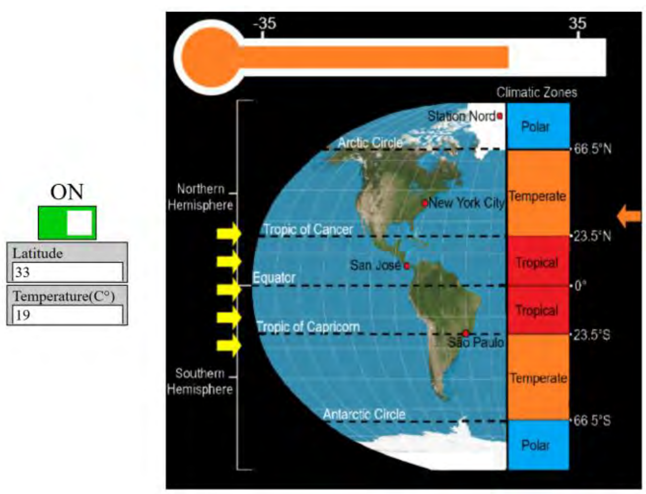


Figure 1: The Climatic Zones Simulation

We also chose to include both a table and a graph in order to explore both students' transitions between these different representations as well as how they would reorganize their systems of instruments to include each new artifact. Therefore, the next artifact we designed is a table which students were asked to use the simulation to complete a list of temperatures for given latitudes. We conjectured that this could support their emergent shape thinking since the table represents different values of the two quantities as ordered pairs. Students were then asked to complete the corresponding graph. This third artifact was designed without any scales, requiring students to create their own intervals on each axis before they could plot the values. We conjectured that this would allow us to examine how students would transition between the three artifacts to organize the data in the table as well as how they might reorganize their systems of instruments as they did so.

Finally, although this design may seem to be a strictly sequential learning activity, we set no restrictions on when or how often students could move back and forth between the artifacts as they worked. We thus expected that the students' actions and reasoning would illustrate the "messiness" of their transitions back and forth between the artifacts.

Data Collection and Analysis

The data presented in this paper was collected during a whole-class design experiment (Cobb et al., 2003) that took place in a sixth-grade classroom in the Northeast of the U.S. We conducted two 25-minute virtual sessions via Google Classroom due to COVID-19 restrictions. The students were paired off into breakout rooms where they engaged in the tasks and were interviewed by the researchers. The students' video and shared screens were recorded during the sessions. These recordings were then transcribed for the analysis.

This paper focuses on the retrospective analysis of the activity of one pair, Jami and Gaelyn. We analyzed the data in three stages. In the first stage, we identified episodes of the students' covariational reasoning and static or emergent shape thinking (Moore & Thompson, 2015). In the second stage, we reviewed each students' data chronologically in order to observe the progressions and reorganizations of their reasoning as they transitioned between the artifacts. In the third stage, we analyzed the features of the artifacts that seemed to support those progressions and reorganizations. We used the Soury-Lavergne (2021) framework of complementarities, redundancies, and antagonisms to characterize how this group of artifacts served as instruments in a system that provided a productive space for the students to reorganize their reasoning.

Findings

In this section we present examples of complementarity, redundancy, and antagonism in the order they emerged from the data.

Complementarity - Callback

We asked the students to explore the simulation and identify the quantities that change. Jami reasoned about the latitude, saying that "as you go up [using the cursor] the latitude is positive and then as you go down, the latitude is negative." She also reasoned about how the temperature changed in each climatic zone, explaining,

> Jami: The polar zone is actually very cold because it is up north [latitude], and the temperate zone since it is right between the equator where it is most hottest and the polar zone, it is pretty warm, like the average temperature, and the tropical zones are very hot because they are more close to the equator.

Her reasoning illustrated emergent shape thinking about how the closer from the equator a zone is, the hotter the temperature.

Next, we asked Jami and Gaelyn to use the simulation to complete the table (Figure 2, left). Jami shared her screen showing the simulation on the left side of the monitor and the table on the right side. The students took turns identifying the temperature using the simulation. For example, Jami stated “[negative] sixty-seven is in the latitude, the air temperature is negative 19.” Then, Gaelyn added “negative 23 is 23 degrees.” Both students transitioned between the table and simulation as they looked for the values of temperature that correspond to the given latitude. This reasoning exhibited the complementarity between these two artifacts as they referenced back to the information from the simulation to complete the table.

Latitude	Air Temperature
-67	- 19
- 47	12
-23	23
-10	24
0	35
10	31
23	26
47	8
67	- 9

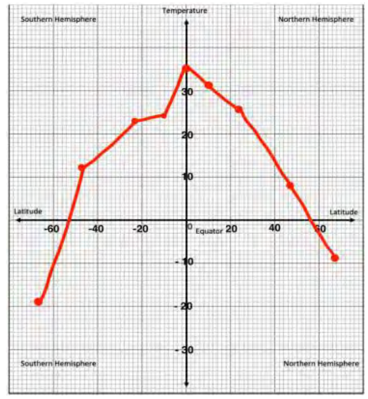


Figure 2: Gaelyn’s Completed Table and Graph

Then we asked the students to use the values in the table to create a graph showing the relationship between latitude and temperature (Figure 2, right). Gaelyn transitioned back and forth between the table and the graph as she plotted the values, reasoning,

> Gaelyn: The first thing was negative 67 to negative 19 [toggling back and forth between the table and graph]. … So that would be about here to [plotting in the first quadrant]. No wait, this is negative, so that would be about here [plotting in the third quadrant] to 19 right above the 20.

Gaelyn’s reorganization of plotting (-67, -19) from the first quadrant to the third quadrant also showed her understanding of graphing negative values. Next, Jami helped Gaelyn by guiding her with the succeeding values from the table as Gaelyn continued plotting them on the graph. Their actions of toggling between the two artifacts showed the complementarity of the table and the graph, as the former provided the information needed to create the latter.

Students’ calling back to the previous artifacts was a necessary action to collect relevant information and create the subsequent artifacts. The students needed to look for the data from the simulation to complete the table and then use the values from table to create the graph.

Antagonism

The students’ reasoning about the simulation, table, and graph also showed an antagonism in how the information in the three artifacts was encoded differently by each. Their reading of data points from the simulation and the table illustrated a straightforward encoding of information about the corresponding values of latitude and temperature. However, the graph required them to coordinate the two quantities and locate them along the x- and y-axes in order to represent the same information. As the students were plotting the points on the graph, Gaelyn moved her cursor along the y-axis and stated, “this is zero [latitude]” (Figure 3, middle). Jami also reasoned about the y-axis by stating that the “equator is right in the middle. It is the origin on the graph” (Figure 3, right).

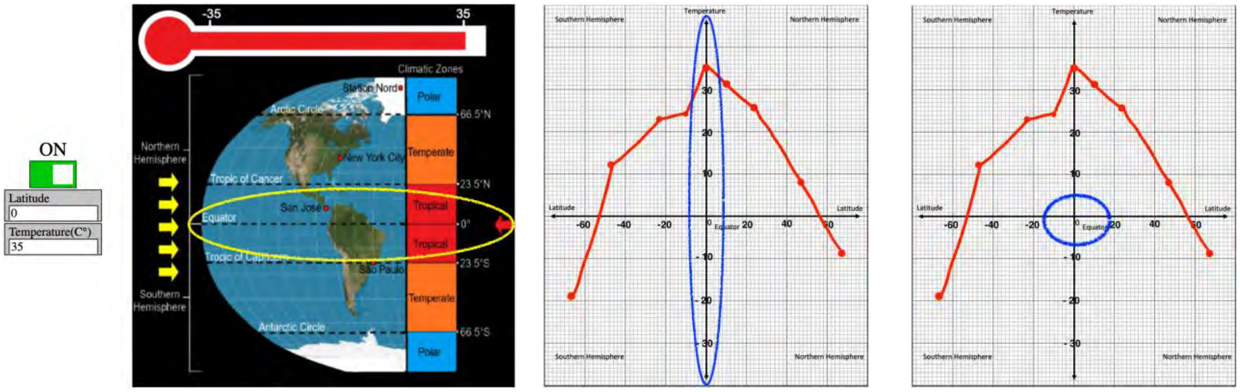


Figure 3: Relation of the Latitude of the Equator in the Simulation (left) to the Y-Axis (Gaelyn, middle) and the Origin (Jami, right) in the Graph

The students' reasoning about the location of the equator being at 0 degrees latitude and in the middle of the graph illustrated their understanding of the y-axis and origin of the graph. This also shows how the students needed to make meaning of the values on the graph, whereas they only needed to read the values from the simulation and the table.

Complementarity - Revision

Next, we asked the students to use the graph to describe the relationship between the latitude and temperature in the northern hemisphere. Gaelyn stated, "the higher the latitude, the lower the temperature," describing the relationship between the two quantities from the equator to the right side of the graph. This statement showed that she was able to identify how the latitude and the temperature in the northern hemisphere covaried in the graph. However, when we asked the students to use the graph to explain the same relationship in the southern hemisphere, Gaelyn wondered, "Would it be the opposite? So, like the lower the latitude, wait, no, never mind." Since Gaelyn was not sure about her response, Jami used the simulation to reason about the values of latitude. She stated that "if you go to the southern, the lower the latitude, because it's a negative, so when you go more negative numbers, on the number line, it gets more like lower, the numbers [moving the arrow downwards]" as shown in Figure 4. Jami probably imagined a "number line" across the different climatic regions in the simulation similar to what can be found on the graph to describe the changes in the values of the latitude. She referred to the values of the latitude as "negative numbers" to reason about "the lower the latitude" as she moved the arrow down in the simulation from temperate to polar climatic zones. In that way, she connected the change in the latitude in the simulation with the x-axis in the graph.

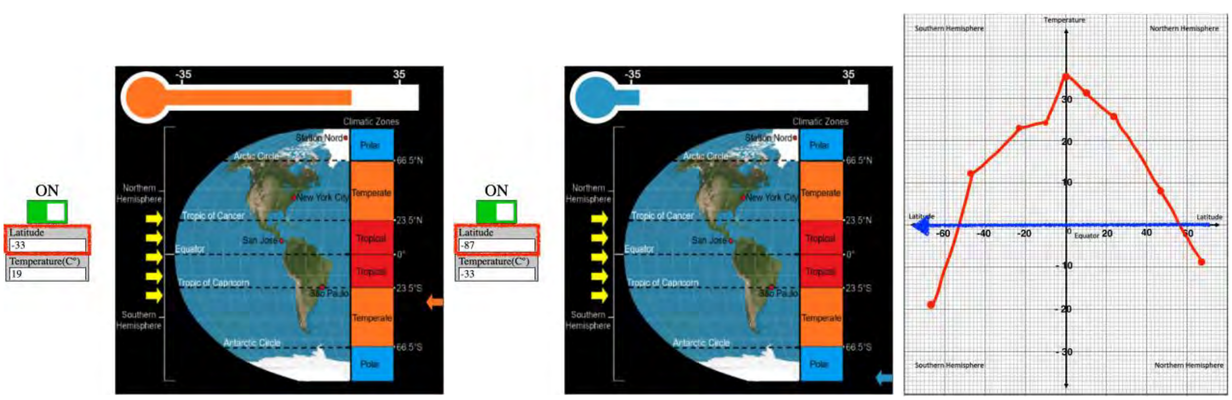


Figure 4: Jami Using the Simulation to Explain Latitude in the Graph

Associating the change in values with the direction on a number line did not seem to make sense for the students because the latitude was decreasing as Jami moved the arrow downwards in the simulation. Instead of reasoning about the negative values of latitude in the south, Gaelyn revised their discussion of latitude in terms of the distance from the equator:

> Gaelyn: … it says the same from the equator, as the distance from the equator increases, the air temperature decreases. Because we can't really say, because they're both going down [referring to the values of the latitude and temperature at the southern hemisphere], so we just have to do the distance.

Jami agreed and restated the relationship between the quantities in the southern hemisphere: "as the distance from the equator increases, the air temperature decreases."

The students' use of the two artifacts to discuss the relationship between latitude and temperature in the two hemispheres led them to revise their understanding of the latitude quantity from a positive or negative number into a measure of the distance from the equator. Since the latitudes on the right side of the graph were positive, it may have been easier for the students to express that covariational relationship than when working with the negative latitudes on the left side of the graph. The simulation allowed them to dynamically explore both regions of this relationship and became a necessary support for their reasoning about the relationship in the graph. Additionally, the students' reasoning exhibited a form of emergent shape thinking in both the graph and simulation. This shows a complementarity between the simulation and the graph as the students moved back and forth between these artifacts in order to revise their reasoning.

Redundancy

The students' activity with the artifacts also showed evidence of the redundancy of the information represented by each. For example, when we asked students to use their graph to describe the location of the highest temperature, Gaelyn reasoned that "the highest temperature, it would be right here [putting the cursor on the y-intercept of the graph] at the equator" (Figure 5). Gaelyn located the highest temperature at the y-intercept on the graph and then related the same point with the equator in the simulation where the highest temperature was also found. Jami added, "because the line is labeled temperature [making a vertical motion with her index finger]." This kind of reasoning showed the redundancy of information offered by the graph and simulation by connecting the information about the y-axis on the graph with both the location of the equator (Gaelyn) and the temperature data (Jami) from the simulation.

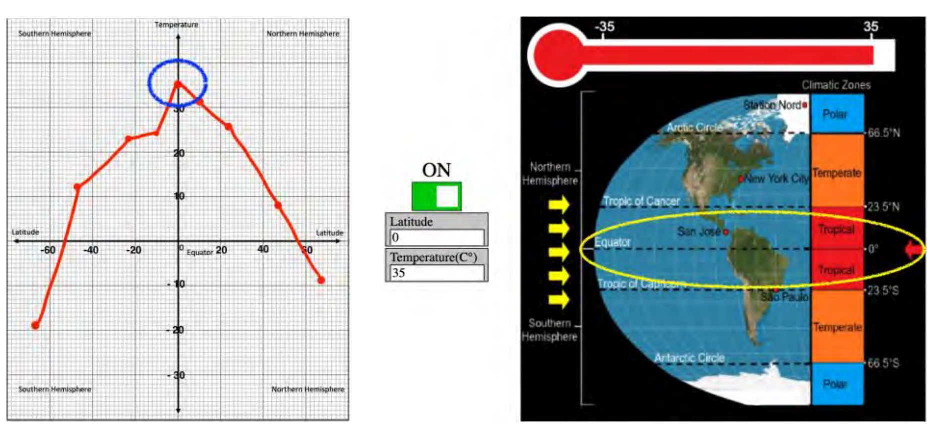


Figure 5: Gaelyn's Relation of the Highest Temperature in the Graph and the Simulation

The students' transitioning between the graph and simulation offered them opportunities to explore the similarities in the features of the two artifacts. Finding such information gave them opportunities to reason about how the quantities and their relationship are represented in both the simulation and the graph.

Conclusions

By examining *how* students transition between artifacts, our work informs the design of artifacts, tasks, and questioning that support them in making connections between different representations as they (re)organize their *systems* of instruments (Rabardel & Bourmaud, 2003). Jami and Gaelyn's transitions between the three related artifacts we designed influenced how they reasoned about the quantities and covariational relationships represented by those artifacts. Specifically, their transitions between the artifacts offered a constructive space for them to reason emergently with the graph instead of thinking about its static shape as described in the literature. Furthermore, they reasoned emergently across all of the artifacts, showing that they constructed a structure of covarying quantities not inherently tied to a particular representation.

Our findings highlight the specific forms of complementarities, antagonisms, and redundancies (Soury-Lavergne, 2021) that supported the synergy (Mariotti & Montone, 2020) between these related artifacts. The need for the students to use each artifact to create the next prompted them to reference back to previous artifacts as they continued to work. These callbacks show one kind of complementarity between the artifacts that supported the students' reasoning. The students' use of one artifact to revise their reasoning about another shows another kind of complementarity. Both of these complementarities worked hand-in-hand with the redundancy of information found between the artifacts, as the students explored and reasoned about the same features, quantities, and relationships across the different artifacts. At the same time, the antagonism between the different ways each artifact encoded this information offered a constructive space for the students to reason emergently (Moore & Thompson, 2015) across these different contexts and especially to make covariational meaning of their graphs.

Our analysis contributes to an expansion of Soury-Lavergne's (2021) work to include two possible subcategories of complementarity: callbacks to previous artifacts to support the creation of subsequent artifacts and revisions of reasoning based on the exploration of other related artifacts. More studies are needed to further investigate these identified subcategories and to look for other possible forms of complementarities, redundancies, and antagonisms in different sets of data. There is also a need to examine the similarities and differences in the progressions of reasoning among other students. For instance, it would be interesting to examine the progressions of other pairs who did not immediately reason emergently as they transitioned to the table and graph, and how the synergy between the artifacts may have influenced such progress.

Acknowledgments

This research was supported by a grant from the National Science Foundation (#1742125). The data presented, statements made, and views expressed are solely the responsibility of the authors. We would like to thank Jay Singh, Debasmita Basu, Sowmith Etikyala, and Anthony Cuviello for their help in the design of the simulation and activities.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. International Journal of Computers for Mathematical

Learning, 7(3), 245-274.
Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 5(33), 352–378.
Castillo-Garsow, C. (2012). Continuous quantitative reasoning. Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context, 2, 55-73.
Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. Educational Researcher, 32(1), 9-13.
Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. Educational Studies in Mathematics, 26(2–3), 135-164.
Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. Journal for Research in Mathematics Education, 26(1), 66-86.
Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In Cai, J. & Knuth, E. J. (Eds.), Early algebraization: A global dialogue from multiple perspectives (pp. 215-238). Springer.
Ellis, A. B., Özgür, Z., Kulow, T., Williams, C. C., & Amidon, J. (2015). Quantifying exponential growth: Three conceptual shifts in coordinating multiplicative and additive growth. The Journal of Mathematical Behavior, 39, 135-155.
Ellis, A., Tasova, H. I., & Singleton, B. (2018). How quantitative reasoning can support graph understanding in algebra. In Hodges, T. E., Roy, G. J., & Tyminski, A. M. (Eds.), Proceedings of the 40th Annual Meeting of PME-NA (pp. 195-198). University of South Carolina/Clemson University.
Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. The Journal of Mathematical Behavior, 31(3), 313-330.
Johnson, H. L., Hornbein, P., & Azeem, S. (2016). Investigating functions with a Ferris wheel. The Mathematics Teacher, 110(5), 344-351.
Lagrange, J.-B., Artigue, M., Laborde, C., Trouche, L., 2001. A meta study on IC technologies in education. Towards a multidimensional framework to tackle their integration. In Van den Heuvel, M. & Panhuizen, M. (Eds.), Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education, vol. 2. Freudenthal Institute, The Netherlands (pp. 111–112).
Mariotti, M. A., & Montone, A. (2020). The potential synergy of digital and manipulative artefacts. Digital Experiences in Mathematics Education, 6(2), 109-122.
Moore, K.C., & Thompson, P.W. (2015). Shape thinking and students' graphing activity. In Fukawa-Connelly, T., Infante, N., Keene, K., & Zandieh, M. (Eds.), Proceedings of the Eighteenth Annual Conference on RUME (pp. 782-789).
Noss, R., & Hoyles, C. (1996). Windows on mathematical meanings: Learning cultures and computers (Vol. 17). Springer Science & Business Media.
Panorkou, N., & Germia, E. F. (2021). Integrating math and science content through covariational reasoning: the case of gravity. Mathematical Thinking and Learning, 23(4), 318-343.
Piaget, J. (2001). Studies in reflecting abstraction. (R. L. Campbell, Ed. &, Trans.). Taylor and Francis. (Original work published 1977).
Rabardel, P., 2000. Ele´ments pour une approche instrumentale en didactique des mathématiques. In Bailleul, M., (Ed.), Les instruments dans la pratique et l'enseignement des mathématiques, Actes de l'école d'été de didactique des mathématiques, IUFM de Caen, France.
Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: a developmental perspective. Interacting with Computers, 15(5), 665-691.
Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen, K. R. Dawkins, M. Blanton, W. N. Coulombe, J. Kolb, K. Norwood, & L. Stiff (Eds.), Proceedings of the 20th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 298-303). Raleigh: North Carolina State University.
Stevens, I., & Moore, K. C. (2016). The Ferris wheel and justifications of curvature. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 644–651). Tucson, AZ: The University of Arizona.
Soury-Lavergne, S. (2021). Duos of Digital and Tangible Artefacts in Didactical Situations. Digital Experiences in Mathematics Education, 7(1), 1-21.
Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking

mathematically. In Cai, J. (Ed.), Compendium for Research in Mathematics Education (pp. 421-456). National Council of Teachers of Mathematics.

Trouche, L. (2004). Managing the complexity of human/machine interactions in computerized learning environments: Guiding students' command process through instrumental orchestrations. International Journal of Computers for Mathematical Learning, 9(3), 281-307.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. European Journal of Psychology of Education, 10(1), 77-101.

TECHNOLOGIES USED TO SUPPORT PROBLEMS OF PRACTICE DURING EMERGENCY REMOTE TEACHING

Rachel Harrington
Western Oregon University
harringtonr@wou.edu

Ann Wheeler
Texas Woman's University
awheeler2@twu.edu

Shannon O. S. Driskell
University of Dayton
sdriskell1@udayton.edu

Steve Rhine
Pacific University
steverhine@pacificu.edu

This report focuses on problems of practices identified by 209 respondents to an electronic survey in spring 2020 and the ways MTEs called upon their TPACK to address these problems. From the data, eight problems of practice emerged: changing pedagogy to account for online contexts, rehearsals of teaching, monitoring in class student learning, student engagement, assessment, sense of community, mathematical manipulatives, and communication of mathematical concepts. Results indicate that MTEs addressed these problems, altering their teaching by adopting many new technologies and functions. These included: video conferencing applications, Desmos, virtual manipulatives, GeoGebra, Google Slides, Google Docs, Flipgrid, videos/webinars, and discussion boards. Implications for MTEs who teach or are interested in teaching online are discussed.

Keywords: Technology, Online and Distance Education, Teacher Educators

In March 2020, mathematics teacher educators (MTEs) were tasked with a rapid transition to emergency remote teaching (ERT) in response to the worldwide COVID-19 pandemic. Drawing from previous experience and accessible online resources, MTEs quickly redesigned courses to be delivered in remote synchronous and asynchronous formats. Using a survey research approach, this study examined MTEs' perceptions of transitioning their teaching to ERT, and how it impacted the quality of instruction they delivered. In the analysis of the data, we asked, what problems of practice emerged from this experience and what types of technologies were used to address them? Our work aims to describe the ways MTEs transitioned to ERT and to identify technologies that can be carried forward into future instruction of all course delivery formats (in-person, remote synchronous, remote asynchronous, hybrid).

Literature Review

Technological Pedagogical Content Knowledge (TPACK) is constructed from the interaction of individual contributing knowledge domains: Pedagogy, Content, and Technology (Angeli & Valanides, 2009). This interaction is often modeled by a Venn diagram of overlapping circles (see Figure 1) associated with each domain (Mishra & Koehler, 2006; Niess, 2005).

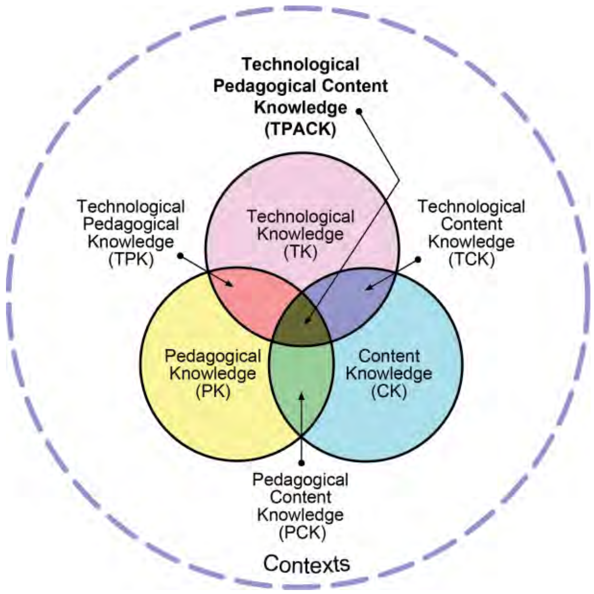


Figure 1. Mishra's (2019) updated TPACK framework.

While there are sub-sections of TPACK represented by the different shaded regions (PCK, TPK, etc.), TPACK should be thought of as a total package, more than the sum of its parts (Niess, 2005). Furthermore, Koehler and Mishra (2008) discussed how classroom contexts vary greatly, and thus there is a wide variation in the ways teachers implement their TPACK. Mishra (2019) later changed the Context in the TPACK framework to ConteXtual Knowledge (XK) to highlight that all of the knowledge domains exist in a space enclosed by the Contexts of the learning.

Porras-Hernandez and Salinas-Amescua (2013) focused on how Context influences the TPACK and represented context as concentric circles surrounding the whole, with micro (in-class learning conditions), mezzo (building or district conditions), and macro (societal conditions) factors. Much has been written about TPACK and its impact on instruction. Rosenberg and Koehler (2015) noted that only 36% of the literature on TPACK mentions Contexts, and less (14%) discusses macro-level factors. However, in Spring 2020, MTEs experienced an almost universal impact on the macro-level ConteXtual factors and this directly influenced the mezzo and micro factors. Our work leveraged this opportunity to study the ways their TPACK brought MTEs through this change.

Methods

Using the most updated understanding of TPACK (Mishra, 2019), we developed survey questions aimed at understanding how the TPACK ConteXtual change of ERT was experienced by MTEs. The Mathematics Teacher Educators' Migration to Online Teaching in Response to COVID-19 survey was for MTEs' classes that support a mathematics teacher education program for undergraduate and/or graduate students. Seven expert MTEs, an instructional technology expert, and a survey construction expert all worked to establish content validity of the survey. Two pilot surveys were given to MTEs to validate the questions. The revised survey was sent to members of the Association of Mathematics Teacher Educators in May 2020. The survey included nine demographic questions, 14 Likert scale questions, and three open-response questions.

The results of the Likert responses were compiled and all mentions of technologies were tallied and classified by type. The short answer responses were analyzed consistent with a modified version of the Qualitative Hypothesis-Generating approach (Auerbach and Silverstein, 2003). In each open response, all Relevant Text to the research question was identified. The list of Relevant Text was sorted into Repeating Ideas. From the Repeating Ideas, Themes emerged. When placed alongside the MTEs' Likert responses, the Themes told the story of MTEs instructional experiences during Spring 2020.

Results

Mentioned technologies were classified as either conveyance technologies or mathematical action technologies (Dick & Hollenbrands, 2011). C*onveyance technologies* are those used to present, communicate, share, and assess information. In contrast, *mathematical action technologies* are those like computer algebra systems and dynamic geometry software. Table 1 details the most commonly mentioned technologies in the survey results.

Table 1: Frequency and Types of Technology Used

Technology	Unique Mentions	Technology Type
Video Conferencing App	46	Conveyance
Virtual Manipulatives	14	Mathematical Action
Desmos	11	Mathematical Action
GeoGebra	9	Mathematical Action
Google Slides	7	Conveyance
Flipgrid	5	Conveyance
Google Docs	5	Conveyance
Discussion Boards	5	Conveyance
Webinars/Videos	4	Conveyance

While conveyance technologies were most mentioned overall, there was a strong presence of mathematical action technologies ($n = 34$) among the most commonly mentioned technologies. The results indicate that both types were critical to supporting MTEs as they transitioned to ERT.

In the open-ended responses, participants shared Repeating Ideas that we classified into eight problems of practice. These problems were: Changing pedagogy to account for online context, Rehearsals of teaching, Monitoring in-class student learning, Student engagement, Assessment, Sense of community, Mathematical manipulatives, and Communication of mathematical concepts. For example, one participant shared challenges related to *Monitoring in-class student learning*:

> My classes are designed around small group activities. We were able to continue these activities with Zoom breakout rooms, but my ability to monitor and interact deeply with each group was really slowed by Zoom. Unlike a physical classroom, it's impossible to catch elements of every group's conversation at once, and there is a bit of time needed to switch from breakout room to breakout room (and joining a new room can be quite disruptive to the students). (Participant #57)

To further illustrate, another participant identified challenges related to *Rehearsals of teaching*:

> The changes for my elementary methods course were significant and mostly negative. We no longer were able to complete 3 visits to work with an elementary school student and students

weren't able to practice teaching in front of their peers. Also, I had to replace classroom discussions with short written reflection papers due to my class being asynchronous, which meant that students no longer had the benefit of hearing their peers' perspectives. (Participant #104)

Through our analysis, we began to connect the problems of practice with technologies implemented to address those challenges. In these connections, we were able to see how MTEs used their TPACK to carry them through the ConteXtual changes in Spring 2020. Table 2 details these connections and can act as a resource for MTEs facing similar challenges.

Table 2: Problems of Practice Correlated with Technology Used

Problems of Practice	Mentioned Technologies
Changing pedagogy to account for online context	Flipgrid, Screencastify, Zoom videos
Rehearsals of teaching	Discussion rooms, Flipgrid
Monitoring in-class student learning	Google Docs, Jamboard, Desmos, Google Slides, Padlet, Breakout rooms
Student engagement	Breakout rooms, chat, Google Docs
Assessment	Flipgrid, Canvas, Blackboard
Sense of community	Chat, Breakout rooms
Mathematical manipulatives	Virtual manipulatives, Desmos, Zoom polls, Google Slides, Google Docs, GoodNotes
Communication of mathematical concepts	Virtual manipulatives, Desmos, Zoom polls, Google Slides, Google Docs, GoodNotes

Discussion

The rapid transition to ERT had a universal impact on the TPACK Context of MTEs. While some of this transition was negative, there is much to be learned from our collective experience. Using their TPACK, MTEs implemented technologies to solve problems in new and innovative ways. Problems of practice will persist in different forms as we move forward, and this work helps us identify the technologies that can mediate these problems. We can use these identified technologies as a resource for solutions, as inspiration for professional development, and as ways to address new problems that we are yet to encounter. The pandemic pushed us into isolation, but this work brings to light new innovations as we have gathered together again.

References

Auerbach, C. F., & Silverstein, L. B. (2003). *Qualitative data: An introduction to coding and analysis*. New York University Press.

Angeli, C., & Valanides, N. (2009). Epistemological and methodological issues for the conceptualization, development, and assessment of ICT–TPCK: Advances in technological pedagogical content knowledge (TPCK). *Computers & Education*, *52*(1), 154–168.

Dick, T., & Hollebrands, K. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making (pp. xi–xvii)*. Reston, VA: National Council of Teachers of Mathematics.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A new framework for teacher knowledge. *Teachers College Record*, 108(6), 1017–1054.

Nesbary, D. (2000). *Survey research and the world wide web*. Boston, MA: Allyn & Bacon.

Niess, M. L. (2005). Preparing teachers to teach science and mathematics with technology: Developing a technology pedagogical content knowledge. *Teaching and Teacher Education*, 21, 509–523. doi.org/10.1016/j.tate.2005.03.006

SUPPORTING COMMUNITIES OF INQUIRY IN ASYNCHRONOUS, ONLINE MATHEMATICS PROFESSIONAL DEVELOPMENT

Angela Knotts
WestEd
aknotts@wested.org

Nanette Seago
WestEd
nseago@wested.org

Jill Neumayer DePiper
WestEd
jdepipe@wested.org

Asynchronous, online mathematics teacher professional development (PD) was designed to align with research on teacher professional learning as well as to support Communities of Inquiry (e.g., Garrison et al., 2000). The intervention included two actively facilitated formats and a structured independent condition, where facilitation was integrated into the design of the intervention. Participants' responses to intervention activities were analyzed using indicators of Garrison et al.'s Community of Inquiry framework, seeking to understand the ways in which the intervention enabled participant learning across facilitation formats. Analysis has implications for building the CoI framework into subsequent online asynchronous mathematics teacher PD as a way to increase teacher learning, build community, and effectively scale interventions.

Keywords: Professional Development, Teacher Knowledge, Online and Distance Education, Technology

Significance and Purpose

High-quality professional learning is widely accepted as a core component of effective teaching and learning and meaningful school reform (Borko et al., 2014; DeMonte 2013; Heller et al. 2012; Polly et al. 2015; Yoon et al. 2007). Online teacher professional development (PD) opportunities are poised to play a key role in bringing meaningful, effective professional learning to teachers on a broad scale (Killion, 2013), with experts in the field as well as policy makers advocating for more teacher PD to be delivered online (Dede et al., 2009, 2016; US Department of Education, 2010). While the range of potential designs for online PD is vast, offerings can generally be categorized as synchronous (all participants interacting in real time) asynchronous (participants work with materials at their own pace for submission and later feedback), or hybrid (involving both synchronous and asynchronous components) (Fishman, 2016).

A general consensus has emerged within the field regarding the design features of PD most likely to impact teacher and student outcomes, both in general and in mathematics specifically, such as sustained duration, content area-specificity, coherence, active/practice-based learning, collaboration, access to models of effective practice, and expert facilitation/coaching (Elmore, 2002; Garet et al., 2001; Loucks-Horsley et al., 2010; Darling-Hammond et al., 2017); how to integrate these elements in an online setting is less clear. Teachers report that the ability to access online PD anytime is very or extremely important (Parsons et al., 2019), and asynchronous forms of online PD have resulted in positive findings related to teachers' attitude and self-efficacy (An, 2018) as well as high satisfaction and relatively high levels of information sharing (Yoon et al., 2020).

This paper focuses on the ways that careful design choices support the development of a community of inquiry in an online, asynchronous video-based PD—*Video in the Middle: Flexible Digital Experiences for Mathematics Teacher Education* (VIM). A major goal of the VIM project was to demonstrate that online, asynchronous video-based teacher PD can be designed and implemented that both embodies the features recognized by the field as characterizing effective, high-quality professional learning and also democratizes such PD by increasing teacher access to engaging, meaningful practice-based professional learning.

Theoretical Framework

To integrate attention to key learnings about mathematics teacher PD with research on asynchronous online learning, the VIM project design and development is theoretically grounded in key literature on Communities of Inquiry (CoI; Garrison et al., 2001). The CoI framework emphasizes three crucial elements of a successful educational experience: cognitive presence, social presence, and teaching presence. According to the CoI framework, worthwhile learning occurs through the interaction of these three core elements.

Cognitive presence is “the extent to which the participants in a community of inquiry are able to construct meaning through sustained communication” (Garrison et al., 2001, p. 89; Anderson et al., 2001; Garrison & Archer, 2000). Garrison has argued extensively that “asynchronous online learning can create a rich cognitive presence capable of supporting effective, higher-order learning” (2003, p. 47). He suggests that this is best accomplished through a cycle of practical inquiry where participants move through four phases: (1) a *triggering* event where a problem is identified for further inquiry; (2) *exploration*, where an individual explores the issue; (3) *integration*, where learners make meaning from ideas generated during the exploration phase; and (4) *resolution*, where participants apply new skills and knowledge.

Social presence is defined as the ability of participants to project their personal characteristics into the community, thereby presenting themselves to the other participants as ‘real people,’ facilitating both enjoyment of the experience as well as the process of critical thinking they undertake (Rourke et al., 1999; Anderson et al., 1999; Garrison et al., 2001). It supports cognitive presence (and therefore cognitive objectives) through its ability to instigate, sustain, and support critical thinking among a community, and also plays a role in participant retention by ensuring that participants find interactions enjoyable and fulfilling. Richardson and Swan (2003) found that social presence positively affects both student and instructor course satisfaction, and participants who perceived high social presence learned more than those who perceived low social presence (Richardson & Swan, 2003).

Teaching presence—the intentional design, facilitation, and direction of cognitive and social processes—has three components: (1) instructional design and organization; (2) facilitating discourse; and (3) direct instruction (Garrison et al., 2000; Anderson et al., 2001; Garrison & Arbaugh, 2007). It is theorized to support and enhance social and cognitive presence.

Video in the Middle (VIM) Project Design

Drawing on the existing face-to-face PD *Learning and Teaching Linear Functions: Videocases for Mathematics Professional Development* (Seago et al., 2004), the VIM project created 40 asynchronous online mathematics PD modules and researched their effectiveness. Each two-hour module is centered around a purposefully selected video clip from a real mathematics classroom and designed to facilitate teachers’ noticing skills (van Es & Sherin, 2002) and mathematical knowledge for teaching linear functions (Ball & Bass, 2002). VIM modules are designed to be rich, research-based, open education resources that can be used in a wide variety of settings with less up-front time and resource investment than many other high-quality offerings. The modular nature of the design means that school/district PD providers can select particular modules or series of modules they feel meet the needs of their teachers, or they can choose pre-selected “pathways” of modules sequenced by the VIM research and design teams to address specific learning goals. Modules can be implemented in either a facilitated format where a PD leader responds to participants' contributions asynchronously, or in a structured independent format where participants engage in activities and respond to fellow participants' contributions without monitoring or active facilitation from a PD leader.

VIM Module Design

Each module includes the same series of common activities across three phases. In the pre-video phase, participants are introduced to the learning goals related to mathematical content knowledge and pedagogy (Seago et al., 2018). They then solve and reflect on a math task, post their work on a Community Wall, and comment on others' work. In facilitated formats, facilitators also respond to posts. Participants also consider a variety of solution strategies for the task and review a brief "Mathematician's Commentary" on the task. In the video phase, participants review the context of a video clip where students explore the task, watch the video, and analyze it through the lens of a set of framing questions. They also re-examine the solution methods in connection to the video clip and annotate an excerpt from the video transcript. They then re-watch the video with a different framing question, annotate moments in the video, and respond to annotations from fellow participants, expert math educators, and, in facilitated formats, facilitators. In the post-video phase, participants complete a reflection in response to a prompt related to the module learning goals, post their reflection on a Community Wall, and respond to others' posts. The module concludes with teachers engaging in a "Bridge to Practice" activity that asks them either to consider how they might incorporate new strategies and ideas into their own classroom practice or respond to an artifact of practice through the lens of what they have learned.

Pre-video Activities	Video in the Middle	Post-video Activities
0. Introduction to learning goals 1. Explore math task and reflect in journal 2. Share your work on the math task 3. Consider other solutions and perspectives	4. Review the context of the lesson 5. Watch video and reflect 6. Reflect on the lesson graph and solution methods 7. Annotate video transcript 8. Watch video with math educator commentary	9. Module reflection 10. Bridge to practice

Figure 1: Overview of VIM module common learning activities

Cognitive Prescence, Social Prescence, and Teaching Presence in VIM Module Design

VIM modules were designed to embody the three core elements of a community of inquiry as described by Garrison et al. (2000, 2001). Cognitive presence was built into each VIM module in the form of multiple practical inquiry cycles—for example, around the math task activity and the central video phase (table 1). Each module comprises a larger-scale practical inquiry cycle as well.

Table 1: Alignment of elements and categories of Community of Inquiry framework (Garrison et al., 2001) with VIM module activities

Categories	*Opportunities in VIM Modules*	
Triggering Event	Introduction to math task	Introduction to video (context and frame)
Exploration	Solving & reflecting on math task; sharing & comparing solution strategies	Exploring other ideas about video (math educator, facilitator, other participants)

Integration	Responding to/discussing solution strategies	Responding to/discussing video commentary
Resolution	Reconciling alternative solution methods & mathematician commentary with own ideas & method	Bridge to Practice activity

VIM modules promote social presence through Community Walls where participants post and respond to others' math task solutions and reflections as well as activities where they analyze, comment on, and respond to others' comments about the video and transcript. Prompts included:

- "After you have posted your work, take some time to look at the solutions of others. What do you notice? What do you wonder? Add comments to the posts of your peers and/or create a new post to add a general comment on the wall. Please check back later if you are the first to post."
- "As you watch, post comments at the points during the video that highlight how Debra probes Siri to explain further and how Debra's moves support Tiffany to make her explanation clearer for the class. Feel free to pause the video when reading comments and respond to them if you'd like."

In facilitated formats, facilitators also respond to posts.

The video task was also designed to foster social presence; Sherin (2004) points out that video is similar to authentic experience in that it positively affects motivation and interest. Multiple studies have found that there exists a higher level of satisfaction when teacher education courses use video rather than narrations of experience (Choi & Johnson, 2007; Moreno & Valdez, 2007).

Table 2: Alignment of social presence categories of Community of Inquiry framework (Garrison et al., 2001) with VIM module activities

Categories	*Opportunities in VIM Modules*	
Affective, Interactive, and Cohesive Responses	Math task Community Wall interactions	Video commentary interactions

Teaching presence was built into each VIM module with a team of veteran math educators, researchers, and educational technology designers who worked on the design and organization, without the assumption that a member of the design team would facilitate the module. Available pre-sequenced "pathways" of VIMs are another example of instructional design and organization. While in online, asynchronous settings, facilitation of discourse is often carried out by an instructor or facilitator commenting on responses, raising questions to move discussions in a particular direction, keep discourse moving efficiently, and balance participation, Garrison et al. (2000) note that facilitation is a responsibility that may be shared among some or all participants in the learning experience, particularly with adult populations. To facilitate discourse, the VIM modules, and particularly the framing and reflection questions that wrap around learning activities, guide and focus participants' interaction around the module artifacts and materials. Facilitator training materials further support PD leaders in responding to posts and guiding discussion to achieve the module learning goals. In VIM modules, subject matter knowledge is injected strategically as direct instruction through pre-selected and sequenced artifacts such as the Solution Methods document, the Mathematician's Commentary on the task, pre-embedded expert commentary on the video and transcript, and professional readings used to frame Bridge to Practice activities. To increase interaction and help participants to recognize the developmental progression of the inquiry process (Garrison & Arbaugh, 2007), VIM modules include metacognitive journal and Community Wall activities that prompt participants to consider how the module activities influenced their understand and beliefs about mathematics teaching and learning (for example, "I used to think… Now I think…").

Table 3: Alignment of teaching presence categories of Community of Inquiry framework (Garrison et al., 2001) with VIM module activities

Categories	*Opportunities in VIM Modules*	
Instructional Design and Organization	Selection of module goals, task, & prompts; sequencing of task-related activities	Selection of module goals, video, & prompts; sequencing of video-related activities
Facilitating Discourse	Community Wall prompts and interactions	Video analysis prompts and interactions
Direct Instruction	Solution Methods pdf; Mathematician's Commentary slides	Math educator pre-entered prompts

Research Design

California middle and high school teachers were recruited to participate in a Spring 2020 pilot study to address the following research question: *Which elements of the CoI framework were evident in teacher and facilitator responses?* All teachers experienced the same sequence of four, two-hour modules for a total of eight hours of professional development over the course of twelve weeks (February through April 2020). Modules were offered in three formats: (1) project staff-facilitated (n = 17), (2) district leader-facilitated (n = 18), and (3) structured independent (n = 26). Teachers in each of the two district leader-facilitated cohorts were all from the same district, while the other two groups included teachers from various districts.

All three formats included a content focus, the same asynchronous opportunities for sharing solution methods and written reflections with colleagues, and attention to bringing what they learned into practice. Facilitated formats differed from the structured independent format in two ways: 1) pace (project-staff and district-facilitated formats asked teachers to complete one module every two weeks, while teachers in the independent format worked at their own pace, and 2) the role of the facilitator. As a clear and well-defined structure to each VIM module was established during the design phase reflecting research on teacher PD, attention to student thinking, and the importance of teacher reflection, the role of the facilitator in both the project staff- and district-facilitated formats focused on commenting on teachers' posts, encouraging teachers to complete modules, and responding to journal reflections. Project and district facilitators participated in a 90-minute video-conference orientation with project staff, including an overview of the study and timeline, VIM module structure, and online tools. Facilitators also had access to a web-based facilitator guide and video tutorial demonstrating how to use the online tools to respond to participants.

Participants

Participating teachers taught middle school math, Algebra 1, or a first-year high school math course such as Math 1 or Integrated 1.

Teachers in the district leader-facilitated condition were recruited by mathematics leaders from each of two school districts; each of the two leaders then served as the facilitator for their district group. Additional teachers were recruited from across California and randomized into either the structured independent condition or the project staff-facilitated condition. Where multiple teachers were recruited from the same district, teachers were randomly split between the two conditions. Where single teachers were recruited from a site, singleton teachers were matched by similar site location or demographics; matched pairs were then randomized into the two conditions. Of the 68 teachers who began the study, 82% completed all or nearly all study activities across the four modules.

Data Collection

In both facilitated and independent VIM cohorts, participants had eight opportunities (two opportunities per module) to engage with others' posts via Community Walls. Rourke et al. (2001) identify three categories of social presence indicators: **affective** responses (expressions of emotion, feelings, and mood), **interactive** responses (threaded interchanges of socially appreciative messages), and **cohesive** responses (serving to build and sustain a sense of group commitment). Posts and replies from all Community Walls in all cohorts were analyzed for evidence of social presence. Each post and reply showing evidence of social presence was categorized as an affective response, an interactive response, or a cohesive response. Posts and replies were categorized as *affective responses* if they included emoticons, exclamation points, or other text intended primarily to communicate mood or emotion (Kuehn, 1993; Gunawardena & Zittle, 1997). Replies were categorized as *interactive responses* if they expressed social appreciation for the original post or a previous reply. Posts and replies were categorized as *cohesive responses* if they included phatics (communication which serves to establish or maintain social relationships rather than to impart information), salutations, vocatives (addressing participants by name), or addressing the group as "we," "our," or "us." Percentages of posts showing evidence of each category of social presence were calculated for each of the three facilitation formats.

All participants also had four opportunities to engage with others' posts as well as pre-entered "Mathematics Educator" posts in the video analysis activity, an example of teaching presence through direct instruction. Responses to these posts were reviewed for evidence that they had impacted participants' thinking about the video clip or prompted them to engage in deeper reflection around moments or interactions referenced by the Mathematics Educator posts.

Findings and Discussion

Evidence of Social Presence in VIM Modules

Analysis of participants' Community Wall responses across both facilitated and independent cohorts revealed examples of affective, interactive, and cohesive responses. Analysis of Community Wall posts found that numerous posts in all three participation formats either contained or inspired replies/rejoinders that contained evidence of affect (14.8% locally facilitated, 37% project-facilitated, 10.5% structured independent):

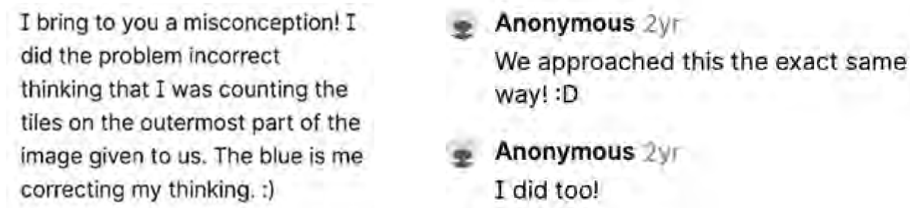


Figure 2: Affective responses in facilitated (left) and independent (right) VIM cohorts

Interactive responses were the most common type of response across all three cohorts (43.7% locally facilitated, 65.2% project-facilitated, 33.7% structured independent), with participants routinely commenting in socially appreciative ways on others' math work or reflections in response to what they found interesting or raised questions for them.

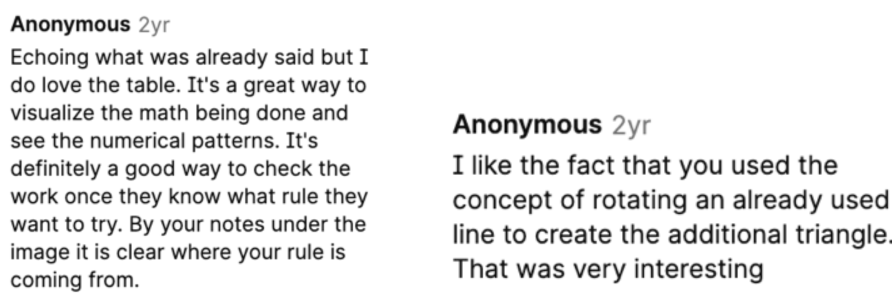


Figure 3: Interactive responses in facilitated (left) and independent (right) VIM cohorts

Cohesive responses were the least common type of response across all cohorts, with only a few examples in each (5.2% locally facilitated, 3.7% project-facilitated, 4.4% structured independent). We hypothesize that this was due in part to technological limitations that defaulted to anonymous posts and replies.

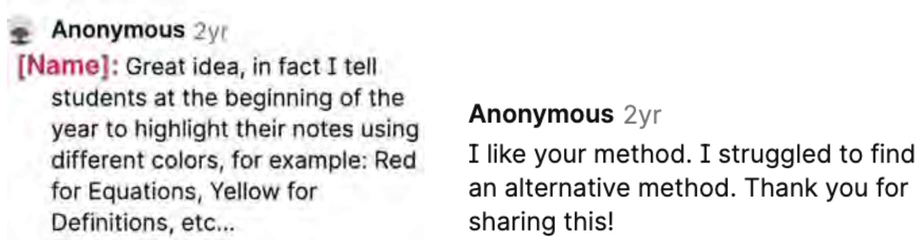


Figure 4: Phatics and vocatives as indicators of cohesive responses in facilitated (left) and independent (right) cohorts

Evidence of Teaching Presence in VIM Modules

Of the three core elements, it is perhaps teaching presence which is the least obvious in terms of how effectively it is accomplished in VIM modules given that they are designed to be used with or without active facilitation; even when used with a facilitator, interaction is significantly lighter touch as compared to the role a facilitator might play in other PD experiences. We have already seen the manner in which the intentional design and organization of VIM modules contributes to both cognitive and social presence through the selection of tasks and video, module goals, framing and reflection questions, and Community Wall prompts.

We can see evidence of the impact of direct instruction when participants are asked to comment on moments in the video and respond to other comments, including comments pre-inserted by the design team to draw teachers' attention to key moments in the video and encourage discussion around them (figure 5).

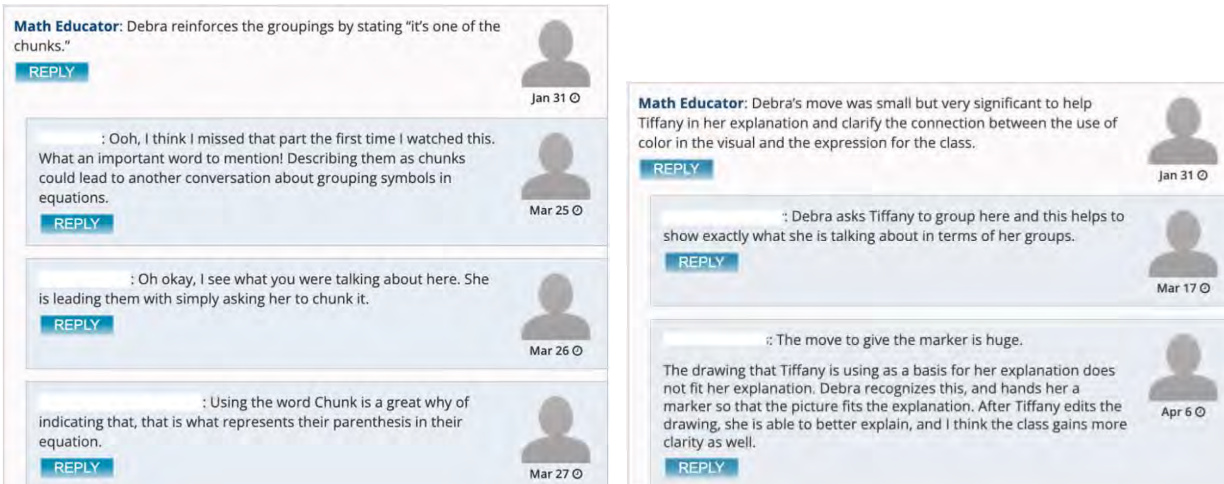


Figure 5: Positive participant responses to pre-inserted video comments from facilitated (left) and independent (right) cohorts

In facilitated cohorts, facilitation of discourse was moderately provided by human facilitators monitoring and responding to posts in order to guide discussion and probe participants' thinking. In both facilitated and structured independent cohorts, a level of in-the-moment facilitation was also provided by participants interacting with module elements/resources such as the mathematical task work, others' perspectives on the mathematics, and most critically, the central video clip. We hypothesize that the unique nature of authentic classroom video described by Sherin (2004) and the intrinsic motivation and interest it inspires for teachers in combination with the prompts, framing questions, and reflection questions fosters a level of discourse that may not otherwise be achieved in an asynchronous, online PD experience.

Conclusion

If schools and districts are to scale quality PD in a cost-effective and widely accessible manner, innovative tools and strategies that do not rely on individual providers spending extensive face-to-face time with small groups of teachers are needed (Cai et al., 2017). Highly structured online asynchronous PD opportunities—with or without active facilitation—may have a role to play in bringing effective, flexible professional learning to a significantly wider audience if they are designed in a way that reflects features of effective PD recognized by the field and supports the development of communities of inquiry. The VIM module design and pilot study provide promising evidence that this can be accomplished, while also pointing us to possible new directions for future development and research.

In particular, greater social presence may be accomplished through technology that avoids anonymous posts and allows for participant profile information and photos. Specific prompts could also be added to encourage participants to revisit previous Community Walls to review and engage with new posts. Additional opportunities for interaction with fellow participants could also be created, for example in relation to the "Bridge to Practice" activities.

Acknowledgments

This project is supported by the National Science Foundation (NSF), through NSF #1720507. Any opinions, findings, and conclusions or recommendations expressed are those of the author(s) and do not necessarily reflect the views of the NSF.

References

An, Y. (2018). The effects of an online professional development course on teachers' perceptions, attitudes, self-efficacy, and behavioral intentions regarding digital game-based learning. *Educational Technology Research and Development, 66,* 1505–1527.

Anderson, T. D., Rourke, L., Garrison, D. R., & Archer, W. (1999). *Social presence in computer conferencing: Chewing the phat (ic)* [Unpublished manuscript].

Anderson, T., Rourke, L., Garrison, D. R., & Archer, W. (2001). Assessing teaching presence in a computer conference environment. *Journal of Asynchronous Learning Network,s 5*(2), 1-17.

Ball, D.L. & Bass, H. (2002). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2001 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3–14).

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *The Journal of Mathematical Behavior, 33*, 149–167.

Cai, J., Morris, A., Hwang, S., Hohensee, C., Robinson, V., & Hiebert, J. (2017). Improving the impact of educational research. *Journal for Research in Mathematics Education, 48*, 2–6.

Choi, H., & Jonson, S. (2007). The effect of problem-based video instruction on learner satisfaction, comprehension and retention in college courses. *British Journal of Educational Technology, 38*(5), 885e895.

Darling-Hammond, L., Hyler, M. E., & Gardner, M. (2017). *Effective teacher professional development.* Learning Policy Institute. https://learningpolicyinstitute.org/sites/default/files/product-files/Effective_Teacher_Professional_Development_REPORT.pdf

Dede, C., Eisenkraft, A., Frumin, K., & Hartley, A. (2016). *Teacher learning in the digital age: Online professional development in STEM education.* Cambridge, MA: Harvard Education Press.

Dede, C., Ketelhut, D.J., Whitehouse, P., Breit, L., & McCloskey, E.M. (2009). A research agenda for online teacher professional development. *Journal of Teacher Education, 60*(1), 8-19.

DeMonte, J. (2013). *High-quality professional development for teachers: Supporting teacher training to improve student learning.* Washington, DC: Center for American Progress.

Elmore, R. F. (2002). *Bridging the gap between standards and achievement: The imperative for professional development in education.* Washington, DC: Albert Shanker Institute.

Fishman, B. (2016). Possible futures for online teacher professional development. In C. Dede, A. Eisenkraft, K. Frumin, & A. Hartley (Eds.), *Teacher learning in the digital age: Online professional development in STEM Education.* Camrbidge, MA: Harvard Education Press.

Garrison, D. R. (2003). Cognitive presence for effective asynchronous online learning: The role of reflective inquiry, self-direction and metacognition. In J. Bourne & J. C. Moore (Eds.), *Elements of quality online education: Practice and direction* (pp. 47–58). Sloan-C.

Garrison, D.R., Anderson, T., & Archer, W. (2000). Critical inquiry in a text-based environment: Computer conferencing in higher education. *The Internet and Higher Education, 2*(2), 87-105.

Garrison, D.R., Anderson, T., & Archer, W. (2001). Critical thinking, cognitive presence, and computer conferencing in distance education. *American Journal of Distance Education, 15*(1), 7-23.

Garrison, D.R., & Arbaugh, J.B. (2007). Researching the community of inquiry framework: Review, issues, and future directions. *The Internet and Higher Education, 10*(3), 157-172.

Garrison, D.R., & Archer, W. (2000). *A transactional perspective on teaching and learning: A framework for adult and higher education.* Oxford, UK: Pergamon.

Garet, M.S., Porter, A.C., Desimone, L., Birman, B.F., & Yoon, K.S. (2001). What makes professional development effective? Results from a national sample of teachers*. American Educational Research Journal, 38*, 915–945.

Gunawardena, C.N., & Zittle, F.J. (1997). Social presence as a predictor of satisfaction within a computer-mediated conferencing environment. *The American Journal of Distance Education, 11*(3), 8-26.

Heller, J.I., Daehler, K.R., Wong, N., Shinohara, M., and Miratrix, L.W. (2012). Differential effects of three professional development models on teacher knowledge and student achievement in elementary science. *Journal of Research in Science Teaching, 49*(3): 333–362.

Killion, J., & Learning Forward. (2013). *Meet the promise of content standards: Tapping technology to enhance* *professional learning*. Learning Forward. https://learningforward.org/wp-content/uploads/2017/09/tapping-technology-to-enhance-professional-learning.pdf

Kuehn, S. (1993). Communication innovation on a BBS: A content analysis. *Interpersonal Computing and Technology: An Electronic Journal for the 21st Century, 1*(2). http://jan.ucc.nau.edu/~ipctj/1993/n2/kuehn.text

Loucks-Horsley, S., Love, N., Stiles, K., Mundry, S., & Hewson, P. (2010). *Designing professional development for teachers of science and mathematics (3rd Edition).* Thousand Oaks, CA: Corwin Press.

Moreno, R., & Valdez, A. (2007). Immediate and delayed effects of using a classroom case exemplar in teacher education: the role of presentation format. *Journal of Educational Psychology, 99*(1), 194-206.

Parsons, S. A., Hutchison, A. C., Hall, L. A., Parsons, A. W., Ives, S. T., & Leggett, A. B. (2019). U.S. teachers' perceptions of online professional development. *Teaching and Teacher Education, 82*, 33–42.

Polly, D., McGee, J., Wang, C., Martin, C., Lambert, R., and Pugalee, D. (2015). Linking professional development, teacher outcomes, and student achievement: The case of a learner-centered mathematics program for elementary school teachers. *International Journal of Educational Research, 72*, 26–37.

Richardson, J. C., & Swan, K. (2003). Examining social presence in online courses in relation to students' perceived learning and satisfaction. *Journal of Asynchronous Learning Networks, 7*(1), 68−88.

Rourke, L., Anderson, T. Garrison, D. R., & Archer, W. (2001). Assessing social presence in asynchronous, text-based computer conferencing. *Journal of Distance Education, 14*(3), 51-70.

Rourke, L., Anderson, T., Garrison, D. R., & Archer, W. (1999). Assessing social presence in asynchronous text-based computer conferencing. *Journal of Distance Education, 14*(2), 50-71.

Seago, N., Koellner, K. Jacobs, J. (2018). Video in the middle: Purposeful design of video-based mathematics professional development. *Contemporary Issues in Technology and Teacher Education, 18*(1).

Seago, N., Mumme, J., & Branca, N. (2004). *Learning and teaching linear functions.* Portsmouth, NH: Heinemann.

Sherin, M. G. (2004). New perspectives on the role of video in teacher education. In J. Brophy (Ed.), *Using video in teacher education* (pp. 1-28). Oxford, UK: Elsevier.

U.S. Department of Education (2010). *Transforming American education: Learning powered by technology; National education technology plan.* Washington, DC: U.S. Department of Education, Office of Educational Technology.

van Es, E.A., & Sherin, M.G. (2002). Learning to notice: Scaffolding new teachers' interpretation of classroom interactions. *Journal of Technology and Teacher Education, 10,* 571-596.

Yoon, K.S., Duncan, T., Lee, S.W.-Y., Scarloss, B., and Shapley, K. (2007). *Reviewing the evidence on how teacher professional development affects student achievement (Issues and Answers Report, REL 2007-No. 033).* National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southwest, Institute of Education Sciences, U.S. Department of Education. Washington, DC.

Yoon, S.A., Miller, K., Richman, T., Wendel, D., Schoenfeld, I., Anderson, E. & Shim, J. (2020). Encouraging collaboration and building community in online asynchronous professional development: Designing for social capital. *International Journal of Computer-Supported Collaborative Learning 15*, 351–371.

MODELING AMBITIOUS TEACHING IN SYNCHRONOUS ONLINE MATHEMATICS EDUCATION COURSES

Nicholas Kochmanski
UNC Greensboro
nmkochmansk@uncg.edu

Terrie M. Galanti
University of North Florida
terrie.galanti@uncf.edu

Alees Lee
Weber State University
aleeslee@weber.edu

Jonathan D. Watkins
Ball State University
jdwatkins@bsu.edu

There is a prevailing notion that online course modalities present challenges to teaching in ambitious ways, and thus to modeling ambitious teaching for pre-service and in-service teachers. In this paper, we report on the results of an exploratory analysis of one mathematics teacher educator's efforts to model ambitious teaching in a synchronous, online mathematics education course. We report on both the online tools central to this effort and the teaching moves for using those tools effectively. In so doing, we highlight how the teacher educator's decision to be transparent about the challenges involved in adapting instruction to an online environment presented further opportunities to model the kind of instructional experimentation central to ambitious teaching.

Keywords: Online and Distance Education, Technology, Pre-service Teacher Education

A primary goal of many mathematics education courses is to support participating teachers in learning to teach in ambitious and equitable ways, such that they can support students' attainment of rigorous mathematical learning goals (Kazemi et al., 2009). Ambitious and equitable teaching empowers students to engage in complex problem-solving and to reason mathematically. As teachers elicit and respond to students' contributions, they position students as being mathematically competent (Lampert et al., 2009) and broaden opportunities for discourse (Ghousseini et al., 2017). Ambitious and equitable teaching is, however, challenging and complex work. Mathematics teacher educators (MTEs) can support teachers in learning to enact ambitious and equitable teaching by modeling this type of teaching in education courses (Lampert et al., 2010), thereby providing teachers with opportunities to see what it looks like to enact this type of teaching effectively. Developing such a vision is an essential step in teaching in ambitious ways with students (Munter & Correnti, 2017).

Several issues, including the ongoing COVID-19 pandemic and declining enrollments in teacher education programs, have prompted many colleges and universities to shift their programs online. Many teacher educators have claimed that this shift in course delivery method presents challenges for teaching in ambitious ways (Thompson et al., 2021), and thus for modeling ambitious teaching for pre-service and in-service teachers. In many cases, these presumptions have been based on the very real challenges teacher educators faced when they rapidly transitioned courses previously designed for face-to-face instruction to emergency remote teaching (Hodges et al., 2020). Yet, it is entirely possible that carefully planned online learning experiences in mathematics teacher education can provide opportunities for teachers to see ambitious teaching in action. For example, Jung and Brady (2020) highlight the value of breakout rooms and shared Google Docs in building social norms and dialogic interactions that support reasoning and problem solving within a community of learners. Unfortunately, very few studies have closely examined the ways in which MTEs can use online teaching tools to model specific forms of ambitious teaching for teachers.

In this paper, we report on the results of an exploratory analysis of one MTE's (hereafter referred to as Professor K) efforts to model ambitious teaching in a synchronous, online mathematics education course. We focused our analysis on Professor K's efforts because the participating pre-service and in-service teachers indicated on course exit tickets and end-of-course evaluations that they had ongoing opportunities to experience ambitious teaching, despite the online modality of the course. We therefore determined that it would be illuminating to examine how Professor K provided those opportunities for the teachers in the synchronous online course. We investigated the following research questions: How did Professor K model ambitious teaching in an online, synchronous mathematics education class? What technologies did he use to model ambitious teaching in a synchronous, online mathematics education course? What instructional moves were associated with his use of the technologies?

The Synchronous Online Mathematics Education Course

The mathematics education course that is the focus of this analysis took place in a school of education at a large, public university in the Mid-Atlantic of the United States. Four pre-service teachers and four in-service teachers were enrolled in the 14-week course. Each of the weekly sessions was conducted online and synchronously over the Zoom video conferencing platform. Almost all students kept their cameras on for each class period, even though it was not required. Notably, our analysis of Professor K's teaching was retrospective, meaning Professor K was unaware of the focus of our research at the time of his teaching.

The course was designed to support teachers in deepening their mathematical knowledge for teaching in the middle grades. To accomplish this, Professor K broke the course into two-week units, each of which focused on the processes of students' learning, as well as the means of supporting that learning, for different mathematical domains relevant to teaching middle grades mathematics (e.g., ratios and proportions or expressions and equations). In other words, instead of focusing each lesson solely on mathematics, Professor K designed the lessons such that the class engaged with mathematics as teachers. While the focus of the lessons changed over the course of the semester, the general structure of each lesson stayed relatively consistent. Professor K organized each lesson around a set of editable Google slides. The slides included links to Jamboards, so that students could represent their mathematical thinking; links to Google documents that included pictures or representations of students' work; and designated writing spaces through which teachers could record their thinking and share it publicly over Zoom.

Because all eight of the participants were also in their student teaching internships, Professor K designed the course such that the activities in which teachers engaged mirrored the kinds of activities they might implement with their students. This meant he designed each lesson to provide the participating teachers with opportunities to see ambitious teaching. For example, in almost all lessons, Professor K engaged the teachers in several problem-solving tasks, each of which followed a general structure. First, he would introduce the class to a task, and thus model how teachers can support broad and active engagement. Professor K would then instruct the class to work on the task in small groups using breakout rooms. During these small groups, he would ask assessing and advancing questions and thus model listening to and responding to students' intellectual contributions. Finally, he would bring the class together to discuss their work in the primary Zoom room. In these discussions, Professor K routinely asked teachers to restate or build on previously shared ideas, in the process positioning them as being mathematically and pedagogically competent.

Conceptualizing Ambitious Teaching

In this paper, we define ambitious teaching as teaching that supports all students in not only acquiring knowledge, but also understanding, using, and generating new knowledge to solve authentic problems (Lampert & Graziani, 2009). This kind of teaching is extremely complex. It requires adaptations to practice as teachers elicit, make use of, and respond to students' thinking. Because of this complexity, it is important for MTEs to represent ambitious teaching for teachers, as doing so can clarify what ambitious teaching looks like in practice (Lampert et al., 2013). Teacher educators can then decompose the work of ambitious teaching into learnable units, often called teaching practices (e.g., launching instructional tasks), and support teachers in learning to enact those practices effectively. Thus, we see it as essential for MTEs to model ambitious teaching as part of their work with pre- and in-service teachers in both face-to-face and online settings.

Methods

Data Collection

We collected the following data from the online, synchronous mathematics education course described above: (1) participants' responses to weekly class exit tickets, (2) participants' responses to mid-semester and end-of-course evaluations, (3) video recordings of Professor K's weekly lessons, and (4) artifacts from the weekly lessons, including the participating teachers' written work as they responded to questions on Google slides, their bi-weekly homework assignments, and teachers' work as they solved mathematics tasks using Google's Jamboard application. We also interviewed Professor K to clarify his decision-making.

Data Analysis

We conducted two phases of analysis to understand how Professor K used online tools to model ambitious teaching. In the first phase of analysis, we identified a representative episode of a lesson that (a) included the use of the online tools most typical of Professor K's instruction and in which he attempted to model ambitious teaching. This involved first reviewing the weekly Google Slide sets for the course to document the types of activities in which Professor K typically engaged the teachers. We then met as a research team to discuss the nature of the activities and to look for any commonalities among them. After this meeting, we wrote a brief memo in which we outlined a general structure for the types of activities that were most typical of Professor K's teaching. Almost all the course activities followed the general structure outlined above. We then looked across the video recordings for an activity in which Professor K used this typical structure and in which he clearly intended to model aspects of ambitious teaching. This activity became the focal episode for the second phase of the analysis.

In the second phase of analysis, we watched the representative episode and coded for the forms of ambitious teaching modeled and the online tools and technologies Professor K used when modeling those forms of teaching. We then conducted a second round of coding to understand how Professor K made use of the online tools and technologies. To conduct this finer-grained coding, we drew on the idea of teaching moves. Following Jacobs and Empson (2016), we define teaching moves as directly observable units of teaching activity that share a common purpose. One type of teaching move, for example, might involve a teacher (or, in this case, teacher educator) asking a question to elicit a mathematical strategy. In this example, the move features an observable form of teaching activity (asking a question) and a specific purpose for that activity (eliciting a mathematical strategy). Focusing on teacher moves enabled us to foreground the observable forms of teaching activity that appeared central to Professor K's use of the online tools to model ambitious teaching. We engaged in consensus coding for both rounds of coding. To conclude our work, we recorded the results of our coding in a table.

Findings

Over the course of the selected episode, the participating teachers had several opportunities to experience ambitious teaching in an online, synchronous classroom environment. For example, Professor K modeled how teachers can encourage broad and active engagement in conceptually rich problem-solving tasks by introducing (i.e., launching) the instructional task in such a way as to ensure everyone could get started but without telling them exactly how to engage in the task. As the teachers explored the instructional task during small-group work time, Professor K also asked questions and listened closely to their explanations, thereby modeling how teachers can elicit and respond to students' intellectual contributions. Additionally, Professor K modeled how to broaden opportunities for discourse and position students as mathematically competent, both by asking assessing and advancing questions in small-group work time to encourage previously quiet teachers to participate *and* by calling on specific teachers to share their thinking during the whole-group discussions.

These opportunities to experience ambitious teaching appeared to rely, in part, on Professor K's implementation of several online teaching tools. Further, we found that Professor K made several teaching moves that were central to his use of these online teaching tools to model ambitious teaching for the participating teachers. Table 1 shows the online teaching tools and associated moves as organized by the phases of the instructional episode in which they occurred (i.e., the task introduction, independent and small-group work time, and concluding whole-class discussion).

In what follows, we describe the online teaching tools and many of the associated moves for each of the three phases of instruction, in the process outlining how they enabled Professor K to model ambitious teaching.

Task Launch Phase

In this phase of the episode, Professor K introduced the focal instructional task and told the teachers that they would work on the task in small groups. Professor K leveraged three online teaching tools to launch the task: (1) screen sharing Google Slides with participants in Zoom, (2) editable Google Slides that included hyperlinks to Google Docs, and (3) editable Google Slides that included templates for recording thinking during the instructional task. He used these tools to clarify expectations for what the teachers were expected to do on the instructional task and how they should participate in their small groups.

Clarifying Features of Instructional Task. Professor K shared his screen and showed the teachers the task prompt, which was written into an editable Google Slide. As shown in Figure 1, the instructional task asked the teachers to analyze students' work for a mathematics problem involving ratios and proportions. Professor K made both verbal and visual teaching moves to guide teachers through the task prompt. First, he employed in-the-moment text highlighting to emphasize his verbal instructions as he read aloud the task. Because the teachers had both shared access to the Google slides *and* Professor K was sharing his screen, the teachers were able to follow along visually with his verbal instructions.

Professor K then demonstrated how the teachers were expected to click a link in the editable Google Slides to access the students' work (see Figure 1). This ensured that the teachers were aware of how they could access the student work, and were thus able to begin working on the instructional activity. Finally, in showing the teachers the student work, Professor K noted that the student work might look different from how the teachers would typically solve such a mathematics problem.

Table 1: Teaching Moves Associated with Modeling Ambitious Teaching in Online Education Course

Phase of Episode	Online Teaching Tools	MTE Teaching Moves
Launch Introducing task to teachers	● Expectation Google Slides to establish activity goals and norms for group participation ● Hyperlinks to Google Docs for viewing student work ● Task Templates in Google Slides assigned to each small group to prompt conversation	● Share screen with task information, highlighting text on slides for emphasis ● Set expectations for interacting in the online environment ● Acknowledge difficulty of task ● Give instructions both verbally and on Google Slides ● Assign random groups ● Set up the Explore phase by suggesting group roles (Notetaker, Spokesperson, Timekeeper) ● Set an 8-minute timer
Explore Facilitating small group analysis of student work	● Zoom Breakout Rooms for small group conversations ● Task Templates in Google Slides for monitoring student progress in real-time and recording student thinking	● Give teachers 3 minutes of think time as a group ● Listen without guiding the direction of teacher conversations ● Ask purposeful content questions about proportional reasoning ● Set up Summarize phase by validating, suggesting, and encouraging participation
Summarize Synthesizing emergent themes in a whole group discussion	● Task Templates in Google Slides for instructors and teachers to compare and contrast noticings, wonderings, and connections ● Screen sharing to focus teachers' attention on individual group contributions ● Chat for participating non-verbally ● Chat for monitoring and acknowledging teacher questions and comments	● Frame whole-group discussion by referring to task templates ● Remind teachers about the goal of synthesizing themes ● Expect each group to share their thinking ● Encourage groups to question each other ● Validate class-related conversations in the chat ● Summarize big/ideas without over-contributing

He then explained that the differences between the teachers' own strategies and the students' strategies were intended to provide them with an opportunity to make sense of students' thinking on their own, thereby highlighting that the teachers were engaging in a challenging task that involved instructional problem solving. By spending time showing the teachers how they could access the instructional task, Professor K ensured that they felt prepared to engage in a complex task in an online learning environment.

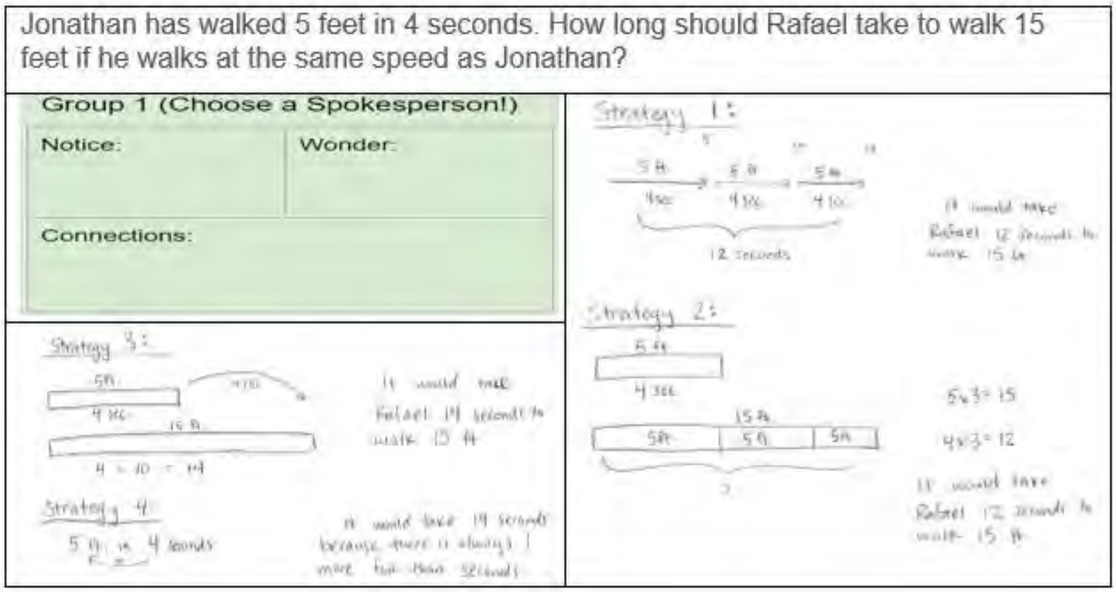


Figure 1: Walking Speeds Task

Clarifying Group Work Expectations. In addition to explaining the features of the instructional task, Professor K briefly outlined expectations for how the teachers should engage with one another as they worked in small groups. Prior to opening up the breakout groups, he encouraged the groups to assign their own roles (notetaker, spokesperson, and timekeeper to move conversation forward). This allowed the teachers to define their own participation and suggested ways in which the teachers could participate productively. He also assigned each group an editable Google Slide with a template for how the small groups could record the ideas they discussed. He then instructed the teachers to record their observations about the student work on their assigned slide. This move enabled Professor K to model a process for eliciting and making visible the teachers' thinking, and thus was an opportunity to model ambitious teaching. Further, by providing opportunities for both verbal and written communication, Professor K leveraged technology to broaden the teachers' access to the intellectual activity. As an aside, Professor K also paused to openly reflect on his efforts at continued growth in his own teaching, saying, "[I'm] trying to work on those directions - I have been trying to push myself on this." This also modeled how an improvement mindset is central to ambitious teaching.

Task Exploration Phase

In the exploration phase, Professor K supported broad and active participation in the small-group work time. He also modeled how teachers can elicit and respond to thinking. Professor K leveraged two online teaching tools to do so: (1) the editable Google Slides, which included templates for the teachers to record the results of their small-group work time and (2) the Zoom breakout rooms. He used the Google Slides to monitor all of the groups' thinking simultaneously, and visited the breakout rooms to elicit, listen to, and respond to that thinking.

Monitoring Progress in Google Slides. After releasing students to their breakout rooms, Professor K waited in the main Zoom room for approximately three minutes before checking in with the groups. This teaching move appeared to implicitly position the teachers as mathematically and pedagogically capable. It also appeared to encourage their engagement in small group discussions. As he waited, Professor K monitored the teachers' thinking by watching as they recorded their ideas on the editable Google Slides. When Professor K joined the breakout rooms, he was prepared to engage in the conversation, meaning he could more effectively model how teachers elicit and respond to thinking. Thus, within the exploration phase, the teachers had opportunities to both make meaning and simultaneously build virtual artifacts of their thinking, both of which are tenets central to ambitious teaching (Ghousseini et al., 2017).

Moving Between Breakout Rooms to Elicit, Listen to, and Respond to Thinking. After monitoring the teachers' written thinking on the Google Slides for three minutes, Professor K then entered each breakout room in turn. He spent approximately one minute in each room actively engaged in discussing the teachers' interpretations of the student work samples. The short duration of these breakout room visits communicated Professor K's confidence in the ability of each small group to make sense of the mathematics and the students' thinking.

In each visit, Professor K asked questions to elicit the teachers' thinking, listened actively as the teachers explained their thinking or discussed ideas with one another, and then asked additional assessing and advancing questions when appropriate. At times Professor K affirmed the productivity of the teachers' discourse with phrases like, "Yeah, kids think like that. Be sure to add it to your wonderings" and "You should feel like you can add that to your spokesperson's thoughts." At times he also challenged them to be ready to justify their mathematical thinking when sharing with the whole class, asking questions such as, "What do you mean by grouping?" He also prompted specific teachers to share ideas with the whole group. For example, when one of the small groups brought up a point of contention in their small-group discussions, he noted, "You should ask [the whole group] about that." Once he had checked in with each group, Professor K then moved back to the main room and continued to monitor the teachers' thinking by watching their written work on the editable Google Slides. This again positioned the teachers as both mathematically and pedagogically capable.

Summarizing the Task

In the summarizing phase, Professor K brought the teachers together to discuss their ideas as a whole group. Professor K made use of several online teaching tools during this phase of the episode, including: (1) the editable Google Slides, (2) screen sharing, and (3) the chat feature in Zoom. Throughout the summarizing discussion, Professor K reminded students of the expectations for discussing ideas in an online classroom. Specifically, he encouraged students to listen to the other groups by considering what was similar to or different from what they discussed in their own small groups. As he facilitated the summarizing discussion, Professor K attempted to both synthesize big ideas without over-contributing and also validate conversations between teachers that were occurring in the Zoom chat.

Synthesizing Big Ideas without Over-Contributing. To facilitate the discussion, Professor K shared his screen to display each group's slide in turn. This teaching move oriented the teachers to focus on one group's ideas at a time. By making this move, Professor K modeled how a teacher can facilitate discussion towards a learning goal while also avoiding the pitfall of over-contributing or over-facilitating. In other words, he chose which group would share and in what order, but he then left the subsequent discussion up to the teachers. Additionally, by sharing his screen, Professor K ensured the sharing groups did not have to worry about managing the online tools themselves, meaning they could focus on the ideas they wished to communicate.

Professor K also encouraged the teachers to question one another after the groups shared. Further, when this happened, Professor K pointed out that this was occurring in an online, synchronous class. For example, Professor K said, “We are starting to hear pushing on each other’s thinking - back and forth - which is exceptionally important for our class to be successful moving forward.” By making this comment, Professor K modeled how teachers can negotiate productive social norms, which are central to the success of ambitious teaching.

Validating Chat Conversations. Professor K also took advantage of the Zoom chat feature in the online learning environment to support broader student participation. For example, during the whole group discussion, Professor K paused, checked the chat, and noted that the teachers were “having a great conversation in the chat” without him. He highlighted this as a positive, saying, “It’s good to see people using that feature to their advantage.” This teaching move also modeled the importance of attending to varied means of participation in a lesson.

Discussion and Implications

This exploratory analysis of an online, synchronous teaching episode underscores that it is possible to model ambitious teaching in an online learning environment. We clarify how MTEs can model ambitious teaching in online mathematics education courses by highlighting several online teaching tools and instructional moves involved in the effective use of those tools. Our findings serve as a starting point for describing the work of ambitious online teaching and also point to strategies that other teacher educators might use in their own practice.

Further, the results of this analysis contribute to research aimed at characterizing effective online teaching approaches more generally (e.g., Carillo et al., 2020). Researchers often define effective online instruction as teaching that supports and facilitates a community of inquiry in which learners construct new forms of knowledge (Garrison et al., 2000). However, MTEs are obligated to not only support learners in constructing knowledge but also support them in developing as teaching professionals. In looking closely at Professor K’s instructional episode, we were struck by the very real *affordances* of an online learning environment for modeling the kind of adaptation and experimentation that is central to ambitious teaching. During instruction, Professor K was open about the intellectual challenge involved in teaching ambitiously in the online environment. He leveraged the complexities of modeling ambitious teaching in a synchronous online environment as an opportunity to lay bare his own developing practice in order for teachers to learn from his efforts to improve his teaching. In doing so, he normalized the importance of experimenting to improve, all in service of growing as a teaching professional.

Finally, while it is important for teachers to experience ambitious teaching as learners, it is also beneficial for teachers to see and consider the very real challenges involved in *learning* to teach ambitiously in a new context (Kazemi et al., 2018). Prior research indicates that supporting teachers to learn *from* models of teaching involves an explicit effort to press and support teachers to notice the instructional decisions and teaching moves represented in those models (van Es & Sherin, 2008). Though Professor K made occasional reference to his instructional decisions, these references were often spontaneous or general in nature. Future research might investigate strategies that MTEs can use to purposefully draw teachers’ attention to specific instructional decisions or moves when modeling in an online learning environment.

References

Carrillo, C., & Flores, M. A. (2020). COVID-19 and teacher education: A literature review of online teaching and learning practices. *European Journal of Teacher Education, 43*(4), 466-487. https://doi.org/10.1080/02619768.2020.1821184

Garrison, R., Anderson, T., and Archer, W. (2000). "Critical Inquiry in a Text-based Environment: Computer Conferencing Higher Education." *The Internet and Higher Education 2,* (2–3), 87–105. https://doi.org/10.1016/S1096-7516(00)00016-6

Ghousseini, H., Beasley, H., & Lord, S. (2017). Using generative routines to support learning of ambitious mathematics teaching. In E. Galindo, E., & Newton, J., (Eds.). *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education,* pp. 869 - 876. Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Hodges et al., (2020)

Jacobs, V. R., & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM, 48*(1), 185-197.

Jung, H., & Brady, C. (2020). Maintaining rich dialogic interactions in the transition to synchronous online learning. *Information and Learning Sciences*, 121(5/6), 391-400.

Kazemi, E., Franke, M., & Lampert, M. (2009, July). Developing pedagogies in teacher education to support novice teachers' ability to enact ambitious instruction. In Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia (Vol. 1, pp. 12-30).

Kazemi, E., Gibbons, L. K., Lewis, R., Fox, A., Hintz, A. B., Kelley-Petersen, M., & Balf, R. (2018). Math Labs: Teachers, teacher educators, and school leaders learning together with and from their own students. *Journal of Mathematics Education Leadership, 19*(1), 23-36.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., ... & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*(3), 226-243.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. The Elementary School Journal, 109(5), 491-509.

Thompson, A. D., Lindstrom, D., & Schmidt-Crawford, D. A. (2020). COVID-19 Era: A time for us to lead! *Journal of Digital Learning in Teacher Education, 36*(4), 204–205.

Van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers'"learning to notice" in the context of a video club. *Teaching and teacher education*, *24*(2), 244-276.

INTERACTION TYPES IN ONLINE AND HYBRID MATHEMATICS INSTRUCTION

Erin Krupa
NC State University
eekrupa@ncsu.edu

Michael Hoyes, Jr.
NC State University
mbhoyes@ncsu.edu

Karen Hollebrands
NC State University
kfholleb@ncsu.edu

Engagement in the mathematics classroom through interactions with the instructor, peers, and content are necessary for an effective learning experience. As such, it is important to understand the types of interactions that teachers utilize to engage students, especially as they have had to shift from a complete face-to-face setting to various remote modalities. Utilizing four interaction types (learner-content, learner-instructor, learner-learner, and learner-interface) this paper analyzes 35 videos of classroom instruction with the purpose of describing the interactions that take place throughout the course of the mathematics lesson. While there was not a significant difference in the type of interaction and the modality of instruction, there was a significant difference in the type of interaction enacted and the modality of instruction.

Keywords: High School Education, Instructional Activities and Practices, Technology, Online and Distance Education

Introduction

While historically providing education through distance and remote learning at the K-12 level was reserved for gifted older students (Campbell & Storo, 1996), the technological advances have allowed for platforms and delivery systems that have made it a feasible means of stretching the walls of the classroom to reach all students (Offir et al., 2003). Teachers have had to contend with the preeminent mindset that distance education works best for independent, self-motivated learners (Chou et al., 2010; Corso et al., 2021) as they try to create learning opportunities that engage all learners with varying ability levels and technological skills.

Teachers tend to try to duplicate their face-to-face instruction when teaching online, which includes trying to replicate different types of classroom interactions (Battalio, 2007; Garrison & Cleveland-Innes, 2005). Interaction is seen as essential to the online educational experience and as such is a primary focus for online learning (Garrison & Cleveland-Innes, 2005). To explore the interactions secondary mathematics teachers have utilized in their online and hybrid teaching, we investigate the research question: *What types of interactions are teachers facilitating in high school mathematics classes across different modalities of online teaching?*

Theoretical Framework

Learner-content interaction is “intellectually interacting with content to bring about changes in the learner’s understanding, perspective, or cognitive structures” (Hillman et al., 1994, p. 30). Moore characterized this form of interaction as a one-way flow of information from the subject matter to the student (Lin et al., 2017) and stated that “without it there cannot be education” (Moore, 1989, p. 2). This interaction includes: conjecturing mathematics, solving problems, confirming one's work, interpreting mathematics, and making predictions.

Moore (1989) claims that learner-instructor interaction is regarded as “highly desirable by many learners-is interaction between the learner and the expert who prepared the subject material, or some other expert acting as instructor” (pp. 1–2). Offir et al. (2003) study refined the five categories of this interaction. These categories are: social, which pertains to classroom climate; procedural, which is dissemination of administrative instructions about the course; expository, which is presenting information by the teacher; explanatory, which is the teacher’s use of student input to explain content; and cognitive task engagement, which is the teacher presenting a question/task for the learners to engage in processing information.

Learner-learner interaction is the interaction between learners. Moore (1989) characterized this as the least focused by many within education but essential for younger learners where peer group interactions often aid motivation. We categorized this interaction as the presentation of a solution, responding to another learner, collaborative problem-solving, and small group discussion.

Hillman et al. (1994) recognized that another interaction dimension should be considered in technology-mediated learning environments. This type of interaction is necessary because the students must navigate technology to complete the educational task. While this could be thought of as manipulation of a mathematical object and thus learner-content, the difference lies in that content knowledge is a non-factor during the interface interaction. "Regardless of the proficiency of the learner, inability to interact successfully with the technology will inhibit his or her active involvement in the educational transaction" (Hillman et al., 1994). We operationalized this as responding to a poll or providing an answer via features of a technology medium.

Participants, Data, and Methods

Nineteen experienced high school mathematics teachers from seven districts participated in this study. Teachers had between six and 34 years of experience, with eight being the median years of experience. They had completed a 30-credit hour Master's degree or a 12-hour graduate certificate in mathematics education in Summer 2020. There were 35 videos analyzed for this study. Teachers submitted a video recording of one of their classes from the fall of 2020 (n=19) and an additional video from the spring of 2021 (n=16). Teachers either submitted videos of pre-recorded (Fall-13, Spring-4), synchronous online (Fall-13, Spring-4, hybrid (Fall-13, Spring-4), or face-to-face (Fall-13, Spring-4) instruction. These videos captured the entire class period, including breakout rooms, and were not edited before submission. COVID restrictions in each district dictated the mode of instruction. In the fall of 2020, most teachers recorded a video in advance of their class to post on their learning management system for students to view asynchronously. The videos ranged in length from five to 54 minutes in the fall of 2020 and 15 to 71 minutes in the spring of 2021.

Three levels of coding were applied to the analysis of each video. First, the type of interaction was identified (learner-content, learner-instructor, learner-learner, learner-interface) and coded. Then the second level of interaction was identified and coded. Finally, it was noted if technology was used during the activity.

The researchers, the authors of this paper, met to view a video together, clarify the unit of analysis, and define each of the codes. The unit of analysis was considered a level 2 interaction type. When there was a change in the activity type, that was treated as a new unit to code. For example, a teacher interacting with students to solve a problem would be coded (learner-instructor, solve a problem). If the teacher then provided students with directions about submitting their solution, that would be considered a new unit and coded learner-instructor, procedural (administrative). There were 1,962 total turns for this data, 366 for fall 2020 and 1,596 for spring 2021.

One video was coded together, and then six videos were assigned to pairs of researchers to determine inter-rater reliability. Agreement on units of analysis was determined (87.1%) and agreement on codes on the units that were in common was calculated (86.7%). Once inter-rater reliability was established, one researcher coded the remainder of the videos.

Data were analyzed using frequencies and Chi-square Test of Independence to determine if there were any significant associations between two categorical variables. Chi-square tests were run to determine if there were significant differences in the type of instruction and level 1 interaction types, semester of video and interaction types, and level 1 interaction and the specific level 2 interaction.

Results

First, an analysis of the number of unique interactions by the type of instruction shows that there were many more interactions in face-to-face instruction (n=928) over the two semesters than in the other types hybrid (n=525), synchronous (n=426), and pre-recorded (83).

During the 2020-2021 school year, mathematics teachers overwhelmingly utilized learner-instructor interactions (74.31%), predominately cognitive task engagement or expository. Of the learner-instructor interactions, 36.87% were cognitive task engagement, 26.32% were expository, and 13.64% were explanatory. Overall, 25.28% of the interactions were learners engaging with mathematics content. Most learner-content interactions involved students solving a problem (52.02% of learner-content interactions) or making a conjecture (44.15%). In addition,

there is evidence that teachers provided a few opportunities for their students to confirm their answers (n=18), predict (n=1), and interpret their solutions (n=0). There were very few learner-learner (n=6) and learner-interface (n=1) interactions of any type across the two videos from the participants. Of the six learner-learner interactions, all were collaborative problem solving and came from three teachers' classes. Of these six learner-learner interactions, five were in face-to-face instruction, and one was from hybrid instruction.

Due to the low numbers of learner-learner and learner-interface, only learner-content and learner-instructor were included in the next analyses. There was a significant association between the type of instruction and semester ($\chi^2(3)=246.57$, $p<0.001$). This suggests teachers provided more interactions in the spring of 2021 than in the fall of 2020. Interestingly, though there were more interactions in the spring of 2021, there was no significant association between the semester and interaction type ($\chi^2(1)=0.057$, $p=0.811$). Therefore, teachers provided the same types of interactions during the entire year, so there were no differences in instances of learner-content and learner-instructor interactions. There was also no significant association between the type of instruction and interaction type ($\chi^2(3)=2.873$, $p=0.412$). This means that teachers provided the same types of level 1 interactions regardless of the mode of delivery.

Looking at the learner-instructor interactions (n=1459), there was a significant association between the type of instruction and level 2 interaction types ($\chi^2(24)=121.638$, $p<0.001$). This suggests differences in the type of specific interactions based on the mode of instructional delivery. First, students had minimal opportunity to engage in inquiry, justification, or social interactions across all modalities, as evidenced by having percentages less than five. The percent of interactions of cognitive task engagement was similar for face-to-face (38.9%), hybrid (36.7%), and synchronous online (35.3%), but lower for pre-recorded videos (22.2%). Teachers engaged in explanations more frequently in face-to-face instruction (10.0%) than in the other modalities. Further, the percentage of expository interactions was the highest for pre-recorded videos (55.6%) and similar for face-to-face (21.4%), hybrid (28.8%), and synchronous online (28.0%). Finally, pre-recorded videos had the most procedural interactions (14.3%), followed by synchronous online (8.2%).

Similarly, of the learner-content interactions (n=496), there was a significant interaction between the type of instruction and Level 2 interaction types ($\chi^2(9)=99.842$, $p<0.001$).

Confirmation, or checking work, was most prevalent in pre-recorded videos. In fact, 25% of the learner-content interactions for pre-recorded videos were teachers having students confirm correct answers. No pre-recorded video had students conjecture. However, conjecturing was present in hybrid (28.2%), face-to-face (47.0%), and synchronous online (67.0%). Teachers very rarely had students make predictions, there was only one such learner-content interaction, and that was on a pre-recorded video. Finally, students solved a problem most often in hybrid or pre-recorded instruction (70% for each) but also had opportunities to solve problems in face-to-face (50.2%) and synchronous online (29.9%) instructional settings.

Conclusion

In this study, teachers tend to prefer to engage students in their classrooms via learner-instructor interactions. This is shown through this interaction mode utilized in 74.31% of the interactions. However, teachers also tended to rely heavily on asking questions about the material, lecturing, and answering student questions. This implies that teachers utilize whole class interaction patterns such as the I-R-E (initiation-response-evaluation) (Vogler et al., 2018), where they provide information to solicit a response, receive a response from the student, and then react to those responses to either offer more content or ask a follow-up clarification question. This whole class interaction model requires real-time feedback not available during pre-recorded lessons. The significant association between the type of instruction and the level 2 learner-instructor interaction types reflects that teachers utilized specific methods to cater to the modality of instruction. Teachers could not enact the whole class interaction patterns when using a pre-recorded video, so they relied heavily on exposition to convey information. Alternatively, once students could interact in real-time with teachers, students' responses were evaluated through explanatory interactions. Not being able to interact directly with students also probably led to an increase in having to provide procedural interactions so that the students could navigate the course materials and remain aligned with the course schedule. Once teachers and students occupied the same space, procedural interactions were less commonplace.

Though whole class interactions are a common practice, there were deviations in a quarter of the interactions to promote a more student-centered approach. Teachers allowed students to engage in the content to either perform mathematics (solve a problem 52.02%) or articulate an opinion about the nature of the content (conjecture 44.15%). While these interaction types were prevalent, there were still significant differences based on the type of instruction. There were more interactions focused on students generating correct answers in pre-recorded videos and fewer interactions focused on formulating opinions or conclusions.

While differences in the interactions that took place when teachers were teaching online or in a hybrid format were noted, future research is needed to understand better why teachers made the choices they did when teaching in these different settings. Several factors may have contributed to these differences - local policies, familiarity with technology, professional development, beliefs about mathematics teaching, time to transition courses to an online environment, and classroom management concerns, to name a few. Answers to this question may enable mathematics teacher educators to better prepare prospective and practicing teachers to support students' engagement through different types of interactions while learning online.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1852837. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Battalio, J. (2007). Interaction online: A reevaluation. *The Quarterly Review of Distance Education*, *8*(4), 339–352.

Campbell, P. B., & Storo, J. (1996). Reducing the distance: Equity issues in distance learning in public education. *Journal of Science Education and Technology*, *5*(4), 285–295. https://doi.org/10.1007/BF01677125

Chou, C., Peng, H., & Chang, C.-Y. (2010). The technical framework of interactive functions for course-management systems: Students' perceptions, uses, and evaluations. *Computers & Education*, *55*(3), 1004–1017. https://doi.org/10.1016/j.compedu.2010.04.011

Corso, M. J., Bundick, M., Quaglia, R., & Haywood, D. (2021). *Where student, teacher, and content meet: Student engagement in the secondary school classroom*. 13.

Garrison, D. R., & Cleveland-Innes, M. (2005). Facilitating cognitive presence in online learning: Interaction is not enough. *American Journal of Distance Education*, *19*(3), 133–148. https://doi.org/10.1207/s15389286ajde1903_2

Gunawardena, C. N., Lowe, C. A., & Anderson, T. (1997). Analysis of a global online debate and the development of an interaction analysis model for examining social construction of knowledge in computer conferencing. *Journal of Educational Computing Research*, *17*(4), 397–431. https://doi.org/10.2190/7MQV-X9UJ-C7Q3-NRAG

Hillman, D., Willis, D., & Gunawardena, C. (1994). Learner-interface interaction in distance education: An extension of contemporary models and strategies for practitioners. *American Journal of Distance Education*, *8*(2), 30–42.

Hirumi, A. (2002). A framework for analyzing, designing, and sequencing planned elearning interactions. *Quarterly Review of Distance Education*, *3*(2), 141.

Lin, C.-H., Zheng, B., & Zhang, Y. (2017). Interactions and learning outcomes in online language courses: Online interactions and learning outcomes. *British Journal of Educational Technology*, *48*(3), 730–748. https://doi.org/10.1111/bjet.12457

Moore, M. G. (1989). Editorial: Three types of interaction. *American Journal of Distance Education*, *3*(2), 1–7. https://doi.org/10.1080/08923648909526659

Offir, B., Barth, I., Lev, Y., & Shteinbok, A. (2003). Teacher–student interactions and learning outcomes in a distance learning environment. *The Internet and Higher Education*, *6*(1), 65–75. https://doi.org/10.1016/S1096-7516(02)00162-8

Vogler, A.-M., Prediger, S., Quasthoff, U., & Heller, V. (2018). Students' and teachers' focus of attention in classroom interaction—Subtle sources for the reproduction of social disparities. *Mathematics Education Research Journal*, *30*(3), 299–323. https://doi.org/10.1007/s13394-017-0234-2

DESIGNING AN ONLINE VIDEO-BASED ENVIRONMENT FOR PROMOTING MATHEMATICAL ARGUMENTATION

Ho-Chieh Lin
The Ohio State University
Lin.2532@osu.edu

During the past two years, the COVID-19 pandemic has forced teachers to shift their instruction online, further exacerbating the challenges for teachers in orchestrating rich mathematics discussions. To tackle this issue, this study explored an approach that integrates online video-sharing culture (e.g., YouTube) into mathematical practices. Specifically, I designed an environment to engage children in creating and sharing mathematics how-to videos online for subsequent peer discussions. Guided by design research principles, I recruited four upper elementary children in the United States. The analysis of this qualitative study showed that the environment created offered unique opportunities for the children to communicate their ideas in a multimodal way and engage in mathematical argumentation. The findings provide insights into how an online environment can be designed to reshape children's argumentative discourse.

Keywords: Design Experiment; Instructional Activities and Practices; Online and Distance Education; Reasoning and Proof

Mathematical argumentation, a type of discourse that involves communication of mathematical ideas with reasoning, is central to the development of mathematical knowledge (Conner et al, 2014). This discourse fosters greater conceptual understanding (Wood et al., 2006) and advances academic achievement (Chapin et al., 2009; Kazemi & Stipek, 2001). Although benefits of mathematical argumentation are evident, enacting such practices in classrooms challenges many teachers (Conner et al., 2014). This problem has been exacerbated during the past two years. The COVID-19 pandemic forced in-person classes to be conducted remotely, while most teachers have little experience or support with how to engage children in mathematics discussions online. Indeed, the literature on mathematical argumentation in fully online contexts is relatively scarce as compared to that of in-person contexts. By designing a video-based online learning environment, this study addresses the necessity to investigate ways to engage learners in mathematical argumentation.

Video technologies have been used for mathematical teaching and learning in various ways, such as anchored instruction (Cognition and Technology Group at Vanderbilt, 1992) and flipped instruction (De Araujo et al., 2017). However, little is known about how videos produced by learners themselves can be leveraged for promoting mathematical argumentation. Indeed, research that explores the use of video production as a means for mathematics learning is just emerging. In a recent study that investigated how middle school students collaboratively produced mathematics tutorial videos, Oechsler and Borba (2020) found that this activity changed the classroom dynamic and engaged students in communicating mathematical ideas with multiple modes (e.g., language, gesture, image, and music). They argued that video production is a new way of expressing mathematics and its multimodal affordance prompts students to reorganize their thinking. Their findings suggest video production has the potential to foster children's mathematics discourse. It is therefore reasonable to further investigate ways to design an environment involving video production activities for promoting mathematical argumentation.

This study explored an approach that integrates children's online video-sharing culture (e.g., YouTube and TikTok) into a problem-based learning model. Specifically, this approach uses a process during which children solve math problems, create online videos to explain their strategies, and then discuss strategies after watching peers' videos. I hypothesize that when learners use templates from online video platforms to communicate their mathematical ideas, they likely attend to the validity of mathematical statements and thus engage in mathematical argumentation. The proposed approach allows an investigation into the development of an online video-based learning ecology and children's mathematics discourse. The guiding research questions are: (a) *How can an online video-based environment be designed to promote mathematical argumentation?* (b) *In what ways do children participate in mathematical argumentation in the designed environment?*

Theoretical Framework

This study is framed by the emergent perspective (Cobb & Yackel, 1996) and humans-with-media perspective (Borba & Villarreal, 2005). The emergent perspective is a version of social constructivism that attempts to explore the complementary area between sociocultural and constructivist constructs. The premise of this perspective is that learning is a constructive process that emerges when learners participate in and contribute to socially organized activities (Cobb & Yackel, 1996). According to Cobb (2007), the emergent perspective agrees with the constructivist view of learning as an adaptive process; however, it is informed by a distributed cognition view of learning as a social activity supported by cultural tools and thus expands the meaning of a learning activity. Cobb et al. (2001) argued that the tools and symbols used in the contexts are not additions to but constituent parts of learners' activity. Therefore, this view connects with the notion of semiotic mediation in the sociocultural theories, and learner's engagement in mathematics activities entails reasoning with tools and symbols (Cobb, 2007).

The idea of reasoning with tools and symbols in the emergent perspective aligns with the premise of humans-with-media perspective (Borba & Villarreal, 2005). That is, collective efforts of human (e.g., students, teachers) actors and non-human actors (e.g., computers, mobile phones) contribute to the reorganization and production of human knowledge. In this perspective, human and media technologies influence and shape each other through interaction, negotiation, and reflection. As claimed by Borba and Villarreal (2005), discourse supported by digital media is qualitatively different from oral or written language, primarily because it involves a combination of visual and audio information, such as images and music. It is therefore reasonable to assume this type of discourse can shape human thinking in a distinct way.

In this study, participants' discourse and learning are mediated by online videos they themselves created. When producing a video, a participant coordinates their verbal explanations and visual representations in a coherent way. This requires them to externalize and reorganize their mathematical ideas. When viewing a video, a participant attempts to make sense of a mathematical strategy through the verbal explanations and visual representations in the video. These online interactions are therefore multimodal through technologies, which distinguishes it from typical classroom interaction. I posit that, with an appropriate design of activities, video-based discourse has the potential to drive learners to attend to the validity of mathematical statements, leading them to productive mathematical argumentation.

To develop a video-based environment for promoting mathematical argumentation, it is important to understand how learners engage in communicating their mathematical thinking with technologies. Since this process involves collective efforts of the designer, participants, and technologies, taking the emergent and humans-with-media perspectives together enables me to conceptualize discourse and learning that occur during human-media interaction.

Methodology

In this qualitative study, I aimed to develop an environment and learning theories for video-based mathematical argumentation in an online environment. I adopted principles of design research (Cobb et al., 2003) because using video production as a means for learning mathematics has not been an established practice in typical classrooms (Oechsler & Borba, 2020). Therefore, these principles make this study interventionalist and characterized by an iterative cycle of design, enactment, analysis, and refinement. In addition, I drew on the conjecture mapping technique (Sandoval, 2014) for conceptualizing and carrying out my design. Figure 1 shows a conjecture map I have developed at the current stage of the research design process. My design conjecture is that if learners participate in the discursive practices guided by an online video-based activity structure, then the mediating processes will emerge. My theoretical conjecture is that if these mediating processes take place, they will bring about the desired learning outcomes-that is, constructing viable arguments and reorganizing mathematical ideas. The proposed activity structure will be introduced in the next section.

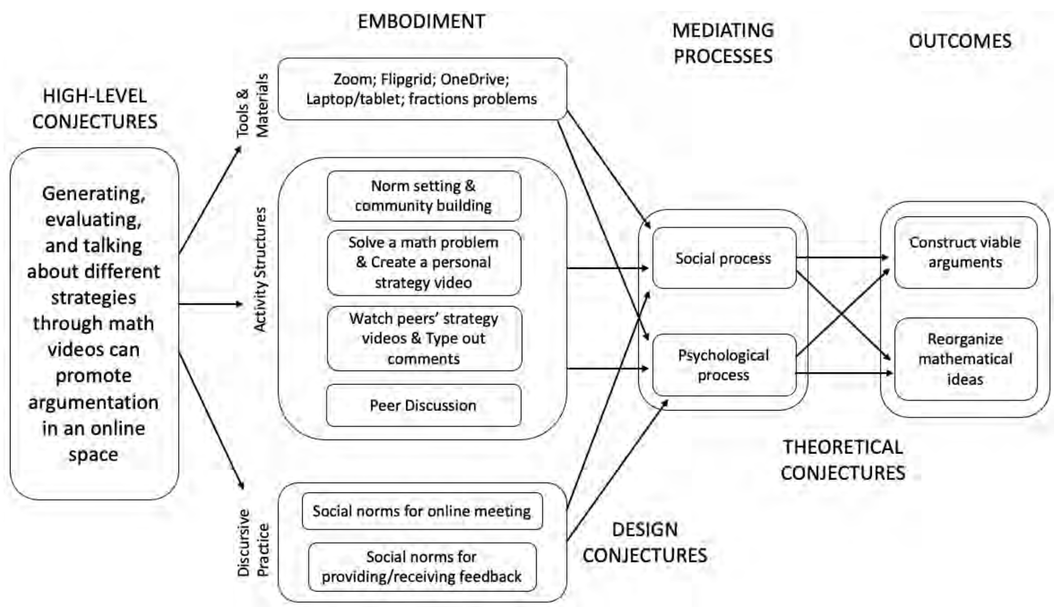


Figure 1. Conjecture map for conceptualizing this design study.

VBAS Environment

Currently, I have developed a four-phase video-based activity structure (VBAS) that aims to promote mathematical argumentation for small groups in an online space. This VBAS environment involves participants using three online technological tools: Zoom, OneDrive, and Flipgrid. Particularly, Flipgrid is a video platform that allows users to produce personal videos and disseminate them in a designated virtual community for viewing. Supported by these technologies, learners participate in a sequence of four activities: (a) setting up norms and building community, (b) independently solving a math problem and creating a strategy video, (c) watching strategy videos and taking notes, (d) discussing the strategies with a partner. Following this design, I worked with a pair of participants at a time. Since I situated my design in a problem-based model, each cycle of research sessions started with a problem-solving task. The mathematics problems employed in this study were about fraction comparisons. An example is shown in Figure 2.

> Imagine that you were invited to a home party. You saw the same pies being served at three tables of your friends: one table serves two pies with two people, another table serves three pies with three people, and the third table serves five pies with six people. Friends at each of the tables call out to you to pull up a chair and share the pies. You are to join one of the tables. Is there any helpful way that you can use math to help with your decision-making?

Figure 2. Fraction comparison problem used during a design cycle.

Rationale of the Designed Environment

The proposed environment features video production activities. When children produce a mathematics video, they realize that their explanations need to make sense to themselves first so that the viewers can understand. Such a self-explanation process can help individuals modify their existing knowledge with new information and promote their problem-solving skills (Chi et al., 1994). This may also help the children see themselves as capable mathematical thinkers.

Participants & Settings

This study is part of a larger project examining children's video-based mathematics discourse in an online environment. I recruited two fifth graders and two sixth graders in the United States, who were female Asian Americans. The participants were grouped into pairs since peer discourse is the focus of this study. During each research session, I met with one pair remotely via Zoom as they used their own computers or tablets at home. Each pair participated in a series of 60–90-minute sessions. In addition, I conducted semi-structured interviews with each participant after they completed the research sessions.

Data Collection & Data Analysis

Data collected for this study came from multiple sources: (a) video recordings of each Zoom session, (b) online math videos created by participants, (c) feedback on videos typed by participants, and (d) the researcher's memo.

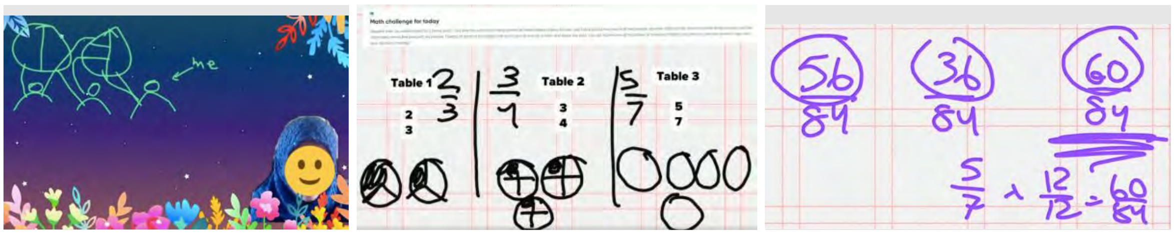
To analyze the data, I adopted the constant comparison method (Corbin & Strauss, 2014), which enables me to identify themes of significant phenomena. My analysis involved examining mathematics videos created by the participants and video recordings of research sessions to highlight and transcribe episodes in which participants generated mathematical arguments.

Results

Here I present some findings from a research cycle of VBAS that employed the problem introduced in Figure 2 and involved a pair of sixth graders, named Alissa and Nancy. Prior to this cycle, the pair had participated in two cycles of research sessions. The analysis here focuses on how the children engaged in three VBAS activities: (a) problem solving and strategy video production, (b) video viewing & commenting, and (c) peer discussion.

Problem Solving & Strategy Video Production Activity

This activity asks participants to independently solve a math problem and create a short video, less than two minutes, in which they explain their strategies. The goal is to activate their existing math knowledge and construct viable argument. As shown in Figure 3, Nancy and Alissa used different strategies in solving the problem that involves comparing fraction magnitudes. Nancy only considered the situation in the first table of the problem. She first cut each whole into halves, creating four halves. Then she distributed each half to each person. Next, she cut the remaining half into three smaller parts and shared them with three people. In contrast, Alissa considered all the situations. On the first and second table, she divided each whole into equal parts based on the number of sharers, and then each person would take one part from each whole. However, she stopped using this strategy for the third table. She probably noticed the drawing would not help her compare the magnitudes, so she switched to a more formal strategy-that is, first finding a common denominator for the three fractions, next converting all fractions to share the common denominator, and finally comparing the numerators.



(a) Nancy's work. (b) Alissa's work-part 1. (c) Alissa's work-part 2.

Figure 3. Screenshots of two participants' strategy videos.

In addition, Nancy and Alissa used different features afforded by Flipgrid to represent their video contents. Nancy's video was stylish as she put herself in the corner with a flowery frame and a starry night scene as background. In contrast, Alissa's video is more simplistic, focusing on the mathematical representations. Moreover, the pair used different background music.

Video Viewing & Commenting Activity

The third activity asks participants to individually watch each other's video twice and jot down their thoughts in a table with three sentence starters on OneDrive. The goal is to prompt a participant to examine their partner's strategy video carefully or even critically. Figure 4 displayed Nancy's contributions.

In the "I notice…" row, Nancy expressed her observation of the difference between Alissa's strategy and hers, claiming Alissa's strategy was complicated and her speech pace was fast in the video. In addition to the comments on the strategy video, Nancy stated that she understood Alissa had limited time to explain such a complicated method.

In the "I wonder…" row, Nancy expressed her inquiry about why Alissa kept using this strategy. This inquiry probably comes from the fact that Alissa had used this strategy during the previous two research cycles.

In the "I would suggest…" row, Nancy suggested Alissa try a different strategy that would make sense to lower graders. The reason why Nancy offered this suggestion is worth further investigation.

I notice…	that she is using factions in the more complicated way, and she is going fast (I understand this though, she had limited time)
I wonder…	why she always uses fractions with inequivalent denominators then changes them to common denominator fractions
I would suggest	to try a different technique, like, using a less complicated way for maybe 5th or 4th graders?

Figure 4. Screenshot of Nancy' comments.

Peer Discussion Activity

During this activity, participants' notes are screenshared, and they are encouraged to talk about their comments with their partner and respond to each other. The goal is to see if the pair would engage in mathematical argumentation. Presented below is an episode in which Nancy first talked about her comments. Note that Alissa did not show her face during this interaction.

Nancy: you always use like fractions like that, you find fractions with unequivalent denominators, and you find them equivalent denominators. How do you always… how do you always find a way to do that?

Alissa: The reason I think is because like, it's easier to compare that way and how I like, what I do to find the multiples that sometimes I'll like, multiply the number by like the other number to get multiple. Or like, I try to find, like, more like the least common multiple and I...

Nancy: How do you find a number… How do you find a number to multiply with?

Alissa: For example, like, if I try to find the common multiple of three and seven, what I would do is, I like, I do three times seven. So I would find 21.

In the above vignette, Nancy first described what she noticed about Alissa's consistent use of an algorithm that converted fractions with unlike denominators into fractions with the same denominators. She asked Alissa how she could always identify a number (i.e., common multiple) that she could convert different denominators to. Alissa first responded with a general rule, but it seemingly did not fully convince Nancy. Therefore, Nancy rephrased her question in a more specific way, and then Alissa described the rule by using an example. Their conversations continued as follows.

Nancy: But do you always like a certain technique or something? Like, try to find something they'll have in common or something?

Alissa: Oh, yeah. So like, sometimes, like, what I'll do is that I'll also try to find if I think that maybe like a number is like, maybe like there's a smaller number than that. What I tried to do with that, uhh…I'll..umm

This vignette demonstrated that Nancy went back to her original inquiry about Alissa' consistent use of such method. Alissa probably did not think about the question before because her response revealed some level of hesitance. Speculating that writing down her thoughts might help process her thinking, I thus intervened and asked if she would like to share her screen and use the Zoom whiteboard. Alissa seemed to agree, so she shared her screen immediately. She then began writing down mathematical symbols on the whiteboard while explaining a different method (see Figure 6).

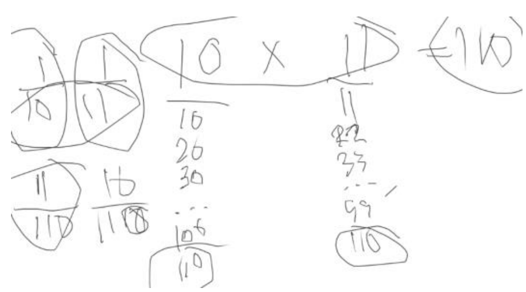


Figure 5. Alissa's writing on the whiteboard.

Alissa: [shares her screen immediately] So like, let's say I have like the numbers 10 and maybe 11. So what I would do is I would list maybe the multiples until, or what I would do is [talking while writing on the board] I would go like that and then 110, or what I would do is that I'll go 100. And like I will go umm so times one it's 10, times two, times three, and then so on and then 100 and then 110. And then I would go to 11. And then I would go like times one, times two, times three and then I don't know, then like 99, and then I get…

Nancy: 110. One thing that both have in common, okay.

Alissa: Yeah, something like that.

Nancy: So go multiply them until you find a way to find the number, okay.

Alissa: Yeah, like, so I don't usually do that. I just usually will be…uh… I'll like multiply these numbers together to get this number [circling numbers on the board].

The above vignette showed that writing might help Alissa externalize her thinking. She proposed another example and explained another way to find a common multiple for two different numbers. This method made sense to Nancy as she replied with an answer to Alissa's example and replied with her interpretations. After talking about the rule, Nancy inquired about Alissa's decision of using the rule.

Nancy: Um, okay. But why do you always use this one way?

Alissa: Because like, let's say I want to compare like 1/10 and 1/11. So they're not equal [circling 10 and 11]. I wouldn't be able to properly compare them. And like just looking at them and then like maybe deciding, I think 1/11 is bigger, really? Or 1/10 is bigger, then so it's better to go like 11 over 110, and then 10 [writes 10/110]. This one is bigger [circling 11/110].

In this vignette, Nancy 's question had potential to shift their discussion from procedure-focused to concept-focused. However, Alissa did not articulate the underlying concepts. As demonstrated in her response and Figure xx, she applied her previous example to a fraction case and showed how the algorithm worked. Nancy seemed to accept Alissa's response and proposed her suggestions as follows.

Nancy: Ok. So, umm…So if you had to explain this problem to maybe a fourth grader, how, would you use a different technique or use the same technique? with more explanation?

Alissa: I would use the same technique with more explanation. So that they would understand better because …yup!

The above episode indicates that Nancy and Alissa engaged in asking questions and justifying their ideas with limited intervention from the facilitator. Particularly, Nancy's questioning showed that she attended to Alissa's responses, and her questions prompted Alissa to develop a more comprehensive account for finding a common multiple for denominators when comparing fraction magnitudes. However, their discussions were focused on procedures. This finding is evident in a few other episodes when participants talked about their strategies toward other fraction comparison problems.

Discussion

This study investigated how an online video-based environment can be designed to support mathematical argumentation and the way children engage in mathematical argumentation in such an environment. The results indicate that the proposed environment created unique opportunities for the participants to communicate their mathematical thinking in multiple modes and engage in mathematical argumentation. The video production activity involves a multimodal process of constructing viable mathematical arguments as the children individually coordinated their verbal explanations and relevant visual representations. Moreover, the video viewing and commenting activity encouraged the children to carefully examine their partner's arguments. Since the children were asked to type comments based on their observation, they would re-watch the part they did not understand and reorganize their thinking. This process is also multimodal as the children attended to the mathematical ideas that were verbally and visually transmitted. Furthermore, the peer discussion activity allowed the children to talk about their ideas to each other in real time. Because the previous two activities have engaged participants in thinking through their arguments and their partners' arguments, they may come to this activity with inquiries or revised ideas. Therefore, a productive mathematical argumentation like the one presented in the results section could likely happen, and in humans-with-media terms (Borba & Villarreal, 2005), the videos play an acting role as important as human participants during this discourse.

The video production activity that involves participants designing math videos based on their own strategies is featured in this study. This process involves a great deal of autonomous problem-solving and decision making since participants need to figure out affordances of the video platform, activate their existing mathematics knowledge, and arrange presentation contents in a way that makes sense to themselves and their peers. In addition, the participants often added pleasant backdrops and cheerful music to their videos, which makes them appealing to the viewers. In this sense, the math videos are not merely pertaining to mathematics, but also communication and personal identity.

Much of existing literature on mathematical argumentation focuses on teacher's moves, such as questioning (e.g., Conner et al., 2014). In contrast, this study attempted to lessen the need for the facilitator to directly guide children's discourse by proposing a sequence of learner-centered activities. The VBAS design offers scaffolds to prepare children to lead their discourse by themselves. This way of positioning children as capable agents is different from that of using adult-created videos for flipped classrooms or using ready-made dialogic videos for vicarious learning, which usually involve obedience-oriented ways of problem-solving strategies and explanations. Indeed, the proposed design encourages children to not only engage in video-based discourse but also come to take ownership of their mathematical learning.

While the findings seem promising, there are limitations to this study. For instance, the participants of this study were upper elementary children who knew each other, had some experiences with video production, and felt comfortable sharing their mathematical ideas online. It is unknown how learners with different backgrounds engage in the proposed environment. Additionally, this study has not analyzed how the children's knowledge on fraction comparison developed over time throughout the sessions. These are important areas that need further research in order to understand how to better support children's mathematical argumentation in an online space.

Acknowledgment

This material is based upon work supported by 2021 Ministry of Science and Technology Taiwanese Overseas Pioneers Grants (TOP Grants). Any opinions, findings, and conclusions or recommendations stated here are those of the author and do not necessarily reflect the views of the Ministry of Science and Technology in Taiwan.

References

Borba, M. C., & Villarreal, M. E. (2005). *Humans-with-media and the reorganization of mathematical thinking: Information and communication technologies, modeling, visualization and experimentation* (Vol. 39). Springer Science & Business Media.

Chapin, S. H., O'Connor, C. and Anderson, N. C. 2003. *Classroom discussions: Using math talk to help students learn, grades 1–6*, Sausalito, CA: Math Solutions.

Chi, M. T., De Leeuw, N., Chiu, M. H., & LaVancher, C. (1994). Eliciting self-explanations improves understanding. *Cognitive science*, *18*(3), 439-477.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple theoretical perspectives. In F. Lester (Ed.), *Handbook of research on teaching and learning mathematics* (2nd edition). Greenwich CT: Information Age Publishing.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher*, 32(1,), 9–13.

Cobb, P., Stephan, M., McClain, K., & Gravemeijer, K. (2001). Participating in classroom mathematical practices. *Journal of the Learning Sciences, 10*(1-2), 113-163.

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational psychologist*, *31*(3-4), 175-190.

Cognition and Technology Group at Vanderbilt. (1992). The Jasper series as an example of anchored instruction: Theory, program description, and assessment data. *Educational Psychologist*, *27*(3), 291-315.

Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429.

Corbin, J., & Strauss, A. (2014). Basics of qualitative research: Techniques and procedures for developing grounded theory. *Sage publications*.

De Araujo, Z., Otten, S., & Birisci, S. (2017). Conceptualizing "Homework" in Flipped Mathematics Classes. *Educational Technology & Society, 20* (1), 248–260.

Kazemi, E. and Stipek, D. 2001. Promoting conceptual thinking in four upper-elementary mathematics classrooms. *Elementary School Journal*, 102(1): 59–80.

Oechsler, V., & Borba, M. C. (2020). Mathematical videos, social semiotics and the changing classroom. *ZDM*, 52(5), 989–1001.

Rasmussen, C., & Stephan, M. (2008). A methodology for documenting collective activity. *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics learning and teaching*, 195.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the learning sciences*, *23*(1), 18-36.

Toulmin, S. E. (1958/2003). *The uses of argument* (updated ed.). New York: Cambridge University Press. Original work published 1958.

Woods, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for research in mathematics education*, *37*(3), 222-255.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for research in mathematics education*, *27*(4), 458-477.

RELACIÓN ENTRE EL RAZONAMIENTO COVARIACIONAL Y LOS ESQUEMAS DE USO EN UNA TAREA DE APROXIMACIÓN

RELATIONSHIP BETWEEN COVARIATIONAL REASONING AND USE SCHEMES IN AN APPROACH TASK

Mihály A. Martínez-Miraval
Pontificia Universidad Católica del Perú
martinez.ma@pucp.edu.pe

Martha L. García-Rodríguez
Instituto Politécnico Nacional
mlgarcia@ipn.mx

El estudio realizado tiene por objetivo analizar el cambio en las acciones mentales de un estudiante asociadas con los esquemas de uso que moviliza al resolver una tarea de aproximación al área de una región con GeoGebra. Se consideraron elementos teóricos relacionados con el enfoque del razonamiento covariacional y con el enfoque instrumental. El estudiante, luego de generar, movilizar y refinar sus esquemas de uso, hizo visibles comportamientos que se asocian con acciones mentales, como dibujar un polígono para aproximar un área y reconocer que, a más puntos sobre la gráfica, la distancia entre estos es menor y se obtiene una mejor aproximación. Es decir, los comportamientos observables pueden ser asociados con niveles de razonamiento o bien con cambios en las acciones mentales asociadas, que indican un tránsito de un nivel de razonamiento a otro.

Palabras clave: Resolución de Problemas; Tecnología; Razonamiento y Demostraciones

En la literatura, el razonamiento covariacional se identifica como una habilidad esencial para la comprensión de conceptos de Cálculo, como es el caso de la integral definida (Carlson et al., 2002), contar con esta habilidad significa entender las coordinaciones que se dan entre los valores de dos o más variables que se ponen en juego al resolver un problema (Thompson y Carlson, 2017). Kouropatov y Dreyfus (2014) manifiestan que el concepto de integral se puede desarrollar a partir de una estructura jerárquica de cuatro elementos, donde, la aproximación al área de una región con objetos geométricos es el primer nivel de dicha estructura.

Tratar de que los estudiantes relacionen el concepto de integral definida con el área bajo la curva, promueve la construcción de conocimiento matemático y de significados de los conceptos involucrados en la resolución de problemas, estas construcciones se pueden apoyar en el uso de tecnologías digitales (Santos-Trigo y Aguilar, 2018).

Las tecnologías digitales permiten experimentar con cantidades que cambian dinámicamente en tiempo real (Carlson et al., 2002). En palabras de los autores: “estas tecnologías ofrecen valiosas herramientas para (…) aplicar el razonamiento covariacional y analizar e interpretar funciones asociadas con situaciones dinámicas” (Carlson et al., 2002, p. 374). GeoGebra es un ejemplo de un sistema de geometría dinámica que, aparte de dar movimiento a las construcciones realizadas, genera un entorno de aprendizaje interactivo y visual que permite explorar cómo se relacionan las variables con la solución geométrica (Zengin, 2018).

Para conocer sobre el razonamiento covariacional que ponen en juego los estudiantes cuando utilizan GeoGebra se realizó una investigación que tuvo como objetivo *analizar el cambio en las acciones mentales de un estudiante asociadas con los esquemas de uso que moviliza al resolver una tarea de aproximación al área de una región con GeoGebra.*

Aproximación conceptual

Se consideraron elementos teóricos relacionados con el enfoque del razonamiento covariacional propuesto por Thompson y Carlson (2017) y con el enfoque instrumental desarrollado por Rabardel (1995).

Las imágenes de covariación funcionan como vehículos de operaciones mentales, expresadas a través de representaciones e impresiones de un individuo en un contexto dado, y como estas evolucionan, se pueden categorizar por niveles que surgen de forma ordenada (Carlson et al., 2002), y la evidencia de tal evolución se manifiesta en la forma en que una persona coordina los cambios en los valores de dos cantidades (Saldanha y Thompson, 1998). Una imagen de covariación está sustentada en las acciones mentales que una persona realiza, y estas pueden asociarse con el conjunto de comportamientos observables en un sujeto que resuelve un problema (Carlson et al., 2002). El constructo teórico de Thompson y Carlson (2017) consta de seis niveles de razonamiento covariacional, en este trabajo se consideran del segundo al cuarto (tabla 1) y se asocian con acciones mentales correspondientes a cada nivel en el sentido descrito por Carlson et al. (2002).

Tabla 1: Niveles de razonamiento covariacional. Adaptado de Thompson y Carlson (2017)

Niveles	Descripción
Coordinación de valores	AM4: Coordina los valores de una variable (x) con los valores de otra variable (y) con la anticipación de crear una colección discreta de pares (x, y).
Coordinación gruesa de valores	AM3: Forma una imagen general de los valores de las cantidades que varían juntos, como "esta cantidad aumenta mientras que la cantidad disminuye".
Precoordinación de valores	AM2: Imagina que los valores de dos variables varían, pero de forma asincrónica: una variable cambia, luego la segunda variable cambia, luego la primera, y así sucesivamente.

Por otro lado, Rabardel (1995) indica que los esquemas de uso se relacionan con la ejecución de tareas de índole funcional que pueden tener objetivos propios y que están relacionadas con el reconocimiento de características, propiedades y actividades específicas de GeoGebra. En este trabajo se realizaron tareas como dibujar un polígono para aproximar un área, determinar su área, hacer zoom para refinar una aproximación, entre otras. Los estudiantes tenían conocimiento de diversas herramientas de GeoGebra: función, polígono, área, zoom, utilizadas para otros temas y otros contextos, por lo que ya habían generado esquemas de uso de ellas, por lo que, en algunos casos los movilizan o refinan y, en otros, generan nuevos.

Métodos y procedimientos

En este documento se presenta el trabajo de un estudiante de los primeros ciclos universitarios al que llamaremos Juan, quien contaba con 17 años de edad. Las evidencias obtenidas fueron recolectadas mediante documentos escritos, archivos digitales y una entrevista semiestructurada que fue videograbada, así como reportes realizados por el investigador. Juan contaba con los antecedentes de haber trabajado con funciones reales de variable real, había estudiado el tema de límites, reconocía la continuidad de una función a partir de su gráfica, o de forma analítica al utilizar la definición y había realizado el cálculo del área de figuras geométricas. La tarea diseñada incluyó dos ítems y está relacionada con el concepto de integral definida mediante el área de polígono como una aproximación al área que se forma bajo la gráfica de una función y el eje X en un intervalo cerrado. El estudiante trabajó esta tarea en GeoGebra, mediante un enlace que les fue proporcionado para ello.

La figura 1 muestra una región R limitada por la gráfica de una función f y el eje X para $x \in [0; 8]$. Utilice el enlace para sus exploraciones: https://www.geogebra.org/m/mmcyf6pr.

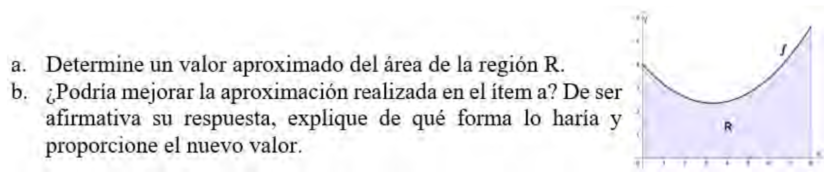


Figura 1: Aproximación con polígonos

Análisis y discusión

El estudiante movilizó su **esquema de uso polígono** para trabajar en la tarea y lo utilizó para dar una primera aproximación del área de la región R. Para el inciso (a), Juan mencionó que dibujó un primer polígono con 5 puntos sobre la gráfica de la función y al observarlo se percató de que las regiones del polígono que exceden a la región R se podían reducir al colocar más puntos sobre la gráfica de la función, por lo que dibujó un nuevo polígono con 9 puntos. Determinó su área movilizando su **esquema de uso área** y respondió al ítem (a). Cuando Juan dibuja un polígono y coloca puntos sobre la curva, se hacen visibles un conjunto de variables: (i) número de puntos sobre la gráfica de la función (vértices del polígono), (ii) distancia entre dos puntos consecutivos (lados del polígono), (iii) área del polígono y (iv) áreas excedentes.

Al visualizar los valores de las variables número de puntos y áreas excedentes muestra comportamientos de una AM2, coloca un número de puntos y observa las áreas excedentes, coloca más puntos y observa nuevamente las áreas excedentes, es decir, de acuerdo con Thompson y Carlson (2017), relaciona estas variables de manera asincrónica. El hecho de variar el número de puntos para aproximar el área solicitada es evidencia de que a Juan no le pareció suficiente colocar 5 puntos y decidió aumentar el número a 9, con lo que se aproxima mejor al área de la región R, rasgos asociados con una AM3.

La evidencia encontrada sugiere que Juan primero movilizó el **esquema de uso polígono** e hizo visible una **AM2** cuando relacionó las variables número de puntos y áreas excedentes. El movilizar nuevamente el **esquema de uso polígono** y dibujar más puntos se asocia con rasgos de una AM3, y también induce la movilización del **esquema de uso área**, con lo que enfoca su atención en una nueva variable que es el área del polígono (figura 2).

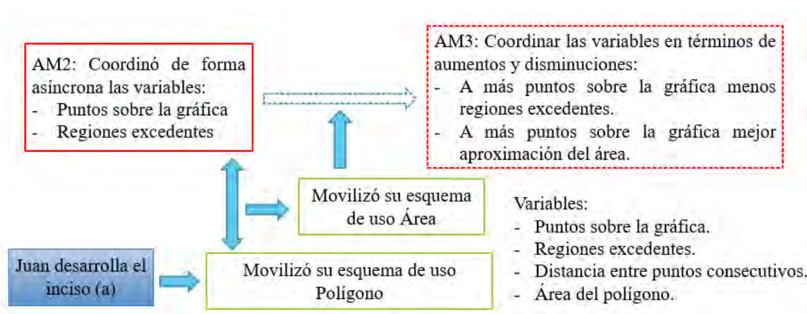


Figura 2: Relación entre las acciones mentales y la movilización de esquemas de uso

Para el inciso (b), Juan explicó que dibujó un polígono con 12 puntos para obtener una mejor aproximación, con lo que generó un **esquema de uso partición**, ya que Juan había identificado que al colocar más puntos la distancia entre ellos disminuía y se lograba una mejor aproximación, aquí Juan había puesto en juego una nueva variable distancia entre dos puntos consecutivos. Juan identificó que el aumento del número de puntos corresponde a una mejor aproximación del área de la región R y a la disminución de la distancia entre puntos consecutivos, comportamientos que corresponden a una AM3. También refinó su **esquema de uso partición** al querer aumentar el número de puntos e identificar que, para hacerlo, necesitaba modificar la escala de la gráfica utilizando la herramienta Zoom de GeoGebra. Las conjeturas de Juan las realiza a partir de la modificación del número de puntos y del área del polígono que se forma con ellos, al parecer esto lo llevó a crear una colección discreta de pares que relacionan estos valores, que es un rasgo de un comportamiento asociado con una AM4.

Conclusiones

El aporte de este trabajo al campo de la Matemática Educativa es el uso de dos aproximaciones teóricas: el razonamiento covariacional de Thompson y Carlson (2017) y el enfoque instrumental de Rabardel (1995), para generar un modelo explicativo de la evolución del razonamiento covariacional de Juan y su interrelación con los esquemas de uso cuando trabaja en una tarea con GeoGebra.

En este modelo los elementos teóricos de cada enfoque quedan entrelazados y en momentos parece que actuaran de manera simultánea. Al mismo tiempo que se pone en juego un esquema de uso, se cambia a otra acción mental, por lo que resulta difícil distinguir qué ocurre primero.

Se identifican comportamientos que ya no están asociados con una acción mental específica, por lo que pensamos que están en un tránsito de una acción mental a otra.

Agradecimientos

A la línea de investigación Tecnologías y Visualización en Educación Matemática del Instituto de Investigación en Educación Matemática IREM-PUCP de la Pontificia Universidad Católica del Perú.

Al Instituto Politécnico Nacional, CICATA Legaria a través de los proyectos de investigación 20201191 y 20210725 por el apoyo brindado para llevar a cabo esta investigación.

Referencias

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S. & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal for Research in Mathematics Education, 33*(5), 352–378.

Kouropatov, A. & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *ZDM Mathematics Education, 46* (4), 533–548.

Rabardel, P. (1995). Les hommes et les technologies: aproche cognitive des instrumentns contemporains. Paris: Armand colin.

Saldanha, L. A. & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berenson & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America* (Vol 1, pp. 298-304). Raleigh, NC: North Carolina State University.

Santos-Trigo, M. & Aguilar, D. (2018). Resolución de problemas matemáticos: del trabajo de Pólya al razonamiento digital. En A. Ávila (Ed.), *Rutas de la Educación Matemática* (pp. 148-167). Sociedad Mexicana de Investigación y Divulgación de la Educación Matemática, A. C., SOMIDEM.

Thompson, P. W. & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). Reston, VA: National Council of Teachers of Mathematics.

Zengin, Y. (2018). Incorporating the dynamic mathematics software GeoGebra into a history of mathematics course. *International Journal of Mathematical Education in Science and Technology, 49*(7), 1083-1098.

RELATIONSHIP BETWEEN COVARIATIONAL REASONING AND USE SCHEMES IN AN APPROACH TASK

The study aims to analyze the change in the mental actions of a student associated with the use schemes that he mobilizes when solving an approximation task to the area of a region with GeoGebra. Theoretical elements related to the covariational reasoning approach and the instrumental approach were considered. The student, after generating, mobilizing and refining his use schemes, made visible behaviors that are associated with mental actions, such as drawing a polygon to approximate an area and recognizing that, the more points on the graph, the distance between them is smaller and a better approximation is obtained. That is, observable behaviors can be associated with levels of reasoning or with changes in associated mental actions, which indicate a transition from one level of reasoning to another.

PRESERVICE TEACHERS' TASK CHOICE IN TECHNOLOGY-RICH ENVIRONMENTS

Michael S Meagher	M. Todd Edwards	Asli Özgün-Koca
Brooklyn College - CUNY	Miami University	Wayne State University
mmeagher@brooklyn.cuny.edu	edwardm2@miamioh.edu	aokoca@wayne.edu

A group of 27 preservice teachers (PSMTs) from two different institutions ranked and provided a rationale for their rankings of three versions of a task presented on three different technology platforms (CODAP, Geogebra, spreadsheet). The PSMTs then worked through the tasks in detail, reranked the tasks and provided a rationale for why they either changed or maintained their ranking. More than half the PSMTs changed their ranking and almost half changed their first choice of task. Rationales for change showed increased attention to the mathematical content of the task and the possibilities for "math action technologies" (Dick and Hollebrands, 2011) available in each platform.

Keywords: Teacher Education - preservice, Technology, Teacher Beliefs

Introduction

Choosing tasks to use in lessons in order to teach mathematical concepts is a basic activity of mathematics teachers. Therefore, teachers' ability to choose tasks is critical. The purpose of this study is to compare how pre-service teachers justify their ranking of three versions of the same task presented on three different technology platforms (CODAP, Geogebra, spreadsheet), first after reviewing the instructions for the task, and then, after working through each version of the task in detail using each platform.

Literature review and relationship to research

As Sullivan et al. (2013) and others have noted, teachers generally do not have time to develop significant numbers of their own tasks and generally rely on pre-existing tasks for use in their lesson plans. Therefore, evaluation of existing tasks is an important skill for preservice mathematics teachers (PSMTs) to develop. Sullivan et al. suggest three levels of decisions to be made in evaluating and choosing a task for use: (i) whether it matches curriculum goals, (ii) whether the demand of the task is appropriate for the students, and (iii) whether the task can accommodate a variety of learners.

Dick and Hollebrands (2011) argue that advanced digital technologies available for use in mathematics education typically fall into two categories namely "conveyance technologies" and "mathematical action technologies" (MATs). McCulloch et al. (2018) in a study involving 21 mathematics teachers with experience in teaching with advanced digital technologies pose the questions "Why do teachers choose to use technology to teach mathematics? What technology tools do teachers choose to use and why? and What general factors do teachers consider when selecting a particular technology tool to use?" They conclude that ease of use, for both the teachers themselves and their students is key factor in choosing particular technologies.

At the level of task choice, advanced digital technologies add another level of complexity to the decision making in planning lessons. Building on a study by Laborde (2002) involving teachers designing activities using an advanced digital technology (in this case Cabri), Kasten & Sinclair (2009) posit that teachers choosing tasks might first be drawn to activities that simply provide "visual amplification," (p. 134), could, after some experience with technology choose tasks involving generation of data for analysis and finally reach a stage where they are choosing tasks that take advantage of unique affordances of the technology.

McCulloch et al. (2018) provide many instances of their participants arguing for the value of MATs in particular tasks that were effective in promoting student learning. McCulloch et al. (2021) go further and argue that, not only can MATs be an effective driver of student exploration and learning, but that such use is a matter of digital equity.

In our study with relatively inexperienced pre-service mathematics teachers we sought to examine the issue of evaluation of materials by studying differences between PSMTs evaluation of tasks based on a relatively cursory "static" review followed by an extended period of actually doing the task as a learner. We were particularly interested in their attention to the suitability of the tasks to the teaching of the content and, per Dick & Hollebrands (2021) their attention to the extent to which the different versions of the task exemplified MATs

Methods and Methodologies

Setting and Participants

Two groups of Pre-service Mathematics Teachers (PSMTs) (n=15 & n=12) at two different institutions were presented with the following task (see Figure 1a):

Suppose you are teaching an 8th or 9th grade class about comparing the mean and median of a set of data. You would like to give them an investigation where they "play" with some different data sets. This link: https://tinyurl.com/bdfvx5ep provides three examples of how such a task could be presented using three different technology platforms: CODAP, spreadsheet, and a pre-built Geogebra applet.

Your task is to read the example activities at the link above and rank the versions of the task from best (the one that you are the most likely to use) to worst (the one that you are least likely to use) along with a rationale for your ranking. You will submit your responses using this Google form: https://forms.gle/Tw9m6UE7caNvrgEEA during class.

a

CODAP b | Spreadsheet c | Pre-built Geogebra App d

Figure 1. Comparing Mean and Median Tasks

The three tasks posed exactly the same questions for users to answer but in the context of different technological platforms. The images of each platform (Figure 1b, c,and d) were included for PSMTs to gauge the capabilities and interactivity level that each platform might have to offer. They worked on the task for 10 minutes in the class. Then, as a homework exercise the PSMTs were asked to revisit the examples but this time work through the examples in detail and to reconsider their ranking of the tasks and complete a form ranking the tasks one more time. They were also asked to list pros and cons of each technology platform.

Data Collection

The data collected consisted of: (i) The PSMTs initial rankings and rationales for their rankings; (ii) Their work on the tasks; (iii) Their reranking of the tasks with rationale for changing (or not); (iv) Their thoughts on the pros and cons of each platform.

Data Analysis

The participants' original critiques were analyzed using the constant comparative method (Glaser & Strauss, 1967) to establish trends in the data and to develop a code book. The code book was then applied to the PSMTs explanation for either maintaining or changing their rankings and to their "pros and cons" of each technology platform.

Results

Initial analyses/critiques

Change of ranking

The first result to note is that almost two-thirds (17 of 27 (63%)) of the PSMTs changed their ranking of the activities after having worked through them in detail. Furthermore, of the 17 PSMTs who changed their ranking more than 70% (12 of 17) changed their *first* choice. Therefore, after working through the activities in detail almost half the PSMTs 12 of 27 (44.4%) changed their first choice of activity.

Table 1. PSMTs' rankings by platform

	Initial ranking			Second ranking		
	1st Choice	2nd Choice	3rd Choice	1st Choice	2nd Choice	3rd Choice
CODAP	15	8	4	18	5	4
Geogebra	8	10	9	7	12	8
Spreadsheet	4	9	14	2	10	15

Table 1 above shows that in the final rankings there was a clear overall preference for CODAP with 18 of the 27 (66.7%) participants ranking it first. This represents a change in the direction of CODAP up from 15 out of 27 in the initial rankings. It should be noted as well that of the 12 participants who changed their first choice more than half (7 out of 12) made CODAP their first choice and in only 2 instances was CODAP moved out of first place.

Another result of note was the "demotion" of spreadsheets. The spreadsheet activity was initially ranked last by 14 of 27 (51.9%) of the PSMTs of the remaining 13 PSMTs, 9 of 13 (69.2%) gave the spreadsheet a lower ranking after working through the activities.

Rationales for changing or not changing

What accounts for the change by those who changed their rankings? Before considering the rationales for the second ranking it is instructive to consider the PSMTs' rationales for the first rankings i.e. what were they telling us about what they valued before working through the task in detail. In analyzing those rationales a number of patterns emerged:

A number of PSMTs expressed concern about difficulty of usability: "With Codap, you have to teach students how to use the applet, which takes up substantial time." (I1P7 - Institution 1, PSMT 7); "I believe what will scare off the students is that the other technology platforms requires more clicking. So instead of actually learning about mean and median I think a good portion of the class will involve learning on how to use the software" (I2P10).

Interestingly some support was expressed for spreadsheets on the grounds of its utility outside of the particular task or even a school setting "Spreadsheet may not be the best but it allows for interaction with excel, which is commonly used in professions post schooling" (I1P9), "Spreadsheet is the most applicable and widely used in all kinds of jobs and tasks" (I1P10).

Usability and applicability outside the task are general issues and, in the rationales before the PSMTs worked through the task there was relatively little attention to the content of the task itself. Many PSMTs mentioned visualisation in a general sense but one PSMT wrote "I like the visualizations that CODAP and Geogebra provide. This allows students to have a visual understanding of the values. CODAP will also allow students to highlight and isolate various ideas for students to look more deeply at the ideas of mean and median" (I1P1). This level of explicit attention to the context of the task was not common.

Analysis of the rationales submitted by the PSMTs "answered" the three issues raised above. Using the software convinced some PSMTs that CODAP was "Not as hard to use as I thought" (I1P7), "I thought CODAP was complicated to operate, but it turned out to be unexpectedly convenient" (I2P3). It was also noted that "the [Geogebra] applet provided does not allow for much engagement with the software itself" (I1P9).

In addition, coding of the data showed a shift in attention towards the specifics of the task at hand. The most significant pattern to emerge, in terms of "math action technologies" Dick and Hollebrands (2011) is the number of PSMTs who noted that CODAP "combines features from both spreadsheets and Geogebra" (I2P12) to make the mathematical content of the task more visible and so "makes the comparison easier to see" (I2P6).

In the rationales for rankings after doing the tasks the PSMTs also started to pay attention to the suitability of each platform for the task. A number of PSMTs commented on the constraint placed by pre-made Geogebra applet on "math action technologies": "[The Geogebra applet] does a lot of the work for you" (I1P10); "The creativity in this applet compared to the rest is more limited" (I1P8). Many PSMTs got specific about how the platform enabled or constrained understanding mean and median, for example "even though Geogebra . . . provides visual representation of the concept . . . the comparison is my easier to see in CODAP . . . in the case of . . . two data set[s], all you need to do is . . . drag each data set to the x-axis of the graph and see the measurement of Median and Mean in numeric and graphical representation, whereas in Geogebra when you drag the data (change and manipulate data) you would not be able to see the previous data and result, unless taking a screenshot or opening a different browser" (I2P6).

In terms of the "demotion" of spreadsheets, the rationales noted the lack of visuals provided by the spreadsheet "The spreadsheet was moved to last because the only way data is displayed is a table" (I2P9), or noted that while visuals could be generated they would require some work "I find the spreadsheet rather limiting . . . If we want a visual, we have to put together a graph, which is something CODAP already has" (I1P7).

Overall, a comparison of the rationales for the pre- and post-rankings shows a shift away from generalised statements to statements that were more engaged with the understanding of the mathematics in the task.

Conclusion

Evaluating and choosing tasks for implementation in their classrooms is a vital task for teachers. Pre-service teachers, moreover, need experience in a trajectory of choosing a task, working through a task in detail and implementing the task to see the differences in their own understanding of the task at each stage of that trajectory. The research presented above engaged pre-service teachers in the first two stages of this trajectory. The results of the study were that than half the participants changed their ranking of three versions of the same task after they had worked through the tasks in detail showing how much difference there can be in understanding of a task before and after working through the task in detail. Furthermore, it was encouraging to see how much the focus of attention shifted from general statements about tasks and technology to specific attention to the mathematical content of the task and the possibilities for "math action technologies" in various platforms.

References

Dick, T. P., & Hollebrands, K. F. (2011). *Focus in high school mathematics: Technology to support reasoning and sense making* (pp. xi-xvii). Reston, VA: National Council of Teachers of Mathematics.

Glaser, B., & Strauss, A. (1967). The discovery of grounded theory. 1967. Weidenfield & Nicolson, London, 1-19.

Kasten, S. E., & Sinclair, N. (2009). Using Dynamic Geometry Software in the Mathematics Classroom: A Study of Teachers' Choices and Rationales. *International Journal for Technology in Mathematics Education*, *16*(4).

Laborde, C. (2002). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning*, *6*(3), 283-317.

McCulloch, A. W., Hollebrands, K., Lee, H., Harrison, T., & Mutlu, A. (2018). Factors that influence secondary mathematics teachers' integration of technology in mathematics lessons. *Computers & Education*, *123*, 26-40.

McCulloch, A. W., Lovett, J. N., Dick, L. K., & Cayton, C. (2021). Positioning Students to Explore Math with Technology. *Mathematics Teacher: Learning and Teaching PK-12*, *114*(10), 738-749.

Sullivan, P., Clarke, D., Clarke, D., & Roche, A. (2013). Teachers' Decisions about Mathematics Tasks When Planning. *Mathematics Education Research Group of Australasia*.

PROMOTING COVARIATIONAL REASONING WITH THE AID OF DIGITAL TECHNOLOGY

Helen Mariel Pérez Martínez
Cinvestav-IPN
helen.perez@cinvestav.mx

Carlos A. Cuevas-Vallejo
Cinvestav-IPN
ccuevas@cinvestav.mx

Erasmo Islas Ortiz
Cinvestav-IPN
erasmo.islas@cinvestav.mx

José Orozco-Santiago
Benemérita Universidad
Autónnoma de Puebla
jose.orozco@fcfm.buap.mx

In this paper, we present the development of an investigation on the promotion of covariational reasoning in high school students (14-15 years old) in Mexico. The study consists of designing and applying a sequence of didactic activities that simulate a real situation virtually. The activities are organized through a Hypothetical Learning Trajectory supported by digital technology and elements of Cuevas-Pluvinage didactics. The activities were evaluated according to the levels of covariation proposed by Carlson and colleagues, categorizing students' achievements and difficulties for each level of understanding. The results show that the activities favor students' progress by moving from the context situation to the different representations, establishing the relationship between the variables, and identifying their functional dependence.

Keywords: Secondary Education, Technology, Algebra, and algebraic thinking.

Introduction

Variational thinking has been interpreted as a way of modeling the covariation between quantities and magnitudes, according to Vasco (2003):

> a dynamic way of thinking, which attempts to mentally produce systems that relate their internal variables so that they covary like the patterns of covariation of quantities of the same or different magnitudes in the cut-out subprocesses of reality. (p. 6)

In various investigations (Carlson et al., 2002; García, 2016; Tompson & Carlson, 2017), it has been found that the concept of covariation between variables leads to the concept of function because the covariation relationship is often expressed as a functional relationship. Despite being present from elementary algebra to calculus, it is still a topic that is a source of difficulties for students. They fail to perceive and relate the patterns between the quantities involved in the different mathematical situations presented to them. There are even studies, such as that of Aldon and Panero (2020), that have reported cases where the interpretation of the mathematical situation proposed when analyzing the variables involved, students confuse a graph with the trajectory of a moving object, from which a lack of clarity in the meaning of the axes and the interpretation of the graphical representations is inferred.

Given this problem, some studies, such as Vasco (2006), recommend that the teaching of variation should occur in various contexts that represent real situational problems for the student, that are adapted to the proposed learning objectives, and that lead to the concept of function. However, these contexts must have specific characteristics that make it possible to achieve these objectives; as mentioned by Hitt (2021), the characteristics of the environment that can contribute to the development of conceptual understanding include the teachers' approaches, the types of tasks given to students, and the use of a variety of representations.

For his part, Avila (2018) warns of the importance of variational thinking for the concept of a real function since "there is a need to establish activities in the classroom, in which school alternatives that promote the recognition of variation in situations in which the concept of function is present are raised" (p. 198).

Our study approaches activities from a variational-covariational approach with digital technology support. In the NCTM standards (2000), technology enriches the range and quality of investigations since it allows viewing mathematical ideas from multiple perspectives. The feedback enhances student learning that technology provides, and it gives teachers options for adapting instruction to student's special needs.

Theoretical Framework

Variational thinking

Since Moore and Carlson (2012) report the importance of variational and covariational reasoning for students' ability to model dynamic situations, the position we adopt regarding variational thinking corresponds to that of Maury and colleagues (2012), which is complemented by that of Moore and Carlson, since they conceive this thinking as the ability to identify phenomena of variation and change, to interpret them, describe them, quantify them, model them, transform them and predict their consequences. Developing variational thinking allows students to naturally identify phenomena of variation and change and to be able to model and transform them, which will contribute to the development of mathematical thinking processes linked to algebra, functions, and calculus.

Considering that the study has as part of its learning trajectory to involve the analysis of the coordination and functional relationship between variables, we assume as one of its central axes *the covariate reasoning,* which is defined by Carlson et al. (2002) as “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 357).

The framework on covariation proposed by Carlson and colleagues consists of mental actions, which provides a taxonomy to classify and analyze the behaviors that students present when engaging in covariation activities. According to the authors, there are five levels of development; however, for this research and considering the academic level, the activities were designed according to the first three levels: Level 1 (N1) Coordination, Level 2 (N2) Direction, and Level 3 (N3) Quantitative Coordination; which are presented in terms of mental actions supported by images.

Cuevas and Pluvinage Didactics

The implementation of this didactic framework is considered since it establishes didactic engineering for the planning of a mathematics program at a post-elementary level; proposing didactic principles that guide the design of tasks; for example, starting the activities with a conceptually rich problem of interest to the student, where the student is the one who solves it, and once the problem is solved, his answer must be verified—allowing the student to be an agent of action and leading him gradually towards the mathematical concept (Cuevas & Pluvinage, 2003).

Objectives

This study's objective is to design a didactic sequence that promotes the understanding of the processes of variation-covariation as an antecedent for the concept of function in high school students (14-15 years old). Activities are proposed in digital didactic environments to achieve this, which address mental actions linked to recognizing variables and their relationships. These activities aim to identify what level of covariation students reach when interacting with mathematical models that allow them to interpret, predict, describe and explain situations of functions, both linear and nonlinear. In this way, we will be able to inquire about the difficulties faced by students, categorizing them and associating them to each level of covariation.

Research Question

What levels of covariational reasoning do students achieve when interacting with scenarios designed for linear and nonlinear function covariation situations?

Methodology

This study is developed using Design-Based Research (DBR) proposed by (Bakker, 2018), consisting of the phases of preparation and design, teaching experiment, and analysis of results. For the design of our sequence of activities, we used a Hypothetical Learning Trajectory (HLT) (Simon 1995) and elements of Cuevas-Pluvinage didactics.

Phase 1: Preparation and design

The role of technology in this study is as a means of support for the simulation of our context of variation and covariation. The context used to carry out each activity consists of simulating the route of a cable car, where the student is asked to analyze the movement of the cabins in different sections of the way.

The objective of each activity is described below:

Activity #1. For this activity, mental action one (MA1) is taken into account, considering the coordination of the value of one variable with the changes in the other, i.e., in this case, it is considered whether the student goes through a process where they first try to identify the different magnitudes involved in the phenomenon: constants, parameters, and variables. Once they have identified which magnitudes vary and remain constant; students are expected to establish a functional relationship involving the magnitudes through different forms of representation (symbolic, tabular, and graphical) (see Figure 1a).

Activity #2 part 1. This activity takes into account mental action two (MA2), in which the coordination of the direction of change of one variable with the changes in the other variable is considered; that is, it is observed if it can identify the direction of change if it presents a growth or a decrease in the output value while considering the changes in the input value (see figure 1b).

Activity #2 part 2. In this activity, we consider the fourth principle of the Cuevas-Pluvinage didactics: try as much as possible every time operations that lead us to mathematical concepts are performed, to implement the inverse process.

This activity is implemented to observe if the students, by providing them with a graphical representation, can present AM1 behaviors (coordination of the two variables) and MA2 behaviors (identify that by changing one of the variables, the other one shows a decrease or an increase).

Activity #3 and #4. These activities focus on mental action three (MA3), the quantification of change. The coordination of the amount of change in the independent variable with the amount of change in the dependent variable is carried out. In this case, the student must identify accelerating (increasing) or decelerating (decreasing) variational behavior (see Figures 1c and 1d).

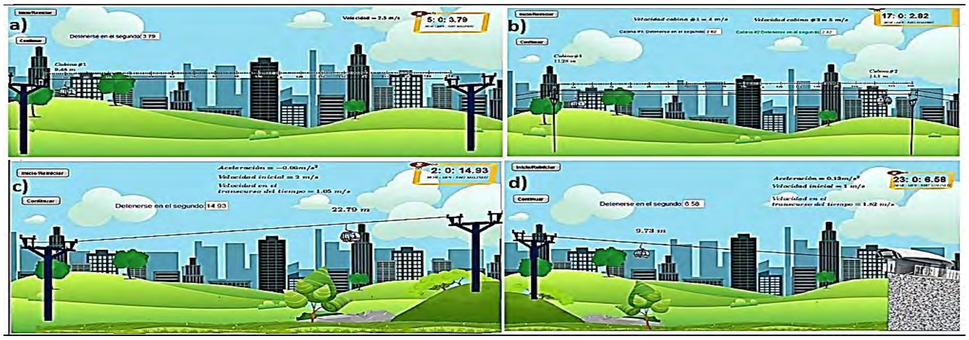


Figure 1. Scenarios used in each activity

For the particular context of this study, we have adapted the levels of covariation proposed by Carlson et al. (2002); the corresponding descriptors are shown below.

Table1. Descriptors of mental actions are used for the evaluation

Level	Description	Behaviors
Level 1 (L1) Coordination	At the coordination level, the images of covariation can support the mental action of coordinating the change of one variable with changes in the other variable (MA1)	In this case, the student must identify the different magnitudes involved in the context of the ropeway (time-hours, time-seconds, distance, speed, height), typifying the magnitudes involved in: constant, parameter, or variable. Once identified which magnitudes vary or remain constant, the student must establish a functional relationship involving the magnitudes through different forms of representation (symbolic, tabular, or graphical), which model the situation of the travel of a cabin in a section of the ropeway.
Level 2 (L2) Direction	At the direction level, the images of covariation can support the mental actions of coordinating the direction of change of one variable with changes in the other variable. The mental actions identified as MA1 and MA2 are both supported by L2 images.	In addition to the students showing Level 1, they should identify the direction of change, i.e., whether there is an increase or decrease in the output value (distance) while considering changes in the input value (time). When analyzing the graphical representations, the student must identify the functional relationships between the distance and time quantities. Subsequently, the student must determine which graphical representation grows or decreases faster than the other and justify such behavior.
Level 3 (L3) Quantitative Coordination	At the quantitative coordination level, the images of covariation can support the mental actions of coordinating the amount of change in one variable with changes in the other	The student identifies that the value of the distance decreases or increases (considering the difference between the distances) for each increment of time by intervals. Considering what magnitudes make them behave that way and how they behave (in our case, they consider what is

	variable. The mental actions identified as MA1, MA2, and MA3 are supported by L3 images.	happening with acceleration and velocity at that moment).

Phase 2: Teaching experiment

The research was conducted with 56 students in two groups (14-15 years old) in Mexico. Of which we considered 21 students since they were the ones who completed all the activities. Each student had a computer, Guided Learning and Exploration Sheets (GLES), and Interactive Virtual Didactic Scenarios (VIDS). The HLT comprises four activities, which correspond to four sessions of 50 minutes each; at the end of each session, the students handed in the GLES, and their answers were discussed as a group.

Phase 3: Analysis and results

In this article, we present the results of a student with the pseudonym Kenny, who was randomly selected from the sample. We will focus on making a detailed evaluation of his evolution in the levels of covariation during the activities presented. Kenny's case is an example of how the analysis was carried out for each student; reporting the results for all 21 students corresponds to later work in this report.

Activity #1:

In this first activity, Kenny partially achieved level 1 (MA1) because he failed to identify that the magnitude of the wire does not vary (see figure 2a); he also showed difficulties to identify what type of variation some magnitudes presented (see figure 2b).

It does not present any difficulty regarding the behavior of establishing a functional relationship in symbolic, tabular, and graphical form.

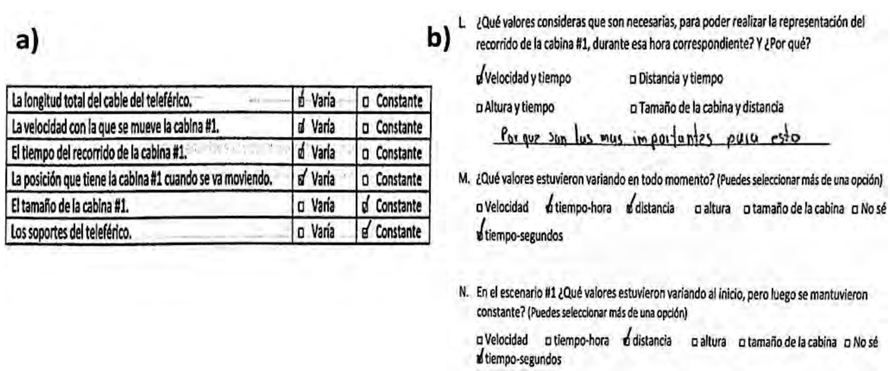


Figure 2. Kenny's response to Activity #1

Activity #2-part 1 and 2:

Kenny presents minor improvements in the behavior of MA1, identifying which magnitudes vary or which remain constant (see Figure 3a). He also shows progress in identifying magnitudes according to their type of variation. The behavior of establishing a functional relationship using different forms of representation is still present; however, we noticed that when asked to try to abstract the shape of the graph of the behavior of the phenomenon he was observing (the movement of two cabins during their ride on the cable car), Kenny opted to place points on the Cartesian plane (see Figure 3b), which shows that he needed values to be able to make this graphical representation. Therefore, Kenny has shown signs of achieving level 1.

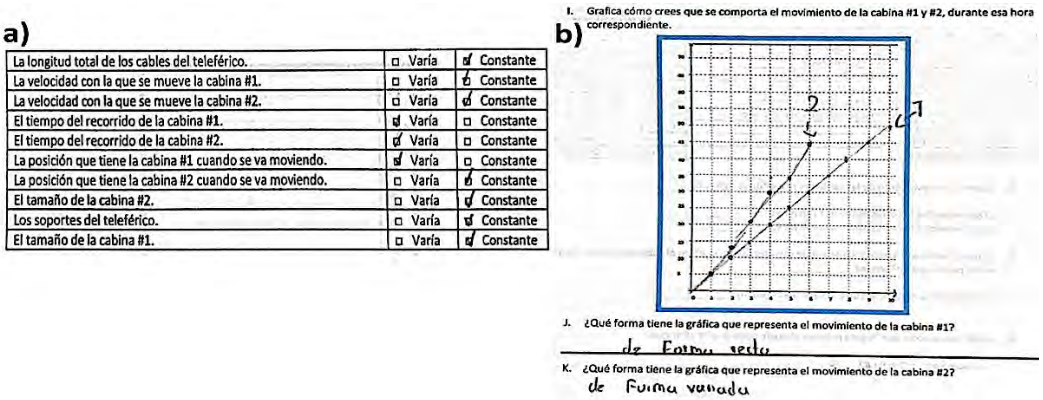


Figure 3. Kenny's response to activity #2-part 1.

Concerning MA2 behaviors, we note that Kenny manages to extrapolate and predict the distance that the cabin will travel at a specific time, which is not visible graphically (see figure 4b, section b_1). In activities #2-part 1, sections T and U (see figure 4a), and in the activity #2-part 2 (see figure 4c), Kenny manages to identify the growth pattern with the support of the graphs; we confirm the above even though in item C_1 (see figure 4c) he inverts the order of the answers. As shown in questions b_1 and b_2 (Figure 4b), it is clear that he only had confusion when exchanging the line between Booth #1 and Booth #2.

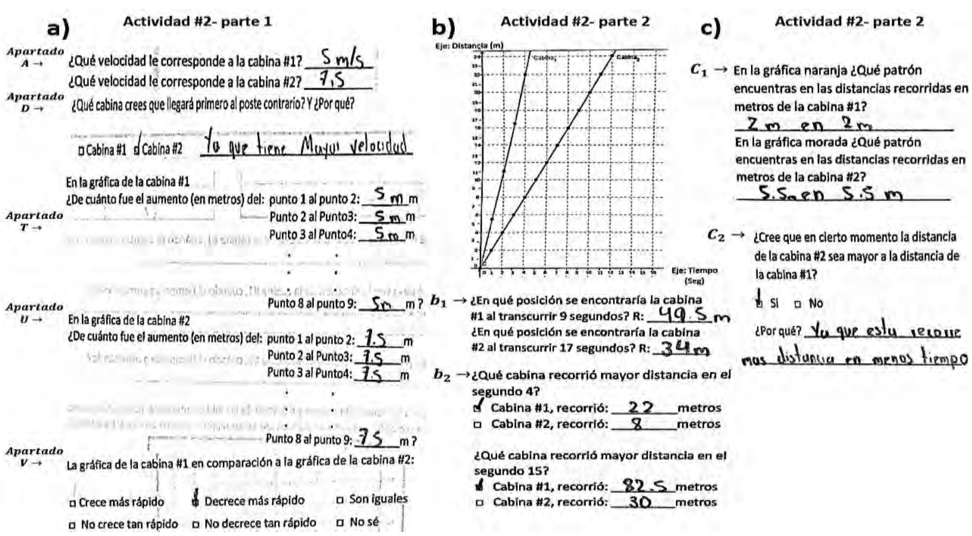


Figure 4. Kenny's response to activity #2-part 1 and 2.

Considering Kenny's confusion in the line of answer C_1 (see figure 4c), we consider as correct answer C_2 (see figure 4c) since it is sequential with answer C_1; therefore, we observe that both figure 4a and figure 4c support that Kenny identifies that one of the magnitudes causes another one to present changes of increase or decrease (see figure 4a-sections A and D and 4c-section C_2). Kenny presents difficulties when trying to select the behavior of a graph since we found that Kenny does not recognize the terms "grows or decreases faster" (see figure 4a, section V). We can conclude at this stage that Kenny shows signs of being able to reach Level 2.

Activity #3 and #4:

The evolution that Kenny has had concerning previous behaviors that presented difficulty in MA1 can be observed in activity #3 sections C, D, and E (see figure 5a), where he was able to correctly identify the type of variation that certain magnitudes presented at certain times of the cabin route, we also observe in activity #4 (see figure 5b), that Kenny shows behavior that allows him to identify the magnitudes and classify them into: constant, parameter or variable; and even manages to determine which variables are dependent on others.

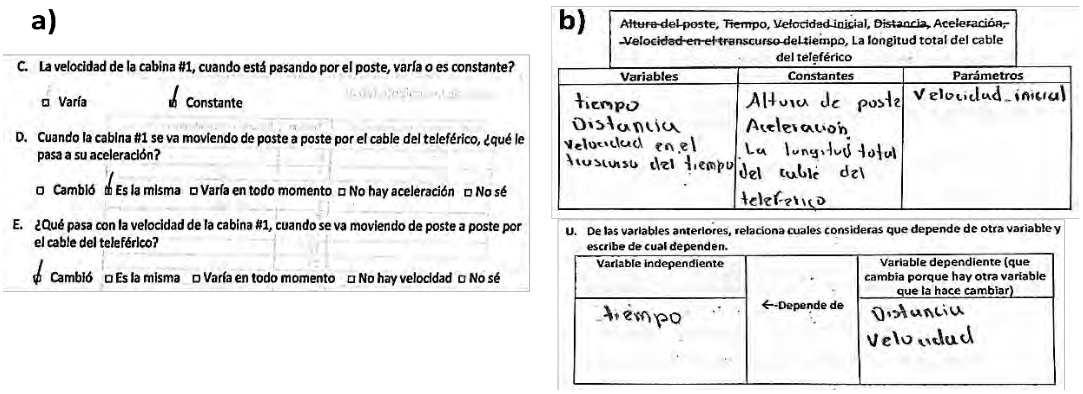


Figure 5: Kenny's response to activities #3 and #4

Another Progress of Kenny concerning MA2 behaviors is that he shows to be more familiar with the expressions "grows or decreases" and what they imply (see Figure 6a). It is worth noting that he still maintains the other MA2 behaviors previously described in the results of Activity #2-part 1 and 2 (see Figure 6b).

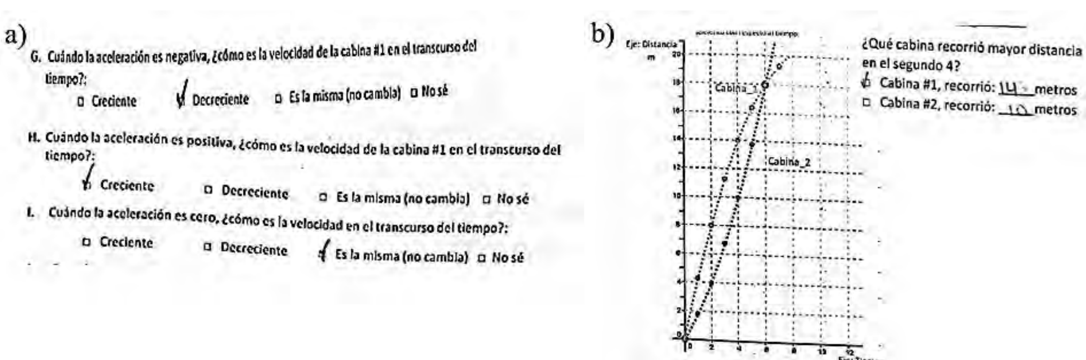


Figure 6: Kenny's response to activity #4

With all of the above, we can affirm that Kenny presents the necessary behaviors of MA1 and MA2 to place him in Level 2 of covariation.

During the analysis, we did not obtain evidence that Kenny managed to reach Level 3, despite presenting certain behaviors that allow him to identify that certain magnitudes influence others and that these cause them to behave in a certain way in certain intervals (see Figure 7a and 7b); he is not able to identify that the value of the distance decreases or increases (considering the difference between the distances) for each increment of time, by intervals (see Figure 7c).

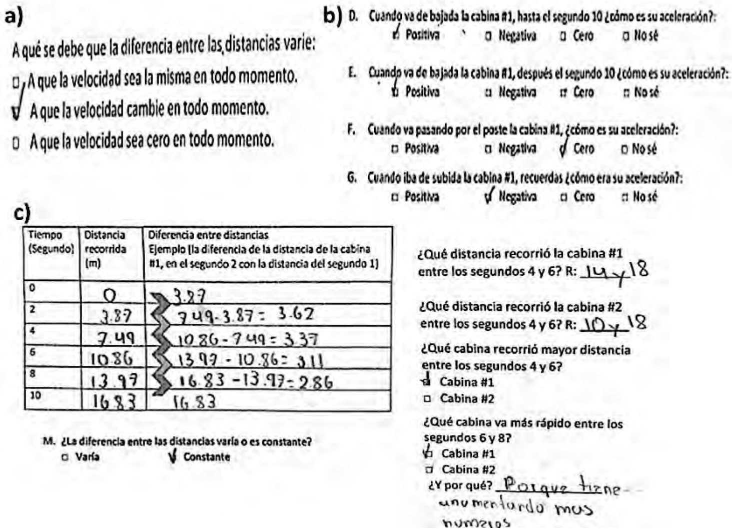


Figure 7: Kenny's response to activity #4

Discussion and conclusions

During the analysis, difficulties were observed in Kenny throughout the activities concerning some behaviors of the MAs, such as in the case of MA3, since there was no strong evidence of being able to coordinate the amount of change in one variable with changes in the other variable in certain intervals, the student could not advance to level 3 of covariation. These difficulties and the evolution of the behavior of each MA allowed us to determine that the student was able to reach level 2 of covariation. Overall, of the 21 students, only 71% managed to get N1, and of that group, only 24% achieved N2, thus answering our research question.

It is worth mentioning that the activities were developed face-to-face after a long period of sanitary confinement, which is a variable to consider. However, the VIDS instrumentation process did not present difficulty for the students in discovering the actions executed by each button, input box, and dynamic text.

The results of this research in progress are encouraging since the development of the activities allowed us to observe the evolution of Kenny's MA behaviors when interacting with the VIDS and making use of GLES; these MAs were designed in a graduated way, comparing and analyzing his improvements concerning some behaviors that, at the beginning of the first activity represented an obstacle for him. Our research is still in progress, and we consider these observations to project the redesign of activities and the learning trajectory towards a second research cycle.

Acknowledgments

To Conacyt and Cinvestav-IPN for funding the research project

References

Aldon, G., & Panero, M. (2020). Can digital technology change the way mathematics skills are assessed? ZDM Mathematics Education. 52(7), 1333–1348.https://doi.org/10.1007/s11858-020-01172-8.

Avila, P. (2011). Razonamiento covariacional a través de software dinámico. El caso de la

Variación lineal y cuadrática. Tesis de Maestría. Medellín: Universidad Nacional de Colombia.

Bakker, A. (2018). Design Research in Education: A Practical Guide for Early Career Researchers (1st ed.). Routledge. https://doi.org/10.4324/9780203701010

Carlson, M., Jacobs, S., Coe, E., Larsen, S., y Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events A framework and a study. EMA, 8 (2), 121-156. Journal for Research in Mathematics Education, 33(5), 352-378. https://doi.org/10.2307/4149958

Cuevas, C., & Pluvinage, F. (2003). Les projets d action pratique, elements d une ingenierie denseignement des mathematiques. . . In Annales de Didactique et de Sciences Cognitives, 8, 273–293

García, J. (2016). Desarrollo del razonamiento covariacional, en la conceptualización de la función lineal a través de software interactivo (Doctoral dissertation), Universidad de Medellín.

Hitt, F., & Dufour, S. (2021). Introduction to calculus through an open-ended task in the context of speed: representations and actions by students in action. ZDM - Mathematics Education, 53(3), 635–647. https://doi.org/10.1007/s11858-021-01258-x

Maury, E., Palmezano, G., Cárcamo, S., Sistema de tareas para el desarrollo del pensamiento variacional en 5° grado de educación básica primaria, Escenarios, 10(1), 7-16 (2012)

Moore, K., & Carlson, M. (2012). Students' images of problem contexts when solving applied problems. The Journal of Mathematical Behavior, 31, 48–59. doi:10.1016/j.jmathb.2011.09.001

National Council for Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. Journal for Research in Mathematics Education, 26(2), 114-145. https://doi.org/10.2307/749205

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. Compendium for research in mathematics education, 421-456. Reston, VA: National Council of Teachers of Mathematics.

Vasco, C.. (2003). El pensamiento variacional y la modelación matemática. In Anais eletrônicos do CIAEM– Conferência Interamericana de Educação Matemática, Blumenau, 9, 2009-2010.

PROMOVIENDO EL RAZONAMIENTO COVARIACIONAL CON APOYO DE LA TECNOLOGÍA DIGITAL

Helen Mariel Pérez Martínez
Cinvestav-IPN
helen.perez@cinvestav.mx

Carlos A. Cuevas-Vallejo
Cinvestav-IPN
ccuevas@cinvestav.mx

Erasmo Islas Ortiz
Cinvestav-IPN
erasmo.islas@cinvestav.mx

José Orozco-Santiago
Benemérita Universidad
Autónnoma de Puebla
jose.orozco@fcfm.buap.mx

En este artículo presentamos el desarrollo de una investigación sobre la promoción del razonamiento covariacional en estudiantes de secundaria (14-15 años) en México. El estudio consiste en el diseño y la aplicación de una secuencia de actividades didácticas que simulan de forma virtual una situación real. Las actividades se organizan mediante una trayectoria hipotética de aprendizaje con apoyo de la tecnología digital y elementos de la didáctica Cuevas-Pluvinage. Las actividades se evaluaron de acuerdo con los niveles de covariación propuestos por Carlson y colegas, categorizando los logros y las dificultades de los estudiantes para cada nivel de comprensión. Los resultados muestran que las actividades favorecen el progreso de los estudiantes al transitar de la situación de contexto hacia las diversas representaciones logrando establecer la relación entre las variables e identificar su dependencia funcional.

Palabras clave: Educación Secundaria, Tecnología, Álgebra y pensamiento algebraico.

Introducción

El pensamiento variacional ha sido interpretado como una forma de modelación de la covariación entre cantidades y magnitudes, de acuerdo con Vasco (2003):

> una manera de pensar dinámica, que intenta producir mentalmente sistemas que relacionen sus variables internas de tal manera que covaríen en forma semejante a los patrones de covariación de cantidades de la misma o distintas magnitudes en los subprocesos recortados de la realidad. (p. 6)

En diversas investigaciones (Carlson et al. 2002; García, 2016; Tompson y Carlson, 2017), se ha encontrado que el concepto de covariación entre variables conduce al concepto de función, debido a que la relación de covariación con frecuencia se expresa como una relación funcional. A pesar de estar presente desde el álgebra elemental hasta el cálculo, sigue siendo un tema que es una fuente de dificultades para los estudiantes; lo anterior debido a que no logran percibir y relacionar los patrones que surgen entre las cantidades involucradas en las diferentes situaciones matemáticas que se les presentan. Incluso hay estudios, como el de Aldon y Panero (2020), que han reportado casos donde la interpretación de la situación matemática propuesta al analizar las variables implicadas, los estudiantes confunden una gráfica con la trayectoria de un objeto en movimiento, de lo cual se infiere una falta de claridad en el significado de los ejes y en la interpretación de las representaciones gráficas.

Dada esa problemática, algunos estudios, como el de Vasco (2006), recomiendan que la enseñanza de la variación debe darse en una diversidad de contextos que representen para el estudiante problemas situacionales reales, que se adapten a los objetivos de aprendizaje propuestos y conduzcan al concepto de función. Sin embargo, estos contextos deben contar con ciertas características que posibiliten el alcance de dichos objetivos, tal como lo menciona Hitt (2021), las características del entorno que pueden contribuir al desarrollo de la comprensión conceptual incluyen los enfoques de los profesores, los tipos de tareas que se dan a los alumnos y el uso de una variedad de representaciones.

Por su parte, Ávila (2018) advierte de la importancia del pensamiento variacional para el concepto de función real, ya que "se observa la necesidad de establecer actividades en el aula de clases, en las que se planteen alternativas escolares que promuevan el reconocimiento de la variación en situaciones en las cuales el concepto de función está presente" (p. 198).

En nuestro estudio abordamos actividades desde un enfoque variacional-covariacional con el apoyo de la tecnología digital. En los estándares del NCTM (2000) se menciona que la tecnología enriquece la gama y la calidad de las investigaciones, ya que permite ver las ideas matemáticas desde múltiples perspectivas. El aprendizaje de los alumnos se ve favorecido por la retroalimentación que la tecnología proporciona, además ofrece a los profesores opciones para adaptar la enseñanza a las necesidades especiales de los alumnos.

Marco referencial

Pensamiento variacional

Dado que Moore y Carlson (2012) reportan la importancia del razonamiento variacional y covariacional para la capacidad de los estudiantes de modelar situaciones dinámicas, la postura que adoptamos con respecto al pensamiento variacional corresponde a la de Maury y colegas (2012), la cual se complementa con la de Moore y Carlson, ya que conciben a este pensamiento como la capacidad de identificar fenómenos de variación y cambio, para interpretarlos, describirlos, cuantificarlos, modelarlos, transformarlos y predecir sus consecuencias. Desarrollar el pensamiento variacional permite a los estudiantes identificar de manera natural fenómenos de variación y cambio; y que sean capaces de modelarlos y transformarlos, lo que contribuirá a desarrollar procesos de pensamiento matemático ligados al álgebra, las funciones y el cálculo.

Considerando que el estudio tiene como parte de su trayectoria de aprendizaje involucrar el análisis de la coordinación y la relación funcional entre variables, Asumimos como uno de sus ejes centrales *el razonamiento covariacional* el cual es definido por Carlson et al. (2002) como "las actividades cognitivas implicadas en la coordinación de dos cantidades que varían mientras se atiende a las formas en que cada una de ellas cambia con respecto a la otra" (p.357).

La propuesta de Carlson y colegas consiste en el marco de las acciones mentales, el cual proporciona una taxonomía para clasificar y analizar los comportamientos de los estudiantes al abordar actividades de covariación. De acuerdo con los autores, son cinco niveles de desarrollo; sin embargo, para los fines de esta investigación y considerando el nivel académico, las actividades se diseñaron de acuerdo con los tres primeros niveles: Nivel 1 (N1), Coordinación; Nivel 2 (N2), Dirección y Nivel 3 (N3), Coordinación Cuantitativa. Estos niveles se presentan en términos de las acciones mentales sustentadas en imágenes.

Didáctica Cuevas y Pluvinage

Se considera la implementación de este marco didáctico dado que establece una ingeniería didáctica para la planeación de un programa de matemáticas en un nivel post-elemental; proponiendo principios didácticos que guían el diseño de tareas; por ejemplo, iniciar las actividades mediante un problema conceptualmente rico y de interés para el estudiante, en donde sea él quíen lo resuelva, y una vez resuelto el problema se debe verificar su respuesta. Permitiendo al estudiante ser un agente de acción y conduciéndolo de manera gradual hacia el concepto matemático (Cuevas y Pluvinage, 2003).

Objetivo

Nuestro objetivo es el diseño de una secuencia didáctica que promueva la comprensión de los procesos de variación-covariación, como antecedente para el concepto de función en estudiantes de secundaria (14-15 años). Para lograrlo se proponen actividades en entornos didácticos digitales, los cuales abordan acciones mentales que estén ligados al reconocimiento de las variables y las relaciones entre ellas. Por medio de estas actividades se pretende identificar qué nivel de covariación logran alcanzar los estudiantes al interactuar con modelos matemáticos que le permitan interpretar, predecir, describir y explicar situaciones de funciones, tanto lineales como no lineales. De esta manera, podremos indagar acerca de las dificultades a las que se enfrentan los estudiantes, categorizándolas y asociándolas a cada nivel de covariación.

Pregunta de investigación

¿Qué niveles de razonamiento covariacional logran alcanzar los estudiantes al interactuar con los escenarios diseñados para situaciones de covariación de funciones lineales y no lineales?

Metodología

Este estudio se desarrolla mediante una investigación basada en diseño (IBD) propuesta por (Bakker, 2018), la cual consiste en las fases de preparación y diseño, experimento de enseñanza y análisis de resultados. Para el diseño de la secuencia de actividades, utilizamos una trayectoria hipotética de aprendizaje (Simon, 1995) y elementos de la didáctica Cuevas-Pluvinage.

Fase 1: Preparación y diseño

El papel de juega la tecnología en este estudio es como medio de apoyo para la simulación de nuestro contexto de variación y covariación. El contexto que se utiliza para llevar a cabo cada actividad consiste en simular el trayecto de un teleférico, en donde se le solicita al estudiante analizar el movimiento de las cabinas en distintos tramos del recorrido.

A continuación, describimos el objetivo de cada actividad:

Actividad #1. Para esta actividad se toma en cuenta la acción mental uno (AM1), considerando la coordinación del valor de una variable con los cambios en la otra, es decir, en este caso se atiende si el estudiante transita por un proceso en donde primero trate de identificar las diferentes magnitudes involucradas en el fenómeno: constantes, parámetros y variables. Una vez que han identificado cuáles magnitudes varian y cuales se mantienen constantes, se espera que los estudiantes establezcan una relación funcional que involucre a las magnitudes mediante diferentes formas de representación (simbólica, tabular y gráfica) (ver figura 1a).

Actividad #2 parte 1. En esta actividad se toma en cuenta la acción mental dos (AM2), en la que se considera la coordinación de la dirección del cambio de una variable con los cambios en la otra variable; es decir, se observa si puede identificar la dirección del cambio, si este presenta un crecimiento o un decrecimiento en el valor de salida, mientras se consideran los cambios en el valor de entrada (ver figura 1b).

Actividad #2-parte 2. En esta actividad consideramos el cuarto principio de la didáctica Cuevas-Pluvinage: intentar en lo posible, cada vez que se realicen operaciones que nos lleven a conceptos matemáticos, implementar la operación inversa.

Se implementa está actividad con el fin de poder observar si los estudiantes, al proporcionarles una representación gráfica, pueden presentar comportamientos de la AM1 (coordinación de las dos variables) y comportamientos de la AM2 (identificar que al cambiar una de las variables la otra presenta una disminución o un aumento)

Actividad #3 y #4. Estas actividades se enfocan en la acción mental tres (AM3), en la cuantificación del cambio. Se lleva a cabo la coordinación de la cantidad de cambio en la variable independiente, con la cantidad de cambio en la variable dependiente. En este caso, el estudiante debe identificar el comportamiento variacional acelerado (cada vez más grande) o desacelerado (cada vez más pequeño) (ver figura 1c y 1d).

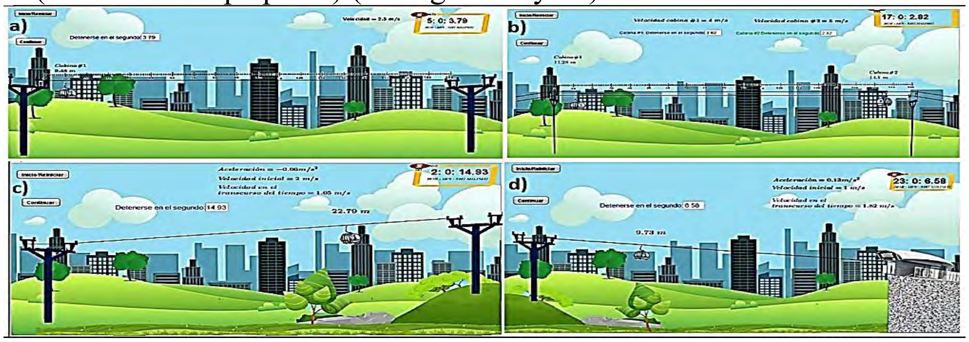


Figura 1. Escenarios empleados en cada actividad

Para el contexto de este estudio realizamos una adaptación de los niveles de covariación de Carlson et al. (2002), los descriptores correspondientes se muestran a continuación.

Tabla1. Descriptores de las acciones mentales utilizados para la evaluación

Nivel	Descripción	Comportamiento
Nivel 1 (N1) La Coordinación	En el nivel de coordinación, las imágenes de la covariación pueden sustentar a la acción mental de coordinar el cambio de una variable con cambios en la otra variable (AM1)	El estudiante identifica diferentes magnitudes involucradas en el contexto del teleférico (tiempo-horas, tiempo-segundos, distancia, velocidad, altura), clasificandolas en constante, parámetro o variable. Una vez que identifica qué magnitudes varian y cuáles se mantienen constantes, el estudiante establece una relación funcional que las involucra mediante distintas formas de representación (simbólica, tabular o gráfica), las cuales modelan el recorrido de una cabina en un tramo del teleférico.
Nivel 2 (N2) La Dirección	En el nivel de dirección, las imágenes de la covariación pueden sustentar a las acciones mentales de coordinar la dirección del cambio de una de las variables con cambios en la otra. Las acciones mentales identificadas como AM1 y AM2 ambas son sustentas por imágenes de N2.	Además de que el estudiante muestre el Nivel 1, debe identificar la dirección del cambio; i.e. si presenta un crecimiento o decrecimiento en el valor de salida (distancia) mientras considera los cambios en el valor de entrada (tiempo). Al analizar las representaciones gráficas el estudiante debe identificar las relaciones funcionales entre las magnitudes de distancia y tiempo. Posteriormente debe identificar cuál representación gráfica crece o decrece más rápido en comparación con la otra, y justificar dicho comportamiento.
Nivel 3 (N3) Cuantitativa	En el nivel de la coordinación cuantitativa, las imágenes de la covariación pueden	El estudiante identifica que el valor de la distancia disminuye o aumenta (considerando la diferencia entre las distancias) por cada incremento de tiempo, por intervalos.

	sustentar a las acciones mentales de coordinar la cantidad de cambio en una variable con cambios en la otra. Las acciones mentales identificadas como AM1, AM2 y AM3 son sustentadas por las imágenes de N3.	Considerando qué magnitudes hacen que se comporten de esa manera y su forma de comportarse (en nuestro caso que consideren que está pasando con la aceleración y la velocidad en ese momento).

Fase 2: Experimento de enseñanza

La investigación se realizó con 56 estudiantes (14-15 años) en dos grupos en México. De los cuales consideramos 21 estudiantes, que fueron los que completaron todas las actividades. Cada estudiante contaba con una computadora, hojas impresas de exploración y aprendizaje guiado (HEAG), y escenarios didácticos virtuales interactivos (EDVI). La THA está conformada por cuatro actividades, las cuales corresponden a cuatro sesiones de 50 minutos cada una, al finalizar cada sesión los estudiantes entregaban las HEAG y se discutían grupalmente sus respuestas.

Resultados

En este artículo presentamos los resultados de un estudiante con el seudónimo de Kenny, el cual fue seleccionado aleatoriamente de la muestra y nos enfocaremos en hacer una evaluación detallada de su evolución en los niveles de covariación durante las actividades presentadas. El caso de Kenny es un ejemplo de cómo se llevó a cabo el análisis para cada estudiante, reportar los resultados de los 21 estudiantes corresponde a un trabajo posterior a este informe.

Actividad #1:

En esta primera actividad, Kenny logró parcialmente alcanzar el nivel 1 (AM1), debido a que no logró identificar que la magnitud del cable no varía (ver figura 2a), también mostraba dificultades para identificar qué tipo de variación presentaban algunas magnitudes (ver figura 2b)

En lo que respecta al comportamiento de establecer una relación funcional de forma simbólica, tabular y gráfica, no presenta dificultad.

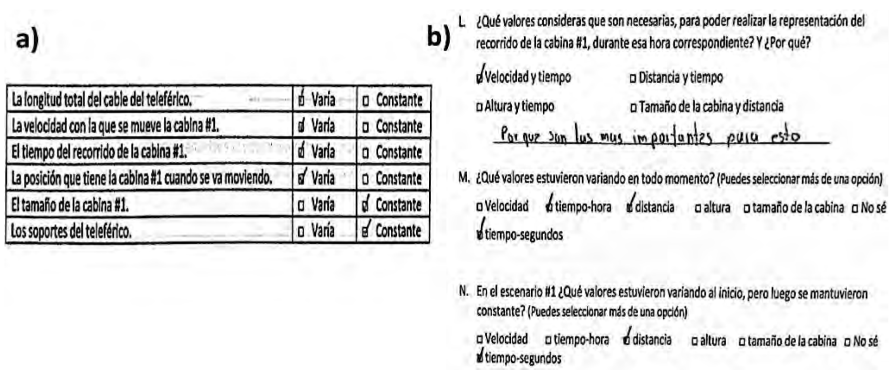


Figura 2. Respuesta de Kenny a la actividad #1

Actividad #2-parte 1 y 2:

Kenny presenta pequeñas mejorías en el comportamiento de la AM1, al identificar cuáles magnitudes varían o cuáles se mantienen constantes (ver figura 3a). También presenta un progreso en identificar magnitudes según su tipo de variación. El comportamiento de establecer una relación funcional haciendo uso de diferentes formas de representación aún sigue presente, sin embargo, notamos que al solicitarle que tratara de abstraer la forma de la gráfica del comportamiento del fenómeno que estaba observando (el movimiento de dos cabinas durante su recorrido por el teleférico), Kenny opta por colocar puntos en el plano cartesiano (ver figura 3b), lo cual muestra que precisó de valores para poder realizar esa representación gráfica. Por lo tanto, Kenny ha mostrado indicos de lograr el nivel 1.

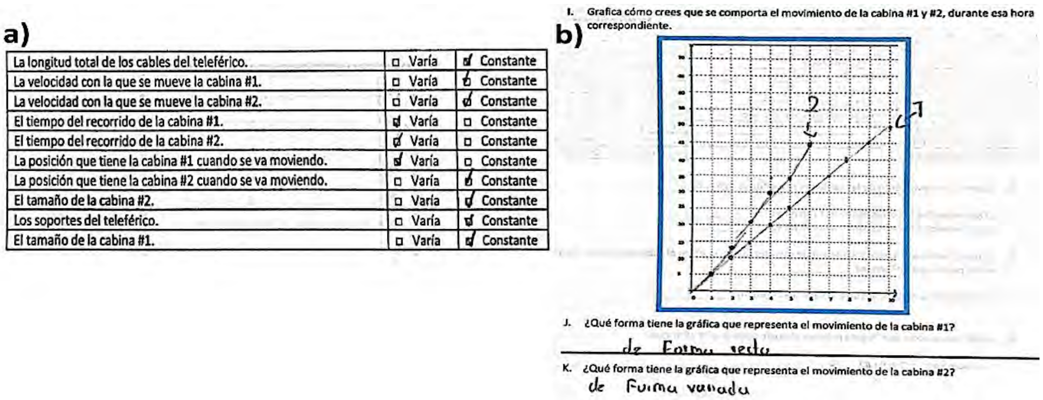


Figura 3. Respuesta de Kenny a la actividad #2-parte 1

Con respecto a los comportamientos de la AM2, notamos que Kenny logra extrapolar y predecir la distancia que recorrerá la cabina en un determinado momento, el cual no es visible de forma gráfica (ver figura 4b, apartado b_1).En las actividades #2-parte 1, apartado T y U (ver figura 4a) y en la actividad #2-parte 2 (ver figura 4c), Kenny logra identificar el patrón de crecimiento con el apoyo de las gráficas, confirmamos lo anterior a pesar de que en el item C_1 (ver figura 4c) invierte el orden de las respuestas. Por lo mostrado en las preguntas b_1 y b_2 (figura 4b) es claro que solo tuvo una confusión al intercambiar la linea entre cabina #1 y cabina #2.

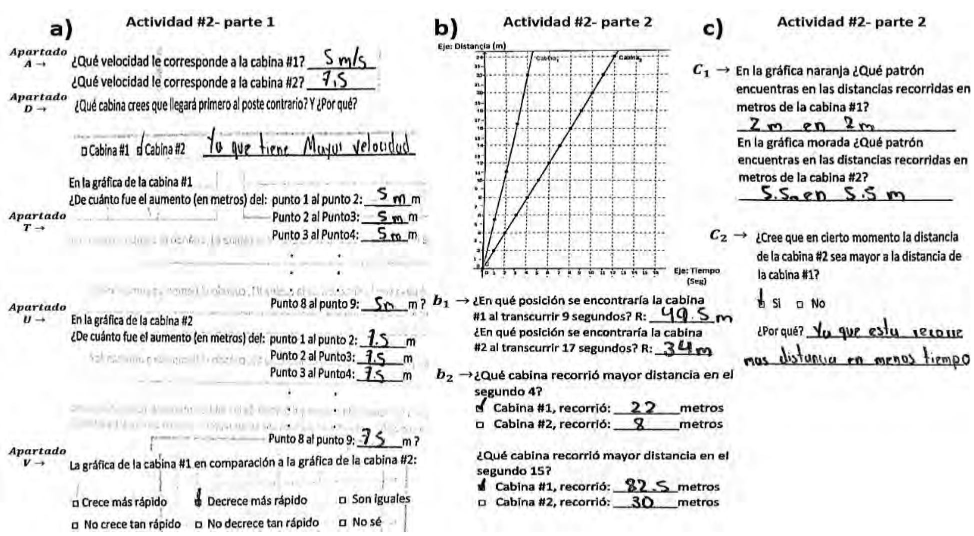


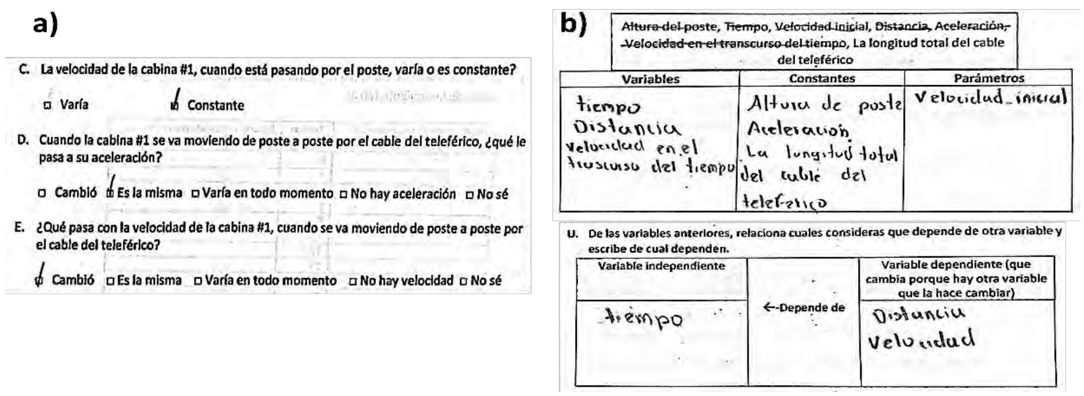
Figura 4. Respuesta de Kenny a la actividad #2-parte 1 y 2

Considerando la confusión de Kenny en la línea de la respuesta C_1 (ver figura 4c), consideramos como correcta la respuesta C_2 (ver figura 4c), puesto que es secuencial con la respuesta C_1, por lo tanto, se observa que tanto la figura 4a como la figura 4c avalan que Kenny identifica que una de las magnitudes hace que otra presente cambios de aumento o disminución (ver figura 4a -apartados A y D y 4c-apartado C_2).

Kenny presenta dificultades al tratar de seleccionar el comportamiento de una gráfica, ya que, comprobamos que Kenny no reconoce los términos“crece o decrece más rápido” (ver figura 4a, apartado V). Podemos concluir en esta etapa que Kenny, presenta indicios de poder alcanzar el Nivel 2.

Actividad #3 y #4:

La evolución que ha tenido Kenny con respecto a los comportamientos previos que presentaban dificultad en la AM1, se puede observar la actividad #3 apartados C, D y E (ver figura 5a), donde pudo identificar correctamente el tipo de variación que presentaban ciertas magnitudes en determinados momentos del recorrido de la cabina, también observamos en la actividad #4 (ver figura 5b), que Kenny presenta un comportamiento que le permite identificar las magnitudes y clasificarlas en: constante, parámetro o variable; e incluso logra determinar que variables son dependientes de otras.


a)

C. La velocidad de la cabina #1, cuando está pasando por el poste, varía o es constante?
□ Varía ✓ Constante

D. Cuando la cabina #1 se va moviendo de poste a poste por el cable del teleférico, ¿qué le pasa a su aceleración?
□ Cambió ✓ Es la misma □ Varía en todo momento □ No hay aceleración □ No sé

E. ¿Qué pasa con la velocidad de la cabina #1, cuando se va moviendo de poste a poste por el cable del teleférico?
✓ Cambió □ Es la misma □ Varía en todo momento □ No hay velocidad □ No sé

b)

~~Altura del poste~~, ~~Tiempo~~, ~~Velocidad inicial~~, ~~Distancia~~, ~~Aceleración~~, ~~Velocidad en el transcurso del tiempo~~, La longitud total del cable del teleférico

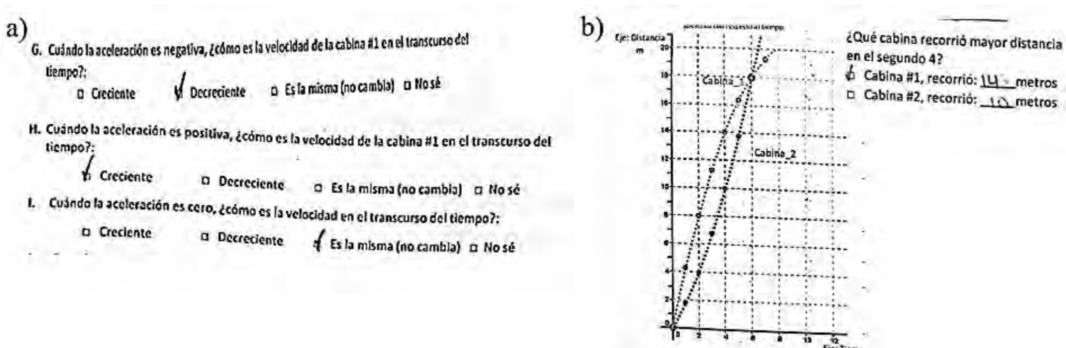
Variables	Constantes	Parámetros
tiempo Distancia Velocidad en el trascurso del tiempo	Altura de poste Aceleración La longitud total del cable del teleférico	Velocidad inicial

U. De las variables anteriores, relaciona cuales consideras que depende de otra variable y escribe de cual dependen.

Variable independiente		Variable dependiente (que cambia porque hay otra variable que la hace cambiar)
tiempo	←Depende de	Distancia Velocidad

Figura 5: Respuesta de Kenny a la actividad #3 y #4

Otro Progreso de Kenny con respecto a los comportamientos de AM2 es que desmuestra estar más familiarizado con las expresiones “crece o decrece” y lo que implica cada una de ellas (ver figura 6a). Cabe destacar que aún sigue manteniendo los demás comportamientos de la AM2, anteriormente descritos en los resultados de la Actividad #2-parte 1y 2 (ver figura 6b)


a)

G. Cuándo la aceleración es negativa, ¿cómo es la velocidad de la cabina #1 en el transcurso del tiempo?:
□ Creciente ✓ Decreciente □ Es la misma (no cambia) □ No sé

H. Cuándo la aceleración es positiva, ¿cómo es la velocidad de la cabina #1 en el transcurso del tiempo?:
✓ Creciente □ Decreciente □ Es la misma (no cambia) □ No sé

I. Cuándo la aceleración es cero, ¿cómo es la velocidad en el transcurso del tiempo?:
□ Creciente □ Decreciente ✓ Es la misma (no cambia) □ No sé

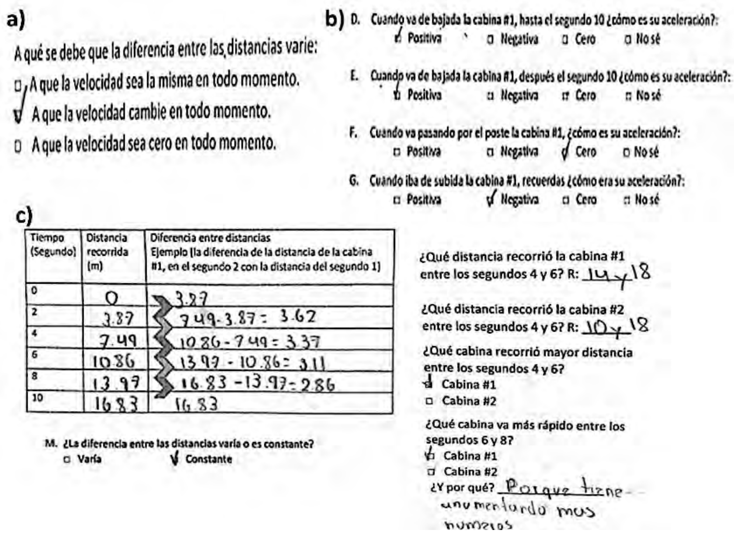
b)

¿Qué cabina recorrió mayor distancia en el segundo 4?
✓ Cabina #1, recorrió: 14 metros
□ Cabina #2, recorrió: 10 metros

Figura 6: Respuesta de Kenny a la actividad #4

Con todo lo expresado anteriormente podemos afirmar que Kenny presenta los comportamientos necesarios de AM1 y AM2 para poder ubicarlo en el Nivel 2 de covariación.

Durante el análisis no encontramos que Kenny alcanzara el Nivel 3, a pesar de presentar ciertos comportamientos que le permiten identificar que ciertas magnitudes influyen en otras, y que estas causan que se comporten de cierta forma en determinados intervalos (ver figura 7a y 7b), no logró identificar que el valor de la distancia disminuye o aumenta (considerando la diferencia entre las distancias) por cada incremento de tiempo, por intervalos (ver figura 7c).



Tiempo (Segundo)	Distancia recorrida (m)	Diferencia entre distancias Ejemplo (la diferencia de la distancia de la cabina #1, en el segundo 2 con la distancia del segundo 1)
0	0	3.87
2	3.87	7.49-3.87 = 3.62
4	7.49	10.86-7.49 = 3.37
6	10.86	13.97-10.86 = 3.11
8	13.97	16.83-13.97 = 2.86
10	16.83	16.83

Figura 7: Respuesta de Kenny a la actividad #4

Discusión y conclusiones

Durante el análisis se observaron dificultades que se presentaron en Kenny a lo largo de las actividades con respecto a algunos comportamientos de las AM, tal es el caso de la AM3, dado que no se encontraron indicios contudenetes de poder coordinar la cantidad de cambio en una variable con cambios en la otra variable en determinados intervalos, el estudiante no pudo avanzar hacia el nivel 3 de covariación. Estas dificultades y su evolución de comportamientos de cada AM, nos permitieron determinar que el estudiante logró alcanzar el nivel 2 de covariación. De forma general de los 21 estudiantes solo el 71% logro alcanzar el N1 y de ese grupo solo el 24% logro el N2 con lo cual damos respuesta a nuestra pregunta de investigación.

Cabe mencionar que las actividades se desarrollaron de forma presencial después de un largo periodo de confinamiento sanitario, lo cual es una variable a considerar. Sin embargo, el proceso de instrumentación de los EDVI no presentó dificultad en los estudiantes al descubrir las acciones ejecutadas por cada botón, casillas de entrada y textos dinámicos.

Los resultados de esta investigación en proceso son alentadores, ya que el desarrollo de las actividades permitió observar la evolución de los comportamientos de las AM de Kenny al interactuar con los EDVIs, haciendo uso de HEAG, estas AM se diseñaron de forma graduada, comparando y analizando sus mejorías con respecto a algunos comportamientos que, al inicio de la primera actividad representabán un obstáculo para él. Nuestra investigación sigue en desarrollo y consideramos estas observaciones para proyectar el rediseño de actividades y de la trayectoria de aprendizaje hacia un segundo ciclo de investigación.

Agradecimientos

A Conacyt y a Cinvestav-IPN por financiar el proyecto de investigación

Referencias

Aldon, G., y Panero, M. (2020). Can digital technology change the way mathematics skills are assessed? ZDM Mathematics Education. 52(7), 1333–1348.https://doi.org/10.1007/s11858-020-01172-8.

Avila, P. (2011). Razonamiento covariacional a través de software dinámico. El caso de la

Variación lineal y cuadrática. Tesis de Maestría. Medellín: Universidad Nacional de Colombia.

Bakker, A. (2018). Design Research in Education: A Practical Guide for Early Career Researchers (1st ed.). Routledge. https://doi.org/10.4324/9780203701010

Carlson, M., Jacobs, S., Coe, E., Larsen, S., y Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events A framework and a study. EMA, 8 (2), 121-156. Journal for Research in Mathematics Education, 33(5), 352-378. https://doi.org/10.2307/4149958

Cuevas, C., y Pluvinage, F. (2003). Les projets d action pratique, elements d une ingenierie denseignement des mathematiques. . . In Annales de Didactique et de Sciences Cognitives, 8, 273–293

García, J. (2016). Desarrollo del razonamiento covariacional, en la conceptualización de la función lineal a través de software interactivo (Doctoral dissertation), Universidad de Medellín.

Hitt, F., y Dufour, S. (2021). Introduction to calculus through an open-ended task in the context of speed: representations and actions by students in action. ZDM - Mathematics Education, 53(3), 635–647. https://doi.org/10.1007/s11858-021-01258-x

Maury, E., Palmezano, G., Cárcamo, S., Sistema de tareas para el desarrollo del pensamiento variacional en 5° grado de educación básica primaria, Escenarios, 10(1), 7-16 (2012)

Moore, K., y Carlson, M. (2012). Students' images of problem contexts when solving applied problems. The Journal of Mathematical Behavior, 31, 48–59. doi:10.1016/j.jmathb.2011.09.001

National Council for Teachers of Mathematics. (2000). Principles and standards for school mathematics. Reston, VA: Author.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. Journal for Research in Mathematics Education, 26(2), 114-145. https://doi.org/10.2307/749205

Thompson, P. W., y Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. Compendium for research in mathematics education, 421-456. Reston, VA: National Council of Teachers of Mathematics.

Vasco, C.. (2003). El pensamiento variacional y la modelación matemática. In Anais eletrônicos do CIAEM–Conferência Interamericana de Educação Matemática, Blumenau, 9, 2009-2010.

CLIMATE CHANGE BY THE NUMBERS: HOW NUMERICAL ESTIMATION CAN SUPPORT SCIENCE LEARNING

Ian Thacker
University of Texas, San Antonio
ian.thacker@utsa.edu

Prompting people to estimate climate change numbers before showing them the true value can shift learners' attitudes and conceptions. Yet, interventions created for such a learning experience are not easily accessible to the general public. The purpose of this preregistered study was to address this research gap by developing and testing an openly accessible online intervention that presents participants with novel numbers about climate change after they estimate those numbers. An experimental online study design was used to investigate the impact of the intervention on undergraduate students' climate change understanding and plausibility perceptions. Findings revealed that posttest climate change knowledge was higher among those randomly assigned to use the app compared with those assigned to a control condition, and that supplementing this experience with numeracy instruction was linked with more robust gains.

Keywords: climate change education; conceptual change; numerical estimation

Misconceptions about climate change are widespread in the USA. For example, as of September 2020, only 55% of adults in the USA correctly believed that most climate scientists think that climate change is happening, suggesting that the remaining 45% hold a serious misconception (Marlon et al., 2020). Fortunately, many approaches exist that have the potential to shift these misconceptions.

Numbers found in the news or online can be a powerful tool for science learning. For example, I encourage the reader to take a moment to estimate in the following quantity: What is the percentage change in the world's ocean ice cover since 1960? The true value may surprise you (see footnote).[1] Presentation of novel data after people first estimate that data can elicit more explicit reflection on the novel evidence and integration of supported claims (Richter & Maier, 2017) and can subsequently shift people's attitudes, beliefs, and misconceptions to be more aligned with scientists, particularly with regards to climate change (Ranney & Clark, 2016; Ranney et al., 2019; Rinne et al., 2006; Thacker & Sinatra, 2022). These findings suggests that just a handful of numbers can incite conceptual change. Findings also show conceptual change occurring as a result of such interventions are moderated by people's willingness and openness to reason with new evidence (Thacker & Sinatra, 2022). However despite these promising findings, the interventions created for these studies are not easily accessible to the general public and the extent to which conceptual changes endure over time is not well known.

The purpose of this preregistered conceptual replication study (Plucker & Makel, 2021) was to address this research gap by developing and testing a novel and openly accessible online intervention that presents participants with novel numbers about climate change after they estimate those numbers. The study uses an experimental design to investigate the impact of the intervention on undergraduate students' science learning and test whether affective and motivational constructs that are hypothesized to moderate this change, as hypothesized in models of conceptual change.

[1] The change in the world's ocean ice cover since 1960 is a 40% decrease (Ranney & Clark, 2016).

Theoretical Framework

Plausibility Judgments for Conceptual Change

The Plausibility Judgments for Conceptual Change model (PJCC), proposes that novel information (such as surprising data) can be a catalyst for conceptual change because it can prompt learners to appraise or reappraise the plausibility of their existing beliefs (Lombardi, et al., 2016). When people encounter novel information such as a surprising number about climate change, they first process the information for validity—perhaps by considering the credibility of the source and estimating whether the information seems reasonable—and then make a judgment of the plausibility of the conception supported by the new information. The extent to which people explicitly evaluate the plausibility of a conception depends, in part, on their motivation, emotion, and views about knowledge (or epistemic dispositions). More explicit plausibility evaluations are thought to leads to increased potential for conceptual change. For example, when a person estimates a number, they may draw from their prior knowledge or apply quantitative reasoning skills. Such an explicitly crafted estimate may better prepare the learner to interpret and assess the validity of a scientifically accepted value when later presented with it (c.f., Lombardi et al., 2016; Richter & Maier, 2017).

Numerical Estimation

Numerical estimation is an educated guess for a quantity that can draw from a person's prior experiences and understanding of number and operations (Dowker, 2005). Understanding and estimating magnitudes of quantities is considered to be central to the development of number concept (Siegler, 2016) and represents an important skill that is emphasized in both mathematics and science K–12 standards (Cheuk, 2012).

Of the common categories of numerical estimation skills (e.g., computational estimation and numerosity; Reys & Reys, 2004), research on measurement estimation is the most relevant for this study. Measurement estimation concerns the explicit estimation of real-world measures (Sowder & Wheeler, 1989) and is useful for understanding factors that help people judge whether real-world quantities are reasonable and valid. Findings suggest that peoples' estimation accuracy and judgments of reasonableness improve when they use measurement estimation strategies, such as a tolerance for error and impression in estimates (Shimizu & Ishida, 1994; Thacker et al., 2021), flexible techniques for rounding digits (Joram et al., 1998), and use of the benchmark strategy—the use of known and given values to estimate unknown values through processes of analogy, association, mental iteration, and/or proportional reasoning (Brown & Siegler, 2001; Joram et al., 1998, Siegler, 2016). For example, when asked to estimate a "real world" quantity, a skilled learner might draw from their prior experiences, recall related quantities, and mathematically manipulate them to more accurately arrive at an estimate. Such an explicitly crafted estimate might better prepare the learner to interpret and assess the validity of a scientifically accepted value when later presented with it (c.f., Lombardi et al., 2016; Richter & Maier, 2017).

Numerical Estimation for Conceptual Change

Experimental research has found that presenting people with novel climate change numbers after they estimate them can support climate change learning, and that supplementing such an experience with instruction on numerical estimation strategies and prompts to activate reflection can positively impact learning (Ranney & Clark, 2016; Thacker & Sinatra, 2022). For example, Thacker & Sinatra (2022) found that asking people to estimate climate change numbers before presenting them with the true value led to increased climate change knowledge when compared with a control group that read an expository text about the greenhouse effect. Further, although they found no main effects of modifying the intervention with either estimation instruction or prompts to activate reflection, they did find learning gains when including *both* modifications among individuals who were willing and open to reason with new evidence. Yet, despite these promising findings, this intervention was used in a closed setting and was not openly accessible to the public. Further, the extent to which conceptual changes endured over time was not well documented. Also, no explanation or justification for the validity of the "true values" were provided, which may have dampened people's perceptions that climate change is plausible.

Preregistered Research Questions and Hypotheses

The current study aimed to test whether the findings of Thacker & Sinatra (2022) would replicate when the intervention was modified to be (a) openly accessible, (b) include estimation accuracy feedback, (c) include justification for each number estimated, and (d) when the posttest assessment was completed ten days after the intervention. Namely, I sought to answer the following preregistered research questions (for the full anonymous preregistration, see https://bit.ly/3rP2m9c):

- RQ1. To what extent would estimation of and exposure to novel climate change data using an online learning intervention improve learners' climate change knowledge and plausibility judgments compared with reading an expository text?
    - (H1) I hypothesized that people assigned to the intervention conditions would have greater knowledge at posttest when compared with people assigned to read an expository text. I anticipated no significant differences in climate change plausibility perceptions between the intervention groups and the comparison group at posttest or delayed posttest.
- RQ2. To what extent do warm constructs (i.e., mathematics anxiety and epistemic dispositions) moderate the effects of the interventions on knowledge and plausibility?
    - (H2.1) Math anxiety: I anticipated that individuals with high math anxiety would benefit less from the mathematics skills instruction. That is, I expected that math anxiety would negatively moderate the effects of the intervention modified with estimation instruction when the outcome is climate change knowledge or plausibility perceptions.
    - (H2.2) Epistemic dispositions: I anticipated that individuals with higher levels of active open-minded thinking would benefit more from the intervention and modified intervention compared with the comparison condition when the outcome was climate change knowledge. That is, I expected that active open-minded thinking would positively moderate the effects of the intervention. I also anticipated that there would be a significant main effect of active open-minded thinking on plausibility perceptions, but no significant interactions with the experimental conditions.
- RQ3. To what extent does enhancing the intervention with instruction on estimation strategies change learners' climate change knowledge and plausibility?
    - (H3) I hypothesized that supplementing the intervention with instruction on estimation skills would lead learners to report more scientific knowledge about climate change compared with those who are assigned to the intervention but without estimation instruction. I expected no significant differences in climate change plausibility between these conditions.

One additional research question was posed in the preregistration concerning whether self-reported estimation strategies differed across conditions. Related analyses are still underway.

Methods

Participants and Procedure

I formed a national online Qualtrics panel of $N = 605$ undergraduate students to participate in an experimental online survey. To obtain this sample, Qualtrics representatives initially used multiple platforms to widely share a survey link online, 2651 people initially clicked on the link to participate, but 2046 were dropped from the analysis because they either did not meet the eligibility criteria (over 18 and a full-time undergraduate student), did not pass an attention check, or because they were flagged as a "speeder" by the algorithm created by Qualtrics. There was no missing data at pretest or posttest. Of the 605 students who met this criteria, 57% agreed to participate in a followup study, and 88 (15% of the analytic sample) completed a short followup survey 10 days after completing the initial intervention.

Participants in the main analytic sample were 20.3 years old on average, 74% Female, 15% Male, 3% nonbinary/other, 56% White, 19% African American, 17% Asian American, 5% Two or More Races, and 2% American Indian or Alaskan Native. All participants (a) completed a pretest to measure their misconceptions about climate change, plausibility judgments about climate change, mathematics anxiety, and prior epistemic dispositions, (b) were directed to a web app that randomly assigned them to one of three conditions (control group that read an expository text about the greenhouse effect, the intervention, and the intervention supplemented with estimation instruction), (c) were directed back to the original survey where they completed an identical posttest of knowledge, plausibility perceptions, and a demographics questionnaire, and (d) were contacted ten days later to complete the knowledge and plausibility perceptions measured again (but only if they had initially indicated that they agreed to be contacted in the demographics section of the posttest). For a summary of the procedures, see Figure 1.

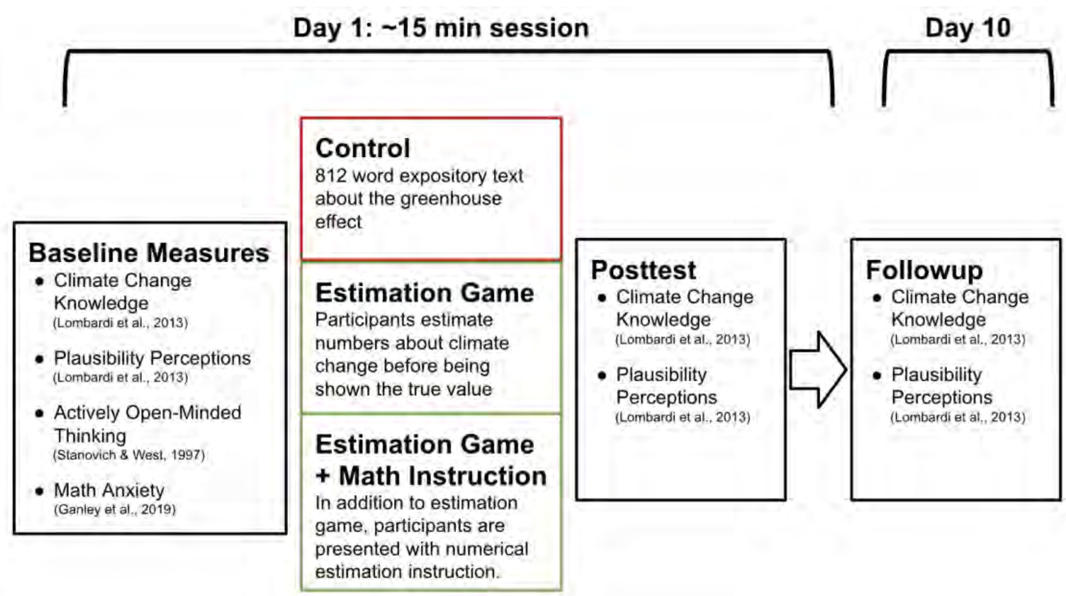


Figure 1. Visual Representation of the Survey Flow, Materials, and Procedures

Materials

All survey materials and intervention texts are available in the supplemental materials (https://bit.ly/3rP2m9c), also see Appendix A for select excerpts.

Conditions

There were three experimental conditions: the intervention group, modified intervention group, and control group. Students in the baseline intervention (also referred to as the "Estimation Game") estimated 12 climate change related numbers before being presented with the scientifically accepted answer, a short explanation to justify the scientifically accepted answer, and accuracy feedback. Students in the modified intervention condition completed the Estimation Game, but prior to the game, they engaged with instructional text that emphasized three numerical estimation strategies—tolerance for error, the benchmark strategy, and flexible rounding—with examples and two checks for understanding. These three strategies were found in prior research to be productive for this specific task (Thacker et al., 2021). Students in the control group were presented with an 812-word expository text about the greenhouse effect adapted from Lombardi et al., (2013). See Figure 1 for a summary of the procedures and Appendix A for more detailed text excerpts and screenshots). All three experimental conditions were presented in an openly accessible, open-source online web app [http://143.110.210.183/; also see Thacker et al., 2021].

Dependent variables

Climate change knowledge. Knowledge of climate change was measured using seven items adapted from the Human Induced Climate Change Knowledge measure (HICCK; Lombardi et al., 2013). Participants responded as to whether they believed that *climate scientists* would believe that certain statements are true (e.g., "Most of the world's glaciers are decreasing in size. This is evidence of climate change"). Responses were on a seven-point agreement scale. Participants completed this scale pre-intervention, post-intervention, and at the 10-day followup. All were reliable at conventional levels ($\alpha_{pre} = .65$, $\alpha_{post} = .74$, $\alpha_{followup} = .77$).

Plausibility perceptions. Perceptions of plausibility that humans are responsible for climate change were measured using four items adapted from the Plausibility Perceptions Measure (PPM; Lombardi et al., 2013). These items were intended to capture participant's *personal* positions on whether humans are responsible for climate change as they responded to statements ("Evidence from around the world shows that the climate is changing in many regions") on a six point agreement scale from 1 = *Highly Implausible (or even impossible)* to 6 = *Highly Plausible*. This scale was also completed at pretest, posttest, and during the followup and was reliable at conventional levels ($\alpha_{pre} = .81$, $\alpha_{post} = .85$, $\alpha_{followup} = .83$).

Estimation strategy reports. Participants in intervention conditions also provided open-ended descriptions of strategies that they used to estimate numbers. Coding and analysis of this variable is still underway.

Covariates

Epistemic dispositions. Baseline epistemic dispositions were measured using the Actively Open-Minded Thinking scale (AOT; Stanovich & West, 1997) that captures participants' willingness to reason with novel evidence using seven items (e.g., "People should take into consideration evidence that goes against their beliefs") using a seven point agreement scale ranging from 1 (completely disagree) to 7 (completely agree; $\alpha = .71$).

Math Anxiety. Participants also completed a Mathematics Anxiety Questionnaire (Ganley et al., 2019) consisting of nine items (e.g., "I get a sinking feeling when I think of trying to solve math problems") with five response options ranging from 1 (Not true of me at all) to 5 (Very true of me; $\alpha = .93$).

Analytic Strategy

To assess the effects of the interventions on the knowledge and plausibility outcome variables (RQ1 & RQ3), I used ordinary least squares regression with robust standard errors using a separate model for knowledge and plausibility perceptions. Predictors were the treatment condition and pre-test scores. To assess moderating effects (RQ2), I repeated these analyses after adding math anxiety and actively open-minded thinking as moderators of the treatment condition, with separate models for each moderator.

To test whether learning was retained ten days after the pre-test (an exploratory question), followup knowledge and plausibility scores were used as main outcomes in two separate regression models with experimental condition as a predictor and pretest scores as covariates.

Results

All coefficients, standard errors, and *p*-values from regression models are presented in Table 1.

RQ1: Main Effects of the Intervention. With regards to knowledge as the main outcome, participants in both intervention conditions outperformed the control group. This difference was significant for the modified intervention before and after adjusting for moderating variables and interactions, whereas for the unmodified intervention, the effect was only significant before adjusting for moderators. When plausibility perceptions were the main outcome, no significant main effects of the intervention were found. As such, H1 was confirmed.

I also used contrasts to test whether findings replicated those of Thacker & Sinatra (2022), revealing that posttest climate change knowledge was indeed higher among those in either intervention condition compared with those assigned to a control condition ($d = 0.30$, $p < .001$).

Table 1. Effects of Experimental Conditions on Posttest Knowledge and Plausibility and The Moderating Effects of Math Anxiety and Actively Open-Minded Thinking (N = 605).

	Posttest Knowledge			Posttest Plausibility Perceptions		
	No Moderator	Math Anxiety as Moderator	AOT as Moderator	No Moderator	Math Anxiety as Moderator	AOT as Moderator
	b (SE) p	*b* (SE) p	*b* (SE) p	*b* (SE) p	*b* (SE) p	*b* (SE) p
Estimation Game	**0.232******	**0.270~**	0.016	0.232	0.190	-0.521
	(0.046)	**(0.139)**	(0.244)	(0.141)	(0.430)	(0.791)
	p < .001	**p = .054**	p = .946	p = .102	p = 0.659	p = .511
Estimation Game + Estimation Instruction	**0.168******	**0.387****	**0.527***	0.161	0.249	-0.027
	(0.046)	**(0.137)**	**(0.234)**	(0.142)	(0.423)	(0.760)
	p < .001	**p = .005**	**p= .025**	p = .260	p = .556	p = .972
Moderator		-0.014	**0.146******		0.015	**0.268***
		(0.032)	**(0.034)**		(0.098)	**(0.111)**
		p = .670	**p < .001**		p = .877	**p = .016**
Intervention * Moderator		-0.013	-0.040		0.014	0.146
		(0.045)	(0.049)		(0.139)	(0.157)
		p = .774	p = .408		p = .919	p = .353
Modified Intervention * Moderator		**-0.077~**	**0.142****		-0.031	0.037
		(0.045)	**(0.047)**		(0.138)	(0.152)
		p = .084	**p = .003**		p = .824	p = .807
R2	0.464	0.472	0.537	0.626	0.626	0.638

Note: AOT = Actively Open-Minded Thinging. The comparison condition is the control condition in which participants read an expository text about the greenhouse effect. All models include pretest scores as a covariate, namely, posttest knowledge is adjusted for pretest knowedge and posttest plausibility is adjusted for pretest plausibility perceptions. Boldfaced values indicate significant results for predictors.
~p<0.1; *p<0.05; **p<0.01;***p<0.001

RQ2: Moderating Effects of Math Anxiety and Actively Open-Minded Thinking. Findings revealed significant main effects of Actively Open-Minded Thinking when knowledge and plausibility were the main outcome variable, and that it moderated the effect of estimation instruction on knowledge. Thus, I found support for H2.1 and partial support for H2.2.

RQ3: Comparison Between Intervention and Modified Intervention. Comparing only the two intervention groups revealed no significant differences after controlling for pretest scores. This was true for both outcomes: posttest knowledge (p = .180) and plausibility (p = .623).

Followup Analysis. As noted, I followed up with willing participants approximately ten days after the intervention. Because only 15% of the analytic sample (n = 88) completed the followup survey, pre-planned (but exploratory) multilevel analyses were underpowered. I therefore ran two separate regression models with followup knowledge and plausibility scores as the main outcomes, experimental condition as the main predictor, and pretest scores as covariates. Neither model revealed significant effects of either intervention or modified intervention groups when compared with the control (all conditional p > .248).

Significance

I sought to investigate whether the learning that occurs when people encounter novel statistics was enhanced with additional instruction on estimation strategies. Consistent with prior research, I found that students who learned from novel statistics performed about a third of a standard deviation better than a control group on a posttest of climate change knowledge (c.f., Ranney & Clark, 2016; Thacker & Sinatra, 2022).

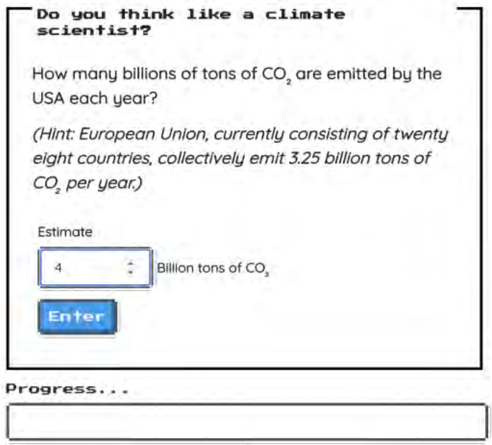
Findings also revealed that students' willingness to reason with new evidence was a predictor of learning and moderated the effects of numerical estimation instruction. These effects of Actively Open-Minded Thinking provide support for the Plausibility Judgments for Conceptual Change Model (Lombardi et al., 2016); instruction that emphasizes the explicit evaluation of evidence appears to be most effective among those willing to consider belief-discrepant information (also see Richter & Maier, 2017). Educators might thus consider pairing estimation instruction with special emphasis on the importance of keeping an open mind when examining new types of numerical evidence, even if the evidence is contrary to students' current beliefs.

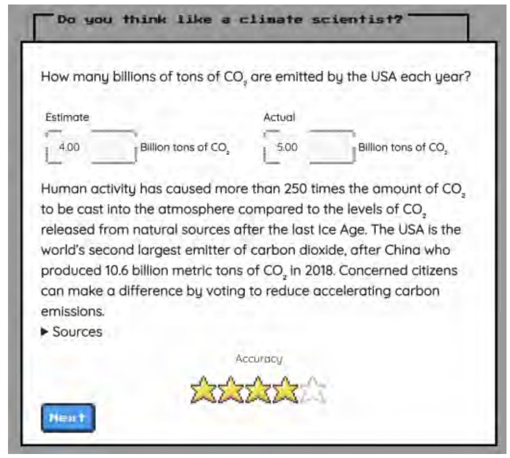
I also found no significant main effects of supplementing the Estimation Game with instruction that emphasized three key estimation strategies (tolerance for error, flexible rounding, and the benchmark strategy). This may suggest that the baseline intervention may be equally effective at encouraging explicit evaluation of quantities for most learners. It could also be that the outcome measures were not sensitive to capture the learning that occurred from this short micro-intervention. While I did find that the effects of the modified intervention was more robust to inclusion of covariates, future research should explore measuring alternate learning outcomes that are more sensitive to whether and which estimation skills were applied to support meaning-making from climate change quantities.

Another contribution of this study is that it provides mathematics instructors with a tool that enables applications of mathematical skills to learn about relevant science topics. The central intervention provides educators concerned with public understanding of science with an easily accessible learning application that can be shared and adapted to provide opportunities for students to apply numeracy skills towards making of key numbers that shape our changing environment.

Appendix A. Examples of The Experimental Conditions

Intervention Group (Screenshots from the "Estimation Game")





Modified Intervention Group (Excerpts from estimation instruction that preceded Estimation Game)

Sometimes people estimate everyday numbers in their head. For example, you might quickly estimate the cost of tax and tip in your head before ordering a meal at a restaurant. These calculations are naturally very rough and imprecise, and it is okay if your guess is not perfect…

REFERENCE NUMBERS

Numbers that you already know (reference numbers) can help you estimate numbers that you do not know. For example, if you know that about 300 pennies fit in a small, 8oz milk carton, you can use this information to estimate the number of pennies that fit in a gallon…

SIMPLIFYING NUMBERS

When using reference numbers, you may want to round values to make mental computation easier. For example, let's estimate the population of California given that the population of Kentucky is 4.47 million. Before making our estimate, we first round the Kentucky population to 4 million to make the math easier, and scale this value according to our beliefs about the size of California compared to Kentucky. If you were to guess that California…

Control Group (Excerpt of Expository Text adapted from Lombardi et al., 2013)

THE ENHANCED GREENHOUSE EFFECT

Many people have heard of the "greenhouse effect", but not everyone knows what the "greenhouse effect" is exactly. The greenhouse effect refers to the way that certain gasses in earth's atmosphere keep the planet warmer than it would otherwise be. The earth's greenhouse effect is a natural occurrence…

Note. All materials appeared within the Estimation Game environment (http://143.110.210.183/). For complete intervention materials, see the supplemental materials (https://bit.ly/3rP2m9c).

References

Brown, N. R., & Siegler, R. (2001). Seeds aren't anchors. *Memory & Cognition*, *29*(3), 405–412.

Cheuk, T. (2012). *Relationships and convergences found in the Common Core State Standards in mathematics (practices), Common Core State Standards in ELA/literacy (student portraits), and a framework for K-12 science education (science & engineering practices)*. http://nstahosted.org/pdfs/ngss/explanationofvenndiagram.pdf

Dowker, A. (Ed.). (2005). A good guess: Estimation and individual differences: Implications for psychology, neuroscience and education. *Individual differences in arithmetic* (pp. 141–151). Psychology Press.

Ganley, C. M., Schoen, R. C., LaVenia, M., & Tazaz, A. M. (2019). The construct validation of the math anxiety scale for teachers. *Aera Open, 5*(1), 2332858419839702.

Joram, E., Subrahmanyam, K., & Gelman, R. (1998). Measurement estimation: Learning to map the route from number to quantity and back. *Review of Educational Research, 68*(4), 413–449.

Lombardi, D., Nussbaum, E. M., & Sinatra, G. M. (2016). Plausibility judgments in conceptual change and epistemic cognition. *Educational Psychologist*, *51*(1), 35–56.

Lombardi, D., Sinatra, G. M., & Nussbaum, E. M. (2013). Plausibility reappraisals and shifts in middle school students' climate change conceptions. *Learning and Instruction*, *27*, 50–62.

Marlon, J., Howe, P., Mildenberger, M., & Leiserowitz, A. (2020). Yale climate opinion maps–US 2016. *Yale Program on Climate Change Communication*. Retrieved from https://climatecommunication.yale.edu/visualizations-data/ycom-us/

Plucker, J. A., & Makel, M. C. (2021). Replication is important for educational psychology: Recent developments and key issues. Educational Psychologist, 56(2), 90-100.

Ranney, M., & Clark, D. (2016). Climate change conceptual change: Scientific information can transform attitudes. *Topics in Cognitive Science*, *8*(1), 49–75.

Ranney, M. A., Shonman, M., Fricke, K., Lamprey, L. N., & Kumar, P. (2019). Information that boosts normative global warming acceptance without polarization: Toward JS Mill's political ethology of national character. In D. Wilkenfeld & R. Samuels (Eds.) *Anthology on the experimental philosophy of science* (pp. 61-96). (In Bloomsbury's Advances in Experimental Philosophy series). New York: Bloomsbury

Rinne, L. F., Ranney, M. a, & Lurie, N. H. (2006). *Estimation as a catalyst for numeracy: Micro-interventions that increase the use of numerical information in decision-making.* Proceedings of the Seventh International Conference of the Learning Sciences, 571–577.

Reys, R. E., & Reys, B. J. (2004). Estimation in the mathematics curriculum: A progress report. *Beyond Written Computation*, *29*, 101–112.

Richter, T., & Maier, J. (2017). Comprehension of multiple documents with conflicting information: A two-step model of validation. *Educational Psychologist, 52*(3), 148-166.

Shimizu, K., & Ishida, J. (1994). The cognitive processes and use of strategies of good Japanese estimators. In R. E. Reys & N. Nohda (Eds.), *Computational Alternatives for the twenty-first century: Cross-cultural perspectives from Japan and the United States* (pp. 161-178). Reston, VA: National Council of Teachers of Mathematics.

Siegler, R. S. (2016). Magnitude knowledge: The common core of numerical development. *Developmental Science, 19*(3), 341-361.

Sowder, J. T., & Wheeler, M. M. (1989). The development of concepts and strategies used in computational estimation. *Journal for Research in Mathematics Education*, 130–146.

Stanovich, K. E., & West, R. F. (1997). Reasoning independently of prior belief and individual differences in actively open-minded thinking. *Journal of Educational Psychology, 89*(2), 342–357.

Thacker, I., Broadway, R., & Feder, S. (2021). Estimating climate change numbers: How tolerance for error can support science learning. In Olanoff, D., Johnson, K., & Spitzer, S. M. (Eds.). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1738-1743). Philadelphia, PA.

Thacker, I. & Sinatra, G. M. (2022). Supporting climate change understanding with novel data, estimation instruction, and epistemic prompts. Advance online publication. *Journal of Educational Psychology.* https://doi.org/10.1037/edu0000729

LEVERAGING TECHNOLOGY TO SUPPORT SOCIOPOLITICAL DISPOSITIONS

Nicholas E. Witt
Western Michigan University
nicholas.e.witt@wmich.edu

In this theoretical paper, it is argued that a crucial component of a critical literacy (Rubel & McCloskey, 2021) learning progression focused on Gerrymandering was the role of mathematical action technology (Dick & Hollebrands). I contend that the technology provided access to the relevant mathematics content, which contributed to an understanding of how the mathematics connects to the context. Empirical support from the implementation of the technology-based critical literacy lesson with prospective elementary and secondary mathematics teachers illustrates the emergence of a learning progression model. Implications regarding teacher education, curriculum development in K-12 classrooms, and integrating technology-based mathematics lesson set within critical literacy contexts will be discussed.

Keywords: technology, equity, social justice

The integration of innovative tools and curricular reform ideas is a challenge for many teachers. It is often easier and more efficient to rely on techniques and tools that have already been vetted and utilized previously. Thus, the need to better understand how technological tools such as *Mathematical Action Technology* (MAT) (Dick & Hollebrands, 2011), or digital technology, "…that can perform mathematical tasks and/or respond to the user's actions in mathematically defined ways," (p. xii) can be utilized effectively may be a necessary first step to encouraging more teachers to take up these practices. In addition to new tools, teachers also face challenges implementing new curricular reform efforts (Schoenfeld, 2004). In recent years, advocacy focused on teaching for the development of a sociopolitical disposition in students has become more common (NCTM, 2018; Bartell et al., 2017). Despite advocacy for integrating MAT and incorporating *critical literacy* (Rubel & McCloskey, 2021) contexts to support more equitable mathematics teaching practices, little to no research has been conducted on how these dual challenges interact with each other in practice. In this paper, I argue that we can create learning environments in which students are afforded opportunities to do mathematics which supports their ability to engage in discourse surrounding social injustices in their communities and society. I aim to describe how leveraging MAT to create a critical mathematical environment enabled prospective mathematics teachers (PMTs) enrolled in a Technology, Pedagogy and Content (TPC) course to engage with a mathematics lesson about Gerrymandering in new ways. In closing, I will describe some potential implications for pursuing the dual goals of teaching to develop a sociopolitical disposition and integrating MAT.

A Technology-Based Critical Literacy Learning Progression

In framing this discussion, I offer a learning progression which models the sequence of events that led to my students being able to articulate the mathematical concept of *compactness* and the role that it plays in instances of Gerrymandered legislative districts. The model below (see Figure 1) provides an approach to sequencing a unit or lesson to leverage the affordances of MAT to build mathematical knowledge which can then help students articulate connections between abstract mathematics and real-life applications.

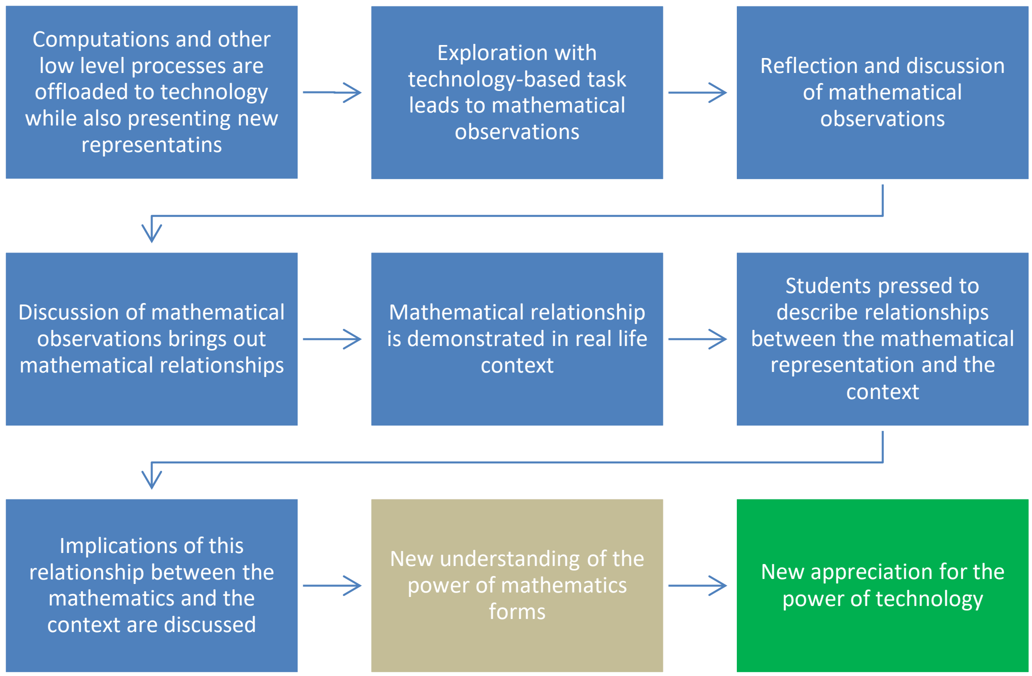


Figure 1: Technology-based critical literacy lesson progression

In Figure 1 above we see how this progression is dependent upon being able to make mathematical observations. Yet, as I will describe below, these observations may not have been possible without a dynamic representation of the polygons. In this sense, *low-cognitive demand tasks* (Stein, Grover, & Henningsen, 1997) such as computing area and perimeter and sketching each shape were offloaded onto the technology. This enabled more focus on describing important characteristics of the mathematical objects which guided the classroom discussion. The last two boxes in the figure represent hypothesized components that could serve as important extensions of this work. I highlight the final box in green because this aligns with a potential impact on the instructor whereas the gray box mostly applies to students. Though this progression above may bear some resemblance to other frameworks for teaching mathematical modelling set within real-life contexts (e.g., Cirillo, Bartell, & Wager; 2016), I aim to focus on how the leveraging of MAT, can make these lessons more viable for engaging in mathematical practices even when set withing real-life contexts.

Technology In Support of Equity

Though the ways in which technology can support equitable mathematics classrooms have been discussed by a number of scholars (e.g., Clements & Sarama, 2017; Kitchen & Berk, 2016; Dunham & Hennssey, 2008), I focus this paper on how technology can promote access to important mathematical content knowledge by supporting engagement in mathematical practices when the impracticality or tediousness of executing numerous low-level computations is offloaded onto the technology. In particular, the integration of MAT can support students' ability to do mathematics by allowing them to explore dynamic mathematical objects represented in novel ways leading them to make and test conjectures. When a large number of cases and multiple dynamic representations can efficiently be explored by students, this may support their ability to generalize their mathematical observations. In doing so, this can open up the possibility of student thinking being more focused on relationships between decontextualized mathematics and real-life applications of the mathematics. The idea that utilizing *technology as reorganizer* (Pea, 1985) can free the user to engage in higher order thinking skills has been discussed in a number of frameworks for integrating technology (e.g., Sherman & Cayton, 2015), however, it's connection to teaching within a critical literacy context is a novel contribution of the current discussion. When new mathematical understandings are developed, this can lead to a better understanding of real-life contexts that can be understood with respect to the mathematical relationships present within the context.

Teaching Mathematics in Support of Critical Literacy

Although the reasons that a teacher may choose to contextualize their mathematics lesson varies (Rubel & McCloskey, 2021), I continue this discussion by focusing on contextualization that may support the development of a sociopolitical disposition (Bartell et al., 2018). These *critical literacy* (Rubel & McCloskey, 2021) contexts may support what has been described as *teaching mathematics for social justice* (Gutstein, 2006) and may encourage students to see the power of mathematics in a new light. *Critical literacy contexts* include those which utilize mathematics to bring attention to social injustices which arise from: imbalances in power dynamics, a lack of or disproportionate allocation of resources, or unfair or discriminatory practices that target specific marginalized groups. An example of such a context might be using mathematics to understand the impacts of Gerrymandering in the U.S. redistricting process. (e.g., Berry III et al., 2020; Safi, Bush, & Desai, 2018; Rubel, Driskill, & Lesser, 2012). By engaging in a mathematics lesson devoted to helping students engage in discourse surrounding Gerrymandering, students can see how mathematics can help detect instances where a legislative district's boundary lines have been manipulated to benefit a particular political power. When this is done, voters are less likely to be represented by a person who will advocate for their needs. Gerrymandering often impacts communities of color by either breaking up high concentrations of voters of color or by ensuring these voters only have the minimum number of representatives by putting a high concentration of these voters into the same district. Thus, Gerrymandering, though practiced by both major political parties in the U.S., can have serious impacts in society due to the disenfranchisement of voters that results.

A number of challenges arise when teachers attempt to contextualize their mathematics lessons to promote critical literacy. One issue is that these topics are often seen as controversial and this may deter teachers because they fear the backlash that may occur. Another issue is that teachers may feel as though they must sacrifice mathematics goals to meet social justice goals or vice versa (Bartell, 2013). With this challenge in mind, I describe how I attempted to alleviate some of this perceived sacrifice by offloading low-level skills to the technology as I implemented an adaptation of Berry III et al.'s (2020) Gerrymandering unit.

Background on the Intervention Unit

In this section, I describe how prospective mathematics teachers (PMTs) enrolled in a technology, pedagogy, and content (TPC) course at a large midwestern university engaged in an intervention unit where they simultaneously developed their understanding of mathematics content, engaged in mathematical practices, and were able to make connections from their mathematical activity to a critical literacy context lesson based on the Gerrymandering section found in Berry III et al.'s (2020), *High School Mathematics Lesson to Explore, Understand, and Respond to Social Injustice*. In years past, the TPC course has focused on developing PMTs' *technological pedagogical content knowledge* (TPACK) (Mishra & Koehler, 2009). This ability to teach with technology and design lesson materials that leverage the affordances of MAT is developed through class discussions and readings, course assignments focused on finding and creating technology-infused resources, and a field experience where PMTs have the opportunity to design and teach a lesson that integrates MAT. The intervention unit took place during the Spring of 2021 and due to COVID-19 protocols, it was taught through a Web conferencing application. This intervention was developed to demonstrate how technology can be utilized to enhance a critical literacy context and then PMTs were asked to revise the lesson to include their own perspective to the lesson detailed below.

Throughout the transition to synchronous distance education, I frequently utilized Desmos Activity Builder (DAB) to facilitate mathematical discussions while also allowing students to engage with MAT. Desmos Activity Builder allows teachers to create activities that are similar to a set of presentation slides with the addition that these slides can contain MAT such as Desmos' Graphing Calculator application, Desmos' Dynamic Geometry Environment, and Input Boxes that allow students to provide responses that teachers can see update in real time through their Dashboard (see Figure 2 below).

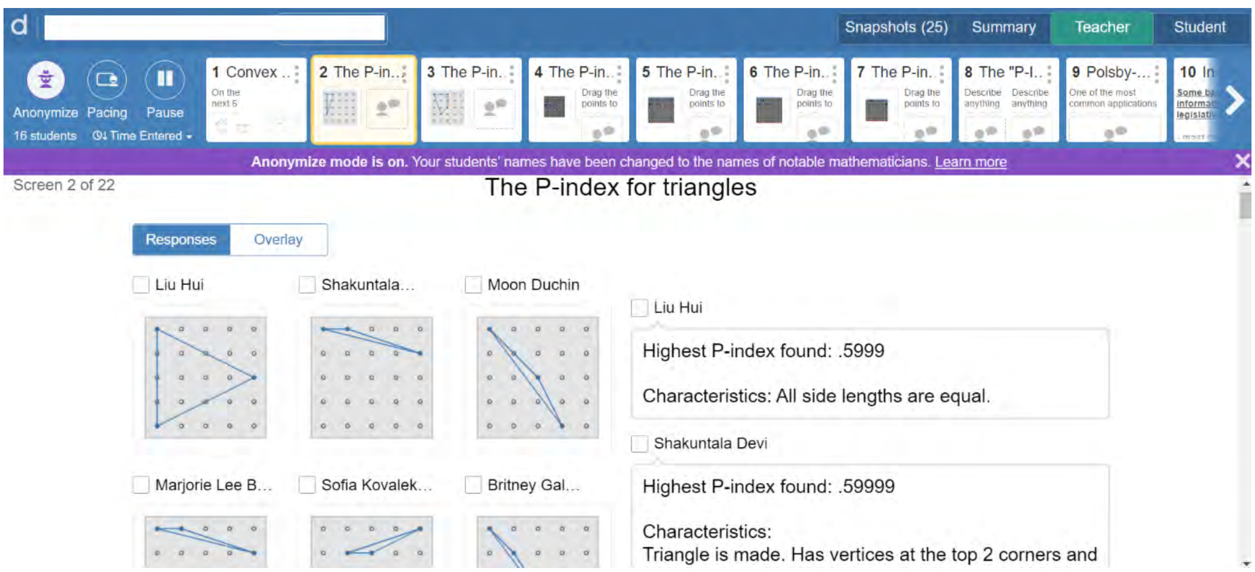


Figure 2. Desmos Activity Builder Dashboard View

As shown in the figure above, teachers can anonymize students' names (they are randomly assigned the names of mathematicians), capture their work as a Screenshot, and then select and present student work to facilitate classroom discussions. This was a common practice in the course and in particular, this lesson.

A Technology Based Lesson on Gerrymandering

In this section, I provide a detailed account of the sequences of activities that the PMTs in my course engaged in and some results of how PMTs engaged with the technology-based Gerrymandering lesson. A learning goal for the implementation of this lesson was to equip my students with the ability to answer the question of: *What connections between the mathematical context of exploring the effects of manipulating vertices of polygons can be made to the critical literacy context of Gerrymandering?*

During the implementation of the technology-based Gerrymandering lesson, PMTs accessed a set of Desmos Activity Builder slides (*available in unblinded draft*) where the first seven slides allow them to manipulate the vertices of various polygons while observing how a numerical "p-index" changes as they create different polygons. This "p-index" is commonly known as the Polsby-Popper Index, which assigns a value between zero and one that approaches one as the shape becomes closer to being a circle. This can be realized through the exploration by noting how the shapes that have the highest "p-index" are those that are as close to being a regular polygon as possible. To add a level of exploration and avoid students searching for the term "Polsby Popper" on the Web, I chose to rename this "p-index" for the first group of seven slides.

A second group of slides after the set exploring the Polsby Popper Index, provided some context for why this metric may be of interest. The second group of slides (see Figure 3 below), allows students to place a set of vertices around some maps of legislative districts in the United States. In doing so, the PMTs were later asked to consider why some districts may have a low Polsby Popper Index and some a large Polsby Popper Index.

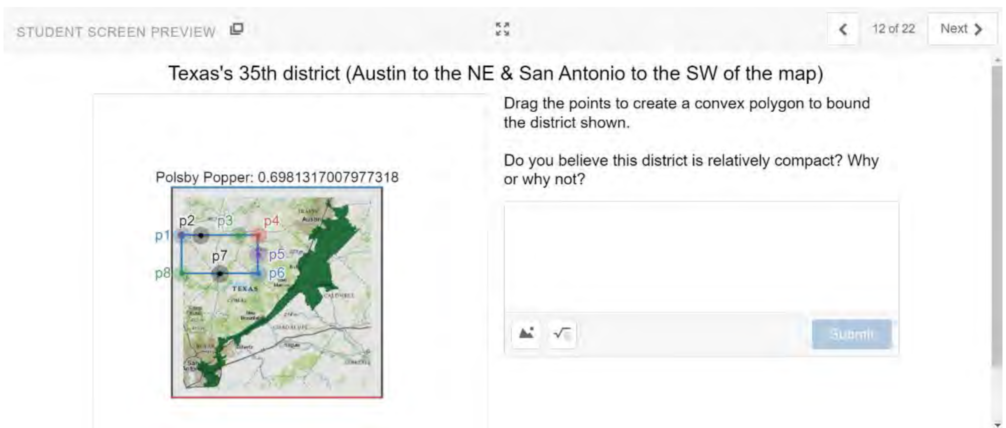


Figure 3. Estimating Compactness of U.S. Districts

This group of slides was then followed by a third group of slides where a hypothetical state's districts can be created by manipulating points surrounding sets of X's and O's representing a concentration of voters that typically vote for political party X and O. In engaging with this group of slides, PMTs were able to manipulate district lines in such a way that despite a state being composed of a majority of X's the districts could be drawn so that the districts elect the same number of representatives from the O political party (see Figure 4 below).

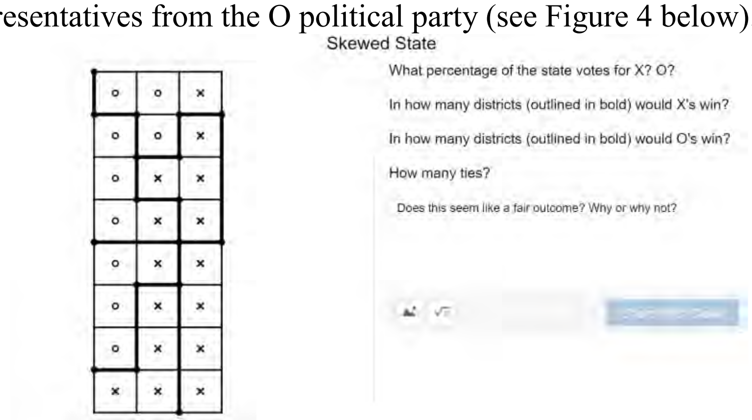


Figure 4. Gerrymandering in a Hypothetical State

Though the three groups of slides were implemented during the same class session, given the time constraints of K-12 classrooms, a similar lesson would be better suited as a sequence of three lessons with the first group of slides devoted to one lesson focused on building some intuition surrounding compactness of shapes as measured by the Polsby Popper Metric.

After the in-class activity facilitated through Desmos Activity Builder, PMTs were instructed to read a news article (https://truthout.org/articles/35-states-at-risk-of-rigged-districts-due-to-gerrymandering-report-finds/) about Gerrymandering's potential impact in the year to come and then reflect on why this is should be considered a social injustice. In addition to the reflection, they were provided with the following post-lesson prompt:

> "The Polsby-Popper Index, used to determine compactness of shapes, is found by the formula:
>
> $$PP(D) = \frac{4\pi(Area\ of\ Shape\ D)}{(Perimeter\ of\ Shape\ D)^2}$$
>
> Using this formula and what you now know about Gerrymandering, please explain why a Gerrymandered district might give a smaller value of PP(D)?"

How the MAT Supported Mathematical Exploration

In this section, I provide some description of what mathematical practices the PMTs engaged in, what new mathematical understanding appeared to emerge, and what connections they were able to make between the context of thinking about the compactness of an abstract polygon to thinking about the compactness of a shape that represents a legislative district.

During the first group of slides, PMTs explored the nature of the Polsby-Popper Index by dragging vertices of abstract polygons. As the points were dragged, the value of a "p-index" changed simultaneously. On these slides, PMTs recorded observations about what characteristics they observed in the shapes that yield the highest and lowest "p-index." At the close of this group of slides, they were asked to record observations about the shapes that occurred across all six polygon exploration slides in the group. This group of slides was effective in supporting PMTs to engage in conjecturing about the nature of the Polsby Popper Index which were then revised after further exploration and classroom discussion. Conjectures about the nature of the Polsby Popper Index were seen as a number of PMT responses were initially focused on superficial observations or less relevant properties such as the angle measures, but later responses indicate a shift in attention from angle measures to the regularity of the shape. For example, some responses regarding the hexagon exploration are shown in Table 1 below.

Table 1: Responses to the prompt "Drag the points to create the highest and lowest P-index that you can and then describe some characteristics of the shape you created"

Student (Pseudonym)	Initial Observation
Rey	"making as small of angles as I could helped to make P-index smaller"
Kris	"…a normal hexagon"

Later in the lesson these ideas appear to have been refined through the exploration and discussion so that their ideas now align more closely with what my goals for this part of the lesson were. In Table 2 below we see how these same students above refined their initial ideas above to observe something about the regularity of the shape or the ratio of the area to the perimeter.

Table 2: Refined understandings of the "p-index"

Student (Pseudonym)	Refined Understanding
Rey	"I think compactness measures how close or far apart the vertices of a shape are. Which makes the shape either close to equal in sides and angles (regular), or narrow in shape (irregular)."
Kris	"normal polygons had the highest p-index values, whereas the ones with the smallest area-to-perimeter ratio had the lowest p-index values."

Responses and ideas such as these could not have been organically developed by students in this lesson without the aid of technology. In the activity, students were able to simultaneously generate a vast number of polygons with different dimensions and have the corresponding Polsby Popper Index calculated for them. This allowed them to focus their attention on the properties of the shapes such as the regularity or the ratio of the area to the perimeter. In order for a paper-based lesson to bring out the similar ideas without being explicitly presented with these ideas, it might require students to construct a number of different polygons on either the same grid which would end up being very cluttered or it may require multiple sheets of paper. Students might then use a dissection method for calculating the area of each triangle formed through a dissection so that they could find the area of the polygon or they may count the number of grid squares within each shape. Both of these methods would be both inefficient and may be considered low cognitive demand.

Connecting the Abstract Mathematics to the Critical Literacy Context

Connecting decontextualized mathematics to contextualized problems is often challenging for students as the context can obscure the underlying concepts previously explored. In Table 3 below, the responses from Rey and Kris came from their post-lesson prompt described above. One can see that Rey mentions the role of area and perimeter, but why this may help to detect Gerrymandering is less clear. Rey does appear to see some connection between "balance" and the "weight" of votes, which I believe to be a step towards seeing a connection between district lines being drawn to produce a greater *efficiency gap* (Stephanopoulos & McGhee, 2015) and how this could disenfranchise voters when their votes are seen as having less weight. On the other hand, we see Kris make strong connections between boundaries being "stretched" and how this results in a greater perimeter for the district while not necessarily changing the area or even making the area smaller.

Table 3: Post-lesson prompt responses connecting the mathematics to Gerrymandering

Student (Pseudonym)	Response
Rey	"This formula shows that the P-index is the relationship between the area and perimeter of districts. A gerrymandered district might give a smaller value of PP(D) because they are drawn so that some districts don't "matter" as much as others. To my understanding, a smaller PP(D) shows that they have less weight in their votes. When districts are more compact, they are more balanced in their voting. When the PP(D) of all districts are closer to one

	another, this will make votes and the districts themselves more balanced. Lastly, it would reduce the racial gerrymandering that is happening.
Kris	"Gerrymandered districts are not very compact because they are often stretched far and thin and encompass regions that are far apart from each other despite other districts having closer proximity to themselves. As a shape is stretch or distorted, the relative area on the inside becomes a much smaller value in comparison to the perimeter that surrounds it. Much like how flattening a basketball maintains a constant surface area, but significantly reduces the internal volume. Therefore, the more gerrymandered a district is, the larger the denominator will be under the district's area"

The results from the previous two sections together exemplify a rather typical trajectory in the TPC course where initial ideas were developed through exploration with MAT, which led to a refinement of these initial ideas about the mathematics. This refined understanding of the mathematics supported a relatively accurate description of the connection between the mathematics discussed in the class lesson to the context of Gerrymandering.

Discussion

Within this *Technology-Based Critical Literacy Mathematics Lesson,* the role of MAT was critical in enabling my students to develop new understandings about the critical literacy context. This was done by equipping them with an ability to communicate the connections between compactness and Gerrymandering. The empirical support with prospective mathematics teachers demonstrates that with MAT, it is possible to simultaneously teach cognitively demanding mathematics content while also bringing about new understandings of social injustices. Much more work can to be done to provide support for the notion that leveraging the affordances of technology can generate novel approaches to exploring critical literacy contexts within a K-12 classroom. Future research may focus on how this type of lesson may support developing K-12 students' mathematical and technological identities as well as a critical eye towards sociopolitical contexts. Future curriculum work may focus on identifying, generating, an implementing other *Technology-Based Critical Literacy Mathematics Lessons*. Finally, the effect on teachers' dispositions regarding technology use and teaching mathematics through critical literacy contexts is also needed to understand how teachers could be better supported in implementing equitable teaching practices.

References

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. Journal for Research in Mathematics Education, 48(1), 7-21.

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. Journal for Research in Mathematics Education, 44(1), 129-163.

Berry III, R. Q., Conway IV, B. M., Lawler, B. R., & Staley, J. W. (2020). High school mathematics lessons to explore, understand, and respond to social injustice. Corwin Press.

Cirillo, M., Bartell, T. G., & Wager, A. A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. R. Hirsch (Ed.), Annual perspectives in mathematics education 2016: Mathematical modeling and modeling mathematics (pp. 87–96). Reston, VA: National Council of Teachers of Mathematics

Clements, D. H., & Sarama, J. (2017). Valid issues but limited scope: A response to Kitchen and Berk's research commentary on educational technology. Journal for Research in Mathematics Education, 48(5), 474-482.

Dick, T. P., & Hollebrands, K. F. (2011). Focus in high school mathematics: Technology to support reasoning and sense making (pp. xi-xvii). Reston, VA: National Council of Teachers of Mathematics.

Dunham, P., & Hennessy, S. (2008). Equity and use of educational technology in mathematics. Research on technology and the teaching and learning of mathematics, 1, 345-418.
Gutstein, E. (2006). Reading and Writing the World with Mathematics: Toward a Pedagogy for Social Justice. Taylor & Francis.
Kitchen, R., & Berk, S. (2016). Research Commentary: Educational Technology: An Equity Challenge to the Common Core. Journal for Research in Mathematics Education, 47(1), 3-16.
Koehler, M., & Mishra, P. (2009). What is technological pedagogical content knowledge
(TPACK)?. Contemporary issues in technology and teacher education, 9(1), 60-70.
Pea, R. D. (1985). Beyond amplification: Using the computer to reorganize mental functioning. Educational psychologist, 20(4), 167-182.
Safi, F., Bush, S. B., & Desai, S. (2018). Gerrymandering: When Equivalent Is Not Equal!. Mathematics Teaching in the Middle School, 24(2), 82-89.
Schoenfeld, A. H. (2004). The math wars. Educational policy, 18(1), 253-286.
Sherman, M., & Cayton, C. (2015). Using appropriate tools strategically for instruction. The Mathematics Teacher, 109(4), 306-310.
Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. American educational research journal, 33(2), 455-488.
Stephanopoulos, N. O., & McGhee, E. M. (2015). Partisan gerrymandering and the efficiency gap. U. Chi. L. Rev., 82, 831.
Rubel, L. H., & McCloskey, A. V. (2021). Contextualization of mathematics: which and whose
world?. Educational Studies in Mathematics, 1-22.
Berry III, R. Q., Conway IV, B. M., Lawler, B. R., & Staley, J. W. (2020). High school
mathematics lessons to explore, understand, and respond to social injustice. Corwin Press.
Rubel, L. H., Driskill, M., & Lesser, L. M. (2012). Decennial redistricting: Rich mathematics in context. The Mathematics Teacher, 106(3), 206-211.
Rubel, L.H., McCloskey, A.V. (2021). Contextualization of mathematics: which and whose world?. Educational Studies in Mathematics (107) 383–404. https://doi.org/10.1007/s10649-021-10041-4

EMBODYING COVARIATION THROUGH COLLABORATIVE INSTRUMENTATION

Toni York	Steven Greenstein	Denish Akuom
Montclair State University	Montclair State University	Montclair State University
yorka1@montclair.edu	greensteins@montclair.edu	akuomd1@montclair.edu

Covariational reasoning and the creation and interpretation of graphs of covariational situations are important skills in math and science. Unfortunately, research shows that students often struggle to make meaningful connections between graphs and the covariational situations they represent. Educational activities designed to help students overcome this struggle tend to use either student-generated or automatically-generated graphs, and have students either act out covariational situations or more passively observe them. In this paper, we present the design of a tool and task that enabled two students to simultaneously embody both the creation of a graph and the covariational actions that the graph represents. Through a process of collaborative instrumentation, the students made meaningful connections between their motions and the embodied traces they created as they reasoned about the covarying quantities of height and time.

Keywords: Algebra and Algebraic Thinking, Middle School Education, Technology, Instructional Activities and Practices

Covariational reasoning, or the ability to coordinate changes in two covarying quantities (Carlson et al., 2002; Thompson & Carlson, 2017), is an important part of how students understand and use graphs. However, research shows that students of all ages often struggle to make connections between graphs and the covariational situations they represent (Oehrtman et al., 2008). For example, students commonly view graphs more as wireframe images of a situation rather than records of covariation, seeing static shapes rather than emerging traces (Moore & Thompson, 2015).

Many different educational activities have been designed and tested that attempt to support students in making these connections. One of the many decisions faced by the designers of these activities is a choice about whether to use student-generated or automatically-generated graphs. Students may create their own graphs and thus see how they are built, but many students struggle with this process and the underlying covariational meaning is lost (e.g., Mevarech & Kramarsky, 1997). Conversely, graphing utilities allow students to focus on the covariation without having to worry about the graphing process itself, but students then tend to blindly trust these automatically-generated graphs without thinking more deeply about what they represent (e.g., Cavanagh & Mitchelmore, 2000).

Similarly, designers must also choose whether students will act out the covariational situation being graphed or more passively observe it. For example, designs using motion sensors paired with automatically-generated graphs allow students to literally walk their covariation and see the resulting graph appear in real time (e.g., Duijzer et al., 2019). In contrast, designs that provide animations of a covariational situation allow students to observe or even control a covariational situation but not to physically act it out themselves (e.g., Stevens et al., 2017).

We categorize the designs resulting from these two decisions by whether or not the covariational situation and the corresponding graphing are each embodied (e.g., Lakoff & Johnson, 2000; Núñez, et al., 1999; Wilson, 2002) by the students. Some designs have tried to support student embodiment of both covariation and graph creation. For example, designs aimed at teaching kinematics to college students have used ball-and-track activities paired with student-generated graphs (e.g., McDermott et al., 1987). In these activities, students both generate the motion of a physical ball in a track they have assembled and then also graph that motion. Along with instruction and practice, McDermott et al. found that this design supported improvement in college students' covariational understanding of kinematics graphs.

However, we found no design in the literature that has students simultaneously embodying both the act of graph creation and the covariational situation being graphed. We therefore conjecture that leveraging the benefits of embodying both covariation and the corresponding graph creation at the same time may help students to make these connections between graphs and the covariational situations they represent. Our research question is thus: *What kinds of graph thinking and covariational reasoning might students engage in while working with a tool designed to support collaborative instrumentation and the simultaneous embodiment of both graphing and covariation?* Accordingly, we address this question through research framed by the theories of covariational reasoning, embodied design, instrumented activity, and intersubjectivity.

Theoretical Framework

When students coordinate two varying quantities (Thompson, 1993, 2011) by attending to and making sense of the ways the quantities change in relation to each other, they engage in *covariational reasoning* (Carlson et al., 2002; Thompson & Carlson, 2017). Covariational reasoning is fundamental to students' understandings and uses of graphs precisely because *graphs* in two dimensions are representations, or traces, of the relationship between two covarying quantities. In order to support the development of students' covariational reasoning, Moore and Thompson (2015) endorse a particular way of thinking about graphs as emergent relationships called emergent shape thinking. Whereas *static shape thinking* relies on perceptual cues and global properties of graphs as static objects, *emergent shape thinking* involves understanding a graph simultaneously as what is made (a trace in progress) and how it is made (as a record of the relationship between covarying quantities). These ways of thinking about graphs will be useful in our analysis of students' graphing activity as we aim to help them make connections between graphs and the covariational situations they represent.

As we describe below, the first author embarked on a design experiment (Cobb et al., 2003) to produce a tool to mediate the formation of these connections. The tool used in this project emerged from a design cycle informed by the theory of embodied cognition. Rather than consisting solely of computational operations on symbolic propositions that occur entirely within a disconnected mind, *embodied theories of cognition* hold that cognitive processes "must be understood in the context of [the mind's] relationship to a physical body that interacts with the world. … Hence, human cognition, rather than being centralized, abstract, and sharply distinct from peripheral input and output modules, may instead have deep roots in sensorimotor processing" (Wilson, 2002, p. 625). From this perspective, even the most complex ideas are grounded in and emergent from the lived reality of bodily experiences in the world.

Empirical evidence in support of embodiment theories of cognition (e.g., Lakoff & Núñez, 2000) has impacted mathematics education research (e.g., Abrahamson & Lindgren, 2014). In that context, mathematics must be reconceptualized and accounted for not as "an objective mathematics, independent of human understanding" but rather "in terms of the human bodily-based and situated conceptual systems from which it arises" (Núñez et al., 1999, p. 47). Accordingly, such an account should be useful for devising more effective instruction through grounded learning situations that foster meaningful mathematical understanding. Abrahamson's (2014) pedagogical framework of *embodied design* provided us with a template with which to do so as it gave language and structure to the process by which the *Graph Tracer* tool (presented below) was designed. Through *phenomenalization*, we hypothesized about the embodied schema that underlie the coordination and graphical tracing of two covarying quantities. Through *concretization*, we digitally designed and 3D printed a physical artifact with which two learners could collaboratively enact their solutions to problems that would have them enact those schema. And through *dialog*, we elicited the learners' informal enactments of covariational reasoning and supported their formalization by engaging them in reflective activity.

Lastly, in order to analyze the contribution of the collaborative aspect of two students' graphing activity, we drew on the theories of instrumented activity and intersubjectivity. Vérillon and Rabardel's (1995) theory of *instrumental genesis* provides a framework for analyzing student learning in the process through which objective artifacts become tools to be used to accomplish a task. From this perspective, an artifact (e.g., a physical or digital tool) becomes an instrument (e.g., tool, sign) when a subject (e.g., actor, learner) has integrated it into a conceptual scheme for a specific implementation of the artifact in the context of their goal-directed activity. Thus, an instrument is a *psychological* construct established by the subject's instrumental relation with the artifact. In practice, an instrumented activity is characterized by individual subject-object, subject-instrument, and object-instrument interactions.

In our case, however, with two subjects working collaboratively, we will be interested in what we refer to as the subjects' *collaborative instrumented activity*. These are captured in the range of $subject_1$-$subject_2$ instrumented interactions. Then, in tandem with this analysis, we leverage Vygotsky's (1978) notion of *intersubjectivity* to examine the interpsychological process by which the two subjects negotiate and co-construct a shared conceptual scheme for their instrumented activity through the verbal and nonverbal communications in their graphing activity. Gillespie and Cornish's (2010) notion of intersubjectivity adds some texture here by considering "the mutual awareness of agreement or disagreement" (p. 3) in addition to shared understandings and even misunderstandings. Thus, for the sake of our analysis, intersubjectivity manifests itself in participants' actions and intentionalities towards each other and in their joint movement toward negotiated and developed meanings and actions.

Methodology

In a design experiment (Cobb et al., 2003), researchers iteratively design, test, and revise both an instructional design and their theories about how that design supports student learning. The designs of the tool, task, and questioning presented in this paper emerged from the third iteration of an ongoing design experiment that seeks to produce a tool that can be used to help students make meaningful connections between height/time graphs and the kinds of real covariational situations that these graphs represent. We briefly describe the second iteration design in order to provide context for the third iteration presented in this paper.

In the second iteration, the design involved a video of simultaneous covariational motion and graph creation that students could view and explore. Specifically, a video of a vertically moving object (Figure 1, left) was played on a cell phone while the phone was slid horizontally from left to right (Figure 1, right). This action was then itself recorded to create a second video in which a height/time graph of the relationship between the original object's height and the trace of its motion could be seen. Students could then use familiar video playback controls to drag the video's progress bar back and forth and observe the graph being emergently traced by the object's motion. However, even when they were provided with an animated graphical overlay depicting the graph in the final version of the video, students still struggled to connect the horizontal movement to the vertical motion in the original video. Moreover, they were unable to imagine similar sliding videos they could rely on to create or interpret other height/time graphs.

Figure 1: Vertical Video Screenshots (left) and Sliding Video Screenshots (right)

In the current iteration of this design experiment, we sought to build on that earlier iteration to create a more grounded and embodied experience for the students than the sliding video seemed to offer. While the sliding video situated the covariational motion and the graph creation as simultaneous processes, it did not directly leverage students' own simultaneous embodiment of both of these processes. The framework of embodied design thus informed the third iteration, which yielded a design that includes a tool called the Graph Tracer (Figure 2, left) along with a set of tasks using that tool. The Graph Tracer is a 3D-printed rectangular prism with two cutouts: one on the bottom so that a paper can be slid beneath it, and one down through the top so that a marker can be inserted upright and moved up and down to draw on the paper below.

Figure 2: The Graph Tracer (left) and Collaborative Graph Tracer Use (right)

The Graph Tracer is intended to be used by a pair of students, one of whom controls the speed of the paper sliding beneath it while the other controls the up and down motion of the marker (Figure 2, right). One student thus embodies the quantity of time while the other embodies the quantity of height, and we hypothesize that through their collaborative intersubjective engagement, they both embody time and height as they work together to use the Graph Tracer. The result is the creation of a visual trace of the marker's motion over time, in other words, a height/time graph representing that motion.

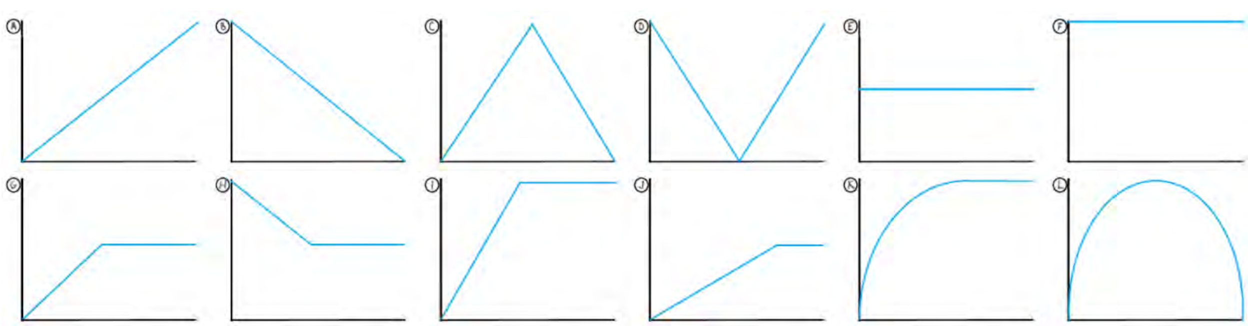


Figure 3: The Ordered Set of Traceable Graphs A-L

The task designed for use with the Graph Tracer includes a set of papers with an ordered group of preprinted graphs involving a variety of different slopes (Figure 3). A pair of students are asked to work together to use the Graph Tracer tool to trace each graph in order, reflect on their experience of doing so, and compare and contrast the different graphs. Examples of specific questions include “What did you have to do to make this graph?” and “Compare this graph to your previous graph, how is it similar or different?” The provided papers also include blank graphs, on which students can experiment with making their own traces.

As part of the design experiment, this design was tested in a task-based clinical interview (Clement, 2000; Ginsburg, 1981) with two students, “Eri” and “Robinson.” At the time of data collection, Eri was a twelve-year-old in the 7th grade who had some experience with graphing points in a coordinate plane but expressed that she was not familiar with graphing lines. Robinson, Eri’s younger brother, was a ten-year-old in the 4th grade who described having no graphing experience. The interview took place over Zoom with the designer/researcher (DR, first author) leading the interview and a partner researcher (PR, the students’ parent) in the room with the students. Collected data includes the video recording of the 45-minute interview, scanned copies of the students’ graphs, and the researchers’ field notes. Through a retrospective analysis (Cobb et al., 2003) of that data, we examined the students’ collaborative graphing activity using the Graph Tracer from both instrumented activity and covariational reasoning perspectives.

Findings

Here we present two excerpts from Eri and Robinson’s activity. The first exemplifies the intersubjective process through which the Graph Tracer became a collaborative instrument for them. The second shows how Eri leveraged her instrumented use of the Graph Tracer to revise her reasoning and apply it in a more sophisticated graphing scenario.

When they were first handed the Graph Tracer, neither Eri (Figure 4, left) nor Robinson (right) had ever seen it before or had any idea how to use it. Although we started the interview by explaining the goal of making a picture that represented height changing over time, we did not demonstrate the use of the Graph Tracer. The students thus began their explorations by enacting various ways they imagined the tool might be used, first tracing the y-axis of Graph A and then turning the Graph Tracer diagonally to trace the graph without moving the paper. At this point in their collaboratively instrumented activity, the students were interacting with each other and the Graph Tracer to develop a shared scheme for its use (Vérillon & Rabardel, 1995) but not yet with the object of the Graph A and its covariational meaning.

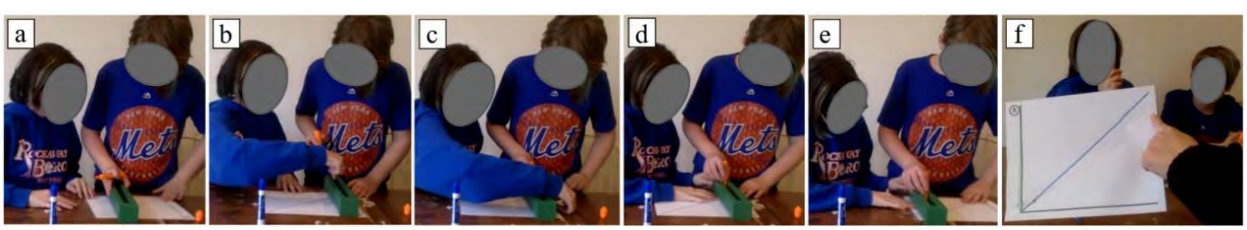


Figure 4: First Trace of Graph A

Once we clarified the constraint that the Graph Tracer must remain parallel to the y-axis, the students then started to explore how they could trace the graph by moving the paper and the marker simultaneously. Robinson first demonstrated how he thought the paper could be slid beneath the Graph Tracer (Figure 4a). Eri then used her finger to show the marker’s possible motion (Figure 4b) as she wondered, “What if you do [that slide of the paper] and you start moving [the marker] upward?" Figures 4c-e show how the pair attempted to implement their idea, first with Eri reaching over to pull the paper and then pushing from her side. They had reorganized themselves so that one person could hold the Graph Tracer while the other pulled the paper. Thus, the tool's intended (Malafouris, 2013) utilization scheme (Verillon & Rabardel, 1995) and the associated task prompted the pair's coordination of initially individual (i.e., uncoordinated movements of the pen and paper) and eventually collaborative instrumentation.

As they drew this first trace, the students began to interact with the graph as well as with each other and the Graph Tracer. In this attempt, Eri embodied the quantity of time by controlling the paper while Robinson simultaneously embodied height by controlling the marker. The pair was able to complete their trace of Graph A, but were not happy with their result (the orange line in Figure 4f; see also Figure 6). Eri had pulled the paper relatively quickly and Robinson remarked afterwards: "that was not traced." We interpret their dissatisfaction with the result as though they struggled to negotiate a collaborative scheme to enact a smooth trace.

The pair then decided to try tracing Graph A again on the same page with a different color marker and a different tracing technique. Eri asked to switch roles so that she could control the marker as Robinson controlled the paper (Figure 5a). Robinson offered a suggestion in response: "You move up slowly while I pull the paper this way." With this slower attempt and some help from the PR in steadying the paper (Figure 5b), they were better able to coordinate their actions to achieve the trace they wanted. In Figure 5c, the pair successfully completed a trace of Graph A using this new technique.

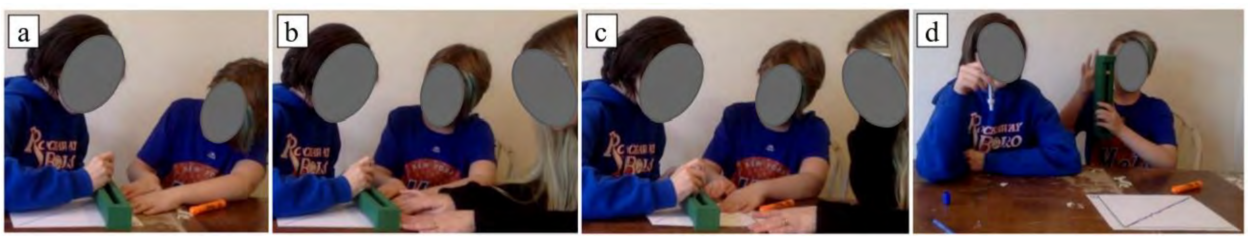


Figure 5: Second Trace of Graph A

Finally, the trace that emerged from this second attempt was placed on the table in front of the pair (Figure 5d; see also Figure 6) as they responded to the DR's question: "How did you have to move the marker to get that picture?" Robinson enacted his answer nonverbally by holding the Graph Tracer up in the air in front of himself and moving his finger slowly upwards inside the marker track. Eri motioned similarly in the air with the marker she was still holding in her hand as she also answered out loud, "I just went like... very slowly upwards."

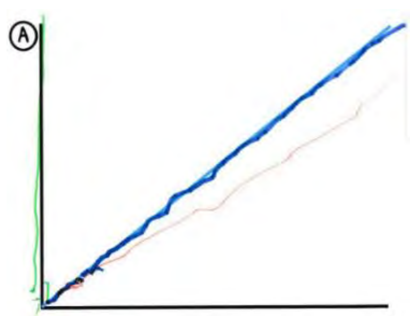


Figure 6: Eri and Robinson's Three Graph A Traces:
The *y*-Axis (green), The First Attempt (orange), and The Second Attempt (dark blue)

The students' actions and reflections as they enacted their three traces of Graph A (Figure 6) show their process of collaborative instrumentation. As the pair negotiated how to use the Graph Tracer and attended to details of paper and marker speed in their tracing actions, the graph and Graph Tracer became an instrument with which they intersubjectively reasoned covariationally about height and time. Both their successful trace and their demonstration of the marker's motion as a change in height over time show how they had connected their enacted tracing of the graph with its covariational meaning. The completion of this trace and the students' articulations of their activity are evidence of the first major step the pair took towards enacting a shared understanding of how marker speed affects graph shape when using the Graph Tracer.

As the pair continued to enact traces of other graphs, they refined their collaborative scheme for creating smooth traces as they reflected on how their individual actions influenced their collaborative traces in different ways. For instance, when asked to compare and contrast Graphs G, I, and J by ordering them in increasing speeds of the marker motions required to trace them, Eri and Robinson initially disagreed. Robinson's intuition was that faster motion of the marker would make steeper graphs, which we interpret as an *emergent* (Moore & Thompson, 2015) view of the graph as a trace of the marker's motion. However, Eri based her reasoning on how long her subjective experience of drawing each graph had felt and the maximum height a graph reached, which we interpret as a *static* view of the graph based on its visual features.

The disagreement was resolved as they experimented by drawing three different lines at three different speeds on the same blank graph and then comparing their speeds and the steepness of those lines (Figure 7, left). This analysis convinced Eri to change her mind as she came to Robinson's way of thinking. We interpret that she did so by considering Robinson's movement of the paper in addition to the time she felt as she drew the graph. Thus, the slope of the graph seemed to have become a multiplicative object for Eri, composed of both embodied actions. As the episode continued and the pair began to develop collaborative schemes for creating and interpreting their own height/time graphs, Eri's initial static reasoning gave way to more emergent reasoning about the graph as a trace of the marker's motion over time.

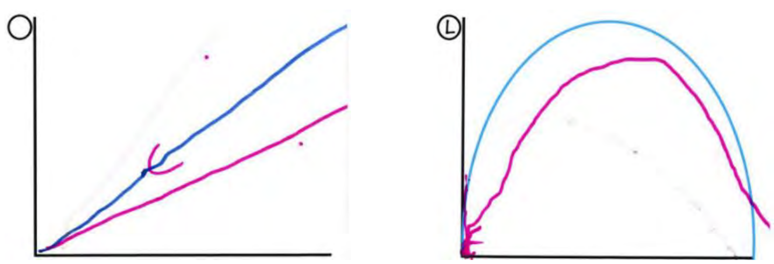


Figure 7: Eri's Marker Speed Experiment (left) and Eri's Trace of Graph L (right)

Although Robinson grew tired of the activity and eventually left the interview, Eri remained and took some time to consider Graph L. She used the Graph Tracer (with the PR's help) to attempt to trace the curve of Graph L and produced an imperfect but structurally similar result (Figure 7, right). She then expressed emergent shape thinking as she explained what she had done: She moved the marker the fastest at the ends of the curve and slowest along the top. Although Robinson's departure put an end to the pair's collaboratively instrumented activity, Eri's continued solo instrumentation via this activity seemed to support her new awareness of how the marker's speed impacted the shape of the resulting trace. This completed her process of enacting her new understanding of that relationship and showed further evidence of her emergent reasoning about the graphs she had created with the Graph Tracer.

Concluding Discussion

Using our analytic lens of collaborative instrumentation, the findings presented in this paper show how Eri and Robinson were able to move together from complete unfamiliarity with height/time graphs and the Graph Tracer tool towards a more fluent and collaborative stage in their instrumented use of the Graph Tracer and in the shared meanings they made of graphs given to or constructed by them. The Graph Tracer's requirement for simultaneous embodiment (e.g., Lakoff & Johnson, 2000; Núñez, et al., 1999; Wilson, 2002) of both the covariational situation and the creation of the corresponding graph also provided the students with a rich opportunity to reason about how these two acts are meaningfully related. Through their joint activities and their discussion of the relationship between marker speed and graph shape, both students worked together to enact their growing transformation of those artifacts into instruments they could use to reason emergently (Moore & Thompson, 2015) about height/time graphs. Eri also displayed a more sophisticated understanding of how marker speed influenced the steepness of the graph, thus beginning to interact with the idea of slope, an important graphing concept.

To continue this design experiment, we are interested in refining the design of both the Graph Tracer tool as well as the task, traceable graphs, and questioning used in this iteration. Eri and Robinson encountered physical difficulties in bracing the Graph Tracer and sliding the paper smoothly. A solution as simple as working on a smoother surface might mitigate the issue, but this solution places the burden on the user rather than the designer. Instead, the addition of a tray to brace the Graph Tracer and make the paper easier to slide could be considered. Alternately, it might be worth laminating the traceable graphs and using dry erase markers. Furthermore, modifications to the physical dimensions of the Graph Tracer like narrowing its width might improve students' ability to see their trace as they work, but this would need to be tested to determine the effects on durability and stability during use. Future iterations might also refine the questioning in order to elicit deeper and more elaborate descriptions of students' reasoning as they work. Possible directions that could benefit from deeper questioning include what students feel they are struggling with and why they choose certain actions as they use the Graph Tracer.

One limitation of the study is the messiness of the interview setting. The online setting presented barriers to conducting an effective interview, including difficulties seeing each other and the students' work as well as the challenge of two researchers conducting an interview together remotely. However, from our analysis, we conclude that the current design of the Graph Tracer has strong potential for use in the learning of covariation and graphing, thereby challenging the status quo of how these topics are traditionally taught. Future work might include refining the design of the Graph Tracer as well as gathering more data and performing further analysis to examine students' collaborative use of the Graph Tracer to reason about slope. We would also like to develop new tasks and gather new data to explore how students might use the Graph Tracer to reason about other graphing and function concepts such as rate of change, concavity, and points of inflection. Implementation of this design in a classroom setting could also support an investigation into how the Graph Tracer might be used as part of the regular middle school curriculum. Finally, it would be interesting to explore other designs that could also support the simultaneous embodiment of both covariation and covariational graphs and to further assess the analytic value of the collaborative instrumentation concept.

Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. 1812887. We would also like to thank our colleagues Helene Leonard and Eileen Fernández for their help with this project.

References

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction, 2*(1), 1-16.

Abrahamson, D., & Lindgren, R. (2014). Embodiment and embodied design. In R. K. Sawyer (Ed.), *The Cambridge Handbook of the Learning Sciences* (2 ed., pp. 358-376). Cambridge University Press.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-378.

Cavanagh, M., & Mitchelmore, M. (2000). Student misconceptions in interpreting basic graphic calculator displays. In *PME CONFERENCE* (Vol. 2, pp. 2-161).

Clement, J. (2000). Analysis of Clinical Interviews: Foundations and Model Viability. In A. E. Kelly & R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547-589). Lawrence Erlbaum Associates.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Duijzer, C., Van den Heuvel-Panhuizen, M., Veldhuis, M., & Doorman, M. (2019). Supporting primary school students' reasoning about motion graphs through physical experiences. *ZDM, 51*(6), 899-913.

Gillespie, A., & Cornish, F. (2010). Intersubjectivity: Towards a dialogical analysis. *Journal for the Theory of Social Behaviour, 40*(1), 19-46.

Ginsburg, H. (1981). The Clinical Interview in Psychological Research on Mathematical Thinking: Aims, Rationales, Techniques. *For the Learning of Mathematics, v1 n3* p4-11 Mar 1981.

Lakoff, G., & Núñez, R. E. (2000). *Where Mathematics Comes From: How the embodied mind brings mathematics into being*. Basic Books.

Mevarech, Z. R., & Kramarsky, B. (1997). From verbal descriptions to graphic representations: Stability and change in students' alternative conceptions. *Educational Studies in Mathematics, 32*(3), 229-263.

McDermott, L. C., Rosenquist, M. L., & Van Zee, E. H. (1987). Student difficulties in connecting graphs and physics: Examples from kinematics. *American Journal of Physics, 55*(6), 503-513.

Moore, K. C., & Thompson, P. W. (2015). *Shape thinking and students' graphing activity*. Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*, 45-65.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, D.C.: Mathematical Association of America.

Paoletti, T., Lee, H. Y., & Hardison, H. L. (2018). *Static and emergent thinking in spatial and quantitative coordinate systems.* Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Stevens, R. (2011). The Missing Bodies of Mathematical Thinking and Learning Have Been Found. *Journal of the Learning Sciences, 21*(2), 337-346.

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B., & Hardison, H. (2017). Principles for Designing Tasks that Promote Covariational Reasoning. In *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education*.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*, 165–208.

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education. WISDOMe Monographs* (Vol. 1, pp. 33–57). Laramie, WY: University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 421-456).

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review, 9*(4), 625-636.

Chapter 17:

Working Groups and Colloquia

FUTURE DIRECTIONS FOR STATISTICS EDUCATION

Travis Weiland
University of Houston
tweiland@uh.edu

Christopher Engledowl
New Mexico State university
chriseng@nmsu.edu

In continuing from past working groups on statistics education, this working group met for PME-NA 43 to present new findings in the field and to make new connections, as well as work toward future directions. Each day of the working group was based on a different theme including measurement, issues of equity and social justice, and data science. Each day included short presentations of a few projects in line with the theme, followed by discussions of the theme in statistics education, and ended with discussions of future directions related to the theme.

Keywords: Data Analysis and Statistics, Measurement, Equity, Inclusion, and Diversity

The recent data science revolution in K-12 curriculum as well as the pandemic laying bare issues with the statistical literacy of the citizenry have made statistics education gain traction and attention in educational research. The pandemic has brought data and data-based arguments front and center in the media and people are increasingly able to openly access data for themselves (Ancker, 2020). Data science has also found a new hold in the K-12 mathematics curriculum. This can be seen in added focus on data science in the new GAISE II report (Bargagliotti et al., 2020) and based on curriculum reforms in California that seem to be reinvigorating the old math wars. The goal of this working group is aimed at continuing discussions begun at previous meetings, and to create space for new work and collaborations. In an effort to create an open environment but also focus on specific elements of the field, the working group was be structured around three themes. The working group consisted of the leaders (authors) as well as approximately 6-10 participants each day of the conference. We had the largest number of returning participant this year of any of the three years we have done this working group, which was exciting. Furthermore, the group had participants from a wide range of expertise including university faculty, doctoral students, and researchers, which allowed for rich and diverse conversations during the conference. One of the greatest challenges in statistics education is creating community and growing the field and those are exactly what this group aims to do. Our primary product each year is increased connections between researchers, faculty, and doctoral students and creating new spaces for collaboration and discussion. In this post conference report, we focus on highlighting some of the major ideas and topics that came up in our discussions during the working group.

Discussion in Relation to Themes

The working group was structured around three main themes: measurement, issues of equity and social justice, and data science. The structure allowed people to freely flow in and out of the group based on interests from day to day. The themes also helped us to focus our discussions and create an environment for considering tangible future directions.

Measurement

Recently, there have been calls for increased focus on not only the need for developing instruments backed by strong validity arguments, but also for examination of validity evidence for existing instruments and closer attention to instruments being used for research purposes (Lavery et al., 2019). The NSF-funded Validity in Measurement in Mathematics Education

Statistics Education synthesis group has spent the past two years documenting instruments used in statistics education research, as well as the types of validity evidence that accompany them. In this theme, we discussed findings and implications of the ongoing project, and broadly discussed issues of validity in measurement. A big question that came up from people participating in the discussion on this day was, *how do people in the field help others know what instruments are available and what they measure and for what populations*? The VMED project is working towards a solution to this issue, but it will continue to be an issue of advertising beyond people in just statistics or mathematics education as other disciplines continue to research the teaching and learning of mathematics. This is particularly true in the case of the growing data science movement that is transdisciplinary.

During our discussions one of the questions we centered around was, *what instruments does statistics education need?* We present here some of the main discussion points that came up in hopes to spark future discussion and work in these directions:

- There seems to be a need for statistics education measurements at the K-12 grade levels including measures of statistical knowledge and practice as well as measures around identity, beliefs, affect, and dispositions.
- As the importance of preparing teachers to teach important understandings from statistics and data science continues to increase it will be important for researchers and teacher educators to have measures of those understandings to develop and measure the effectiveness of teacher preparation and education programs and experiences.
- Observation protocols are needed for researching the practice of teaching statistics. Such protocols could be useful for preparing teachers of statistics in teacher preparation programs at universities and for site supervisors there to focus on important aspects of teaching statistics particularly those that are potentially different than teaching mathematics as Cobb and Moore (1997) discuss. Furthermore, such protocols would be helpful in teacher education for use in lesson studies and other professional development experiences. They could also be useful for teachers who use research-based teaching practices to defend their practices that may seem out of line with administrators visions of teaching statistics
- How do we work with other disciplines to develop measures that support multiple disciplines? In particular, disciplines like computer science, civics, and science. This need for interdisciplinary work is increasingly important given the current data science revolution.

Another issue that came up in conversations around the use of measures was, *who do we want using these measures?* As researchers, our main focus is on measures for educational research. However, when a measure is made public it is open for anyone's use. Do we want measurements for policy makers and administrators to use in schools? A measure created for the best of intentions and used in ethical ways by researchers can easily be used in ways they were not designed or intended for. A particular worry is policy makers attempting to use such measures to evaluate teachers and make personnel decisions based on their results, which we have seen with Value-Added Models (VAMs), to the consternation of those that created them (American Statistical Association, 2014). Measurement will continue to be an ongoing area of much needed research in statistics education particularly as what it means to do statistics and data science continues to evolve rapidly with developments in technology.

Issues of Equity and Social Justice

The centrality of context to statistical practice makes the discipline well positioned to interrogate issues of equity and social justice and creates spaces in the often neutrally positioned mathematics curriculum for interrogating such issues. Thought there are several decades of past work in relation to the topic of equity and social justice in statistics education (Frankenstein, 1994; Lesser, 2007; Rubel et al., 2016) it is an area that has not received wide attention in the field. This however seems to be shifting. For example, in our working group every doctoral student that attended showed some level of interest in this theme and most were studying issues relate to this theme in their dissertation work. In searching recent National Science Foundation funding awards, we also notice a notable presence of funded work in this theme particularly over the past three years. That said what was really notable was it's presence at all. Another thing we noticed in our search is a significant influence of data science on these projects with several focusing explicitly on data science. We discussed the fruitful connections we see possible in interdisciplinary work leveraging affordances from mathematics education, statistics education, and data science education to creating meaningful opportunities for students to engage in investigations of issues of equity and social justice in school mathematics classrooms (see Table 1 for details).

Table 1: Affordances each field offers for engaging in investigations of issues of equity and social justice in statistics and data science education in school mathematics classrooms.

Mathematics Education	Statistics Education	Data Science Education
• Strong location in the school curriculum • Growing sociopolitical turn • Developed frameworks around equity • Decades of work on critical mathematics education (CME) and Teaching Mathematics for Social Justice (TMfSJ)	• Strong basis in mathematics curriculum • Simulation-based approaches to inference • Literature base around teaching and learning • Educational models • Context is central throughout • Centered around investigations driven by questions • Inductive approach • Focus on variability (diversity) • Methodological discipline	• Focus on cleaning, managing and structuring data commonly needed in publicly available datasets • Focus on use of open-source technology • Focus on data visualization • Data stories

There are also interesting shifts in the world of methodologies that relate to this theme. For example, in the past decade there have been discussions about critical quantitative methods (Stage & Wells, 2014). More recently there has been a strong push in educational research to look at using quantitative methodologies using critical race theory lenses under the moniker of QuantCrit (Garcia et al., 2018; Sablan, 2019). There are even research institutes focused on rethinking quantitative methods and methodologies all together to create new quantitative ways of knowing the world such as the Institute for Critical Quantitative, Computation, and Mixed Methodologies, whose mission it is to, “to advance the presence of scholars of color among those using data science methodologies, and challenge researchers to use those methods in ways that can dismantle the structural barriers to enable human flourishing for underrepresented communities, professionals, and young people” (https://www.icqcm.org/). We are encouraged by the growing body of work in this area. However, there is still much work to be done. Addressing issues of equity and social justice in statistics education is still predominantly being done by a small subset of the field. Furthermore, such considerations have not yet made there way into the recommendation of major policy documents such as the Statistical Education of Teachers (Franklin et al., 2015 or the GAISE II report (Bargagliotti et al., 2020).

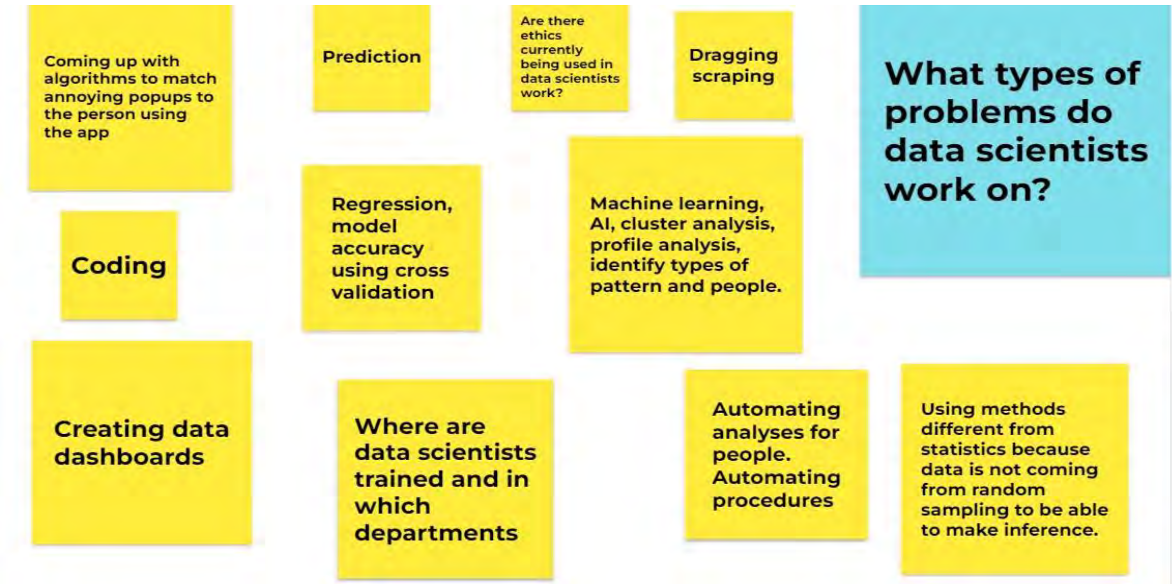
Data Science

Data Science is increasingly gaining traction and attention in the K-12 curriculum. California in particular has been making huge strides at the policy level to create spaces for data science in the mathematics curriculum. Yet we have little research in data science education relative to what are the epistemic practices are that students should learn, how students learn those practices, or how to prepare teachers to teach data science. Furthermore, data science itself is an ill-defined field that does not yet have a clear identity or community itself. Questions for us to consider then are *how do mathematics and statistics education relate to data science education?* Also, *what could be fruitful points of collaboration?*

In some ways it seems like data science is sucking the air out of the room in new initiatives in the K-12 mathematics curriculum. Big name scholars like Jo Boaler are actively working to create significant space for data science in the high school mathematics curriculum in California as part of recent policy changes in the mathematics curriculum in that state that has sparked a reinvigoration of the old math wars. However, this work is moving very quickly and there is little research in data science education currently to guide curricular reforms. Data science itself is still rather ill-defined as a field making it difficult to then state what students should learn or understand from the field. In our working group, one of our main conversations in this theme was, *what is data science?* We found a wide range of models and definitions during our discussion and search, but little common grounding other than it involves interdisciplinary work drawing from computer science and statistics and that data wrangling and visualization are central aspects of data science work. In thinking about data science, we brainstormed from our collective understanding of what data science is, to reflect on the question: *What types of problems do data scientists work on?* You can see the main points of our discussion on the Jamboard we used to organize our thoughts shown in Figure 1.

We ended our discussion of data science much as we started in being unsure of what it was and how it should unfold in the school mathematics curriculum. We have more questions than answers, but we hope that as interest in and demand for data science in the school curriculum increases more scholars will actively investigate issues in this theme to help guide the rapid development of curriculum.

Figure 1: Jamboard of thoughts around the question of What types of problems do data scientists work on?



Author 1 encourages scholars to also be cautious in reflecting on what is data science for and who is it serving in the curriculum when working on scholarship in this area. Data science education initiatives are being well funded by tech companies and the common rationale for the need for data science relies on the common neoliberal claims of work force needs. In reflecting upon Labaree's (1997) discussion of the American struggle over educational goals we must consider is the goal of education to serve the workforce needs of private corporations or is it to nourish democratic equality and citizenship? What does data science education look like with the goal of nourishing democratic equality and citizenship? I think this is an important connection and one not being considered widely.

Concluding Thoughts

Each year we end the working group feeling exhausted and excited. We are exhausted from the rigor of planning three days of engaging 90-minute sessions that are highly flexible and from engaging in 90 minutes of intense and invigorating conversations. It is those very conversations that excite us too. It is exciting to be in community with people that are deeply interested in the things we are interested in. We continue to carry on this working group because of that, because of the community that is developed and the interactions that nourishes us from being together. Statistics educators that do rigorous educational work in spaces around K-12 teaching and learning are a very small group who are spread across many organizations, institutions, disciplines, and locations. In mathematics education spaces, we miss out on conversations around the content we love. In statistics spaces, we miss out on the conversations about how people learn and teach. In statistics education spaces, it is not uncommon for much of the focus to be on post-secondary statistics education. This working group at PME-NA is a rare opportunity for those of us that transverse all of those spaces, educational research, mathematics education, statistics education, and statistics. We hope that the highlights we present here help to spark more conversations around statistics education in K-12 settings and that more people come join in those conversations next year.

References

Ancker, J. (2020). The COVID-19 pandemic and the power of numbers. *Numeracy*, *13*(2). https://doi.org/10.5038/1936-4660.13.2.1358

American Statistical Association (2014). ASA statement on using value-added models for educational assessment. https://www.amstat.org/policy/pdfs/ASA_VAM_Statement.pdf

Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. (2020). *Pre-K-12 guidelines for assessment and instruction in statistics education II (GAISE II)* (Second edition). American Statistical Association.

Ben-Zvi, D., Makar, K., & Garfield, J. (Eds.). (2018). *International handbook of research in statistics education*. Springer.

Frankenstein, M. (1994). Understanding the politics of mathematical knowledge as an integral part of becoming critically numerate. *Radical Statistics*, *56*, 22–40.

Franklin, C., Bargagliotti, A., Case, C., Kader, G., Scheaffer, R., & Spangler, D. (2015). *Statistical Education of Teachers*. American Statistical Association. http://www.amstat.org/education/SET/SET.pdf

Garcia, N. M., López, N., & Vélez, V. N. (2018). QuantCrit: Rectifying quantitative methods through critical race theory. *Race Ethnicity and Education*, *21*(2), 149–157. https://doi.org/10.1080/13613324.2017.1377675

Labaree, D. F. (1997). Public goods, private goods: The American struggle over educational goals. *American Educational Research Journal*, *34*(1), 39–81.

Lavery, M. R., Jong, C., Krupa, E. E., & Bostic, J. (2019). Developing an instrument with validity in mind. In *Assessment in Mathematics Education Contexts: Theoretical Frameworks and New Directions*. Routledge.

Lesser, L. M. (2007). Critical values and transforming data: Teaching statistics with social justice. *Journal of Statistics Education*, *15*(1), 1–21.

Rubel, L. H., Hall-Wieckert, M., & Lim, V. Y. (2016). Teaching mathematics for spatial justice: Beyond a victory narrative. *Harvard Educational Review*, *86*(4), 556–579.

Sablan, J. R. (2019). Can you really measure that? Combining critical race theory and quantitative methods. *American Educational Research Journal*, *56*(1), 178–203. https://doi.org/10.3102/0002831218798325

Stage, F. K., & Wells, R. S. (2014). Critical quantitative inquiry in context. *New Directions for Institutional Research*, *2013*(158), 1–7. https://doi.org/10.1002/ir.20041

TEACHING AND LEARNING WITH DATA INVESTIGATION

Sunghwan Byun
North Carolina State University
sbyun2@ncsu.edu

Christopher Engledowl
New Mexico State University
chriseng@nmsu.edu

Travis Weiland
University of Houston
tweiland@uh.edu

Susan O. Cannon
Mercer University
cannon_so@mercer.edu

With a renewed focus on data investigation and its relationship to issues of equity and justice, we build on past statistics education working groups to co-develop a syncretic space in which new knowledge can emerge while embracing the tension between critical and dominant views. To do so, we invite a wide range of discussions and collaborations within the domain of statistics and data science education. The primary goal of this working group is to initiate cross-institutional collaborative subgroups based on overlapping research interests. The first two sessions will focus on identifying mutual interests among participants, and the third session will be devoted to planning for collaborative activities over the coming year.

Keywords: Data Science, Statistics, Equity, Social Justice, Measurement

Data science has begun to appear in the K-12 mathematics curriculum in various locations (Bargagliotti et al., 2020). Moreover, discourse around data science has become a political hot potato in curriculum reform in certain regions in the world (e.g., Fortin, 2021), resembling that of the earlier Math Wars (Schoenfeld, 2004). Against this backdrop, this working group aims to spark multiple research collaborations with a focus on *teaching and learning with data investigation*. Our word choice of data investigation is intentional; we invite wider perspectives on teaching and learning with real-world data, such as growing literature on data science education, while building on the continuing development of curriculum and research in statistics education. Teaching with data investigation requires a drastic shift toward identifying heuristic solutions for real-world problems *in context*; this contrasts with teaching mathematics, which often focuses on generalization and abstraction (Lee et al., in press; Wild & Pfannkuch, 1999). The ongoing pandemic has also brought meaning-making with data to the fore of determining everyday practices like wearing a mask and getting vaccines. Developing data-enabled citizenry with awareness of ethical implications of data use is a key concern to promote democratic social processes. However, it is well-documented that teaching and learning with data investigations is marginalized in the mathematics curriculum (Lovett & Lee, 2017), and teachers often do not feel prepared to teach statistics and data science in K-12 classrooms (Banilower et al., 2018).

PME-NA has been an international space to examine issues about statistics and data science education from a wide range of perspectives. For example, Morgado and Sánchez (2021) presented a use of Geogebra to support students' statistical covariational reasoning, and Rubel et al. (2021) applied a critical lens to examine the GAISE II report (Bargagliotti et al., 2020). These contributions reflect that statistics and data science cross both disciplinary and theoretical boundaries (Ben-Zvi et al., 2018). This working group will deepen discussions begun at previous meetings, and also germinate seeds for new collaborations. In an effort to create a syncretic environment that can also focus on specific elements within the domain, the working group is structured to develop sub-groups based on research interests and to engage in dialogic processes across the sub-groups.

Emerging Interests from the Past Working Group

The discussions in the past working group suggest some broad themes of interests that participants may bring. We briefly discuss these themes and propose this working group to deepen these discussions and form collaborative sub-groups.

Issues of Equity and Social Justice

The centrality of context to data investigation makes statistics and data science education an optimal space to examine social issues and participate in democratic social processes. The past working group has identified important affordances of mathematics, statistics, and data science education in terms of centering equity and social justice in students' learning experiences. Discussions may include applying the current understanding of teaching mathematics for social justice to bring data investigation to the fore of political participation. Discussions may also include the role of data science education in widening students' access to public datasets and communicating their findings with multiple stakeholders.

Data Science Education in K-12 Settings

Teaching and learning data science in a K-12 setting poses both promises and challenges. It is an opportunity to lower the boundaries among the subject areas that are often siloed in art, language, mathematics, science, and social studies. For example, students may investigate US census data to identify important trends in the US demographics. This kind of interdisciplinary study brings together domain content knowledge in social studies and data investigation practices and allows students to question and understand important sociopolitical issues. One of the associated challenges is offering resources and professional development opportunities that are needed to integrate data investigation across multiple disciplinary learning settings.

Measurement

There have been recent calls for careful examinations of kinds of validity evidence and arguments for the research instruments that are used to advance the field of mathematics education (Lavery et al., 2019). The interdisciplinary nature of teaching and learning with data investigation adds complexity to this foundational work. This working group will discuss these issues and how they present different challenges in various research contexts related to teaching and learning with data investigation.

Structure of Sessions

Through sessions, participants will gradually form sub-groups with shared research foci. We purposefully structured whole group discussions to facilitate inter-group dialogues.

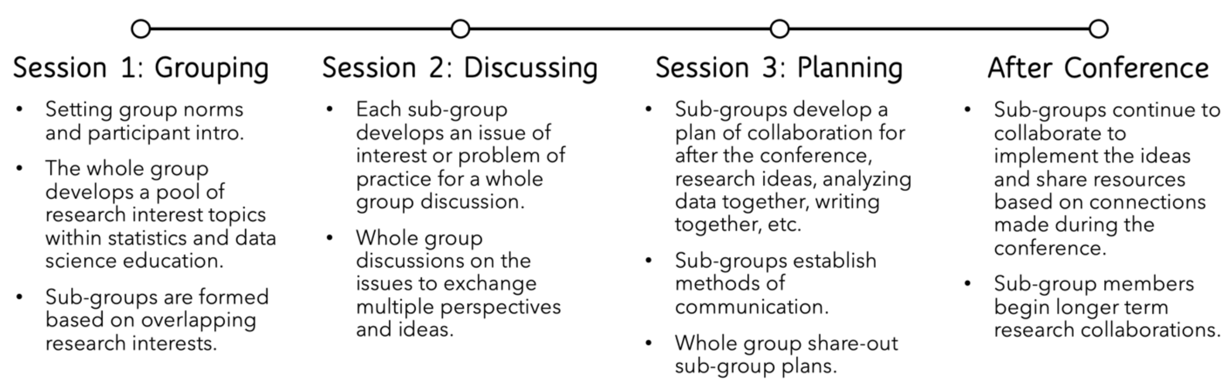


Figure 1: Session Plans and Goals

References

Banilower, E. R., Smith, P. S., Malzahn, K. A., Plumley, C. L., Gordon, E. M., & Hayes, M. L. (2018). *Report of the 2018 NSSME+.* Horizon Research.

Bargagliotti, A., Franklin, C., Arnold, P., Gould, R., Johnson, S., Perez, L., & Spangler, D. (2020). *Pre-K-12 guidelines for assessment and instruction in statistics education II (GAISE II)*. American Statistical Association.

Ben-Zvi, D., Makar, K., & Garfield, J. (Eds.). (2018). *International handbook of research in statistics education*. Springer.

Fortin, J. (Nov. 17, 2021). California tries to close the gap in math, but sets off a backlash. *The New York Times*. Retrieved from: https://www.nytimes.com/2021/11/04/us/california-math-curriculum-guidelines.html?auth=login-google1tap&login=google1tap

Lavery, M. R., Jong, C., Krupa, E. E., & Bostic, J. (2019). Developing an instrument with validity in mind. In J. D. Bostic, E. E. Krupa, & J. C. Shih (Eds.), *Assessment in mathematics education contexts: Theoretical frameworks and new directions.* Routledge.

Lee, H., Mojica, G., Thrasher, E., & Baumgartner, P. (in press). Investigating data like a data scientist: Key practices and processes. *Statistics Education Research Journal*.

Lovett, J. N., & Lee, H. S. (2017). New standards require teaching more statistics: Are preservice secondary mathematics teachers ready? *Journal of Teacher Education*, *68*(3), 299-311.

Morgado, C. N., & Sánchez, E. (2021). The informal covariational statistical reasoning: Focus on the notion of aggregate using digital technology. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), P*roceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1659-1667). Philadelphia, PA.

Rubel, L. H., Peralta, L. M., Herbel-Eisenmann, B., Jiang, S., Khan, J. B., & Lim, V. Y. (2021). Theorizing data science education: An intersectional feminist perspective on data, power, and "playing the game." In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 217-221). Philadelphia, PA.

Schoenfeld, A. (2004). The math wars. *Educational Policy*, *18*(1), 253-286.

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, *67*(3), 223–248.

WORKING THROUGH DISSONANCE: ADDRESSING TENSIONS THAT ARISE WHEN STUDYING MATHEMATICS TEACHER PREPARATION USING AN EQUITY LENS

Dr. Jennifer B. Chauvot
University of Houston
jchauvot@uh.edu

Dr. Jeannette D. Alarcón
University of Houston
Jdalarcon2@uh.edu

We view critical dissonance as key for equity work. As such, this working group session focuses on developing appropriate methodological approaches for a systematic study of phenomena related to the intersection of equity work and mathematics teaching and learning in university-based teacher preparation; we call this approach equity-focused research. We will create a collegial, scholarly space for unpacking and addressing persistent issues that arise when research design in mathematics education research is driven from an equity perspective. The goal is to engage with fellow researchers to discuss and/or develop appropriate methods and designs for studying the ways that preservice teachers understand, implement, and/or resist culturally responsive/sustaining teaching practices after learning about them in their mathematics methods coursework.

The main topic of this session is tensions that arise when designing research using an equity lens. It is imperative that research integrate an equity focus in all aspects of study design, but this can be particularly challenging when working with large sets of numerical data that can be available within teacher preparation programs. In keeping with the spirit of the conference call for proposals, this working group session will use facilitated "critical reflection and conversation" to engage participants across the three sessions.

Our work is situated as critical education research. As such we draw from scholarship that raises critical consciousness in understanding and addressing persistent social issues such as racism in education (Banks, 1993; Gay, 2000; Ladson-Billings, 1995; Nieto, 2004; Villegas and Lucas, 2002). Building from the concept of critical pedagogy we understand dialogue as the key to building critical consciousness about content and systems (Darder, et al, 2003). We draw from Hernandez et al. (2013) model of five thematic categories that can be useful for framing research in mathematics teaching and learning using an equity lens. They are (1) content integration, (2) facilitating knowledge construction, (3) prejudice reduction, (4) social justice, and (5) academic development to guide curriculum development. Given time constraints for the working group session and the urgency for equity work, categories three and four will be highlighted as a framework for the session.

Category 3, prejudice reduction, must be included in curriculum design and implementation for mathematics teacher preparation because mathematics, as a subject area in school, has long been cast in a gate-keeping role. As such, many pre-service teachers enter their programs believing that certain types of people are better at mathematics than others. Despite durable attempts to frame this idea as a byproduct of personal preference for school subjects, the reality is that positioning mathematics in this way is and has always been racially motivated. In the working group, we will explore ideas for disrupting this notion via mathematics teacher preparation coursework and experiences. Category 4, social justice, is an often-misused concept in education. Because social justice has its beginnings in the labor rights movement, we believe that the mathematics curriculum could be a lesser tapped space for helping students use mathematics to uncover root causes for pressing social issues such as patterns of economic disparity, especially along racial lines.

The session structure and expected outcomes are as follows:

- Working Group Structure
    - Session One: Establish the working group environment with an emphasis on collaboration and building community
    - Session Two: Engaged action planning
    - Session Three: Wrapping up and planning for the next steps.
- Time structure for 3 90-minute sessions
    - Opening work (10 minutes)
    - Planned presentation (15 to 20 minutes)
    - Work time (45 to 50 minutes)
    - Closing work (10 minutes)
- Planned presentations
    - Day one: brief presentation to define the notion of interdisciplinary, collegial work and situate it as relevant for the working group (Alarcón et al, 2021).

In this work, we share our approach to teacher educator professional development exploring problems of practice together, and how collaborative planning for the teacher preparation coursework played an integral role in helping us, as teacher educators, understand culturally responsive teaching as a pathway to actualizing equity and justice in education.

- Day two: brief presentation of an exemplar study to situate our conversation about the tensions that can arise when using large datasets to explore equity pedagogy in teacher preparation (Alarcón & Chauvot, under review).
- Day three: presentation of artifacts and other work products (all participants)

- Collaboration activities and preliminary ideas for artifacts

    - Asset Mapping; Community Commitments; Community Check-in; Closing Circle activities are all structured, facilitated activities; they will result in artifacts that will serve as organizing documents for the group
    - The work time segment will utilize principles of cooperative learning with an emphasis on group sense-making and consensus-building. Artifacts produced could include a research statement for conducting equity-focused research; an action plan for revising an existing project with an equity focus; and/or an action plan for future collaboration among working group members.

References

Alarcón, J.D., Chauvot, J., Gronseth, S. & Cutler, C. (2021). Applying Critical Pedagogy and UDL to Synchronous Online Course Design to Equip Pre-Service Teachers in Mathematical Discourse. In K. Hollebrands, R. Anderson, & K. Oliver (Eds.), *Online Learning in Mathematics Teacher Education* (pp. 115-130), Springer. https://doi.org/10.1007/978-3-030-80230-1

Alarcón, J. & Chauvot, J. B. (under review). Unpacking survey data within critical autoethnographic study: A move toward critical mixed methods, *The Teacher Educator*

Banks, J. A. (1993). Multicultural education: Historical development, dimensions, and practice. *Review of Research in Education*, 19, 3-49.

Darder, A., Baltodano, M., & Torres, R. D. (Eds.). (2003). *The critical pedagogy reader* (p. 1). New York: Routledge Falmer.
Gay, G. (2000). *Culturally responsive teaching: Theory, research, and practice*. Teachers College Press.
Hernandez, C.M., Morales, A.R. & Shroyer, M.G. (2013). The development of a model of culturally responsive science and mathematics teaching. *Cultural Studies of Science Education, 8*, 803–820. https://doi.org/10.1007/s11422-013-9544-1
Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal, 32*(3), 465–491 https://doi.org/10.3102/00028312032003465
Nieto, S. (2004). Black, white, and us: The meaning of brown v. board of education for Latinos. *Multicultural Perspectives, 6*(4), 22-25. https://doi.org/10.1207/s15327892mcp0604_7
Villegas, A. M., & Lucas, T. (2002). Preparing culturally responsive teachers: Rethinking the curriculum. *Journal of Teacher Education, 53*(1), 20-32.

WORKING GROUP REPORT: CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

Michelle Cirillo
University of Delaware
mcirillo@udel.edu

Megan Staples
University of Connecticut
megan.staples@uconn.edu

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Jill Newton
Purdue University
janewton@purdue.edu

AnnaMarie Conner
University of Georgia
aconner@uga.edu

Carlos Nicolas Gomez Marchant
The University of Texas Austin
nico.gomez@utexas.edu

Argumentation, justification, and proof are conceptualized in many ways in extant mathematics education literature. At times, the descriptions of these objects and processes are compatible or complementary; at other times, they are inconsistent and even contradictory. Regardless of the descriptions of these processes, however, given the importance of argumentation, justification, and proof to the discipline of mathematics, as well as their valued roles as learning practices, it is critical to query the relationship between engaging students in such processes and the promotion of equitable learning spaces and outcomes. The Argumentation, Justification, and Proof Working Group met for the sixth time at PME-NA 43 in 2021 in Philadelphia, PA. The goal of the 2021 sessions was to facilitate discussions and collaborations among researchers to advance our collective understanding of argumentation, justification and proof through an equity and inclusion lens. A concluding discussion that looked across the sessions occurred on Day 3.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking; Equity, Inclusion, and Diversity

History of the Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) met for the first time at PME-NA 37 in 2015 (Cirillo et al., 2015). The group then met for three additional years in 2016, 2017, and 2018 (Staples et al., 2016; Conner et al., 2017; Conner et al., 2018). Topics covered across the four years included: definitions and concept images of argumentation, justification, and proof (AJP), research implications of using different definitions to study the constructs, positionality of researchers studying AJP, how definitions and descriptions are used by researchers and practitioners within particular contexts and applications, and how issues of equity, social justice, and marginalized populations intersect with the teaching and learning of AJP. The AJP-WG sessions were well-attended each year, and the group has been active between meetings. Following the 2018 meeting, AJP-WG members began work on an edited book (i.e., Bieda et al., 2022). Across the authoring teams of the AJP-WG conference papers, the book, and the white papers (i.e., Cirillo et al., 2016; Staples et al., 2017), dozens of scholars have been involved in the group's work, including many graduate students. A Research Colloquium, where the team shared analyses from the book, was facilitated at the 2020 conference (Cirillo & Bieda, 2020). In 2021, we returned to the Working Group format with a focus on AJP through an equity and inclusion lens.

Working Group 2021: Goals and Focus

This sixth meeting of the working group focused on the degree to which AJP may or may not promote equity and inclusion in mathematics classrooms. To explore this idea, we engaged participants in explorations of classroom artifacts and data focused on the interrelationship between each construct of AJP and specific aspects of equity and inclusion. Day 1 began with introductions of the leadership team and the participants in the room and on Zoom (i.e., the conference was hybrid due to the COVID-19 pandemic). M. Staples and C. N. Gomez Marchant then introduced our theoretical framing for the session. Specifically, the team focused on the constructs of *access* and *agency* as we analyzed classroom artifacts to consider how justification can support access and agency for students, and thus potentially more equitable outcomes. We then explored how justification can be an *equity practice* in classrooms (in addition to a mathematical and learning practice). Day 2 began with a presentation by K. Kosko of elementary classroom data and an interactive discussion regarding the role of relational and cultural factors in promoting (or hindering) mathematical argumentation and concluded with time for networking among participants. On Day 3 we shifted the focus to proof. Two classroom video clips were shared by M. Cirillo to consider the ways in which proof may or may not promote equity and inclusion in mathematics classrooms. The Day 3 session concluded with interactive activities wherein participants reflected on the three days of WG activities and engaged in discussions aimed at making connections across the WG themes.

Theoretical Background

Given that argumentation, justification and proof (AJP) are important both to the discipline of mathematics and to supporting student sensemaking and understanding, it is critical to investigate potential relationships between engaging students in such processes and promoting equitable and inclusive learning spaces and outcomes. Access and agency are constructs that are potentially crucial for examining the role of argumentation, justification, and proof in creating more equitable outcomes (Gutiérrez, 2002), because these practices can provide students with access to powerful mathematics and opportunities to understand themselves as knowers and doers of mathematics, as well as having their voices matter and influence the classroom. As we seek to promote strong mathematics learning and rehumanize mathematics classrooms, we need to further examine and theorize potential relationships between equitable engagement and the constructs of AJP. We assert this against a backdrop where mathematics, and its ways of knowledge development, have a history of exclusion that not only impacts participation in mathematics education (Louie, 2017) but also in the discipline more broadly. For example, proof has been positioned as a high-status process and thus can be positioned as exclusionary (see, e.g., Knuth, 2002; Otten et al., 2020).

In framing equity for the working group, we followed Unterhalter's (2009) definition of equity as "equality turned into an action" (p. 416). This means we have to consider the consequences of our actions in the learning and teaching of mathematics as working towards the construction of equitable mathematics learning spaces or perpetuating current myths about who can and cannot do mathematics. Equity, therefore, is about who has power in the classroom, in the community, and in our global society (Gutiérrez, 2009). For our goals and purposes in guiding conversation about the role of equity within AJP research, we used Gutiérrez's four dimensions of equity to encourage conversation about differing elements of an equitable classroom (see Figure 1). Access reflects the opportunity students have to learn. Achievement focuses on tangible measures like standardized test scores, enrollment in Advanced Placement courses, and participation in courses. These two elements, which make up the dominant axis, are needed to be seen as successful in our capitalistic society. On the other axis, the identity dimension focuses on teachers' concentration on capital and resources students bring into the mathematics classroom; while the power dimension "takes up issues of social transformation" (Gutiérrez, 2009, p. 6). The identity and power dimensions make up the critical axis. The critical elements work to change society while the dominant ones will only maintain the larger systems in place.

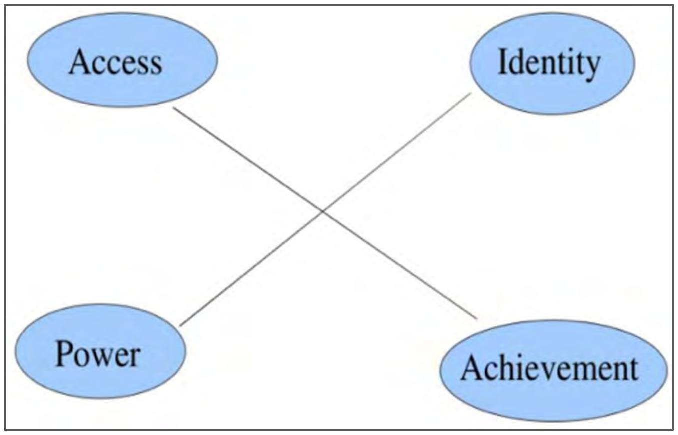


Figure 1: Dimensions of Equity (Gutiérrez, 2009)

Another important construct that motivated our conversations about equity in AJP was *agency*. Agency was defined as, "The ability to influence and make decisions about how and what is learned in order to expand capabilities" (Adair & Colegrove, 2021, p. 8). Agency is related to the racial and social identities of learners because agency influences how people position themselves in relation to micro and macro systems of power. While a broad definition of agency involves the notion of a willingness to act, we also drew on the work of González-Howard and McNeill (2020) who specify the construct of *epistemic agency*, which encompasses how students are positioned, participate, and shape the knowledge construction process in the classroom.

Working Group 2021 Sessions

In the sections that follow, we report on the activities of each of the three days of the working group. We conclude this section with a summary of the reflective discussion that took place on Day 3, where we looked across the activities of the three sessions.

Day 1 – Introductions and Justification through an Equity and Inclusion Lens

We began Day 1 by leveraging a google slide template to have people introduce themselves to the group and facilitate connections (see materials in AJP, 2022). Our goal was to help people get to know others "in the room," particularly as we were spread across in-person and online spaces. After providing context for the group's past work and activities, we turned to the 2021 WG focus by sharing a quotation from our proposal:

> Given the importance of argumentation, justification and proof to the discipline of mathematics, as well as their valued roles in supporting student sense making and understanding, it is critical to query the relationship between engaging students in such processes and the promotion of equitable and inclusive learning spaces and outcomes. (Cirillo et al., 2021, p. 1922)

This launched our work, considering the relationship between argumentation, justification, and/or proof and equity. Participants did a turn-and-talk to share their initial thoughts on this. In the share back, many issues were raised, from who determines acceptable reasons, to classroom interactions and how AJP might serve to either distribute authority - when used to elicit students' ideas and thinking or monopolize it (power) - for example, if a teacher concentrates the authority to determine what counts as a proof or not.

C. N. Gomez Marchant then described the framing of equity we wanted to begin with to guide our work, as described above. Our group proceeded to consider *justification* as an *equity practice.* We offered the following as a definition of justification: "Mathematical justification is the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (Bieda & Staples, 2020, p. 103).

Leveraging some of the constructs from the discussion of equity and an equity lens, we posed the following question, highlighting the constructs of *access* and *agency:* "How does justification create opportunities to access important mathematical ideas and highlight students' capacities as thinkers and doers of mathematics (agency)?" To the degree that it promotes access and agency, we can consider justification as a potentially equity-promoting practice and explore how it might promote equity outcomes. In small groups, participants then had the opportunity to review two transcripts - one from a middle school and one from a high school classroom. Participants were asked to look for instances of justification and discussed whether justification, in those instances, was serving to promote equity and inclusion.

In the follow-up discussion, it seemed clear to participants that justification was a practice teachers could (and should!) organize in their classrooms and that it could serve to promote equity goals. It was noted that widespread or regular engagement in justification by students is dependent on norms and a particular environment that would center students' ways of thinking and position students as mathematical thinkers. Classrooms with such norms that had lots of student contributions *but did not include justification activity* would likely not be as equitable in terms of promoting access to mathematics and agency, particularly epistemic agency. Students' contributions to knowledge development were seen as a crucial element.

A question that was prompted, and not resolved, was: Is it the act of justification that makes [justification] an equity-promoting practice (the practice itself) or is it the pedagogical work that makes it part of the classroom that makes it an equity-promoting practice? In considering justification itself, we might further consider the unique nature of the practice (as compared to other practices like modeling, problem solving, and conjecturing) that allows for authority to be taken up by any individual, which may allow it to play a different role in how equitable outcomes are supported.

Day 2 – Argumentation through an Equity and Inclusion Lens

Day two of the working group focused on exploration of argumentation through an equity and inclusion lens. After facilitating participant sign-ins, attention was drawn to the focal question: "In what ways are students positioned to access important mathematical ideas and highlight their capacities as thinkers and doers of mathematics?"

A. Conner, C. N. Gomez Marchant, and K. Kosko asked WG members to consider this focal question through the lens of mathematical argument and argumentation as they engaged with artifacts from an elementary classroom. After illustrating how students' own conceptions of argument may vary (see Figure 2 which contains the teacher prompt: "It is rude to argue"), the working group was introduced to a 360-video of an informal activity focusing on the Commutative Property of Multiplication in a third-grade classroom (see Kosko et al., 2022). The recorded students had just completed an initial task where they were asked to cover a 3cm by 4cm array with Cuisenaire rods (length-based concrete manipulatives where each color of rod represents a specific length). The video (Extended Reality Initiative, 2022) began after this initial task, with the teacher asking students to cover a new array (7cm by 8cm) with exactly one color of rod (i.e., one set length of rod). After students engaged in the task, there was a class discussion about why commutativity works.

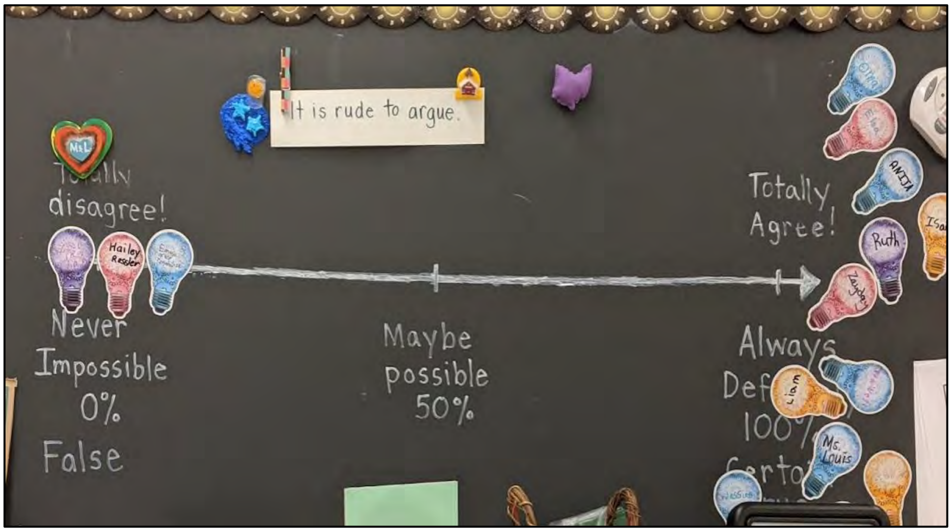


Figure 2: Example of elementary students' initial reaction to whether "It is rude to argue."

Working group participants watched the video to familiarize themselves with the situation and then were asked to view it again while considering two questions: How are students positioned to access important mathematical ideas? How are their capacities for being thinkers/doers of mathematics highlighted (or not)? In the subsequent discussion, a common theme emerged both from the online (Zoom) and face-to-face discussions among the WG participants. There seemed to be a tension between supporting access and agency to students as they were thinking through the mathematics while also attending to specific learning goals. For example, within the online conversation, one working group member noted that by asking students in the front-right side of the classroom to keep the strategies they had come up with to share with the whole class, the teacher effectively "shut down the conversation in the group." However, another member noted that "if you have specific learning goals you want to give all [students] access to…" then such decisions may need to occur. As reflection and discussion of the 360-degree video continued, M. Staples summarized the theme of the discussion by stating that "as we're trying to provide access for mathematical ideas, we have to shape the environment" but then left the group to ponder the question of what mathematical trade-offs occur in such cases.

Following WG members' reflections on access to mathematics in the 360 video, A. Conner shifted the group's attention to data related to how preservice teachers (PSTs) engaged with the same video (see slides 9-10 of Day 2 in AJP, 2022). Participants noticed that a PST who described students' mathematics was more focused on students than the PST who focused on the classroom teacher. Participants also noted that this method of examining preservice teachers' point of view while watching a 360 video seems promising for accessing PSTs' beliefs or current views of teaching and learning. One participant reflected that these two videos and reflections demonstrated that the classroom teacher can attempt to *give* agency to students, or they might *allow* students to have such agency. Extending this point, an online participant commented that one PST may be attempting to "make [agency] happen" versus "making a moment" to allow for students' agency. After this discussion, WG members were provided with opportunities to network during the time that remained in the session.

Day 3 – Proof through an Equity and Inclusion Lens

The primary activities for Day 3 were: (1) considering proof through an equity, inclusion, and access lens and (2) looking across the three WG sessions (described in the next section of this paper). M. Cirillo began Day 3 reminding participants of the access and agency aspects of the WG's equity and inclusion framework (described earlier in this paper). Cirillo then introduced Bass's (2015) Cycle of Inquiry and Justification by sharing a representation she developed to capture these ideas (see Figure 3). In this framework, the "reasoning of inquiry," which could be characterized as inductive reasoning, includes Exploration, Discovery, and Conjecture; while the "reasoning of justification," which could be characterized as deductive reasoning, includes Proof and Verification. Cirillo contrasted the Cycle of Inquiry and Justification with the notion of Proof as Show-and-Tell (see Cirillo et al., 2017). She then posed the following question: How do we create opportunities for student agency within the process of proving?

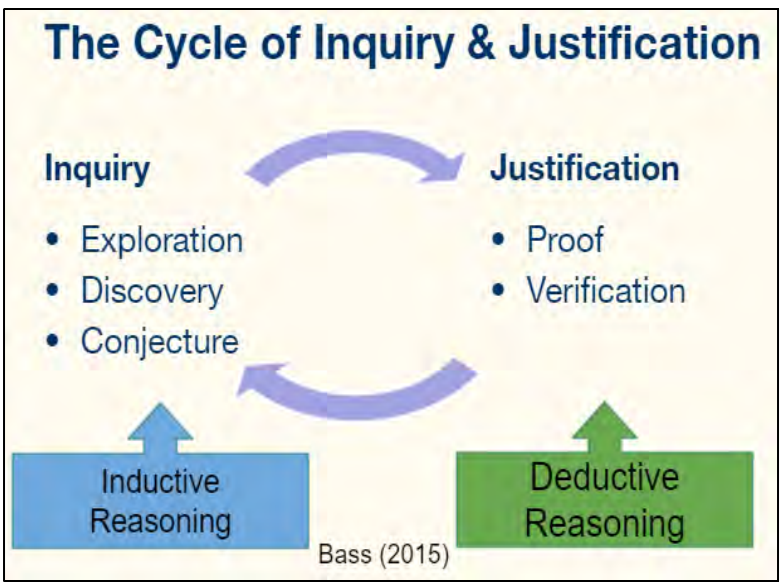


Figure 3: Bass's (2015) Cycle of Inquiry and Justification as represented by M. Cirillo

To explore these ideas, Cirillo shared two videos from her *Proof in Secondary Classrooms* study (Cirillo, 2015-2020) and posed a focal question inspired by Bieda and Staples' (2020) paper: In what ways are students positioned to access important mathematical ideas and highlight their capacities as thinkers and doers of mathematics? In Video 1, a secondary student shared a proof that she had written during groupwork with her class. In the video we can see the students' classmates applauding her work and directing their comments about the proof directly to her. In Video 2, two secondary students are interviewed about their choices and rationales for choosing to work on proofs in different formats (e.g., two-column or flow proof). Participants and WG leaders reflected on when and where they noticed student access and agency in the videos. Herbst and Chazan's (2012) notion of (competing) professional obligations was cited in a discussion about tensions between honoring the students' agency and honoring the obligation to the mathematics. C. N. Gomez Marchant offered some commentary about the videos, which touched on the lack of constraint in students' communication of their mathematical ideas and their agency with respect to making choices about how to represent their work (e.g., in a proof format that aligns with their preferences rather than one dictated by the teacher). Last, Cirillo raised questions about who, in school mathematics, has opportunities to engage in proof?

Day 3 - Looking Across the Three Days

In a synthesizing activity on Day 3, we asked participants to engage with the following question via a Jamboard activity (see AJP, 2022): "Over the past few days, what questions have come up for you with respect to AJP through an equity and inclusion lens?" Themes related to conceptual framings, classroom practice, disciplinary conceptions of rigor and what is valued, and the voices present in these conversations emerged across the question posed.

Several participants asked questions about the framework used for exploring the equity considerations and implications of AJP. Referencing the choice of Gutiérrez's (2002) access and achievement constructs, one participant asked, "While I understand the importance of helping students play the game, how are we empowering them to change the game? How are we changing the game ourselves?" Another participant wondered about other potential framings, asking "What are good ways to examine equity issues in relation to AJP? We used access and agency. What other approaches could be helpful?"

Many participants posed questions about teachers and students enacting AJP as an equity practice in K-12 mathematics classrooms, asking, for example: (a) " How can teachers incorporate Equitable Mathematics Teaching Practices (proposed by NCTM) in teaching mathematical argumentation, justification, or proof?" (b) "Across middle and high school mathematics, is there a trajectory of sorts for AJP?" (c) "How do we transition from researcher-implemented interventions to equipping class[room] teachers broadly to implement these practices in their own classes?" and (d) "Can we see each of A, J, and P as related to equity as practices or is it really more related to how the teacher engages students in the process?"

Participants also shared their wonderings about important disciplinary tensions related to what counts as rigor and what is important and valued in AJP. In particular, the following questions were posed: (a) "I'm continuing to think more about the tension between students producing complete "airtight" proofs and honoring students' ideas and where they currently are in their proof understanding. Do all class proofs need to be airtight, especially at the beginning?" (b) "Does an emphasis on formal proof foreground the math discipline and therefore lead teachers (and students) to think that proof is only for those going into math fields?" (c) "Do we need to question the definitions of rigor and proof as well as attending to the things that we've been attending to the last few days? (d) "While I see the importance of A & J... Aside from (as Nico says) helping students get through the "mathematics" gateway, what other reasons do we have to enculturate students into rigorous proofs? Can enculturation happen without oppression of some sort (and hence implicit gatekeeping)?" (e) "How can students be both well prepared for university work related to AJP while simultaneously respecting their thinking and sensemaking?"

Finally, questions were raised about the limited voices in current conversations about AJP research and practice. For example, one participant inquired: "In what ways are we open to and inviting of diverse voices to the discussion of AJP in research?" Another participant asked a related, but more specific question, "How can we ensure value for teacher and student voices in our AJP research? How can we respectfully conduct research with teachers?"

Conclusion

Across the three days of the AJP-WG, we considered separately—justification, argumentation, and then proof —through an access, equity, and inclusion lens. During the last portion of Day 3, the group had the opportunity to reflect on the discussions that took place over the course of the three sessions. Through participants' contributions to the Jamboard and the subsequent discussion, questions and comments related to conceptual framings, classroom practice, the role of the teacher in AJP, disciplinary conceptions of rigor, and so forth, surfaced. These discussions gave the AJP-WG and its leadership team much to ponder and consider for AJP-WG discussions at future PME-NA meetings.

Building on our previous work, if accepted, the 2022 AJP-WG will consider questions surrounding how research on argumentation, justification, and proof uses differing frameworks and tools to investigate these constructs (see Bieda et al., 2022). We will question if the frameworks and methods currently in use allow us to investigate the problems of practice related to AJP in today's classrooms, and, attending to questions raised at the end of the 2021 working group session, we will consider what is needed in frameworks and methods that will facilitate the investigation of AJP in meaningful and inclusive ways.

Acknowledgement

The AJP team would like to thank all WG participants for their thoughtful participation in the 2021 sessions (see participant list in the Day 1 "Welcome" slide links in AJP, 2022). The research reported in this paper was supported with funding from the National Science Foundation (DRK12 #1453493, PI: Michelle Cirillo; DRK12 #1908159, PI: Karl Kosko). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

Adair, J. K., & Colgrove, K. S.-S., (2021). *Segregation by experience: Agency, racism, and learning in the early grades*. Chicago, IL. The University of Chicago Press.

Argumentation, Justification, and Proof Working Group [AJP]. (2022, January 20). *Published resources*. https://drive.google.com/drive/folders/1T3zENt7xdbnVkZpv-PpVeUXrTHdKKOAt?usp=sharing

Bass, H. (2015). Mathematics and teaching. In P. Cassazza, S. G. Krantz, & R. D. Ruden (Eds.), I, Mathematician (pp. 129-139). Washington, DC: Mathematical Association of America.

Bieda, K. N., Conner, A., Kosko, K. W., & Staples, M. (Eds.). (2022). Conceptions and consequences of mathematical argumentation, justification, and proof. Springer.

Bieda, K. N., & Staples, M. (2020). Justification as an equity practice. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(2), 102-108. https://doi.org/10.5951/MTLT.2019.0148

Cirillo, M. (2015-2020). CAREER: Proof in secondary classrooms: Decomposing a central mathematical practice (DRL No. 1453493) [Grant]. National Science Foundation. https://www.nsf.gov/awardsearch/showAward?AWD_ID=1453493&

Cirillo, M. & Bieda, M. (2020). Conceptions and consequences of what we call argumentation, justification, and proof. Research Colloquium in A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), *Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 156-158). Mazatlán, Sinaloa, Mexico: Cinvestav/AMIUTEM/PME-NA.

Cirillo, M., Kosko, K. W., Newton, J., Staples, M., & Weber, K. (2015). Conceptions and consequences of what we call argumentation, justification, and proof. Working Group Paper in Bartell, T.G., Bieda, K.N., Putnam, R.T., Putnam, R.T., Bradfield, K., & Dominguez, H. (Eds.). *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education.* (pp. 1343-1351). East Lansing, MI: Michigan State University.

Cirillo, M., Kosko, K. W., Newton, J., Staples, M., Weber, K., Bieda, K., Conner, A., Mejía-Ramos, P. Otten, S., Creager, M., Hummer, J., Singh, R., & Strachota, S. (2016). *Conceptions and consequences of what we call argumentation, justification, and proof.* [White paper] https://www.researchgate.net/publication/300088457_White_Paper_2016_Conceptions_and_Consequences_of_What_We_Call_Argumentation_Justification_and_Proof

Cirillo, M., Murtha, Z., McCall, N., & Walters, S. (2017). Decomposing mathematical proof with secondary teachers. In L. West (Ed.), Reflective and collaborative processes to improve mathematics teaching (pp. 21-32). Reston, VA: NCTM.

Cirillo, M., Staples, M., Kosko, K.W., Newton, J., Conner, A., & Gomez, C.N. (2021). Conceptions and consequences of what we call argumentation, justification, and proof. In Olanoff, D., Johnson, K., & Spitzer, S.M. (2021). *Proceedings of the 43rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. (pp. 1922-1925). Philadelphia, PA.

Conner, A., Kosko, K. W., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2017). Conceptions and Consequences of What We Call Argumentation, Justification, and Proof. In E. Galindo & J. Newton, (Eds.), *Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1464-1473). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Conner, A., Gomez, N., Park, H., Foster, J. K., Zhuang, Y., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2018). Conceptions and consequences of what we call argumentation, justification, and proof. In T. E. Hodges, G. J. Roy, & A. M. Tyminski, (Eds.), *Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1402-1411). Greenville, SC: University of South Carolina & Clemson University.

Extended Reality Initiative. (2022, January 20). *Grade 3 commutative property of multiplication* [Video]. YouTube. https://youtu.be/ojhzS7n8ggo

González-Howard, M., & McNeill, K. L. (2020). Acting with epistemic agency: Characterizing student critique during argumentation discussions. *Science Education*, *104*(6), 935–982.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Toward a new equity research agenda. *Mathematical Thinking and Learning*, *4*(2-3), 145-187.

Gutiérrez, R. (2009). Framing equity: Helping students "play the game" and "change the game." *Teaching for Excellence and Equity in Mathematics*, *1*(1), 5–7.

Herbst, P., Chazan, D. (2012) On the instructional triangle and sources of justification for actions in mathematics teaching. *ZDM Mathematics Education* 44, 601–612 (2012). https://doi.org/10.1007/s11858-012-0438-6

Knuth, E. (2002). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal of Mathematics Teacher Education*, 5, 61-88. https://doi.org/10.1023/A:1013838713648

Kosko, K. W., Heisler, J., & Gandolfi, E. (2022). Using 360-degree video to explore teachers' professional noticing. *Computers and Education, 180*, 1-13. https://doi.org/10.1016/j.compedu.2022.104443

Louie, N. L. (2017). The culture of exclusion in mathematics education and its persistence in equity-oriented teaching. *Journal for Research in Mathematics Education*, *48*(5), 488-519. https://doi.org/10.5951/jresematheduc.48.5.0488

Otten, S., Wambua, M. M., & Govender, R. (2020). Secondary teachers differing views on who should learn proving and why. In A. I. Sacristan, J. C. Cortes-Zavala, & P. M. Ruiz-Arias (Eds.), *Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 996-1000). Mexico: Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Staples, M., Newton, J., Kosko, K., Conner, A**,** Cirillo, M., & Bieda, K. (2016) Conceptions and consequences of what we call argumentation, justification, and proof. In Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education.* (pp. 1704-1712). Tucson, AZ: University of Arizona.

Staples, M., Newton, J., Kosko, K. W., Conner, A., Cirillo, M., Bieda, K. N., Yopp, D., Zaslavsky, O., Hummer, J., Strachota, S., Singh, R., An, T., Going, T., Zhuang, Y. (2017). *Using artifacts to explore conceptions and consequences of argumentation, justification, and proof.* [White paper] Retrieved June 22, 2020, from Research Gate at: https://www.researchgate.net/publication/317267228_White_Paper_Using_Artifacts_to_Explore_Conceptions_and_Consequences_of_Argumentation_Justification_and_Proof

Unterhalter, E. What is equity in education? Reflections from the capability approach. *Studies in Philosophy and Education*, *28*, 415–424.

Wood, T. (1999). Creating a context for argument in mathematics class. *Journal for Research in Mathematics Education, 30*(2), 171-191. https://doi.org/10.2307/749609

CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF: INTERROGATING OUR FRAMEWORKS

AnnaMarie Conner
University of Georgia
aconner@uga.edu

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Michelle Cirillo
University of Delaware
mcirillo@udel.edu

Megan Staples
University of Connecticut
megan.staples@uconn.edu

Kristen Bieda
Michigan State University
kbieda@msu.edu

Argumentation, justification, and proof are essential constructs in the field of mathematics and teaching and learning in mathematics classrooms. Research involving these constructs draws on a variety of definitions and analytic frameworks that include discursive, social, and content factors. This year, we examine frameworks used in research involving these constructs and engage working group participants in critical conversations about how these have been used, with what impact, as well as considering future directions for addressing new questions and making further progress on long-standing ones.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking

History of Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) met for the first time during the 37th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education (PME-NA) in 2015 (Cirillo et al., 2015). The group met from 2016 to 2018 (Staples et al., 2016; Conner et al., 2017; Conner et al., 2018). AJP-WG sessions were well-attended each year, involving over 50 scholars, including many graduate students. In addition, the group has been active between meetings. Following the 2018 meeting, AJP-WG members began work on an edited book, published in 2022 (Bieda et al., 2022). A Research Colloquium, where we shared analyses from the book, was facilitated at the 2020 conference (Cirillo & Bieda, 2020). Across authoring teams of the AJP-WG conference papers, the book, and white papers (i.e., Cirillo et al., 2016; Staples et al., 2017), many scholars have contributed to the work. The working group was re-convened at the 2021 Annual Meeting of PME-NA, with a focus on AJP through an equity and inclusion lens.

Topic and Goal

In the concluding chapter of *Conceptions and Consequences of Mathematical Argumentation, Justification, and Proof*, Kosko and Bieda (2022) relate the results of asking authors to carefully consider the consequences of definitions in their analyses of given data related to argumentation, justification, and proof (A, J, and P). They conclude that both definitions and analytic frameworks are essential components in understanding how researchers interpret data. Furthermore, these definitions and frameworks are essential determinants of what data are collected and examined within research studies and how these studies are interpreted into classroom practice. Kosko and Bieda highlight a question from an earlier iteration of this working group that continues to influence our conversations: How does analyzing an artifact (e.g., classroom episode, interview response) from a particular lens (e.g., A, J, or P) influence what we say about it?

In 2022, our working group builds upon this earlier question by examining the influence, efficacy, and usefulness of existing frameworks for research on AJP and considers needed improvements to frameworks that examine both the products and processes of AJP. Current and future questions, including those discussed during the 2021 working group (see Cirillo et al., this volume), suggest a need to critically examine frameworks in use and potentially revise currently used frameworks, adapt frameworks from other areas of mathematics education, or theorize and construct new frameworks for examining processes and products of AJP in classrooms.

Theoretical Background

Argumentation, justification, and proof are considered central to the discipline of mathematics and, as such, have been part of research in mathematics education for decades (see, e.g., Harel & Sowder, 2007; Stylianides et al, 2017; Tall, 1992). However, while proof has been explicitly named and examined in reviews of research, argumentation and justification have been slower to be acknowledged as explicit areas of research (see Stylianides et al.'s (2016) review of PME research on proof and argumentation). Despite their ubiquitous presence in mathematics classrooms, mathematics education researchers have defined these constructs in numerous ways (see, e.g., Cirillo et al., 2016), and have used a wide range of theoretical and analytic constructs to frame their work. In the book that originated in this working group, the editors asked authors to engage in a thought experiment by using a given definition of a construct within their analysis of data across grade levels. Authors were free to supplement the definition with a compatible analytic framework. Within the 12 chapters explicitly focused on specific constructs, authors used frameworks focused on discursive analysis, social factors, and content or task analysis. For instance, all of the argumentation and one of the justification chapters used discursive frameworks; at least one chapter addressing each construct brought social factors into their analysis, including several explicitly addressing the role of the teacher; and five chapters (at least one for each construct) included an analysis of the task related to AJP.

Plan for the Working Group

In the 2022 Working Group sessions, we provide time for presentations related to commonly-used frameworks for research on AJP, presentations that raise questions about frameworks used across A, J, and P (or that could be extended to do so), group discussions that brainstorm the kinds of frameworks that are needed for new questions and new perspectives on AJP. Questions that will frame our sessions include (1) What frameworks for AJP have been used, how have they been used, and what have those frameworks allowed us to accomplish in past research? (2) How does analyzing an artifact (e.g., classroom episode or interview response) from the lens of argumentation, justification, or proof through a particular framework influence claims we can make? (3) What frameworks for analyzing AJP are still needed to help move the field forward with respect to teaching and learning AJP, classroom research, fostering reasoning and sensemaking, promoting equitable participation, classroom practice, and other aspects of researching AJP? Concerning session activities, Day 1 will focus on current frameworks used in AJP, with focused discussion surrounding discursive analysis (i.e., Toulmin's Scheme) and content or task analysis. Day 2 will continue this thread with attention to analytic frameworks illuminating social factors (i.e., sociomathematical norms). Although time on Days 1 and 2 will be devoted to needed or novel analytical frameworks, Day 3 will focus explicitly on this topic and developing action items for the working group and the field as a whole. We will also intersperse time for networking during the sessions.

References

Bieda, K. N., Conner, A., Kosko, K. W., & Staples, M. (Eds.). (2022). Conceptions and consequences of mathematical argumentation, justification, and proof. Springer.

Cirillo, M., Kosko, K. W., Newton, J., Staples, M., & Weber, K. (2015). Conceptions and consequences of what we call argumentation, justification, and proof. Working Group Paper in Bartell, T.G., Bieda, K.N., Putnam, R.T., Putnam, R.T., Bradfield, K., & Dominguez, H. (Eds.). Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education. (pp. 1343–1351). East Lansing, MI: Michigan State University.

Cirillo, M., Kosko, K. W., Newton, J., Staples, M., Weber, K., Bieda, K., Conner, A., Mejía-Ramos, P. Otten, S., Creager, M., Hummer, J., Singh, R., & Strachota, S. (2016). Conceptions and consequences of what we call argumentation, justification, and proof. [White paper] https://www.researchgate.net/publication/300088457_White_Paper_2016_Conceptions_and_Consequences_of_What_We_Call_Argumentation_Justification_and_Proof

Cirillo, M. & Bieda, M. (2020). Conceptions and consequences of what we call argumentation, justification, and proof. Research Colloquium in A. I. Sacristán, J. C. Cortés-Zavala, & P. M. Ruiz-Arias (Eds.), Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. (pp. 156–158). Mazatlán, Sinaloa, Mexico: Cinvestav/AMIUTEM/PME-NA.

Conner, A., Kosko, K. W., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2017). Conceptions and Consequences of What We Call Argumentation, Justification, and Proof. In E. Galindo & J. Newton, (Eds.), Proceedings of the 39th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1464–1473). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

Conner, A., Gomez, N., Park, H., Foster, J. K., Zhuang, Y., Staples, M., Cirillo, M., Bieda, K., & Newton, J. (2018). Conceptions and consequences of what we call argumentation, justification, and proof. In T. E. Hodges, G. J. Roy, & A. M. Tyminski, (Eds.), Proceedings of the 40th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 1402–1411). Greenville, SC: University of South Carolina & Clemson University.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester (Ed.), Second handbook of research on mathematics teaching and learning, (pp. 805–842). Information Age Publishing.

Kosko, K. K. & Bieda, K. N. (2022). Conclusion: Considering the consequences of our conceptions of argumentation, justification, and proof. In K. N. Bieda, A. Conner, K. K. Kosko, & M. Staples (Eds.), Conceptions and consequences of mathematical argumentation, justification, and proof (pp. 313–323). Springer.

Staples, M., Newton, J., Kosko, K., Conner, A, Cirillo, M., & Bieda, K. (2016) Conceptions and consequences of what we call argumentation, justification, and proof. In Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology Mathematics Education. (pp. 1704–1712). Tucson, AZ: University of Arizona.

Staples, M., Newton, J., Kosko, K. W., Conner, A., Cirillo, M., Bieda, K. N., Yopp, D., Zaslavsky, O., Hummer, J., Strachota, S., Singh, R., An, T., Going, T., Zhuang, Y. (2017). Using artifacts to explore conceptions and consequences of argumentation, justification, and proof. [White paper] Retrieved June 22, 2020, from Research Gate at: https://www.researchgate.net/publication/317267228_White_Paper_Using_Artifacts_to_Explore_Conceptions_and_Consequences_of_Argumentation_Justification_and_Proof

Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematics education research. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), The second handbook of research on the psychology of mathematics education (pp. 315–351). Brill Sense.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 237–266). National Council of Teachers of Mathematics.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 495–512). National Council of Teachers of Mathematics.

WORKING GROUP ON MATHEMATICS CURRICULUM RECOMMENDATIONS FOR ELEMENTARY TEACHER PREPARATION: FIRST REPORT

Julien Corven
University of Delaware
jcorven@udel.edu

Joseph DiNapoli
Montclair State University
dinapolij@montclair.edu

Lynsey Gibbons
University of Delaware
lgibbons@udel.edu

Valerie Long
Indiana University of Pennsylvania
vlong@iup.edu

Rachel Starks
Boston University
rnstarks@bu.edu

This report summarizes the work and conversations of the first meeting of the working group on Mathematics Curriculum Recommendations for Elementary Teacher Preparation at the PME-NA 2021 conference. We share a theoretical framework that proved especially salient to our discussions and synthesize the outcomes of small group conversations that we facilitated. We will also share our plans for the continuation of this working group at the AMTE 2022 conference and our proposed next steps for designing research to address questions that arose.

The organizers of the working group on Mathematics Curriculum Recommendations for Elementary Teacher Preparation had several goals for our initial meeting. First, we wanted to bring together a community of researchers interested in improving how future elementary teachers are prepared to teach mathematics. Second, we wanted to better understand the current state of teacher preparation programs. In particular, we wanted to know what participants perceived to be obstacles to the effective preparation of elementary teachers of mathematics, specifically in regards to following the recommendations from the Conference Board of the Mathematical Sciences (2012) and the Association of Mathematics Teacher Educators [AMTE] (2017) for PSTs to receive twelve credit hours of instruction in elementary mathematics. Such discussions also resulted in gathering preliminary data about the number and sequencing of elementary mathematics content and methods courses in participants' programs. Finally, we wanted to initiate a discussion about what researchers thought were the most important topics, practices, and ideas to be included in such courses. Since the PME-NA 2021 conference, the organizers of the working group have met to debrief and think about our next steps for continuing this work. In this report, we will first share the content of our conversations and work at the PME-NA 2021 conference, and then we will discuss our planned next steps for continuing this work at the PME-NA 2022 conference and beyond.

What We Learned at the PME-NA 2021 Conference

Prior Research

On the first day of the conference, we shared some research results to situate the need for this working group. First, we discussed results from Corven et al. (accepted) regarding the relationship between the number of instructional minutes on mathematics topics in teacher preparation and the specialized content knowledge (SCK; Ball et al., 2008) program graduates demonstrated for those topics. An important additional result shared was that incorporating other potential sources of learning about the topics (e.g., professional development, experience teaching the topic) did not change the strength of the relationship and appeared to be additive to prior instruction for the purpose of predicting SCK demonstrated (Corven et al., accepted). Participants later stated that these results increased their comfort with the idea of not including every elementary mathematics topic in their courses; however, other obstacles to making such a change (e.g., the need for PSTs to pass licensure exams) emerged. Next, we shared curriculum design principles from the Elementary Mathematics Project (EMP; Chapin et al., 2021), a curriculum that also chooses to focus on high-leverage mathematics topics and instructional practices. Several participants were excited to learn about the availability of the curriculum materials and the built-in professional development opportunities for mathematics teacher educators (MTEs), such as videos of expert instructors enacting the lessons with pre-service teachers (PSTs). One of the results of research on this curriculum that was shared related to the importance of wanting PSTs to engage in math communities and reimagine their relationship with mathematics (Gibbons et al., 2018). The EMP curriculum emphasizes using mathematical discussions in small groups and MTEs modeling how to facilitate whole-class discussions using student thinking (Chapin et al., 2021).

A Guiding Theoretical Framework

Although there are many similarities between the two programs that we discussed during the conference, the research presented on them exhibited a key theoretical framework that helped guide the remainder of our discussions. This framework was the knowledge-oriented vs. thinking-oriented approaches to teacher preparation proposed by Li and Howe (2021). Knowledge-oriented approaches emphasize teaching mathematical understandings (i.e., content) that PSTs will need to know to teach mathematics effectively, whereas thinking-oriented approaches emphasize PSTs learning to reason about, explain, and make sense of the mathematics they will teach. Thinking-oriented approaches do not treat mathematical knowledge as a fixed product that is given to students, and they often help teachers build skills for analyzing student thinking. The research from Corven et al. (accepted) looked at outcomes that are more knowledge-oriented, whereas the published research shared on EMP looked primarily at thinking-oriented outcomes. However, there is research on both programs to look at other outcomes. For example, Corven (2021) considered how PSTs in the first program understood student solutions to a division story problem, and Starks (2021) explored PSTs' knowledge of decimals within programs using the EMP curriculum. Thus, although Li and Howe's framework relates to the design of teacher preparation programs, we recognized that the distinction in the framework is not binary, and teacher preparation programs generate outcomes that are aligned with both teacher knowledge and teacher thinking.

Obstacles to Ensuring PSTs are Well-Prepared to Teach Elementary Mathematics

In our initial discussions on the first meeting of the working group, we generated a list of obstacles that participants identified as perceived barriers to the effective preparation of elementary teachers in mathematics. Some of these potential obstacles are listed below.

- Assumption that PSTs already know elementary mathematics content
- Lack of consistency between instructors of the same course
- Lack of guidance or support for new instructors of the course (especially for adjunct faculty and graduate students)
- Lack of cohesion or coherence across courses (especially when courses can be taken in any sequence)
- Too few courses/credit hours, especially in particular program structures (e.g., a one-year Master's degree program)
- Too many topics or grade bands to cover, particularly when PSTs need to pass licensure exams

- Lack of cooperation and collaboration between mathematics and education department faculty

Although the number of obstacles identified was disheartening (and we note that the above list is not complete), we observed that creating documents with specific recommendations for elementary teacher preparation courses could provide a first step towards addressing several of these obstacles. Specifically, recommendation documents could improve consistency between instructors in terms of which topics to teach and how much time to spend on each topic and, at the same time, support instructors new to teaching the course. Additionally, if we can obtain further research support to justify teaching fewer topics, then more institutions would likely feel comfortable promoting courses that feature deeper engagement on fewer mathematical topics. Although this change might have an impact on PSTs' performance on content-based licensure exams, there could be other ways to address such an obstacle (e.g., a one-credit supplemental course that covers additional material specific to such an exam). However, such decisions would need to be made on an institution-by-institution basis. We also believe that producing these documents would support MTEs in advocating for additional courses at institutions that currently allot fewer than twelve credit hours to elementary mathematics coursework.

Preliminary Ideas for Course Designs

On the second day of the working group meeting, we broke into three small groups to brainstorm ideas for outlines of elementary content courses. Two of the small groups met in person, and one met virtually over Zoom. After working for about an hour, we came back together to share what we had come up with in small groups.

Two of the groups developed similar topic outlines for two elementary mathematics content courses. Both groups developed a first course that focused on number and operations with whole numbers and decimals (i.e., in base 10), and a second course that focused on fractions and rational numbers. One group also elected to include measurement within this second course to link these critical topics. Another similarity is that both groups discussed ideas to bring in some thinking-oriented approaches within a course intended to promote content knowledge. Two specific ideas mentioned were (1) activities about sorting or categorizing student work and connecting students' strategies to developmental progressions and (2) using number talks as a pedagogical technique to build PSTs' number sense and help PSTs think about different strategies students may use for various operations. One group brought up an important question about wanting evidence that certain topics were easier or harder for PSTs to learn or were worth skipping in teacher preparation programs. Such research would help inform decisions about how much time to spend on each topic, or whether the topic should even be included at all. This idea was a key element of our debriefing conversation on the final day of the working group.

The third group took a different tack by thinking about broader goals for PSTs' mathematical experiences and considering thinking-oriented approaches to course design recommendations. As an example, this group supported using deep, whole-class discussions to deepen PSTs' SCK and allow the instructor to understand PSTs' current reasoning. They also called for explorations that allow PSTs to reconsider what it means to be "smart" in mathematics and to broaden their view of mathematics. Furthermore, they suggested designing tasks to engage PSTs in the Common Core Standards for Mathematical Practice to give them models for what they could do in their future classrooms. Like the first two groups, this group also mentioned the importance of moving away from the goals of covering lots of material and "exposing" PSTs to concepts (i.e., introducing them to PSTs without promoting mastery of them). One advantage of this group's approach is that the activities they developed could be implemented across many different mathematical topics. Additionally, this group wanted to connect the course content to what is currently being taught in elementary schools, specifically by using activities from reform-oriented elementary mathematics textbooks. After discussion among the members of the working group, participants agreed that the different approaches the small groups took was valuable. Recommendation documents could include both a list of mathematical topics and suggested timeframes alongside pedagogical approaches instructors should consider when teaching the courses, allowing teacher preparation courses to attend to both knowledge- and thinking-oriented outcomes simultaneously.

Next Steps

As part of the debriefing conversation on the final day of the working group, we discussed potential next steps. As a preliminary note, three of the organizers of this working group will be running a working group session at the AMTE 2022 conference using some of the same activities we engaged in at PME-NA. Our goal is to get a broader perspective on these issues from practicing MTEs who may have a different perspective on or level of involvement with the mathematics education research community. Additionally, we will be encouraging attendees at that session to bring ideas generated from our conversations back to meetings of their local AMTE affiliates. We hope that such continuations of our work will help working group members gain a better understanding of MTEs' needs for the documents we hope to produce. Two of the organizers have already conducted a discussion session at a local AMTE affiliate meeting (Long & Corven, 2021); one idea that arose from that conversation (more discussion of pedagogy and teaching methods within content courses) aligns with integrating knowledge- and thinking-oriented approaches. However, we anticipate that other ideas could emerge from these conversations to complement the discussions we had last year. For example, the idea of following PSTs into the field post-graduation was not explicitly brought up in our discussions at the PME-NA 2021 conference. MTEs at regional institutions who may be members of AMTE (and/or a local affiliate), but not PME-NA, may have relationships with local schools at which their graduates teach that afford increased access to conduct such research. We see extensions to local AMTE affiliates as an opportunity to grow the membership of this working group, the mathematics education research community, and the PME-NA organization.

As a working group, we developed some broad ideas for what types of research we would need to conduct to move forward. For example, additional research on the effect of eliminating a particular topic from teacher preparation coursework might be needed to convince institutions that removal is warranted. Thus, our goals for the PME-NA 2022 conference will be not only to continue with the drafting of recommendations documents, but also to spend an equal amount of time on designing specific research studies that we could conduct at our own institutions to inform this work. We also hope to engage in studies that are collaborative across institutions to examine the effects of different teacher preparation program structures, while accounting for potential differences in PST population. Alternatively, some participants may wish to conduct action research, for example, by teaching two sections of the same course with different emphases on topics and comparing outcomes. Our discussions at the PME-NA 2022 conference would focus on refining participants' ideas for designing and conducting such studies. Additionally, the organizers of the working group have had preliminary discussions about applying for NSF grant funding to support collaborative research to meet the working group's goals. Obviously, designing such studies, writing a grant proposal, and applying for such funding would take time, but we see the continuation of this working group as a resource to engage more individuals in such a proposal.

Additionally, at future meetings of the working group, we wish to engage more diverse and international (across North America) perspectives on the curriculum and development of elementary teacher preparation programs. We invite colleagues from all nations and institutions with an interest in this research area to participate in this working group and provide additional perspectives on the questions this working group is addressing. Furthermore, we have received interest from additional researchers in joining the organization team for this working group; we affirm that we enthusiastically support new members to join this working group in either a participant or leadership role. Our meeting at the PME-NA 2021 conference was an excellent first step towards addressing the questions we are focused on, and we look forward to applying the lessons we learned to our future meetings over the next several years.

References

Association of Mathematics Teacher Educators [AMTE]. (2017). *Standards for preparing teachers of mathematics*. https://amte.net/standards

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Chapin, S. H., Gibbons, L. K., Feldman, Z., Callis, L. K., & Salinas, A. (2021). The Elementary Mathematics Project: Supporting preservice teachers' content knowledge for teaching mathematics. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 89–113). Springer. https://doi.org/10.1007/978-3-030-68956-8_4

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. American Mathematical Society and the Mathematical Association of America. Retrieved March 7, 2021, from https://www.cbmsweb.org/archive/MET2/met2.pdf

Corven, J. (2021). Prospective elementary teachers' evaluations of student solutions to division story problems. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1134–1135).

Corven, J., DiNapoli, J., Willoughby, L., & Hiebert, J. (accepted). Long-term relationships between mathematics instructional time during teacher preparation and specialized content knowledge. *Journal for Research in Mathematics Education*.

Gibbons, L., Feldman, Z., Chapin, S., Batista, L. N., Starks, R., & Vasquez-Aguilar, M. (2018). Facilitation practices in mathematics teacher education and the mathematical identities of preservice elementary teachers. *Mathematics Teacher Education and Development*, *20*(3), 20–40. https://eric.ed.gov/?id=EJ1195989

Li, Y., & Howe, R. E. (2021). Toward a thinking-oriented training in mathematics for elementary school teachers. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 13–49). Springer. https://doi.org/10.1007/978-3-030-68956-8_2

Long, V. N., & Corven, J. (2021, May 12–13). *Curriculum recommendations for elementary mathematics teacher preparation* [Working group]. 15th Annual Symposium of the Pennsylvania Association of Mathematics Teacher Educators, Online.

Starks, R. (2021). Exploring relationships between prospective teachers' decimal models and their procedural and conceptual decimal knowledge. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1072–1077).

MATHEMATICS CURRICULUM RECOMMENDATIONS FOR ELEMENTARY TEACHER PREPARATION WORKING GROUP: PHASE II

Julien Corven
Illinois State University
jccolvi@ilstu.edu

Joseph DiNapoli
Montclair State University
dinapolij@montclair.edu

Lynsey Gibbons
University of Delaware
lgibbons@udel.edu

Kim Johnson
West Chester University
KJohnson2@wcupa.edu

Valerie Long
Indiana University of Pennsylvania
vlong@iup.edu

Dana Olanoff
Widener University
dolanoff@widener.edu

Rachel Starks
Boston University
rnstarks@bu.edu

In our first year of work at the PME-NA and AMTE conferences, we explored the current state of elementary teacher preparation programs and generated questions about how to create a viable recommendations document for the mathematics curriculum of elementary teacher preparation. Our current goal is to build upon this work by designing research studies to generate data and answer questions that underlie our future recommendations for elementary teacher preparation.

Keywords: Preservice Teacher Education, Teacher Educators, Mathematical Knowledge for Teaching, Elementary School Education

Teacher preparation programs vary substantially in what mathematics elementary pre-service teachers (PSTs) learn and to what degree across Canada (NCEE, 2016), Mexico (Hrusa et al., 2020), and the United States (Malzahn, 2020). We confirmed these findings at the 2021 PME-NA conference (see Corven et al., this volume, for details); many participants' programs fell short of the recommended 12 credit hours in elementary mathematics (Conference Board of the Mathematical Sciences, 2012). The organizers of this working group believe experiences during teacher preparation affect how teachers teach elementary mathematics; thus, more guidance on how to effectively prepare elementary PSTs to teach mathematics may reduce undesirable variation between preparation programs and improve the average quality of graduates' teaching. To reach these goals, we need to generate recommendations for the (re)design of elementary teacher preparation in mathematics that are concrete, actionable, and based on research.

Limiting the number of mathematics topics covered in elementary teacher preparation may be both desirable and necessary. Corven et al. (in press) conclude the amount of instructional time PSTs receive on mathematics topics is strongly and positively associated with specialized content knowledge (SCK; Ball et al., 2008) novice teachers demonstrate for those topics. Further, since the amount of instructional time required to generate an appreciable increase in an average PST's SCK for a topic seems to be substantial, attempting to cover every topic in elementary mathematics during teacher preparation is not feasible (Corven et al., in press). There has been some movement towards limiting the number of topics taught in elementary teacher preparation programs. One example is the Elementary Mathematics Project (Chapin et al., 2021), which also invites PSTs to build a new relationship with mathematics as members of dual communities of mathematics learners and future mathematics teachers (Gibbons et al., 2018). At our previous meeting, members of the working group wanted to know if research existed that addressed whether certain mathematical topics were more or less important for PSTs to know and/or were easier or harder for PSTs to learn. Without answering such questions, content recommendations would only reflect individuals' opinions, and therefore might be disregarded.

At this working group's meeting at the 2022 Association of Mathematics Teacher Educators conference, not only did some of the same questions arise, but participants shared different ideas for how to craft recommendations for teacher preparation programs based on local constraints. The organizers drew upon these conversations to revise a long-term plan for the working group, which we will enact this year. We see our meetings this past year as comprising Phase I of the working group; our goals were to get a sense of the landscape of elementary teacher preparation programs and to think about obstacles to creating program recommendations. This year, we move into Phase II, which consists of designing research to inform our future recommendations.

Organization and Presentation Plan

The first session will re-orient members to the work that has already been done so as not to be duplicative; however, members will have opportunities to ask questions about and make additions to the previously generated list of concerns about elementary teacher preparation. We will also review frameworks that inform our work, such as knowledge- vs. thinking-oriented approaches to teacher preparation (Li & Howe, 2021). We will then brainstorm research questions that would be important to answer to make informed recommendations for teacher preparation programs. We anticipate members asking research questions concerning the impact of inclusion and/or omission of certain mathematical topics in content courses, the perceptions of novice elementary teachers regarding their preparation to teach mathematics topics, and mathematics professional development opportunities for novice elementary teachers.

In the second session, participants will work in groups to design small-scale research studies that they could conduct in their own contexts to generate data that will help answer one of the research questions from the previous day. We would encourage participants to consider methods like action research and survey research that can be conducted with minimal financial support and within each participant's capacity. Additionally, participants may decide to collaborate on a multi-site study to compare or contrast outcomes from different approaches to teacher preparation or examine impacts of contextual factors. We expect participants to provide each other with feedback on the research plans and to help each other develop appropriate instruments (e.g., interview protocols, survey questions) for the collection of relevant data. We will ask participants to document their ideas in a shared Google Drive folder.

During the final session, we will ask participants to share their proposed studies and receive additional feedback and ideas from the group. Ideally, participants could support each other in data collection efforts to allow for collaboration on specific research questions. Regardless, all study plan documents will be kept in the shared Google Drive to facilitate submission to local Institutional Review Boards. We expect this meeting of the working group will lead to ongoing productive collaborations among working group members prior to our next meeting.

The New Long-Term Plan

In Phase III, members will conduct the research designed in Phase II, and the working group will support each other in synthesizing the results of the studies and writing research reports for future PME-NA conferences. Thus, we expect Phase III to take two years to complete. In Phase IV, we will organize a conference to draft and refine the recommendations based on the Phase III research results. We will also disseminate the recommendations and the research supporting them via an edited book. Through this plan, we intend to support PME-NA members in the conduct of research to improve elementary teacher preparation throughout North America.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Chapin, S. H., Gibbons, L. K., Feldman, Z., Callis, L. K., & Salinas, A. (2021). The Elementary Mathematics Project: Supporting preservice teachers' content knowledge for teaching mathematics. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 89–113). Springer. https://doi.org/10.1007/978-3-030-68956-8_4

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. American Mathematical Society and the Mathematical Association of America. Retrieved March 7, 2021, from https://www.cbmsweb.org/archive/MET2/met2.pdf

Corven, J., DiNapoli, J., Gibbons, L., Long, V., & Starks, R. (this volume). Working group on mathematics curriculum recommendations for elementary teacher preparation: First report.

Corven, J., DiNapoli, J., Willoughby, L., & Hiebert, J. (in press). Long-term relationships between mathematics instructional time during teacher preparation and specialized content knowledge. *Journal for Research in Mathematics Education*.

Gibbons, L., Feldman, Z., Chapin, S., Batista, L. N., Starks, R., & Vasquez-Aguilar, M. (2018). Facilitation practices in mathematics teacher education and the mathematical identities of preservice elementary teachers. *Mathematics Teacher Education and Development*, *20*(3), 20–40. https://eric.ed.gov/?id=EJ1195989

Hrusa, N. A., Much Islas, P., Schneider, J. A., & Vega, I. J. (2020). Policies for teacher professionalization in Mexico's education reform. In F. M. Reimers (Ed.), *Empowering teachers to build a better world: How six nations support teachers for 21st century education* (pp. 63–85). Springer. https://doi.org/10.1007/978-981-15-2137-9_4

Li, Y., & Howe, R. E. (2021). Toward a thinking-oriented training in mathematics for elementary school teachers. In Y. Li, R. E. Howe, W. J. Lewis, & J. J. Madden (Eds.), *Developing mathematical proficiency for elementary instruction* (pp. 13–49). Springer. https://doi.org/10.1007/978-3-030-68956-8_2

Malzahn, K. A. (2020). *2018 NSSME+: Trends in U.S. mathematics education from 2012 to 2018.* Horizon Research, Inc. Retrieved March 7, 2021, from http://horizon-research.com/NSSME/wp-content/uploads/2020/11/Math-Trend-Report-Final.pdf

The National Center on Education and the Economy [NCEE]. (2016). *Canada: Diversity and decentralization*. Author. Retrieved June 25, 2021, from http://ncee.org/wp-content/uploads/2017/02/CanadaCountryBrief.pdf

MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

David Plaxco
Clayton State University
davidplaxco@clayton.edu

Paul Reimer
AIMS Center
paul@aimscenter.org

Caro Williams-Pierce
University of Maryland iSchool
carowp@umd.edu

Amy Ellis
University of Georgia
amyellis@uga.edu

Susanna Molitoris-Miller
Kennesaw State University
smolitor@kennesaw.edu

Amber Simpson
Binghamton University
asimpson@binghamton.edu

Michelle Zandieh
Arizona State University
zandieh@asu.edu

Matt Mauntel
Florida State University
mmauntel@fsu.edu

Muhammed Fatih Dogan
Adiyaman University
mfatihdogan@adiyaman.edu.tr

For the third time in four years, the Mathematical Play Working Group met to discuss the state of research into the role that play can have in mathematics education research. This meeting provided an opportunity for members within the group to share recent progress in their respective areas of focus. The group also continued theoretical discussions regarding areas of focus for new research and productive framings and perspectives for addressing mathematical play. Additionally, the working group dedicated a portion of the meeting to collaborating with the Embodied Mathematical Imagination and Cognition [EMIC] *Working Group.*

Keywords: Instructional activities and practices; Affect, emotion, beliefs & attitudes; Informal education

For the past four years, the Mathematical Play working group has convened to investigate and advance the field's understanding of mathematical play. Our collaborative work has been characterized by: 1) experiencing mathematical play together, 2) identifying the features and affordances of mathematical play for learning and doing mathematics, 3) and discussing implications for designing mathematical play learning experiences across ages and contexts. Broadly, our work has spanned both theoretical and practical worlds; for example, we seek to understand what aspects of mathematical activity might allow it to be characterized as "play" while simultaneously considering the extent to which mathematical play is possible and productive in classrooms. Our explorations have ranged in context from early childhood preschool classrooms to university mathematics courses, and have examined the use of block play, videogames, partner activities, simulations, board games, and making/tinkering activities.

Summary of 2021 Meeting

Our goal for the 2021 working group meeting was to examine mathematical play along two dimensions - activity grounded in instruction and activity grounded in play - with specific exploration of activities that might support shifts along these dimensions in two specific ways (Figure 1). On the first day, we explored how traditional mathematical instructional tasks might be altered to afford greater opportunity for play. On the second day, we investigated ways play-based activities might be adapted to better support meaningful learning toward specific conceptual outcomes. Working group sessions engaged participants in conceptualizing these two shifts through sharing and explorations of existing projects and discussions of perspectives to support such shifts. Finally, we spent the third day of the meeting collaborating with the Embodied Mathematical Imagination and Cognition [EMIC] Working Group.

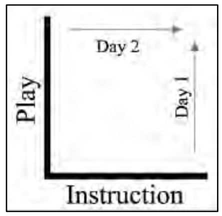


Figure 1: Organization of two shifts we will explore – increasing the playfulness of high instructional tasks (Day 1) and increasing the instructional utility of high play tasks (Day 2)

Day 1: Adapting Mathematical Tasks to Increase Opportunities for Play

On the first day of the 2021 working group, Ellis and Plaxco each presented examples from their respective projects demonstrating efforts to shift existing activities grounded in instruction toward more playful settings. Ellis presented a new (as of the conference) project in which her research team is modifying existing tasks designed to support covariational reasoning about linear and quadratic relationships for play activities. This new area is based on Ellis' extensive work supporting students' sensemaking regarding rates of change and accumulating growth (e.g., Ellis, Ely, Singleton, & Tasova, 2020). In her group's attempt to play-ify existing tasks, Ellis described scenarios in which players (with experience creating accumulation graphs from a given growing shape) would build accumulation graphs and for other players who were then tasked with reconstructing the growing shape. Working group attendees participated in a paired activity to play this game with each other.

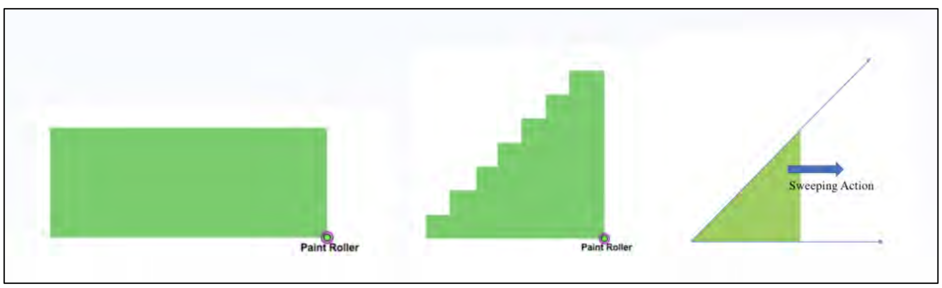
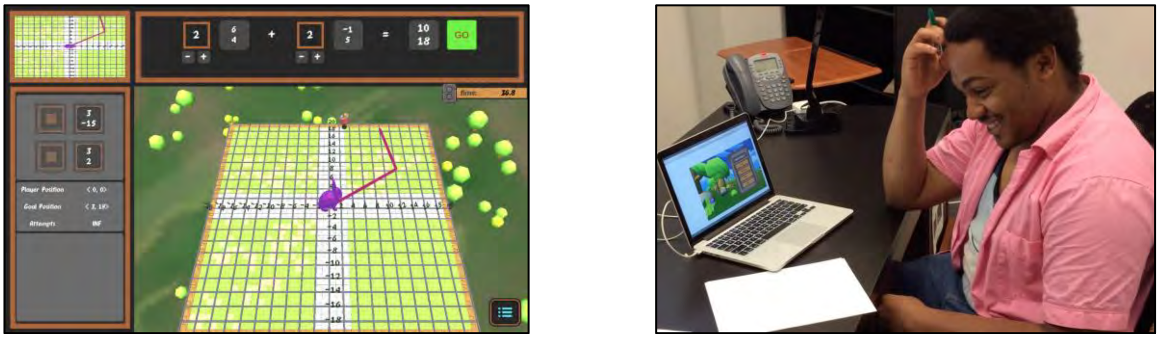


Figure 2: Screenshots of growing figures from Ellis' presentation

Following this, Ellis had group participants share and discuss and then shared out their experiences among the whole group. Ellis then presented brief examples of student data that exemplified some of the phenomena that the group had experienced. Following this presentation, Ellis outlined her team's working definition for playful math as instances in which students engage in activity with (a) agency in exploration, (b) self-selection of goals, (c) self-direction in how to accomplish the goals. The ensuing discussion addressed several topics, notably the importance of focusing on student agency and the role it plays in authentic play as well as the tension between structure (toward an intended learning goal) and autonomy (the natural progression of the player's activity).

Plaxco then presented examples from his research collaboration in designing and creating the videogame *Vector Unknown* (Figure 3a; Mauntel, Levine, Plaxco, & Zandieh, 2021) for supporting students' Linear Algebra understanding that is based on the existing Inquiry-Oriented Linear Algebra curriculum [IOLA; Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013]. In his presentation, Plaxco provided an additional instance of the importance of layer agency by describing a player (Lance, Figure 3b) deciding to play what Lance described as a "meta" game of achieving an *additional* goal beyond the goal of the game intended by the design team. The resulting discussion connected Ellis' presentation with Plaxco's, specifically focusing on the role that level design can have in shifting players' focus toward more authentic activities that can also tend to lead to richer mathematical discussions.

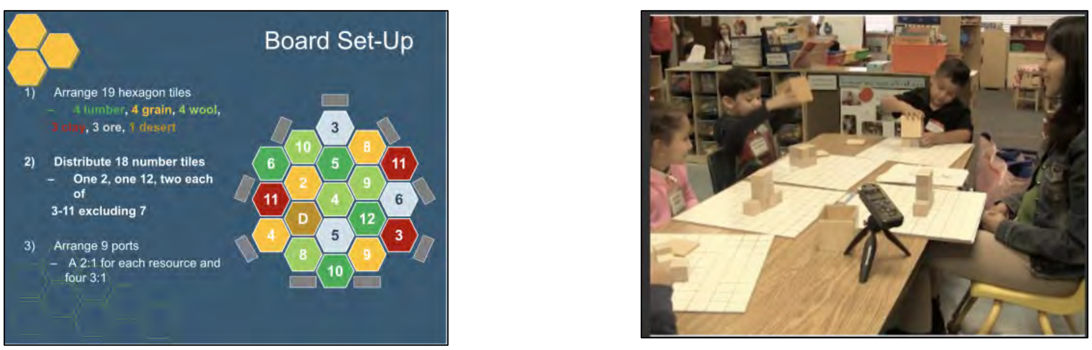


(a) (b)

Figure 3: A screenshot of the *Vector Unknown* (a) and photo of Lance deciding to play a "meta" game within the videogame (b)

Day 2: Adapting Play Activities to Support Mathematics Learning

On the second day, Molitoris-Miller, Reimer, and Simpson presented playful contexts that provide opportunities for mathematical thinking and learning. Molitoris-Miller suggested ways undergraduates encountered moments of mathematical play through explorations with the board game Catan (Austin, Kronenthal, & Miller, 2021). In particular, discussions focused on how board and game setup, players' strategies and solution methods, and students' mathematical arguments traversed the traditional boundaries of play and mathematics.



(a) (b)

Figure 4: Molitoris-Miller's described board setup for the game Catan (a) and Reimer shared analysis of guided play episodes in preschool classrooms (b).

Reimer then presented his research conducted in preschool classrooms to explore how young children and teachers engaged in spatial reasoning through play (Reimer, 2021). Reimer described guided play episodes that encouraged children to explore a variety of contexts and materials. In particular, discussion focused on ways teachers and children participated in play together, with specific attention to the agency and authorship afforded to children through co-participation in play.

Finally, Simpson engaged the working group in a tinkering task (Simpson, Zhong, & Maltese, 2022) using common materials to design a delivery mechanism for sharing objects with a friend next door. The group used popsicle sticks, elastic bands, straws, and other materials to construct their mechanisms. Simpson guided discussions around the potential of low-notation, playful environments to provide opportunities for youth to experience, express, and build upon mathematical practices in a context that is not bounded by standardized tests and textbooks.

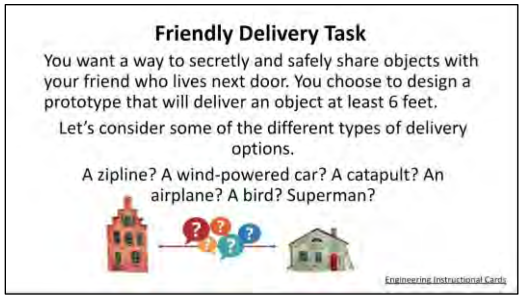


Figure 5: Simpson's engineering task offered opportunities to legitimize informal ways of thinking about mathematics.

Day 3: Intra-working-group Discussion with EMIC

The final day, the Mathematical Play and EMIC working groups met together for a session focused on exploring areas of overlapping interest. In particular, the two working groups converged to explore play as an embodied approach to mathematical learning, and embodied activities as opportunities for mathematical play. EMIC group members, led by Hortensia Soto, guided participants through an activity focused on supporting an embodied sense of the geometric properties of triangles, specifically, the fact that the interior angles of any triangle sum to 180° and external angles of any polygon sum to 360° (Soto, 2021). The groups also discussed how this activity might be generalized for any closed polygon. The following discussion centered around the importance of embodied activities for mathematical learning and discussed potential commonalities between the play, embodied mathematics, and creativity.

Ongoing and Future Work

Since the conference, several subsets of attendees have continued collaborations and intend to develop these into extensive projects. For example, co-author Ellis and meeting attendee Rob Ely have secured internal grants at each of their universities, have continued to analyze their play data, and have developed analytical framings for mathematical play, which they intend to present at the next PME-NA conference. Co-authors Reimer, Williams-Pierce, and Simpson have continued collaborations, meeting to develop their focus on failure and feedback in free-play environments. Co-authors Plaxco, Zandieh, and Mauntel have continued development of a 3-dimensional version of *Vector Unknown* and are collecting data with that version of the game during Spring 2022. In reflecting on the Working Group, co-authors Plaxco, Reimer, Williams-Pierce, and Ellis identified agency as a critical aspect of mathematical play that warrants further development. This aspect of players' experiences emerged across every discussion and presentation during the working group session as a critical component for not only successful play, but also deeper mathematical meaning to emerge during play. In light of this, we have chosen this as the central focus of the next Working Group, which we intend to shift in format from years' past. Specifically, with our next Working Group, we intend to spend the time collaboratively developing theoretical framings related to Mathematical Play. The primary goal of this work is to develop a collective touchstone for: (1) defining mathematical play in a way that situates all of the group members' work within the field, (2) identifying and characterizing the role of agency in mathematical play, and (3) identifying meaningful structural connections between mathematical practices and playful disposition.

References

Austin, J., Kronenthal, B. G., & Miller, S. M. (2021). Off to a Good Start: Using Mathematics to Evaluate Settlement Locations in Catan. *Teaching Mathematics Through Games*, *65*, 137.

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. *Educational Studies in Mathematics*, *104*(1), 87-103.

Mauntel, M., Levine, B., Plaxco, D., & Zandieh, M. (2021). The Characterization and Evolution of Strategies About Vector Equations in the Game Vector Unknown. *Digital Experiences in Mathematics Education*, *7*(3), 453-476.

Reimer, P. N. (2021). Serious work through play: Teaching and learning spatial reasoning in early childhood (Order No. 28644442). Available from ProQuest Dissertations & Theses A&I; ProQuest Dissertations & Theses Global. (2570444250). Retrieved from https://www.proquest.com/dissertations-theses/serious-work-through-play-teaching-learning/docview/2570444250/se-2?accountid=10139

Simpson, A., Zhong, Q., & Maltese, A. (2022). Spontaneous Mathematical Moments Between Caregiver and Child During an Engineering Design Project. *Early Childhood Education Journal*, 1-12.

Soto, H. (2021). Acting Out Euclidean Transformations. *PRIMUS*, 1-15.

Wawro, M., Zandieh, M., Rasmussen, C., & Andrews-Larson, C. (2013). Inquiry oriented linear algebra: Course materials. Available at http://iola.math.vt.edu. This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

PLAYFUL MATHEMATICS AND LEARNING

Coordinators: Robert Ely, University of Idaho, ely@uidaho.edu; Amy Ellis, University of Georgia, amyellis@uga.edu

In this colloquium we build on findings established in the working group *Mathematical Play: Across Ages, Context, and Content*, which met in 2018, 2019, and 2021 (e.g., Plaxco et al., 2021). In those meetings, the working group developed definitions and criteria for characterizing mathematical play and for exploring multiple contexts in which mathematical play occurs. The work presented in this *playful mathematics and learning* colloquium extends these characterizations to examine relationships between play and classroom-based learning.

Playifying classroom mathematics offers opportunities to center student voices, particularly those voices that may be underrepresented in traditional classroom settings. It also offers spaces for challenging status-quo classroom norms, which can increase student agency and autonomy (Gresalfi et al., 2018). At the same time, there is a tension between the open-ended nature of play and the highly directed nature of typical classroom instruction. We explore this tension by identifying design principles for infusing play into classroom spaces, and by discussing theoretical and methodological constructs for characterizing the mathematical play that can arise both in and out of classroom spaces. In doing so, we explore mathematical play phenomena across ages (pre-kindergarten through college), contexts (informal and classroom settings), and domains (early number, algebra, geometry, probability, and calculus).

Session 1: Playifying Mathematics in the Classroom

The bulk of research on mathematical play investigates the mathematics that emerges in children's natural play or in informal spaces. We instead introduce the construct of "playful math", exploring a set of four design principles to playify classroom-based activities: (a) allow for free exploration within constraints; (b) allow the student to act as both designer and player; (c) engender task-based anticipation; and (d) enable task-based feedback. We share results from two related projects that enacted these principles in middle-school, high-school, and undergraduate settings in algebra, geometry, and probability. We found that the design principles were successful in eliciting playful engagement, which corresponded to increases in student agency, goal selection, and sustained engagement.

Amy Ellis, Anna Bloodworth, & Dru Horne (30 mins): We share the results of two small-group teaching experiments and one whole-classroom teaching experiment investigating rates of change through a task called *Guess My Shape*. In shifting from typical covariation activities to a game in which students became shape designers, we found that students introduced novel shapes and graphs that represented more challenging mathematical ideas than what had occurred in prior non-play versions of the tasks. We discuss the resulting mathematics that emerged and discuss implications for task design and student engagement.

Rob Ely & Annelise Nielsen (30 mins): We present findings from a set of classroom activities in which undergraduates played the *Double Spin* probability game and a *Fold-and-Cut* geometry activity. We share characteristics of a form of novel engagement during playful math, the Explore-Strategize Cycle, in which exploration of phenomena in the task environment gave rise to strategic goal-directed activity, the outcome of which changed the space of potential phenomena explored next, and so forth in a cyclical manner. Such cycles were not present in

Lischka, A. E., Dyer, E. B., Jones, R. S., Lovett, J. N., Strayer, J., & Drown, S. (2022). Proceedings of the forty-fourth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Middle Tennessee State University.

non-play activity. We present on Explore-Strategize Cycle and discuss resulting task design strategies.

Discussion (30 mins): Participants will discuss in groups (a) the relationship between design principles for playifying tasks and the resulting student play, and (b) the relationship between student play and mathematical learning goals and outcomes.

Session 2: Drawing Classroom Mathematics from Play

Janet Walkoe & Mari Levin (30 mins): We claim that the insights children gain when playing create a rich body of experiences that are potential resources for later algebraic reasoning. Drawing on a systems approach of learning, we theorize movement toward expertise as a reorganization of sub-conceptual knowledge elements children gain as they interact with their world. We call these elements *seeds of algebraic thinking* (Walkoe & Levin, 2020). Here we provide examples of seeds and illustrate how they can be taken up in a range of experiences. We elaborate how our perspective highlights the relationship between children's early play and their engagement in formal algebra instruction.

Anita Wager, Candice Love, Madison Knowe, Melissa Gresalfi, & Amy Noelle Parks (30 mins): In the *Playful Learning Project*, kindergarten teachers are engaging in professional development around integrating play in mathematics teaching. This involves recognizing the mathematical content embedded in spontaneous play and the possibilities for designing mathematical engagement that embody a spirit of playfulness. We discuss how the PD design elements supported teachers to: (a) recognize and elicit early number and geometric concepts in free play and design playful lessons to allow children to explore these concepts; (b) identify the key classroom mathematical routines and their emotional and mathematical heartbeats (e.g., free play, center-based play, choral counting, problem-solving); and, (c) identify structures in adopted curricula that support playful, exploratory mathematics and structures (and structures that do not).

Discussion (30 mins): The discussion will address theoretical principles for recognizing and fertilizing the mathematical seeds found in children's informal play.

Session 3: Play and Intellectual Need

Aaron Weinberg (30 mins): When a student encounters a problem they cannot solve using their current knowledge, they experience a need for a concept or tool to resolve it. Harel (2013) called this an intellectual need, and proposed that such a need is required for students to learn a concept or tool. We can design problematic situations that we hope will engender students' intellectual need for particular concepts and tools we want them to learn. Yet, sometimes students respond with confusion rather than curiosity when faced with such problematic situations. Playifying mathematics tasks could help with this. I describe a theoretical framework for designing and enacting tasks that can provoke intellectual need, and I discuss ways in which playful math might help students persist in their mathematical exploration to experience intellectual need.

Discussion: We include a full hour of discussion in order to integrate ideas across all three sessions of the colloquium. We will first coordinate small-group discussions to conceptualize and resolve the tension between the open-ended nature of play and the directed nature of classroom learning (30 minutes). We will then convene a whole-group discussion to identify principles for capitalizing on findings from informal and formal mathematical play in order to playify classroom spaces.

References

Gresalfi, M., Horn, I., Jasien, L., Wisittanawat, P., Ma, J.Y., Radke, S.C.,...,Sfard, A. (2018). Playful mathematics learning: Beyond early childhood and sugar-coating. In J. Kay & R. Luckin (Eds.), *13th International Conference of the Learning Sciences* (pp. 1335 – 1342). London, UK: University College London.

Harel, G. (2013). Intellectual need. In K.R. Leatham (Ed.), *Vital Directions for Mathematics Education Research* (pp. 119 – 151). Springer Science + Business Media.

Plaxco, D., Reimer, P.N., Williams-Pierce, C., Ellis, A.B., Molitoris-Miller, S., Simpson, A., Zandieh, M., Mauntel, M., & Dogan, M.F. (2021). Mathematical play: Across ages, context, and content. In Olanoff, D., Johnson, K., & Spitzer, S. (Eds.), *Proceedings of the Forty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1913 – 1915), Philadelphia, PA.

Walkoe, J., & Levin, M. (2020). Seeds of algebraic thinking: Towards a research agenda. *For the Learning of Mathematics, 40*(2), 27 – 31.

FINAL REPORT: PRESERVICE TEACHER LEARNING OF PRACTICE THROUGH SIMULATED TEACHING EXPERIENCES BEFORE, DURING, & AFTER COVID

Liza Bondurant
Delta State University
lbondurant@deltastate.edu

Heather Howell
ETS
hhowell@ets.org

Minsung Kwon
California State University, Northridge
minsung.kwon@csun.edu

Carrie Lee Wilkerson
East Carolina University
leecarr16@ecu.edu

Yvonne Lai
University of Nebraska-Lincoln
yvonnexlai@unl.edu

Bima Sapkota
Purdue University
bsapkota@purdue.edu

This working group was a continuation of a 2019 PME-NA working group focused on the challenges and opportunities of using simulations of teaching practice as an educative tool for preservice teachers. Initially, we shared our diverse experiences designing and implementing simulations before and during the COVID-19 school closures. This discussion inspired us to consider the various dimensions of simulations and the continuums of choices along each dimension. We aim to meet again at PME-NA 2022 to refine our definitions of the dimensions of simulations, discuss research collaborations, and develop an NSF grant proposal to continue our work.

Keywords: Simulation, Preservice Teacher Education, Online and Distance Education, Technology, Classroom Discourse.

During their professional preparation preservice teachers (PTs) must obtain authentic practice-based experiences (Ball & Forzani, 2009; Forzani, 2014). According to Grossman, Hammerness, and McDonald (2009) approximations of practice are one of the three key components of professional practice and they provide “opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity” (Grossman et al., 2009, p. 283). Through approximations of practice PTs obtain opportunities to learn from their mistakes, try different instructional approaches, and improve their pedagogy (Girod & Girod, 2008). PTs have the opportunity to enhance their skills in a safe environment and have a minimal chance to negatively impact actual students’ learning. A plethora of professions, such as medicine, aviation, and the military have expanded their use of simulations. Several researchers have investigated the impact of simulations on teacher professional learning (e.g., Bondurant & Amidon, 2021; Howell & Mikeska, 2021; Lee et al., 2021; Mikesa et al., 2021; Straub et al., 2014). An endorsement from the American Association of Colleges for Teacher Education ([AACTE], 2020) and the COVID-19 pandemic, the use of digital simulations has expanded. We see the expanded use of digital simulations as an opportunity to learn from our experiences and develop best practices.

Focus of Work

We entered the working group with a set of guiding questions oriented around how our collective work, thinking, or use cases might have been affected by COVID-19, both in the short-term in contrast to prior use of simulations, and in the longer term thinking about what parts of those changes are likely to persist in a post-pandemic world. In contrast to the 2019 working group that focused on the theory and design of simulations themselves, this 2021 session was focused more directly on the contextualization of simulation use, particularly the contrast case provided by the context of working within COVID-19 inducted constraints. Our guiding questions, described in the main proceedings document, were as follows:

1. What affordances and/or challenges in using, adapting, and integrating digital simulations existed before the pandemic?
2. How and why did the affordances and/or challenges change during the pandemic?
3. What affordances and/or challenges do you see in using, adapting, and integrating digital simulations after the pandemic?

Organization for Active Participant Engagement

We began, as we had proposed, by grounding the discussion in the results of the prior working group. This was, in part, to orient the group as some presenters and participants had attended in 2019 while others had not. It also served to establish a common vocabulary, drawing on a provisional list of dimensions of authenticity that had been the core of the 2019 group's discussion. We then went on to describe each of our own experiences before and after COVID-19, beginning with the working group organizers and then inviting the participants to share similarly. While originally conceptualized as a one to one mapping of session dates to experiences with simulations before, during, and after COVID-19, we found that beginning with the contrast of prior experience to current experience was a more productive orientation for most participants, which ended up bridging both sessions 1 and 2.

Subsequently, we discussed the commonalities among our work and attempted to determine the key design decisions or dimensions of simulations. Finally, we closed by deciding that we wish to meet again at PME-NA 2022 to refine our definitions of the dimensions of simulations, discuss research collaborations, and develop an NSF grant proposal to continue our work.

Building on Prior Work

Initially, the 2019 PME-NA organizers, Heather, Yvonne, and Carrie, shared the takeaways from their working group. The 2019 organizers discovered that it would be helpful to develop a common vocabulary and identify design characteristics of simulations. They identified the following dimensions of simulations: boundedness of set-up, complexity of intended response, role of feedback, extemporaneity, linearity of time, individuality, and elaboration of student character(s). They discussed how the researcher's goals impact their design decisions. A key distinction the organizers uncovered among their work was whether the effectiveness of the simulations was the primary focus of the research.

Contrasting Simulation Use Cases

Next, each participant shared their work using simulations, beginning with the five 2021 organizers. The first four shared their experiences within their own courses and the final organizer shared her experiences across multiple sites. New participants to the group were then invited to share their work using simulations. The aim was to only share how they used simulations before COVD-19. However, many of the organizers' work continued or expanded during COVID-19. Therefore, the organizers shared their experiences and takeaways before and during COVID-19. These presentations inspired discussions that lasted into the second session.

Liza studied elementary and secondary PTs' use of equitable teaching practices in virtual simulations. Her takeaways were that overall PTs found the virtual simulations very authentic and beneficial to their pedagogical content knowledge and skills. She also explored the PTs' implicit biases, perceptions of the student avatars, and quantity and quality of the opportunities that the PTs' provided students of different social markers (gender, race) to participate. She found that unless a random method was deliberately used, PTs did not call on students equitably. She also found that PTs typically asked closed questions that were not cognitively demanding. Finally, Liza noticed that the student avatars' behaviors and words often reinforced stereotypes.

Minsung studied elementary PTs' experiences in non-digital simulations and digital-simulations. In using simulations during the pandemic, Minsung noticed how PTs sequenced student work differently, how PTs provided examples/counterexamples to address errors and misconceptions, whether PTs asked students to critique each other's work, and the quantity and quality of PTs' interactions with the students. Minsung also noticed the interactor effect (referred to as the interviewer effect in survey research). She noticed that each avatar's dispositions varied depending who the interactor was. In the simulation facilitated by the female interactor, avatars shared that they were so excited with math. Minsung found that the affordances of using digital simulations include exploring some key pedagogical practices, analyzing student work samples in a shared work space, less nerve-wracking but feeling more real, conferencing their teaching performances with peers in a public space by reducing variability across different field placement settings/contexts, receiving content-specific feedbacks from the instructor, identifying their strengths and areas of improvement on the targeted teaching practices, and recording/assessing/researching teaching in a safe environment.

Yvonne explored secondary PTs' written and video-recorded responses to samples of students' work in math content courses as a part of the Mathematics Of Doing Understanding Learning and Educating for Secondary Schools (MODULES[2]) project. She examined differences in Foundation and Contingency skills across 27 pairs of video and written simulations of practice (SoPs). Foundation was defined as whether or not math knowledge is harnessed productively in teaching and contingency was defined as responding to potentially unexpected student contributions (Rowland, 2013). She found 7 pairs with higher Foundation in the video SoP and zero pairs with higher Foundation in the written SoP.

Carrie investigated elementary PTs' facilitation of number talks in peer-to-peer rehearsals and virtual simulations. Carrie noticed that the foci of peer-to-peer feedback varied from understanding of properties to personal solution strategies. Virtual rehearsal simulation (VRS) feedback centered on positioning of students, understanding student solution strategies, and pressing for connections between strategies. In addition to variation in feedback foci, the development of elicitation strategies was studied. By coding different elicitation prompts, analysis of teaching episodes found that PTs who rehearsed in virtual simulations increased in their use of more closed-ended prompts during rehearsals, as compared to PTs who rehearsed with peers. However, when comparing change in elicitation strategies when teaching in classroom settings, PTs that rehearsed with virtual simulations increased in their use of more open-ended prompts while PTs that rehearsed with peers decreased in their use of more open-ended prompts (Lee et al., 2021). The study of how PTs develop in their use of elicitation prompts during rehearsals seeks to better understand how to support stronger discourse practices.

Heather's perspective is a bit different as her work has been as a researcher studying elementary PTs' facilitation of classroom discussions in virtual simulations across multiple sites, giving her a broad perspective on simulation use across very different contexts but a less intimate one as she has not been working with her own students. The focus across her projects has largely been on formative assessment, relying on a theory of action that supposes that some learning is produced by engagement alone, but that the primary learning mechanism is reflecting and receiving feedback and support. Her more recent work has focused on scaffolding across simulation types.

Heather reported several high level noticings from her experiences with math teacher educators (MTEs) and PTs. Heather's first noticing was that before the pandemic people were hesitant and cautious about using simulations. While some of the caution may have been well-founded, she suspected that some were also fear of the new. Heather has always acknowledged the affordances, but also that simulations have a place among other activities and are neither designed as nor well-used as replacements for interactions with real children. Heather noticed that the field has swung toward simulations with enthusiasm as a result of the COVID-19 school closures, which increased both their popularity and the number of cases in which they were, of necessity, used as replacements for field work rather than complements to it. While the enthusiasm is encouraging, she noted this loss of hesitance as potentially problematic, as it might produce more acceptance of off-label use of simulation in the future, which might be appropriate but, like any off-label use, merits careful consideration of unintended consequences.

Heather also observed that using simulations in online courses was easy and seamless. She conjectured that we may be more likely to accept interactions through a screen as legitimate human interactions after the abundance of online classes and meetings that we participated in during COVID-19. Like all online classes, simulations opened up access to PTs whose lives were less complicated with the removal of long distance commutes, but it limited access in other ways, exacerbating inequality for PTs with inadequate technology access. None of this is particular to simulations per-se, as distance learning forced all of us to consider more closely the affordance of distance-learning in ameliorating some sources of inequity while amplifying others, but the processor-intensive requirements of simulation certainly brought it into sharp focus in this context.

Heather also reflected on PTs' feedback on surveys that raised questions about transfer to classroom practice, in some cases related to simulation but in some cases more broadly. Many PTs commented that teaching five students in a small group on a screen was very like teaching in a zoom breakout room with real students. Some complained that it seemed unnatural in the simulation to see only the students' faces and not themselves on video, and that not seeing themselves on screen made it difficult to regulate their facial expressions. While raised in the context of the simulation, this critique only makes sense in contrast to zoom teaching, as both Mursion simulation and real-life do not afford visual feedback of this kind. And it points to a larger set of competencies that PTs training during COVID-19 might struggle with, having spend most of their formative practice in an environment where facial regulation, body language, and even the ability to 'hide' by turning the camera off for a moment may have encouraged teaching practices that are unavailable in real life. Many PTs reported simulations as more relevant to PTs who will eventually teach online. However, Heather considers connecting through a screen to be a surface feature of the experience. PTs also reported that the simulation helped them facilitate either a small group or a whole-class discussion, depending on the design of the simulation.

Heather noticed that across the studies it was clear to her that intensity of use led to more PT learning. Heather conjectured that there may be a threshold of use to reach a minimum comfort level, and wondered how to normalize the experience for PTs so their first experience was not focused solely on getting acclimated to the environment. PTs knowledge of and skills with leading argumentative discussions improved when they had more simulation experiences. PTs with less experience reported a desire to be better prepared for the simulation, because they were unable to support unpredictable student responses. Heather observed that the amount of learning available also varied by context and degree of support available from the MTEs, with those who were able to provide more structure for preparation and reflection eliciting more learning. She was not surprised by this, as MTEs always mediate learning from experience. However, Heather pointed out that this finding hints at an affordance of simulation in allowing this direct mediation, which is not always available in traditional field work.

Lastly, Heather noticed that MTEs paid most attention to what they had available to pay attention to. When ETS staff scored the videos and provided written feedback and scores, few MTEs viewed their PTs' videos. When ETS staff only provided the videos, most MTEs watched the videos despite the time commitment required to do so. These MTEs uniformly reported that they needed to watch the videos to understand their PTs' strengths and weaknesses. The videos also provided the MTE insight about the degree to which their own instruction had been successful, both with respect to personally defined goals and overall. Finally, most of the MTEs owned what they saw, attributing their PTs' performance to their own actions as teachers, and using it to reflect on how to grow professionally.

Next, the working group participants were given the opportunity to share their experiences. The participants included: Kyle Turner, a graduate student from the University of Texas at Arlington, Bima Sapkota, a graduate student from Purdue University, Zeynep Arslan, a graduate student from Trabzon University, and Neet Priya Bajwa, from Illinois State University. Kyle, Bima, and Zeynep chose to share their experiences.

Before the pandemic Kyle participated in the Mathematics Association of America (MAA) META Math research project. The goals of the META Math project were to "increase faculty capacity to guide undergraduate pre-service teachers in making explicit connections between undergraduate mathematics and secondary school mathematics, and developing deep, sophisticated understanding of mathematics taught in grades 7-12" (META Math). The investigators and assistants developed lessons, a year-long faculty development program to train faculty on how to use of the lessons, researched PT learning that resulted from the use of these lessons, and evaluated the effectiveness of the faculty development program. The in person simulations in this project involved PTs analyzing hypothetical student work samples that contained common misconceptions. PTs were asked to identify and using guiding questions to help the hypothetical student better understand the material.

As an instructor of an elementary mathematics methods course, Bima used simulations during COVID as an alternative to field experiences. Using online conferencing, PTs rehearsed teaching elementary math lessons with their peers. PTs also video-recorded their peer-teaching sessions. Moreover, Bima is using the approximations of practice framework as her guiding framework for her ongoing dissertation study. In particular, Bima is investigating how teacher educators could assess secondary PTs' Mathematical Knowledge for Teaching when they engage in approximations of practice, including lesson plan, implementation, and reflections on teaching. The focus of Bima's dissertation research is on how to best support the PTs' discipline-specific practices, including posing purposeful questions and selecting mathematical tasks for students.

Zeynep used both in person and virtual simulation with PTs. She found the simulations beneficial to the PTs content knowledge (of solving equations) and instructional practices.

Looking to the Future

Finally, we discussed the components of simulation experiences that are worth preserving after the pandemic and our future collaborations. The participants agreed that although the research using simulations is growing, the results have been promising, and there are many unresolved questions. All participants plan to continue to use simulations and expressed interest in meeting again for a working group at PME-NA 2022. Liza expressed a desire to develop an agreed upon framework that captures the dimensions of simulations. She would like to solidify the design decisions that exist as well as best practices (when to do what, with whom, and why). Heather suggested continuing to use simulations as formative assessments (for PTs and MTEs) and as a 'boundary condition' to facilitate conversations. Heather expressed a need for a broader infrastructure, more vendors, and more equity of access. Yvonne expressed interest in working on a research commentary. Heather also suggested developing an NSF grant proposal to support a conference, noting that an affordance of the PME working group format is allowing a group of scholars to work toward such a goal over time and to allow graduate student members access to this type of collaborative experience. All attendees expressed interest in these future endeavors.

References

American Association of Colleges for Teacher Education. (2020). Teaching in the Time of Covid-19: State Recommendations for Educator Preparation Programs and New Teachers. AACTE Report Series. Washington D.C.

Bondurant, L. (2020). Reflections on Changes in Field-related Experiences during COVID-19. AMTE Connections, 30(2), 1-4.

Bondurant L., Amidon J. (2021) Virtual Field Experiences as an Opportunity to Develop Preservice Teachers' Efficacy and Equitable Teaching Practice. In: Hollebrands K., Anderson R., Oliver K. (eds) Online Learning in Mathematics Education. Research in Mathematics Education. Springer, Cham. https://doi.org/10.1007/978-3-030-80230-1_16

Ball, D. L., Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. Journal of Teacher Education, 60(5), 497-511.

Forzani, F. M. (2014). Understanding "core practices" and "practice-based" teacher education: Learning from the past. Journal of Teacher Education, 65(4), 357-368.

Girod, M., & Girod, G. R. (2008). Simulation and the need for practice in teacher preparation. Journal of Technology and Teacher Education, 16(3), 307-337.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching: Theory and Practice, 15(2), 273-289.

Howell, H. & Mikeska, J. N. (2020). Approximations of practice as a framework for understanding the role of authenticity in simulations of teaching. Journal of Research on Technology in Education.

Lee, C. W. & Freas, H. Utilizing Teaching Simulations for Small Group Mathematics Discussions in the Void of Field Placement Opportunities. In Ferdig, R.E., Baumgartner, E., Hartshorne, R., Kaplan-Rakowski, R. & Mouza, C. (2020). Teaching, technology, and teacher education during the COVID-19 pandemic: Stories from the field (pp. 335-341). Association for the Advancement of Computing in Education (AACE).

Lee, C., Lee, T., Dickerson, D., Castles, R., & Vos, P. (2021). Comparison of Peer-to-Peer and Virtual Simulation Rehearsals in Eliciting Student Thinking Through Number Talks. Contemporary Issues in Technology and Teacher Education, 21(2), 294-324.

META Math. Mathematical Association of America. (n.d.). Retrieved November 27, 2021, from https://www.maa.org/programs-and-communities/curriculum%20resources/meta-math

Mikeska, J.N., Webb, J., Bondurant, L., Kwon, M., Imasiku, L., Domjan, H., & Howell, H. (2021). Using and adapting simulated teaching experiences to support preservice teacher learning. In P.H. Bull and G.C. Patterson (Eds.), Redefining teacher education and teacher preparation programs in the post-COVID-19 era. IGI Global. DOI: 10.4018/978-1-7998-8298-5

Reinholz, D.L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. Journal for Research in Mathematics Education, 49(2), 140-177.

Rowland, T. (2013). The Knowledge Quartet: the genesis and application of a framework for analysing mathematics teaching and deepening teachers' mathematics knowledge. Sisyphus—Journal of Education, 1(3), 154-43.

Schwartz, C., Lee, C.W., & Gonzalez, M., & Belford, L. Using Virtual Simulations and Videoconferencing to Rehearse and Enact Number Talks in Online Settings. In Ferdig, R.E., Baumgartner, E., Hartshorne, R., Kaplan-Rakowski, R. & Mouza, C. (2020). Teaching, technology, and teacher education during the COVID-19 pandemic: Stories from the field (pp. 729-736). Association for the Advancement of Computing in Education (AACE).

Straub, C., Dieker, L., Hynes, M., & Hughes, C. (2014). Using virtual rehearsal in TLE TeachLivE™ mixed reality classroom simulator to determine the effects on the performance of mathematics teachers. 2014 TeachLivE™ National Research Project: Year 1 Findings.

CONCEPTUALIZING ETHICS, AUTHENTICITY, AND EFFICACY OF SIMULATIONS IN TEACHER EDUCATION

Carrie Lee Wilkerson East Carolina University leecarr16@ecu.edu	Liza Bondurant Delta State University lbondurant@deltastate.edu	Yvonne Lai University of Neb.-Lincoln yvonnexlai@unl.edu
Heather Howell ETS hhowell@ets.org	Bima Sapkota Purdue University bsapkota@purdue.edu	Minsung Kwon Cal. St. Univ, Northridge minsung.kwon@csun.edu

This working group explores the ethics of simulations of practice in teacher education. As the pedagogy of simulations of practice becomes increasingly popular in content and methods courses, we step back and ask: How are we equipping preservice teachers to engage with students and content in equitable ways? How are K-12 students portrayed and perceived in simulations? How may the need for efficiency run up against ethics? The working group expands on the work of a 2019 and 2021 PME-NA working group with a goal of disseminating conversations around the ethics of simulations of practice. We have reached out to different publishers with the goal of collaboratively editing a book. Working group participants will be invited to contribute book chapters. During our time together we will construct common themes with participants and develop an action plan for the completion of the book.

Keywords: Simulation of Practice, Preservice Teacher Education, Online and Distance Education, Technology, Classroom Discourse

Preservice teachers (PSTs) need ways to connect their content preparation to their future teaching practice. To do so, they need environments where they can experiment with teaching moves–including those that involve attending to power dynamics and handling student conceptions–prior to entering the field (Association of Mathematics Teacher Educators [AMTE], 2017; Ball & Forzani, 2009; McDonald, Kazemi, & Kavanagh, 2013). One common approach to providing PSTs with these environments are *simulations of practice;* opportunities for PSTs to engage in practices that are proximal to actual teaching (cf. Grossman, 2009). For instance, in a simulation of orchestrating a discussion, a PST may lead a discussion with students acted by avatars with particular characteristics (e.g., Bondurant & Amidon, 2021); with fellow PSTs acting as students; or PSTs may be provided student work ahead of time and asked to describe different potential strategies for connecting student contributions. Simulations provide a variety of opportunities for teachers to develop their teaching knowledge and practices (Girod & Girod, 2008).

During COVID-19, the use of digital simulations, which are particularly amenable to online and hybrid applications, came into the mainstream and received both positive and negative attention at a broader scale than ever before. The American Association of Colleges for Teacher Education ([AACTE], 2020) endorsed digital simulation and collaborated with a technology provider, Mursion™; the first time a professional organization had provided such a strong endorsement. At the same time, Mursion™ received negative press attention around the practice of having a single actor represent characters with a different race or ethnicity than their own (Baker-White, 2021).

In short, digital simulations seem to be entering a new phase in teacher education; one in which their potential is beginning to be realized at scale, supported by educational theory and research, but with much work remaining to be done to design simulations that are ethical, authentic, and useful. Furthermore, more attention is needed to the ways in which digital and non-digital approaches are complementary, overlapping, and mutually informative (Mikeska et. al 2021) and how attending simultaneously to the interactive and non-interactive working of teaching can help in the development of broader design principles and approaches, as we have argued for through this series of working groups.

More generally, especially in the current climate, where pandemic-related school closures may have impacted mathematics learning (Kuhfeld & Tarasawa, 2020), and school closures may exacerbate already widening achievement gaps (Dorn, Hancock, Sarakatsannis, & Viruleg, 2020), PSTs need experiences where they can attend to students and content in equitable ways. Teacher educators need ways to notice and provide feedback for such practices, as well as evaluate the explicit and implicit ethics informing the design of simulations, whether or not the simulations are digital. This working group seeks to expand this conversation in the field.

Focus of the Work

This working group seeks to explore the following questions: What common ethical concerns emerge from our work in the space of designing, implementing, and assessing authentic simulations in teacher development? What are design principles for digital and non-digital simulations that promote equitable teaching? How can we disseminate our collective work in the space of simulations in teacher development?

Organization and Plan for Active Engagement

This continuing group will expand the conversation around simulations by critically examining the ethics, authenticity, and efficacy of our collective work in simulations with the goal of dissemination. Prior to convening in Nashville, we will inquire into different dissemination opportunities and select the best avenue to present to our members (edited volume or special journal issue).

The working group will consist of three sessions during the conference followed by virtual meetings through the following year. Sessions will focus on creating a collective vision for the dissemination of work and formulating groups and a timeline.

Session 1. Collective Understanding: Conceptualization and Common Language

In this session, we will begin by briefly sharing the takeaways from our 2019 and 2021 PME-NA sessions with a focus on ethical concerns, authenticity, and efficacy of simulation work. We will also engage new participants by asking them to share their experiences. From this brief review we will break into groups to further conceptualize issues around ethics and design principles. Within group discussions, a common discourse will be negotiated as a group. This common discourse will ground our approach to dissemination of work.

Session 2. Identifying Key Topics for Dissemination

From conceptualization and common discourse, we will then converge on key topics for a special call or edited volume on simulations in teacher education. It is anticipated this work will involve negotiations and exploration of essential investigators in each of the spaces identified.

Session 3. Timeline

Finally, we aim to design a call for chapter proposals and construct a feasible timeline for the writing and editing process. The organizers will brainstorm outlets for sharing the call for chapters. Additionally, they will compile a list of roles and responsibilities and delegate them to individuals. They will also add meeting dates and zoom links to their calendars to ensure the plan is effectively executed.

References

American Association of Colleges for Teacher Education. (2020). Teaching in the Time of Covid-19: State Recommendations for Educator Preparation Programs and New Teachers. AACTE Report Series. Washington D.C.

Association of Mathematics Teacher Educators [AMTE]. (2017). Standards for preparing teachers of mathematics. Author. https://amte.net/standards

Baker-White, E. (2021, December 1). "This Is Blackface": White actors are playing Black characters in virtual reality diversity training. BuzzFeed News. https://www.buzzfeednews.com/article/emilybakerwhite/diversity-training-mursion-vr-white-actors

Ball, D. L., Forzani, F. M. (2009). The work of teaching and the challenge for teacher education. Journal of Teacher Education, 60(5), 497-511.

Bondurant, L. (2020). Reflections on Changes in Field-related Experiences during COVID-19. AMTE Connections, 30(2), 1-4.

Dorn, E., Hancock, B., Sarakatsannis, J., & Viruleg, E. (2020, June 1). COVID-19 and student learning in the United States: The hurt could last a lifetime. McKinsey & Company. Retrieved from https://www.mckinsey.com/industries/education/our-insights/covid-19-and-student-learning-in-the-united-states-the-hurt-could-last-a-lifetime

Forzani, F. M. (2014). Understanding "core practices" and "practice-based" teacher education: Learning from the past. Journal of Teacher Education, 65(4), 357-368.

Girod, M., & Girod, G. R. (2008). Simulation and the need for practice in teacher preparation. Journal of Technology and Teacher Education, 16(3), 307-337.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching: Theory and Practice, 15(2), 273-289.

Kuhfeld, M., & Tarasawa,B. (2020). The COVID-19 slide: What summer learning loss can tell us about the potential impact of school closures on student academic achievement. Portland, OR: Collaborative for Student Growth at NWEA. Retrieved from https://www.nwea.org/content/uploads/2020/05/Collaborative-Brief_Covid19-Slide-APR20.pdf

McDonald, M., Kazemi, E., & Kavanagh, S. S. (2013). Core practices and pedagogies of teacher education: A call for a common language and collective activity. Journal of Teacher Education, 64(5), 378-386.

Mikeska, J., Howell, H., Dieker, L. & Hynes, M. (2021). Understanding the Role of Simulations in K-12 Mathematics and Science Teacher Education: Outcomes From a Teacher Education Simulation Conference. Contemporary Issues in Technology and Teacher Education, 21(3) 781-812

Mikeska, J.N., Webb, J., Bondurant, L., Kwon, M., Imasiku, L., Domjan, H., & Howell, H. (2021). Using and adapting simulated teaching experiences to support preservice teacher learning. In P.H. Bull and G.C. Patterson (Eds.), Redefining teacher education and teacher preparation programs in the post-COVID-19 era. IGI Global. DOI: 10.4018/978-1-7998-8298-5erica.

COMPLEX CONNECTIONS WORKING GROUP: OUTCOMES AND PROGRESS

Beth L. MacDonald
Utah State University
beth.macdonald@usu.edu

Steven Boyce
Portland State University
sboyce@pdx.edu

Diana L. Moss
University of Nevada, Reno
dmoss@unr.edu

Claudia M. Bertolone-Smith
Chico State University
cmbertolone-smith@csuchico.edu

Cameron Byerley
University of Georgia
cameron.byerley@gmail.com

Karen Zwanch
Oklahoma State University
karen.zwanch@okstate.edu

Sarah Kerrigan
Virginia Tech
stk123@vt.edu

Christopher Roman
Portland State University
roman7@pdx.edu

Jeffery Grabhorn
Portland State University
jeff4@pdx.edu

The Complex Connections: Reimagining Units Construction and Coordination working group began at PMENA 2018, with the aim of facilitating collaboration amongst researchers and educators sharing Steffe's (2017) concerns about units construction and coordination for all learners. This working group made dramatic progress in their ability to conceptualize their work and impact the mathematics education field in the form of publications and task development.

Keywords: Learning Theory, Learning Trajectories and Progressions, Mathematical Knowledge for Teaching, Number Concepts and Operations

The Complex Connections: Reimagining Units Construction and Coordination working group began at PMENA 2018, with the aim of facilitating collaboration amongst researchers and educators sharing Steffe's (2017) concerns about units construction and coordination for all learners. The main goals of the working group were to extend research around units construction and units coordination to new grade levels and to facilitate collaborations with researchers with different backgrounds and experiences. In this report, we describe how the goals of the working group have evolved, including progress and products up through the working group meetings at PMENA 2021.

Past Two Years of Progress

Given that this Working Group has been formed over the past four years, we thought it might be helpful to provide context to the progress made in this working group thus far. When this working group was formed in 2018, we had five leaders and a skeletal webpage. Our goals were two-fold: (1) Extending Units Coordination, and (2) Widening Units Coordination. Essentially, this meant considering how units coordination could frame mathematics education research in grade levels not examined (i.e., prekindergarten and secondary education). By widening the frame used for Units Coordination, we envisioned carrying this frame into mathematics education intersections (i.e., special education and cognitive psychology) not yet explored in mathematics education research.

Since switching to a two-page proposal, our group has not provided context for progress made. So, we wanted to provide a brief discussion of the progress made in the past two years. By doing so, we hope to illustrate how many goals have been met and how our group continues to examine innovative approaches to examine nuances in individual's units coordination development, as well as scale-up this work to examine broad trends in individual's units coordination in a variety of contexts and for a variety of purposes.

Progress Made in 2019

In 2019, our leadership team changed to include one new leader and our attendance grew dramatically (see Figure 1). Given this growth, we began examining Units Coordination in more depth, allowing for tasks to be created intentionally for online course designs (used for both research and teaching), how covariational reasoning contexts explain students' reasoning through their units coordination, and details surrounding the intersection between mathematics education and special education. Our webpage (https://unitscoordination.wordpress.com/) began including a robust member list and tasks used in teaching experiments, interviews, and high education course designs.

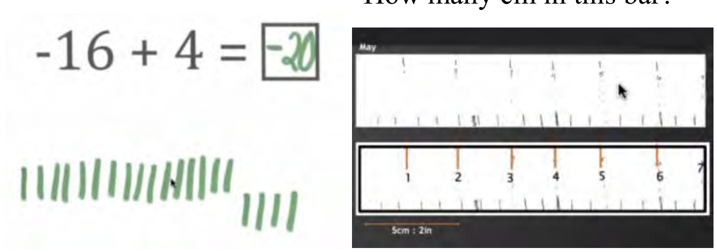
Figure 1: Attendees at the 2019 Complex Connections Working Group

This group began to create a connected group of scholars, students, and teachers. Intellectual outcomes included 10+ publications bridging from prekindergarten to Calculus students. Moreover, unique modalities and online formats for engaging pre-service teachers began to emerge in this group's work, setting the stage for important and innovative pedagogical tools for an unimagined world-wide pandemic. This group spent much of their time examining tasks that many were using in their research and teaching and often revisiting the question, "what constitutes as a unit?" and "what does it mean to coordinate units in these new contexts and with these new tools?". These reflective questions seemed quite broad but allowed the group to refine and adjust their language and task development while designing new courses and research studies. In short, in 2019 this working group was better positioned to extend and widen the Units Coordination frame to more areas and with unique contexts. Outcomes included publications, and intellectual cohesiveness in the mathematics education field.

Progress Made in 2020 (June 2021)

In June of 2021, our leadership lost one leader to the PME-NA Steering Committee and gained one new leader. Due to COVID19, our working group moved to a hybrid format wherein many participants were in an online synchronous format. Considering the reduced attendance and engagement during this conference, our working group continued to grow in members by making innovative accommodations for non-English speakers. In total, we had 17 participants with six of these serving also as group leaders. Outcomes from this working group included refined foci in how units and units coordination are used/defined in areas such as, integer reasoning, combinatorial reasoning, covariational reasoning, pre-service teachers' fraction reasoning, and algebraic reasoning. The term "unit" was now considered to encompass several types of units: (1) discrete units, (2) continuous units, (3) static units, and (4) dynamic units. This allowed the group to again reflect on, "what constitutes as a 'unit'?", and "how can these new units be coordinated in such novel contexts?". This group also examined novel contexts and tasks being developed for research and teaching purposes. Some of these tasks (see Figure 2) were not designed with units coordination frameworks, promoting the discussion around the question, "how might these tasks be helpful or limiting in their design and enactment?". By refining how foci and questions, our groups' intellectual growth continued to progress in a cohesive manner.

How many cm in this bar?



Consider the following statements.

A. I travel 12 meters in 5 seconds.
B. I travel at a constant rate of 12 meters per 5 seconds.
C. I travel at a constant rate of 12/5 meters per second.
D. It takes me 5/12 seconds to travel 1 meter.

Pick a pair of statements that **don't mean exactly the same thing**. Write a statement that would convince anyone else in the class that those two statements don't mean exactly the same thing.

Pick a pair of statements that **mean exactly the same thing**. Write a statement that would convince anyone else in the class that those two statements do mean exactly the same thing.

Figure 2: Examples of some tasks discussed that broached novel ways to frame Mathematics Education with Units Coordination Learning Theories.

In short, in June 2021 this working group was better positioned to extend and widen the Units Coordination frame to particular areas and with more refined language and tasks. Outcomes included five new publications, two new software apps designed to measure individuals' units coordination, and a more comprehensive set of language around what constitutes as a unit and how units can be coordinated in several novel contexts.

Progress Made Throughout the October 2021 Working Group

In the fall of 2021, the PME-NA conference held a second hybrid conference, allowing attendees to participate via online synchronously and/or in-person. The difference in this conference is that many more participants attended in-person. This only increased our participation from June of 2021. In particular, we had a total of 19 attendees and seven of these served as leaders. Many of these attendees were ones who participated in the Complex Connections Working Group each year they attended PME-NA. However, three of these 19 participants were new participants and joined our group to both learn and contribute in unique ways. In the subsequent sections, we will elaborate on progress made during each day of the working group, explaining details of this group's intellectual growth and associated outcomes.

Day One Progress and Outcomes

The first day of the working group began by using language from past working groups to define and characterize a unit, units coordination, and contexts participants used or envisioned using units coordination in their research and teaching. We divided into two in-person groups and one online group, capturing ideas in three separate Google documents. Participants also recorded several questions they still had in their discussion. By the end of this day's working group, we were positioned to examine tasks from one research group and make note of tasks and apps we could examine in subsequent working groups.

The group had a wide variety of ways to characterize units (see Figure 3). For example, one participant wrote that a unit is defined differently dependent upon the type of unit one is constructing (i.e., perceptual, figurative, abstract) whereas another participant implicitly suggests an individual's formation of collections, suggesting discrete, static units. These variations suggested how often researchers and teachers use this term differently and why the progress this group has made in reflecting on these variations is important.

- "things" like angle degree measure
- a "thing" that can be iterated
- what is a unit depends...is distinguished between perceptual / figurative units. Forming collections versus not. Abstract unit.
- unit for me: constructed via the unitizing operation
  discrete 1s and measurement units

Figure 3: Participant notes characterizing what constitutes as a unit

The group described units coordination in one broad manner, providing a more coherent characterization of this frame. For instance, one participant connected the units variation discussion to units coordination by stating that units coordination refers to, "how units change over time and how the relations and actions with units change in different situations." Moreover, this group added that, "You have to at least distribute the elements of one composite unit across another composite unit." This statement was echoed in each small group, providing a coherent frame of reference for the remainder of the time in the working group.

The group members also described the contexts they used units coordination. This variation included variety in age of person (prekindergarten to college level) and variety of units in varying contexts (e.g., fractional units, whole number units, algebraic units, measurement units, covariational units). This wide variety explained the variation in how units were characterized but also provided means to coherently synthesize units coordination in a more in-depth and comprehensive manner.

Finally, the groups explicated many questions they still had regarding their own work and this framework (see Figure 4). These questions served as the basis for the remainder of day one's work and the basis for progress we would make in day two, day three, and beyond.

- what do we each mean by "quantity" in relation to "units?" - Drew from Thompson's explanation of how "units" are specific types of quantities.
- can a unit be within a measurement model?
- what does a situation mean? Is this a recognition template?
- what about division of fractions? How would different color help with recognizing the units? Jerry says "use word "disembedding" quite a bit. Perhaps color theory would be helpful?
- do you see overlap with Units Coordination and proportional reasoning?
- does units coordination have layers or levels?

Figure 4: Examples of questions participants had in small group discussions

Many of these questions pressed participants to reflect on their language and the underlying scheme theory used to explain how units coordination develops and changes over time.

Moreover, some participants were examining characteristics of tasks that they found leveraged students' mathematical development but had been developed for only practical purposes.
To examine units, units coordination, and to help participants answer some of these questions, we explored fraction tasks (designed for pre-service teachers - PST) and their responses. We discussed the limitations many of these tasks inherently had in their design and enactment.
In short, progress from day one, allowed the working group to further refine their definitions and characterizations for units and units coordination. Novel contexts and questions surrounding tasks fundamentally provided more nuanced progress on task development, task enactment, participants' responses, and analyses of these responses.

Day Two Progress and Outcomes

The second day of the working group continued the examination and discussion of written assessment items. Both the in-person and remote participants analyzed the work of pre-service teachers (PSTs) on static fractions tasks, discussing the following prompts in small groups:

- Is there indication that the PST coordinated units?
- Is there counter-indication that the PST coordinated units?
- What are the obstacles to overcome in assessing units coordination with this item?
- What are the ways that you have assessed units coordination?

One of the takeaways from the following discussion was the difficulty with making an assessment of an adult's units coordination with fractions solely from their responses to written items. Participants discussed myriad ways that a student might make marks and representations on the paper to determine a normative size or number without coordinating fractional units. We discussed how such assessments, though useful for making inferences of children's reasoning, may not be sensitive enough to assess adults' fraction schemes. Yet, they may be useful in other ways. For instance, we shared a PST's response to a task that was intended to evoke their use of l (see Figure 5). After project leaders shared this response, the members discussed how the PST's converting the fraction 9/6 to a mixed number was not necessarily a counter-indication of their construction of an iterative fraction scheme. In fact, the members discussed that it might indicate a PST's tendency to focus on part-whole relationships when developing appropriate pedagogical content knowledge for elementary school instruction.

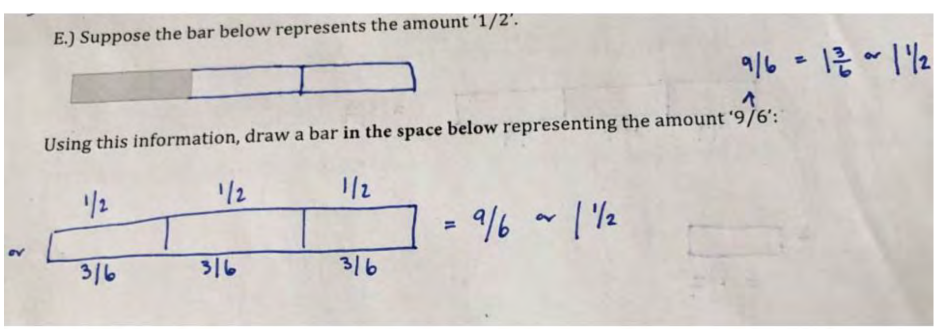


Figure 5: Screenshot of PST fraction task response shared by Working Group leaders

The second half of the Working Group was led by Dr. Teruni Lamberg, who participated in-person. Dr. Lamberg shared a sequence of fractions tasks she had used with in-service elementary and middle grades teachers' professional development. We continued Day 1 discussions regarding relationships between fractions understandings, such as part-whole schemes versus iterative or partitive schemes, and their constitutive operations. Dr. Lamberg's tasks were not designed to assess units coordination, rather, they were designed to assess fractions understandings as described in the Common Core State Standards. Thus, consideration of these tasks provided opportunities to make connections between units coordination and fractions curricula. Of particular import was the discussion of the disembedding operation. Disembedding is a hallmark of coordination of two levels of units. It involves mentally "removing" an item or amount from a whole extent or collection, without modifying the size or numeracy of the extent or collection. We discussed Dr. Lamberg's idea that intentionally varying color may be helpful in fostering students' disembedding composite units, particularly in the context of fraction division.

Day Three Progress and Outcomes

The third day of the working group focused on how units coordination is assessed. The in-person attendance, of six, was lower than the previous days, which is typical for the last working group day due to early flights. There were eight individuals who participated remotely. The first half of the session the discussion was led by Dr. Amy Hackenberg, who streamed a video of a teaching experiment session with a sixth-grade student as well as a written task (and rubric) that she had used in a clinical interview setting. Dr. Hackenberg explained how the contrasts in the student's responses to the tasks in the video were useful for understanding the distinction between a student not coordinating units and a student coordinating units "in activity". We discussed the details of the rubric for the interview task she shared, and several participants noted that this level of detail in the types of expected responses for students at different stages of units coordination could be an area of improvement in assessments with PSTs.

One important benefit of this Working Group is to provide a space for discussion about units coordination research presented elsewhere in the program. In addition to Dr. Hackenberg's discussion of her research, two participants in the working group from James Madison University elaborated on their research report investigating the validity of written instruments for assessing PSTs' units coordination. Their research team presented their report at a session earlier that morning, and they attended the Working Group meeting to elaborate on their findings and to discuss potential collaborations. Several of the members of the Working Group who have been focused on PSTs' fractions understandings expressed interest in collaborating with the design of methods to validate and improve upon written assessments.

Two additional Working Group members shared applets that they have designed. As shown in Figure 6, Sarah Kerrigan created a Geogebra version of the Bars Tasks (https://www.geogebra.org/m/zhgajwzp).

Figure 6: Screenshot of applet shared by Sarah Kerrigan

Although this applet was created out of necessity, due to the need to assess students' reasoning remotely because of the COVID19 pandemic, several participants noted its potential benefits over traditional paper and pencil assessments. We discussed the potential for students to complete the assessment via Geogebra and screen-record their activities, improving inferences of their process for solving the tasks. The Bars Tasks were originally validated with sixth-grade students by comparing researchers' inferences based solely on the paper with other researchers' inferences based on watching video-recordings of the students' solving the task (Norton et al., 2015). Kerrigan's applet could potentially be used to either validate the written assessment with older students, such as with PSTs, or to replace the written assessment for use with other groups of students.

Unlike Sarah Kerrigan's applet, the applet that Dr. Cameron Byerley shared was not designed specifically to assess units coordination. She discussed an applet (see Figure 7) that was designed to help individuals construct proportional reasoning related to the risk of contracting, being hospitalized, or dying from COVID19. Units coordination is involved throughout students' mathematical development, and one of the goals of the Working Group is to extend our understanding of its roles and its relationships with quantitative, covariational, and probabilistic reasoning. Dr. Byerley discussed how her knowledge of units coordination informed and influenced applet design decisions, such as the use of horizontal, left-aligned rectangular bars to represent and compare the size of COVID risks with everyday probabilities such as getting in a car accident or winning the lottery. Her discussion of interviews with citizens on their use of the applet was heartening, in that some of the participants expressed a deepened understanding of the risks they faced with not getting vaccinated, but disappointing in that others expressed either skepticism of the statistics or a disconnect between the sizes of the bars and the relative sizes of the risks.

Figure 7: Screenshot from applet shared by Working Group Leader Cameron Byerley

Progress Made Following the October 2021 Working Group

Since the members of the working group met, work has been done to develop a Special Issue, reflection of tasks relative to the granular size of studies and modalities, and updates of the working group website. The website (https://unitscoordination.wordpress.com/) now features new members, tasks, rubrics and publications. Thus, the working group has connected new resources, for different purposes, and connected scholars, so as to form new research questions and research foci. It is our hope to leverage the energy from these connections for more actionable work on a Special Issue in the fall of 2022.

Conclusion

Throughout the past four years, this working group has refined and housed publications, tasks, discussions, and virtual tools to examine units construction and coordination. In the past year, this work has become more refined and intentional, prompting members to conceptualize their work with this frame in a more precise manner. Moreover, the working group website (now the second search item in a Google search for "units coordination") has developed dramatically, bridging scholarship across grade levels, within the intersections of special education, mathematics education, early childhood education, and research in undergraduate mathematics education. Evidence of engagement with these tools is allowing scholars and teachers to develop rubrics, allowing more coherent foci with this frame across the nation. Thus, we measure progress through the conceptual development of the working group and aim to evidence this with a special issue in the next two years.

References

Norton, A., Boyce, S., Phillips, N., Anwyll, T., Ulrich, C., & Wilkins, J. L. M. (2015). A written instrument for assessing students' units coordination structures. *IEJME-Mathematics Education, 10*(2), 111-136.

Steffe, L. P. (2017). Psychology in Mathematics Education: Past, Present, and Future. (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 27-56). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

COMPLEX CONNECTIONS: REIMAGINING UNITS CONSTRUCTION AND COORDINATION WITH ALGEBRAIC AND COVARIATIONAL REASONING

Beth L. MacDonald
Illinois State University
blmacd1@ilstu.edu

Karen Zwanch
Oklahoma State University
karen.zwanch@okstate.edu

Sarah Kerrigan
Virginia Tech
stk123@vt.edu

Steven Boyce
Portland State University
sboyce@pdx.edu

Cameron Byerley
University of Georgia
cameron.byerley@gmail.com

Diana L. Moss
University of Nevada, Reno
dmoss@unr.edu

Claudia Marie Bertolone-Smith
California State University, Chico
cmbertolone-smith@csuchico.edu

Christopher Orlando Roman
Portland State University
roman7@pdx.edu

Jeffery Grabhorn
Portland State University
jeff4@pdx.edu

A powerful aspect of using neo-Piagetian frames to examine students' construction and coordination of units is their potential to apply across mathematical domains. This working group aims to facilitate collaborations between scholars of algebraic and covariational reasoning, with the particular aim of extending research on units coordination and construction across these contexts. An ultimate goal of this working group will be to develop a Special Issue.

Keywords: Algebra and Algebraic Thinking, Learning Theory, Number Concepts and Operations, Learning Trajectories and Progressions

Background

This Complex Connections working group, focusing on units construction and coordination learning theories, will provide collaborative discussions for participants examining, assessing, and teaching students' algebraic and covariational reasoning. Thus, this paper delineates findings from these two constructs, set in units construction and coordination learning theories.

A connection between middle-grades students' units construction and coordination and their algebraic reasoning is emerging in the research. Notably, Hackenberg and her colleagues (Hackenberg et al., 2017) have determined that to meaningfully reason about unknowns relies cognitively on students' operations on composite units. This suggests that construction of assimilatory composite units is a consequential prerequisite to much of the algebraic reasoning that is typical in middle-grades curricula. Additionally, Olive and Çaglayan (2008) demonstrated ways that assimilating with three levels of units supported middle-grades students' solutions to a word problem that could be modeled algebraically. In their investigation, assimilating with three levels of units supported students to utilize a substitution method for solving the system of equations because they could anticipate an equation as a three-level unit structure and operate on that structure in mental activity. Additional connections have been drawn between students' units construction and coordination and their algebraic reasoning as it relates to generalizing figural linear patterns (Hackenberg, 2013; Zwanch, 2022b), their equation writing (Hackenberg, 2013; Hackenberg & Lee, 2015; Hackenberg et al., 2021; Zwanch, 2019), and their guess and check strategies to solving algebraic equations (Zwanch, 2022a).

Similar to connections being drawn between units construction and coordination with algebraic reasoning, researchers have started to investigate connections between units coordination and covariational reasoning. Carlson and colleagues (Carlson et al., 2002) built a framework for capturing covariational reasoning and defined it as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Johnson (2015), directly built on Carlson and colleagues' framework in her investigations of secondary students' quantification of rate and ratio. Although these frameworks provide a way to categorize students' covariational reasoning, there has been little work investigating connections between construction and coordination of units and levels of covariational reasoning in these frameworks. Given these advances in this field, members of the working group will discuss points of collaboration.

Session 1: Extending to Algebraic Reasoning

<u>Goal: Discuss connections between algebraic reasoning and units construction/coordination</u>

Activities: In a large group, participants will solve algebraic reasoning tasks and analyze their solutions, focusing on the potential solution strategies of students. Participants will form small groups based on their own interests in (1) early and middle-grades algebra, (2) elementary algebra reasoning, and (3) secondary grades of algebraic reasoning. Within these small groups, participants will view and analyze videos, transcript excerpts, and written work taken from empirical studies. Participants will discuss theoretical and pragmatic connections between the students' units construction and coordination and aspects of their solution strategies.

Session 2: Extending to Covariational Reasoning

<u>Goal: Discuss connections between covariational reasoning and units construction/coordination</u>

Activities: In a large group discuss the theoretical connections between the two bodies of research before engaging in some covariational tasks. In small groups, participants will solve covariational tasks and analyze their solutions to identify what units constructions and coordination emerged in their work. After coming back together to share our ideas, we will break into groups again based on own interests (1) middle-grades covariational reasoning, (2) secondary-grades covariational reasoning, and (3) undergraduate covariational reasoning. Participants will have the opportunity to analyze student data from clinical interviews, discuss task design and assessment, and theoretical and teaching applications.

Session 3: Extending to Algebraic and Covariational Reasoning

<u>Goal: Discuss connections between algebraic and covariational reasoning, and units construction/coordination</u>

Activities: In a large group, participants will solve algebraic reasoning and covariational reasoning tasks and analyze their solutions, utilizing units construction and coordination to orient to the potential students' strategies. Participants will form small groups based on their own interests in algebra and covariational reasoning (1) early and middle-grades, (2) elementary grades (3) secondary grades. Within these small groups, participants will view and analyze videos, transcript excerpts, and written work taken from clinical interviews and teaching experiments. Participants will discuss theoretical and pragmatic connections between the students' units construction and coordination, and aspects of their solution strategies. We will wrap this session up with a review of our drafted call for a Special Issue proposal. We plan to explore journal possibilities and language in the call. The details for this Special Issue will be advertised widely with our listserv and on our units coordination webpage (https://unitscoordination.wordpress.com/). Future work will be discussed to allow for opportunities for future collaboration. Additional information will be added to the group's webpage, allowing for current research and resources to be made available.

References

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for research in mathematics education*, *33*(5), 352-378.

Hackenberg, A. J. (2013). The fractional knowledge and algebraic reasoning of students with the first multiplicative concept. *The Journal of Mathematical Behavior*, *32*(3), 538-563.

Hackenberg, A. J., Aydeniz, F., & Jones, R. (2021). Middle school students' construction of quantitative unknowns. *The Journal of Mathematical Behavior*, *61*, 100832.

Hackenberg, A. J., Jones, R., Eker, A., & Creager, M. (2017). "Approximate" multiplicative relationships between quantitative unknowns. *The Journal of Mathematical Behavior*, *48*, 38-61.

Hackenberg, A. J., & Lee, M. Y. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, *46*(2), 196-243.

Johnson, H. L. (2015). Secondary students' quantification of ratio and rate: A framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*, *17*(1), 64-90.

Olive, J., & Çağlayan, G. (2008). Learners' difficulties with quantitative units in algebraic word problems and the teacher's interpretation of those difficulties. *International Journal of Science and Mathematics Education*, *6*(2), 269-292.

Zwanch, K. (2022a). Examining middle grades students' solutions to word problems that can be modeled by systems of equations using the number sequences lens. *Journal of Mathematical Behavior, 66*, 1-16. https://doi.org/10.1016/j.jmathb.2022.100960

Zwanch, K. (2022b). Using number sequences to account for differences in students' generalizations. *School Science and Mathematics, 122*, 86-99. DOI: 10.1111/SSM.12516

Zwanch, K. (2019). Using number sequences to model middle-grades students' algebraic representations of multiplicative relationships. In Otten, S., Candela, A. G., de Araujo, Z., Haines, C., & Munter, C. (Eds.) *Proceedings of the 41st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 177–185). St. Louis, MO.

PROGRESS REPORT: THE EMBODIED MATHEMATICAL IMAGINATION AND COGNITION WORKING GROUP

INFORME DE PROGRESO: EL GRUPO DE TRABAJO DE COGNICIÓN E IMAGINACIÓN MATEMATICA INCORPORADA

Mitchell J. Nathan	Hortensia Soto	Avery Harrison Closser
U. of Wisconsin - Madison	Colorado State Univ.	Worcester Polytechnic Inst.
mnathan@wisc.edu	hortensia.soto@colostate.edu	ahclosser@wpi.edu

Erin Ottmar	Dor Abrahamson	Janet Walkoe
Worcester Polytechnic Inst.	Univ. of California, Berkeley	University of Maryland
erottmar@wpi.edu	dor@berkeley.edu	jwalkoe@umd.edu

The Embodied Mathematical Imagination and Cognition (EMIC) Working Group was established in 2015 at the annual PME-NA conference. As an interdisciplinary group of researchers, teachers, students, and practitioners, EMIC has strived to advance our understanding of how our physical experiences and body-based resources impact the ways we learn mathematics. Over the past several years, EMIC's founding members have collaborated on initiatives to a) expand this area of research through grants, workshops, and publications, b) share these findings with teachers and practitioners, and c) grow the community through new members and opportunities for participation. This progress report highlights the progress and accomplishments of EMIC and its members.

Keywords: Technology; Cognition; Informal Education; Learning Theory.

A Brief History and Motivation of the EMIC Working Group

The first meeting of the EMIC working group took place in East Lansing, MI during PME-NA 2015. It has a somewhat longer origin, however, growing out of several earlier collaborative efforts to review the existing literature, document embodied behaviors, and design theoretically motivated interventions. One early event was the organization of the 2007 AERA symposium, "Mathematics Learning and Embodied Cognition." This and other gatherings led to a funded NSF "catalyst" grant to explore a Science of Learning Center, which was to involve scholars from multiple institutions and countries. Though unfunded, those SLC efforts shaped a subsequent 6-year NSF-REESE grant, "Tangibility for the Teaching, Learning, and Communicating of Mathematics," starting in 2008. Interest from the International PME community in this topic grew and led to special issues of *Educational Studies in Mathematics* (2009), *The Journal of the Learning Sciences* (2012), and an NCTM 2013 research pre-session keynote panel, "Embodied cognition: What it means to know and do mathematics," along with a series of academic presentations, book chapters, and journal articles, as well as several masters' theses and doctoral dissertations. By now, several research programs have formed to investigate the embodied nature of mathematics (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards, Ferrara, & Moore-Russo, 2014; Lakoff & Núñez, 2000; Radford 2009), demonstrating a "critical mass" of projects, findings, senior and junior investigators, and conceptual frameworks to support an on-going community of like-minded scholars within the mathematics education research community.

It was within this historical context that approximately 22 members of PME-NA 2015 came together for three 90-min sessions of semi-structured activities. On **Day 1,** the organizers engaged attendees in some of the body-based math activities used in their research on proportional reasoning and geometry. One activity (Figure 1b) engaged participants in a projection activity using Alberti's Window, which helped experience the role of body perspective on form and representation. We discussed how embodied theories are advancing our understanding of mathematical thinking, and how these ideas are shaping a new class of educational interventions. During **Day 2**, we used hands-on activities to expand our own understanding of topology. We then built on the emerging rapport among the group to hold a facilitated discussion of the potential intellectual benefits of forming a self-sustaining Working Group on embodied cognition, along with the necessary infrastructure it would need to maintain. Several concrete proposals led to the list of Future Steps on **Day 3**. However, before we tackled those matters, participants began the session doing math games and activities in small groups, including Spirograph, Set, Rush Hour, Tangrams, and Mastermind. We reflected on how some games and activities draw people into rich mathematical thinking and actions, and how we naturally engage in math through these activities. Day 3 culminated in an organized list of Future Steps, with some working group members assigned to specific tasks. We have convened annually since.

Figure 1: A small selection of embodied activities created by EMIC organizers and experienced by EMIC participants. *Clockwise from top left*: experiencing geometric transformations, acting out geometry conjectures, constructing icosahedra first as small, then at human scale, and enacting topological relations.

Founding Members

The EMIC community has welcomed several new members since its inception in 2015 and the leaders of our working group at PME-NA have shifted each year to reflect the focus of each annual meeting. Below are the founding members of EMIC who have led the charge for many of this group's collaborations within and between PME-NA convenings.

- Candace Walkington, Southern Methodist University
- Caro Williams-Pierce, University at Albany, SUNY / University of Maryland
- David Landy, Indiana University / Netflix
- Dor Abrahamson, University of California, Berkeley
- Erin Ottmar, Worcester Polytechnic Institute
- Martha W. Alibali, University of Wisconsin-Madison
- Mitchell J. Nathan, University of Wisconsin-Madison
- Ricardo Nemirovsky, San Diego State University
- Rogers Hall, Vanderbilt University

Collective Efforts and Accomplishments

Beyond PME-NA, EMIC members have actively collaborated over the past several years to continue advancing this field of research, bridging the gap between research and classroom practice, and expanding the EMIC community. Below, we provide overviews of the progress and accomplishments shared by members of our group (see Figures 1 and 2).

Table 1: Events Organized by the EMIC Community Since PME-NA 2015

2015	EMIC Working Group at **PME-NA 37**. Theme: Embodied Mathematical Imagination and Cognition. The launch of the working group and community at PME-NA.
2016	EMIC Working Group at **PME-NA 38**. Theme: Finding Common Interests for Creating an Online Community of Scholars.
2017	EMIC Working Group at **PME-NA 39**. Theme: Embodiment at the Crossroads of Mathematical Collaboration. Topics included the power of collaborative gestures.
2018	EMIC Working Group at **PME-NA 40**. Theme: Extending Theoretical Frameworks of Embodied Cognition to Assessing and Assisting Mathematics Learning.
2019	EMIC Working Group at **PME-NA 41**. Theme: Co-design of Novel Embodied Instructional Activities for Mathematics Education. Topics included the influences of eliciting movement in physical and video game-based learning activities.
2019	(May) NSF-funded EMIC I Workshop.
2020	(May). EMIC Virtual Panel #1. Hosted by embodiedmathematics.com.
2020	(October). EMIC Virtual Panel #2. Hosted by embodiedmathematics.com.
2021	(June; postponed from June 2020). EMIC Virtual Working Group at **PME-NA 42**. Theme: How Embodied Mathematics Can Bridge Cultural Divides. Topics included varieties of cultural and linguistic ways of expressing mathematical ideas.
2021	(October; postponed from June 2020). NSF-funded EMIC II Workshop.
2021	(October). EMIC Research Colloquium at **PME-NA 43**. Theme: Promoting Inclusive Mathematics Education Research and Practices. Topics included exploring varieties of physical inclusivity in different virtual stations and immersive technology.

Highlights: Funded Grant Proposals and Projects

1. Creation and maintenance of the website https://www.embodiedmathematics.com/. This is a portal for resources, events (including workshops and webinars), and announcements.

2. NSF EMIC I. A successful proposal to host a three-day NSF Synthesis and Design workshop, May 20-22, 2019, hosted by University of Wisconsin-Madison. The theme was "The Future of Embodied Design for Mathematical Imagination and Cognition" with 8 members serving as organizers. There were 50 attendees, comprised of international researchers, educational practitioners, and graduate students. All attendees presented. Keynotes from Dr. Brian Bottge and Dr. Maxine McKinney de Royston. This resulted in a white paper (Nathan et al., 2021) and an open access journal article (Abrahamson et al., 2020).
    a. Follow-up collaboration between a WG faculty member (Abrahamson) and workshop participant (Danny Luecke), a Secondary Math Education Developer/Instructor at Turtle Mountain Community College, North Dakota, to create pre-/in- service teacher courses, Ojibwe Math I, II, and III, as well as a course titled Math In Context.
3. NSF EMIC II. A successful proposal to host a second NSF three-day Workshop in 2020 (delayed to Sept. 25-26, 2021), hosted by Colorado State University. The theme was: "EMIC: Professional Development for Undergraduate Mathematics Instructors" with 7 member organizers. Keynotes from Dr. Nathalie Sinclair, Dr. Brooke Ernest, Dr. Francisco Ortega, & Dr. Molly Kelton.
4. Several grants awarded by IES CASL program to study embodied cognition, including: the role of action in pre-college proof performance in geometry (Funded 2016-2020 for Nathan & Walkington); the use of perceptual learning technology to study algebra learning (Ottmar & Landy, 2018), and use of augmented and virtual reality (AR/VR) to foster collaborative embodiment in immersive learning environments (Walkington & Nathan, 2020).

Products and Dissemination

1. Invitation by Springer to produce an edited book on our collective work on Embodied Mathematical Imagination and Cognition for the *Research in Mathematics Education* Series (Jinfa Cai, series editor).
2. Exhibit presented by Abrahamson and Rosenbaum at an NSF-funded WG meeting (EMIC I, Madison, WI) has led to a research project (Palatnik, WG faculty), a journal paper (Benally, Kimiko, Palatnik, Abrahamson, in press), and a Spencer proposal (item).
3. A joint symposium on Embodied Cognition with 6 members at the 2018 APA Technology, Mind & Society Conference (Harrison et al., 2018)
4. Publication of Nathan (2021), a textbook on embodied cognition for education.

Community Engagement/Broadening Participation

1. Expanding a group website using the Google Sites platform to connect scholars, support ongoing interactions throughout the year, and regularly adding additional resources/activities https://sites.google.com/site/emicpmena/home
2. Several members creating and teaching Embodied Cognition and Gesture seminars at their institutions.
3. Extending the embodied-design agenda into special education in dialogue with Universal Design for Learning (Abrahamson, Flood, Miele, & Siu, in press)
4. Some senior members joining junior members' grant proposals as Co-PIs and advisors.
5. Interest from Head of Education R&D (Dr. Ilona łowiecka-Tańska) of the largest STEM museum in Europe (the Copernicus Hall of Science in Warsaw, Poland) to incorporate

exhibits that had been developed for/at an NSF-funded WG meeting (EMIC I, Madison, WI) by Abrahamson and Rosenbaum.

6. EMIC Virtual Workshop #1, May 19, 2020. Theme: “Instructional Gestures for Classrooms and On-Line Mathematics Learning.” Designed for an audience of K-16 teachers, parents, and mathematics education researchers. Panelists: Dr. Mitchell J. Nathan, Dr. Martha Alibali, & Dr. Rebecca Boncoddo. Discussant: Dr. Hortensia Soto. Moderator: Dr. Caro Williams-Pierce. Hosted by embodiedmathematics.com.
7. EMIC Virtual Workshop #1, October 12, 2020. Theme: “Reading Group: The Future of Embodied Design for Mathematics Teaching and Learning.” Convener: Dr. Dor Abrahamson, with a panel of authors from the (Abrahamson et al., 2020) paper.

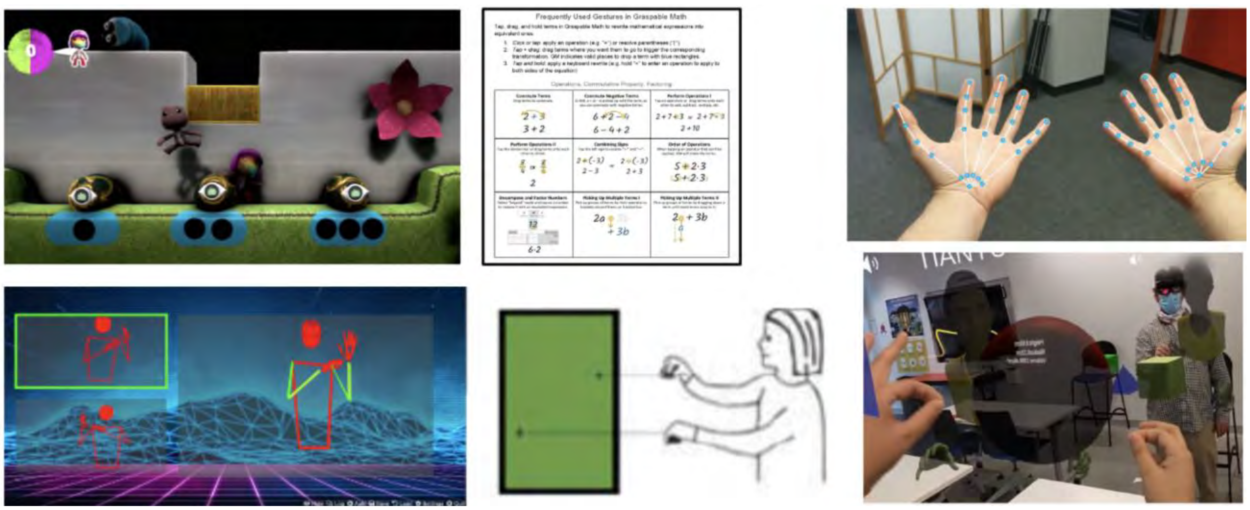


Figure 2: Embodied technologies designed by EMIC members that engage body based processes in support of mathematical reasoning. Clockwise from top left: The collaborative video game, Rolly’s Adventure; Graspable Math; hand joints and holographic math objects in the augmented reality/virtual reality (AR/VR) shAR Geometry environment; Mathematical Imagery Trainer for Proportion (MIT-P); The Hidden Village-Online.

Conclusions and Looking Ahead

Embodied accounts of reasoning, learning, and instruction are proving to be highly generative frameworks for understanding and cultivating mathematics learning and teaching, and for informing a new generation of designs for interactive technologies, curricula, and assessments. The true potential for these ideas and practices to lead to transformational mathematics education depends on a thriving, informed, interactive community of scholars and practitioners who engage with these approaches, evaluate their impacts, and improve theory and practice. Thus, we see predictably recurrent events such as the PME-NA working groups and research colloquia as a cornerstone for maintaining a highly educated and engaged core community. We will continue to offer outreach events and pursue external funding for specialized workshops when the resources are available. Future research colloquium topics that offer particular promise to the mathematics education community are: Broadening participation in mathematics and STEM through embodied design for learning, teaching, and assessment; the growing access to augmented and immersive technology, such as AR and VR, as a means to transform learners’ collaborative interactions with mathematical objects and phenomena

(Walkington et al., 2021); and the potential for multimodal learning analytics that use machine learning methods to integrate complex data streams that include channels of speech, gesture, eye gaze, facial expressions, mouse clicks, and so on, to support more sophisticated descriptions of learners' behaviors that can lead to real-time formative assessment and predictive models of educationally important student outcomes (Abrahamson et al., 2021; Closser, 2021).

Acknowledgments

We wish to acknowledge the scholars and educators who have been the backbone of these interactive working groups, workshops, and panels by providing the stimulating intellectual environment for these ideas to take root and blossom. We also wish to acknowledge the generous funding from NSF-DCL Grant #1824662 and NSF-DUE Grant #1835409.

References

Abrahamson, D. (2014). The monster in the machine, or why educational technology needs embodied design. In V. Lee (Ed.), *Learning technologies and the body: Integration and implementation* (pp. 21-38). New York: Routledge (Taylor & Francis Group).

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning [Original Research]. *Frontiers in Education, 5*(147). https://doi.org/10.3389/feduc.2020.00147

Abrahamson, D., Worsley, M., Pardos, Z. A., & Ou, L. (2021). Learning analytics of embodied design: Enhancing synergy. *International Journal of Child-Computer Interaction*, 100409.

Alibali, M. W. & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*. (Special Issue on Embodiment in Mathematics.), *21*(2), 247-286. DOI: 10.1080/10508406.2011.611446.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Benally, J., Palatnik, A., Ryokai, K., & Abrahamson, D. (in press). Learning through negotiating conceptually generative perspectival complementarities: The case of geometry. *For the Learning of Mathematics*.

Closser, A. H., Erickson, J. A., Smith, H., Varatharaj, A., & Botelho, A. F. (2021). Blending learning analytics and embodied design to model students' comprehension of measurement using their actions, speech, and gestures. *International Journal of Child-Computer Interaction*, 100391.

De Freitas, E., & Sinclair, N. (2014). Mathematics and the body: Material entanglements in the classroom. Cambridge University Press.

Edwards, L., Ferrara, F., & Moore-Russo, D. (Eds). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. Charlotte, N.C.: Information Age.

Harrison A., Ottmar, E., Arroyo, I., Rosenbaum, L., Bakker, A., Abrahamson, D., Hulse, T., Manzo, D., & Landy, D. (2018, April). Embodiment and action in mathematics games. Symposium conducted at the *American Psychological Association's conference on Technology, Mind & Society*, April 5-7, 2018, Washington, DC.

Lakoff, G., & Núñez, R. E. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic books.

Nathan, M. J. (2021). *Foundations of embodied learning: A paradigm for education*. Routledge.

Nathan, M.J., Williams-Pierce, C., Walkington, C., Abrahamson, D., Ottmar, E., Soto, H., & Alibali, M.W. (2021). *Embodied Design for Mathematical Imagination and Cognition*. In ISLS Rapid Community Report Series. https://repository.isls.org//handle/1/6852.

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111-126.

Walkington, C., Gravell, J., Velazquez, J. A., He, T., Hickey, G., Nathan, M. J., & Cuevas, A. (2021). Collaborative Virtual Learning in the shAR Geometry Simulation Environment. *Grantee Submission*.

Walkington, C., Gravell, J., Velazque, J., He, T., Hickey, G., Nathan, M. J., & Cuevas, A. (2021, June). Collaborative virtual learning in the shAR geometry simulation environment. In A. Wichmann, H. U. Hoppe, & N. Rummel (Eds.), *General Proceedings of the 1st Annual Meeting of the International Society of the Learning Sciences 2021* (pp. 17-20). Bochum, Germany: International Society of the Learning Sciences.

EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) RESEARCH COLLOQUIUM

COLOQUIO DE INVESTIGACIÓN: COGNICIÓN E IMAGINACIÓN MATEMATICA INCORPORADA

COLLOQUE DE RECHERCHE: IMAGINATION ET COGNITION MATHÉMATIQUES INTÉGRÉES

Mitchell J. Nathan
U. of Wisconsin - Madison
mnathan@wisc.edu

Candace Walkington
Southern Methodist University
cwalkington@mail.smu.edu

Avery Harrison Closser
Worcester Polytechnic Inst.
ahclosser@wpi.edu

Erin Ottmar
Worcester Polytechnic Inst.
erottmar@wpi.edu

Martha Wagner Alibali
U. of Wisconsin - Madison
Martha.alibali@wisc.edu

Hannah Smith
Worcester Polytechnic Inst.
hsmith2@wpi.edu

The Embodied Mathematical Imagination and Cognition (EMIC) Research Colloquium offers hands-on individual and collaborative mathematics activities; technological, curricular, and pedagogical demonstrations; and open spaces for exploration and discussion regarding the embodied nature of mathematics education. At PME-NA 44 we will use the conference theme of "critical dissonance and resonant harmony" to invite the community to directly experience some of the many ways embodied perspectives challenge the status quo through expanded notions of mathematical activity and assessment, and to explore some current contributions, future potential, and challenges for more inclusive educational experiences for learners and teachers.

Keywords: Cognition; Learning Theory; Social Justice; Systemic Change; Technology

Theoretical Background

It is increasingly apparent that effective mathematics education systems need to address the embodied needs of learners by designing educational experiences that: 1) are grounded in people's natural propensities for movement, sensation, perception, and collaborative interactions with cultural and scientific tools; 2) support "multiple linguistic repertoires," (McKinney de Royston et al., 2020, p. 10) including verbal, nonverbal, and multimodal ways of knowing, communicating, and assessment; and 3) facilitate learners' participation in systems that support the distributed nature of knowledge and reasoning (e.g., Abrahamson 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards et al., 2014; Radford 2009; Walkington et al., 2021). Evidence-based design principles are informing innovations in material-based curricula and digital technologies that support embodied mathematical imagination and cognition (EMIC) using platforms such as physical and video games, skits and virtual reality, craft making and tactile computing, motion capture, and participatory simulations (e.g., Abrahamson et al., 2020).

A Brief History and Motivation of the EMIC Working Group

The EMIC working group officially started in East Lansing, MI during PME-NA 2015. However, it emerged from a variety of earlier initiatives, including a 2007 AERA symposium, "Mathematics Learning and Embodied Cognition," a 6-year NSF-REESE grant (Nemirovsky et al., 2008), special issues of *Educational Studies in Mathematics* (2009), *The Journal of the Learning Sciences* (2012), and an NCTM 2013 research pre-session keynote panel, "Embodied cognition: What it means to know and do mathematics." A group of PME-NA members recognized a “critical mass” of investigators involved in related projects, and contributed findings and conceptual frameworks to form a stable and growing mathematics education research community to discuss, explore, and advance this scholarly program. We have convened annually at PME-NA ever since, organizing events that investigate the embodied nature of mathematics through physical and digital games, art and craft, individual and collaborative dance and movement, language, perception, immersive experiences, and more. EMIC members have continued to explore new technologies (e.g., Harrison et al., 2018; Walkington et al., 2021) and research methods (e.g., Abrahamson et al., 2021; Closser et al., 2021). They go beyond scholarly publications and presentations to offer: novel embodied technologies, including some that have gone to scale, for classroom use, special education, and emerging bilinguals; teacher learning programs for instructional uses of gestures; and webinars for teachers and parents. We are redesigning our website so it can continue to serve as a portal for resources, events (including workshops and webinars), teaching materials (e.g., syllabi), and announcements.

EMIC 2022: Critical Dissonance and Resonant Harmony

The organizers (including the authors; plus Dr. Janet Walkoe, U. MD and coordinator M. Nathan) have several years of experience co-designing and facilitating the PME-NA sessions. In 2022, we will organize hands-on activities and group discussions around the central questions that frame this year’s conference theme. On **Day 1**, we will use these prompts in small-group interactions to guide design activities with participants around (a) developmentally appropriate math topics (e.g., number, operations & algebra; geometry & measurement; probability & data analysis; functions & calculus; etc.); (b) research questions and designs for exploring embodiment of that topic; and (c) considerations of embodiment (vs. traditional) approaches to lesson design, instruction, and assessment. The first day will explore “How does this approach challenge a settled mathematics learning status quo?” As one example, an EMIC perspective might reframe high school geometry proofs from static two-column schemas to a deductive process grounded in movement- and object-based transformations. At least one organizer will join each small group. Day 1 will also include open-ended Q&A on the nature of embodiment theory, research methods, and design principles.

On **Day 2**, participant-groups (membership can be fluid) will explore “How does this approach help to create more socially just contexts for learning and teaching mathematics?” and “Whose voice does your work center in the mathematics learning process?” We will start this off by sharing findings and videos from the EMIC community demonstrating examples of support for emerging bilingual students and students with physical disabilities. An example is describing expansive forms of assessment practices that include ways students exhibit nonverbal (e.g., gesture-based) and distributed ways of knowing. **Day 3** reconfigures the small groups around common themes that arose in topic-specific groups on Days 1 & 2. We will invite these groups to reflect on who math education experiences are designed for, who is systematically marginalized, and what is gained in a truly inclusive approach. We then shift to a facilitated whole-group discussion, inviting members of each group to share ways an embodied framework can improve learning conditions for every mathematics learner, and benefit society more broadly.

Conclusions and Looking Ahead

Embodied approaches to design, instruction, and assessment show promise for fostering systemic change and harmony in mathematics education for amplifying inclusion of historically marginalized communities. There is much more to do to understand these impacts and realize their potential. An engaged community is vital for such transformative work to occur.

References

Abrahamson, D. (2014). The monster in the machine, or why educational technology needs embodied design. In V. Lee (Ed.), *Learning technologies and the body: Integration and implementation* (pp. 21-38). New York: Routledge (Taylor & Francis Group).

Abrahamson, D., Nathan, M. J., Williams-Pierce, C., Walkington, C., Ottmar, E. R., Soto, H., & Alibali, M. W. (2020). The future of embodied design for mathematics teaching and learning [Original Research]. *Frontiers in Education, 5*(147). https://doi.org/10.3389/feduc.2020.00147

Abrahamson, D., Worsley, M., Pardos, Z. A., & Ou, L. (2021). Learning analytics of embodied design: Enhancing synergy. *International Journal of Child-Computer Interaction*, 100409.

Alibali, M. W. & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*. (Special Issue on Embodiment in Mathematics.), *21*(2), 247-286. DOI: 10.1080/10508406.2011.611446.

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Closser, A. H., Erickson, J. A., Smith, H., Varatharaj, A., & Botelho, A. F. (2021). Blending learning analytics and embodied design to model students' comprehension of measurement using their actions, speech, and gestures. *International Journal of Child-Computer Interaction*, 100391.

De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.

Edwards, L., Ferrara, F., & Moore-Russo, D. (Eds). (2014). *Emerging perspectives on gesture and embodiment in mathematics*. Charlotte, N.C.: Information Age.

Harrison A., Ottmar, E., Arroyo, I., Rosenbaum, L., Bakker, A., Abrahamson, D., Hulse, T., Manzo, D., & Landy, D. (2018, April). Embodiment and action in mathematics games. Symposium conducted at the *American Psychological Association's conference on Technology, Mind & Society*, April 5-7, 2018, Washington, DC.

McKinney de Royston, M., Lee, C., Nasir, N. I. S., & Pea, R. (2020). Rethinking schools, rethinking learning. *Phi Delta Kappan*, *102*(3), 8-13.

Nemirovsky, R., Hall, R., Nathan, M. J., Alibali, M. W., & Leander, K. (2008). Tangibility for the teaching, learning, and communicating of mathematics. Grant proposal to the National Science Foundation Research and Evaluation on Education in Science and Engineering (NSF-REESE; DRL-0816406; 2008-2014).

Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics, 70*(2), 111-126.

Walkington, C., Gravell, J., Velazque, J., He, T., Hickey, G., Nathan, M. J., & Cuevas, A. (2021, June). Collaborative virtual learning in the shAR geometry simulation environment. In A. Wichmann, H. U. Hoppe, & N. Rummel (Eds.), *General Proceedings of the 1st Annual Meeting of the International Society of the Learning Sciences 2021* (pp. 17-20). Bochum, Germany: International Society of the Learning Sciences.

MYTHS OF OBJECTIVITY IN MATHEMATICS EDUCATION

MITOS EN TORNO A LA OBJETIVIDAD EN LA EDUCACIÓN MATEMÁTICA

Sofía Abreu
Michigan State University
abreusof@msu.edu

Aida Alibek
University of Georgia
aida.alibek@uga.edu

David M. Bowers
U. of Tennessee–Knoxville
dbower14@utk.edu

James Drimalla
University of Georgia
james.drimalla@uga.edu

Beth Herbel-Eisenmann
Michigan State University
bhe@msu.edu

Alexander S. Moore
Virginia Tech
asm1@vt.edu

Lee Melvin Peralta
Michigan State University
peralt11@msu.edu

This Working Group was occasioned by a convergence of concerns over the widespread de facto acceptance of notions of objectivity in the field of mathematics education. Because the ideologies of the field continue to be based largely in elements of scientism, we propose that the mathematics education arena familiar to researchers must shift and take up the work of philosophy and critique for the field to remain relevant in current and future times. This scientism structures, for example, the common belief in the usefulness of mathematics and the subjectivity of the soft science research in which we engage (e.g., using inter-rater reliability). This working group provides participants the opportunity to engage in active learning experiences and critical discussions about the ways in which objectivity infiltrates aspects of our field, and to expose oneself to new viewpoints about the philosophy of mathematics education.

Keywords: Doctoral Education, Research Methods, Systemic Change.

Dialectically engaging with the idea of objectivity, the nature of knowledge in mathematics education, and the philosophy of mathematics education research is a task that is long overdue, particularly in the U.S. context. This Working Group captures this urgency whilst also capturing the anxiety of the present historical moment apropos politics, global health crises, and new ways of popularly understanding "facts." Indeed, engaging with this issue at this particular point in the history of our field is an ethical imperative, which many mathematics education researchers continue to disavow by focusing instead on the production of qualified labor power in students (see Baldino & Cabral, 2013, 2015) as the implicit, neoliberal goal of mathematics education.

This Working Group will examine various myths of objectivity in mathematics and mathematics education research, problematizing ideas that are taken for granted and often are glossed over. However, we first and foremost approach this Working Group with the guiding question, "Is our field prepared to ask and engage with these questions?" Myths are stories we tell ourselves and tell others, and as such, they have their own agency. A myth performs or creates an effect, such as trying to say that something is the "truth" of the world. In that sense, myths are "real," but they do not necessarily preclude alternative myths from existing or being mutually exclusive with other myths. Conventional notions of objectivity, however, seek to kill alternative myths, and as such, strive to delineate a clear line between what is objective and what is not. As a consequence, the act of leaving myths of objectivity unrecognized is an exercise in hegemony and the reproduction of dominant epistemologies—what are sometimes called "common sense." An example of the ways in which myths of objectivity infiltrate mathematics education research is the work of Piaget (1970), whose *Structuralism* can be read as a resolution of the (ostensible) duality of objectivity and subjectivity into pure objectivity, claiming that differences in psychological structures should be simultaneously considered infinite in variability yet deterministic in universality. Another example can be seen in "equity" research that does not, in fact, achieve its espoused goal of equity but rather results in a meritocratic fetish of mathematics success and failure that implicates the primacy of school credit system even within research that claims to dismantle it (Baldino & Cabral, 2013, 2015; Bullock, 2013; Gutiérrez, 2008; Martin, 2003; Moore & Johnson, under review; Pais, 2012; Straehler-Pohl & Pais, 2014). Another is the unquestioned understanding of dominant mathematics as necessary to be a "successful" member of society or as the only path towards solving present crises. In particular, even as there is more research that embraces students' subjectivities, the mathematics that is taught—one that emerges from a dominating culture—is still framed as universal while the overlooked historical and material interests driving it continue to reinforce oppressive practices and structures (Bullock, 2018; Fasheh, 2012; Gutiérrez, 2017). As Martin et al. (2010) ask, whose mathematics are we teaching? "For whom? and for what purposes" (p. 14)? We do not posit that objectivity is a false narrative, but rather we posit that objectivity is *a narrative* into which the field has largely fallen whilst simultaneously assigning it ultimate epistemic authority. To be clear, we do not wish to throw objectivity out of researchers' discourse, but we desire to shift participants' discourse towards a dialectical engagement with objectivity and subjectivity.

To the above ends, we organize this working group in a way that: (1) is welcoming to newcomers to the constituent perspectives surfaced while simultaneously digging deeply within and across in a manner that will push the thinking of veterans, (2) explicitly surfaces and honors a diversity of perspectives, (3) takes seriously the possibility of improving mathematics education practice in the face of the irreconcilable dialectic of objectivity and subjectivity that exists across the wholistic fabric of our work, and (4) honors and employs research-based and anti-hierarchical pedagogy. Thus, each day is comprised of a series of task- and inquiry-based explorations and discussions. Each task is designed and implemented by a different subset of contributors, rooted in our own humanity and experiences with/against uncritical notions of objectivity. We—the facilitators—are ourselves varied in terms of race, gender, sexuality, dis/ability, and career stage, as well as in terms of our particular areas of specialized expertise (including activism, art, media and critical discursive analysis, philosophy, and psychoanalysis), all of which contribute to a rich collection of tasks and discussions.

The sequence of discussions is designed to give attention to as many aspects of our practice as possible. On Day 1, our two key areas of focus are *Disciplinary Knowledge* (e.g., Mathematics) and *Research Design*. On Day 2, our two key areas of focus are *The Publication/Outreach Process* and *Final Products of Research*. We conclude on Day 3 with an *Overarching Conversation* reaching across these interrelated areas as well as time spent *Looking Ahead* to implications for our own work as well as to possibilities for subsequent working groups, collaborations, and articles. Tasks vary substantively across the Working Group, including time spent: doing/reflecting on mathematics, critically examining unstated or uninterrogated assumptions in research design, examining research products through etic frames such as art, and examining our work towards meaningful social/educational change through etic frames such as activism. Each task gives rise to open-ended opportunities for discussion within and across topics of, for example, unearthing values, questioning reality and existence, examining what we mean by "knowing," and surfacing tacit aspects of practice.

References

Baldino, R. R., & Cabral, T. C. B. (2013). The productivity of students' schoolwork: An exercise in Marxist rigour. *Journal for Critical Educational Policy Studies*, *11*(4), 70–84.

Baldino, R. R., & Cabral, T. C. B. (2015). Profitability of qualified-labour-power production. *Journal for Critical Educational Policy Studies*, *13*(1), 67–82.

Bullock, E. C. (2013). Conducting "good" equity research in mathematics education: A question of methodology. *Journal of Mathematics Education at Teachers College*, *3*(2).

Bullock, E. C. (2018). Intersectional analysis in critical mathematics education research: A response to figure hiding. *Review of Research in Education*, *42*(1), 122–145.

Fasheh, M. J. (2012). The role of mathematics in the destruction of communities, and what we can do to reverse this process, including using mathematics. In O. Skovsmose & B. Greer (Eds.), *Opening the cage: Critique and politics of mathematics education* (pp. 93–106). Sense.

Gutiérrez, R. (2008). A "gap-gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education*, *39*(4), 357-364.

Gutiérrez, R. (2017, October 5–8). Living mathematx: Towards a vision for the future [Paper presentation]. *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Indianapolis, IN.

Hegel, G. W. F. (1977). *Phenomenology of spirit* (A. V. Miller, Trans.). Oxford University Press. (Original work published 1807)

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Martin, D. B. (2003). Hidden assumptions and unaddressed questions in mathematics for all rhetoric. *The Mathematics Educator*, 13(2), 7-21.

Martin, D. B., Gholson, M., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education*, 3(2), 12–24.

Moore, A. S., & Johnson, E. (under review). Gender as a reflex-category in mathematics active learning and lecture.

Pais, A. (2012). A critical approach to equity. In O. Skovsmose & B. Greer (Eds.), *Opening the cage: Critique and politics of mathematics education* (pp. 49–92). Sense.

Piaget, J. (1970). *Structuralism* (C. Maschler, Trans.). Basic Books.

Straehler-Pohl, H., & Pais, A. (2014). Learning to fail and learning from failure: Ideology at work in a mathematics classroom. *Pedagogy, Culture, & Society*, *22*(1), 79–96.

Wenger, E. (1998). *Communities of practice: Learning, meaning, identity*. Cambridge University Press.

WORKING GROUP REPORT: GENDER AND SEXUALITY IN MATHEMATICS EDUCATION

Julia Przybyla-Kuchek
University of Georgia
juliapk@uga.edu

Brent Jackson
Michigan State University
bjackson@msu.edu

Katrina Piatek-Jimenez
Central Michigan University
k.p.j@cmich.edu

Ana Dias
Central Michigan University
dias1al@cmich.edu

Jennifer Hall
Monash University
jennifer.hall@monash.edu

Elizabeth Kersey
University of N. Colorado
elizabeth.kersey@unco.edu

Alexander Moore
Virginia Tech
asm1@vt.edu

Weverton Ataide Pinheiro
Indiana University
wpinheir@iu.edu

Lynda Wiest
University of Nevada
wiest@unr.edu

In this report, we describe the goals, organization, and outcomes of the Gender and Sexuality Working Group at the 2021 PME-NA annual meeting. We discuss the general organization and format of the group meetings and provide summaries of the papers, commentaries, and discussions that occurred. We then describe the ongoing projects and plans that have resulted from collaborations formed through the working group.

Progress Made at the 2021 PME-NA Conference

During the 2021 PME-NA conference, the Gender and Sexuality Working Group convened to discuss emerging conceptual and theoretical frameworks currently being utilized by scholars to explore topics related to gender and sexuality in mathematics education. The journal *Mathematics Education Research Journal* (MERJ) recently published a special issue on the topic of innovations within this area, and members of this working group contributed four of the seven articles published in this special issue. Authors of these four papers presented brief summaries of their work and another member of the working group provided a short commentary guided by the following two questions: (a) How does this article relate to your own work? (b) How do the theories and/or methodologies stimulate and/or “push” your own thinking? These presentations sparked lively discussions regarding the proposed conceptual and theoretical frameworks. This format allowed all members of the working group to become familiar with the emerging and developing perspectives, and to consider how these perspectives may inform their own research.

We begin with Hall’s overview of the MERJ special issue. Then, the remaining structure of this report follows the structure of the presentations as they occurred: *paper presentation* followed by a *commentary*.

Mathematics Education Research Journal Special Issue Overview

Jennifer Hall provided an overview of the recent Special Issue of the *Mathematics Education Research Journal*, entitled Innovations in ‘Gender Issues’ Research in Mathematics Education, which she co-edited with Eva Norén (Stockholm University). Jennifer and Eva were the Co-Convenors of the International Organization of Women and Mathematics Education (IOWME) from 2016 to 2021, and they aimed to lead a project that would have a meaningful impact on the field.

Recently, there has been a resurgence of interest in 'gender issues' research, with scholars pushing the field in various ways. Hence, the Special Issue was an outlet for such researchers to share their work. Papers with the following foci were sought: (a) Non-binary/non-homogenous conceptions of gender; (b) Feminist and/or queer theory frameworks; (c) Transgender, non-binary, and other gender minority participants, and (d) Gender-sensitive research designs.

The call for abstract submissions was made from September to November of 2019 on channels such as the IOWME Facebook group and the Mathematics Education Research Group of Australasia mailing list. In total, 14 abstracts were received from authors from nine countries, and seven abstracts from authors in two countries were accepted and developed into papers.

Each paper was reviewed by one of the co-editors of the Special Issue, an author of another article in the Special Issue, and an external reviewer, and went through the usual process for revision. The resulting papers comprised two short communications (up to 3,000 words) and five full-length articles (up to 9,000 words). The short communications were both position papers, whereas two of the full-length articles were theoretically-focused papers and three were empirically-focused papers. The published papers can be accessed at https://link.springer.com/journal/13394/volumes-and-issues/33-4.

Many authors drew on feminist, queer, and/or poststructural theories in their work. When empirical research was discussed, qualitative methodologies and methods, such as narrative inquiry, interviews, and observations, were commonly used and/or recommended. Overall, there was an acknowledgement by the authors and co-editors of the Special Issue of the complexity of gender as a construct and the ever-changing nature of 'gender issues' research. Recommendations were made to look to other fields for inspiration and for examples of innovative research, as well as to increase awareness and focus on intersectionality in 'gender issues' research. Furthermore, researchers—both those focused on gender and those for whom gender is just one 'variable' in their research—were cautioned to clearly operationalize the term *gender* in their work and to use consistent terminology. It is hoped that the Special Issue will serve as a valuable resource for researchers in the coming years.

Paper Presentation by Przybyla-Kuchek

In alignment with our goals as a working group to share emerging theories and methodologies in gender and sexuality in mathematics education, Przybyla-Kuchek (2021) suggested Baxter's (2003) Feminist Poststructural Discourse Analysis (FPDA) as a possible methodological approach to researching gender and its intersections with other social identities in mathematics classrooms. Although significant work on gender and sexuality in mathematics education takes identity into consideration, researchers commonly use methodological approaches that tend to focus on analysis of interviews with participants. Much has been learned from speaking to girls and women about their mathematical experiences, and this work continues to be important, but researchers should also consider how identities are experienced, enacted, or performed as they are being created. Researchers taking up poststructural theory and feminism together theorize gender as performative, and thus gives space to consider how identity is performed in and outside of classrooms.

Przybyla-Kuchek highlighted the transcript of a seventh-grade mathematics classroom interaction that appears in her MERJ article, shared the video clip of the interaction, and briefly summarized the gendered discourses that could be producing the interaction. The interaction is of a spontaneously formed small group of students. Two White male students attempted to "help" Georgia, a White female student, answer a question on a worksheet. Gendered and racialized discourses played out in the interaction as the complex subject position of "damsel in distress" was performed by two participants. One female student voluntarily took up a subject position in a more powerful way to distract one of the boys from "helping" Georgia. The boys' subject positions as "super smart" were consistent and solidified in the interaction. Although the interaction was complex and multiple discourses were working to position the participants in relation to mathematics, Przybyla-Kuchek identified the interaction as a moment of resistance, a speech event in which gendered discourses were disrupted and participants created new and powerful subject positions.

With such analyses, researchers can consider how powerful discourses (re)produce gendered subject positions for girls in mathematics classrooms. More importantly, using this approach, researchers and participants could identify the everyday, local ways in which powerful and even harmful discourses that produce girls as less capable in mathematics are disrupted. FPDA uncovers the significant work that participants are already doing to tear holes the fabric of discourses that constitute who can do mathematics and what it means to do mathematics. Furthermore, Przybyla-Kuchek considers how research is conducted and written to produce truths about girls and mathematics, and challenges mathematics educators to be self-reflexive across the entirety of the research process, including the writing and publication process.

Commentary by Piatek-Jimenez

Piatek-Jimenez' commentary surrounded the main themes that were discussed during a virtual meeting with Przybyla-Kuchek. In her paper, Przybyla-Kuchek (2021) discussed arguments by Walkerdine (1998) and Parks (2009) who suggested that the "gender gap" in mathematics is only as real as we have made it, through our designs of educational practices. In other words, it is not that the researchers are discovering an underlying truth, but that a "truth" is being created through the process. This idea resonated with arguments made by Piatek-Jimenez and Dias (2021), related to large-scale assessments, where the authors analyzed recent TIMSS, PISA, and NAEP data, and noted that whether one can claim a gender achievement gap exists depends on how one chooses to analyze the data and interpret the results. This point also related to the plenary address presented by Dr. Toya Frank during the first night of the PME-NA 2021 Conference, when she discussed conducting quantitative research. During her presentation, Dr. Frank noted "Numbers are not objective, but are socially constructed as well." Piatek-Jimenez argued that such a lens is important when conducting gender and sexuality research, as it is important to remember that the work that we do is affected by socially constructed perceptions and there is no "truth" that we are "discovering."

In addition to the lenses that are used while conducting research, the formats used to present the work influence how scholars in the field interact with and interpret the results. Przybyla-Kuchek (2021) discussed two innovative ways to present results, both of which disrupt the traditional structures used within mathematics education research. For example, Walshaw (2005) used a "split-text format," where the entire transcript of an interview and Walshaw's own writing and analysis were shown simultaneously on the page. Utilizing this format restructures the relationship between the author and the reader. It also allows the reader to conduct their own analysis of the text, removing "truth" from the analysis of the author. Such a format is not only unusual in our field, but Piatek-Jimenez argued that it is discouraged. For instance, Piatek-Jimenez recalled being told by a mentor that the entirety of transcripts should never be made public. As such, Walshaw's usage of split-text formatting is one way of interrupting common epistemological and research traditions.

Additionally, Przybyla-Kuchek (2021) interrupted such norms by sharing a portion of one transcript prior to sharing the gender and racial/ethnic background of the participants. Based on pseudonyms that the participants chose for themselves, one might make assumptions about the participants prior to learning their actual gender/racial/ethnic identities. By presenting the results using this format, Przybyla-Kuchek forced the reader to check their own assumptions and reread the interview transcript with alternate lenses. This method of presenting data provides the reader with a deeper understanding of the results of the study and how the reader's own biases may influence their interaction with the article itself.

Paper Presentation by Kersey

Kersey presented the methods from two studies reported in a single article in the MERJ special issue. In the article, Kersey and Voight (2021) focused on the experiences of queer and/or transgender postsecondary students in science, technology, engineering, and mathematics (STEM) fields. Based on the paucity of research within and outside of mathematics education on this population, Kersey & Voigt inferred that STEM fields may be less welcoming than the humanities and social sciences and investigated this possibility through their studies. In the first study, narrative inquiry was employed with postsecondary transgender students, whereas the second study employed a grounded theory narrative study with undergraduate queer students. The authors reported that transgender students who had transitioned indicated that they were subjected to lower expectations when presenting as female. Transgender women experienced this change as a net positive, since their treatment by others was no longer accompanied by gender dysphoria. Queer students experienced mathematics and other STEM fields as objective and independent of identity, yet simultaneously exclusionary of their queer identities. Many of the queer students in these studies found strength and resilience in queer communities, but there were some transgender women who did not view being queer as a central facet of their identity and did not feel the same sense of community. In general, participants who were more gender-nonconforming felt a greater need for community with other queer people. Kersey presented a tool used in their data collection that helped to draw connections between gender category oppression and gender transgression oppression.

Commentary by Jackson

Jackson commented on the article and presentation by remarking on three aspects: the use of queer theory, theory, and methods. Regarding queer theory, Jackson highlighted his appreciation for the way that queer theory was employed throughout the research project and article. For instance, in the methodology the authors "avoided placing boundaries around STEM, queer, or transgender." The researcher did not themself decide who was a STEM student. Jackson encouraged the working group to pause and to think about how, in our own research, what counts as in/outside of "math ed" and to what extent this is in conflict with some foundations of queer theory is defined by researchers. In the studies discussed in the article, the authors allowed the participants to identify with STEM, queer, transgender, or any other affiliation. There is a tension because this also brought forward some issues related to the construct of identity that the working group has been exploring and discussing. Throughout the article, the authors returned to the notion of identities and the way that identities are able to shift: Identities do not have to be stable, or have an interior expression. Jackson reflected on the notion that identity, in some regard, is about a claim to an affiliation; one can claim, but the claim isn't always "settled" as there is always the contingency that another person can accept/reject that affiliation. This brought forward the notion of *identity work*, for whom we perform identities, and/or how we mobilize our identities.

Methodologically, Jackson discussed the use of narrative inquiry and narrative analysis. Jackson highlighted the distinction between a narrat*ed* event and a narrat*ive* event. The analysis in the paper was focused on narrated events, those from the past. Whereas narrative events could also be analyzed to highlight the way that discourses are used in the telling of events itself. Jackson discussed how this distinction challenged him to think about methodologies that would tighten the time frame between the narrated event and the narrative event. Jackson noted that at times the participants' stories that were represented in the article seemed vague in the relationship to STEM contexts specifically, and that it seemed the participants themselves were analyzing/generalizing their own experiences. Jackson wondered about the insights that might be available through methods in which participants journaled immediately after each class to describe "unsettling" events or things that made them more or less comfortable.

Jackson also commented that participants in the study used word such as masculine and feminine. Jackson wondered what methods would help us to accurately capture what is perceived as masculine or feminine since these meanings change in space and time. Furthermore, Jackson discussed the language of privilege and oppression used in the paper and how careful researchers have to be to use such terms. Jackson highlighted how the authors noted that some participants may not have been comfortable in naming an experience, or their set of experiences as oppressive. Hence, researchers need to capture rich details of experiences.

Paper Presentation by Wiest

The purpose of this paper was to call for research on sexual identity in mathematics education, which is starkly absent (Dubbs, 2016; Fischer, 2013). Wiest discussed that the little research that has been done has shown differences in academic outcomes based on sexual orientation in spatial skills (Xu et al., 2020) and mathematics and science grades and course taking (Pearson & Wilkinson, 2017). Individuals in academic STEM (science, technology, engineering, and mathematics) pursuits tend to report uncomfortable and unsupportive environments that take a toll on psychological well-being, career satisfaction, productivity, and persistence (e.g., Barres et al., 2017; Institute of Physics, Royal Astronomical Society, and Royal Society of Chemistry, 2019; Mattheis et al., 2019; Stout & Wright, 2016). Wiest argued that targeted research is needed on mathematics education and sexual identity. Wiest suggested research questions might be formulated within these broader questions:

- What academic and social experiences and outcomes do lesbian, Gay, Bisexual, plus (LGB+) students have in general and in relation to individual and intersecting identities? How might these experiences and outcomes differ by age/grade level and academic setting? Which factors favorably and unfavorably influence student performance, dispositions, and success? How might these experiences and outcomes differ by specific fields and subfields of study, as well as by individuals' degree of disclosure of LGB+ identities, and why?
- What might "queer" curriculum and pedagogy look like? How might instruction be inclusive of LGB+ students and LGB+-headed families? How can the effectiveness of these instructional approaches and programs be evaluated?
- What are the workplace experiences and roles of LGB+ school staff, including teachers, administrators, and counsellors? What are the roles of non-LGB+ staff?
- How can education efforts best succeed within the broader sociopolitical climate, which can vary widely in relation to LGB+ acceptance? How might the field serve an advocacy role in that regard?

Commentary by Dias

Dias highlighted that the purpose of Wiest's (2021) paper is to call for research on sexual identity in mathematics education, summarize the existing research on sexual identity in STEM/mathematics education, and propose research questions that would advance the work in this area. Dias reflected on the absence of attention to sexual minorities in mathematics education research and calls for equity.

Dias related the paper and presentation to her own work by discussing the following points. First, Weist (2021) highlighted that sexual identity affects learning in the mathematics classrooms, as do many other aspects of students' subjectivities. These identities are important for educators, who are called upon to teach mathematics equitably to all students. Wiest wrote that teacher educators must prepare teachers with pedagogies that are inclusive of sexual minorities and humanize minoritized sexualities in their classrooms and in schools in general. Dias stated that this argument is consistent with her own arguments which has been criticized as being mathematics education without mathematics. Dias affirmed that her goal is that teachers are prepared to humanize LGB+ students, as well as students who don't conform to the gender binary, and learn how to work with parents of LGB+ students for the sake of the students' well-being, when families are not supportive of their sexual identities.

Second, Dias commented that Wiest uses LGB+ to indicate sexual orientations that include lesbian, gay, bisexual, and beyond (e.g., asexual, pansexual), essentially, all sexual orientations that are not strictly heterosexual.

Third, Dias reflected on the point in Wiest's paper regarding students' challenges and resilience. Although some students exhibit resilience in response to challenges, many LGB+ students are negatively affected by their experiences both in and out of school (e.g., by physical and verbal victimization, by exclusion from school curricular materials and classroom discussions) that can result in a number of self-harmful outcomes, including poorer psychological well-being and lower educational engagement, performance, and aspirations. Dias noted the important contribution of this paper to call for more research that can inform teaching practices and school policies to help protect and support sexual minorities.

Lastly, Dias noted the importance of disseminating research that shows that positive school support (and other factors such as openly gay teachers, gay-straight alliances, and participation in queer youth centers) are important for LGB+ students to feel comfortable in school. This idea encouraged a project that Dias is a part of in which spaces and moments for LGB+ student populations in campuses in the state of Sao Paulo, Brazil are being investigated. Weist mapped a variety of questions that can advance the field. We must call on journals and publishers to print this work. Dias stated that she will take the questions to her students and research group. In conclusion, Dias argued that Wiest's publication is very important in that it calls attention of the mathematics education community to the need of research in this area clearly, comprehensively, and in a nuanced manner.

Paper Presentation by Moore

In his presentation, Moore centered on the implications of sexuality for mathematics education researchers. Moore's motivation to write the paper was to synthesize a theoretical basis for queer research in mathematics education—as one did not seem to have existed before. These implications can be summarized in three points:

First, theory is important. Theory in our field is too often seen as a methodological tool, a setting that we must choose to best fit the situation and our insights. In this way, theory becomes a slave to context. The trouble with this approach is that if the contextual basis that we consider mathematics education to be is always the same, then the theoretical impositions we make as researchers will always turn out in the same ways.

Second, we must make our thinking, understandings and discourses about queerness in mathematics education as complicated as possible. This recommendation is not meant to be obtuse. It is rather to see that queerness is not only about identity and subjective positions but is rather about our research and the ways in which we go about doing it. The queer project rejects a suspiciously understandable approach to recognizing our subjects. Our recognitions of each other are always-already failed recognitions. So then why would we be so naïve as to think that the goal of research is to be as organized and scientific as possible? On the contrary, the goal is to make research as complicated as possible. Only then might we approximate the inherent failures that we have committed in attempting to recognize anything in our data.

Third, we must take seriously the ways in which mathematics implicates and is implicated by gender and sexuality. Mathematics education cannot escape the anxieties, the identifications, the questions, and the desires that are produced by sexuality, and therefore also by gender as a sexual construct. Taking up the task of queer research in mathematics education means recognizing our enterprise as one of radical subjectivity.

A queer understanding of our field leads us to understand the critical importance that Moore calls for in the paper: if we are to ask questions about identity, subjectivity, gender, and sexuality, then the queer project in mathematics education is precisely the project of making life livable—even for the heterosexual and cisgender. The queer project is expansive in this way, just as the Queer Identity Intersection (a diagram that Moore introduced in the paper) gives possibilities for emergent, reconstructive centers that do not exist now. To briefly recapitulate: If we are engaged in "gender and sexuality" research, then we must queer what we think about those terms, and see that in order to engage in such research means to be as complicated as possible, imaginative and philosophical with our approach to theory, creating new understandings that are not extensions of existing "classic" knowledge that we have inherited over the last 50 years, and seeing ourselves as engaged in a process of humanizing not only the research that we perform, but also the field of mathematics education, and the mathematics itself.

Commentary by Pinheiro

Moore's (2021) review of theoretical literature provides lenses to understand the ways that research frameworks can contribute to our understanding of the intersection between mathematical and queer identities. In his article, Moore reviewed the theoretical framework of 28 articles in which queer identities in mathematics education were discussed. Through the literature review, Moore proposed the framework, The Queer Identity Intersection (QII) of Mathematics Education. Such a framework is composed of two roads in which one can get situated and analyze through perspectives where you are in the roads and what you need to get to the intersection. Moore's intentions with the QII framework were to provide researchers with an analytical tool that provides inciteful ideas for researchers' conceptualizations of what they need to have a strong theoretical framework that can analyze queer students mathematical and queer identities at the intersections. For Moore, queer identities and mathematical identities are situated in the borderlands, a place that he suggests is uncomfortable because of the multitude of realities (e.g., queer identity reality and mathematical identity reality). Moore argues that both queer and mathematical identities, when enacted, contribute to the conceptualization of the learning processes of one's understanding of becoming part of a community of practice. Therefore, one's mathematical identity is constantly affected by one's queer identity, which influences students' experiences with learning and knowing mathematics, and being part of the community of practice that encompasses mathematics. Keeping in mind that queer identities are marginalized in schools, and that mathematical identities can be marginalizing forces that limit who is capable of learning and doing mathematics (Reinholz & Shah, 2018), Moore concludes that "to fully understand notion of queer identity in mathematics education, [he] suggests [the] intersection of queer identity and mathematical identity" (p. 18), since in the intersection, we are trying to understand the experience of the individual itself, not looking specifically at one or the other identity, but at the "element of identity that […] becomes the unit of analysis" (p. 18). Moore implicitly suggests that mathematics for social justice lessons must be implemented by considering students' experiences in the mathematics classroom, considering the intersection between students' queer identities and mathematical identities.

After pointing to these important highlights, Pinheiro discussed how Moore's (2021) article relates and contributes to his own research. Pinheiro also uses intersectionality theory to explore the ways that gender, sexuality, and mathematics play a role in queer high school students' experiences with the teaching of mathematics for social justice. Pinheiro stated that the major push of Moore's article to his understanding of issues of gender and sexuality intersected with mathematics comes from Moore's idea that mathematical identity is also a social identity. Therefore, intersectionality, as a theory of social identities, applies directly to our understanding of how at the intersections, one's experience in the mathematics classroom is defined and shaped specifically by the individual's sexual, gender, and mathematical identities.

Ongoing Projects

After the conference concluded, the authors created a WhatsApp group with the intent of having an efficient channel for continuing our conversations. Everyone whose contact information was on the sign-up sheet created during the conference was given a choice to join the group. In the WhatsApp group we have identified common research interests and concrete projects upon which we are establishing new collaborations. As a result of this newly formed social media group, three members of the Working Group (Pinheiro, Dias and Moore) are planning to analyze data collected by Pinheiro on women's experiences in an undergraduate mathematics program. Two other members of the group (Piatek-Jimenez and Hall) are also working on a research collaboration and are currently preparing a Spencer Foundation grant application to fund their work.

Future Directions

The working group participants during the 2021 PME-NA expressed a desire to explore applications of these emerging theoretical and methodological perspectives. Particularly, there was interest in the application of different theories to data analysis, and what that application of methods might reveal and how it might impact students, teachers, and classrooms. To move forward in the 2022 PME-NA, we plan to share interview and classroom data with participants, and as a group we will analyze the data through various theoretical and methodological perspectives as a way to consider what researchers might conclude from the data using different theoretical approaches. Our hope is that the discussions will make space for participants to compare what different methods with various theoretical lenses are able to capture or limit in their own research in gender and sexuality in mathematics education.

References

Barres, B., Montague-Hellen, B., & Yoder, J. (2017). Coming out: The experience of LGBT+ people in STEM. *Genome Biology*, *18*(62).

Baxter, J. (2003). *Positioning gender in discourse: A feminist research methodology*. Palgrave: Macmillan.

Dubbs, C. (2016). A queer turn in mathematics education research: Centering the experience of marginalized queer students. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1041–1048). The University of Arizona.

Fischer, D. J. (2013). *Out 4 math: The intersection of queer identity and mathematical identity*. Drexel University.

Institute of Physics, Royal Astronomical Society, & Royal Society of Chemistry. (2019). *Exploring the workplace for LGBT+ physical scientists*. Institute of Physics.

Kersey, E., & Voigt, M. (2021). Finding community and overcoming barriers: Experiences of queer and transgender postsecondary students in mathematics and other STEM fields. *Mathematics Education Research Journal*, *33*(4), 733–756.

Mattheis, A., De Arellano, D. C. R., & Yoder, J. B. (2019) A model of queer STEM identity in the workplace. *Journal of Homosexuality* 1–25.

Moore, A. S. (2021). Queer identity and theory intersections in mathematics education: A theoretical literature review. Mathematics Education Research Journal, 33(4), 651–687.

Parks, A. N. (2009). Doomsday device: Rethinking the deployment of the 'achievement gap' in equity arguments. *For the Learning of Mathematics, 29*(1), 14–19.

Pearson, J., & Wilkinson, L. (2017). Same-sex sexuality and educational attainment: The pathway to college. *Journal of Homosexuality, 64*(4), 538–576.

Piatek-Jimenez, K., & Braz Dias, A. L. (2021). Gender equity in mathematics? In D. A. Peralta & J. A. Pacheco (Eds.), *Currículo e avaliação externa: entre políticas internacionais e práticas nacionais* (pp. 125-147). Vila Nova de Famalicão: Edições Húmus.

Przybyla-Kuchek, J. (2021). The possibilities of feminist poststructural discourse analysis as an approach to gender research in the mathematics classroom. *Mathematics Education Research Journal,* 33, 689–711 (2021).

Reinholz, D. L., & Shah, N. (2018). Equity analytics: A methodological approach for quantifying participation patterns in mathematics classroom discourse. *Journal for Research in Mathematics Education, 49*(2), 140–177.

Stout, J. G., & Wright, H. M. (2016). Lesbian, gay, bisexual, transgender, and queer students' sense of belonging in computing: An intersectional approach. *Computing in Science and Engineering, 18*(3), 24–30.

Walkerdine, V. (1998). *Counting girls out: Girls and mathematics*. New York, NY: Routlege Falmer.

Walshaw, M. (2005). Getting political and unraveling layers of gendered mathematical identifications. *Cambridge Journal of Education, 35*(1), 19–34.

Wiest, L. R. (2021). Prying open the closet door: a call for research on sexual identity and mathematics education. *Mathematics Education Research Journal*, *33*(4), 641–650.

Xu, Y., Norton, S., & Rahman, Q. (2020). Sexual orientation and cognitive ability: A multivariate meta-analytic follow-up. *Archives of Sexual Behavior, 49*(2), 413–420. https://doi.org/10.1007/s10508-020-01632-y

GENDER AND SEXUALITY WORKING GROUP: APPLYING THEORY TO DATA

Julia Przybyla-Kuchek
University of Georgia
juliapk@uga.edu

Brent Jackson
WestEd
bjackso@wested.org

Katrina Piatek-Jimenez
Central Michigan University
k.p.j@cmich.edu

Jennifer Hall
Monash University
jennifer.hall@monash.edu

Ana Lúcia Braz Dias
Central Michigan University
dias1al@cmich.edu

Weverton Ataide Pinheiro
Texas Tech University
wpinheir@iu.edu

Gender research in mathematics education continues to be important work for gender equity in mathematics, as well as in STEM fields more broadly. Additionally, authors recently called for research on sexuality, identified a dearth of research in this area, and a need to theorize and operationalize such research (Dubbs, 2016; Moore, 2021; Wiest, 2021). Building upon working group discussions of emerging theories and methodologies during PME-NA 2021, in the 2022 working group sessions, we will apply these theories and methods to interview and classroom data to explore concrete ways of studying gender and sexuality in mathematics education that utilize different perspectives. Our goals are to (a) explore how gender and sexuality may be "seen" through different methods of data collection, focusing on how different meanings of gender and sexuality, as they are described and performed, may lead to new insights; and (b) produce a list of research questions and potential projects to consider for future research.

Keywords: Gender; Equity, Inclusion, and Diversity; LGBTQIA+

Throughout the progression of gender research from achievement and participation to broader systemic, cultural, and societal barriers, researchers have identified a plethora of barriers for girls and women in mathematics (Leyva, 2017; Lubienski & Ganley, 2017). They have also demonstrated the importance of mathematics education research to gender equity in mathematics, as well as in STEM more broadly. Despite a lull of gender research, in the past 5 years there has been renewed interest in gender research with the emergence of new research paradigms (e.g., queer theory and postmodernism), and the consideration of intersectionality (Leyva, 2017; Lubienski & Ganley, 2017; Lubienski & Ataide Pinheiro, 2020; Stinson & Walshaw, 2017). For example, one important shift has been for researchers to move beyond male/female, men/women, and boys/girls binaries in terminology usage and throughout gender research methodology. Additionally, authors have recently called for research on sexuality in mathematics education, identified a shortage of research in this area and argued a need to theorize and operationalize such research (Dubbs, 2016; Moore, 2021; Wiest, 2021). For example, one area of continued interest is expanding notions of identity. Building upon the decades of gender research and expanding sexuality research in mathematics education, our working group continues to explore new approaches to gender and sexuality studies and identify paths for progress towards gender equity in the many ways it can be defined.

History and Goals

The Gender and Sexuality Working Group has convened for the past 4 years. In the past two conferences, we focused largely on establishing shared theoretical backgrounds, conceptualizing identity, and exploring methods for engaging in gender and sexuality research. During the PME-NA 2021 working group sessions, we explored emerging theories and methodologies through presentations and commentaries on articles in the Special Issue of *Mathematics Education*

Research Journal titled: Innovations in 'Gender Issues' Research in Mathematics Education. Authors of four of the seven articles in the issue presented brief overviews of their ideas, highlighting new insights on gender and sexuality theories, methods, and trends. Emerging theories discussed included queer theory and feminist poststructural theory. Furthermore, as an editor of the issue and working group member, Hall provided insights on gender and sexuality research internationally, reflecting on Special Issue submissions.

The working group participants during the PME-NA 2021 sessions expressed a desire to explore applications of these emerging theoretical and methodological perspectives. Particularly, there was interest in applying different theories to data analysis, what that application of methods might reveal, and how such research might impact students, teachers, and classrooms. To move forward, at PME-NA 2022, we plan to share interview and classroom data with participants. As a group, we will analyze data through various theoretical and methodological perspectives as a way to consider what researchers might conclude about gender and sexuality from the data. We expect to use these discussions to make space for participants to better understand how taking up different methods with various theoretical lenses can capture or limit their own research in gender and sexuality in mathematics education. Hence, our goals are to (a) explore how gender and sexuality may be "seen" through different methods of data collection, focusing on how different meanings of gender and sexuality, as they are described and performed, may lead to new insights; and (b) produce a list of research questions and potential projects to consider in our own future research.

Working Group Structure

Broadly, the working group structure will progress from jointly analyzing post-secondary interview data to analyzing K–12 classroom data. The sessions will culminate in group discussions about how participants operationalize gender and sexuality based on the types of data we collect and which theoretical perspectives we bring to the data.

Session 1

In the first session, participants will analyze a more familiar form of data for researchers studying gender and sexuality in mathematics education: interviews. The organizers will share several transcripts and audio/video from individual and focus group interviews with post-secondary participants. The interviews focused on participants mathematical experiences and were collected as part of ongoing projects conducted by members of the working group.

Session 2

Building from discussions about the analysis of interview data from the first day, the organizers will share whole-class and small-group video data from K–12 mathematics classrooms. Participants will discuss, as a group, what they noticed relating to gender and sexuality in the data and generate research questions to pose to the group. There will be a focus on what meanings gender and sexuality take and how gender and sexuality are understood in analyzing the data, considering small-group and whole-class settings.

Session 3

On the last day of the working group, participants will reflect on the analysis done in the two previous sessions. Discussions will focus on how we might approach data from various perspectives including how interview data (i.e., how a participant discusses their experiences) highlights or even requires different theoretical orientations or understanding of gender and sexuality compared to classroom data (i.e., performances of gender and sexuality in the moment). Participants will identify potential new research questions to pose to the field of mathematics education and potential projects that could seek to answer such research questions.

References

Dubbs, C. (2016). A queer turn in mathematics education research: Centering the experience of marginalized queer students. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1041–1048). The University of Arizona.

Leyva, L. A. (2017). Unpacking the male superiority myth and masculinization of mathematics at the intersections: A review of research on gender in mathematics education. *Journal for Research in Mathematics Education*, *48*(4), 397–433. https://doi.org/10.5951/jresematheduc.48.4.0397

Lubienski, S. T., & Ganley, C. M. (2017). Research on gender and mathematics. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 649–666). National Council of Teachers of Mathematics.

Lubienski, S. T., & Pinheiro, W. A. (2020). Gender and mathematics: What can other disciplines tell us? What is our role? *Journal of Urban Mathematics Education*, *13*(1), 1–14. https://doi.org/10.21423/jume-v13i1a377

Moore, A. S. (2021). Queer identity and theory intersections in mathematics education: A theoretical literature review. *Mathematics Education Research Journal*, *33*(4), 651–687. https://doi.org/10.1007/s13394-020-00354-7

Stinson, D. W., & Walshaw, M. (2017). Exploring different theoretical frontiers for different (and uncertain) possibilities in mathematics education research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 128–155). National Council of Teachers of Mathematics.

Wiest, L. R. (2021). Prying open the closet door: A call for research on sexual identity and mathematics education. *Mathematics Education Research Journal*, *33*(4), 641–650. https://doi.org/10.1007/s13394-021-00397-4

AESTHETIC AND AFFECTIVE DIMENSIONS OF MATHEMATICS LEARNING

V. Rani Satyam
Virginia Commonwealth University
vrsatyam@vcu.edu

Leslie Dietiker
Boston University
dietiker@bu.edu

Meghan Riling
Vanderbilt University
meghan.riling@vanderbilt.edu

Students' aesthetic and affective responses are interrelated and both central to mathematics learning. This new working group will bring together researchers as a community to explore the connection between affect and aesthetic, and how this connection can help to understand how students experience mathematics. The goals of this working group are to evaluate the state of the field, build shared terms, and identify research questions for further inquiry.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Curriculum; Instructional Activities and Practices

Theoretical Background

The role of affect in mathematical experiences and learning has become abundantly clear over time (e.g., Cheeseman & Mornane, 2014; DeBellis & Goldin, 2006; Malmivuori, 2001). *Affect* has been defined as all aspects of experience that involve feeling (McLeod, 1988); this ranges from deeply held, long duration constructs such as beliefs, attitudes, math anxiety, and motivation, to shorter-term and in-the-moment feelings such as emotions and engagement (Grootenboer & Marshman, 2016; McLeod, 1992; Middleton et al., 2017). In math, where success versus failure is starkly visible, students' affective responses can be quite strong (Boaler, 2015) and impactful (Grootenboer & Marshman, 2016; Op 't Eynde et al., 2006).

Yet, similar to how we would also examine a piece of art for explanations for an individual's gasp in a museum, we argue that researchers also need to attend to the nature of the mathematical experience for explanations for how it potentially impacted its students (e.g., inspiring a question). We refer to the way a lesson supports the felt impulses that compel (or impede) a student to continue to progress (or not) through an experience as its *aesthetic dimensions* (Dietiker, 2015). Some researchers have recently begun to study the aesthetic potential of mathematical learning environments (e.g., Dietiker, 2016; Sinclair, 2001), learning for example how the design of technological tools can offer surprise and appeal (Sinclair et al., 2009). While still emerging, the field is learning how to design and enact what Sinclair (2001) calls "aesthetically-rich" mathematical experiences, which she describes as those that "enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride" (p. 26).

This working group will explore the linkage between affect and aesthetic to tackle multiple dimensions of one of the most significant dilemmas facing mathematics classroom practice today: poor student experiences with mathematics that fuel negative student dispositions. Through our working group design, we plan to bring together graduate students, early career and senior faculty together, to solidify the current state of the field and help form informal and formal connections for future research endeavors.

Working Group Goals and Strategies

The goals of this working group are threefold: (a) we seek to bring together researchers in order to evaluate the state of the field in regards to aesthetic and affective issues, (b) we aim to build community, through structured but also informal conversation and sharing each other's work, and (c) we will build shared terms (given the various constructs for similar ideas) and articulate common research questions, to push for inquiry in these areas.

This is a new working group to address the continued need for focus on students' mathematical dispositions, and the potential enabled by identifying the shared interests in aesthetics and affect. Our aim is to enable new solutions to a persistent problem that can become possible by bringing together these two domains. We plan to continue this working group next year, expanding our focus to useful tools and methodologies.

Session Organization

Prior to the conference, we will survey our participants to gather information on their research interests specific to aesthetics and affect, student age group (elementary, secondary, postsecondary), and whether they are already doing work in this area or interested in it as a new topic. We will welcome participants to bring an artifact they would like to work on with the group for feedback, e.g., a task they wish to refine, to improve the aesthetic or affective response.

Day 1: We will start by having participants introduce themselves and share their research interests (15 min). We will give two short interactive presentations, one on aesthetics and one on affect, with whole-group discussion on the linkages for 60 minutes. We'll end with an overview of the goals of this working group and will ask participants to bring an item they are working on or would like to work on, to share for the next session (15 min).

Day 2: We will form mixed subgroups based on the research foci shared on Day 1. These groups will analyze artifacts shared by participants to answer the question: *How do these two perspectives (affect and aesthetic) inform what we know about this artifact?* (45 min) Within each subgroup, we allot 45 minutes for participants to share an item they've been working on or their research interests and goals. To enable participants to meet each other, learn each other's interests, bond informally, we will then have a jigsaw (reforming new groups) to share what was learned by considering multiple perspectives. An underlying goal is for participants to notice the common ideas across researchers, via differently named constructs.

Day 3: Having shared our interests in the previous session, this day will focus on forming directions for the future. For the first 30 min, we will again form subgroups where participants brainstorm important research questions and potential future directions for this collective group. Each group will type into a shared Google Doc, for easy sharing with the whole group (40 min). We will maintain a Google Drive folder, where the Google Docs will be kept as a record after the conference. There will also be a folder for participants to share their published work with the group to facilitate spread of ideas–especially from new scholars. We will provide space for participants to list their names and contact information next to any research questions/directions they'd like to pursue. In the last 20 minutes, we will present our plans for the future of the working group, to focus on tools, instruments, methodologies, etc. helpful for studying the affective and aesthetic dimensions of mathematical learning.

References

Boaler, J. (2015). *What's math got to do with it?: How teachers and parents can transform mathematics learning and inspire success*. Penguin.

Cheeseman, J., & Mornane, A. (2014). Primary students' perceptions of their mathematics learning. In J. Anderson, M. Cavanaugh, & A. Prescott (Eds.), Curriculum in focus: Research guided practice (*Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia*) (pp. 135-142). Sydney: MERGA.

DeBellis, V. A., & Goldin, G. A. (2006). Affect and meta-affect in mathematical problem solving: A representational perspective. *Educational Studies in Mathematics, 63*, 131–147.

Dietiker, L. (2015). What mathematics education can learn from art: The assumptions, values, and vision of mathematics education. *Journal of Education, 195*(1), 1–10.

Dietiker, L., Richman, A. S., Brakoniecki, A., & Miller, E. R. (2016). Woo! Aesthetic variations of the "same" lesson. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education (PME-NA)* (pp. 66–73).

Grootenboer, P., & Marshman, M. (2016). The affective domain, mathematics, and mathematics education. In *Mathematics, affect and learning* (pp. 13-33). Springer, Singapore.

Malmivuori, M. L. (2001). The dynamics of affect, cognition, and social environment in the regulation of personal learning processes: The case of mathematics. Doctoral dissertation.

McLeod, D. B. (1988). Affective issues in mathematical problem solving: Some theoretical considerations. *Journal for Research in Mathematics Education*, *19*(2), 134-141.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. *Handbook of research on mathematics teaching and learning*, *1*, 575-596.

Middleton, J. A., Jansen, A., & Goldin, G. A. (2017). The complexities of mathematical engagement: Motivation, affect, and social interactions. *Compendium for research in mathematics education*, 667-699.

Op 't Eynde, P., Corte, E. D., & Verschaffel, L. (2006). "Accepting emotional complexity": A socio-constructivist perspective on the role of emotions in the mathematics classroom. *Educational Studies in Mathematics*, *63*(2), 193-207.

Sinclair, N. (2001). The aesthetic "is" relevant. *For the learning of mathematics*, *21*(1), 25-32.

Sinclair, N., Healy, L., & Sales, C. O. R. (2009). Time for telling stories: Narrative thinking with dynamic geometry. *ZDM*, *41*(4), 441-452.

COMING OUT ON THE OTHER SIDE OF "THE EQUITY GROUP": FROM CHOQUE TO HARMONY

Juanita M. Silva
Texas State University
jsilva@txstate.edu

Melissa Adams Corral
California State University-Stanislaus
madamscorral@csustan.edu

Ricardo Martinez
University of Nebraska-Lincoln
martinez21@unl.edu

Gladys Krause
William & Mary
ghkrause@wm.edu

Marrielle Myers
Kennesaw State University
mmyers22@kennesaw.edu

Luz A. Maldonado Rodríguez
Texas State University
l.maldonado@txstate.edu

Joel Amidon
University of Mississippi
jcamidon@olemiss.edu

This working group, which is a continuation of a regular working group space dedicated to goals of equity at PME-NA since 2009, invites scholars actively engaged in antiracist research and teaching practices to come together to strategize around how to respond to choques and dissonance to work towards harmony. This year, the aim is to dismantle or deconstruct instances of dissonance and harmony that have emerged from a choque, or a confrontation within our field.

Keywords: Systemic Change, Equity, Inclusion, and Diversity

> Choques represent both moments of contestation and creative production. (Torre & Ayala, 2009, p. 390)

Our work in this session aims to provide a space for discussing a choque, an instance when we crash into something in our field and have to find new ways to think, act, and be. Choques could include revisions on a paper, moments in the classroom, interactions within a department or with colleagues, hitting a research wall, or recent events affecting your work because of the pandemic. We are also opening up a space for sharing moments of joy, especially those arising after a choque; those moments that help you find equilibrium.

This working group is designed as a space for any who attend to engage in reciprocal practices and relationship building. There is an explicit expectation that all those who attend this working group will come prepared to share artifacts of their current efforts and also learn from and through the efforts of other attendees. To be explicit, the outcome is not to produce a product but to make the space for the building of relationships, strategic knowledge, and awareness of those looking to sustain this work. Moving from dissonance to harmony requires collectivity and relationality--you cannot form a harmony with only one voice. As such, relationships are central to the formation and sustainability of research collectives engaging in antiracist research and teaching that seeks to make systemic change.

Within a context that includes the never-ending COVID-19 pandemic, our ability to plan and to go about things as usual seems more and more absurd. Yet, we also watched society quietly step back from the reckonings of Spring 2020 in the face of an ongoing white backlash (Gomez et al., 2022; Anderson, 2016). In other words, in the face of growing recognition that we cannot go on in this way, rather than leading to harmony, some *choques* lead us right back to a status quo of dissonance. This reflects the need for us to embrace *choques*, to challenge ourselves to push through dissonance and discomfort with intention, commitment, and actions towards something new.

History of the Working Group

Originally created as a space to bring attention to issues of equity and diversity to mathematics teaching and learning (Foote et al., 2009), this working group has evolved as the field of mathematics education has changed over the past 13 years. Part of the evolution, as well as the continued need to hold a space at yearly PME-NA conferences, is evident in the shared leadership of this group since its initial meeting in Atlanta in 2009. After one year of interruption in 2020, the work of building space to discuss making change and working towards justice in mathematics education was renewed at PME-NA 43. Presently, the scholars involved in this group have emphasized a focus on bringing together PME-NA attendees committed to antiracism.

Organization of the Working Group

The organization of the working group embraces *choques* through pivots between harmony and dissonance. Each session will build on each other with the following format:

- Session 1-Artifact Unpacking: The goal of this session is to share and learn from artifacts of *choques*, either those resulting in present dissonance, or those that eventually led to a sense of harmony. Participants are expected to bring an artifact (e.g., an example of a *choque* that they either are or have been working through). Through structured collaboration, participants will share the artifact, describing the *choque* and where it has led them. In this session, we hope to help participants work towards initiating their goal (i.e., write an article addressing an equity issue, build a collaborative mentorship team to address social justice issues; discuss an article to gain and share knowledge) Finally, questions of interest will be collected, organized, and posted for use during the following sessions of the working group.
- Session 2-Interconnectedness & Common Ground: Using participant-generated goals emerging from their artifacts, participants will be organized into small groups to engage, share knowledge and practices, and imagine new possibilities. Small groups will engage in structured activities to help synthesize thinking around their artifacts towards productive collaborations and plans of action.
- Session 3-Encouragement & Action: The final day of the working group will be a space for working group participants to continue the work of session two and more open-ended conversation and dialog on areas of interest. Participants will be encouraged to talk about next steps, with the emphasis being this is only the start of the conversation, but the connections made extend beyond the limits of any conference and organization

References

Anderson, C. (2016). White rage: The unspoken truth of our racial divide. Bloomsbury Publishing USA.

Foote, M. Q., Wager, A., Bartell, T., Edwards, A. (2009). Addressing equity and diversity issues in mathematics education. In. S. L. Swars, D. W. Stinson, and S. Lemons-Smith (Eds.) Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Georgia State University.

Gómez, R. F., Guy, M. C., & Cammarota, J. (2022). The psychological rages of Whiteness: acceptable casualties and the White American identity. Ethnic and Racial Studies, 1-20.

Torre, M. E., & Ayala, J. (2009). Envisioning participatory action research entremundos. Feminism & Psychology, 19(3), 387-393.

REPORT ON 2022 WORKING GROUP: CREATING SPACE FOR PRODUCTIVE STRUGGLE TOWARD A MORE EQUITABLE FUTURE

Ricardo Martinez
University of Nebraska - Lincoln
rmartinez21@unl.edu

Melissa Adams Corral
The Ohio State University
adams.2153@osu.edu

Luz A. Maldonado Rodríguez
Texas State University
l.maldonado@txstate.edu

Gladys Krause
William & Mary
ghkrause@wm.edu

Marrielle Myers
Kennesaw State University
mmyers22@kennesaw.edu

Joel Amidon
University of Mississippi
jcamidon@olemiss.edu

REPORT ON: This working group is a consistent space for equipping, informing, and challenging mathematics education researchers to "frame equity as a continually evolving process of growth". Since its inception this working group has continued to productively orient, inspire, and organize mathematics education researchers to move toward outcomes in our field that prioritize anti-racist mathematics education as a mechanism for change. The challenge of this working group remains one of moving from collective reflections around issues of equity and diversity in mathematics education to actions that become catalysts for change. We acknowledge that this year's call for "productive struggle" is necessary, however it needs people and community to support such efforts.

Keywords: Equity, Inclusion, and Diversity, & Social Justice

"Healing is a process that we navigate for a lifetime." (Ginwright, 2018, p. 6)

This working group is a consistent space for equipping, informing, and challenging mathematics education researchers to "frame equity as a continually evolving process of growth". Since its inception, this working group has continued to productively orient, inspire, and organize mathematics education researchers to move toward outcomes in our field that prioritize anti-racist mathematics education as a mechanism for change. The challenge of this working group remains one of moving from collective reflections around issues of equity and diversity in mathematics education to actions that become catalysts for change. The challenge of equity work is that it is contextual and complex. More so, the solutions are rarely simple and require sustained commitment to collective action.

The Equity Working group seeks to push back on traditional views of measuring equity. Insofar to say that no one outcome can ever lead to equity. We are not defined by the products we create but instead we create change by being in community with each other. Equity becomes a reality for all through collective action. The Equity Working Groups is needed each year because each year the participants are different and the issues of inequity present in both mathematics education and in the greater North American context is always in flux. As the opening quote to this report states "healing is a process that we navigate for a lifetime" (Ginwright, 2018, p. 6) and this working groups is one way we can consistently navigate challenging and changing inequities in the field. This report will restate a brief history of the working group followed by an overview of the previous three years. Additionally, will point to the future of the Equity Working Group.

A History of the PMENA Equity Working Group

This Working Group originates from the Diversity in Mathematics Education (DiME) Group, one of the Centers for Learning and Teaching (CLT) funded by the National Science Foundation (NSF). DiME scholars graduated from one of three major universities (University of Wisconsin-Madison, University of California-Berkeley, and UCLA) that comprised the DiME Center. The Center was dedicated to creating a community of scholars poised to address critical problems facing mathematics education, specifically with respect to issues of equity (or, more accurately, issues of inequity).The DiME Group (as well as subsets of that group) has engaged in important scholarly activities, including the publication of a chapter in the *Handbook of Research on Mathematics Teaching and Learning* which examined issues of culture, race, and power in mathematics education (DiME Group, 2007), a one-day AERA Professional Development session examining equity and diversity issues in mathematics education (2008), a book on research of professional development that attends to both equity and mathematics issues with chapters by many DiME members and other scholars (Foote, 2010), and a book on teaching mathematics for social justice (Wager & Stinson, 2012) that also included contributions from several DiME members. In addition, several DiME members have published manuscripts in a myriad of leading mathematics education journals on equity in mathematics education. This working group provides a space for continued collaboration among DiME members and other colleagues interested in addressing the critical problems facing mathematics education.

It is important to acknowledge some of the people whose work in the field of diversity and equity in mathematics education has been important to our work. Over time, the Working Group has encouraged building on and featuring senior scholars' work, including Jo Boaler (Boaler, 2002), Walter Secada (Secada, 1992), Marta Civil (Civil, 2007; Civil & Bernier, 2006; González, Andrade, Civil, & Moll, 2001), Eric Gutstein (Gutstein, 2003, 2006; Gutstein & Peterson, 2013), Jacqueline Leonard (Leonard, 2007; Leonard & Martin, 2013), Danny Martin (Martin, 2000, 2009, 2013), Judit Moschkovitch (Moschkovich, 2002), Rochelle Gutiérrez (2002, 2003, 2008, 2012, 2013) and Na'ilah Nasir (Nasir, 2002, 2011, 2013; Nasir, Hand & Taylor, 2008; Nasir & Shah, 2011). We have as well been building on the work of our advisors, Tom Carpenter (Carpenter, Fennema, & Franke, 1996), Geoff Saxe (Saxe, 2002), Alan Schoenfeld (Schoenfeld, 2002), Megan Franke (Kazemi & Franke, 2004), and Anita Wager, (Wager & Stinson, 2012) as well as many others outside of the field of mathematics education.

PMENA 2019: Heal, Connect, Advance

During the 2019 PMENA working group we began by sharing three words; Heal, Connect, and Advance. These three words became the underpinnings of our activities. During the first session we asked participates to brainstorm what additional words are missing from the three we shared. We then had participants discuss the following question:

> What are some examples, movements, policies, or directions that have occurred in mathematics education that some may have intended as expansion, displacement, or growth that have actually served to privilege some and marginalize others?

Both questions were used to pull generative themes from those attending the working group. Generative themes were used the second day to help groups identify the root cause(s) of the themes they identified. The activity allowed participants to co-construct knowledge together making it easier for them to connect with each other and provide outlets to advance their work related to equity. Figure 1 provides two examples of work done during the conference.

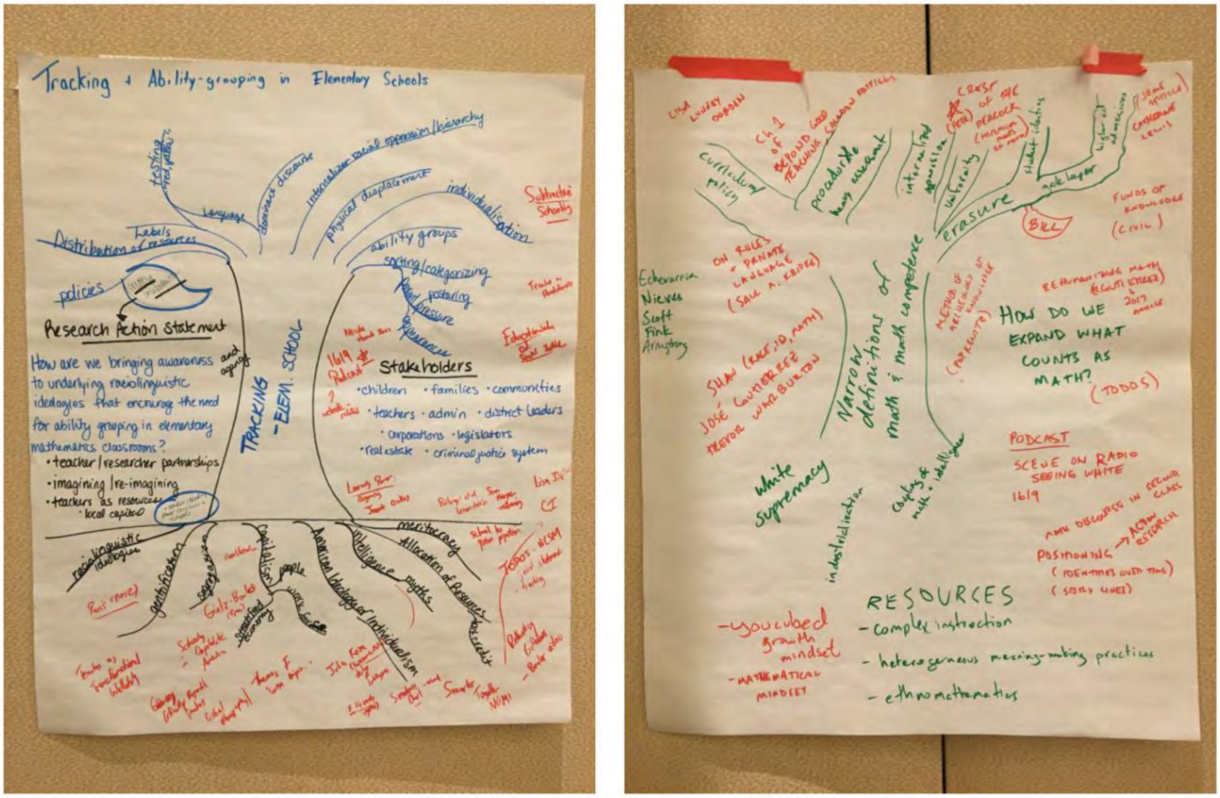


Figure 1: PMENA 2019 Problem Trees]

PMENA 2021: Joy as a Form of Resistance

Building on the work of PMENA 2019, we wanted to continue to advance an ecology of healing by centering Joy as a catalyst for action. We began the first day by asking participants to share what brings them joy because as Desmond Tutu and the Dalai Lama explain in The Book of Joy, “being joyful is not just about having more fun. We’re talking about a more empathetic, more empowered, even more spiritual state of mind that is totally engaged with the world” (2016, p. 63). For those engaged in challenging inequities in academia, moments of resistance are examples of experienced joy. On the second day of the working group, we focused on how the theories we use in mathematics education can and should align to what we value. Multiple theories, traditionally not seen in mathematics education were shared with participants as they were left with these questions:

- What might that have to do with our work as researchers?
- What does it mean for us as researchers to respect ourselves and recognize our power, as well as respect those who choose to work with us?
- What does giving AND receiving mean in our roles and in our work? Who gets to determine that?
- Our research is historically situated, as are we--what work are we continuing? Whose work are we continuing? Why? To what end?
- If resilience is collective, shared between us and others, what does that look like when we are acting as researchers? When we are working with teachers and pre-service teachers?

PMENA 202X: A Sustained Movement

Build on the previous year it is evident that the equity working group (and all working groups) should embrace a virtual component. We seek to continue to provide both a virtual and physical space for people to engage in cognitive dissonance around understanding equity in mathematics education. Harmony begins when multiple voices come together to challenged and ultimately change issues of inequity. The Equity Working Groups seeks to sustain itself by continuing the working done at PMENA, for PMENA members.

References

Boaler, J. (2002). Paying the price for" sugar and spice": Shifting the analytical lens in equity research. *Mathematical Thinking and Learning*, *4*(2-3), 127-144.

Cammarota, J., Berta-Ávila, M., Ayala, J., Rivera, M., & Rodriguez, L. (2016). PAR Entremundos: A practitioner's guide. *Growing critically conscious teachers: A social justice curriculum for educators of Latino/a youth*, 67-89.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal, 97*(1), 3-20.

Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nassir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom.* (pp. 105-117). New York, NY: Teachers College Press.

Civil, M., & Bernier, E. (2006). Exploring images of parental participation in mathematics education: Challenges and possibilites. *Mathematical Thinking and Learning, 8*(3), 309-330.

DiME. (2007). Culture, race, power, and mathematics education. In F. Lester (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (Second ed., pp. 405-433). Charlotte, NC: Information Age Publishing.

Foote, M. Q. (Ed.). (2010). *Mathematics teaching & learning in K-12: Equity and professional development*. New York: Palgrave.

Ginwright, S. (2018). The future of healing: Shifting from trauma informed care to healing centered engagement. *Occasional Paper*, *25*, 25-32.

González, N., Andrade, R., Civil, M., & Moll, L. (2001). Bridging funds of distributed knowledge: Creating zones of practices in mathematics. *Journal of Education for Students Placed at Risk, 6*(1&2), 115-132.

Gutiérrez, R. (2002). Enabling the practice of mathematics teachers: Towards a new equity research agenda. *Mathematical Thinking and Learning, 4*(2&3), 145-187.

Gutiérrez, R. (2003). Beyond essentializing: The complex role of language in teaching Latina/o students mathematics. *American Educational Research Journal, 39*(4), 1047-1088.

Gutiérrez, R. (2008). A "gap gazing" fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education, 39*(4), 357-364.

Gutiérrez, R. (2012). Embracing "Nepantla:" Rethinking knowledge and its use in teaching. *REDIMAT-Journal of Research in Mathematics Education, 1*(1), 29-56.

Gutiérrez, R. (2013). Why (Urban) mathematics teachers need political knowledge. *Journal of Urban Mathematics Education, 6*(2), 7 – 19.

Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education, 34*(1), 37-73.

Gutstein, E. (2006). *Reading and writing the world with mathematics*. New York: Routledge.

Gutstein, E., & Peterson, B. (Eds.). (2013). *Rethinking mathematics: Teaching social justice by the numbers* (2nd Ed.). Milwaukee, WI: Rethinking Schools, Ltd.

Heid, M. K. (2010). Where's the math (in mathematics education research)? *Journal for Research in Mathematics Education, 41*, 102–103.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*, 203-235.

Leonard, J. (2007). *Culturally specific pedagogy in the mathematics classroom: strategies for teachers and students.* New York, NY: Routledge.

Leonard, J., & Martin, D. B. (Eds.) (2013). *The brilliance of Black children in mathematics: Beyond the numbers and toward new discourse.* Charlotte, NC: Information Age Publishers.

Martin, D. B. (2000). *Mathematics success and failure among African American youth: The roles of sociohistorical context, community forces, school influence, and individual*

agency. Mahwah, NJ: Lawrence Erlbaum Assoc. Inc.
Martin, D. B. (2009). *Mathematics teaching, learning, and liberation in the lives of Black children.* London: Routledge.
Martin, D. B. (2013). Race, racial projects, and mathematics education. *Journal for Research in Mathematics Education*, *44*(1), 316-333.
Martin, D. B., Gholson, M. L., & Leonard, J. (2010). Mathematics as gatekeeper: Power and privilege in the production of knowledge. *Journal of Urban Mathematics Education, 3*(2),12–24.
Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning, 4*(2&3), 189-212.
Nasir, N. (2011). *Racialized Identities: Race and achievement for African-American youth.* Stanford University Press.
Nasir, N. S. (2002). Identity, goals, and learning: Mathematics in cultural practice. *Mathematical Thinking and Learning, 4*(2/3), 213-247.
Nasir, N. S. (2013). *Why should math educators care about race and culture?* Plenary address at the 35th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education. Chicago, Il.
Nasir, N., Hand, V., & Taylor, E. (2008). Relevant knowledge in school mathematics: Boundaries and intersections between cultural and domain knowledge in mathematics classrooms. *Review of Research in Education, 32*(1), 187-240.
Nasir, N. & Shah, N. (2011). On Defense: African American males making sense of racialized narratives in mathematics education. *Journal of African American Males in Education, 2*(1), 24-45.
Parks, A. N. (2020). Creating Joy in PK–Grade 2 Mathematics Classrooms. *Mathematics Teacher: Learning and Teaching PK-12*, *113*(1), 61-64.
Saxe, G. B. (2002). Children's developing mathematics in collective practices: A framework for analysis. *The Journal of the Learning Sciences, 1*(2&3), 275-300.
Schoenfeld, A. H. (2002). Making mathematics work for all children: Issues of standards, testing and equity. *Educational Researcher, 31*(1), 13-25.
Secada, W. G. (1992). Race, ethnicity, social class, language, and achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics*(pp. 623-660). New York, NY, England: Macmillan Publishing Co, Inc.
Wager, A. A., & Stinson, D.W. (Eds.). (2012). *Teaching mathematics for social justice: Conversations with educators*. Reston, VA: National Council of Teachers of Mathematics.

CONTINUOUS IMPROVEMENT LESSON STUDY WORKING GROUP REPORT

REPORTE DEL GRUPO DE TRABAJO EN LA MEJORAMIENTO CONTINUA DEL ESTUDIO DE LECCIONES

Dittika Gupta
Midwestern State University
dittika.gupta@msutexas.edu

Lara K. Dick
Bucknell University
lara.dick@bucknell.edu

Melissa M. Soto
San Diego State University
melissa.soto@sdsu.edu

Mollie H. Appelgate
Iowa State University
mollie@iastate.edu

This is a report on the working group aimed at providing a structure for Mathematics Teacher Educators (MTEs) to engage in Continuous Improvement Lesson Study (CILS), a process of MTE professional development that involves working collaboratively to design a lesson, improve preservice teacher (PST) learning, and MTE practice.

Background

The CILS process integrates lesson study (Lewis & Hurd, 2011) and the Continuous Improvement process (Berk & Hiebert, 2009). Lesson study is a collaborative process of investigating instruction with the goal of improving student learning through a single lesson. This teacher-driven professional development focuses on analyzing student learning and leads to enhanced instructor knowledge (e.g. Watanabe, 2004; Demir et al., 2013). Hiebert and colleagues at the University of Delaware developed a model "continuous improvement" for MTEs to reflect through a process of "planning, enactment, analysis, and revision" (Berk & Hiebert, 2009, p. 339). We have integrated these two methods of researching and improving instruction and student outcomes, lesson study and continuous improvement to inform an integrated process that we call Continuous Improvement Lesson Study (CILS) and is shown in Figure 1.

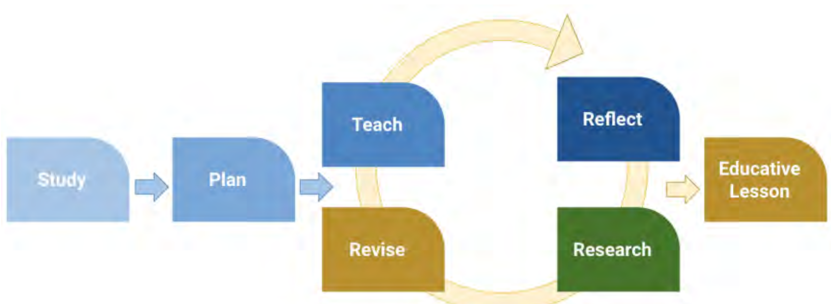


Figure 1: The Continuous Improvement Lesson Study (CILS) Process

Working Group Summary

We began the working group by discussing the CILS along with how we as MTEs improved our own practice while engaging in the process. Because CILS serves as professional development for MTEs, we had participants complete a Jamboard noting challenges they face in teaching preservice teachers (Figure 2) and used the responses when discussing the CILS process to illustrate how CILS helped us address these challenges in our own classroom.

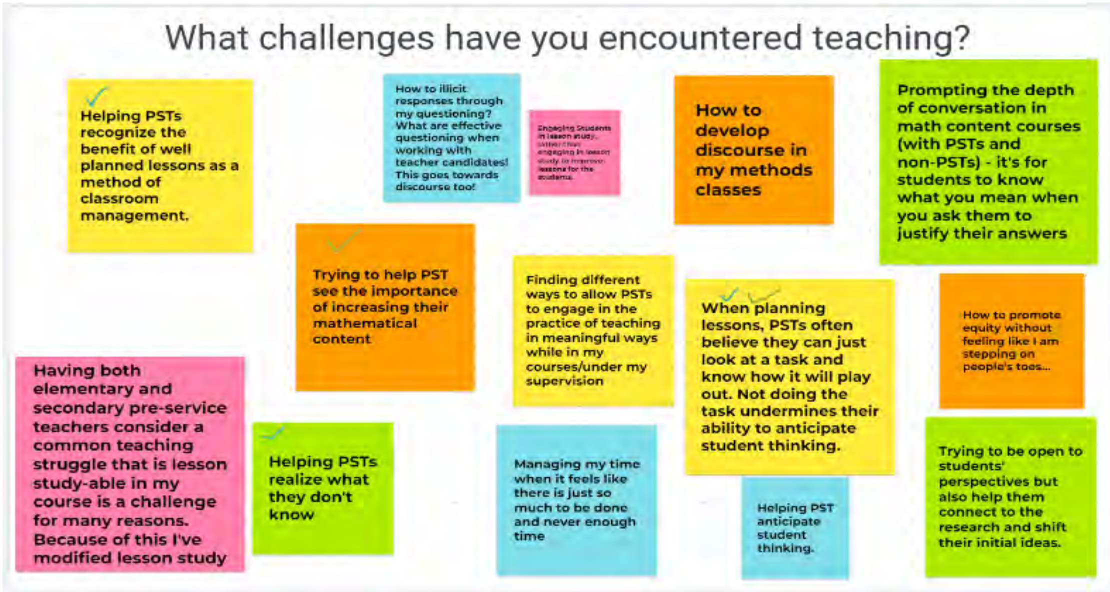


Figure 2: Brainstorming from Participants on the first day of the Working group

The expressed goal of the working group was to have mathematics teacher educators form CILS teams to study a research-based lesson MTEs taught to preservice teachers within their methods or content courses. While the participants were interested in our work and asked insightful questions about the process, it quickly became evident that most of the participants who attended the working group were interested in having their preservice teachers (PSTs) act as teachers while engaging with a lesson study with children and not in using CILS as a means to support their own professional development.

Because of the diverse interests related to lesson study, we invited the working group participants to share how their work intersects with lesson study more broadly. We had good conversations about lesson study, the various ways lesson study has been adapted in higher education, and let the participants know our goal was still to work on CILS in preservice teacher coursework. For the second day, only one participant interested in working with CILS returned to the working group. Our team had conversations with the participant about their work but did not form a CILS team with the participant since the participant was not attending the third day of the conference.

Challenges

Due to the Covid-19 pandemic, our working group members chose to present virtually but with the option of having conference participants join virtually or participate in person while at the conference. We believe that presenting online was one of the main reasons for the limited participation within the working group. When people walked by the working group room, there were no presenters standing at the front or outside the room to share more about the aims of the session and we were not physically at the conference which greatly decreased networking we had hoped to do to share more about our working group. Thus unfortunately, we were unable to achieve the set goals of the working group.

New Modifications Proposed

We propose two modifications moving forward for a refocused working group proposal for PME-NA 44 in Nashville, 1) partner with other researchers who engage in lesson study and 2) broaden the theme of the working group to be around lesson study adaptations, including those that support equity. New collaborators that have expressed interest in the newly adapted working group include Dr. Catherine Lewis, Professor at Mills College and President of the World Association of Lesson Study (WALS), and a leading U.S. expert in lesson study; Dr. Susie Hakansson, Mathematics Education Consultant and member of The California Action Network for Mathematics Excellence and Equity (CANMEE), and other members of the CANMEE network. CANMEE uses lesson study to improve mathematics learning, instruction, and the lesson study process with an equity lens.

The adaptations are meant to address the wide variety of lesson study participants from different levels, such as PSTs and in-service teachers conducting lesson study with children, to mathematics teacher educators engaging in lesson study with other higher education faculty members as a form of professional development (Appelgate et al., 2020; Soto et al., 2019). Some specific adaptations specific to equity that will be discussed include the addition of an equity commentator role in lesson study as well as selecting focal students to monitor and assess student learning, teaching mathematics through problem-solving, and developing and revising lessons focused on teaching mathematics for social justice and analyzing student and teacher learning.

References

Appelgate, M. H., Dick, L., Soto, M., & Gupta, D. (June 2020). Growing a greater understanding of multiplication through lesson study: Mathematics teacher educators' professional development. In A. Appova, R. M. Welder, and Z. Feldman, (Eds.), Supporting Mathematics Teacher Educators' Knowledge and Practices for Teaching Content to Prospective (Grades K-8) Teachers. Special Issue: *The Mathematics Enthusiast*, *17*(2 & 3), pp. 583–613. ScholarWorks: University of Montana. Retrieve (open access) from: https://scholarworks.umt.edu/tme

Berk, D., & Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers, one lesson at a time. *Teachers and Teaching: Theory and Practice*, *15*(3), 337-356.

Demir, K., Czerniak, C. M., & Hart, L. C. (2013). Implementing Japanese lesson study in a higher education context. *Journal of College Science Teaching*, *42*(4), 22.

Lewis, C. C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction.* Portsmouth, NH: Heinemann.

Soto, M., Gupta, D., Dick, L., & Appelgate, M. (2019). Bridging distances: Professional development for higher education faculty through technology-facilitated lesson study. *Journal of University Teaching & Learning Practice*, *16*(3). Retrieved from https://ro.uow.edu.au/jutlp/vol16/iss3/7/

Watanabe, T. (2018). Japanese lesson study in the United States. *Educational Designer*, *3*(11), 1-13.

ADAPTATIONS TO LESSON STUDY TO SUPPORT EQUITABLE TEACHING PRACTICES WORKING GROUP

Melissa M. Soto
San Diego State University
melissoto@gmail.com

Dittika Gupta
Midwestern State University
dittika.gupta@msutexas.edu

Catherine C. Lewis
Mills College
clewis@mills.edu

Susie W. Håkansson
TODOS: Mathematics for ALL
shakans@g.ucla.edu

Mollie H. Appelgate
Iowa State University
mollie@iastate.edu

Lara K. Dick
Bucknell University
lara.dick@bucknell.edu

This working group is a follow-up initiative that will allow the mathematics education community to engage with adaptations of lesson study designed to address equity in the mathematics classroom. This year we will examine four adaptations (Teaching Through Problem-solving, Focal Students and Equity Commentator from the California Action Network for Mathematics Excellence and Equity (CANMEE), Rights of the Learner, and Continuous Improvement Lesson Study (CILS)), identifying impacts on equitable teaching practices (i.e, practices that involve all students in cognitively demanding mathematics). Participants will view a video-recorded research lesson that grounds the discussion of the adaptations. The group aims to publish an edited volume on equity-focused lesson study, with a broader goal of supporting participants to develop lesson study that integrates equity across different contexts (e.g. K-12 teachers, preservice teachers, graduate students, and mathematics teacher educators).

Keywords: Equity, Inclusion, and Diversity, Teacher Educators, Professional Development

Lesson Study

Lesson study (LS) is a collaborative process of investigating instruction that serves as a teacher-driven professional development that focuses on analyzing student learning and leads to enhanced instructor knowledge (Watanabe, 2018). LS traditionally consists of four phases: 1) study the curriculum and formulate goals, 2) plan the lesson, 3) teach the researched lesson, and 4) reflect on learning (Lewis & Hurd, 2011) and has been used both in K-12 and post-secondary classrooms. In the introduction to their 2019 book on LS in mathematics, Huang et al. (2019) noted that the field has not yet published "how to use LS as a tool to address equity of student learning opportunity" and called for this focus (p. 10). This working group will examine the following four equity-focused LS adaptations.

Teaching Through Problem-solving

In Teaching Through Problem-solving, widely used in Japan, *students* develop each new mathematical concept and procedure in the curriculum, by solving a problem that requires the new concept/procedure (Takahashi, 2021). Student reflective mathematics journals, organized board use and student-led discussion routines make student thinking visible and actionable during instruction, allowing students to take agency as authors (not simply consumers) of mathematics (LSGAMC, 2022). School-wide LS joined with Teaching Through Problem-solving has shown a strong positive impact on the mathematics achievement and perseverance of historically underserved learners (Lewis et al., 2022a,b).

California Action Network for Mathematics Excellence and Equity (CANMEE)

CANMEE incorporates equity into the lesson study process by including focal students and equity commentators. Selecting, interviewing, and writing assets-based descriptions of focal students allows teachers to learn about and know their students. Attending to focal students operationalizes equity in the classroom by increasing students' mathematical agency and identity, transforming teachers' beliefs and perceptions of focal students, shifting teachers' instructional practices (Bartell et al., 2017) that impact all students, and creating a culture of professionalism. Equity commentators bring new knowledge to the team, connect theory and practice, make claims about teaching and learning, and model reflection on teaching and learning. They focus on focal students' access to high cognitively demanding grade level mathematics. Emerging evidence suggests impact on both students and teachers.

Rights of the Learner (RotL) Framework

Rights of the learner framework is based on the premise of students engaging in democratic sharing of ideas (Kalinec-Craig, 2017; Torres, 2020). The four Torres's RotL are: "1) the right to be confused; 2) the right to claim a mistake and revise the thinking, 3) the right to speak, listen, and be heard; and 4) the right to write, do and represent what makes sense to them" (Kalinec-Craig, 2017, p. 1). By focusing on the rights of the learner during LS, a teacher provides a more equitable classroom environment where students feel comfortable engaging in the content through their own unique understanding lens.

The Continuous Improvement Lesson Study (CILS) Process

The CILS (Dick et al., 2022) process integrates LS (Lewis & Hurd, 2011) and the Continuous Improvement process (Berk & Hiebert, 2009) to support equitable teaching practices of mathematics teacher educators (MTEs). The incorporation of continuous improvement within LS focuses on both preservice teacher learning and lesson revision as a lesson is taught multiple times by each participating MTE. We found that engaging in the CILS process supports MTEs to develop an educative curriculum and collaborative environment that supports equitable teaching.

Working group Organization

This working group is an extension of a previous working group, with additional lesson study researchers and a structure that enables participants to engage with various lesson study adaptations related to equitable teaching practices. Table 1 contains the structure of the working group. The plan is to continue meeting at future PME-NAs to conceptualize and test equity-focused adaptations and share challenges/learnings with the goal of understanding/improving these adaptations and sharing them in an edited volume.

Table 1: Overview of the Proposed Working group Session

	Activities	Take-Aways
Session One	1. Introductions and Agenda 2. History of Lesson Study (LS) with Four Adaptations 3. Questions and Small Group Discussions Regarding Equitable Teaching Practices	1. Resources regarding the four adaptations. 2. Homework (HW): Watch public lesson video
Session Two	1. Debrief videoed public lesson and unpack lesson through the four equity-focused LS adaptations 2. Small group discussions, on chosen adaptation(s)	1. Debrief lesson 2. HW: Draft LS plan with equity-focused adaptation
Session Three	1. Share draft plans for testing equity-focused adaptations through LS implementation 2. Subgroup work time for refinement of plans 3. Subgroup brief share-out 4. Final reflections, future work plan	1. Continue planning for LS implementation 2. Future plans for collaboration and sharing

References

Bartell, T., Wager, A., Edwards, A., Battey, D., Foote, M., & Spencer, J. (2017). Toward a framework for research linking equitable teaching with the standards for mathematical practice. *Journal for Research in Mathematics Education*, 48(1), 7-21.

Berk, D., & Hiebert, J. (2009). Improving the mathematics preparation of elementary teachers, one lesson at a time. *Teachers and Teaching: Theory and Practice*, *15*(3), 337-356.

Dick, L. K., Appelgate, M. H., Gupta, D., & Soto, M. M. (2022). Continuous improvement lesson study: A model of MTE professional development. *Mathematics Teacher Educator*, *10*(2), 111-128.

Huang, R., Takahashi, A., & da Ponte, J. P. (2019). *Theory and Practice of Lesson Study in Mathematics around the World* (pp. 3–12). Retrieved from https://doi.org/10.1007/978-3-030-04031-4_1

Kalinec-Craig, C. A. (2017). The rights of the learner: A framework for promoting equity through formative assessment in mathematics education. *Democracy and Education*, *25*(2), 1-11.

Lewis, C., Henrick, E., Friedkin, S., & McGee, S. (2022a). Model variation in inquiry processes. In D. Peurach, J.L Russell, L. Cohen-Vogel & W. Penuel (Eds.). *The Foundational Handbook on Improvement Research in Education*, 347-372. Rowman & Littlefield Publishers.

Lewis, C., & Hurd, J. (2011). *Lesson Study Step by Step: How Teacher Learning Communities Improve Instruction*. Heinemann.

Lewis, C., Takahashi, A., Friedkin, S., Liebert, S., & Houseman, N. (2022b). Sustained, effective school-wide Lesson Study: How do we get there? *Vietnam Journal of Education, 6*.

LSGAMC [Lesson Study Group at Mills College]. (2022). Teaching through problem-solving: Overview. Retrieved from: https://lessonresearch.net/teaching-problem-solving/overview/

Takahashi, A. (2021). *Teaching Mathematics Through Problem-Solving*. Routledge.

Torres, O. (2020). Equity in education webinar series: rehumanizing schools - Rights of the learner. Retrieved from https://www.youtube.com/watch?v=_UndpNUCAqw&t=3211s

Watanabe, T. (2018). Japanese lesson study in the United States. *Educational Designer*, *3*(11), 1-13.

WORKING GROUP REPORT: USING SELF-BASED METHODOLOGIES TO UNPACK MATHEMATICS TEACHER EDUCATORS' WORK

Elizabeth Suazo-Flores
Purdue University
esuazo@purdue.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Melva Grant
Old Dominion University
mgrant@odu.edu

Jennifer Ward
Kennesaw State University
jward105@kennesaw.edu

Sue Ellen Richardson
University of Indianapolis
srichardson@uindy.edu

Olive Chapman
Calgary University
chapman@ucalgary.ca

In 2021 we met at PME-NA Mexico and Philadelphia. At PME-NA Mexico, the working group met virtually. Facilitators introduced central tenets of the self-based methodologies (SBMs) and relationships between SBMs and MTEs' work (Chapman, 2020). SBMs are useful in describing MTEs' living awareness, living contradictions, empathetic relations, growing relationships, practical knowledge, and change in self and practice (Chapman et al., 2020). Working group facilitators narrated their methodological decisions and provided examples of SBM studies in video form with captions to accommodate English and Spanish speakers. Attendees asked questions about the similarities and differences among SBMs. To close the working group, proposals for PME-NA 43 and the Association of Mathematics Teacher Educators (AMTE) 2022 conference were discussed.

At PME-NA Philadelphia, the working group met in person and virtually. Two facilitators attended in person with the remainder attending virtually. The hybrid modality allowed MTEs attending the conference to meet virtually as needed alongside MTEs who were attending the conference in person. Facilitators shared the history of our working group and our collaborative project to create a special issue for the *Philosophy of Mathematics Education Journal*. The primary goal of the special issue is to create views of MTE work, their practices and from their unique perspectives, while the editors will focus on ways to encourage discussion about the philosophical underpinnings of the MTEs' stories, narratives, and empirical studies. Researchers using SBMs study their lived experiences by writing stories (i.e., using narrative inquiry), identifying a practice they want to improve (i.e., using self-study), or unpacking the influence of the socio-cultural and political contexts in their work (i.e., using autoethnography). We hypothesize that SBMs emphasize self-understanding, self-exploration, and contribute to understanding, conceptualizing, and improving MTE practices. SBM research processes produce empirical research about MTE practices while supporting MTE professional learning and unearthing how MTE knowledge, instructional approaches, beliefs, and identity inform their relationships with mathematics, pedagogy and practice.

In preparation for the 2021 working group, we reviewed empirical studies using SBMs published between 2014 and 2021 to identify themes in findings. Findings from the SBM studies included two themes: “exploring MTEs’ work in different career stages” and “MTEs incorporating new practices” (Suazo-Flores et al., 2022, under review). These themes documented how MTEs’ knowledge is constituted and informed by experiences, and is therefore dynamic, constantly evolving according to the situation and time.

As a research group, we have been using conference spaces, such as PME-NA and AMTE, to build a critical mass of scholars using SBMs and to invite others to explore the possibilities of SBMs for their inquiry. Drawing from a review of the use of SBMs (Chapman et al., 2020), the working group is identifying how findings from the use of SBMs inform the field of mathematics teacher education. Our goal is to unpack how the use of SBMs benefits MTEs' development and contributes knowledge to the field of mathematics teacher education.

Early reports on MTE practice (Chauvot, 2009; Tzur, 2001) described what MTEs know and how they develop their knowledge. Yet, *how* MTEs know in their interactions with teachers is less studied. In 2008, Jaworski stated that "co-learning between teachers and educators can be seen as the way ahead for developing mathematics teacher education practice" (p. 11). Within this context, we portray MTEs as learners of teaching about teaching mathematics. In conducting studies using SBMs, MTEs document interactions with teachers that yield new practices and intellectual growth, contributing knowledge to mathematics teacher education. When conducting such studies, MTEs draw from either implicit or explicit philosophical considerations (Stinson, 2020). Our task at PME-NA Philadelphia was to invite MTEs to describe their ways of knowing in interactions to enable the facilitators of the working group to hypothesize about MTEs' philosophical positions.

At PME-NA Philadelphia, attendees analyzed data from one facilitator's study. The goal of the activity was to identify evidence of ways of acting and being with a teacher candidate. Working group members' analysis of the data resulted in an awareness of MTE's ways of knowing, i.e., how her knowledge of mathematics teacher education and her identity informed her interactions with the teacher candidate. Building from the discussion of the data, attendees shared preliminary topics and data that might be of interest to the working group and for the special issue. Finally, the group closed the meeting with plans for future meetings. In 2022 we will meet at the AMTE conference. Since most PME-NA 43 attendees were interested in contributing articles for the special issue at the *Philosophy of Mathematics Education Journal,* facilitators created writing groups to meet bi-monthly, virtually between the PME-NA 43 meeting and the AMTE 2022 meeting. In addition, facilitators have been meeting weekly to develop a timeline and plan for the special issue.

References

Chapman, O. (2020). Mathematics teacher educators' use of narrative in research, learning and teaching. *For the Learning of Mathematics, 40*(0), 21-27.

Chapman, O., Kastberg, S. E., Suazo-Flores, E., Cox, D., & Ward, J. (2020). Mathematics teacher educators' inquiry into their practice. In Beswick, K. & Chapman, O. (Eds.), *International Handbook of Mathematics Teacher Education*. 2nd Edition. Volume 2: The mathematics teacher educator as a developing professional (pp. 157-187). Brill-Sense Publishers.

Chauvot, J. B. (2009). Grounding practice in scholarship, grounding scholarship in practice: Knowledge of a mathematics teacher educator–researcher. *Teaching & Teacher Education*, *25*(2), 357-371. https://doi.org/10.1016/j.tate.2008.09.006

Guba, E. G., & Lincoln, Y. S. (1994). Competing paradigms in qualitative research. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp. 105–117). Sage Publications, Inc.

Jaworski, B. (2008). Development of the mathematics teacher educator and its relation to teaching development. In Jaworski & Wood (Eds.), *International Handbook of Mathematics Teacher Education*: Volume 4 (pp. 333-361). Brill Sense.

Stinson, D. (2020). Philosophical considerations always already entangled in mathematics education research, In Bicudo et al. (Eds.), M*athematics Teaching Research Journal, Special Issue on Philosophy of Mathematics Education, 12*(2), 8-23.

Suazo-Flores, E., Kastberg, S., Grant, M. (2022 under review). Humanizing mathematics teacher educators' work through self-study. *AMTE professional book series, Volume 5*.

Tzur, R. (2001). Becoming a Mathematics Teacher-Educator: Conceptualizing the Terrain through Self-Reflective Analysis. *Journal of Mathematics Teacher Education*, *4*(4), 259-283. https://doi.org/10.1023/A:1013314009952

PHILOSOPHICAL UNDERPINNINGS OF MATHEMATICS TEACHER EDUCATORS' WORK

Elizabeth Suazo-Flores
Purdue University
esuazo@purdue.edu

Signe E. Kastberg
Purdue University
skastber@purdue.edu

Melva R. Grant
Old Dominion University
mgrant@odu.edu

Olive Chapman
Calgary University
chapman@ucalgary.ca

Mathematics teacher educators (MTEs) are turning research lens on themselves to explore their knowledge and practices and with that contribute knowledge to the field of mathematics teacher education. In this working group we build from our exploration of MTEs' work. MTEs will describe their work and their views of knowledge and being in their work as MTEs. We invite MTEs to join our working group and assert that MTEs' discussions of their work will provide opportunities for professional learning that reveals how their knowledge and identity inform their practice.

Keywords: Mathematics Teacher Educators, Philosophy, Practice, Qualitative Research.

Qualitative methodologies employed in the study of mathematics teacher education have empowered mathematics teacher educators (MTEs) to turn the research lens on themselves and gain nuanced perspectives. The goals of self-based methodology ([SBM], Chapman et al., 2020) include improving one's own practice, fueling ideas for others' practices and contributing to dialogues about the complex work of MTEs. Published findings from inquiry into MTE practice have illustrated how studies of self can complement findings from explorations of instructional activities and their impacts on mathematics teachers, leaders and learners as well as their environments and contexts (e.g., Grant, 2019; Grant & Ferguson, 2021; Kastberg et al., 2020). Mathematics education as a field benefits when the complexity of the work of MTEs is laid bare.

We aim to continue creating spaces at mathematics education conferences to support MTEs in the writing of articles of SBM studies (Suazo-Flores et al., 2018, 2019, 2020, 2021, 2022). Our focus at PME-NA 44 is the edition of a special issue for the *Philosophy of Mathematics Education Journal*. We learned that MTEs engaged in scholarly inquiry of their practices communicate their diverse ways of knowing, their views of what is real, and their perceptions of others' knowledge, such as the ways they discover or construct knowledge (Guba, 1990; Paul & Marfo, 2001; Stinson, 2020). Moreover, MTEs research pursuits may be constrained by disciplinary cultural norms, research training and/or editorial preferences. Such inquiry makes explicit MTE's role in conducting and reporting research (Guilfoyle et al., 2004), which creates trustworthiness in their reports (Grant & Lincoln, 2021).

The working group at PME-NA 44 will include the work of authors studying MTEs' lived experiences empirically, theoretically, and philosophically informed using SBM or some other intimate approach (Hamilton et al., 2016). This work contributes to the recognition and understanding of the complexity of mathematics teacher education. These papers delve into the "particularities and intricacies" of MTE's work as informed by emotional, social, relational, and organizational contexts (Hamilton et al., 2016). We want to use this working time as a meeting space to continue supporting each other in the writing of such articles.

Session Information

We have regularly met to continue creating professional development spaces to engage more MTEs in writing about their practice from their perspective (Suazo-Flores et al., 2018, 2019, 2020, 2021, 2022). At PME-NA 44, MTEs are invited to join our working group to learn about the philosophical underpinnings of SBM studies (Chapman et al., 2020). Working group activities will include on Day 1, engaging the audience in thinking about philosophical underpinnings involved when conducting research studies that focus on the self. On Day 2, we will invite MTEs to present their studies and to identify philosophy and trustworthiness/authenticity (Grant & Lincoln, 2021). On Day 3, we will develop action items and discuss new projects such as writing a proposal for ICME-15.

References

Chapman, O., Kastberg, S. E., Suazo-Flores, E., Cox, D., & Ward, J. (2020). Mathematics teacher educators' inquiry into their practice. In Beswick, K. & Chapman, O. (Eds.), *International Handbook of Mathematics Teacher Education.* 2nd Edition. Volume 2: The mathematics teacher educator as a developing professional (pp. 157-187). Brill-Sense Publishers.

Grant, M. R. (2019). An autoethnography: So, you want to attract and retain diverse faculty???. *Taboo, 18*(1), 126-140.

Grant, M., & Ferguson, S. (2021). Virtual microteaching, simulation technology & curricula: A recipe for improving prospective elementary mathematics teachers' confidence and preparedness. *Journal of Technology and Teacher Education, 29*(2), 137-164. Waynesville, NC USA: Society for Information Technology & Teacher Education. https://www.learntechlib.org/p/218262

Grant, M. & Lincoln, Y. (2021). A conversation about rethinking criteria for qualitative and interpretive research: Quality as trustworthiness. *Journal of Urban Mathematics Education, 14*(2), 1-15.

Guba, E. (1990). The paradigm dialog. Sage Publications.

Guilfoyle, K., Hamilton, M. L., Pinnegar, S., & Placier, P. (2004). The epistemological dimensions and dynamics of professional dialogue in self-study. In J.J. Loughran, M.L. Hamilton, V.K. LaBoskey, & T. Russell (Eds.), *International handbook of self-study of teaching and teacher education practices* (pp. 1109-1167). Springer. https://doi.org/10.1007/978-1-4020-6545-3_28

Hamilton M.L., Pinnegar S., Davey R. (2016) Intimate scholarship: An examination of identity and inquiry in the work of teacher educators. In Loughran J., Hamilton M. (eds) *International handbook of teacher education*. Springer. https://doi.org/10.1007/978-981-10-0369-1_6

Kastberg, S. E., Lischka, A. E., & Hillman, S. L. (2020). Characterizing mathematics teacher educators' written feedback to prospective teachers. *Journal of Mathematics Teacher Education, 23*. 131-152. Retrieved from https://doi.org/10.1007/s10857-018-9414-6

Paul, J. & Marfo, K. (2001). Preparation of educational researchers in philosophical foundations of inquiry. *Review of Educational Research, 71*(4), 525-547.

Stinson, D. (2020). Philosophical considerations always already entangled in mathematics education research, In Bicudo et al. (Eds.), *Mathematics Teaching Research Journal, Special Issue on Philosophy of Mathematics Education*, *12*(2), 8-23.

Suazo-Flores, E., Kastberg, S., & Grant, M. (2022). *Mathematics teacher educators' professional development on self-based methodologies [working group].* Association of Mathematics Teacher Educators, Las Vegas, Nevada.

Suazo-Flores, E., Kastberg, S., Grant, M., Ward. J., Richardson, S. E., & Chapman, O. (2021). Using self-based methodologies to unpack mathematics teacher educators' work. In D. Olanoff, K., Johnson, & S.M., Spitzer (Eds.). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1907-1910). Philadelphia, PA. http://www.pmena.org/pmenaproceedings/PMENA%2043%202021%20Proceedings.pdf

Suazo-Flores, E., Ward. J., Richardson, S. E., Grant, M., Cox, D., Kastberg, S., & Chapman, O. (2020). Mathematics teacher educators using self-based methodologies. In Sacristan, A.I., Cortes-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.) *Proceedings of the 42nd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 19181). Cinvestav / AMIUTEM / PME-NA. https://doi.org/10.51272/pmena.42.2020

Suazo-Flores, E., Kastberg, S. E., Cox, D., E., Ward, J., Chapman, O., & Grant, M. (2019). Mathematics teacher educators' exploring self-based methodologies. In Otten, S., de Araujo, Z., Candela, A., Munter, C., & Haines, C. (Eds.) *Proceedings of the 41st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. University of Missouri. http://www.pmena.org/pmenaproceedings/PMENA%2041%202019%20Proceedings.pdf

Suazo-Flores, E., Kastberg, E., Ward, J., Cox, D., & Chapman, O. (2018). Mathematics teacher educators' inquiry into their practice: Unpacking methodologies for professional and personal growth. In Hodges, T. E., Roy, G.J., & Tyminski, A. M. (Eds.) *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1469-1477). University of South Carolina & Clemson University. http://www.pmena.org/pmenaproceedings/PMENA%2040%202018%20Proceedings.pdf

TAKING UP MATHEMATICS EDUCATION RESEARCH SPACES AS RESISTANCE: TOWARD DISABILITY JUSTICE

Paulo Tan
Johns Hopkins University
ptan14@jhu.edu

Cathery Yeh
Chapman University
yeh@chapman.edu

James Sheldon
University of Arizona
jsheldon@email.arizona.edu

Kai Rands
Independent Scholar
randske@gmail.com

Katherine Lewis
University of Washington
kelewis2@uw.edu

Amber Candela
University of Missouri-St. Louis
candelaa@umsl.edu

Rachel Lambert
UC Santa Barbara
rlambert@ucsb.edu

Jessica Hunt
North Carolina State
Jhunt5@ncsu.edu

Erica Mason
University of Illinois
enmason@illinois.edu

Alexis Padilla
University of New Mexico
apadilladiv@gmail.com

Anette Bagger
Örebro University
Anette.Bagger@oru.se

Alison Mirin
University of Arizona
amirin@math.arizona.edu

Juuso Nieminen
Hong Kong University
juuso@hku.hk

"Taking up space as a disabled person is always revolutionary" (Ho, 2020, p. 115). We are mathematics education researchers and practitioners demanding radical changes within and beyond our field to center issues of/with/by the disabled. Our working group draws upon critical theories such as Disability Studies in Education, Critical Race Theory, and DisCrit in order to offer a justice-oriented vision of mathematics education based on conceptualizations of disability and ableism as it relates to other forms of oppression and identities (e.g., race, class, gender, queerness, citizenship, carcerality, material precarity, etc). Traditionally, research on mathematics and disabilities has been conducted within a special education paradigm. Under this paradigm, research often, implicitly or explicitly adopts a deficit and dehumanizing approach which locates the "problem" within the individual student rather than in the social, discursive, political, and structural context of education and other systemic dynamics of racialization (Meghji, 2022) and ableism (Goodley et al., 2019). We resist the continued dehumanization of and harms inflicted on disabled communities. We demand that mathematics education researchers and practitioners address this critical dissonance by taking up equity-oriented approaches to understanding disability and its intersections through Black Feminists and Disability Justice methodologies. Likewise, we call for more resonant harmony frameworks such as Multiple Mathematics Knowledge Bases and Culturally Responsive and Sustaining Practices that honor the voice, agency, and leadership of those most impacted. The purpose of this working group is to build upon our previous working group and to take up and move forward towards solidarity, interdependence, and collective action with the disabled community in our research agenda.

Struggle Toward Disability Justice

Consisting of faculty, graduate students, disability activists, and classroom teachers, this PME-NA working group met from 2016 to 2018,. Some of the major accomplishments of this working group include (a) establishing and maintaining an email list-serv, (b) co-editing a special issue in the journal *Investigations in Mathematical Learning* (Lambert et al., 2018), and (c) publishing an NCTM book titled *Humanizing Disability in Mathematics Education: Forging New Paths* (Tan et al., 2019).

Moving forward, we center Disability Justice, a theoretical framework developed through the grassroots efforts of disabled queer activists and disabled people of color (Sins Invalid, 2019). It focuses on the relationship between ableism and other interlocking systems of oppression, such as racism, sexism, imperialism, settler colonialism, and capitalism. We specifically draw upon the 10 principles for disability justice created by Patty Berne and others from Sins Invalid: intersectionality, leadership of the most impacted, anti-capitalist politics, cross-movement solidarity, recognizing wholeness, sustainability, cross-disability solidarity, interdependence, collective access, and collective liberation. Given the legacy of harm done to disabled communities, Disability Justice, its centralization of the needs and experiences of folx experiencing intersectional oppression, allows researchers to recognize the intersecting legacies of white supremacy, colonial capitalism, gender oppression, and ableism to understand how people's bodies and minds are labeled "deviant", "unproductive" and/or invalid. (Sins Invalid, 2019). Traditionally, researchers determine the questions asked, the methods of data collection, and the meaning made of the data, with no input from the disabled individuals of color. Acknowledging disabled students and communities of color have gifts (Annamma & Morrison, 2018) such as unique mathematical ideas and perspectives and solutions to challenge current structures of ableism (Tan & Kastberg, 2017) and bringing them into the research process to shift power and decolonialize the research process. For example, members of the working group have published an emancipatory research study (Lewis & Lynn, 2018) where Lewis (disabled researcher) was positioned as the inquirer and Lynn (student with dyscalculia) as the expert. We aim to build out theories of intersectional disability justice in and through mathematics education.

Plan for Resistance and Co-conspiring

Session 1: Community, Caring, & Co-learning - Introductions and community building (40min), and discussing an agenda for the group for the upcoming year (50min); Session 2: Generating, Struggling, & Planning - Identify potential collaborative endeavors (30min), break up into subgroups based on areas of interest (40min), and regroup as a large to share out (20min); Session 3: Planning for Next Steps - Subgroup work (40min), large group share out and conversations (20min), and closing and identifying concrete next steps (30min)**.**

References

Annamma, S., & Morrison, D. (2018). DisCrit classroom ecology: Using praxis to dismantle dysfunctional education ecologies. *Teaching and Teacher Education, 73*, 70–80. https://doi.org/10.1016/j.tate.2018.03.008

Goodley, D., Lawthom, R., Liddiard, K., & Runswick-Cole, K. (2019). Provocations for critical disability studies. *Disability & Society, 34*(6), 972-997.

Ho, S. (2020). Canfei to Canji: *The freedom of being loud*. In A. Wong (Ed) Disability visibility: First-person stories from the twenty-first century (pp 112–116): Penguin Random House

Lambert, R., Tan, P., Hunt, J., & Candela, A. G. (2018). Rehumanizing the Mathematics Education of Students with Disabilities; Critical Perspectives on Research and Practice. *Investigations in Mathematics Learning*, 1–4. https://doi.org/10.1080/19477503.2018.1463006

Lewis, K. E. & Lynn , D. M. (2018) An insider's view of a mathematics learning disability: Compensating to gain access to fractions, *Investigations in Mathematics Learning*, *10*(3), 159-172

Meghji, A. (2022). *The racialized social system: Critical race theory as social theory*. Cambridge, UK: Polity.

Sins Invalid. (2019). *Skin, tooth, and bone: The basis of movement is our people (2nd ed.)*.

Tan, P., & Kastberg, S. (2017). Calling for research collaborations and the use of dis/ability studies in mathematics education. *Journal of Urban Mathematics Education, 10*(2). Retrieved from http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/321

Tan, P., Padilla, A., Mason, E. N., & Sheldon, J. (2019). *Humanizing disability in mathematics education: Forging new paths*. NCTM.

Chapter 18:
Doctoral Consortium

MATHEMATICS, MATHEMATICS EDUCATION, AND CITIZENSHIP: CONCEPTIONS, PROFESSIONAL PRACTICE VIEWS, AND FIGURED WORLDS

Amy Brass
Pennsylvania State University
aqb6161@psu.edu

Keywords: Affect, Emotion, Beliefs, and Attitudes; Preservice Teacher Education; Teacher Educators

The United States was recently categorized a "backsliding democracy" (International Institute for Democracy and Electoral Assistance, 2021), and commitment to citizen development in American schools has been inconsistent (Torney-Purta & Lopez, 2006). Citizenship scholars believe that one way to foster civic practices is through interdisciplinary connections and scholarship (Lee et al., 2021). This interest in interdisciplinarity is consistent with recent suggestions by mathematics education organizations, such as the National Council of Teachers of Mathematics (NCTM) (2018, 2020) which encourages students to investigate civic and social issues through mathematics in ways that foster their development as informed and engaged citizens. Additionally, the Association of Mathematics Teacher Educators (AMTE) (2017) also supports this interdisciplinarity by calling for mathematics teacher education programs to "uncover [teacher] candidates' passion for and commitment to mathematics as a discipline and an essential component of an effective citizenry" (p. 42).

Yet, extant research centered on relationships across mathematics, mathematics education, and citizenship is scarce. The purpose of this study is to explore conceptions held by mathematics teacher educators (MTEs) and by middle-level preservice teachers (PSTs) regarding the relationships across mathematics, mathematics education, and citizenship. The goal is to learn how these conceptions relate to MTEs' and PSTs' views on teaching, including how their conceptions do and do not align with current calls for developing democratic citizens, so that educational stakeholders can respond to K-12 schools' and the university's role in sustaining democracy. These research questions guide the study: (1) How do MTEs and middle-level PSTs conceptualize the relationships across mathematics, mathematics education, and citizenship? (2) What relationships exist between the connections MTEs and PSTs make across mathematics, mathematics education, and citizenship and their views of their professional practice?

Conceptual and Theoretical Framings

Drawing on citizenship literature (e.g., Castro, 2013; Westheimer & Kahne, 2004), four multifaceted, overlapping conceptions were created to frame participants' views on citizenship. *Conforming citizens* value consensus, established community norms, and patriotism; *cognizant citizens* are aware and informed of civic issues; *contributing citizens* participate in their communities; and *critical citizens* act as agents of change and focus on societal structures causing and perpetuating injustices. All of these citizens draw on specific intellectual and participatory civic skills (e.g., Patrick, 2002) and civic dispositions (e.g., Stitzlein, 2021). In relation to mathematics, these civic skills and dispositions connect to NCTM's (2000) Process Standards: communication, connections, representations, reasoning and proof, and problem solving. Together, these citizenship conceptions, civic skills and dispositions, and NCTM's Process Standards will serve as a framework for exploring the relationships across mathematics, mathematics education, and citizenship in the study.

The theoretical framework for this study draws on sociocultural theory, specifically the notion that people and their social, historical, and cultural contexts are inseparable (Rogoff, 2003) and the use of figured worlds (Holland et al., 1998). A figured world is "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al., p. 52). The cultural context of the figured world becomes a means for interpreting the members, activities, discourse, and artifacts of that world. It is through these aspects of a figured world that the positioning and identities of the members of that world emerge. The figured worlds that PSTs create about their identities as classroom citizens in a mathematics methods course will be explored through this study, and these figured worlds will be compared and contrasted with PSTs' citizenship in other contexts.

Methods

This study consists of two phases. Phase 1 of the study focuses on views held by MTEs; Phase 2 of the study focuses on the views of PSTs. Methods for each phase are described below.

Phase 1

Phase 1 utilizes an open-ended Qualtrics survey with responses from 90 MTEs and semi-structured interviews with 20 MTEs. Both the survey protocol and the interview protocol consist of prompts to elicit participants' views of citizenship, views of mathematics and mathematics education for citizenship, and connections to views of their professional practice. Qualitative data analysis will involve a mixture of deductive and inductive coding techniques (Miles et al., 2014). Deductive codes represent different citizenship conceptions, civic skills, civic dispositions, and NCTM's (2000) Process Standards. Inductive analysis will be utilized (Bhattacharya, 2017) to identify inductive codes, categories, and themes related to MTEs' views of mathematics, mathematics education, and citizenship.

Phase 2

Phase 2 utilizes an ethnographically-informed case study (de Freitas et al., 2017) that focuses on 22 middle-level PSTs; their conceptions of the relationships across mathematics, mathematics education, and citizenship; and how those conceptions relate to their views of teaching. Data collection methods include observations of PSTs in their educational methods courses; course artifacts, surveys, and video-recording class sessions when PSTs engage in mathematics-related tasks focused on civic issues; and interviews. Like the Phase 1, qualitative data analysis in this phase will use a mixture of deductive and inductive coding techniques (Miles et al., 2014). Connections across emerging codes, categories, and themes that reflect the main elements of figured worlds – artifacts, discourse, and identity – will help me identify the figured worlds at play related to the PSTs' identities, participation, and positioning as classroom citizens in the mathematics methods course and how those worlds relate to their citizenship in other areas.

Anticipated Contributions

Anticipated contributions of this study include strengthening interdisciplinary scholarship between mathematics education and citizenship. This scholarship has the potential to engage student citizens in civic issues and the potential to explore how mathematics can be used to foster the development of informed and engaged citizens. Further, by recognizing how MTEs and PSTs view mathematics and mathematics education for citizenship, efforts can be made to explore ways in which educational stakeholders can respond to K-12 schools' and the university's role in advancing the preservation of democracy.

References

Association of Mathematics Teacher Educators (AMTE). (2017). *Standards for preparing teachers of mathematics*. Author.

Bhattacharya, K. (2017). *Fundamentals of qualitative research: A practical guide*. Routledge.

Castro, A. J. (2013). What makes a citizen? Critical and multicultural citizenship and preservice teachers' understanding of citizenship skills. *Theory & Research in Social Education*, *41*, 219–246. https://doi.org/10.1080/00933104.2013.783522

de Freitas, E., Lerman, S., & Parks, A. N. (2017). Qualitative methods. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 159–182). National Council of Teachers of Mathematics.

Holland, D., Lachicotte Jr., W., Skinner, D., & Cain, C. (1998). *Identity and agency in cultural worlds*. Harvard University Press.

International Institute for Democracy and Electoral Assistance. (2021). *The global state of democracy 2021: Building resilience in a pandemic era*. Author.

Lee, C. D., Nasir, N. S., Smirnov, N., & Carrington, A. (2021). Civic reasoning and discourse: Perspectives from learning and human development research. In C. D. Lee, G. White, & D. Dong (Eds.), *Educating for civic reasoning & discourse* (pp. 53-107). National Academy of Education.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook*. Sage.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. NCTM.

National Council of Teachers of Mathematics (NCTM). (2018). *Catalyzing change in high school mathematics: Initiating critical conversations*. Author.

National Council of Teachers of Mathematics (NCTM). (2020). *Catalyzing change in middle school mathematics: Initiating critical conversations*. Author.

Patrick, J. J. (2002). *Defining, delivering, and defending a common education for citizenship in a democracy* [Paper presentation]. Summit on Civic Learning in Teacher Preparation, Boston, MA.

Rogoff, B. (2003). *The cultural nature of human development*. Oxford University Press.

Stitzlein, S. M. (2021). Defining and implementing civic reasoning and discourse: Philosophical and moral foundations for research and practice. In C. D. Lee, G. White, & D. Dong (Eds.), *Educating for civic reasoning & discourse* (pp. 23-52). National Academy of Education.

Torney-Purta, J., & Lopez, S. V. (2006). *Developing citizenship competencies from kindergarten through grade 12: A background paper for policymakers and educators*. Education Commission of the States.

Westheimer, J., & Kahne, J. (2004). Educating the "good" citizen: Political choices and pedagogical goals. *PS: Political Science and Politics, 37*(2), 241-247.

EXAMINING CRITICAL FACTORS IN PARENT-CHILD MATH ENGAGEMENT

Anastasia Betts
University at Buffalo SUNY
albetts@buffalo.edu

Parent-child engagement in shared math activities has a significant positive impact on children's mathematics achievement, yet studies of parent motivations and decision-making related to shared math activity with their children are largely absent from the research literature (Betts & Son, Accepted for Presentation). Furthermore, many students arrive at school unready to take full advantage of mathematics learning in school, quite often due to low prior knowledge borne of insufficient exposure to early math concepts and skills in the home prior to beginning kindergarten (Betts, Thai, Jacobs, & Li, 2020; Duncan et al., 2007; Nguyen et al., 2016). Parents are children's first teachers and are often the most well-positioned in terms of proximity (both physical and relational) to provide children with the modeling, scaffolding, and expectations that lead to learning and success (Bandura, 1977; Bloom, 1984; Bronfenbrenner, 1992; Marjoribanks, 1976; Mowder, 2005; Vygotsky, 1986). Parents are also one of the biggest influences in the home learning environment prior to the onset of formal schooling (Blevins-Knabe, 2016; Cankaya & LeFevre, 2016).

Dissertation Study Goals

The purpose of the proposed study is to examine the factors that influence parents' cognitions and decision-making around shared math-activity with their young children by exploring how parents' perceptions of their role, expectations for their child, personal math skills, self-efficacy, and perceptions of time and energy influence and interact with one another. These factors of Role, Expectations, Skills, Efficacy, and Time have been assembled into the RESET Framework, which was then used to create the RESET survey instrument. A further section of the survey was developed to examine the self-reports of Math Activity Participation of Parents (MAPP). This study provides an opportunity to develop tools and approaches for studying the home numeracy environment (i.e., the RESET Framework, the RESET-MAPP survey instrument), which may inform the design of better parent-engagement programs that can increase parent-child math activity and better outcomes for children. Research questions include: (1) To what extent are parents' mathematical parenting beliefs and perceptions represented by the RESET framework? (2) What relationships exist between parents' mathematical parenting beliefs and perceptions, and their self-reports of mathematical parenting behaviors? and (3) What mathematical parenting beliefs, perceptions, and behaviors characterize different groups of parents?

Theoretical Framework

No current literature exists for specifically understanding the interaction of factors that influence the motivations and decision-making of parents with respect to the mathematics education of their children (Mowder, 2005). However, Hoover-Dempsey and Sandler's (1997) parent education involvement model, Bandura (1977), Marjoribanks (1976), and Vygotsky's (1986) social theories of learning, as well as Bronfenbrenner's (1992) Ecological Systems Theory of Learning, and Mowder's (2005) Parental Development Theory provide a rich conceptual framework upon which the present study was grounded. The domains of the RESET Framework were derived from the work of Hoover-Dempsey and Sandler (1995, 1997; Walker et al., 2005; Walker et al., 2010) as well as Mowder (2005) and others. Additionally, Bandura, Marjoribanks, and Vygotsky in their individual work describe the critical importance of the parent/caregiver in the role of the child's development through providing models, establishing expectations, and acting as more knowledgeable others who both guide and "draw forth" the child's development and learning. Lastly, Bronfenbrenner's Ecological Systems Theory of Learning informs the RESET Framework by describing how parents formulate their perceptions along the RESET domains, through proximal influences in the micro- and meso-systems (e.g., family, parent groups, community groups, etc.) and distal influences in the exo-system (e.g., economics, politics, education), and macrosystem (e.g., cultural values, attitudes, beliefs, and ideologies, etc.).

Methodology

This exploratory parallel mixed-methods study proposes to use the RESET framework to examine parents' mathematics parenting cognitions and behaviors using the digital RESET-MAPP survey and follow-up interviews. The sample (n = 829) for the study includes parents of 4- and 5-year-old children who have not yet begun kindergarten. The sample was provided by a commercial sample provider, Innovate MR, which creates samples of participants from their nationwide database of hundreds of thousands of individuals (Innovate MR, n.d.). Samples are created using a blend of diverse sources including using online and offline strategies, large-scale advertising networks, specialty websites that cater to specific demographics (e.g., different ethnicities, cultural backgrounds, interests, educational levels, etc.), as well as using mail and TV advertisement strategies. The sample drawn for this study was diverse in terms of geography (e.g., participants will be drawn from a national database of research participants), ethnicity/cultural background, socio-economic status, education level, marital and employment statuses, age, and gender, etc. Purposive sampling will be used to create the secondary sample ($n \geq 18$) for the follow-up interviews, drawn from the population of survey participants, and selected based on their ability to provide additional depth or elaboration to phenomena that emerge from the survey data. Qualitative and quantitative data collected from the digital survey and qualitative data collected from the follow up interviews will be analyzed for insights into the specific research questions (noted earlier). Quantitative analysis will include descriptive statistics (e.g., measures of frequency, central tendency, dispersion/variation), as well as other tests to identify relationships and correlations. Qualitative analysis measures will include discourse analysis through coding frequency analysis. Analytical coding strategies, including a priori coding using the RESET framework, as well as other methods of inductive and deductive coding will be used. Qualitative data will be transformed into quantitative data as appropriate, and then triangulated with the existing quantitative data for means of deriving insights, drawing conclusions, and answering the specific research questions guiding this study.

Significance of this Study

The development of the RESET Framework is theoretically significant, as it has the potential to fill a gap in the research corpus by establishing a clear and useful conceptual framework for examining factors that influence parent involvement in their child's mathematics education. Understanding parents' beliefs, attitudes, and practices across the RESET domains provides practical significance through empowering the design of better parent engagement programs that that may increase parent-child interactions and improve child early mathematics knowledge. Lastly, this study will provide detailed information about the ways in which parents most need support that can empower them to better support their children.

References

Bandura, A. (1977). Social learning theory. Prentice Hall.

Betts, A., & Son, J-W. (2022; Accepted for Publication). Why parents do what they do: Developing and validating a survey for the mathematical lives of parents and children. *Proceedings of the Paris Conference on Education (PCE2022), Paris, France 2022*. The International Academic Forum (IAFOR).

Betts, A., Thai, K. P., Jacobs, D., & Li, L. (2020). Math readiness: Early identification of preschool children least ready to benefit from formal math instruction in school. *Proceedings of the IAFOR International Conference on Education–Hawaii 2020*. The International Academic Forum. Open access online: http://25qt511nswfi49iayd31ch80-wpengine.netdna-ssl.com/wp-content/uploads/papers/iicehawaii2020/IICEHawaii2020_55552.pdf

Blevins-Knabe, B. (2016). Early mathematical development: How the home environment matters. In Blevins-Knabe & Berghout Austin, A.M. (Eds.) *Early childhood mathematics skill development in the home environment,* (pp. 7-28). Springer International Publishing.

Bloom, B. S. (1984). The 2-sigma problem: The search for methods of group instruction as effective as one-to-one tutoring. *Educational Researcher, 13*(6), 4-16.

Bronfenbrenner, U. (1992). *Ecological systems theory.* Jessica Kingsley Publishers.

Cankaya, O., & LeFevre, J.-A. (2016). The home numeracy environment: What do cross-cultural comparisons tell us about how to scaffold young children's mathematical skills? In B. Blevins-Knabe & A.M. Berghout-Austin (Eds.) *Early childhood mathematics skill development in the home environment* (pp. 87-104). Springer International Publishing.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., & Japel, C. (2007). School readiness and later achievement. *Developmental Psychology, 43*(6), 1428–1446.

Hau, I. (2022). The case for inclusive innovations in early childhood education and care: Whole child, whole family, whole school, and whole community solutions. In A. Betts & K.P. Thai (Eds) Handbook of research on innovative approaches to early childhood development and school readiness (pp. 1-37). IGI Global Publishing. https://doi.org/10.4018/978-1-7998-8649-5.ch001

Hoover-Dempsey, K., & Sandler, H. (1995). Parental involvement in children. *Teachers College Record, 97*(2), 310-331.

Hoover-Dempsey, K.V., & Sandler, H.M. (1997). Why do parents become involved in their children's education? *Review of Educational Research, 67*(1), 3-42.

Innovate MR, (n.d.) ESOMAR 28: *Questions to help buyers of online samples.* InnovateMR. http://hubs.ly/H06gdg20

Marjoribanks, K. (1976). Social learning theory of the family: An analysis. *Psychology in the Schools, 13*(4). 457-462.

Mowder, B. A. (2005). Parent development theory: Understanding parents, parenting perceptions and parenting behaviors. *Journal of Early Childhood and Infant Psychology, 1,* 45-65.

Nguyen, T., Watts, T.W., Duncan, G.J., Clements, D.H., Sarama, J.S., Wolfe, C., & Spitler, M.E. (2016). Which preschool mathematics competencies are most predictive of fifth grade achievement? *Early Childhood Research Quarterly, 36*, 550-560.

Thai, K.P., Betts, A., & Gunderia, S. (2022). The case for inclusive innovations in early childhood education and care: Whole child, whole family, whole school, and whole community solutions. In A. Betts & K.P. Thai (Eds) Handbook of research on innovative approaches to early childhood development and school readiness (pp. 1-37). IGI Global Publishing. https://doi.org/10.4018/978-1-7998-8649-5.ch027

Walker, J.M.T., Wilkins, A.S., Dillaire, J.R., Sandler, H.M., Hoover-Dempsey, K.V. (2005). Parental involvement: Model revision through scale development. *The Elementary School Journal, 106*(2). 85-104.

Walker, J. M., Shenker, S. S., & Hoover-Dempsey, K. V. (2010). Why do parents become involved in their children's education? Implications for school counselors. *Professional School Counseling, 14*(1), 27- 41.

Vygotsky, L. (1986). *Thought and language: Newly revised and edited.* A. Kozulin (Ed). Cambridge, MA: The Massachusetts Institute of Technology.

LEARNERS' COLLABORATIVE USE OF HUMAN-SCALE SPATIAL DIAGRAMS

Camden Bock
The University of Maine
camden.bock@maine.edu

Freudenthal (1971) describes a vision for a didactical exploration of geometry that is "tied to lived reality with strong bonds," where lived reality is a material, spatial world that our bodies "live in," "move in," and "analyze" (pp. 418-419). Fruedenthal (1971) advocates for learners to explore phenomena from their lived reality with a rich mathematical lens. In geometry, diagrams provide an opportunity to connect learners' embodied experiences of the material world to exploration of mathematical phenomena. Gestures can realize "virtual mathematical objects or mathematical relationships" in diagrams — linking the motion of learners' bodies to the inscription — through multimodal, embodied discourse (Chen & Herbst, 2013, p. 302). However, diagrams are commonly inscribed on plane surfaces (*e.g.,* blackboards, paper) at scales smaller than the learners who inscribe them (*e.g.,* Greiffenhagen, 2014; de Freitas, 2012).

Technologies have long existed to bring geometry out of the plane and into the learners' material world (*e.g.,* clay sculpture, wood joinery). However, the time it takes to modify the inscription of a material diagram is typically much greater than their planar counterparts (*e.g., stylus, chalk*). The recent commercial availability of 3D filament pens and immersive virtual, augmented, and extended realities offer new ways to quickly construct and modify diagrams in space (Ng & Sinclair, 2018; Dimmel et al., 2020). Digitally rendered spatial diagrams offer opportunities to bridge the digital affordances of dynamic geometry environments with material affordances that emulate learners' interactions with their lived reality. I ask: How does the scale of a human-scale spatial diagram shape learners' collaborative discourse?

Background

Dynamic geometry environments can connect the movements of learners' bodies to continuous transformations of a digitally rendered figure. Multitouch tablets offer a surface where these maps can be made by the body itself, with the dragging of fingers (Ng & Sinclair, 2015), rather than mediating these interactions through keyboard or mouse input systems. Touch screen tablets thus provide a portal that links the digital worlds of dynamic diagrams directly to movements of learners' bodies. Digitally rendered spatial diagrams can extend this connection between movement and diagramming by allowing learners to pinch and drag components of a figure as they might grasp an object in the real world. The dragging or pinching of fingers to manipulate digitally rendered diagrams on multitouch tablets or digitally rendered spatial diagrams links the movement of the learners' body to smooth transformations of a diagram, which can support students in descriptions of geometric transformations as "continuous and temporal processes" (Ng & Sinclair, 2015, p. 85; Bock & Dimmel, 2021a; 2021b).

Spatial diagrams also offer learners novel ways to relate their body to a diagram. Learners might stand beside, look from above, or step inside a diagram (Dimmel & Bock, 2019; Bock & Dimmel, 2021a; Cangas et al., 2019). In human-scale and larger spatial diagrams, learners might put their head inside, kneel under, reach around, or reach through a diagram (Dimmel et al., 2021; Palatnik & Abrahamson, 2022). Other human-scale embodiments of diagrams have been explored in walking-scale geometry, where learners use their bodies to become vertices or to trace lengths of segments (Ma, 2017). Human-scale spatial diagrams — digitally rendered or physically realized — offer opportunities for networks of learners' bodies to interact with a diagram. Within and among such collaborative, body-based diagrammatic interaction has the potential for *creative mathematical* acts (Sinclair et al., 2013) — moments when learners realize virtual objects and relationships in unscripted ways.

Creative Acts

Creative acts "bring forth" something that "was not present before" in a way that does not "align with current habits and norms" for mathematical behavior, to "change the way language and other signs are used and alter meanings that circulate in a situation" (Sinclair et al., 2013, p. 241). For example, a learner might use a gesture to realize the altitude of a triangle that was not inscribed on a diagram. This could be a creative act if the altitude shaped the learners' mathematical discourse, and if using a gesture to realize the altitude of a triangle was unusual for their mathematical practice. The study of creative acts helps to understand how agency can be distributed in novel and mathematically meaningful interactions among learners, their bodies, and the potentials embedded in the milieu. The lens of creative acts can be applied to investigate the affordances (*e.g.,* scale) of technologies that support creative acts (Sinclair et al., 2013).

To understand the affordance of *scale* in human-scale spatial diagrams, it is important to understand learners' discourse as multimodal (*i.e.,* composed of speech, intonation, rhythm, gesture, body position, and body movement) and in relation with other learners and the spatial inscriptions composing a spatial diagram. The creative acts lens draws attention to the distribution of agency between learners, their bodies', and diagrams. With human scale diagrams, learners' can not only realize virtual mathematical objects and relationships with language, gesturing (above, in front of the diagram) and direct manipulation, but also by moving their body to vary their perspective, by gesturing (around, inside, outside and through the diagram), by starting and stopping their movement, and by directing their gaze.

Research Question, Methods, Design

I will conduct a series of semi-structured interviews where participants collaboratively explore a set of spatial diagrams. Interviewers will ask participants to repeat or clarify statements or actions they make during their exploration. Multimodal archives of participants' exploration of the diagram will be collected, including video of their first person (virtual) perspective of the spatial diagram, video of their bodies in the 'real' world, and audio of the conversation between participants. Within each interview, episodes will be identified where learners relate their bodies to the spatial diagrams in new ways (*e.g., stepping inside)*. Transcripts of verbal elements of discourse will be developed and embedded as subtitles in a composite video archive. Participants' gestures and bodily relationships with the diagram will be spatially annotated on the video archives. Analysis will include a characterization of creative acts within the episode and coding of the relationships between learners' bodies and the scale of the diagram.

This study seeks to describe some of the ways that learners' bodies can relate to human-scale spatial diagrams, which are uncommon in school settings. I am inspired by Benally et al.'s (2022) investigation of the collaborative construction of human-scale physical spatial diagrams and the asymmetric embodied relationships to a diagram. This study will contribute an understanding of what is possible with learners' collaborative explorations of spatial diagrams. More specifically, this study will explore the ways that learners realize the potentials of diagrams by reconfiguring their embodied relationships with a human-scale or larger diagram. This work extends understandings of learners' use of perspective in digital (Bock & Dimmel, 2021a, 2021b; Dimmel et al., 2020) and physical (Benally et al., 2022) spatial diagrams, the inscription of diagrams in space (Ng & Sinclair, 2018; Cangas et al., 2019), and the embodied relationships between learners and diagrams (Sinclair et al., 2013; de Freitas & Sinclair, 2014).

References

Benally, J., Palatnik, A., Ryokai, K., & Abrahamson, D. (2022). Charting our embodied territories: Learning geometry as negotiating perspectival complementarities. *For the Learning of Mathematics*.

Bock, C. G., & Dimmel, J. K. (2021a). Dynamic Spatial Diagrams and Solid Geometric Figures. *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Bock, C. G., & Dimmel, J. K. (2021b). Digital Representations without Physical Analogues: A Study of Body-Based Interactions with an Apparently Unbounded Spatial Diagram. *Digital Experiences in Mathematics Education*, 7(2), 193–221. https://doi.org/10.1007/s40751-020-00082-4

Cangas, D., Morga, G., & Rodríguez, J. L. (2019). Geometry teaching experience in virtual reality with NeoTrie VR. *Psychology, Society, & Education.* 11(3), 355-366.

Chen, C.-L., & Herbst, P. G. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics, 83(2),* 285–307. https://doi.org/10.1007/s10649-012-9454-2

de Freitas, E. (2012). The diagram as story: Unfolding the event-structure of the mathematical diagram. *For the Learning of Mathematics, 32(2),* 27–33.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body material entanglements in the classroom*. Cambridge University Press. https://doi.org/10.1017/CBO9781139600378

Dimmel, J., & Bock, C. (2019). Dynamic Mathematical Figures with Immersive Spatial Displays: The Case of Handwaver. In G. Aldon & J. Trgalova (Eds.), *Technology in Mathematics Teaching: Selected Papers of the 13th ICTMT Conference* (pp. 99–122). Springer International Publishing. https://doi.org/10.1007/978-3-030-19741-4_5

Dimmel, J., Pandiscio, E., & Bock, C. (2020). The Geometry of Movement: Encounters with Spatial Inscriptions for Making and Exploring Mathematical Figures. *Digital Experiences in Mathematics Education.* https://doi.org/10.1007/s40751-020-00078-0

Freudenthal, H. (1971). Geometry between the Devil and the Deep Sea. *Educational Studies in Mathematics, 3(3/4),* 413–435. JSTOR. https://www.jstor.org/stable/3482035

Greiffenhagen, C. (2014). The materiality of mathematics: Presenting mathematics at the blackboard. *The British Journal of Sociology, 65(3),* 502–528.

Ng, O.-L., & Sinclair, N. (2015). "Area Without Numbers": Using Touchscreen Dynamic Geometry to Reason About Shape. *Canadian Journal of Science, Mathematics and Technology Education, 15(1)*, 84–101. https://doi.org/10.1080/14926156.2014.993048

Ng, O.-L., & Sinclair, N. (2018). Drawing in Space: Doing Mathematics with 3D Pens. *Uses of Technology in Primary and Secondary Mathematics Education* (pp. 301–313). Springer.

Sinclair, N., Freitas, E., & Ferrara, F. (2013). Virtual encounters: The murky and furtive world of mathematical inventiveness*. ZDM, 45(2),* 239–252. https://doi.org/10.1007/s11858-012-0465-3

DEVELOPING COMBINATORIAL MEANING FOR ALGEBRAIC STRUCTURE: LESSONS FROM A DESIGN-RESEARCH CYCLE

Lori J. Burch
Indiana University
ljburch@iu.edu

Keywords: Algebra & Algebraic Thinking, Design Experiments, High School Education

Goals of Research

The broad goal of my dissertation study is supporting mathematical thinkers (students and teachers alike) to develop mathematical meanings as personally powerful and to leverage such meanings in the teaching and learning of algebra (Silverman & Thompson, 2008). Correspondingly, the following research questions have guided the design and implementation of this work. First, what affordances does developing algebraic structure as a generalization of one's own reasoning hold for mathematical thinkers? Second, how can teachers, in whole-class instruction, support students to develop personally powerful mathematical meanings related to algebraic structure? Through my dissertation, I report on two studies within a design-research cycle: a teaching experiment with two secondary algebra teachers (as participants) and a subsequent classroom study at the secondary level. Both interventions aimed to investigate the development of mathematical meanings as participants: (a) generalized quantitative relationships out of a combinatorics task (i.e., a counting problem) and (b) reasoned about their generalization in the context of symbolic tasks typical of secondary algebra curricula (Kaput, 2008/2017).

Theoretical Framework

I follow Thompson and colleagues' definition of mathematical meanings as "the space of implications that the current understanding mobilizes" (2014, p. 13). Elsewhere (Burch, under review), I contrasted two potential mathematical meanings for the cubic identity for binomial expansion, i.e., $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, which is the mathematical focus of my dissertation studies. Here, I briefly describe these two meanings—computational and combinatorial—before situating the combinatorial meaning within a theoretical model of combinatorial thinking (Lockwood, 2013). First, mathematical thinkers operating from a *computational meaning* credit equivalence between $(a+b)^3$ and $a^3 + 3a^2b + 3ab^2 + b^3$ to the result of algebraic computation, i.e., sequential two-factor application of the distributive property. This meaning's space of implications often implies a uni-directional nature that allows a thinker to determine an expansion from binomials but does not imply structural relationships that would allow the thinker to use the same kind of reasoning to determine binomial factors given an expanded polynomial (Tillema & Burch, 2021). By contrast, the *combinatorial meaning* emphasizing combinatorial and quantitative relationships (Tillema & Gatza, 2016) can mobilize a bi-directional understanding of the identity. That is, $(a+b)^3$ can be interpreted as three independent events, $(a+b)$ and $(a+b)$ and $(a+b)$, where *a* and *b* each represent a number of possibilities for two categories of outcomes that are available on each event. The expansion, $a^3 + 3a^2b + 3ab^2 + b^3$, represents a case breakdown of all possible three-event outcomes, where cases are organized according to the number of *a*-possibilities selected (Tillema & Burch, in press). A thinker's space of implications when operating from a combinatorial meaning includes opportunities to leverage structural relationships to multiply and factor any three binomials (beyond cubing) and to anticipate structural relationships in any degree of expansion.

Given hypothesized differences in the space of implications, the combinatorial meaning for the cubic identity has the greater potential of being personally powerful. Thus, it is this meaning that I aimed for study participants to develop. To that end, I find useful Lockwood's (2013) model of combinatorial thinking (MCT) in guiding the development of this study. The MCT consists of three aspects on which a person might focus when solving a combinatorics (or counting) problem: a set of outcomes, a counting process, and a formula/expression. When approaching the cubic identity from a combinatorial perspective, a thinker could consider the following question related to the symbolic problem $(a+b)^3$: What are all possible partial products when cubing the binomial $(a+b)$? The *set of outcomes* is the set of all possible partial products, where each partial product (i.e., outcome) contains one factor from each binomial. The *counting process* describes the method the thinker uses to produce (or organize) a set of outcomes (e.g., a case breakdown). Lockwood defined the *formula/expression* as determining the cardinality of the set of outcomes. In this example, $2^3 = 1 + 3 + 3 + 1$ represents the number of partial products in a way that reflects the set's organization. When emphasizing the combinatorial nature of algebraic structure, the formula/expression can represent the identity itself as $(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$. Thinkers with a personally powerful combinatorial meaning operate from an expectation of connectedness between the three MCT components. That is, the *counting process* should be represented consistently in both the *set of outcomes* and the *formula/expression* used to symbolize them. Such a connectedness allows a thinker to mobilize their meaning toward answering broader questions related to relationships between the coefficients of a polynomial and its roots (Viète's formulas). It is this connectedness in combinatorial reasoning that I aimed to support participants across both studies to develop.

Methods

In my dissertation, I report on two studies as part of a design-research program (Cobb et al., 2017). In both studies, I acted as teacher-researcher with a mathematics education researcher collaborating as witness-researcher. The first study, a teaching experiment (Steffe & Thompson, 2000), included two veteran secondary mathematics teachers who participated in six, 90- to 150-minute, teaching episodes over four days. The second study was a classroom study (Cobb et al.), including 18 secondary students in a dual-credit Finite Mathematics course. All students participated in a curricular unit over 10 days of instruction. Four acted as focus students, which included real-time recording of classwork and clinical interviews before and after the classroom intervention. All interviews and classroom instruction were recorded with multiple video cameras capturing student work and researcher/student interaction. Videos have been mixed into single files for use in multiple iterations of retrospective data analysis (e.g., transcription, coding, making conjectures and interpretations) (Steffe & Thompson). Both interventions were anchored by the Card Problem (Tillema & Gatza, 2016) which provided a quantitative and combinatorial context from which to generalize the cubic identity for binomial expansion (Burch et al., 2021).

Preliminary Findings & Anticipated Contributions

Responses to my research questions will contribute to the integration of combinatorics in secondary curricula (Lockwood et al., 2020) in a way that (a) addresses shortcomings of current teaching methods (e.g., Mason, 2017) and (b) supports the development of coherent meanings related to algebraic structure (Warren et al., 2016). The findings from my first study show important differences in teachers' combinatorial meanings that can inform teacher education and research efforts (Burch, under review). My second study will contribute empirically to the MCT literature by applying the MCT framework (Lockwood, 2013) as a model for algebra *instruction*.

References

Burch, L., Tillema, E.S., & Gatza, A.M. (2021). "Counting" on quantitative reasoning for algebra. *Mathematics Teacher: Learning and Teaching PK-12, 114*(6), 452–462.

Cobb, P., Jackson, K., & Dunlap Sharpe, C. (2017). Conducting design studies to investigate and support mathematics students' and teachers' learning. In J. Cai (Ed.). *Compendium for research in mathematics education* (pp. 208–233).

Kaput, J.J. (2017). What is algebra? What is algebraic reasoning? In J.J. Kaput, D.W. Carraher, M.L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–18). Routledge. (Original work published 2008).

Lockwood, E. (2013). A model of students' combinatorial thinking. *Journal of Mathematical Behavior, 32*(2), 251–265.

Lockwood, E., Wasserman, N.H., & Tillema, E.S. (2020). A case for combinatorics: A research commentary. *The Journal of Mathematical Behavior, 59*, 100783.

Mason, J. (2017). Overcoming the algebra barrier: Being particular about the general, and generally looking beyond the particular, in homage to Mary Boole. In S. Stewart (Ed.), *And the Rest is Just Algebra* (pp. 97-117). Spring, Cham. https://doi.org/10.1007/978-3-319-45053-7_6

Silverman, J., & Thompson, P.W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499-511.

Steffe, L.P., & Thompson, P.W. (2000). Teaching experiment methodology: Underlying principles and essential elements. *Handbook of Research Design in Mathematics and Science Education*, 267–306.

Thompson, P.W., Carlson, M.P., Byerley, C., & Hatfield, N. (2014). Schemes for thinking with magnitudes: An hypothesis about foundational reasoning abilities in algebra. In K.C. Moore, L.P. Steffe, & L.L. Hatfield (Eds.), *Epistemic algebra students: Emerging models of students' algebraic knowing*. WISDOMe Monographs (Vol. 4, pp. 1-24). Laramie, WY: University of Wyoming. Retrieved from http://bit.ly/1aNquwz

Thompson, P.W., Carlson, M.P., & Silverman, J. (2007). The design of tasks in support of teachers' development of coherent mathematical meanings. *Journal of Mathematics Teacher Education, 10*, 415–432.

Tillema, E.S., & Burch, L. (2021). Leveraging combinatorial and quantitative reasoning to support the generalization of advanced algebraic identities [Invited paper presentation]. The 14th International Congress on Mathematical Education (ICME-14), Shanghai, China.

Tillema, E.S., & Burch, L.J. (in press). Using combinatorics problems to support secondary teachers' understanding of algebraic structure. *ZDM Mathematics Education.*

Tillema, E.S., & Gatza, A.M. (2016). A quantitative and combinatorial approach to non-linear meanings of multiplication. *For the Learning of Mathematics, 36*(2), 26–33.

Warren, E., Trigueros, M., & Ursini, S. (2016). Research on the learning and teaching of algebra. In Á. Gutiérrez, G.C. Leder, & P. Boero (Eds.) *The second handbook of research on the psychology of mathematics education* (pp. 73–108). Brill Sense.

ACCESS TO COGNITIVELY DEMANDING MATHEMATICAL LEARNING OPPORTUNITIES IN CO-TAUGHT ELEMENTARY MATHEMATICS

Laurel Dias
University of Utah
laurel.dias@utah.edu

Keywords: Elementary School Education, Instructional Activities and Practices

In elementary school classrooms, teachers are expected to deliver instruction to a wide range of learners with diverse backgrounds and prior knowledge. The broad scope of student needs challenges teachers to provide all students with mathematical learning opportunities (MLOs) that are both accessible and cognitively demanding. Although rigorous MLOs are essential to develop students' conceptual understanding, many learners do not have access to the same cognitively demanding MLOs as their peers (Hiebert & Grouw, 2007; Lynch et al., 2018). The success of all learners in mathematics, including those who struggle, requires that teachers work together to provide instruction, support, and interventions that go beyond explaining the content and enable students to become expert learners and doers of mathematics (Hiebert & Grouw, 2007; Lambert, 2020, 2021). Co-teaching is an instructional model that draws on the expertise of more than one teacher as they deliver joint instruction and consider the needs of every learner in the classroom (Cook & Friend, 2010). Research has identified five commonly used co-teaching models; however, we know little about the nuances of these co-teaching models in elementary mathematics and their relationship with the cognitive demand and accessibility of MLOs. The research question that guides this study is what is the relationship between different co-teaching models and the cognitive demand and accessibility of MLOs offered to students? Answering this question is critical to understanding how co-teaching can enrich learning opportunities for all elementary students in mathematics.

Theoretical Framework

The cognitive demand, or level of thinking, required to solve a task is meaningfully connected to students' development of conceptual understanding (Huinker & Bill, 2017). More cognitively demanding tasks allow students to struggle to make sense of mathematics and engage in more conceptual learning (Hiebert & Grouws, 2007; Lambert & Stylianou, 2013; Silver & Stein, 1996). A Task Analysis Guide for determining the cognitive demand required to solve a mathematical task was created using research on characteristics of academic tasks, high-level thinking skills, Professional Standards for Teaching Mathematics, and the examination and categorization of hundreds of tasks used in QUASAR classrooms (Smith & Stein, 1998, p. 348). In addition to the potential cognitive demand of tasks, it is important to consider the implementation of the task. Teachers have control to alter the cognitive demand of tasks both when they set up and when they implement the task (Stein et al., 2009).

Universal Design for Learning (UDL) is a philosophical approach to instruction grounded in the learning sciences and neuroscience, in which teachers redesign curriculum and instruction with student experiences and variability in mind (Meyer, Rose, and Gordon, 2014). UDL Math is a mathematical version of UDL created to support teachers in designing instruction that centers all students as problem-solvers capable of engaging in cognitively demanding mathematical tasks (Lambert, 2021). The six design elements of the UDL Math framework delineate criteria for how the design of MLOs can develop students into mathematical learners (Lambert, 2021).

It's crucial to study the instructional design of co-taught lessons because co-teaching models are not all enacted the same way and may support teachers in enhancing student access to MLOs differently. The lenses of cognitive demand and UDL Math will be applied to co-taught lessons to determine how MLOs are influenced by each co-teaching model teachers enact in this study.

Methods

Data Gathering

The participants in this study are a pair of elementary teachers who designed and enacted co-taught math instruction for twenty students in a 2nd-grade class. I selected the teachers through purposeful sampling of pilot survey data (Patton, 2002). This study consisted of two data collection cycles during the 2021-22 school year at a diverse urban elementary school. Each cycle consisted of 45-minute video-recorded lesson observations 2-3 times a week. The first cycle lasted four weeks and provided the teacher pair time to develop their co-teaching relationship in math and gain familiarity with different co-teaching structures. During the second cycle, teachers delivered grade-level mathematics instruction using various co-teaching models. Instructional artifacts were collected and documented throughout the observations. Additionally, the teacher pair participated in reflective interviews and surveys throughout the data collection process to involve them as sense-makers of their instruction (Savin-Baden & Major, 2013).

Data Analysis

The analysis aims to understand the affordances of each co-teaching model for students' MLOs in elementary mathematics. After determining the co-teaching model used for each co-taught lesson, I will analyze lessons for two criteria. First, I will code the students' access to MLOs with a coding scheme I developed based on evidence of UDL Math design elements in the lesson (Lambert, 2021). Second, the cognitive demand of MLOs during lessons will be analyzed using the Instructional Quality Assessment Rubrics (Boston, 2012). Interrater reliability will be calculated for both coding schemes applied to the data set. Considering both access and cognitive demand will give a rich picture of the affordances of each co-teaching model for student MLOs.

Contributions

Through this research, I seek to better understand how co-teaching structures impact students' access to cognitively demanding mathematics. The need to understand how the cognitive demand of MLOs can be maintained while designing accessible instruction is an essential topic in the field of education today. In today's classrooms, students have unique learning needs that must be considered when designing MLOs. The COVID-19 pandemic has further exacerbated the wide range of student differences in mathematical understanding. This study will produce an empirically grounded understanding of how co-teaching can support elementary mathematics instruction that is cognitively demanding and accessible to all students. In addition, this study is situated in actual classroom practice, allowing for important insights into how co-teaching can be used to plan and enact inclusive mathematics lessons. These findings will also benefit teacher professional learning by providing guidance on the design and enactment of co-taught mathematics instruction.

References

Boston, M. (2012). Assessing instructional quality in mathematics. The Elementary School Journal, 113(1), 76-104.

Cook, L., & Friend, M. (2010). The state of the art of collaboration on behalf of students with disabilities. Journal of Educational and Psychological Consultation, 20(1), 1-8.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. Second handbook of research on mathematics teaching and learning, 1(1), 371-404.

Huinker, D., & Bill, V. (2017). Implement Tasks that Promote Reasoning and Problem Solving. Taking Action: Implementing Effective Mathematics Teaching Practices in K-grade 5 (pp. 37-65). National Council of Teachers of Mathematics.

Lambert, R. (2020). Increasing Access to Universally Designed Mathematics Classrooms. Policy Analysis for California Education, PACE.

Lambert, R. (2021). The Magic Is in the Margins: UDL Math. Mathematics Teacher: Learning and Teaching PK-12, 114(9), 660-669.

Lambert, R., & Stylianou, D. A. (2013). Posing cognitively demanding tasks to all students. Mathematics Teaching in the Middle School, 18(8), 500-506.

Lynch, S. D., Hunt, J. H., & Lewis, K. E. (2018). Productive struggle for all: Differentiated instruction. Mathematics Teaching in the Middle School, 23(4), 194-201.

Meyer, Anne, David H. Rose, and David Gordon. 2014. Universal Design for Learning: Theory and Practice. Wakefield, MA: CAST Professional Publishing.

Patton, M. Q. (2002). Qualitative research and evaluation methods (3rd ed.). Sage.

Savin-Baden, M., & Major, C. H. (2013). Fieldwork. Qualitative Research: The Essential Guide to Theory and Practice (pp. 338-356). Routledge.

Silver, E. A., and Stein, M. K. (1996). The QUASAR project: The revolution of the possible in mathematics instructional reform in urban middle schools. Urban Education, 30(4): 476−521.

Smith, M. S., & Stein, M. K. (1998). Reflections on practice: Selecting and creating mathematical tasks: From research to practice. Mathematics teaching in the middle school, 3(5), 344-350.

Stein, M. K., Smith, M. S., Henningsen, M., and Silver, E. A., (2009). Implementing Standards-Based Mathematics Instruction: A Casebook for Professional Development, 2nd Edition. Teacher's College Press.

MULTILINGUAL INTERNATIONAL STUDENTS' MATHEMATICS IDENTITIES IN PROOF-BASED COLLEGIATE MATHEMATICS COURSES: USING NARRATIVE INQUIRY

Jihye Hwang
Michigan State University
hwangji7@msu.edu

Keywords: Language, Diversity, Equity, and Inclusion, Undergraduate mathematics

Mathematics is not language-free. Researchers have claimed that mathematics has its discourse (Barwell et al., 2007), which includes both linguistic non-linguistic parts, and each language system has different mathematics discourses (Barwell, 2018; Halliday, 1978). The stereotype that "mathematics is language-free" is not only toxic in diminishing roles of language in mathematics but can also make multilingual international students hidden from people's attention and care in their mathematics classrooms. In this study, I refer to multilingual people as those who use one language in some situations and another in other situations (Planas & Setati, 2009). According to this definition, I include people who are traditionally called "English as Second Language," "English as Another Language," or "non-native English speakers" as multilingual.

I believe that multilingual international students' experiences in proof-based mathematics classrooms should be investigated. The following is a summary of my reasoning for the claim: 1) language is a part of one's identity; 2) many multilingual international students experience a transition in their language of instruction (LOI) as they moved to the U.S from their home country, which influences their experiences and identities; 3) transitions from computation-based mathematics to proof-based mathematics courses will influence their identities in mathematics courses; hence, 4) multilingual international students face a two-fold transitions, one linguistic and one mathematical, that influence their mathematics identity in proof-based mathematics courses. Thus, this study tries to answer the following research question and sub-questions: What are multilingual international students' mathematics identities in various linguistic and mathematical contexts?

1. How do multilingual international students leverage their first and/or other languages when the LOI in proof-based mathematics courses is different from their first and/or languages?
2. What kinds of lived experiences do multilingual international students report to have affected their mathematics identities?

In this study, mathematics identity is defined as one's understanding of their relationship to mathematics, experiences that constructed this relationship across time and space, and one's vision for their mathematical possibilities for the future, which I adapted from Norton's (1997) definition of identity. Further, I view language as sources of meaning (Barwell, 2018), which expands the notion that language is a resource that students bring to the mathematics classroom (Moschkovich, 2002), by acknowledging that language is not neutral but stratifying and stratified.

Researcher Positionality

I am a multilingual international student who is a Korean woman and has studied and taught mathematics in the U.S. in my second language, English. Throughout my experiences, I faced a lot of challenges communicating mathematics in English and sometimes felt I was not valued by the community, and in turn, that led me to take some passive positions. I do not think that language is not the only component that intersects with my experiences, but I think it has been an important factor, especially related to my confidence. I tended to attribute the reasons for my challenging experiences solely to my lack of proficiency in speaking English, however, I believe there are a lot of external influences that shaped my view of my experience. This view is enhancing a deficit narrative about multilingual international students, whereas I adopt an anti-deficit perspective toward multilingual international students' experiences as it can be found in my research question and how I view language.

Method

For this study, I use narrative inquiry as a research method. Narrative inquiry is a research method that focuses on understanding human's lived stories (Connelly & Clandinin, 1990). People's experiences are in their stories as well as their beliefs, attitudes, and values, and their experiences are represented in a story as a whole rather than listed facts or figures (Foote & Bartell, 2011). According to the perspective, to understand students' experiences during their education, it is required to listen to people's stories of their education. By studying their stories, researchers can have a powerful tool to communicate, evoke emotions, empathize, and reflect on the experiences as well as provide insights into the factors of such experiences (Foote & Bartell, 2011; Martinie et al., 2016). Narrative inquiry is a useful tool to study anyone's experiences, but in the educational field, it has been more used to study marginalized populations' experiences to invite readers to revisit their own views and experiences in the educational field (Kim, 2012; Martinie et al., 2016).

The study participants will consist of five to six multilingual international undergraduate and graduate students in a large public university in the Midwestern U.S. The collected data will be their mathematics autobiography titled "Me and Math: my relationship with mathematics up to now" (Di Martino & Zan, 2010), one semi-structured synchronous interview based on their autobiography, and a series of email interviews that can provide multilingual international students with sufficient time to reflect and think about their experiences without worrying about a time constraint. Then the data will be analyzed to build a narrative because narratives can communicate who the participants have been in mathematics classrooms and what are their experiences as a mathematics doer better than the patterns of their behaviors.

Anticipated Contributions

In K-16 mathematics education, especially in higher education mathematics education, not much of multilingual international students' experiences and how they view themselves in mathematics is studied even though it is important. Hence, this study will be one addition to a set of studies, as well as a calling for more research on this population to serve them in a better and more equitable way. Moreover, this study will contribute to challenging the deficit narratives about multilingual international students and focus on what they are bringing into the classroom by viewing language as sources of meaning (Barwell, 2018). Narratives of multilingual international students in their mathematics learning and doing that will appear as a result of this study will hopefully provide an opportunity for readers to empathize with their experiences, reflect and challenge their own perspectives toward multilingual international students, and further consider these students' experiences in their teaching and research.

References

Barwell, R. (2018). From language as a resource to sources of meaning in multilingual mathematics classrooms. *Journal of Mathematical Behavior*, *50*(August 2017), 155–168. https://doi.org/10.1016/j.jmathb.2018.02.007

Barwell, R., Barton, B., & Setati, M. (2007). Multilingual issues in mathematics education: Introduction. *Educational Studies in Mathematics*, *64*(2), 113–119. https://doi.org/10.1007/s10649-006-9065-x

Connelly, F. M., & Clandinin, D. J. (1990). Stories of experience and narrative Inquiry. *Educational Researcher*, *19*(5), 2–14. https://doi.org/10.3102/0013189X019005002

Di Martino, P., & Zan, R. (2010). "Me and maths": Towards a definition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education*, *13*(1), 27–48. https://doi.org/10.1007/s10857-009-9134-z

Foote, M. Q., & Bartell, T. G. (2011). Pathways to equity in mathematics education: How life experiences impact researcher positionality. *Educational Studies in Mathematics*, *78*(1), 45–68. https://doi.org/10.1007/s10649-011-9309-2

Halliday, M. A. K. (1978). Sociolinguistic aspects of mathematical Education. *Language as Social Semiotic: The Social Interpretation of Language and Meaning*, *iii*, 195–204.

Kim, J. H. (2012). Understanding the lived experience of a Sioux Indian male adolescent: toward the pedagogy of hermeneutical phenomenology in education. *Educational Philosophy and Theory*, *44*(6), 630–648. https://doi.org/10.1111/j.1469-5812.2010.00733.x

Martinie, S. L., Kim, J. H., & Abernathy, D. (2016). "Better to be a pessimist": A narrative inquiry into mathematics teachers' experience of the transition to the Common Core. *Journal of Educational Research*, *109*(6), 658–665. https://doi.org/10.1080/00220671.2015.1020913

Moschkovich, J. (2002). A situated and sociocultural perspective on bilingual mathematics learners. *Mathematical Thinking and Learning*, *4*(2–3), 189–212. https://doi.org/10.1207/s15327833mtl04023_5

Norton, B. (1997). Language, identity, and the ownership of English. *TESOL Quarterly*, *31*(3), 409–429.

Planas, Ń., & Setati, M. (2009). Bilingual students using their languages in the learning of mathematics. *Mathematics Education Research Journal*, *21*(3), 36–59. https://doi.org/10.1007/BF03217552

PROMOVIENDO EL RAZONAMIENTO PROPORCIONAL CON APOYO DE LA TECNOLOGÍA DIGITAL

Erasmo Islas
Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional (Cinvestav)
erasmo.islas@cinvestav.mx

Los conceptos asociados al razonamiento proporcional, tales como fracción, razón y proporción son clave en la educación matemática en los niveles básicos, ya que son pilares de las matemáticas superiores y su aplicación es transversal a otras ciencias, tales como física, química, economía, entre otras disciplinas (Lamon, 2007; Lobato et al., 2010). Más allá de servir de fundamento, el razonamiento proporcional es crucial en el ámbito cotidiano de las personas.

Dada la relevancia del tema, un sinnúmero de investigaciones ha reportado la problemática en la educación matemática alrededor del razonamiento proporcional; en nuestro estudio, resumimos tales problemáticas en cuatro categorías: 1) La carencia de sentido en el tratamiento de situaciones proporcionales debida a una excesiva aritmetización (Hart, 1988; Lamon, 1994, 2007, 2020; Adjiage y Pluvinage, 2007). 2) Una pobreza en el desarrollo de habilidades proporcionales de los estudiantes al no tener la oportunidad para resolver problemas en contextos que les permitan explorar las múltiples representaciones de la proporcionalidad (Obando et al., 2014; Mutaquin et al., 2017; Weiland et al., 2020). 3) Los estudiantes avanzan en su educación matemática con un concepto de razón vago o inexistente, lo cual conlleva a que la proporcionalidad no se vincule con las funciones lineales (Streefland, 1984; Harel et al., 1994). 4) Incapacidad de los estudiantes para distinguir las relaciones lineales de las relaciones no lineales, provocando la tendencia a aplicar el modelo de la proporcionalidad directa en contextos donde no es aplicable (De Bock et al., 2002, 2007; Fernández y Llinares, 2012).

Tomando en cuenta la problemática planteada, el problema al que se enfrenta la comunidad es cómo enseñar a los estudiantes a razonar proporcionalmente. El acercamiento escolar más frecuente hacia la proporcionalidad se da desde una perspectiva aritmética, separándola de sus vínculos con las relaciones funcionales y sus múltiples representaciones. En nuestra propuesta ponemos a prueba un diseño de enseñanza en el que consideramos que el tránsito entre múltiples representaciones, apoyado en simulaciones digitales, propicia mejoras en las habilidades de razonamiento proporcional en estudiantes de secundaria.

Preguntas de Investigación y Marco Teórico

Una vez planteada la problemática, en este estudio buscamos dar respuesta a las siguientes preguntas de investigación: 1) ¿Cómo diseñar una secuencia didáctica, mediada por la tecnología, que ayude a los estudiantes a promover sus habilidades de razonamiento proporcional y les permita aplicar el conocimiento adquirido en otros contextos? 2) ¿Qué ventajas (o desventajas) se pueden apreciar en el razonamiento proporcional de estudiantes de secundaria, cuando se les plantean tareas de proporcionalidad con múltiples representaciones y enfocadas a la distinción entre situaciones lineales y no lineales?

Respecto al objeto de estudio, el razonamiento proporcional de los estudiantes de secundaria, tomamos como referencia la base teórica construida por Karplus et al. (1983), Lesh et al. (1988) y Lamon (1994). También, aceptamos como valido el modelo de razonamiento proporcional propuesto por Modestou y Gagatsis (2010), quienes consideran a la distinción entre situaciones lineales y no lineales un aspecto clave del razonamiento proporcional. En este sentido, son fundamentales los estudios sobre la sobregeneralización de la linealidad de De Bock et al. (2002, 2007). Nuestro diseño de tareas busca que los estudiantes desarrollen las habilidades de razonamiento proporcional establecidas por Weiland et al. (2020) y Lobato et al. (2010), las cuales incluyen: 1) Atender y coordinar dos cantidades que varían dependientemente, 2) Distinguir las situaciones lineales de las no lineales, 3) Reconocer y utilizar las estructuras de las situaciones proporcionales (por ejemplo, razón unitaria, constante de proporcionalidad, linealidad), 3) Reconocer el invariante en una proporción, 4) Comprender la proporcionalidad desde múltiples representaciones, 5) Distinguir entre razones y fracciones. Consideramos a estas cinco habilidades como clave para nuestro estudio y son las que invocamos con el término habilidades proporcionales.

Enfoque Metodológico

El proyecto se enmarca en un enfoque de investigación basada en diseño (IBD), según la tradición de Gravemeijer y Cobb (2006), y Bakker (2018). La propuesta de enseñanza consiste en un enfoque socioconstructivista, de acuerdo con Cobb et al. (1992) y Simon (1995). En armonía con el enfoque epistemológico descrito, para la secuencia de actividades de enseñanza utilizamos una trayectoria hipotética de aprendizaje (Simon, 1995). A su vez, el diseño de actividades se basa en los principios de la didáctica de Cuevas y Pluvinage (2003). Acorde con el enfoque de diseño, se pretende innovar mediante la simulación de situaciones reales con tecnología digital.

En el análisis retrospectivo del diseño (análisis de datos) se evalúa la adquisición de habilidades con los niveles de razonamiento proporcional propuestos por Langrall y Swaford (2000), asimismo se buscará incrementar la teoría en esa dirección en función de los hallazgos.

Resultados Preliminares

El primer ciclo se realizó mediante una intervención didáctica presencial en dos grupos de una Escuela Secundaria en México. Participaron en el estudio 35 estudiantes (14-15 años) en el invierno de 2021. La instrucción se dividió en tres sesiones de 90 minutos en un salón de cómputo de la escuela. Cada estudiante contaba con una computadora personal que tenía precargados entornos didácticos virtuales interactivos (EDVI's), en Geogebra. A la par de los EDVI´s cada estudiante contaba con hojas de actividades. Durante las sesiones se promovió un aprendizaje colaborativo y se discutieron las actividades de forma grupal. Nuestro primer ciclo de IBD reveló cuestiones sobre la eficacia de las tareas diseñadas y dio pautas para su rediseño. En general, los estudiantes lograron transitar en las múltiples representaciones de la proporcionalidad y aplicaron habilidades desarrolladas en problemas de aplicación; sin embargo, la evidencia fue insuficiente para hacer afirmaciones contundentes sobre la adquisición de habilidades. Por otro lado, debido a la diversidad de contextos se profundizó poco en la generalización de los modelos matemáticos a los que llegaban los estudiantes.

En cuanto a la influencia de la tecnología en el estudio, es claro que su uso permite simular situaciones cotidianas susceptibles de problematizarse, además de brindar ventajas pragmáticas y epistémicas. El uso de entornos didácticos virtuales interactivos (EDVIS) permitió: presentar al estudiante datos aleatorios, interactuar con los contextos de forma dinámica, mostrar los objetos matemáticos en distintas representaciones y validar los resultados. Los datos sobre la experiencia de los estudiantes con los EDVI's permitieron detectar concepciones erróneas que, probablemente en formas de enseñanza tradicionales, pasarían inadvertidas. Por otro lado, el interactuar con los EDVI´s provocó en los estudiantes un interés que iba más allá del contexto matemático planteado, involucrándose con problemas que se derivaban de las tareas.

Referencias

Adjiage, R. & Pluvinage, F. (2007). An Experiment in Teaching Ratio and Proportion. *Educational Studies in Mathematics*, *65*(2), 149–175. https://doi.org/10.1007/s10649-006-9049-x

Bakker, A. (2018). Design Research in Education: A Practical Guide for Early Career Researchers (1st ed.). Routledge. https://doi.org/10.4324/9780203701010

Cobb, P., Yackel, E., y Wood, T. (1992). Interaction and learning in mathematics classroom situations. Educational Studies in Mathematics, 23, 99-122.

Cuevas, C. A. y Pluvinage, F. (2003). Les projets d'action practique, elements d'une ingeniere d'ensigment des mathematiques. Annales de didactique et de sciences cognitives, 8, 273-292.

De Bock, D., Van Dooren, W., Janssens, D. & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics*, *50*(3), 311-334. https://doi.org/10.1023/a:1021205413749

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2007). The Illusion of Linearity. Springer Publishing. https://doi.org/10.1007/978-0-387-71164-5

Fernández, C. y Llinares, S. (2012). Relaciones implicativas entre las estrategias empleadas en la resolución de situaciones lineales y no lineales. Revista latinoamericana de investigación en matemática educativa, 15(1), 9-34.

Gravemeijer, K. y Cobb, P. (2006). Design research from a learning design perspective. En J. Van den Akker, K. Gravemeijer, S. McKenney y N. Nieveen (Eds.), Educational Design Research (pp. 45-85). Routledge. https://doi.org/10.4324/9780203088364

Harel, G., Behr, M., Lesh, R. y Post, T. (1994). Invariance of Ratio: The Case of Children's Anticipatory Scheme for Constancy of Taste. Journal for Research in Mathematics Education, 25(4), 324-345. https://doi.org/10.2307/749237

Hart, K. (1988). Ratio and Proportion. En J. Hiebert y M. Behr (Eds.), Number Concepts and Operations in the Middle Grades (pp. 198-219). National Council of Teachers of Mathematics.

Karplus, R., Pulos, S. y Stage, E. (1983). Early Adolescents' Proportional Reasoning on Rate Problems. Educational Studies in Mathematics, 14(3), 219-233. https://www.jstor.org/stable/3482278

Lamon, S. J. (1994). Ratio and Proportion: Cognitive Foundations in Unitizing and Norming. En G. Harel y J. Confrey (Eds.), The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 89-120). State University of New York Press.

Lamon, S. J. (2007). Rational Numbers and Proportional Reasoning: Towards a Theoretical Framework for Research. En F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629-667). Information Age Publishing.

Lamon, S. J. (2020). *Teaching Fractions and Ratios for Understanding: Essential Content 83 and Instructional Strategies for Teachers*. Routledge, 2020.

Langrall, C. W. y Swafford, J. (2000). Three balloons for two dollars: Developing proportional reasoning. Mathematics teaching in the middle school, 6(4), 254-261.

Lesh, R., Post, T. & Behr, M. (1988). Proportional reasoning. En J. Hiebert y M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93–118). National Council of Teachers of Mathematics.

Lobato, J. E., Ellis, A. B. & Charles, R. I. (2010). *Developing Essential Understanding of Ratios, Proportions, and Proportional Reasoning for Teaching Mathematics in Grades 6-8*. National Council of Teachers of Mathematics.

Modestou, M. y Gagatsis, A. (2010). Cognitive and Metacognitive Aspects of Proportional Reasoning. Mathematical Thinking and Learning, 12(1), 36-53. https://doi.org/10.1080/10986060903465822

Muttaqin, H., Putri, R. I. I. y Somakim, S. (2017). Design Research on Ratio and Proportion Learning by Using Ratio Table and Graph with Oku Timur Context at the 7th Grade. Journal on Mathematics Education, 8(2), 211-222. https://doi.org/10.22342/jme.8.2.3969

Obando, G., Vasco, C. E., y Arboleda, L. C. (2014). Enseñanza y aprendizaje de la razón, la proporción y la proporcionalidad: un estado del arte. Revista latinoamericana de investigación en matemática educativa, 17(1), 59-81.

Simon, M. A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. *Journal for Research in Mathematics Education*, *26*(2), 114-145. https://doi.org/10.2307/749205

Streefland, L. (1984). Search for the roots of ratio: Some thoughts on the long term learning process (towards ... a theory) Part I: Reflections on a teaching experiment. Educational Studies in Mathematics, 15(4), 327-348. https://doi.org/10.1007/bf00311111

Vergnaud, G. (1988). Multiplicative structures. En J. Hiebert y M. Behr (Eds.), Number concepts and operations in

the middle grades (pp. 141–161). National Council of Teachers of Mathematics.
Weiland, T., Orrill, C.H., Nagar, G.G., Brown, R. & Burke, J. (2021) Framing a robust understanding of proportional reasoning for teachers. J Math Teacher Educ, 24, 179–202. https://doi.org/10.1007/s10857-019-09453-0

THE ROLE OF STUDENTS' GESTURES IN OFFLOADING COGNITIVE DEMANDS ON WORKING MEMORY IN PROVING ACTIVITIES

Vladislav Kokushkin
Virginia Tech
vladkok@vt.edu

The role of individual cognitive differences in doing mathematics has become a significant area of research within the psychology of mathematics education (e.g., Bull & Lee, 2014; De Smedt et al., 2009; Ping & Goldin-Meadow, 2010). Prior literature revealed the importance of working memory (WM) – a mental workspace used by individuals to simultaneously maintain and actively manipulate relevant information when they are engaged in a cognitive task (Baddeley & Hitch, 1974; Daneman & Carpenter, 1980; Ma et al., 2014). WM modulates the number of pieces of information (chunks) an individual can operate on. If the task requires managing too many chunks simultaneously, a learner's WM becomes overloaded and they tend to seek ways to offload some of the information onto the environment (Risko & Gilbert, 2016).

A typical example of cognitive offloading is hand gesturing. Several studies investigated the role of students' gestures in lightening the cognitive load on their WM in the context of elementary mathematics (Alibali & DiRusso, 1999; Cook et al., 2012; Goldin-Meadow et al., 2001; Ping & Goldin-Meadow, 2010; Wagner et al., 2004). However, little is known about the effect of gesturing on students' WM when they are engaged in high-level mathematical activities, proving activities in particular.

Research literature shows that novice undergraduate students work with proofs in different ways than experienced mathematicians. Weber and Mejía-Ramos (2011) observed that, when reading a proof, mathematicians tend to skim the proof and grasp its overall framework before attending to the details. In contrast, novice readers may not have enough expertise to decide what is important to read first and tend to check everything, line by line (Selden & Selden, 2003).

The goal of my dissertation research is to investigate how undergraduate students navigate cognitive challenges in working with proofs. In line with previous research findings, I hypothesize that various strategies in reading, constructing, and presenting proofs pose different cognitive load on a prover's WM (Inglis & Mejía-Ramos, 2021; Selden & Selden, 1999). The requirement to simultaneously store and manipulate a growing amount of information contained in proofs will place a considerable burden on an individual's WM. Therefore, a student will need to find available mechanisms for cognitive offloading. Without being allowed to write or draw, hand gesturing should become the primary way to offload the information onto the environment. Thus, the central research question guiding my study is *how do undergraduate students use gestures to offload cognitive demands on WM when they are engaged in various proving activities, such as reading, constructing, and presenting mathematical proofs?*

Theoretical Framework

In addressing this research question, I adopt an embodied (Barsalou, 2008; Lakoff & Johnson, 1999; Lakoff & Nuñez, 2000; Roth & Thom, 2009; Shapiro, 2007; Varela et al., 1991; Wilson 2002) and Piagetian (Piaget, 1970, 1972) perspective.

Specifically, consistent with the Gesture as Simulated Action framework (Holster & Alibali, 2008), I consider gestures as a unique type of action that plays an important role in students' organization of thought and manifest the embodied nature of mathematical cognition (Alibali & Nathan, 2012). Next, I conceptualize the notion of mathematical proof through Harel and

Sowder's (2005, 2007) transformational proof schemes, according to which the prover is actively involved in operating upon mathematical objects, observing the result, and building upon the proof. Hand gestures allow learners to "act on" abstract objects by simulating motion-based transformations as if these objects were physically present (e.g., Nathan et al., 2021; Williams-Pierce et al., 2017). Finally, I use Pascual-Leone's (1970) neo-Piagetian model of WM. Pascual-Leone asserts that human cognitive performances are products of coordinated sensory-motor activity, and depend on the activation of a set of task-relevant mental schemes. WM capacity is then defined as the number of mental schemes one can activate at a time. The centrality of action in the proposed framework will help to explain the beneficial effect that gestures have on proving mathematical conjectures – by offloading cognitive demands on students' WM.

Data and Methods

I recruited ten undergraduate students enrolled in Introduction to Proofs. Each of them participated in two WM screening assessments: Figural Intersection Task (Pascual-Leone & Baillargeon, 1994) and Direct Following Task (Pascual-Leone & Johnson, 2005, 2010). Based on the results of the assessments, the participants were divided into groups with low (three students; scores 5.5 and lower), medium, and high (three students; 6.75 and higher) WM capacity. Students with low and high WM capacities were invited to participate in five more interviews: three proof-based interviews, a stimulated recall interview, and a post-interview assessment of fundamental number theory concepts and proof techniques.

During the three proof-based interviews, the participants were engaged in reading for comprehension, reading for validation, presenting, and constructing mathematical proofs. The participants were not given calculators, pens, scratch paper, or other figurative materials. Each interview consisted of a number-theoretic and a "geometry-based" task (a conjecture involving mathematical objects, the properties of which can be naturally represented via hand gesturing).

The complexity of each task was conceptualized in terms of its M-demand – the number of schemes it requires to be activated (Case, 1978; Pascual-Leone & Goodman, 1979). I conducted a line-by-line analysis of the M-demands of the tasks based on the hypothetical sequence of mental actions that needed to be carried out to complete the proof. In doing that, I first broke the proof into lines, containing 1-2 mathematical statements. For each line, I counted the number of previous lines involved to make sense of the current line, and the number of contextual schemes that a prover may or may not need to activate when working with the proof. The need to activate these schemes places additional cognitive pressure on the participants' WM. The participants' mastery of these schemes was measured separately in a post-interview assessment.

The described methodology allowed me to identify the places of proofs, in which the experienced M-demand exceeded an individual's WM capacity. If the participant produced a hand gesture (or a sequence of gestures) when working on the target places of proofs, the gesture was labeled as having an offloading purpose. The nature of offloading gestures was then discussed with the participants during the stimulated recall interviews. The intra- and inter-subject analysis of students' gestures is being conducted at the current moment.

Anticipated Contributions

In summary, the most significant contribution of my study is closing the gap in understanding the effect of students' hand gesturing on WM in proving activities. Understanding mechanisms underlying the phenomenon of cognitive offloading will inform undergraduate instruction and help to improve learning conditions for students in advanced mathematics classrooms, which addresses one of the major goals of research in mathematics education.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development*, 14, 37–56.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*, 21(2), 247–286.

Baddeley, A. D., & Hitch, G. (1974). Working memory. In *Psychology of learning and motivation* (Vol. 8, pp. 47–89). Academic Press.

Barsalou, L. W. (2008) Grounded cognition. *Annual Review of Psychology, 59*, 617–645.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology* (Trans. W. Mays). New York: Gordon and Breach.

Bull, R., & Lee, K. (2014). Executive functioning and mathematics achievement. *Child Development Perspectives*, *8*(1), 36–41.

Case, R. (1978). Intellectual development from birth to adulthood: A neo–Piagetian interpretation. *Children's thinking: What develops*? *14*(2), 37–71.

Cook, S. W., Yip, T. K., & Goldin-Meadow, S. (2012). Gestures, but not meaningless movements, lighten working memory load when explaining math. *Language and cognitive processes*, *27*(4), 594–610.

Daneman, M., & Carpenter, P. A. (1980). Individual differences in working memory and reading. *Journal of verbal learning and verbal behavior*, *19*(4), 450–466.

De Smedt, B., Janssen, R., Bouwens, K., Verschaffel, L., Boets, B., & Ghesquière, P. (2009). Working memory and individual differences in mathematics achievement: A longitudinal different levels of representing unknowns. *Journal for Research in Mathematics Education*, 52–80.

Goldin-Meadow, S., Nusbaum, H., Kelly, S., & Wagner, S. (2001). Explaining math: Gesturing lightens the load. *Psychological Science*, 12, 516–522.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical thinking and learning*, *7*(1), 27–50.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. *Second handbook of research on mathematics teaching and learning*, *2*, 805–842.

Hostetter, A. B., & Alibali, M. W. (2008). Visible embodiment: Gestures as simulated action. *Psychonomic bulletin & review*, *15*(3), 495–514.

Inglis, M., & Mejía-Ramos, J. P. (2021). Functional explanation in mathematics. *Synthese*, *198*(26), 6369-6392.

Lakoff, G., & Johnson, M. (1999). Philosophy in the flesh: The embodied mind and its challenge to western thought (Vol. 640). New York: Basic books.

Lakoff, G., & Núñez, R. E. (2000). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being. New York, NY, USA: Basic Books.

Ma, W. J., Husain, M., & Bays, P. M. (2014). Changing concepts of working memory. *Nature Neuroscience*, *17*(3), 347–356.

Nathan, M. J., Schenck, K. E., Vinsonhaler, R., Michaelis, J. E., Swart, M. I., & Walkington, C. (2021). Embodied geometric reasoning: Dynamic gestures during intuition, insight, and proof. *Journal of Educational Psychology, 113*(5), 929–948.

Pascual-Leone, J. (1970). A mathematical model for the transition rule in Piaget's developmental stages. *Acta Psychologica*, *32*, 301–345.

Pascual-Leone, J., & Baillargeon, R. (1994). Developmental measurement of mental attention. *International Journal of Behavioral Development*, *17*(1), 161–200.

Pascual-Leone, J., & Goodman, D. (1979). Intelligence and experience: A neoPiagetian approach. *Instructional Science*, *8*, 301–367.

Pascual-Leone, J., & Johnson, J. (2005). Handbook of Understanding and Measuring Intelligence. India: SAGE Publications.

Pascual-Leone, J., & Johnson, J. (2010). A developmental theory of mental attention: Its applications to measurement and task analysis. In *Cognitive Development and Working Memory: A Dialogue Between Neo-Piagetian Theories and Cognitive Approaches* (pp. 27–60). Psychology Press.

Piaget, J. (1970). Structuralism. New York: Basic Books.

Piaget, J. (1972). Intellectual evolution from adolescence to adulthood. *Human development*, *15*(1), 1–12.

Ping R.M., Goldin-Meadow S. (2010). Gesturing saves cognitive resources when talking about nonpresent objects. *Cognitive Science,* 34, 602–619.

Risko, E. F., & Gilbert, S. J. (2016). Cognitive offloading. *Trends in cognitive sciences*, *20*(9), 676-688.

Roth, W. M., & Thom, J. S. (2009). Bodily experience and mathematical conceptions: From classical views to a phenomenological reconceptualization. *Educational Studies in Mathematics*, *70*(2), 175–189.
Selden, A. & Selden, J. (1999). Can you tell me whether this is a proof? Extended version of a poster presentation at PME 23. In O. Zavalavsky (Hg.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, S. 364). Haifa: PME.
Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for research in mathematics education*, *34*(1), 4–36.
Shapiro, L. (2007). The embodied cognition research programme. *Philosophy Compass*, *2*(2), 338–346.
Varela, F.; Thompson, E. & Rosch, E. (1991). *The Embodied Mind*. Cambridge, MA: MIT Press.
Wagner, S. M., Nusbaum, H., & Goldin-Meadow, S. (2004). Probing the mental representation of gesture: Is handwaving spatial? *Journal of Memory and Language*, *50*(4), 395–407.
Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs: An exploratory study. Educational Studies in Mathematics, *76*(3), 329–344.
Williams-Pierce, C. C., Pier, E. L., Walkington, C., Boncoddo, R., Clinton, V., Alibali, M. W., & Nathan, M. J. (2017). What we say and how we do: Action, gesture, and language in proving. *Journal for Research in Mathematics Education*, *48*(3), 248–260.
Wilson, M. (2002). Six Views of Embodied Cognition. *Psychonomic Bulletin & Review*, *9*(4), 625–636.

RULES OF ENGAGEMENT: GRADUATE TEACHING ASSISTANTS' PERCEPTIONS OF THEIR ROLES AS AGENTS OF MATHEMATICS SOCIALIZATION

Kristyn Lue
University of Maryland, College Park
kriselue@umd.edu

Keywords: Undergraduate Education; Equity, Inclusion, and Diversity; Systemic Change

Background and Significance

Over the last few decades, the field of higher education has been concerned with the retention of underrepresented students of Color in the science, technology, engineering, and math (STEM fields). STEM identity development has emerged as a useful analytic framework in this research, as students with stronger STEM identities—students who recognize themselves and are recognized by others as "STEM people"—are more likely to persist in the STEM fields (Carlone & Johnson, 2007; Chemers et al., 2011; Malone & Barabino, 2009). STEM identity develops through the process of socialization, in which *agents of socialization* set and maintain the norms, culture, and values that newcomers in the STEM fields should emulate. At institutions of higher education, instructors act as primary agents of socialization (Garibay, 2018) both implicitly and explicitly signaling who "belongs" in the STEM fields, and who doesn't, with Black, Latinx, and Indigenous students often falling into the latter category (Haynes & Patton, 2019; Johnson, 2007; McGee, 2016).

Although mathematics is a gateway into the broader STEM fields (Adiredja & Andrews-Larson, 2017; Leyva et al., 2021), there is a dearth of literature that focuses on undergraduate *mathematics* socialization. Yet, mathematics courses are places where many underrepresented students of Color may experience socialization from their instructors that leads them to decide that they aren't "math people", and thus, are not "STEM people"—leading to their attrition from the STEM fields. Moreover, the role of graduate teaching assistants (GTAs) as agents of mathematics socialization remains unexamined, despite the large role they play in teaching these lower-level undergraduate mathematics courses that are gateway courses into the broader STEM fields (Ellis, 2014; Speer et al., 2005).

Research Question and Guiding Concepts

Given this background and dearth of research on the socializing influence of GTAs in mathematics departments, the primary question that guides this dissertation study is: ***How do GTAs perceive their role as agents of mathematics socialization for underrepresented undergraduate students of Color*?** At the heart of my research question is how GTAs might serve as *gatekeepers* into the culture of the mathematics community at their institution. How do GTAs define the shared values, beliefs, and practices of this culture? How do they identify the ways that these values and beliefs show up in their teaching practices and interactions with underrepresented undergraduate students of Color? How do they articulate and understand their roles as agents of mathematics socialization, and the power and opportunities they have to help underrepresented undergraduate students of Color feel a sense of belonging in undergraduate spaces? What constraints do they face in doing so?

Besides GTAs' individual beliefs, values, and descriptions of their teaching practices, this study also focuses on how race and power operate at a structural level to influence these beliefs, values, and practices, and how these beliefs, values, and practices may be exclusionary to undergraduate students of Color—despite GTAs' intentions. This focus on structurally-amplified effects—rather than individuals' intentions—reflects the study's grounding in Critical Race Theory (e.g. Bell, 1998; Davis, 2019; Delgado & Stefancic, 2017) and Critical Whiteness studies (e.g. Battey & Leyva, 2016; Leonardo, 2004; Matias et al., 2014), which also influence my choice to study GTAs at a historically white institution of higher education (HWI). With these foundational assumptions in mind, this study also draws on both Martin's (2000) framework on mathematics identity and socialization, and Liu's (2017) framework on power governors and privileged spaces to conceptualize mathematics identity as *membership* into mathematics communities as HWIs, and position GTAs as *gatekeepers* into the culture of this community.

Study Design

Due to the focus on values, beliefs, practices, and culture, this critical qualitative study borrows heavily from critical ethnographic methods (Madison, 2020). This study utilized purposeful sampling (Patton, 2014) to recruit ten mathematics GTAs from a large public flagship university that is a historical white institution. Data collection consisted of: (1) initial fieldwork in the mathematics department at the study site, (2) narrative individual interviews with each of the participants, and (3) six weekly focus groups with all ten participants. The initial fieldwork consisted of informational interviews, document analyses, and observations of the physical space of the study site, in order to help build the historical and current context of where participants teach, study, and work. The interviews and focus groups largely centered on how participants view the norms, culture, and values of mathematics, how they have come to these beliefs through their own socialization processes, and how they may perpetuate or seek to disrupt these norms and beliefs as agents of mathematics socialization for undergraduate students of Color. Data analysis will consist of open coding to look for line-by-line emerging themes to create initial codes. Codes will then be refined by looking for patterns and connections among the themes in order to create a codebook (Glesne, 2016). The codebook, as well as the guiding concepts, will be used to conduct "codeweaving" (Saldaña, 2009) in order to interpret and make sense of the findings through the lens of the guiding concepts.

Preliminary Findings and Significance

While data analysis is still ongoing, preliminary themes include (1) a lack of institutional support and disconnect from faculty values (2) colorblind and power-evasive discourse (Bonilla-Silva, 2018) (3) dissonance between intention and practice, and (4) differentiation by race and nationality in opportunities and willingness to learn about and articulate structural racism in mathematics education. As analysis continues, it is important to note that while this dissertation study alone will not fully answer *how* GTAs act as agents of mathematics socialization for undergraduate students of Color, the findings will provide an important foundation to understand the influences that GTAs may have on undergraduate underrepresented students of Color, the opportunities they may have to enact change in mathematics departments, and how they might be supported in doing so. Ultimately, the goal of this study—and the goal of all of my scholarly work— is to build on the current body of critical mathematics education research that seeks to transform mathematics education spaces to be more inclusive and welcoming of underrepresented students of Color. More specifically, this line of research seeks to illuminate how to disrupt the ways in which mathematics socialization might uphold whiteness and perpetuate racism, so that we might better retain, support, and serve underrepresented students of Color in mathematics and the STEM fields, and work towards furthering racial equity and justice in higher education as a whole.

References

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education*, *9*(2). 49–80.

Bell, D. (1992) *Faces at the bottom of the well: The permanence of racism.* Basic Books.

Bonilla-Silva, E. (2018). *Racism without racists: Color-blind racism and the persistence of racial inequality in America* (5th edition). Rowman & Littlefield.

Carlone, H. B., and Johnson, A. (2007). Understanding the science experiences of successful women of color: Science identity as an analytic lens. *Journal of Research and Science Teaching*, *44*(8), 1187-1218.

Chemers, M. M., Zurbriggen, E. L., Syed, M., Goza, B. K., & Bearman, S. (2011). The role of efficacy and identity in science career commitment among underrepresented minority students. *The Journal of Social Issues*, *67*(3), 469–491.

Davis, J. (2019). Using critical race theory as a pedagogical, theoretical, methodological, and analytical tool in mathematics education for Black students in urban areas. In J. Davis & C. C. Jett (Eds.), *Critical Race Theory in Mathematics Education* (pp.183-205). Routledge.

Delgado R. & Stefancic, J. (2017). *Critical race theory: An introduction* (3rd Edition). New York University Press.

Garibay, J. C. (2018). Beyond traditional measures of STEM success: Long-term predictors of social agency and conducting research for social change. *Research in Higher Education*, *59*(3), 349–381.

Glesne, C. (2016). *Becoming qualitative researchers: An introduction* (5th edition). Pearson Education, Inc.

Haynes, C., & Patton, L. D. (2019). From racial resistance to racial consciousness: Engaging white STEM faculty in pedagogical transformation. *Journal of Cases in Educational Leadership*, *22*(2), 85–98.

Johnson, A. C. (2007). Unintended consequences: How science professors discourage women of color. *Science Education*, *91*(5), 805–821.

Leonardo, Z. (2004). The color of supremacy: Beyond the discourse of “white privilege.” *Educational Philosophy and Theory*, *36*(2), 137–152.

Leyva, L. A., McNeill, R. T., & Marshall, B. L., & Guzmán, O. A. (2021). “It seems like they purposefully try to make as many kids drop”: An analysis of logics and mechanisms of racial-gendered inequality in introductory mathematics. *The Journal of Higher Education*, 92(5), 784-814.

Liu, W. M. (2017). White male power and privilege: The relationship between White supremacy and social class. *Journal of Counseling Psychology*, *64*(4), 349–358.

Madison, D. S. (2020). *Critical ethnography: Methods, ethics, and performance* (3rd edition). SAGE Publications, Inc.

Malone, K. R., & Barabino, G. (2009). Narrations of race in STEM research settings: Identity formation and its discontents. *Science Education*, *93*(3), 485–510.

Martin, D. B. (2000). *Mathematics success and failure among African-American youth: The role of sociohistorical context, community forces, school influence, and individual agency.* Lawrence Erlbaum Associates, Inc.

Matias, C. E., Viesca, K. M., Garrison-Wade, D. F., Tandon, M., & Galindo, R. (2014). “What is critical whiteness doing in OUR nice field like critical race theory?” Applying CRT and CWS to understand the white imaginations of white teacher candidates. *Equity & Excellence in Education: University of Massachusetts School of Education Journal*, *47*(3), 289–304.

McGee, E. O. (2016). Devalued Black and Latino racial identities: A by-product of STEM college culture? *American Educational Research Journal*, *53*(6), 1626–1662.

Patton, M. Q. (2014). *Qualitative research & evaluation methods: Integrating theory and practice (4th ed.).* SAGE Publications.

Saldaña, J. (2009). *The Coding manual for qualitative researchers*. SAGE Publications.

BEYOND PREDETERMINED ANSWERS: WHAT EXPERIENCES AND ENGAGEMENT DO OPEN-ENDED MATHEMATICS TASKS ELICIT FOR PRESERVICE ELEMENTARY TEACHERS?

Emily A. Mainzer
Pennsylvania State University
ead202@psu.edu

Keywords: Preservice Teacher Education; Instructional Activities and Practices; Affect, Emotions, Beliefs, and Attitudes

This dissertation study seeks to qualitatively describe the experiences and engagement of elementary preservice teachers (PSTs) during open-ended tasks. Ball (1990) identified a need to intervene in elementary PSTs' "knowledge, assumptions, and feelings about mathematics and about themselves in relation to mathematics" (p. 13). Open-ended mathematics tasks have the potential to play a role in provoking this break with prior experience. While many tasks in mathematics education have encouraged students to share their own approaches to getting an answer (e.g. Humphreys & Parker, 2015; Parrish, 2014), recently developed open-ended tasks (e.g. Danielson, 2016, 2018; Ray-Riek, 2013) are different in that no set answer exists.

Background and Theoretical Framework

The open-ended mathematics tasks that are the focus of this research include "Which One Doesn't Belong?" (Danielson, 2016), "Notice and Wonder" (Ray-Riek, 2013) and "How Many?" (Danielson, 2018). Practitioner resources have shared anecdotal evidence that these tasks may elicit engagement that is both broader in terms of which students engage and in terms of how they engage (Danielson; Illustrative Mathematics, 2021; Newell & Orton, 2019; Ray-Riek; Rumack & Huinker, 2019). These tasks may elicit similar engagement in PSTs, as it offers them the opportunity to become the "arbiter" (e.g. Dewey, 1934, p. 68) of their own mathematical experience rather than having the outcome set for them from the beginning.

Despite these tasks' popularity in practitioner circles, research into the tasks' impacts is lacking. The use of these tasks meets the criteria for what Matney et. al (2020) called a black hole of research: "an instructional practice for which there is a scarcity of blind-peer-reviewed research evidence supporting its efficacy, yet has attained critical gravity in the teaching field" (p. 247). Existing in a 'black hole' does not mean that an instructional practice or activity has no value; rather, it means that there is a critical mass of something that intuitively attracts practitioners, but research has yet to thoroughly and systematically shed light on it.

Because the use of open-ended tasks have not been thoroughly researched, identifying theory relevant to their potential requires reviewing theoretical and empirical literature regarding phenomena with similar characteristics. This review identifies three theories that provide insight into why these tasks may provoke more frequent and deeper engagement: positioning, aesthetic engagement, and self-determination theory (SDT).

Sociocultural positioning theory posits that at any moment, each member of the classroom (teachers and students alike) is positioning others and being positioned by others (Hand & Gresalfi, 2015; Langer-Osuna & Esmonde, 2017). The teacher's acts of positioning—which includes selection and implementation of tasks—can make certain identities available to students and not others. In other words, "what someone does in a particular activity is always done in relation to what one has opportunities to do" (Hand & Gresalfi, 2015, p. 191).

While positioning helps to explain the opportunities for engagement made possible by open-ended tasks, aesthetic engagement and SDT suggest innate human characteristics with which these opportunities connect. For Dewey (1934), aesthetic engagement was both a universal human need and something which the environment must support in order for it to occur. As Sinclair (2001) noted, "these dual facets—of perception and of action—permit children to become absorbed in and identify themselves with some object or idea" (p. 26). This perspective expands engagement as not only a taking up of, but being "caught up in" (Wong, 2007, p. 209).

The robust empirical basis for SDT (Ryan & Deci, 2017, 2020) strengthens the assertion that there are universal human characteristics and that environmental factors can support the expression of these characteristics. SDT asserts that humans, in addition to having physical needs, have basic psychological needs. When in an environment that supports these needs, "people's curiosity, creativity, productivity, and compassion are most robustly expressed" (Ryan & Deci, 2017, p. 5) The need most relevant to open-ended tasks is autonomy, or the extent to which a person feels regulation over their own experiences and behavior (Ryan & Deci, 2017).

Methodology and Methods

This study seeks to formally research the impacts of a pre-existing phenomenon for which there are already anecdotally reported benefits. As such, it makes sense to not be tied to one methodology, but rather to choose a pragmatic selection (e.g., Coyle, 2010; Frost & Nolas, 2011; Frost et al., 2010; Morgan, 2007) of methodologies and associated methods that are most likely to provide the information needed to address the research question. Drawing upon multiple methodologies—particularly those of ethnography, case study, and grounded theory—allows for seeking participants' perspective of the phenomenon, defining the phenomenon clearly, and providing space to not only understand the phenomenon through the lens of existing constructs but also for new insights to arise.

This study consists of four data collection methods conducted in three sections of an undergraduate elementary math methods course: collection of PSTs' introductory assignments regarding their relationship with math, in-person video-recorded observation of four open task sessions, an open-ended questionnaire, and semi-structured interviews. The instructor for the class will facilitate the PSTs' completion and discussion of the tasks while the researcher observes. After each task implementation, participants will respond to the brief questionnaire. The researcher will interview willing participants to further understand specific instances of their engagement (including video-stimulated recall) as well as their responses to the questionnaire.

Video-recordings, transcripts, and responses to semi-structured questionnaires and interviews will be coded according to existing engagement-related constructs (individual: behavioral, affective, and cognitive; collective engagement; expressions of autonomy; and aesthetic experience). The researcher will also code these items inductively for any new insights that arise. The introductory assignment will be used to provide further background for understanding PSTs' responses on the questionnaire and in interviews.

Importance of Research

This study will have important implications for elementary mathematics teacher education and mathematics education research. It addresses a 'black hole" of research by formally exploring the impacts of an instructional practice lauded by practitioners. In doing so, the study offers a potential opportunity for PSTs to experience mathematics in a different and more holistic way than what they likely experienced in their own K-12 education. Preliminary results suggest that open-ended tasks provoke many PSTs to think and feel about mathematics in a way they have not previously experienced. The openness of the tasks seems to be a significant factor in positioning PSTs in a way that makes these experiences possible.

References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics*, *10*(2), 10–16. https://www.jstor.org/stable/40247978

Coyle, A. (2010). Qualitative research and anomalous experience: A call for interpretative pluralism. *Qualitative Research in Psychology, 7*(1), 79–83. https://doi.org/10.1080/14780880903304600

Danielson, C. (2016). *A shapes book: Which one doesn't belong?* (teacher's guide). Stenhouse.

Danielson, C. (2018). *How many: A counting book* (teacher's guide). Stenhouse.

Dewey, J. (1934). *Art as experience.* Minton, Balch, & Company.

Frost, N. A., & Nolas, S. (2011). Exploring and expanding on pluralism in qualitative research in psychology. *Qualitative Research in Psychology, 8*(2), 115–119. https://doi.org/10.1080/14780887.2011.572728

Frost, N., Nolas, S. M., Brooks-Gordon, B., Esin, C., Holt, A., Mehdizadeh, L. & Shinebourne, P. (2010). Pluralism in qualitative research: the impact of different researchers and qualitative approaches on the analysis of qualitative data. *Qualitative Research, 10*(4), 441–460. https://doi.org/10.1177/1468794110366802

Hand, V., & Gresalfi, M. G. (2015). The joint accomplishment of identity. *Educational Psychologist, 50*(3), 190-203. https://doi.org/10.1080/00461520.2015.1075401

Humphreys, C., & Parker R. (2015). *Making number talks matter: Developing mathematical practices and deepening understanding, grades 3–10*. Stenhouse.

Illustrative Mathematics. (2021, September 15). *Experience IM K-5 math: A focus on student thinking* [webinar recording]. https://illustrativemathematics.org/experience-im-k-5-math-a-focus-on-student-thinking/

Langer-Osuna, J.M., & Esmonde, I. (2017). Identity in research on mathematics education. In J.Cai (Ed.), *Compendium for research in mathematics education*. National Council of Teachers of Mathematics.

Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black holes of research on instructional practice: The case of number talks. *Investigations in Mathematics Learning, 12*(4), 246-260. https://doi.org/10.1080/19477503.2020.1804273

Middleton, J. A. (2013). Introduction/editorial: the problem of motivation in the middle grades. *Middle Grades Research Journal, 8*(1), xi-xiii.

Morgan, D. L. (2007). Paradigms lost and pragmatism regained. *Journal of Mixed Methods Research, 1*(1), 48–76. https://doi.org/10.1177/2345678906292462

Newell, C., & Orton, C. (2018). Classroom routines: An invitation to discourse. *Teaching Children Mathematics, 25*(2), 94-102. https://doi.org/10.5951/teacchilmath.25.2.0094

Parrish, S. (2014). *Number talks: Whole number computation, grades K-5*. Math solutions.

Ray-Riek, M. (2013). *Powerful problem solving: Activities for sense making with the mathematical practices*. Heinemann.

Rumack, A. M., & Huinker, D. (2019). Capturing mathematical curiosity with notice and wonder. *Mathematics Teaching in the Middle School, 24*(7), 394-399. https://doi.org/10.5951/mathteacmiddscho.24.7.0394

Ryan, R. M., & Deci, E. L. (2017). *Self-determination theory*. The Guilford Press.

Ryan, R. M., & Deci, E. L. (2020). Intrinsic and extrinsic motivation from a self-determination theory perspective: Definitions, theory, practices, and future directions. *Contemporary Educational Psychology, 61*. https://doi.org/10.1016/j.cedpsych.2020.101860

Sinclair, N. (2001). The aesthetic is relevant. *For the Learning of Mathematics, 21*(1), 25–33. https://www.jstor.org/stable/40248343

Wong, D. (2007). Beyond control and rationality: Dewey, aesthetics, motivation, and educative experiences. *Teachers College Record, 109*(1), 192–220. https://www.tcrecord.org/Content.asp?ContentId=12740

THE BLACK WOMAN'S TOOLKIT: STORIES OF PERSISTENCE IN UNDERGRADUATE MATHEMATICS COURSES.

Olanrewaju T. Oriowo
University of North Carolina at Charlotte
ooriowo@uncc.edu

Keywords: Equity, Inclusion, and Diversity, Gender, Preservice Teacher Education

This study is an asset-based approach to exploring the experiences of Black women, who are pursuing secondary mathematics licensure, as mathematics learners. The most recent American Community Survey shows that although Black women are making great strides in undergraduate degree attainment, very few were in the field of education and fewer still in mathematics and statistics. Black women accounted for 6.24% of bachelor's degrees in 2020 (NCES, 2022) which is actually about two thirds of the undergraduate degrees awarded to Black students. However, only 3.24% of the degrees awarded to Black women were in the field of education and 0.4% in the field of mathematics and science (NCES, 2022). Since data that drills down to the specific licensure areas for the education degrees is unavailable, we can assume that secondary mathematics education was not a popular choice, implying that the actual percentage of secondary mathematics education degrees might be closer to the proportion of mathematics and statistics degrees.

Research supports the belief that having a teacher or role model who is of the same race or ethnicity has positive implications on students' dispositions, motivation, and achievement (e.g. NCES, 2019; Young et al., 2020). To put this in perspective, 15% of students enrolled in US public elementary and secondary schools are Black or African American (NCES, 2022), 7.5% of public school teachers are assigned to secondary mathematics, and 6.7% of teachers are Black. If we were to assume the proportion of secondary mathematics teachers applies to distributions by race, then we can expect about 0.5% of secondary mathematics teachers to be Black. At this rate, a fraction of a percentage of these students can hope to have a black, female, mathematics teacher over the course of their educational journey.

Young et al. 's (2020) study revealed a discontinuity between the ability of Black girls compared to their participation patterns in Advanced Placement (AP) courses, noting that the mentorship of Black educators can provide the "aspirational role model to emulate" (p. 215). The purpose of this study, therefore, is to 1) contribute to literature that focuses on the successes and persistence of Black women, specifically future secondary mathematics educators, who engage in the study of mathematics at the collegiate level, 2) elucidate the knowledge, skills, and resources that these women leverage in order to find that success, and 3) gain insight into ways that teacher preparation programs can recruit more Black women into mathematics education programs and create learning environments that are inclusive and that support the development of a robust mathematics identity.

I focus on the experience of the first, credit-bearing, undergraduate mathematics course in this study because mathematics courses are very important gatekeepers to continued participation in STEM and, by extension, other mathematics-related programs (such as secondary mathematics education) in many universities (Ellis et al., 2014; Tao & Gloria, 2018) and therefore an important site to examine for critical experiences that help or hinder the uptake of secondary mathematics education as a course of study. I define persistence as continuing the journey to degree attainment as a secondary mathematics education major (or minor, depending on the institutional program structure).

The two overarching research questions guiding this study are as follows:

1. When sharing stories of persistence and success in the first undergraduate mathematics course, which cultural capital do Black girls who are Secondary Mathematics Education majors, believe was instrumental in that success?
2. How do the counternarratives of the participants compare and contrast in terms of their capital?

Using a counternarrative research design, I propose to engage narrators in the reflective telling of mathematics learning experiences before, during, and after a particular event: the first, credit-bearing mathematics course taken at their current 4-yr institution as a requirement for their major. I chose Counternarrative Inquiry, not only because it allows attention to the ways in which marginalized peoples are resisting derogatory dominant narratives (Mertens, 2020), but because it also preserves the voice of the participants, which is vital for this study and is aligned to the theoretical framework. I will use Yosso's (2005) Community Cultural Wealth (CCW) as the analytical framework, applying it through a lens of Critical Race Theory (Bell, 1995; Delgado & Stefancic, 2017) and Black Feminist Thought (Collins, 2015; Combahee River Collective, 1977), to draw attention to what is revealed in the stories told by the narrators.

Preliminary findings from a pilot study reveals that aspirational capital, defined as the "the ability to maintain hopes and dreams for the future, even in the face of real and perceived barriers…this resilience is evidenced in those who allow themselves…to dream of possibilities beyond their present circumstances" (Yosso, 2005, p. 77-78), is a fundamental anchor is the persistence of the two Black women participants. Social capital, which is derived from networks of peers and the community that provide emotional and instrumental support (Yosso, 2005), is notable in its limited use towards success. It is especially evident when participants tell stories about classroom experiences, stories that highlight an isolation and loneliness that has been reported in several research studies (e.g., King & Pringle, 2018; Marra et al., 2009, etc.).

Where there have been several studies documenting the underrepresentation of Black women in undergraduate and graduate STEM and STEM-related disciplines (Alexander & Hermann, 2016; Ellington & Frederick, 2010; Haynes et al., 2016), asset-based studies that attend to the experiences and barriers that begin to offer insight as to sociopolitical and cultural factors that surround how Black women experience mathematics are relatively few (King & Pringle, 2018; Lim, 2008; Young et al., 2018). Per Young et al.'s (2020) recommendation, this study will contribute to the body of research that attends to sociocultural hurdles that Black girls and women overcome en route to success.

References

Alexander, Q. R. & Hermann, M. A. (2016). African-American women's experiences in graduate science, technology, engineering, and mathematics education at a predominantly White university: A qualitative investigation. *Journal of Diversity in Higher Education*, *9*(4), 307–322. https://doi.org/10.1037/a0039705

Bell, D. A. (1995). Who's afraid of critical race theory? University of Illinois Law Review, 1995(4), 893–910.

Clandinin, J. D., & Rosiek, J. (2007). Mapping a landscape of narrative inquiry: Borderland spaces and tensions. In D. J. Clandinin (Ed.), Handbook of narrative inquiry: Mapping a methodology (pp. 35-75). Sage.

Collins, P. H. (2015c). No guarantees: Symposium on Black Feminist Thought. *Ethnic and Racial Studies*, *38*(13), 2349–2354. https://doi.org/10.1080/01419870.2015.1058512

Combahee River Collective. (1977). *The Combahee river collective statement*. https://americanstudies.yale.edu/sites/default/files/files/Keyword%20Coalition_Readings.pdf

Connelly, F. M., & Clandinin, J. D. (2006). Narrative Inquiry. In J. Green, G. Camilli, & P. Elmore (Eds.), Handbook of complementary methods in education research (pp. 375-385). Lawrence Erlbaum.

Delgado, R. & Stefancic, J. (2017). *Critical race theory: An introduction*. New York University Press.

Ellington, R. M. & Frederick, R. (2010). Black high achieving undergraduate mathematics majors discuss success and persistence in mathematics. *The Negro Educational Review*, *61*(1-4), 61–84.

Ellis, J., Fosdick, B. K., & Rasmussen, C. (2016). Women 1.5 times more likely to leave STEM pipeline after calculus compared to men: Lack of mathematical confidence a potential culprit. *PloS One*, *11*(7), e0157447–e0157447. https://doi.org/10.1371/journal.pone.0157447

Haynes, C., Stewart, S., & Allen, E. (2016). Three Paths, One Struggle: Black Women and Girls Battling Invisibility in U.S. Classrooms. The Journal of Negro Education, 85(3), 380–391. https://doi.org/10.7709/jnegroeducation.85.3.0380

King, N., & Pringle, R. (2018). Black girls speak STEM: Counterstories of informal and formal learning experiences. Journal of Research in Science Teaching, 56(5), 539–569. https://doi.org/10.1002/tea.21513

Leonard, J., Walker, E. N., Bloom, V. R., & Joseph, N. M. (2020). Mathematics Literacy, Identity Resilience, and Opportunity Sixty Years since "Brown v. Board": Counternarratives of a Five-Generation Family. Journal of Urban Mathematics Education, 13, 12–37.

Lim, J. H. (2008). Double Jeopardy: The Compounding Effects of Class and Race in School Mathematics. Equity & Excellence in Education, 41(1), 81–97. https://doi.org/10.1080/10665680701793360

National Center for Education Statistics. (2019). Spotlight A: Characteristics of public school teachers by race/ethnicity. https://nces.ed.gov/programs/raceindicators/spotlight_a.asp

National Center for Education Statistics. (2022). Bachelor's degrees conferred by postsecondary institutions, by race/ethnicity and sex of student: Selected years, 1976-77 through 2019-20. https://nces.ed.gov/programs/digest/d21/tables/dt21_322.20.asp

Marra, R. M., Rodgers, K. A., Shen, D., & Bogue, B.(2009). Women engineering students and self-efficacy: A multi-year, multi-institution study of women engineering student self-efficacy. *Journal of Engineering Education (Washington, D.C.)*, *98*(1), 27–38. https://doi.org/10.1002/j.2168-9830.2009.tb01003.x

Mertens, D. M. (2020). *Research and evaluation in education and psychology: Integrating diversity with quantitative, qualitative, and mixed methods* (5th ed.). SAGE.

Tao, K. W. & Gloria, A. M. (2018). Should I stay or should I go? The role of impostorism in STEM persistence. *Psychology of Women Quarterly*, *43*(2), 151–164. https://doi.org/10.1177/0361684318802333

Yosso, T. J. (2005) Whose culture has capital? A critical race theory discussion of community cultural wealth. *Race Ethnicity and Education*, *8*(1), 69-91. https://doi.org/10.1080/1361332052000341006

Young, J., Young, J., & Capraro, R. (2018). Gazing Past the Gaps: A Growth-Based Assessment of the Mathematics Achievement of Black Girls. *The Urban Review*, *50*(1), 156–176. https://doi.org/10.1007/s11256-017-0434-9

Young, J., Young, J., & Capraro, R. (2020). Advancing Black Girls in STEM: Implications from Advanced Placement Participation and Achievement. *International Journal Of Gender, Science And Technology*, 12(2), 202-222. http://genderandset.open.ac.uk/index.php/genderandset/article/view/566

INTEGRATING SPATIAL REASONING INTO EARLY CHILDHOOD CLASSROOMS: AN INTRAPRENEURIAL APPROACH TO SPATIALIZING THE CURRICULUM

Robyn K. Pinilla
Southern Methodist University
rpinilla@smu.edu

Keywords: Geometry and Spatial Reasoning, Early Childhood Education, Instructional Activities and Practices, and Curriculum

Spatial reasoning comprises a vital set of skills for development in science, technology, engineering, arts, and mathematics (STEAM) domains (Lord & Rupert, 1995; Wai et al., 2009) that support mathematical and academic achievement (Duncan et al., 2007) but is rarely taught in early childhood classrooms (Clements & Sarama, 2011; Moss et al., 2015). Many factors may influence teachers enacting spatial reasoning instructional practices, including the professional learning opportunities they have experienced (Horn & Garner, 2022; Putnam & Borko, 2000), the tools to which they have access (Kazemi & Hubbard, 2008; Lave & Wenger, 1991), and their knowledge of and beliefs about teaching spatial reasoning (Ball et al., 2008; Batista et al., 1982; Skott, 2013). This study seeks to address the problem that we do not have a clear picture of how, or if, early childhood teachers enact spatial reasoning instructional practices by considering how professional development (PD), tools, and teachers' knowledge and beliefs interact with teaching practice to develop ways of supporting teachers' situated learning for teaching spatial reasoning.

This research aims to develop a local theory of situated teacher learning specific to learning and enacting spatial reasoning instructional practices (see Figure 1; Cobb et al., 2003) and a suite of pedagogical tools (Putnam & Borko, 2000) to support those enactments. The proposed situated teacher learning and practice model is adapted from Clarke & Hollingsworth's (2002) model of teacher growth. It specifies conjectured domains contributing to teacher learning and practice, and will undergo iterative refinement using design-based research methods (Cobb et al., 2003; Sandoval, 2014) as responsive to the research context. Simultaneously, tools related to enacting spatial reasoning teaching practices will be developed in context as pedagogical tools (Putnam & Borko, 2000). By developing this local theory and tools to support teacher learning, I aim to answer the question: How can we support early childhood educators in enacting spatial reasoning teaching practices?

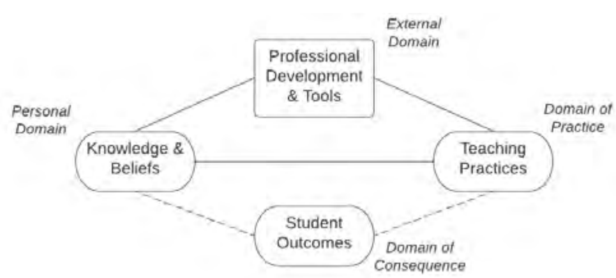


Figure 1: Proposed Model of Situated Teacher Learning & Practice

Methods

This study is conducted at a local STEM-focused elementary school and employs design-based research methods (Cobb et al., 2003; Sandoval, 2014) within a descriptive case study (Merriam, 1998; Yin, 2009). The case will be bounded by grade levels within the school, specifically kindergarten and first-grade, and design iterations will focus on leveraging an open-source curriculum aligned with state and national standards for teaching spatial reasoning. Kindergarten and first-grade teachers will be new to the building in fall 2022, which, combined with the school's inquiry learning framework and adoption of an open-source mathematics curriculum, offers a unique opportunity to spatialize a standards-aligned curriculum by offering a companion guide to typical mathematics instruction. Data collection will occur concurrently with summer and fall semester professional learning opportunities for situated teacher learning about spatial reasoning in which they will collaboratively construct pedagogical tools to enact spatial reasoning teaching practices (Horn & Garner, 2022; Moss et al., 2015). Given the school calendar and professional learning session structures, I anticipate three opportunities for group professional learning with individualized coaching occurring more frequently.

Using semi-structured interviews (Merriam, 1998), I will collect data on in-service teachers' background knowledge and beliefs about teaching spatial reasoning and their experiences with professional development that could support their enactment of spatial reasoning teaching practices. These interviews inform two learning components within the proposed model of situated teacher learning (i.e., the external and personal domains; see Figure 1). I am also creating a companion guide for the kindergarten and first-grade mathematics curriculum with representations and extensions. Pending school district approval, I will observe teachers enacting spatial reasoning instruction within typical mathematics instruction. I will also conduct debrief interviews to elicit feedback after teachers enact recommended practices in the classroom for personal reflections that will be used to triangulate findings and support internal validity and trustworthiness of findings (Merriam, 1998). Data collection and analysis will occur recursively to inform revisions to the research design and suite of tools as I learn how to better support teachers' spatial reasoning instructional practices.

Data Analysis

I will use iterative rounds of qualitative data analysis to explore how researchers can support early childhood teachers in enacting spatial reasoning teaching practices. I will apply a priori provisional codes based on learning components in the proposed bidirectional model of situated teacher learning while reading sources and recording preliminary jottings (Saldaña, 2016). I conjecture that data might inform multiple components and will therefore employ simultaneous coding (Miles et al., 2014) of passages that align with inclusion into multiple a priori descriptive codes. I will then engage in open coding to locate emergent themes using methods most appropriate for each data source type. Interviews will be coded using In Vivo coding methods (Saldaña, 2016) to develop codes and themes that can then be supported by descriptive statistics emerging from survey data of teachers' perceptions about enacting spatial reasoning instruction. Between iterative coding rounds, member checking will ensure accurate capture of teachers' perspectives and provide opportunities to integrate revisions (Merriam, 1998).

Results and Implications

This in-progress research contextualizes Horn and Garner's (2022) work on teacher learning using a sociocultural approach to create a local theory for early childhood teachers' enactment of spatial reasoning instruction. Given that current educational standards minimally mandate spatial reasoning skills instruction (National Research Council, 2006), we must understand how spatial reasoning aligns with our current curricular requirements to provide opportunities for teachers and students to learn these critical skills (Clements & Sarama, 2011). The proposed theoretical framework could guide future research to design professional learning opportunities and tools for teachers' use to leverage their knowledge and beliefs to teach spatial reasoning.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of teacher education, 59*(5), 389-407. https://doi.org/10.1177/002487108324554

Battista, M. T., Wheatley, G. H., & Talsma, G. (1982). The importance of spatial visualization and cognitive development for geometry learning in preservice elementary teachers. *Journal for Research in Mathematics Education*, *13*(5), 332–340. https://doi.org/10.2307/749007

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947-967. https://doi.org/10.1016/S0742-051X(02)00053-7

Clements, D. H., & Sarama, J. (2011). Early childhood teacher education: The case of geometry. *Journal of mathematics teacher education*, *14*(2), 133-148. https://doi.org/10.1007/s10857-011-9173-0

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational researcher*, *32*(1), 9-13.

Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., Pagani, L. S., Feinstein, L., Engel, M., Brooks-Gunn, J., Sexton, H., Duckworth, K., & Japel, C. (2007). School readiness and later achievement. *Developmental Psychology, 43*(6), 1428–1446. https://doi.org/10.1037/0012-1649.43.6.1428

Horn, I., & Garner, B. (2022). *Teacher learning of ambitious and equitable mathematics instruction: A sociocultural approach* (1st ed.). Routledge. https://doi.org/10.4324/9781003182214

Kazemi, E., & Hubbard, A. (2008). New directions for the design and study of professional development: Attending to the coevolution of teachers' participation across contexts. *Journal of Teacher Education, 59*(5), 428-441. https://doi.org/10.1177/0022487108324330

Lave J., & Wegner, E. (1991) *Situated learning: Legitimate peripheral participation*. Cambridge University Press.

Lord, T. R., & Rupert, J. L. (1995). Visual-spatial aptitude in elementary education majors in science and math tracks. *Journal of Elementary Science Education*, 7(2) 47-58. https://doi.org/10.1007/BF03173735

Merriam, S. B. (1998). *Qualitative research and case study applications in education*. John Wiley & Sons.

Miles, M. B., Huberman, A. M., & Saldaña, J. (2014). *Qualitative data analysis: A methods sourcebook* (3rd ed.). Sage.

Moss, J., Hawes, Z., Naqvi, S., & Caswell, B. (2015). Adapting Japanese lesson study to enhance the teaching and learning of geometry and spatial reasoning in early years classrooms: a case study. *ZDM, 47*(3), 377-390. https://doi.org/10.1007/s11858-015-0679-2

National Research Council. (2006). *Learning to Think Spatially*. The National Academies Press. https://doi.org/10.17226/11019.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4-15. https://doi.org/10.2307/1176586

Saldaña, J. (2016). *The coding manual for qualitative researchers* (4th ed.). Sage.

Sandoval, W. (2014). Conjecture mapping: An approach to systematic educational design research. *Journal of the Learning Sciences, 23*(1), 18-36. https://doi.org/10.1080/10508406.2013.778204

Skott, J. (2013). Understanding the role of the teacher in emerging classroom practices: Searching for patterns of participation. *ZDM Mathematics Education, 45*, 547-559. https://doi.org/10.1007/s11858-013-0500-z

Wai, J., Lubinski, D., & Benbow, C. P. (2009). Spatial ability for STEM domains: Aligning over 50 years of cumulative psychological knowledge solidifies its importance. *Journal of Educational Psychology*, *101*(4), 817-835. https://doi.org/10/1037/a0016127

Yin, R. K. (2009). *Case study research: Design and methods* (Vol. 5). Sage.

DESCRIBING THE MATHEMATICAL MODELS UNDERGRADUATE STEM MAJORS CONSTRUCT DURING MODELING TASKS.

Elizabeth Roan
Texas State University
Eaw109@txstate.edu

Keywords: Modelling, Undergraduate Education, Mathematical Representations

Mathematizing (the process of transforming a real-world situation into a mathematical model) receives a lot of attention in modelling literature because it is difficult for students. (Galbraith & Stillman, 2006; Jankvist & Niss, 2020). Many existing studies further our understanding of participants' difficulties with mathematizing by characterizing the "blockages" that occur (Brahmia, 2014; Stillman & Brown, 2014). While these studies adequately describe the blockages in terms of student knowledge, the field would benefit from a study of the mental processes that may be important for participants to productively engage in while mathematizing. Such mental processes would include quantitative reasoning and associated theories such as covariational/multivariational reasoning. Framing *mathematizing* this way is appropriate because previous studies focused on quantitative reasoning have indicated that quantitative reasoning promotes mathematization (Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Mkhatshwa, 2020). Additionally, researchers have suggested that quantitative reasoning is a lens with which to understand participants' mathematical reasoning about real world scenarios (Carlson, Larsen, & Lesh, 2003; Czocher & Hardison, 2021; Larson, 2013). In order to provide theoretical structure for and empirical evidence of the cognitive sources of students' blockages, as well as give consideration to how facilitators could help students overcome them, I will be analyzing STEM undergraduates' mathematization in terms of mental processes rather than actions and behaviors. Thus, the goal of this study is to answer the research question: "How do the mathematical models undergraduate STEM majors construct during mathematization arise from their quantitative, covariational, and multivariational reasoning?" The study will extend the field's understanding of quantitative, covariational, and multivariational reasoning by applying those theories in a novel demographic doing modeling tasks. Additionally, it will extend findings on STEM undergraduates' understandings of differential equations (Arslan, 2010; Rowland & Jovanoski, 2004) by stating a participant's meaning for the differential equation they constructed and its terms. Finally, the results will inform development of instructional materials based on participants' already-existing ways of thinking rather than promoting specific behaviors.

Theoretical Framework

My theoretical framework synthesizes five interrelated constructs/theories: quantitative reasoning, covariational reasoning, multivariational reasoning, schema of action (SoA), and mathematical modeling. Quantitative reasoning is "the analysis of a situation into a quantitative structure- a network of quantities and quantitative relationships" (Thompson, 1993), where a quantity is a triple consisting of an object, attribute, and quantification. Covariational reasoning is defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al., 2002, p. 354). Multivariational reasoning is the extension of covariational reasoning to two or more varying quantities (Jones, 2018). An SoA is defined as an organized pattern of thoughts or behaviors that can be applied to different cognitive objects in different situations (Nunes & Bryant, 2021). Lastly, a mathematical model is a conceptual system consisting of elements, the relationships among elements, operations, and rules of governing interactions. Because a mathematical model is a conceptual system, it is held, at least partially, internally and is expressed into the world through different representations, which are dictated by a participant's use of any external notation systems (Lesh & Doerr, 2003). Synthesizing these theories: (1) The elements of a mathematical model can be characterized by the quantities participants construct/impose onto the task scenario. (2) The rules governing the relationships and interactions between elements can be characterized by an individual's quantitative, covariational, and multivariational reasoning. (3) The operations on the quantities are determined by the SoA's employed by the STEM undergraduates to justify those operations.

Methods

The data collection for this study is part of a larger study of facilitator scaffolding moves that improve participants' mathematical modeling competencies. The data come from task-based interviews designed to evoke complex thought processes where the interviewer's role is to elicit complete coherent reasons the participants give for their own activities (Goldin, 2000). In the larger study, participants will be given at least six tasks over ten sessions. I will curate data from the sessions corresponding to three different dynamics scenarios common to differential equations (Cats & Birds (lotka-volterra predator-prey model), Tropical Fish (net rate inflow-outflow), and Ebola (SIR disease transmission model)). I choose these tasks because they invite different ways of constructing quantities and uses of quantitative operations. Additionally, dynamic tasks elicit dynamic reasoning, which is connected with quantitative reasoning and covariational reasoning (Keene, 2007). Probing questions like "What does <quantity> mean to you?", and "Why did you choose to <add/subtract/multiply/divide> <quantity> and <quantity>?" will bring out the participants' rationales for associating specific elements of the scenarios with the arithmetic operations they choose (that is, mathematization).

In the initial phase of analysis, I will go through the transcripts and written work to identify instances where participants: discuss their conception of a quantity, justify the usage of arithmetic operations, or reason about a quantity's values co-varying/multi-varying with other quantities. For each instance, I will identify the quantities participants construct/impose onto each task scenario by noting the object, attribute, and documenting evidence of quantification using the quantification criteria developed by Czocher and Hardison (2021). I will then document levels of covariational reasoning (Thompson & Carlson, 2017) and types of multivariational reasoning (Jones, 2018; Kuster & Jones, 2019; Panorkou & Germia, 2020) used with the quantities. I will then examine moments when a participant uses an arithmetic operation $(+, -, \cdot, \div)$ and its associated SoA. In some cases, the participant may give an explicit reason for using the operation and in others I may need to infer it based on data from previous tasks. I will use the SoA's documented in Thompson (2011), and Nunes and Bryant (2021) and extend those lists as needed. Each model a participant develops will be characterized by the quantities present in the model, including how each quantity was derived, the covariational/multivariational reasoning used with each quantity, and the SoA the participant used when using arithmetic operations with the quantities they deem relevant to the scenario. This will yield descriptions of how participants transform the quantities they impose onto a situation into mathematical representations via participants' quantitative, covariational, and multivariational reasoning. By performing this analysis across tasks and across participants, I will be able to provide evidence of the mental processes participants engage in while mathematizing different dynamic scenarios.

Acknowledgments

Research that will be reported in this paper is supported by National Science Foundation Grant No. 1750813 with Dr. Jennifer Czocher as principal investigator.

References

Arslan, S. (2010). Do students really understand what an ordinary differential equation is? *International Journal of Mathematical Education in Science and Technology, 41*(7), 873-888.

Brahmia, S. M. (2014). *Mathematization in Introductory Physics.* (Doctor of Philosophy). The State University of New Jersey,

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *33*, 352-378.

Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating a models and modeling perspective with existing research and practice. *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching*, 465-478.

Czocher, J., & Hardison, H. L. (2021). Attending to Quantities through the Modelling Space. In F. Leung, G. Stillman, G. Kaiser, & K. L. Wong (Eds.), *Mathematical Modelling Education in East and West* (pp. 263-272): Springer.

Ellis, A. (2007). The Influence of Reasoning with Emergent Quantities on Students' Generalizations. *Cognition and Instruction, 25*(4), 439-478.

Ellis, A., Ozgur, Z., Kulow, T., Williams, C., & Amidon, J. (2012). Quantifying exponential growth: The case of the Jactus. *Quantitative reasoning and mathematical modeling: A driver for STEM integrated education and teaching in context, 2*, 93-112.

Galbraith, P. L., & Stillman, G. (2006). A Framework for Identifying Student Blockages during Transitions in the Modelling Process. *ZDM: Mathematics Education, 38*(2), 143-161.

Jankvist, U. T., & Niss, M. (2020). Upper secondary school students' difficulties with mathematical modelling.

Jones, S. R. (2018). *Building on Covariation: Making Explicite Four Types of "Multivariation.".* Paper presented at the 21st annual Conference on Research in Undergraduate Mathematics Education.

Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as a parameter. *Journal of Mathematical Behavior, 26*, 230-246.

Larson, C. (2013). Modeling and quantitative reasoning: The summer jobs problem. In R. Lesh, P. L. Galbraith, C. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 111-118). Boston, MA: Springer.

Mkhatshwa, T. P. (2020). Calculus students' quantitative reasoning in the context of solving related rates of change problems. *Mathematical thinking and learning, 22*(2), 139-161.

Rowland, D. R., & Jovanoski, Z. (2004). Student interpretations of the terms in first-order ordinary differential equations in modelling contexts. *International Journal of Mathematical Education in Science and Technology, 35*(4), 503-516.

Stillman, G., & Brown, J. P. (2014). Evidence of implemented anticipation in mathematising by beginning modellers. *Mathematics Education Research Journal, 26*, 763-789.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics, 25*(3), 165-208.

FOSTERING A GROWTH MINDSET IN MATHEMATICS: FACULTY AND STUDENT EXPERIENCES

Yolanda G. Rush
Illinois State University
ygrush@ilstu.edu

Tami S. Martin
Illinois State University
tsmartin@ilstu.edu

Keywords: Affect, Attitude, and Beliefs.

A major hurdle for students at two-year institutions is to complete developmental (remedial) or transfer-level (credit-earning) mathematics courses with a grade of C or better. A national study revealed that about 68% of students at two-year institutions take remedial courses (Chen, 2016). Some students take two or three remedial courses consecutively and only about 11% of those students pass their transfer level “gateway” course (Jaggars & Stacey, 2014). As the name implies, gateway courses are used as a prerequisite to enter many majors. College algebra is one such gateway course (Giang-Nguyen, 2015; Tunstall, 2018) and nearly 1/3 of the college algebra enrollments occur at two-year institutions (Bressoud, 2021). However, college algebra has only about a 50% pass rate (C or better) (Giang-Nguyen, 2015; Okello, 2010). Further, many college algebra students fear mathematics and have their performance influenced by prior experiences in mathematics (Herriot, 2006).

Affective factors—in particular, mindset and self-efficacy—may play a vital role in the success of students at two-year institutions (Center for Community College Student Engagement (CCCSE), 2019). Specifically, researchers have found that mindset interventions can influence students to seek help (Shively & Ryan, 2013), increase perseverance, and lead to lower failure rates in mathematics (Lewis, 2019). In addition, Auten (2013) found that mindsets of students and teachers play a vital role in students’ academic success at community colleges. Although previous work has examined student achievement after mindset interventions (Aronson et al., 2001; Blackwell et al., 2007; Steele, 2007), few studies have documented the experiences of two-year college students and their instructors while they are enrolled in courses that attempt to foster growth mindsets. Knowing more about college students’ experiences, personal views, motivations, attitudes regarding mathematics (Mesa, 2017), and faculty members’ perceptions as they work to implement strategies aimed at fostering growth mindsets in their students would provide evidence about which factors contribute to the effectiveness of mindset interventions. Rather than focusing on the outcome of mindset intervention, I seek to understand the process of attempting to positively affect students mindsets and self-efficacy beliefs.

Thus, my research questions are: (1) How does each instructor approach and implement growth mindset components into their courses? (2) What do two-year college instructors report about their experiences during the process of developing and implementing strategies for fostering growth mindset in their developmental courses and how do they perceive the effects of their approaches on students? (3) What do two-year college students report about their experiences and effects on mindset and self-efficacy beliefs as they engage with their instructors’ who are attempting to foster a growth mindset in their developmental courses?

This will be a qualitative study with a conceptual framework based on a constructivist paradigm and mindset theory. The constructivist paradigm—the belief that learners (students and teachers) will construct their own understanding through experiencing and reflecting on experiences (Lerman, 1989)—will be used as a lens through which to investigate how students and teachers construct their own understanding of growth mindset beliefs as they interact in mathematics classrooms geared toward fostering growth mindset beliefs.

Mindset theory is based on the idea that people hold different beliefs about whether certain attributes (e.g., intelligence and personality) are malleable. People form implicit theories based on their goals and how they interpret the outcomes of their efforts (Bernecker & Job, 2019; Dweck, 2006). A growth mindset (incremental theory) is based on the belief that intelligence is malleable and can be grown through effort and guidance, while a fixed mindset (entity theory) is based on the belief that each person has only a certain amount of talent or intelligence (Dweck, 2006). In learning and performance, people with growth mindsets are open to change; they see failure or adversity as the need to put forth more effort (Bernecker & Job, 2019); and compared to people without growth mindsets, their goals are more aimed at improvement (Dweck, 2006). In this work, mindset theory will serve as a lens through which to view students' and teachers' perspectives about their experiences, as teachers attempt to foster growth mindset beliefs in their students over the course of one semester.

Self-efficacy beliefs can play a major role in the actions of students. Self-efficacy is the belief in one's capability of "organizing and executing" a plan to reach a certain goal (Bandura, 1997, p. 3). This belief can influence one's effort, the amount of time they are willing to spend on a task, and how long they are willing to persist when faced with failure or obstacles (Bandura, 1997). My goal is to determine whether students' self-efficacy beliefs were influenced as they engaged in a classroom that fostered growth mindset beliefs.

In this case study, I will focus on one to three two-year college instructors who are teaching a developmental mathematics course and some of their students. The developmental mathematics course is specifically designed for students whose next mathematics course is college algebra. I will work with the participating instructors to develop approaches for promoting growth mindsets in their students. I will document the instructor collaboration process from my own perspective and instructors will keep weekly journals to document their experiences. Before and after the semester, I will interview the instructors and some of their students, regarding their perceptions about the course, their prior experiences, and their mindsets related to mathematics. For the students only, I will also ask about their mathematical self-efficacies. I will use surveys to assess mindsets of instructors, and mindset and self-efficacy beliefs of students before and after the semester. The survey questions I intend to use are designed to measure mathematical mindsets and mathematical self-efficacy beliefs on a 5-point Likert scale (Rothrock, 2019; Williams, 2015). Student responses will be used to determine if there were shifts in mindset and self-efficacy beliefs during the semester.

This research is important because it builds on past work that has investigated student mindset and self-efficacy at the collegiate level and additional work on effectiveness of growth mindset interventions. In particular, this study will fill a gap in existing research by examining the process of fostering growth mindset beliefs in the mathematics classroom from both instructor and student perspectives. Most community colleges have not implemented the much-needed direct efforts to influence mindset and self-efficacy (CCCSE, 2019), and it is important to learn more about two-year college students' experiences and personal views within the mathematics classroom, and faculty members' strategies (Mesa, 2017). Therefore, my study will capture how instructors experience their work attempting to foster growth mindsets in their students, and how students perceive the effects of those efforts. This study will add to the literature about mindset studies in mathematics at community colleges by documenting the perceptions and experiences of students and instructors in their work on mathematical mindsets.

References

Aronson, J., Fried, C. B., & Good, C. (2001). Reducing the effects of stereotype threat on African American College students by shaping theories of intelligence. *Journal of Experimental Social Psychology 38,*(2*)*, 1–14.

Auten, M. A. (2013). Helping educators foster a growth mindset in community college classrooms. (Publication No. 3591125) [Doctoral dissertation, Walden University] ProQuest Dissertations Publishing.

Bandura, A. (1997). *Self-Efficacy: The exercise of control*. W. H. Freedman.

Bernecker, K., & Job, V. (2019). Mindset theory. In K. Sassenberg & M. L. W. Vlieks (Eds.), *Social psychology in action: Evidence-based interventions from theory to practice* (pp. 179–191). Springer. https://doi.org/10.1007/978-3-030-13788-5_12

Bressoud, D. (2021, November 1). The 2021 CBMS survey of departments in mathematical sciences. *Launchings, Teaching & Learning*. https://www.mathvalues.org/masterblog/the-2021-cbms-survey-of-departments-in-the-mathematical-sciences

Blackwell, L. S., Trzesniewski, K. H., & Dweck, C. S. (2007). Implicit theories of intelligence predict achievement across an adolescent transition: A longitudinal study and an intervention. *Child Development, 78*(1), 246–263. Retrieved from http://dx.doi.org/10.1111/j.1467-8624.2007.00995.x

Center for Community College Student Engagement (2019). *A mind at work: Maximizing the relationship between mindset and student success*. The University of Texas at Austin, College of Education, Department of Educational Leadership and Policy, Program in Higher Education Leadership. https://www.ccsse.org/NR2019/Mindset.pdf

Chen, X. (2016). *Remedial course taking at U.S. public 2- and 4-year institutions: Scope, experiences, and outcomes* (NCES 2016-405). U.S. Department of Education. https://nces.ed.gov/pubs2016/2016405.pdf

Dweck, C. (2006). *Mindset: The new psychology of success*. Random House.

Giang-Nguyen, T. N., (2015). A case study of students' motivation in college algebra courses. *Community College Journal of Research and Practice, 39*(8*)*, 698–707. https://doi.org/10.1080/10668926.2013.824394

Herriott, S. R. (2006). Changes in college algebra. In Hastings, N. B., F. S. Gordon, S. P. Gordon, S. P. Gordon, and J. Narayan (Eds.), *A fresh start for collegiate mathematics: Rethinking the course below calculus*, (pp. 90–100).

Jaggars, S., & Stacey, G. W. (January, 2014). *What we know about developmental education outcomes* (Community College Research Center Report). Teachers College, Columbia University. https://ccrc.tc.columbia.edu/media/k2/attachments/what-we-know-about-developmental-education-outcomes.pdf

Lerman, S. (1989). Constructivism, mathematics and mathematics education. *Educational Studies in Mathematics 20***,** 211–223. https://doi.org/10.1007/BF00579463

Lewis, H. M. (2019). Implementation and effects of university college algebra growth mindset structured assessments in large lectures. [Doctoral Dissertation, Utah State University]. *All Graduate Theses and Dissertations.* 7674.

Mesa, V. (2017). Mathematics education at U.S. public two-year colleges. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 949–967). National Council of Teachers of Mathematics.

Okello, N. P., (2010). Learning and teaching college algebra at university level: Challenges and Opportunities: A case study of USIU. *The Journal of Language, Technology & Entrepreneurship in Africa 2*(1), 204–219.

Rothrock, K. S. (2019). An investigation of first-time college freshmen and relationships among mathematical mindset, identity, self-efficacy, and use of self-regulated learning strategies. (Publication No. 13885687) [Doctoral dissertation, University of Kansas] ProQuest Dissertations Publishing.

Shively, R.L. & Ryan, C.S. (2013). Longitudinal changes in college math students' implicit theories of intelligence. *Social Psychology Education, 16*(2*)***,** 241–256. https://doi.org/10.1007/s11218-012-9208-0

Steele, C. (2007). Stereotype threat and African-American student achievement. In D.B. Grusky & S. Szelenyi (Eds.), The inequality reader: Contemporary and foundational readings in race, class, and gender (2nd ed., 252–257. Routledge. https://doi.org/10.4324/9780429494468-31

Tunstall, S. L. (2018). College algebra: Past, present, and future. *Primus, 28*(7), 627–640. https://doi.org/10.1080/10511970.2017.1388315

Williams, C. (2015). Examining openness to pedagogical change among secondary mathematics teachers: Developing and testing a structural model. (Publication No. 10003482) [Doctoral dissertation, University of North Dakota] ProQuest Dissertations Publishing.

A CASE STUDY APPROACH TO INCONGRUOUS MATHEMATICS SELF-EFFICACY IN UNDERGRADUATES: A SUMMARY OF DISSERTATION RESEARCH

Bridgette E. Russell
Central Michigan University
buse1b@cmich.edu

Christine M. Phelps-Gregory
Central Michigan University
phelp1cm@cmich.edu

We have all encountered students who seem over- or under-confident in their skills. For example, the student who is confident they will do well but fails every exam. Or the student who repeatedly worries of failing yet gets 100% on every exam. These students have incongruous mathematics self-efficacy. Mathematics self-efficacy (MSE) is a person's beliefs about their ability to succeed in mathematical activities. Incongruous math self-efficacy (IMSE) is when a person's MSE and their abilities do not match to a significant extent.

Background and Theoretical Framework

Self-efficacy is defined as a person's beliefs about their ability to complete a particular task (ranging from a problem to a full course) (Bandura, 1986). Much research has been done on self-efficacy and its predictive power in student success (e.g., Bandura, 1986; Hackett & Betz, 1982; MacPhee et al., 2013; Zeldin et al., 2008). This research suggests that a student's self-efficacy is directly correlated to their success in mathematics, their career choices, and their success within their career (e.g., Buschor et al. 2014; Hackett, 1985; Hackett & Betz, 1989; Pajares & Miller, 1994). Research suggests, however, that some students are overconfident, and some are underconfident; that is, some students have IMSE, also called "feeling-of- knowing accuracy" or calibration (Schraw, 1995; Pajares & Miller, 1994). These terms have been used to describe a disconnect between a person's self-efficacy beliefs about a topic and their abilities as measured through assessments (e.g., Stolp & Zabrucky, 2009; Sheldrake, 2016)

More research is needed on IMSE; in particular, qualitative work with undergraduate students is often lacking in the literature. Much of the current literature is quantitative in nature (Labuhn et al., 2010). In addition, much of the work on IMSE is much focused on K – 12 students, as well as teachers (e.g., Chen, 2006; Dassa & Nichols, 2019; Labuhn et al., 2010). When looking at the population of undergraduate students there is even less focused on IMSE (Morán-Soto & Benson, 2018), even though undergraduates differ in significant ways from K – 12 students, particularly in their need for self-motivation.

My dissertation study will focus on undergraduate students in an intermediate algebra course. The following research questions will guide the study. **Research question 1:** What are the global and problem focused mathematics self-efficacy beliefs of undergraduate students in an intermediate algebra course? **Research question 2:** For these students, what is the relationship between problem focused self-efficacy beliefs and performance on a given set of problems? **Research question 3:** For these students, what is the relationship between global mathematics self-efficacy beliefs and performance on an exam in their course? **Research question 4:** For these students, what changes occur in the global and problem focused mathematics self-efficacy beliefs before and after a mastery experience (exam/problems)?

Methods

Using the pilot study as a foundation, students from intermediate algebra will be selected based on an initial survey assessing their MSE beliefs and mathematics history. Fifteen participants will be selected based on their MSE scores, selecting for variation in MSE. They will then be interviewed before their exam. The first interview will follow a semi-structured in-depth format and ask for a more detailed mathematics history focusing on MSE beliefs in the participants' own words. During interview 1 there will also be a problem focused section where the participants will be asked their MSE for three problems from their course work and then be asked to solve the problems. A second survey will be administered after the participants have taken their first exam, but before their grades are returned. This survey will ask participants how they feel about the exam, and their predicted results. After their grades are returned, a second interview will be conducted to assess the participants' predicted results compared to their actual results, and any changes to their MSE beliefs.

Preliminary Findings and Anticipated Results

The purpose of this study is to identify students who display the characteristics of IMSE and to examine their understanding of this misalignment. In doing so, I hope to uncover more about the MSE beliefs of undergraduate students in an intermediate algebra course.

In terms of research question one, in my pilot study I found a range of MSE beliefs, both global and problem focused. Out of the five participants, four expressed moderate to high global MSE beliefs. Across the three problems participants solved in interview one, participants seemed to have more confidence with computational problems, versus word problems, and there was evidence of IMSE in the problem focused self-efficacy ratings, with participants being confident in their answers and getting them wrong, as well as not being confident in their answers, and getting them correct. In my larger dissertation study, I anticipate a broader range of MSE beliefs, as well as more instances of IMSE beliefs. I am also interested in a potential connection between participants' global MSE beliefs and their problem focused MSE beliefs.

In terms of research question two, in my pilot study students' global MSE aligned with their problem performance. There were, however, instances of problem focused IMSE identified in all participants who completed the study. For one participant in particular, Gabrielle, she expressed high confidence on the more difficult computational problem but was incorrect in her solution; she then expressed low confidence in the word problem and was the only participant to get it correct. In my dissertation study, I anticipate finding more cases like Gabrielle.

In terms of research question three, in my pilot study, participants overestimated their performance on the exam. No participant passed the exam; however, all but one participant thought that they would pass. Participants did tend to hedge their predictions. In my dissertation study, I expect to find more instances of IMSE and hope to examine the relationship between global MSE beliefs of students and final course grade. I also hope to find students who underestimate their performance and see how their data compares.

In terms of research question four, in my pilot study there was little to no change in students' global MSE or course grade predictions before and after the exam. There was, however, change in the participants' problem focused MSE especially related to the easier computational problem and the word problem. Before solving these problems, students would rate them as very difficult, but then after solving would rate them as less difficult. In my dissertation study, I anticipate finding similar results. I hope to find more instances of global MSE changes based on exam performance, and how this might be related to final course grade performance.

Implications

This work has important implications for research and practice. In particular, we hope to expand the current understanding of the relationship between performance and MSE by providing qualitative case studies of instances of IMSE. In addition, there are practical implications from having an improved understanding of these relationships, particularly for helping students better judge their own expected performance and their need for additional study.

References

Bandura, A. (1986). The Explanatory and Predictive Scope of Self-Efficacy Theory. Journal of Social and Clinical Psychology, 4(3), 359-373.

Buschor, C. B., Berweger, S., Frei, A. K., & Kappler, C. (2014). Majoing in STEM - What Accounts for Women's Career Decision Making? A Mixed Methods Study. The Journal of Educational Research, 107(3), 167-176.

Chen, P. (2006). Relationship between students' self-assessment of their capabilities and their teachers judgements of students' capabilities in mathematics problem solving. Psychological Reports, 98, 765-778.

Dassa, L., & Nichols, B. (2019). Self-efficacy or Overconfidence? Comparing Preservice Teacher Perceptions of Their Content Knowledge and Teaching Abilities to the Perceptions of Their Supervisors. The New Educator, 15(2), 156-174.

Hackett, G. (1985). Role of Mathematics Self-Efficacy in the Choice of Math-Related Majors of College Women and Men: A Path Analysis. Journal of Counseling Psychology, 32(1), 47-56.

Hackett, G., & Betz, N. (1982). Mathematics Self-Efficacy Expectations, Math Performance , and the Consideration of Math-Related Majors. Annual Meeting of the American Educational Research Association, 1-37.

Hackett, G., & Betz, N. E. (1989). An Exploration of the Mathematics Self-Efficacy/Mathematics Performance Correspondence. Journal for Research in Mathematics Education, 20(3), 261-273.

Labuhn, A. S., Zimmerman, B. J., & Hasselhorn, M. (2010). Enhancing students' self-regulation and mathematics peformance: the influence of feedback and self-evaluative standards. Metacognition Learning, 5, 173-194.

MacPhee, D., Farro, S., & Canetto, S. S. (2013). Acadmeic Self-Efficacy and Performance of Underrepresented STEM Majors: Gender, Ethnic, and Social Class Patterns.

Morán-Soto, G., & Benson, L. (2018). Relationship of mathematics self-efficacy and competence with behaviors and attitudes of engineering students with poor mathematics preparaton. International Journal of Education in Mathematics, Science and Technology, 6(3), 200-220.

Pajares, F., & Miller, M. D. (1994). Role of self-efficacy and self-concept beliefs in mathematical problem solving: A path analysis. Journal of Educational Psychology, 86(2), 195-203.

Schraw, G. (1995). Measures of Feeling-of-knowing Accuracy: A new look at an old problem. Applied Cognitive Psychology, 9, 321-332.

Sheldrake, R. (2016). Differential predictors of under-confidence and over-confidence for mathematics and science students in England. Learning and Individual Differences, 49, 305-313.

Stolp, S., & Zabrucky, K. M. (2009). Contributions of metacognitive and self-regulated learning theoreis to investigations of calibration of comprehension. International Electronic Journal of Elementary Education, 2(1), 7-31.

Zeldin, A. L., Britner, S. L., & Pajares, F. (2008). A Comparative Study of the Self-Efficacy Beliefs of Successful Men and Women in Mathematics, Science, and Technology Careers. Journal of Research in Science Teaching, 45(9), 1036-1058.

ASSESSING PRESCHOOL TEACHERS' SELF-EFFICACY AND ATTITUDES TOWARD STEM THROUGH A PROFESSIONAL DEVELOPMENT PROGRAM

Hannah Smith
Worcester Polytechnic Institute
hsmith2@wpi.edu

Background of the Project

The importance of integrating STEM education into preschool classrooms has been echoed across contexts (National Research Council, 2011; Chesloff, 2013). However, teachers' low confidence and negative attitudes toward STEM, combined with few opportunities for professional development (PD), have resulted in an avoidance of STEM teaching (Timur, 2012). Effective PD programs provide opportunities to improve teachers' confidence and attitudes toward STEM and are crucial to the success of preschool programs (Hamre et al., 2017). Prior studies have shown that effective PD includes developmentally appropriate instruction, individual coaching (Bowman, 2001), group-focused interventions (Ginsburg et al., 2006), and modeling instruction (Wasik et al., 2006). However, PD workshops may provide only one of the aspects mentioned above which is inadequate and may not be high quality (Gomez et al., 2015). Further, common PD programs for preschool teachers lack research-based practices and are not monitored for content and effectiveness (Gomez et al., 2015).

Goals of the Research

The goal of the current project is to assess the effectiveness of a PD program exposing Head Start preschool teachers to STEM. This program uses research-based practices and combines the use of developmentally appropriate activities, group sessions, individualized coaching, and modeling instructional practices. Further the program is comprehensive and long term, which allows for greater growth among our teacher participants and connection with teachers. Through the program, I aim to increase teachers' attitudes and self-efficacy towards STEM teaching, increase their ability to recognize STEM activities in their teaching, and expand current curriculum to incorporate more STEM activities. As part of a research fellowship with the AIMS (Activities Integrating Math and Science) Center that focuses on increasing STEM in classrooms through PD, I am working with a team to design and implement a six-month PD program with preschool teachers focused on increasing STEM activities in their classrooms. For my dissertation, I am leading the research behind the PD program to investigate the following research questions: 1) How do teachers' self-efficacy and attitudes toward STEM change after participating in the PD? 2) In what ways do teachers expand and implement activities presented in the PD? 3) What aspects of the PD lead to the greatest growth in teachers' use of STEM activities in their classrooms?

Methodology

The current project will provide STEM PD for a Head Start preschool site. Participants include 26 California Head Start preschool teachers who teach 131 children in 8 classrooms with each classroom having 2-4 teachers. This PD program will take place over 6 months, beginning in January 2022, with each month covering a different module and focus: patterns, spatial communication, movement, board games, robots, and coding. Each module will follow the same format and consist of a 2-hour professional learning (PL) session, classroom integration of what was learned in the PL session with follow up individualized coaching, and a family engagement section with take home kits for each child around the module's topic. I will be focusing on studying the program, as well as the PL sessions. To study the implementation of the program, I will analyze survey and interview responses. I will also review participants' responses and interactions during the PL sessions, as well as researcher field notes of the sessions. Prior to the beginning, and after completion of the 6-month PD program, teachers will complete a 5-point Likert scale survey to uncover their attitudes and self-efficacy around teaching STEM. This survey was created by combining the Early Childhood Teacher's Attitudes toward Science Teaching Scale (Cho, 2003) and two sections of the Teacher Efficacy and Attitudes Toward STEM (T-STEM) Survey (Friday Institute for Educational Innovation, 2012). Additionally, teachers will fill out a 4-question open-ended reflection worksheet at the end of each PL session asking: what they enjoyed about the session, what was difficult and challenging about the session, how researchers could more effectively lead the session in the future, and a final question specific to each module. For example, the question for the pattern module is: "What patterns are you now noticing in your daily life? Personal experience? Community?" A subset of 10 teachers will also participate in 20-minute exit interviews asking about their overall experience with the program, classroom implementation, expansion of activities, and growth as a preschool teacher of STEM. PL worksheets, interviews and researcher field notes during the sessions will be qualitatively coded to find common themes. I will also include measures of fidelity of the program including number of PL sessions attended, coaching sessions attended, and completion of worksheets. Analyzing these data sources will help us determine the effectiveness of the program. Specifically analyzing the survey responses before and after completing the program will allow us to assess teacher growth in self-efficacy and attitudes toward STEM teaching. By examining the worksheets, interviews and field notes I will be able to detect common themes around how teacher's ability to recognize STEM improved as the program continued and whether they felt as if they had improved in this competency. I will also be able to analyze how teachers were able to implement the activities from the sessions into their classroom and expand on these activities.

Preliminary or Expected Findings

I expect to see increases in teacher attitude and self-efficacy toward STEM as measured by the Early Childhood Teacher's Attitudes toward Science Teaching Scale and the T-STEM Survey. I will run repeated measures t-tests to assess growth in these surveys. To assess whether a teacher's abilities to recognize STEM in their teaching as well as expand their current curriculum to include STEM, the worksheet responses, exit interview responses and field notes will be examined. I also expect to see teachers develop an increased awareness of the ways children learn and experience STEM in the preschool classroom.

Expected Contributions

This dissertation work will have direct contributions to the teachers and their students who participated in this professional development program. As a research practice partnership study, one of the main goals is to provide impact to educators and their students. In addition, this work will examine the effectiveness of this program and provide recommendations for future preschool professional development programs. This will also provide further insights into how we can best increase STEM in preschool classrooms and increase teachers' attitudes and self-efficacy towards STEM.

References

Bowman, D. Burns.(2001). Eager to learn: Educating our preschoolers.

Chesloff, JD. (2013). Why STEM education must start in early childhood. Education Week, 32 (23), 27–32

Cho, H. S., Kim, J., & Choi, D. H. (2003). Early childhood teachers' attitudes toward science teaching: A scale validation study. Educational Research Quarterly, 27(2), 33.

Friday Institute for Educational Innovation (2012). Teacher Efficacy and Attitudes Toward STEM Survey-Elementary Teachers, Raleigh, NC: Author

Ginsburg, H. P., Kaplan, R. G., Cannon, J., Cordero, M. I., Eisenband, J. G., Galanter, M., & Morgenlander, M. (2006). Helping early childhood educators to teach mathematics.

Gomez, R. E., Kagan, S. L., & Fox, E. A. (2015). Professional development of the early childhood education teaching workforce in the United States: An overview. Professional Development in Education, 41(2), 169-186.

Hamre, B. K., Partee, A., & Mulcahy, C. (2017). Enhancing the impact of professional development in the context of preschool expansion. AERA Open, 3(4), 2332858417733686.

Lieber, J., Butera, G., Hanson, M., Palmer, S., Horn, E., Czaja, C., ... & Odom, S. (2009). Factors that influence the implementation of a new preschool curriculum: Implications for professional development. Early education and development, 20(3), 456

National Research Council (2011). Successful K-12 STEM Education: Identifying Effective Approaches in Science,Technology, Engineering, and Mathematics. Washington, DC: The National Academies Press.

Timur, B. (2012). Determination of factors affecting preschool teacher candidates' attitudes toward science teaching. Educational Sciences: Theory and Practice, 12(4), 2997–3009.

Wasik, B. A., & Hindman, A. H. (2011). Improving vocabulary and pre-literacy skills of at-risk preschoolers through teacher professional development. Journal of educational psychology, 103(2), 455.

SECONDARY MATHEMATICS TEACHERS' DECISION-MAKING WITH ONLINE SOCIAL JUSTICE-ORIENTED CURRICULAR MATERIALS

Anita Sundrani
University of Houston
asundrani@UH.edu

Keywords: Curriculum; Equity, Inclusion, and Diversity; Social Justice; Technology

Introduction

As culturally responsive frameworks and anti-bias resources become more commonplace, mathematics teachers across the United States are increasingly seeking content from the internet as they plan their lessons (Gewertz, 2015; Timberlake et al., 2017). However, there is mounting evidence that resources from online spaces may not be standards-aligned, developmentally appropriate, equity-based, or even accurate (Gallagher et al., 2019; Greene, 2016). Still there are teachers who are able to discern high quality and social justice-oriented resources embedded in critical pedagogy that lead to positive student academic, cultural, and affective outcomes. To date, there has been little investigation into secondary mathematics teachers' decision-making processes when selecting which materials to use from online spaces and even less about choosing critical content centered around teaching mathematics for social justice. Therefore, the goal of this project is to explore mathematics teachers' decision-making processes when planning and enacting social justice-oriented mathematics lessons from online spaces.

Background & Theoretical Perspective

Previous studies have investigated how in-service and pre-service mathematics teachers design, implement, and revise lessons centered around social justice (Harper et al., 2021; Myers, 2019). While these studies referenced sources linked to online spaces, they did not interrogate the decision-making processes of the participants as they navigated these spaces.

This study draws from the theoretical frameworks of teacher decision-making and critical mathematics education to investigate how secondary mathematics teachers engage with these spaces, make decisions to use certain materials, and enact these instructional resources in the classroom. Remillard and Heck's (2014) model of the curriculum enactment process provides insight into what factors influence teachers' decisions when using curriculum. A focus on critical mathematics provides a lens into how educators who teach mathematics for social justice approach curricular content from online spaces when planning and enacting their lessons.

Critical Mathematics Educators' Decision-Making

Over the past decade, researchers have begun to explore the decision-making processes of social justice and critical educators. King and Nomikou (2018) linked the ideas of decision-making and teacher agency by suggesting that educators are more apt to take risks in their decision-making if they feel they have more autonomy and agency over their work. MacPherson (2010) investigated eight different decision-making orientations that pre-service teachers take up when teaching culturally and linguistically diverse learners. In her study, Bartell's (2013) participants felt that one of the challenges of TMSJ was making the decision of how to balance social justice with mathematics content. These previous findings suggest that research on decision-making throughout the curriculum enactment process in critical mathematics education still needs to be explored.

Methods

The following research questions will guide my exploration.

1. How do secondary mathematics teachers engage with online spaces when searching for critical mathematics content?
2. What are secondary mathematics teachers' decision-making processes when choosing online materials to teach and/or supplement their curriculum in critical ways?
3. How do secondary mathematics teachers enact critical mathematics content found from online spaces in their classrooms?

Research Design & Data Collection

A sample of secondary mathematics teachers will be recruited from across the United States based on recommendations from critical mathematics education scholars to take part in a multiple-case study (Yin, 2009). As part of phase I of study activities, the participants will be given a demographics survey, asked to provide three social justice-oriented mathematics lesson plans, and interviewed about their interactions with online spaces during lesson planning. During phase II, teachers will be asked to keep a journal as they plan for one social justice-oriented lesson. A semi-structured interview will take place prior to enacting this lesson in the classroom. For phase III, I will visit the participants' schools to observe the lesson plan that was discussed via Microsoft Teams to see how the lesson is enacted in the classroom. At the end of the school day, the teacher will be interviewed to learn how the lesson plan was enacted in the classroom and understand the teacher's in-the-moment decisions.

Data Analysis

To answer RQ1 and RQ2, I will provide a summary of the participants' demographics, including the school characteristics, teacher characteristics, and course information. I will also provide response-level excerpts open coded using an In Vivo approach (Saldaña, 2013) from all interviews and the planning journals. These codes will be analyzed using Remillard and Heck's (2014) model of curriculum enactment and categorized by common characteristics of critical mathematics teachers based on literature (Gutiérrez, 2012; Gutstein, 2006; Ladson-Billings, 1995). Additionally, data gathered from all lesson plans provided will be analyzed using an appropriate lesson plan analysis protocol selected through relevant literature (Hu et al., 2019).

To answer RQ3, I will first open code the data gathered from the lesson planning observations based on teacher actions and verbal communication. I will then categorize this data by in-the-moment decisions identified through a comparison of the intended and enacted lesson plans. I will analyze these moments using a critical framework protocol that has been used with teachers previously to identify strengths with cultural competence and critical consciousness or adapt a critical framework that already exists, as necessary (Brown & Crippen, 2016; Hu et al., 2019).

Anticipated Contributions

Anticipated findings include insights into how social justice topics are embedded in secondary mathematics curriculum. I also hope to understand these educators' rationales for decisions they make when they plan and enact their critical mathematics lessons. This research may also set the stage for further research into critical mathematics educators' decision-making processes. Lastly, findings may be used to inform mathematics methods course content for pre-service teachers and professional development opportunities for in-service mathematics teachers to learn how to critically navigate online spaces when planning for instruction.

References

Bartell, T. G. (2013). Learning to teach mathematics for social justice: Negotiating social justice and mathematical goals. *Journal for Research in Mathematics Education*, *44*(1), 129–163. https://doi.org/10.5951/jresematheduc.44.1.0129

Brown, J. C., & Crippen, K. J. (2016). The Growing Awareness Inventory: Building capacity for culturally responsive science and mathematics with a structured observation protocol. *School Science and Mathematics*, *116*(3), 127–138. https://doi.org/10.1111/ssm.12163

Gallagher, J., Swalwell, K., & Bellows, E. M. (2019). "Pinning" with pause: Supporting teachers' critical consumption on sites of curriculum sharing. *Social Education*, *83*(4), 217–224.

Gewertz, C. (2015, February 17). States ceding power over classroom materials. *Education Week*. https://www.edweek.org/teaching-learning/states-ceding-power-over-classroom-materials/2015/02

Greene, K. (2016). For sale: Your lesson plans. *Educational Leadership*, *74*(2), 28–33.

Gutiérrez, R. (2012). Context matters: How should we conceptualize equity in mathematics education? In B. Herbel-Eisenmann, J. Choppin, D. Wagner, & D. Pimm (Eds.), *Equity in Discourse for Mathematics Education: Theories, Practices, and Policies* (pp. 17–33). Springer. https://doi.org/10.1007/978-94-007-2813-4

Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. Routledge.

Harper, F. K., Maher, E. M., & Jung, H. (2021). Whiteness as a stumbling block in learning to teach mathematics for social justice. *Investigations in Mathematics Learning*, *13*(1), 5–17. https://doi.org/10.1080/19477503.2020.1827662

Hu, S., Torphy, K. T., & Opperman, A. (2019). Culturally relevant curriculum materials in the age of social media and curation. *Teacher College Record*, *121*(14), 1–22.

King, H., & Nomikou, E. (2018). Fostering critical teacher agency: the impact of a science capital pedagogical approach. *Pedagogy, Culture and Society*, *26*(1), 87–103. https://doi.org/10.1080/14681366.2017.1353539

Ladson-Billings, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, *32*(3), 465–491. https://doi.org/10.3102/00028312032003465

MacPherson, S. (2010). Teachers' collaborative conversations about culture: Negotiating decision making in intercultural teaching. *Journal of Teacher Education*, *61*(3), 271–286. https://doi.org/10.1177/0022487109353032

Myers, M. (2019). Learning to teach mathematics for social justice: Helping preservice teachers connect critical topics to standards. In S. Otten, A. G. Candela, Z. de Araujo, C. Haines, & C. Munter (Eds.), *Proceedings of the forty-first annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1087–1095).

Remillard, J., & Heck, D. J. (2014). Conceptualizing the curriculum enactment process in mathematics education. *ZDM - International Journal on Mathematics Education*, *46*(5), 705–718. https://doi.org/10.1007/s11858-014-0600-4

Saldaña, J. (2013). The coding manual For qualitative researchers. In *Sage Publications* (Second).

Timberlake, M. T., Burns Thomas, A., & Barrett, B. (2017). The allure of simplicity: Scripted curricula and equity. *Teaching and Teacher Education*, *67*, 46–52. https://doi.org/10.1016/j.tate.2017.05.017

Yin, R. K. (2009). *Case study research: design and methods* (4th ed.). Sage Publications.

RUPTURING ANTI-BLACKNESS IN MATHEMATICS EDUCATION RESEARCH: QUANTCRIT AS THEORY, METHODOLOGY, & PRAXIS

Blake O'Neal Turner
University of Maryland
bclark37@umd.edu

Keywords: Research Methods, Social Justice, Data Analysis and Statistics, Doctoral Education

Background and Significance

Although mathematics education research is designed to improve mathematics education for *all* students, some scholars may have positioned Black learners of mathematics as historically and systematically inferior to their counterparts thereby furthering the perception that Black learners have deficits (Bullock, 2019; Gardner-Neblett et al., 2021; Gutiérrez, 2008; Martin, 2019). Mathematics education has not fully addressed how racism is related to its teaching, and consequently, the published quantitative research reflects colorblind narratives that fail to consider mathematics as a racialized and gendered space (Battey & Leyva, 2016; Gutiérrez, 2008; Parks & Schmeichel, 2012; Ridgeway & McGee, 2018). Hence there is a dire need for scholars who conduct quantitative mathematics education research to challenge the dominant deficit discourses about Black learners and teachers in mathematics education. To this end, critical scholars in mathematics education have recently put forth calls to advance mathematics education research, policy, and practice toward liberation for Black learners. That is, researchers must attend to the sociohistorical and sociopolitical dimensions of education, including the manifestations of structural racism and antiblackness, especially in posing research questions posed and data collection and interpretations.

Given this background and the urgency for quantitative mathematics education scholars to use research to create, foster, and sustain mathematics spaces as sites of resistance against antiblackness and white supremacy, the primary question that guides this dissertation is: *How can quantitative mathematics education researchers use critical racial research to support and create liberatory mathematics education, particularly for Black learners, teachers, and researchers of mathematics?* I use two theories: Quantitative Critical Theory (QuantCrit; Gillborn et al., 2018) and Black Critical Theory (BlackCrit; Dumas & ross, 2016), and apply them to quantitative education research to illuminate, critique, and dismantle anti-black discourse and practices. This manuscript-style dissertation is motivated by three distinct but interrelated lines of inquiry that explore the possibilities of critical racial research in transforming mathematics education research and education more broadly. Each area of inquiry is reflected in a paper, and each paper represents parts of a complete statement of the need for QuantCrit and BlackCrit practices in mathematics education research.

Research Questions & Design of Studies

Specifically, in my dissertation, I seek to identify the problems of antiblackness in mathematics education research, the possibilities of using critical racial research theories and methodologies in rupturing antiblackness, and how Black mathematics education researchers could learn to use QuantCrit and BlackCrit to expose, critique, and dismantle antiblackness in mathematics education. In Paper 1, I provide a primer on why and how QuantCrit and BlackCrit could be applied to mathematics education research. In Paper 1, I will address the following research question: *Using a Black Critical Theory (BlackCrit) lens, what contributions does*

QuantCrit provide to larger critical conversations centering race and racism in mathematics education? Paper 1 is a conceptual paper that attends to why QuantCrit and BlackCrit are necessary theoretical and methodological frames in quantitative mathematics education research and how QuantCrit could be applied to mathematics education scholarship and praxis.

Critical quantitative education scholars increasingly use secondary data to disrupt deficit discourses about Communities of Color (e.g., Covarrubias, 2011; Gillborn, 2010; Morris & Perry, 2017; Pérez Huber et al., 2018; Young & Cunningham, 2021). In Paper 2, using secondary data analysis of Black New York City Mathematics Teaching fellows' dispositions towards teaching racially and culturally diverse students, I provide an empirical example of BlackCrit and QuantCrit. In Paper 2, I explore *What might a BlackCrit QuantCrit lens uncover about Black mathematics teachers' dispositions towards teaching racially and culturally diverse students?* I will analyze the subset of 42 questions that attend to the participants' dispositions towards teaching racially and culturally diverse students in mathematics classrooms. Undergirding the tenets of QuantCrit, I will be analyzing the data in a three-step process: 1) perform a K-means cluster of the 74 participants who self-selected Black as their racial/ethnic group, 2) use the clusters to create typologies of the participant's dispositions, 3) compare the distinct typologies to interrogate the dispositions of Black NYMTF.

Paper 3 is based on a critical ethnographic study of a six-session collaborative research workshop developed by the researcher that centers on how Black Graduate Students in Mathematics Education Research (BGMER) conceptualize and apply QuantCrit and BlackCrit to their own scholarship. Paper 3 answers the following research questions: *1) What supports do Black emerging mathematics education scholars need to become critical racial researchers who use QuantCrit and BlackCrit in their research? And 2) How do Black graduate mathematics education researchers use QuantCrit and BlackCrit to attend to the specificity of antiblackness in a collaborative research workshop focused on QuantCrit and BlackCrit?*

I aim to not only know *what* critical literacies BGMERs employ and share in the workshops but also *how* BGMERs apply BlackCrit and QuantCrit to their research projects. I will collect field notes, transcribed audio and video from the workshops, and artifacts (such as participants' annotations or notes on the pre-engagement reading materials and participants workshopped research projects), and keep a research journal to reflect on my research processes, as well as, engage in member checking to study the cultural practices and norms of six sessions of the workshop over a period of three months and analyze the data using a constant comparison analysis, allowing me to use the direct experiences and perspectives of the critical scholars to develop a theory on the phenomenon.

Background and Significance

The three studies contribute to the literature by informing how mathematics education researchers can use QuantCrit and BlackCrit in mathematics education research. Specifically, I seek to identify the problems of antiblackness in quantitative mathematics education research, the possibilities of using critical racial research theories and methodologies in rupturing antiblackness, and how Black mathematics education researchers could learn to use QuantCrit and BlackCrit to expose, critique, and dismantle antiblackness in mathematics education. QuantCrit and BlackCrit are relatively new frameworks in education; there is little research specific to mathematics education on how QuantCrit and BlackCrit can be applied to research in mathematics education. The findings from the three studies will lay the groundwork for future research into the possible kinds of questions that researchers need to start asking to dismantle antiblackness and how we execute the inquiry.

References

Battey, D., & Leyva, L. A. (2016). A framework for understanding whiteness in mathematics education. *Journal of Urban Mathematics Education*, *9*(2), 49–80.

Bullock, E. C. (2019). Mathematics curriculum reform as racial remediation. In *Critical Race Theory in Mathematics Education* (pp. 75–97). Routledge.

Covarrubias, A. (2011). Quantitative intersectionality: A critical race analysis of the Chicana/o educational pipeline. *Journal of Latinos and Education*, *10*(2), 86–105.

Dumas, M. J., & Ross, K. M. (2016). "Be Real Black for Me": Imagining BlackCrit in Education. *Urban Education*, *51*(4), 415–442.

Gardner-Neblett, N., Iruka, I. U., & Humphries, M. (2021). Dismantling the Black–White achievement gap paradigm: Why and how we need to focus instead on systemic change. *Journal of Education*, 002205742110319.

Gillborn, D. (2010). The colour of numbers: surveys, statistics and deficit-thinking about race and class. *Journal of Education Policy*, *25*(2), 253–276.

Gillborn, D., Warmington, P., & Demack, S. (2018). QuantCrit: education, policy, 'Big Data' and principles for a critical race theory of statistics. *Race, Ethnicity and Education*, *21*(2), 158–179.

Gutiérrez, R. (2008). Research commentary: A gap-gazing fetish in mathematics education? Problematizing research on the achievement gap. *Journal for Research in Mathematics Education*, *39*(4), 357–364.

Ladson-Billings, G. (2000). Racialized discourses and ethnic epistemologies. *Handbook of qualitative research*, 2, 257-277.

Martin, D. B. (2019). Equity, inclusion, and antiblackness in mathematics education. *Race Ethnicity and Education*, *22*(4), 459–478.

Morris, E. W., & Perry, B. L. (2017). Girls behaving badly? Race, gender, and subjective evaluation in the discipline of African American girls. *Sociology of Education*, *90*(2), 127–148.

Onwuegbuzie, A. J., Dickinson, W. B., Leech, N. L., & Zoran, A. G. (2009). A Qualitative Framework for Collecting and Analyzing Data in Focus Group Research. *International Journal of Qualitative Methods*, *8*(3), 1–21.

Parks, A. N., & Schmeichel, M. (2012). Obstacles to addressing race and ethnicity in the mathematics education literature. *Journal for Research in Mathematics Education*, *43*(3), 238–252.

Pérez Huber, L., Vélez, V. N., & Solórzano, D. (2018). More than 'papelitos:' a QuantCrit counterstory to critique Latina/o degree value and occupational prestige. *Race, Ethnicity and Education*, *21*(2), 208–230.

Ridgeway, M. L., & McGee, E. O. (2018). Black Mathematics Educators: Researching Toward Racial Emancipation of Black Students. *The Urban Review*, *50*(2), 301–322.

Young, J., & Cunningham, J. A. (2021). Repositioning Black Girls in Mathematics Disposition Research: New Perspectives from QuantCrit. *Investigations in Mathematics Learning*, *13*(1), 29–42.

INVESTIGATING STUDENT-CONSTRUCTED THEMES ACROSS THE MATHEMATICS CURRICULUM USING EVERYDAY AESTHETICS

Brady A. Tyburski
Michigan State University
tybursk2@msu.edu

Keywords: Curriculum; Affect, Emotion, Beliefs, and Attitudes; Undergraduate Education

Background and Theoretical Framework

In mathematics curricular documents (e.g., National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; Zorn, 2015), it is common practice to outline cognitive goals, habits of mind, and standards of practice that are intended to organize students' curricular experiences. Correspondingly, there is extensive research on the degree to which students engage with and understand these pre-determined, cross-curricular themes, such as function, equivalence, and abstraction (e.g., Warren & Cooper, 2009; Zandieh et al., 2017). However, we do not often ask students directly about the themes *they* construct based on their experiences within the mathematics curriculum. In mathematics courses, too, we do not tend to provide students with opportunities to engage in thematic reflection. A natural question arises: *How are mathematics students organizing their experiences across the mathematics curriculum?* In other words, which experiences are salient? Which experiences are viewed as (dis)connected? In what ways do students find meaning and continuity across their mathematical experiences?

To address this question, I conceptualize organizing one's experience as an *artistic act*. Such acts are governed in part by cognitive, rational thinking processes; however, they are equally impacted by extra-rational forces, such as emotions, beliefs about (school) mathematics, and one's many identities. To attend holistically to both the rational and extra-rational dimensions of organizing experience, I use the theory of everyday aesthetics (Saito, 2010). This modern branch of aesthetics is grounded in non-Western aesthetic traditions and the classic work of John Dewey (1934/2005). Dewey argued that aesthetics should not be restricted to just evaluations of finished "museum art". Rather, he situated the aesthetic in moment-to-moment experience:

> in order to *understand* the esthetic in its ultimate and approved forms, one must begin with it in the raw; in the events and scenes that hold the attentive eye and ear of man, arousing his interest and affording him enjoyment as he looks and listens. (p. 3, emphasis in original)

Aesthetics in this sense describe impactful experiences that combine emotion, the senses, satisfaction, and understanding. As Sinclair (2009) put it, "Dewey looks for integration with the human being in interaction with the world" (p. 50). By attending to the everyday aesthetics involved in organizing and making sense of one's experiences, then, we gain a holistic and humanistic view on the ways of being and thinking students bring into our mathematics classes.

Although the use of everyday aesthetics to study education is only just emerging (e.g., Marini, 2021; Marini & Merchán, 2021; Sinclair, 2009), aesthetics in general has been a lens used by mathematics education researchers to make sense of important extra-rational dimensions in learning (Dirkx, 2008). In particular, Sinclair (2001, 2008, 2018) has written extensively about the powerful role of aesthetics in mathematical learning and inquiry. Attention to the role of aesthetics in mathematical thinking and sense-making may not be a common topic, but it is one that has appeared in both classic and modern research (e.g., Dreyfus & Eisenberg, 1986; Jasien & Horn, 2022) that continues to shine a light on humanistic aspects of doing mathematics.

Research Questions and Methods

I focus on undergraduate mathematics students—as they have had years to organize and make sense of their K-16 mathematics experiences—and consider two research questions: (1) What are themes students construct across the undergraduate mathematics curriculum? (2) What ways do students organize and make sense of their experiences across the undergraduate mathematics curriculum?

To study these questions, I will recruit ~5 students with at least two semesters of undergraduate mathematics courses completed. Participants will then be invited to reflect on their mathematics experiences by individually creating art that responds to the following prompt: *Consider your journey as an undergraduate mathematics student. This includes your experiences, emotions, people you've interacted with, highs and lows, mathematical content, etc. What are some patterns and themes that you have noticed? Feel free to use any of the artistic resources provided to you to organize and make sense of your experiences. You can also reflect in writing using a story, a poem, or a song, if you wish.* This arts-based approach allows students to reflect individually in a holistic, multimodal, and multisensory way (Bagnoli, 2009; Papoi, 2017). Consistent with the theory of everyday aesthetics, it also provides students the opportunity for reflection on the connective and cohesive forces (Clark & Rossiter, 2008; Sinclair, 2009) they consider to be present across their everyday mathematical experiences.

In line with my goal to learn how *students* organize their experiences, analysis of these artistic artifacts will be carried out via a (video-recorded) discussion led by *students* and governed by their interests and curiosities. To begin, each participant will share their artistic artifact and what it means to them. Follow-up discussion will then concern themes that students identify across each of their artifacts and experiences.

I conceptualize this discussion as a "book club" where the "text" of interest is the mathematics curriculum. Like a book club, the goal of this discussion is for a community of learners to come together to reflect on their opinions, interpretations of, and feelings about a text (e.g., its themes), while being open to the possibility that their views may be transformed and enhanced by other participants' views of the text (Kerka, 1996). In fact, productive book clubs encourage the sharing of alternative (and potentially contradictory) perspectives of the text to facilitate further exploration and discussion of themes (Beach & Yussen, 2011). As the facilitator of this book club, my role is primarily to allow the discussion to be driven by participants, but I will also point out occasional observations and offer suggestions for new topics to consider.

Implications

We know from studies with pre-service teachers that undergraduate mathematics students do not always develop the connections between mathematics content areas we may have expected (e.g., Wasserman, 2018). Research akin to this proposed study has the potential to help us understand the themes students actually come away with on not only a course-by-course basis but also across sequences of courses and the entire mathematics curriculum. It also builds on the calls of researchers (Melhuish et al., 2020; Zandieh et al., 2017) for further study into the extent to which students develop understandings of key themes, such as function, across the undergraduate curriculum. Research of this nature has the potential to offer suggestions for how we might modify the curricula of courses and course sequences to build on patterns, themes, and connections that we know students are likely to construct. In other words, it allows us to integrate a realistic, asset-based view of learners' capabilities (Peck, 2020) into our curricula, a piece of the puzzle essential to improved curriculum design (Lattuca & Stark, 2009).

References

Bagnoli, A. (2009). Beyond the standard interview: The use of graphic elicitation and arts-based methods. *Qualitative Research*, *9*(5), 547–570. https://doi.org/10.1177/1468794109343625

Beach, R., & Yussen, S. (2011). Practices of productive adult book clubs. *Journal of Adolescent & Adult Literacy*, *55*(2), 121–131.

Clark, M. C., & Rossiter, M. (2008). Narrative learning in adulthood. *New Directions for Adult and Continuing Education*, *119*, 61–70. https://doi.org/10.1002/ace.306

Dewey, J. (2005). *Art as Experience*. Penguin Group. (Original work published 1934)

Dirkx, J. M. (2008). The meaning and role of emotions in adult learning. *New Directions for Adult and Continuing Education*, *120*, 7–18. https://doi.org/10.1002/ace.311

Dreyfus, T., & Eisenberg, T. (1986). On the aesthetics of mathematical thought. *For the Learning of Mathematics*, *6*(1), 2–10.

Jasien, L., & Horn, I. (2022). Fixing the crooked heart: How aesthetic practices support sense making in mathematical play. *Journal for Research in Mathematics Education*, *53*(1), 41–64. https://doi.org/10.5951/jresematheduc-2020-0228

Kerka, S. (1996). Book groups: Communities of learners. *New Directions for Adult and Continuing Education*, *71*, 81–90. https://doi.org/10.1002/ace.36719967111

Lattuca, L. R., & Stark, J. S. (2009). *Shaping the college curriculum: Academic Plans in context* (2nd ed.). Jossey-Bass.

Marini, G. (2021). An introduction to everyday aesthetics in education. *Studies in Philosophy and Education*, *40*, 39–50. https://doi.org/10.1007/s11217-020-09740-x

Marini, G., & Merchán, J. D. R. (2021). Reconsidering time in schools: An everyday aesthetics perspective. *Journal of Philosophy in Education*, *55*(4–5), 893–904. https://doi.org/10.1111/1467-9752.12592

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior*, *60*, 1–16. https://doi.org/10.1016/j.jmathb.2020.100806

National Governors Association Center for Best Practices, & Council of Chief State School Officers. (2010). *Common core state standards for mathematics*.

Papoi, K. (2017). Sensemaking through art making: Trash for teaching and visual arts integration with elementary emergent bilingual students. In T. Dell'Angelo, L. Ammentorp, & L. Madden (Eds.), *Using photography and other arts-based methods with English language learners: Guidance, resources, and activities for P-12 educators* (pp. 17–28). Rowman and Littlefield.

Peck, F. (2020). Towards anti-deficit education in undergraduate education: How deficit perspectives work to structure inequality and what can be done about it. *Problems, Resources, and Issues in Mathematics Undergraduate Studies*, *31*(9), 940–961. https://doi.org/10.1080/10511970.2020.1781721

Saito, Y. (2010). *Everyday aesthetics*. Oxford University Press.

Sinclair, N. (2001). The aesthetic *is* relevant. *For the Learning of Mathematics*, *21*(1), 25–32.

Sinclair, N. (2008). Attending to the aesthetic in the mathematics classroom. *For the Learning of Mathematics*, *28*(1), 29–35.

Sinclair, N. (2009). Aesthetics as a liberating force in mathematics education? *ZDM*, *41*(1–2), 45–60. https://doi.org/10.1007/s11858-008-0132-x

Sinclair, N. (2018). An aesthetic turn in mathematics education. *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education*, *1*, 51–66.

Warren, E., & Cooper, T. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. *Mathematics Education Research Journal*, *21*(2), 76–95. https://doi.org/10.1007/BF03217546

Wasserman, N. H. (2018). *Connecting abstract algebra to secondary mathematics, for secondary mathematics teachers*. Springer.

Zandieh, M., Ellis, J., & Rasmussen, C. (2017). A characterization of a unified notion of mathematical function: The case of high school function and linear transformation. *Educational Studies in Mathematics*, *95*(1), 21–38. https://doi.org/10.1007/s10649-016-9737-0

Zorn, P. (2015). *2015 CUPM curriculum guide to majors in the mathematical sciences*. Mathematical Association of America. https://www.maa.org/sites/default/files/CUPM%20Guide.pdf

SYNERGIES AND TAKE-UP WHEN PRACTICE-BASED PROFESSIONAL DEVELOPMENT AND COLLABORATIVE LESSON DESIGN ARE USED TOGETHER

Jennifer Valerio
University of Pennsylvania
valerioj@upenn.edu

Background and Framing Concepts

Designing professional development (PD) that helps teachers enact more ambitious instruction responsive to emergent student thinking is challenging work. Two approaches to this sort of PD, practice-based professional development (PBPD) and lesson study (LS), have different mechanisms for fostering change and different focus points. Although there is substantial research support for each approach (e.g., Grossman et al., 2018; Huang & Shimizu, 2016; Kavanagh et al., 2020; Lewis, 2016), each has limitations that can be characterized as "problems of enactment" (Kennedy, 1999) when used alone.

LS focuses on changing teaching via teacher-led, structured inquiry into problems of practice and by improving the planning process and instructional resources teachers use to enact lessons (Hiebert & Morris, 2012). Inherent within LS's focus on developing knowledge *for* and *of* action is a focus on student learning and the scrutiny and refinement of instructional materials (Watanabe et al., 2008). Lewis's (2016) research on LS over the course of two decades identifies four types of outcomes: changes in teachers' knowledge, beliefs, professional community, and curriculum resources. Missing from this list is a change in teachers' actual teaching practices--what they do when they are teaching. LS, therefore, faces the possibility that without structured opportunities to develop a new set of pedagogical skills, teachers may face the "problem of enactment" described by Kennedy (1999), i.e., vision change without the necessary pedagogical skills to enact the new vision, making teachers unable to execute the lesson plans as intended.

PBPD, by contrast, emphasizes knowledge *in* action, and prioritizes changing teaching through pedagogical training in enacting core teaching practices, i.e., specific instructional skills such as launching problems and facilitating discussions (Ball & Forzani, 2009; Grossman, Hammerness, et al., 2009; Grossman et al., 2018). This model employs pedagogies of enactment, including representation, decomposition, and approximation, (Grossman, Compton, et al., 2009) which require teachers to rehearse teaching practices in situations of increasing complexity in order to develop adaptive expertise (Kavanagh et al., 2020). Challenges arise, however, in the transfer of pedagogical skills to specific educational contexts (Zeichner, 2012; Philip et al, 2018). A parallel "problem of enactment," then, may arise if vision and practice changes occur without the necessary resources to support enactment in teachers' specific learning contexts.

Because the two PD approaches involve different processes, pedagogies, and expertise, they are seldom used together in ways that give equal weight to each approach. Yet, when used in tandem, LS and PBPD could prove complementary in ways that help teachers surmount the "problems of enactment" that can otherwise undermine translation of vision into practice when each approach is used alone.

During the 2020-2021 school year, the Responsive Math Teaching (RMT) Project at Penn Graduate School of Education devoted equal time and resources to both approaches to professional development with the same group of K-8 teachers in a large, urban school district, providing an opportunity to study how the two approaches might work together to foster changes in teaching. This study draws on both constructivist (Piaget, 1973) and socio-cultural (Vygotsky,

1978) frameworks in order to trace individual teachers' learning across time within the context of an inquiry community.

Goals and Research Questions

This study explores how RMT's Collaborative Lesson Design (CLD), a modification of lesson study, and PBPD might have valuable *synergies*, i.e., ways in which they are mutually supportive such that the affordances of one approach complement the challenges of the other. In addition to looking for evidence of synergy between the two approaches, this study also endeavors to better understand and describe whether and how *take-up* occurs as teachers and tools move between the two PD forms. Here, I define *take-up* as the acceptance, adoption, and incorporation of ideas into one's own classroom practice. I have intentionally used the term *take-up* to incorporate both vision and practice changes.

The research questions that follow examine the interplay between PBPD and CLD and the teacher take-up that occurs therein:

1. To what extent does individual teachers' take-up of responsive teaching evolve over the course of the year across both forms of PD and what does this evolution look like?
2. What evidence is there that participation in CLD supports classroom enactment of the practices learned in PBPD?
3. What evidence is there that participation in PBPD supports the enactment of lessons planned in CLD?

Methods

This study focuses on eight purposefully selected participants and employs a qualitative, interpretive, case study approach (Creswell & Poth, 2018), working from the assumption that doing so provides a deeper, more contextualized understanding of the interplay between PBPD and CLD and of individual teacher take-up than a broader, decontextualized look across a larger number of participants. Cases are defined as individual teachers from a large, under-resourced urban school district participating in both forms of professional development during the 2020-2021 school year (Miles, Huberman, & Saldana, 2020; Ravitch & Carl, 2016).

Preliminary Findings

Although it is early in the data analysis process, some tentative findings have emerged. Across participants, the take-up of responsive teaching practices involves an iterative process of building knowledge, deploying that knowledge during enactment, encountering struggles during enactment, and then returning to knowledge-building with a focus on addressing the specific struggles that have emerged. I anticipate that my findings will ultimately include take-up mappings (Valerio, 2021) for illustrative cases, a description of the types of struggles teachers encountered, and a description of the supports that helped teachers overcome these struggles.

Additionally, preliminary data indicate that CLD supports the enactment of practices addressed in PBPD by giving teachers the opportunity to adapt the practices for their specific context and students, to design and adapt high leverage math tasks suitable to their context, and to tap into the expertise of colleagues for advice about a variety implementation challenges. PBPD supports the enactment of lessons developed through CLD by providing space for teachers to decompose and rehearse ways to address enactment challenges that arise and to further hone both their teaching practices and their emergent vision of responsive instruction. Further data analysis will more fully flesh out these findings.

References

Ball, D. L., & Forzani, F. M. (2009). The work of teaching and the challenge of teacher education. Journal of Teacher Education, 60(5), 497-511.

Creswell, J. W., & Poth, C. N. (2018). Qualitative Inquiry & Research Design: Choosing Among Five Approaches. 4th ed. Sage Publications.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. Teachers College Record, 111(9), 2055-2100

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. Teachers and Teaching, Theory and Practice, 15(2), 273-289.

Grossman, P., Kavanagh, S. S., & Pupik Dean, C. G. (2018). The turn towards practice-based teacher education: Introduction to the work of the Core Practices Consortium. In P. Grossman (Ed.), Teaching core practices in teacher education (pp. 1-14). Harvard Education Press.

Hiebert, J., & Morris, A. K. (2012). Teaching, rather than teachers, as a path toward improving classroom instruction. Journal of Teacher Education, 63(2), 92-102.

Huang, R. & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory to practice through lesson study in mathematics: an international perspective. ZDM, 48, 393-409.

Kavanagh, S. S., Conrad, J., & Dagogo-Jack, S. (2020). From rote to reasoned: Examining the role of pedagogical reasoning in practice-based teacher education. Teaching and Teacher Education, 89, 1-11. https://doi.org/10.1016/j.tate.2019.102991

Kennedy, M. M. (1999). The role of preservice teacher education. In L. Darling-Hammond & G. Sykes (Eds.), Teaching as the learning profession: Handbook of teaching and policy (pp. 54-86). Jossey-Bass.

Lewis, C. (2016). How does lesson study improve mathematics instruction? ZDM, 48(4), 571-580.

Miles, M. B., Huberman, A. M., & Saldana, J. (2020). Qualitative Data Analysis: A Methods Sourcebook. 4th ed. Sage Publications.

Piaget, J. (1973). To understand is to invent. Penguin.

Philip, T. M., Souto-Manning, M., Anderson, L., Horn, I. S., Andrews, D. C., Stillman, J., & Varghese, M. (2018). Making justice peripheral by constructing practice as "core": How the increasing prominence of core practices challenges teacher education. Journal of Teacher Education, 70(3), 251-264.

Ravitch, S. M., & Carl, N. M. (2016). Qualitative Research: Bridging the Conceptual, Theoretical, and Methodological. Sage Publications.

Valerio, J. V. (2021). Tracing take-up across practice-based professional development and collaborative lesson design. In D. Olanoff, K. Johnson, & S. Spitzer (Eds.), Productive Struggle - Persevering Through Challenges: Proceedings of the 43rd meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp. 633-641). Philadelphia, PA.

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Harvard University Press.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), Inquiry into mathematics teacher education (pp. 131-142). Association of Mathematics Teacher Educators.

Zeichner, K. (2012). The turn once again toward practice-based teacher education. Journal of Teacher Education, 63(5), 376-382.

USING VIDEO TAGGING TOOLS TO UNDERSTAND FACILITATOR NOTICING

Margaret Walton
University of Maryland
mwalton@umd.edu

Background & Significance

Much of the math education research community has emphasized the importance of teachers *noticing*, or attending to and flexibly responding to student thinking (e.g. Carpenter et al., 1989; Empson & Jacobs, 2008). Like other teaching practices, noticing is a skill that teachers often need to develop (van Es, 2011).

Video clubs (VCs) are a type of professional development (PD) that can help teachers learn to notice (Sherin & Han, 2004; Luna & Sherin 2017; Sherin & van Es, 2005). A VC is a small group of teachers who meet to watch a short video clip of classroom instruction. They then participate in a discussion about aspects of student thinking in the video, which is usually led by a facilitator. The facilitator guides teachers to focus on students and reason about student ideas (van Es et al., 2014; Borko, Jacobs et al., 2014).

A number of researchers argue that facilitators are key to teachers learning to notice (Castro Superfine et al., 2019; van Es, 2011; Goldsmith & Seago, 2008). However, few studies examine how teacher educators learn to facilitate a VC. Studies about facilitator learning have found that facilitators often have difficulty leading teacher discussions about student thinking (Borko, Koellner et al., 2014; Elliott et al., 2009; Jackson et al., 2015). This difficulty might be because facilitators need to develop their own noticing abilities (Amador, 2016; 2020).

The purpose of this study is to support facilitators in learning to lead VC discussions that help teachers learn to notice. I plan to design a facilitator PD (F-PD) that employs a video tagging tool that supports facilitators' own noticing development. I will implement the F-PD with mathematics education graduate students who instruct pre-service teachers (PSTs). My research questions are:

- How can a video-based F-PD support facilitators in learning to lead PSTs in a VC?
    - In what ways can a video-tagging tool support facilitators in noticing student thinking while *participating* in a VC?
    - In what ways can a video-tagging tool support facilitators noticing teacher thinking while *leading* VC discussion?

Background and Theoretical Framing

The processes involved in teacher noticing generally include: (1) attending to important moments of student mathematical thinking, (2) interpreting that thinking using knowledge of content, pedagogy, and students, and (3) determining how to respond (Jacobs et al., 2010; Sherin & van Es, 2009). As teachers' noticing skills develop, they can learn to focus more on the relationship between students' thinking and instruction, and often seek to understand students' mathematical thoughts (van Es, 2011). VCs can support this development (Sherin, 2007).

Facilitators can be key to supporting teachers' noticing learning in PD like VCs, but to do so they may need to develop their own noticing abilities (Amador, 2016; 2021). Noticing is more complex for facilitators than for teachers because they need to notice both student *and* teacher thinking (Amador 2021; Kazemi et al., 2011). Specifically, a facilitator needs to understand how a teacher reasons about instances of student thinking in relation to their classroom instruction. A facilitator must consider several aspects of teacher thinking at once, including: (1) the way teachers understand the math content (Lesseig et al., 2017), (2) the way teachers make sense of student thinking, and (3) how teachers relate content and student ideas to learning goals (Borko, Jacobs et al., 2014).

I intend to develop a series of VCs and VC planning activities to help facilitators with noticing student *and* teacher thinking. One of the features that will make my F-PD different from traditional video clubs is that I will employ a video tagging tool, which allows users to annotate video directly on the screen. This tool could scaffold facilitators' noticing development in several ways, which I explain in the Research Plan.

Research Plan

My dissertation employs design research methods (Cobb et al., 2003; The Design-Based Research Collective 2003; Gravemeijer & Van Eerde, 2009). Consequently, the F-PD design described here is the first iteration of what could be many as I refine and re-implement the design, based on my findings of how facilitators learn to notice.

The F-PD includes five group meetings and two independent work sessions with six math education graduate students called facilitator participants (FPs). These FPs aim to build careers in teacher education. In Part 1 of the F-PD, FPs engage in virtual VCs via Zoom (www.zoom.us) so that they can focus on noticing *student thinking*. During these VCs, FPs annotate the videos that they watch, which could help them identify, reason about, and remember interesting moments of student thinking that they can then discuss with other FPs. Part 2 of the F-PD shifts focus to noticing of *teacher thinking*. Here, FPs use video annotations in a different way. They first analyze example VC annotations by PSTs (Walton & Walkoe, in press) to understand how PSTs think about student thinking. FPs use this analysis to write hypothetical discussion protocols that they could use for VC discussion.

Data collection will include video recordings of the virtual sessions, FPs' video annotations from Part 1, and FPs' video discussion protocols from Part 2. I will code the Part 1 data using similar methods to other research in noticing (Sherin & van Es, 2009) to understand how FPs noticing of student thinking developed. I will focus on noticing of teacher thinking as I examine the data from Part Two. I will analyze FPs' discussion protocol assignments using deductive codes based on a framework I developed that outlines facilitator noticing of teacher thinking.

Impact

Teacher noticing of student thinking has been shown to enhance K-12 learning of mathematics (Carpenter et al., 1989; Empson & Jacobs, 2008). In order for teachers to learn to notice, they need access to high quality PD led by well-prepared facilitators.

I plan to widely disseminate my F-PD design for use with other facilitators. The demand for video-based PD continues to rise (van Es & Sherin, 2017) and my design could create a greater number of experienced facilitators, giving more teachers access to noticing PD.

This project will help researchers further understand and describe the cognitive processes related to noticing, especially from the facilitator perspective. In particular, the exploration of video tagging will help determine how scaffolds can support noticing development. More broadly, the project will also contribute to the conceptualization of facilitator learning, which is an understudied field in education research (Elliott et al., 2009; Tekkumru-Kisa & Stein, 2017), but one that is imperative for developing facilitators who can provide high quality learning experiences to teachers.

References

Amador, J. (2016). Professional noticing practices of novice mathematics teacher educators. *International Journal of Science and Mathematics Education*, *14*(1), 217–241.

Amador, J. M. (2021). Mathematics teacher educator noticing: Examining interpretations and evidence of students' thinking. *Journal of Mathematics Teacher Education*.

Borko, H., Jacobs, J., Seago, N., & Mangram, C. (2014). Facilitating video-based professional development: Planning and orchestrating productive discussions. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction multiple approaches and practices* (pp. 259–281).

Borko, H., Koellner, K., & Jacobs, J. (2014). Examining novice teacher leaders' facilitation of mathematics professional development. *Journal of Mathematical Behavior*, *33*, 149–167.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C.-P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal*, *26*(4), 499–531.

Castro Superfine, A., Amador, J., & Bragelman, J. (2019). Facilitating video-based discussions to support prospective teacher noticing. *Journal of Mathematical Behavior*, *54*, 1–18.

Cobb, P., Confrey, J., Disessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, *32*(1), 9–13.

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, *32*(1), 5–8.

Elliott, R., Kazemi, E., Lesseig, K., Mumme, J., Carroll, C., & Kelley-Petersen, M. (2009). Conceptualizing the work of leading mathematical tasks in professional development. *Journal of Teacher Education*, *60*(4), 364–379.

Empson, S. B., & Jacobs, V. R. (2008). Learning to listen to children's mathematics. In D. Tirosh & T. Wood (Eds.), *International handbook of mathematics teacher education: Volume 2* (pp. 257–281). Rotterdam: Sense Publishers.

Goldsmith, L. T., & Seago, N. (2008). Using video cases to unpack the mathematics in students' thinking. In M. S. Smith & S. N. Friel (Eds.), *Cases in Mathematics Teacher Education: Tools for Developing Knowledge Needed for Teaching*. San Diego, CA: Association of Mathematics Teacher Educators

Gravemeijer, K., & van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *Elementary School Journal*, *109*(5), 510–524.

Jackson, K., Cobb, P., Wilson, J., Webster, M., Dunlap, C., & Appelgate, M. (2015). Investigating the development of mathematics leaders' capacity to support teachers' learning on a large scale. *ZDM Mathematics Education*, *47*(1), 93–104.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Kazemi, E., Elliott, R., Mumme, J., Carroll C. Lesseig, K., & Kelley-Peterson, K. (2011). Noticing leaders' thinking about videocases of teachers engaged in mathematics tasks in professional development. In Sherin, M.G., Jacobs, V.R., & Phillip, R.A. (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 188-203). New York: Routledge.

Lesseig, K., Elliott, R., Kazemi, E., Kelley-Petersen, M., Campbell, M., Mumme, J., & Carroll, C. (2017). Leader noticing of facilitation in videocases of mathematics professional development. *Journal of Mathematics Teacher Education*, *20*(6), 591–619.

Luna, M. J., & Sherin, M. G. (2017). Using a video club design to promote teacher attention to students' ideas in science. *Teaching and Teacher Education*, *66*, 282–294.

Sherin, M.G, (2007). The develop of teachers' professional vision in video clubs. In Goldman, R., Pea, R., Barron, B., Derry, S.J. (Eds.), *Video research in the learning sciences* (pp 383-395). Mahwah, NJ: Lawrence Earlbaum Associates.

Sherin, M. G., & Han, S. Y. (2004). Teacher learning in the context of a video club. *Teaching and Teacher Education*, *20*(2), 163–183.

Sherin, M. G., & van Es, E. A. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of Technology and Teacher Education*, *13*(3), 475–491.

Sherin, M. G., & van Es, E. A. (2009). Effects of video club participation on teachers' professional vision. *Journal of Teacher Education*, *60*(1), 20–37.

Tekkumru-Kisa, M., & Stein, M. K. (2017). A framework for planning and facilitating video-based professional development. *International Journal of STEM Education*, *4*(1).

van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A.

Phillip (Eds.), *Mathematics Teacher Noticing Seeing Through Teachers' Eyes* (pp. 134–151). New York: Routledge.
van Es, E. A., & Sherin, M. G. (2017). Bringing facilitation into view. *International Journal of STEM Education*, *4*(1).
van Es, E. A., Tunney, J., Goldsmith, L. T., & Seago, N. (2014). A Framework for the facilitation of teachers' analysis of video. *Journal of Teacher Education*, *65*(4), 340–356.
Walkoe, J. (2015). Exploring teacher noticing of student algebraic thinking in a video club. *Journal of Mathematics Teacher Education*, *18*(6), 523–550.
Walton M., & Walkoe, J. (in press). Using video-based tools to support facilitators of video-based professional development. *AMTE Professional Book Series, Vol. 5*.

INVESTIGATING SECONDARY PRE-SERVICE TEACHERS' MATHEMATICAL CREATIVITY

Anne Waswa
University of Georgia
anw04326@uga.edu

Mathematical creativity (MC) is important in school and outside school (Chamberlin & Moon, 2005; Panaoura & Panaoura, 2014), it is one of the aims of mathematics (Mann, 2006), and of mathematics education (Andrade & Pasia, 2020; Levenson, 2013). Encouraging creativity in learning has potential to increase student rigor in learning, challenge students to learn more, and promote student engagement in learning (Garfinkel & Albrecht, 2018). On the other hand, lack of support for creativity can result in a decline in learned variability, can hinder radical breakthrough solutions, and restrict academic achievement to narrow sets (Kim, 2016). Unfortunately, research on MC has not been foregrounded (Haavold, 2018; Haylock, 1987; Leikin, 2009, 2011). Among the important areas of research is teachers' MC because as Sheffield (2013) argued, we cannot separate teachers from the development of MC. However, there is less research that focuses on the conceptions of MC of pre-service teachers (PSTs) and the MC of PSTs as they actively engage in problem solving. Therefore, my research seeks to address the following research questions: (1) What is secondary PSTs' mathematical creativity when they engage in problem solving? (2) What are PSTs' conceptions of mathematical creativity in the teaching and learning of mathematics?

Theoretical Frameworks

Radical Constructivism

The theory of knowing and learning that inform my study is radical constructivism (von Glasersfeld's, 1995), whose fundamental principles are also informed by Piaget's constructivist theory of knowing. I will draw from this theory, not to explain PSTSs' MC, but to drive and form the kinds of descriptions that I can provide regarding PSTs' MC. The fundamental principles of radical constructivism include that, "knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the cognizing subject" and that "the function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability; cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality" (von Glasersfeld, 1995, p. 51). The first principle informs my assumptions of PSTs as actively constructing their knowledge as they engaged in PS. The second fundamental principle speaks to the acquisition of knowledge through mental mechanisms. The notion of fit and viability speaks to the construction of knowledge that works for the cognizing subject, as opposed to the construction of knowledge being some movement toward or approximation of reality and truth. It follows that the models I develop in my study are merely viable models, meaning that I cannot develop with certainty models of PSTs' mathematics and MC that provide a true picture of their knowledge.

Framework for MC

I adopt Lithner's (2008) research framework for creative and imitative reasoning. In particular, I use Lithner's framework to define the qualities that characterizes reasoning as creative, and what differentiates creative mathematically founded reasoning (CMR) from general creative reasoning. Lithner (2008) defines reasoning as "the line of thought adopted to produce assertions and reach conclusions in task solving" (p. 257). According to Lithner (2008), CMR should be novel, plausible, and mathematically founded. Other characteristics that I focus on that are indicative of creative thinking include fluency and flexibility (see Lithner, 2008).

Methods

The study is qualitative in nature, with a teaching experiment (TE) (Steffe & Thompson, 2000) and clinical interviews (Ginsburg, 1997) serving as the main methods of data collection. Participants will be six secondary mathematics PSTs.

Teaching Experiment

The main aim of using a teaching experiment (TE) "is for the researcher to experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p.267). From this experience a goal of teaching experiment includes "[constructing] models of students' mathematics" (p. 268), and this aligns with my goal of understanding and developing PSTs' viable models of mathematics in relation to their MC. I plan to conduct tentatively eight one-hour sessions with my participants together with a witness, potentially a graduate student. All sessions will be video-taped using two cameras. One camera will capture PSTs' written work from overhead and their explanations while the other camera will capture their facial expressions, explanations, gestures, and the surrounding environment. Witnesses' notes during the episodes and together with my notes, and participants' scanned written work, will be additional sources of data. The tasks will focus on quadratic growth and linear growth, and I will leverage literature from researcher that focuses on these topics (e.g., Ellis 2011a; 2011b; Fonger et al., 2020; Graf et al., 2018; Lobato et al., 2012). I also leverage literature from quantitative reasoning and/or covariation reasoning (e.g., Carlson et al., 2002; Ellis, 2007; Ellis et al., 2020; Moore, 2013; Moore et al., 2014; Moore & Thompson, 2015; Saldanha & Thompson,1998; Thompson, 2011; Thompson & Carlson, 2017).

Clinical Interview

My goal in conducting clinical interviews is to offer ground for open and reflective thinking and sharing with PSTs, in order to understand PSTs' conceptions of MC. I plan to conduct two clinical interviews with each PST, one before the TE and another after the TE to track and compare their conceptions of MC before and after conducting the TE. I will use an interview protocol as a guide. The protocol contains open-ended questions that focus on PSTs' experiences with MC outside school, in school, and in their teacher preparation program.

Data Analysis

I will use MAXQDA software to organize and analyze my data. My approach to data analysis will entail ongoing and retrospective analysis (Simon, 2019; Steffe & Thompson, 2000) and conceptual analysis (Thompson, 2008).

Anticipated Contributions

My research extends existing research on conceptions of MC by using a different approach of investigating PSTs' conceptions of MC in the context of problem solving. The novelty in this contribution is that PSTs reflect on their own MC after engaging in PS when describing their conceptions of MC. In addition, my research goes beyond identifying characteristics of MC to making connections between PSTs' mathematical meanings and their MC. This a methodological contribution to the field in terms of providing a different approach to studying MC. The findings of these research will potentially inform teacher preparation in terms of contributing ideas on understanding and supporting PSTs' MC and their teaching with and for MC from the developed models of PSTs' mathematical meanings and their MC.

References

Andrade, R. R., & Pasia, A. E. (2020). Mathematical Creativity of Pre-service Teachers in Solving Non-routine Problems in State University in Laguna. Universal Journal of Educational Research, 8(10), 4555-4567.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for Research in Mathematics Education, 33(5), 352-378.

Chamberlin, S. A., & Moon, S. M. (2005). Model-eliciting activities as a tool to develop and identify creatively gifted mathematicians. Journal of Secondary Gifted Education, 17(1), 37-47.

Ellis, A. B. (2007). The influence of reasoning with emergent quantities on students' generalizations. Cognition and Instruction, 25(4), 439-478.

Ellis, A. B. (2011a). Middle school algebra from a functional perspective: A conceptual analysis of quadratic functions. [Paper Presentation]. North American Chapter of the International Group for the Psychology of Mathematics Education(location).

Ellis, A. B. (2011b). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. Journal for Research in Mathematics Education, 42(4), 308–345.

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). Scaling-continuous variation: Supporting students' algebraic reasoning. Educational Studies in Mathematics, 104(1), 87-103.

Fonger, N. L., Ellis, A. B., & Dogan, M. F. (2020). A quadratic growth learning trajectory. The Journal of Mathematical Behavior, 59, 1-58.

Garfinkel, K., & Albrecht, M. (2018). The Art of Curve Stitching: A Creative Mathematics Project for Students. The Mathematics Teacher, 112(1), 60-66.

Ginsburg, H. (1997). Entering the child's mind: The clinical interview in psychological research and practice. Cambridge University Press.

Graf, E. A., Fife, J. H., Howell, H., & Marquez, E. (2018). The Development of a Quadratic Functions Learning Progression and Associated Task Shells. ETS Research Report Series, 2018(1), 1-28.

Haavold, P. Ø. (2018). An empirical investigation of a theoretical model for mathematical creativity. The Journal of Creative Behavior, 52(3), 226-239.

Haylock, D. W. (1987). A framework for assessing mathematical creativity in schoolchildren. Educational Studies in Mathematics, 18(1), 59-74.

Kim, K. H. (2016). The creativity challenge: How we can recapture American innovation. Prometheus Books.

Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), Creativity in mathematics and the education of gifted students (pp. 129–145). Rotterdam: Sense Publishers.

Leikin, R. (2011). The education of mathematically gifted students: On some complexities and questions. Montana Mathematical Enthusiast Journal, 8(1), 167–188.

Levenson, E. (2013). Tasks that may occasion mathematical creativity: teachers' choices. Journal of Mathematics Teacher Education, 16(4), 269-291.

Lithner, J. (2008). A research framework for creative and imitative reasoning. Educational Studies in Mathematics, 67(3), 255-276.

Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. Mathematical Thinking and Learning, 14(2), 85-119.

Mann, E. L. (2006). Creativity: The essence of mathematics. Journal for the Education of the Gifted, 30(2), 236–260.

Moore, K. C. (2013). Making sense by measuring arcs: a teaching experiment in angle measure. Educational Studies in Mathematics, 83(2), 225-245.

Moore, K. C., Silverman, J., Paoletti, T., & LaForest, K. (2014). Breaking conventions to support quantitative reasoning. Mathematics Teacher Educator, 2(2), 141-157.

Moore, K. C., & Thompson, P. W. (2015, February). Shape thinking and students' graphing activity. In Proceedings of the 18th Meeting of the MAA Special Interest Group on Research in Undergraduate Mathematics Education (pp. 782-789). Pittsburgh, PA: RUME.

Panaoura, A., & Panaoura, G. (2014). Teachers' awareness of creativity in mathematical teaching and their practice. Issues in the Undergraduate Mathematics Preparation of School Teachers, 4, 1-11.

Saldanha, L. A., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensen et al. (Eds.), Proceedings of the 20th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 1, pp. 298–303). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Simon, M. A. (2019). Analyzing qualitative data in mathematics education. In Designing, conducting, and publishing quality research in mathematics education (pp. 111-122). Springer, Cham.
Sheffield, L. J. (2013). Creativity and school mathematics: Some modest observations. ZDM – International Journal on Mathematics Education, 45(2), 325-332.
Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. Handbook of research design in mathematics and science education, 267-306.
Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In S. Chamberlin, L.L. Hatfield, & S. Belbase (Eds.), New perspectives and directions for collaborative research in mathematics education: Papers from a planning conference for WISDOM^e, University of Wyoming, Laramie, WY (2011), pp. 33-57
Thompson, P. W., & Carlson, M. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), Compendium for research in mathematics education (pp. 421-456). Reston, VA: National Council of Teachers Mathematics.
von Glasersfeld, E. (1995). Radical constructivism: A way of knowing and learning. Washington, D.C.: Falmer Press.

PROSPECTIVE MATHEMATICS TEACHERS' TECHNOLOGICAL PEDAGOGICAL CONTENT KNOWLEDGE IN PIECES

Nicholas Witt
Western Michigan University
nicholas.e.witt@wmich.edu

Keywords: Technology, Pre-Service Teacher Education, Mathematical Knowledge for Teaching

The nature and form of knowledge that teachers utilize while creating technology-infused curriculum materials is of great importance due to the abundance of technology that is accessible and sometimes necessary for teachers to engage with during their day-to-day tasks. Many teachers still rely on textbooks to guide their instruction (Banilower et al, 2018), but recent research has shown that textbooks seldomly suggest using technology in ways that fully leverage its affordances (Sherman et al, 2020). Thus, it would be of great importance to the mathematics education community to better understand what knowledge is needed for teachers to be able to adapt their curricular materials to better integrate technology.

One framework for thinking about this type of knowledge is the Technological Pedagogical Content Knowledge (TPACK) (Mishra & Koehler, 2006) framework, but many questions remain with respect to theorizing the nature and form of this knowledge (Angeli, Valanides, & Christodoulou, 2016). Attempts at theorizing this type of knowledge in the moment of planning have been conducted (Bibi & Khan, 2017) with university instructors, but how this knowledge is utilized in relative novice mathematics teachers will be a new contribution to the field. Furthermore, while fine-grained analyses of conceptual change have mostly focused on disciplines such as mathematics (Izsak, 2005; Levin, 2018) or physics (diSessa, 1993), studies involving harder to define concepts such as ideology (Philip, 2011) and pedagogical knowledge (Markauskaite & Goodyear, 2014) have also been approached from a Knowledge in Pieces (KiP) (diSessa, 1993) epistemological perspective. However, examining mathematics teachers' knowledge in use during the act of lesson planning with mathematical action technology (MAT) (Dick & Hollebrands, 2011) has not been explored and may provide some guidance to mathematics teacher educators on how they may structure interventions to develop such knowledge.

Theoretical Framework

Given that this line of research is centered around understanding what resources (broadly speaking) are drawn upon by Prospective Mathematics Teachers (PMTs) at a fine-grained level of analysis, I will bring a KiP epistemological framing to better understand the knowledge resources that PMTs utilize when designing technology-infused curriculum materials for middle school students. Much of the data will be analyzed through the lens of the TPACK framework, but I acknowledge that this may need to be extended to better suit the context of my study. The TPACK framework extends Shulman's (1987) Pedagogical Content Knowledge (PCK) to include technological knowledge and has many connections to Mathematical Knowledge for Teaching (MKT) (Hill, Ball, and Schilling 2008), but connecting KiP to TPACK may help to clarify how TPACK may apply in this context.

Methods

Stemming from some earlier projects that focused on analyzing the PMT-designed or PMT-adapted curriculum, I am now focusing on understanding and describing the nature of what the process of designing MAT-infused curriculum looks like in the moment of planning. With a focus on the knowledge that PMTs demonstrate in the moment of planning lessons that incorporate MAT, I am planning to conduct a Microgenetic Learning Analysis (MLA) (Parnafes & diSessa, 2013). In utilizing MLA, I will engage in many of the same techniques that are utilized in grounded theory (Strauss & Corbin, 1990), but my focus is on knowledge with the goal being to gain insight into the ways in which different knowledge systems interact with each other while the PMTs are working on creating technology-infused lesson materials.

At this point, I have collected data from four interviews with five participants. Each participant is enrolled in a course on incorporating technology into middle school mathematics classes. It is composed of both prospective elementary and secondary teachers. The first interview was focused on reflecting on their first lesson plan that was created for the course. The second interview was a task-based interview involving mathematics content that is relevant to their final lesson plan. This interview was conveyed through the same digital platform that they utilized in their final lesson construction. The third interview was a two-hour long interview where they were in the process of creating their final lesson plan. The fourth interview was a reflection on their experience in the course and their final lesson construction.

The interviews were semi-structured with the exception of the third interview where they were free to create their lesson in a manner that they typically worked. During this interview I asked questions about why they made the curricular decisions that I observed and what they were thinking about. I plan to conduct one more round of interviews to serve as a *member-check* (Creswell & Poth, 2016) interview once some initial data analysis has been completed. In addition to the interview video and audio data, I asked each participant to share their computer screen which I also have recorded. Throughout the semester, I observed every class meeting except for one and recorded observation notes of moments in the class session when the participants were speaking about something that may be relevant to how they may think about constructing a lesson involving technology.

For the analysis, I am beginning to do this with an open consideration of the data where I am “tagging” moments in the video of my interviews where I see knowledge related to Technology, Pedagogy, and Content. From here, I plan to look for themes across the four rounds of interviews and then try to construct a model of how each participant’s knowledge changed over the course of the semester. I hope to be able to construct models of how their thinking changed over as it pertains to constructing lessons with technology.

Contributions of the Study

As technology has become more and more omnipresent in mathematics classrooms there is a need to better understand how PMTs and in-service teachers learn how to effectively integrate technology into their curriculum. Methodologically, few mathematics education studies that examine the in-the-moment knowledge that teachers exhibit as they create lesson materials and furthermore even fewer studies do so with the added complexity of integrating technology. It is my hope that this study provides some guidance for future studies on how teachers develop their knowledge of creating lessons that integrate mathematical action technology. Finally, as the knowledge in pieces epistemological frame has typically been utilized in studies of mathematics content knowledge or science content knowledge, this study will be an attempt at expanding the reach of this perspective to teacher knowledge involved in creating curriculum.

References

Angeli, C., Valanides, N., & Christodoulou, A. (2016). Theoretical considerations of technological pedagogical

content knowledge. Handbook of technological pedagogical content knowledge (TPACK) for educators, 11.
Banilower, E. R., Smith, P. S., Malzahn, K. A., Plumley, C. L., Gordan, E. M., & Hayes, M. L. (2018). Report of the 2018 NSSME+. Chapel Hill, NC: Horizon Research, Inc.
Bibi, S., & Khan, S. H. (2017). TPACK in action: A study of a teacher educator's thoughts when planning to use ICT. Australasian Journal of Educational Technology, 33(4).
Creswell, J. W., & Poth, C. N. (2016). Qualitative inquiry and research design: Choosing among five approaches. Sage publications.
Dick, T. P., & Hollebrands, K. F. (2011). Focus in high school mathematics: Technology to support reasoning and sense making (pp. xi-xvii). Reston, VA: National Council of Teachers of Mathematics.
diSessa, A. A. (1993). Toward an epistemology of physics. Cognition and instruction, 10(2-3), 105-225.
diSessa, A. A. (2018). A friendly introduction to "knowledge in pieces": Modeling types of knowledge and their roles in learning. In Invited lectures from the 13th international congress on mathematical education (pp. 65-84). Springer International Publishing.
Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. Journal for research in mathematics education, 39(4), 372-400.
Izsak, A. (2005). " You have to count the squares": applying knowledge in pieces to learning rectangular area. The Journal of the Learning Sciences, 14(3), 361-403.
Levin, M. (2018). Conceptual and procedural knowledge during strategy construction: A complex knowledge systems perspective. Cognition and Instruction, 36(3), 247-278.
Markauskaite, L., & Goodyear, P. (2014). Tapping into the mental resources of teachers' working knowledge: Insights into the generative power of intuitive pedagogy. Learning, Culture and Social Interaction, 3(4), 237-251.
Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. Teachers college record, 108(6), 1017-1054.
Parnafes, O., & diSessa, A. A. (2013). Microgenetic learning analysis: A methodology for studying knowledge in transition. Human Development, 56(1), 5-37.
Sherman, M., Cayton, C., Walkington, C., & Funsch, A. (2020). An analysis of secondary mathematics textbooks with regard to technology integration. Journal for Research in Mathematics Education, 51(3), 361-374.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational researcher, 15(2), 4-14.
Strauss, A., & Corbin, J. (1990). Basics of qualitative research. Sage publications.